TERNARY SUPERRING

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Abstract. We introduce a new algebraic system, which is called ternary semisuperring. A ternary semisuperring is a nonempty set with a binary hyperoperation's addition and a ternary hyperoperation's multiplication. It is a generalization of ternary semihyperring and semisuperring. We also introduce the notions of fuzzy superideal in ternary semisuperrings and give characterizations of fuzzy superideals.

Keywords: ternary semisuperring; ternary hyperoperation; fuzzy superideal; ternary semihyperring; semisuperring.

1. Introduction

The theory of semiring was first developed by H.S. Vandiver. Semiring constitute a fairly natural generalization of ring, with board applications in the mathematical foundation of computer science. Also, semiring theory has many other applications, for example, automata theory, optimization theory, algebra of formal process, combinatorial optimization, Baysian networks and belief propagation.

The literature of ternary algebric system was introduced by D.H. Lehmer [12] in 1932. He investigated certain ternary semigroup called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to S. Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. Dutta and Kar [8] introduced the notion of ternary semiring which is a generalization of the notion of ternary ring.

Hyperstructure theory was first introduced in 1934 by Marty at the 8th Congress of Scandinavian Mathematicians (see [13]). Later on, hyperstructures have been developed in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [2], [3], [6]. Hyperstructures are generalizations of classic structures. For example, hypergroup [2] is a generalization of group, hyperlattice[11] and superlattice[10] are generalizations of lattice and so on.

The concept of fuzzy sets was first introduced by Zadeh [21] in 1965. The theory of fuzzy sets has been developed fast and has many applications in many branches of sciences. In mathematics, the study of fuzzy algebraic structures was first initiated by pioneer paper of Rosenfeld [14]. He first studied the fuzzy subgroup of a group and since then, many researchers have been engaged in extending the concepts and results of abstract algebra based on fuzzy sets.

In a ternary semihyperring [5], the addition is a hyperoperation and the multiplication is a ternary operation. Of course, it is a generalization of semiring. In this paper, we introduce a new algebraic system, which is called ternary semisuperring. A ternary semisuperring is a nonempty set with a binary hyperoperation's addition and a ternary hyperoperation's multiplication. It is a generalization of ternary semihyperring and semisuperring. We also introduce the notions of fuzzy superideal in ternary semisuperrings and give characterizations of fuzzy superideals.

2. Semiring, semihyperring and semisuperring

A *semiring* [5] is a system consisting of a nonempty set S together with two binary operations on S called *addition* and *multiplication* (denoted in the usual manner) such that

(1) $(S, +)$ is a (commutative) semigroup;

(2) $(S, *)$ is a semigroup;

(3) $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all $a, b, c \in S$.

There are many examples of semirings, for example, $(\mathbf{R}, +, \cdot)$, where $\mathbf{R}$ is the set of all real numbers, $+$ and $\cdot$ is the usual addition and multiplication, is a semiring.

A *left* (*right*) *ideal* of a semiring S is a subset I of S such that

(1) $a + b \in I$ for all $a, b \in I$;

(2) $r * a \in I$ $(a * r \in I)$ for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a subset I of S such that it is both a left and a right ideal of S.

For example, under the usual addition and multiplication, $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of all positive integers, is a left (right) ideal of $\mathbf{R}$, it is also an ideal of $\mathbf{R}$.

A *partial hypergroupoid* $(H,*)$ is a nonempty set H with a function from $H\times H$ to the set of subsets of H. I.e.,

$$\begin{aligned} *: H\times H &\longrightarrow \mathbf{P}(H)\\ (x,y) &\longrightarrow x*y. \end{aligned}$$

A *hypergroupoid* is a nonempty set H, endowed with a hyperoperation, that is a function from $H\times H$ to the set of nonempty subsets of H.

If $A,B\in\mathbf{P}(H)-\{\emptyset\}$, then we define $A*B=\cup\{a*b|a\in A, b\in B\}$, $x*B=\{x\}*B$ and $A*y=A*\{y\}$.

A *semihypergroup* is a hypergroupoid $(H,*)$ which satisfies $a*(b*c)=(a*b)*c$.

A *semihyperring* is algebric system $(S,\circ,+)$, where S is a nonempty set and the followings are satisfied:

(1) $(S,+)$ is a (commutative) semihypergroup;
(2) $(S,\circ)$ is a semigroup;
(3) $a\circ(b+c)=a\circ b+a\circ c$ and $(a+b)\circ c=a\circ c+b\circ c$ for all $a,b,c\in S$.

In [1], Ameri and Hedayati studied this algebric system.

In the previous definition, if the first condition is replaced by $(R,+)$ is a (commutative) semigroup and the second condition is replaced by $(R,\circ)$ is a semihypergroup, we obtain the defintion of a *dual semihyperring*.

The following is the definition of a *semisuperring* (Vougiouklis in [15] and Davvaz in [7] studied this system, and called it a semihyperring).

Definition 1 A *semisuperring* is a system consisting of a nonempty set R together with two binary hyperoperations $+$ and $*$ on R such that

(1) $(R,+)$ is a (commutative) semihypergroup;
(2) $(R,*)$ is a semihypergroup;
(3) $a*(b+c)=a*b+a*c$ and $(a+b)*c=a*c+b*c$ for all $a,b,c\in R$.

3. Ternary semihyperring and ternary semisuperring

Let H be a nonempty set and $P^*(H)$ denote the set of all nonempty subsets of H. A map f from $H\times H\times H$ to $P^*(H)$ is called a *ternary hyperoperation*. The pair (H,f) is called a *ternary hypergroupoid*. If $A,B,C\in P^*(H)$, then we define

$$f(A,B,C)=\bigcup_{a\in A,b\in B,c\in C} f(a,b,c).$$

Definition 2 ([4]) The ternary hypergroupoid (H,f) is called a *ternary semihypergroup* if for any $a,b,c,d,e\in H$, we have

$$f(f(a,b,c),d,e))=f(a,f(b,c,d),e)=f(a,b,f(c,d,e)).$$

Definition 3 ([5]) A nonempty set R together with a binary hyperoperation $+$ and a ternary operation f is said to be a *ternary semihyperring* if $(R,+)$ is a (commutative) semihypergroup, (R,f) is a ternary semigroup, and for any $a,b,c,d,e \in R$, the following conditions are satisfied:

(1) $f(a+b,c,d) = f(a,c,d) + f(b,c,d)$,
(2) $f(a,b+c,d) = f(a,b,d) + f(a,c,d)$,
(3) $f(a,b,c+d) = f(a,b,c) + f(a,b,d)$.

Example 1 ([5]) Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation and a ternary multiplication on $\mathbf{Z}$ in the following way: $x \oplus y = \{x,y\}$ and $f(x,y,z) = x \cdot y \cdot z$, for all $x,y,z \in \mathbf{Z}$. Then $(\mathbf{Z},\oplus,f)$ is a ternary semihypergroup.

Definition 4 A nonempty set R together with a binary hyperoperation $+$ and a ternary hyperoperation f is said to be a *ternary semisuperring* if $(R,+)$ is a (commutative) semihypergroup, (R,f) is a ternary semihypergroup, and for any $a,b,c,d,e \in R$, the following conditions are satisfied:

(1) $f(a+b,c,d) = f(a,c,d) + f(b,c,d)$,
(2) $f(a,b+c,d) = f(a,b,d) + f(a,c,d)$,
(3) $f(a,b,c+d) = f(a,b,c) + f(a,b,d)$.

Obviously, ternary semisuperring is a generalization of ternary semihyperring and semisuperring.

Example 2 Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation and a ternary hyperoperation on $\mathbf{Z}$ in the following way: $x \oplus y = \{x,y\}$ and $f(x,y,z) = \{x,y,z\}$, for any $x,y,z \in \mathbf{Z}$. Then $(\mathbf{Z},\oplus,f)$ is a ternary semisuperring.

Example 3 Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation as in the previous example and a ternary hyperoperation on $\mathbf{Z}$ as $g(x,y,z) = \{n_1x + n_2y + n_3z | n_1,n_2,n_3 \in \mathbf{Z}\}$, for any $x,y,z \in \mathbf{Z}$. Then $(\mathbf{Z},\oplus,g)$ is a ternary semisuperring.

Example 4 Let $\mathbf{I}$ be the real interval $[0,1]$. We define $x \oplus y = \{x,y\}$ and $f(x,y,z) = \{t | \min\{x,y,z\} \le t \le \max\{x,y,z\}\}$, for any $x,y,z \in \mathbf{I}$. Then $(\mathbf{I},\oplus,f)$ is a ternary semisuperring.

Example 5 Let $(R,+,*)$ be a semisuperring, we can define a ternary semisuperring $(R,+,f)$ where $f(x,y,z) = x * y * z$ for all $x,y,z \in R$.

Definition 5 Let $(R,+,*)$ be a ternary semisuperring. A is a nonempty subset of R. $(A,+,*)$ is called a ternary sub-semisuperring of R if $a+b \subseteq A$ and $f(a,b,c) \subseteq A$, for all $a,b,c \in A$.

Definition 6 $(I, +, *)$ is a ternary sub-semisuperring of a ternary semisuperring $(R, +, *)$. I is called a

(1) *left superideal* of R if $f(a, b, i) \subseteq I$, for all $a, b \in R$ and $i \in I$,
(2) *right superideal* of R if $f(i, a, b) \subseteq I$, for all $a, b \in R$ and $i \in I$,
(3) *lateral superideal* of R if $f(a, i, b) \subseteq I$, for all $a, b \in R$ and $i \in I$.

If I is both a left and a right superideal of R, then I is called a *two-sided superideal* of R. I is called a *superideal* if it is a left, right and a lateral superideal of R.

4. Fuzzy ternary semisuperring

Let X be a non-empty set. A map $\mu : X \to [0, 1]$ is called a *fuzzy subset* of X. If μ is a fuzzy subset of X, then for any $t \in [0, 1]$, the set $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

For any $a, b \in [0, 1]$, we will use $a \wedge b$ to denote $\min\{a, b\}$ and use $a \vee b$ to denote $\max\{a, b\}$.

Definition 7 A fuzzy subset μ of a ternary semisuperring $(R, +, *)$ is called a fuzzy ternary semisuperring of R if for any $x, y, z \in R$ we have:

(1) $\mu(x) \wedge \mu(y) \leq \inf\limits_{z \in x+y} \mu(z)$;
(2) $\mu(x) \wedge \mu(y) \wedge \mu(z) \leq \inf\limits_{z \in f(x,y,z)} \mu(z)$.

Definition 8 A fuzzy subset μ of a ternary semisuperring $(R, +, *)$ is called a fuzzy superideal of R if for any $x, y, z, v \in R$ we have:

(1) $\mu(x) \wedge \mu(y) \leq \inf\limits_{z \in x+y} \mu(z)$;
(2) $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf\limits_{v \in f(x,y,z)} \mu(v)$.

Theorem 1 *A fuzzy subset μ of a ternary semisuperring $(R, +, f)$ is a fuzzy superideal if and only if each its nonempty level subset is a superideal of R.*

Proof. Let μ be a fuzzy superideal of a ternary semisuperring $(R, +, f)$. If $x, y \in \mu_t$ for some $t \in [0, 1]$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus

$$t \leq \mu(x) \wedge \mu(y) \leq \inf_{z \in x+y} \mu(z),$$

which implies that $\mu(z) \geq t$ for any $z \in x + y$. So $x + y \subseteq \mu_t$.

Suppose that $x, y, z \in R$ and $x \in \mu_t$. Then $\mu(x) \geq t$. And from $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf\limits_{v \in f(x,y,z)} \mu(v)$ we know that $t \leq \mu(x) \leq \inf\limits_{v \in f(x,y,z)} \mu(v)$, which implies that $f(x, y, z) \subseteq \mu_t$. Similarly, we can obtain that $f(y, x, z) \subseteq \mu_t$ and $f(y, z, x) \subseteq \mu_t$. Hence, μ_t is a superideal of R.

Conversely, suppose that every nonempty level subset μ_t is a superideal of R. Let $t_0 = \mu(x) \wedge \mu(y)$ for any $x, y \in R$. Then $x, y \in \mu_{t_0}$, consequently, $x + y \subseteq \mu_{t_0}$. Thus

$$\mu(x) \wedge \mu(y) = t_0 \leq \inf_{v \in x+y} \mu(v).$$

Assume that $\mu(x) = t_1$. Then $x \in \mu_{t_1}$. Since μ_{t_1} is a superideal of R, we have that $f(x, y, z) \subseteq \mu_{t_1}$. Hence, $\mu(x) = t_1 \leq \inf_{v \in f(x,y,z)} \mu(v)$. Similarly, we obtain that $\mu(y) \leq \inf_{v \in f(x,y,z)} \mu(v)$ and $\mu(z) \leq \inf_{v \in f(x,y,z)} \mu(v)$. Thus $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf_{v \in f(x,y,z)} \mu(v)$.

This completes the proof. ∎

Yuan [20] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds λ and μ is also called a (λ, μ)-fuzzy subgroup.

Yao continued to research (λ, μ)-fuzzy normal subgroups, (λ, μ)-fuzzy quotient subgroups and (λ, μ)-fuzzy subrings in [16, 17, 18, 19]. Feng [9] investigated (λ, μ)-fuzzy sublattices and (λ, μ)-fuzzy subhyperlattices. Davvaz studied ternary semihyperring with thresholds in [5].

In the following, we study fuzzy superideal with thresholds of a ternary semisuperring.

Definition 9 Let $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a ternary semisuperring $(R, +, f)$. Then μ is called a *fuzzy superideal with thresholds* of R, if

(1) $\mu(x) \wedge \mu(y) \wedge \beta \leq \inf_{z \in x+y} \mu(z) \vee \alpha$;

(2) $(\mu(x) \vee \mu(y) \vee \mu(z)) \wedge \beta \leq \inf_{v \in f(x,y,z)} \mu(v) \vee \alpha$.

Theorem 2 *A fuzzy subset μ of a ternary semisuperring $(R, +, f)$ is a fuzzy superideal with thresholds (α, β) of R if and only if $\mu_t (\neq \emptyset)$ is a superideal of R for all $t \in (\alpha, \beta)$.*

Proof. Similar to the proof of Theorem 4.14 of [5]. ∎

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