SOME FIXED POINT THEOREMS FOR MULTIVALUED WEAKLY INCREASING OPERATORS

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Abstract. In this paper, some common fixed point theorems for a pair of multivalued weakly increasing operators in partially ordered metric space and in partially ordered Banach space are proved.

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1. Introduction

The study on the existence of fixed points for single valued increasing operators is bounteous and successful, the results obtained are widely used to investigate the existence of solutions to the ordinary and partial differential equations (see [7], [8] and reference therein).

It is natural to extend this study to multivalued case. Recently in [5], [8], [9], the authors have verified some fixed point theorems for multivalued increasing operators. Also in [1], the authors proved some fixed point theorems for multivalued weakly uniform increasing operators.

Motivated by [1], [4], [5], [10], [11], [12] and [13], we prove some fixed point theorems for multivalued operators satisfying some weakly increasing properties in metric space and Banach space, respectively.

2. Preliminaries

In this section some notations and preliminaries are given.

Let X be a topological space and $\leq$ be a partial order endowed on X.

In order to define the multivalued increasing and multivalued weakly increasing operators, we give the following relations between two subsets of X.

Definition 1. ([5]) Let A, B be two nonempty subsets of X, the relations between A and B are defined as follows:

(1) If, for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A \prec_1 B$.
(2) If, for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A \prec_2 B$.
(3) If, $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 1. $\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\leq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 2. $\prec_1$, $\prec_2$and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\leq$ be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X.

We can show that $\prec$ is a partial order on $CC(\mathbb{R})$, which denotes the family of closed and convex subsets of $\mathbb{R}$.

Let us recall some basic definitions and facts from multivalued analysis (for details, see [3]).

Let 2^X denote the family of all nonempty subset of X. A multivalued map $T : X \to 2^X$ is called upper semi continuous (u.s.c.) on X if for each $x_0 \in X$, the set Tx_0 is a nonempty, closed subset of X, and if for each open set V of X containing Tx_0, there exists an open neighborhood U of x_0 such that $T(U) \subseteq V$.

T is said to be completely continuous if $T(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map T is completely continuous with nonempty compact values, then T is u.s.c. if and only if T has a closed graph (i.e. $u_n \to u_0$, $v_n \to v_0$, $v_n \in Tu_n$ imply $v_0 \in Tu_0$).

Definition 2. ([5]) If $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ or $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$, then $\{x_n\}$ is called a monotone sequence.

Definition 3. ([5]) A multivalued operator $T : X \to 2^X \backslash \emptyset$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X, u_n \to u_0, v_n \to v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Definition 4. ([5]) A function $f : X \to \mathbb{R}$ is called order upper (lower) semi continuous, if, for monotone sequence $\{u_n\} \subset X$, and $u_0 \in X$

$$u_n \to u_0 \Longrightarrow \limsup_{n\to\infty} f(u_n) \leq f(u_0), \ \ (f(u_0) \leq \liminf f(u_n)).$$

Remark 3. An operator with closed graph must be an order closed operator and an upper (lower) semi continuous function is an order upper (lower) semi continuous. There exist some examples in [5] showing that the converse is not true.

Let X be a real Banach space with a norm $\|\cdot\|_X$. A non-empty closed subset P of X is said to be an order cone in X if

(i) $P + P \subseteq P$,
(ii) $\lambda P \subseteq P$, for $\lambda \geq 0$, and
(iii) $P \cap -P = \{\theta\}$, where θ denotes the zero element of X.

Then, the relation $x \leq y$ if and only if $y - x \in P$ defines the partial ordering in X. P is called regular, if every nondecreasing and bounded above in order sequence in X has a limit, i.e., if $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y(\, y \in X\,)$, then there exists $x \in X$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$. We do not require any property of the cones in the present discussion, however, the details of order cones and their properties may be found in Guo and Lakshmikantham [6].

Now we introduce the following definition.

Definition 5. Let X be a topological space and $\leq$ be a partial order endowed on X. Two maps $S, T : X \to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Sx \prec_1 Ty$ for all $y \in Sx$ and $Tx \prec_1 Sy$ for all $y \in Tx$. Similarly two maps $S, T : X \to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Ty \prec_2 Sx$ for all $y \in Sx$ and $Sy \prec_2 Tx$ for all $y \in Tx$.

Now, we give some examples.

Example 1. Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $S, T : X \to 2^X$ defined by $Sx = [1, x^2]$ and $Tx = [1, 2x]$ for all $x \in X$. Then the pair of mappings S and T are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Ty = [1, 2y] \prec_2 [1, x^2] = Sx \text{ for all } y \in Sx$$

and

$$Sy = [1, y^2] \prec_2 [1, 2x] = Tx \text{ for all } y \in Tx$$

so S and T are weakly increasing with respect to $\prec_2$ but $S2 = [1, 4] \not\prec_1 [1, 2] = T1$ for $1 \in S2$, so S and T are not weakly increasing with respect to $\prec_1$.

Example 2. Let $X = [0, 1]$ and $\leq$ be usual order on X. Consider two mappings $S, T : X \to 2^X$ defined by $Sx = \{0, 1\}$ and $Tx = [x, 1]$ for all $x \in X$. Then the pair of mappings S and T are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Sx = \{0, 1\} \prec_1 [y, 1] = Ty \text{ for all } y \in Sx$$

and

$$Tx = [x, 1] \prec_1 \{0, 1\} = Sy \text{ for all } y \in Tx$$

so S and T are weakly increasing with respect to $\prec_1$ but $T1 = \{1\} \not\prec_2 \{0, 1\} = S1$ for $1 \in S1$, so S and T are are not weakly increasing with respect to $\prec_2$.

3. Fixed point theorems

In this section, we prove some fixed point theorems for multivalued weakly increasing operators with respect to $\prec_1$and $\prec_2$ in partial order metric space and partial order Banach space.

We will use the following lemma in our theorems.

Lemma 1. ([2]) *Let (X, d) be a metric space, $\varphi : X \to \mathbb{R}$. Define the relation $\leq$ on X as follows:*

$$x \leq y \Longleftrightarrow d(x, y) \leq \varphi(x) - \varphi(y).$$

Then $\leq$ is a partial order on X and X is a partial ordered metric space.

Now we give a fixed point theorem.

Theorem 1. *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}$ be a bounded below function, $\leq$ be a partial order induced by φ and $S, T : X \to 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_1$. Then*

(A1) *there exists a monotone sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...,$ such that $\{x_n\}_{n=1}^{\infty}$ converges to $z \in X$, which is a common fixed point of S and T.*

(A2) *if φ is order lower semi continuous on X, then $\{x_n\}_{n=1}^{\infty}$ satisfies $x_n \leq z$, $n = 1, 2, ...$*

Proof. First, we prove (A1). Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again since S and T are weakly increasing with respect to $\prec_1$ we have $x_2 \in Tx_1 \prec_1 Sx_2$. Thus there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies

$$x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$$

By the definition $\leq$, we have

$$\varphi(x_1) \geq \varphi(x_2) \geq \cdots \geq \varphi(x_n) \geq \cdots.$$

Since φ is bounded from below, then $\{\varphi(x_n)\}_{n=1}^{\infty}$ is convergent. Thus, for $\varepsilon > 0$, there exists N, if $n > m > N$ then

$$d(x_m, x_n) \leq \varphi(x_m) - \varphi(x_n) < \varepsilon$$

which indicates $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Due to completeness of X, we know there is $z \in X$, $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Since S and T are order closed, $\{x_n\}_{n=1}^{\infty}$ monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we deduce that $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T.

Next, we verify (A2). If φ is order lower semi continuous on X, then, for any m, we have

$$\begin{aligned} d(x_m, z) &= \lim_{n\to\infty} d(x_m, x_n) \leq \limsup_{n\to\infty}(\varphi(x_m) - \varphi(x_n)) \\ &= \varphi(x_m) - \liminf_{n\to\infty}\varphi(x_n) \leq \varphi(x_m) - \varphi(z) \end{aligned}$$

which indicates $x_m \leq z$.

The proof of the following theorem carries over in the same manner.

Theorem 2. *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}$ be a bounded above function, $\leq$ be a partial order induced by φ and $S, T : X \to 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. Then*

(B1) *there exists a monotone sequence* $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...,$ *such that* $\{x_n\}_{n=1}^{\infty}$ *converges to* $z \in X$, *which is a common fixed point of* S *and* T.

(B2) *if* φ *is order upper semi continuous on* X, *then* $\{x_n\}_{n=1}^{\infty}$ *satisfies* $z \leq x_n$, $n = 1, 2, ...$

Proof. First, we prove (B1). Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since S and T are weakly increasing with respect to $\prec_2$, we have $Tx_1 \prec_2 Sx_0$. Thus there exist some $x_2 \in Tx_1$ such that $x_2 \leq x_1$. Again since S and T are weakly increasing with respect to $\prec_2$ we have $Sx_2 \prec_2 Tx_1$. Thus, there exist some $x_3 \in Sx_2$ such that $x_3 \leq x_2$. Continuing this process, we will get an decreasing sequence $\{x_n\}_{n=1}^{\infty}$, which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$

By the definition $\leq$, we have

$$\varphi(x_1) \leq \varphi(x_2) \leq \cdots \leq \varphi(x_n) \leq \cdots.$$

Since φ is bounded from above, then $\{\varphi(x_n)\}_{n=1}^{\infty}$ is convergent. Thus, for $\varepsilon > 0$, there exists N, if $n > m > N$, then

$$d(x_m, x_n) \leq \varphi(x_n) - \varphi(x_m) < \varepsilon,$$

which indicates $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Due to completeness of X, we know there is $z \in X$, $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Since S and T are order closed, $\{x_n\}_{n=1}^{\infty}$ monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we deduce that $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T.

Next, we verify (B2). If φ is order upper semi continuous on X, then, for any m, we have

$$d(x_m, z) = \lim_{n\to\infty} d(x_m, x_n) \leq \limsup_{n\to\infty}(\varphi(x_n) - \varphi(x_m)) = \varphi(z) - \varphi(x_m),$$

which indicates $z \leq x_m$.

Now, we give some fixed point theorem in ordered Banach space.

Theorem 3. *Let* E *be a Banach space,* $P \subset E$ *be a regular cone,* $\leq$ *be a partial order induced by* P, $X \subset E$ *be a closed subset and* $S, T : X \to 2^X$ *be two order closed with respect to* $\leq$ *and weakly increasing operator with respect to* $\prec_1$. *If the following condition hold:*

(i) *there exists some* $u \in X$ *such that* $Sx \prec_1 \{u\}$ *and* $Tx \prec_1 \{u\}$ *for all* $x \in X$.

Then, there exists a monotone sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...,$ *such that* $\{x_n\}_{n=1}^{\infty}$ *converges to* $z \in X$, *which is a common fixed point of* S *and* T.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now, since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus, there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again, since S and T are weakly increasing with respect to $\prec_1$, we have $x_2 \in Tx_1 \prec_1 Sx_2$.

Thus, there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^{\infty}$, which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$ By (i), and the definition of $\prec_1$, we have $x_n \leq u, n = 1, 2, ...$ Since P is regular and $\{x_n\}_{n=1}^{\infty}$ is nondecreasing and bounded above in order, there is $z \in X$ such that $x_n \to z$ and so $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Since S and T are order closed, $\{x_n\}_{n=1}^{\infty}$ is monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we get $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T.

The following theorem can be similarly verified.

Theorem 4. *Let E be a Banach space, $P \subset E$ be a regular cone, $\leq$ be a partial order induced by P, $X \subset E$ be a closed subset and $S, T : X \to 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. If the following condition hold:*

(ii) *there exists some $u \in X$ such that $\{u\} \prec_2 Sx$ and $\{u\} \prec_2 Tx$ for all $x \in X$.*

Then there exists a monotone sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$, such that $\{x_n\}_{n=1}^{\infty}$ converges to $z \in X$, which is a common fixed point of S and T.

Remark 1. By Theorems 1, 2, 3 and 4, we give some improved version of Theorems 3.2, 3.3, 4.2 and 4.3 of [5], respectively. See, also, the related theorems in [1].

Now, we introduce the following condition.

Condition I. Let E be a Banach space and let $X \subset E$ be a nonempty subset. Two maps $S, T : X \to 2^X$ are said to satisfy Condition I on X if for any countable subset A of X and for any fixed $a \in X$ the condition

$$A \subseteq \{a\} \cup S(A) \cup T(A)$$

implies $\overline{A}$ is compact; here $T(A) = \cup_{x \in A} Tx$.

Now, we give the following theorems.

Theorem 5. *Let E be a Banach space, $P \subset E$ be a cone (not necessary regular), $\leq$ be a partial order induced by P, $X \subset E$ be a closed subset and $S, T : X \to 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_1$. Further, if S and T are satisfying Condition I on X, then there exists a monotone sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$, such that $\{x_n\}_{n=1}^{\infty}$ converges to $z \in X$, which is a common fixed point of S and T.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now, since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus, there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again, since S and T are weakly increasing with respect to $\prec_1$ we have $x_2 \in Tx_1 \prec_1 Sx_2$. Thus, there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...$

Now let $A = \{x_0, x_1, ...\}$. Since A is countable,

$$A \quad = \quad \{x_0\} \cup \{x_1, x_3, ...\} \cup \{x_2, x_4, ...\} \subseteq \{x_0\} \cup S(A) \cup T(A),$$

and S and T are satisfying Condition I, then $\overline{A}$ is compact. Thus, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence which converges to say $z \in X$. However, $\{x_n\}_{n=1}^{\infty}$ is increasing, so the original sequence $\{x_n\}_{n=1}^{\infty}$ converges to $z \in X$. Since S and T are order closed, $\{x_n\}_{n=1}^{\infty}$ is monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we get $z \in Sz$ and $z \in Tz$, i.e., z is a common fixed point of S and T.

The following theorem can be similarly verified.

Theorem 6. *Let E be a Banach space, $P \subset E$ be a cone (not necessary regular), $\leq$ be a partial order induced by P, $X \subset E$ be a closed subset and $S, T : X \to 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. Further, if S and T are satisfy the Condition I on X, then there exists a monotone sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, ...,$ such that $\{x_n\}_{n=1}^{\infty}$ converges to $z \in X$, which is a common fixed point of S and T.*

Remark 2. By Theorems 5 and 6, we give some improved version of Theorem 3.1 of [4]. See, also, the related theorems in [13].

Now, we give some examples.

Example 3. Let $X = [0, 1]$, $d(x, y) = |x - y|$, $\varphi(x) = -x, x, y \in X$, then X is a complete metric space and the order $\leq$ induced by φ:

$$x \leq y \Longleftrightarrow d(x, y) \leq \varphi(x) - \varphi(y).$$

It is easy to know that this partial order is the usual order on X. Define two mappings $S, T : X \to 2^X$ as $Sx = \{0, 1\}$ and $Tx = [\frac{x}{2}, 1]$ for all $x \in X$. It is easy to verify S and T are satisfies the following:

(1) S and T are weakly increasing operator with respect to $\prec_1$,

(2) S and T are order closed on X.

Hence, we know that S and T have a common fixed point on X according to Theorem 1.

Example 4. Let $E = \mathbb{R}^2$, $X = [0, 1] \times [0, 1], P = \{(x, y) \in X : x \geq 0, y \geq 0\}$, then P is a regular cone. For $(x, y) \in X$, defining two multivalued operators $S, T : X \to 2^X$ as follows:

$$\begin{aligned} S(x, y) &= \{(0, 0), (1, 1)\} \text{ and} \\ T(x, y) &= \begin{cases} [0, y] \times [0, y] & \text{if } \ x \leq y \\ [0, x] \times [0, x] & \text{if } \ y < x \end{cases} . \end{aligned}$$

Since S and T are weakly increasing operator with respect to $\prec_2$, order closed with respect to $\leq$ and $\{(0, 0)\} \prec_2 S(x, y)$ and $\{(0, 0)\} \prec_2 T(x, y)$ for all $(x, y) \in X$, then S and T have a common fixed point on X according to Theorem 4.

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