ON HOLOMORPHICALLY DECOMPOSABLE FREDHOLM OPERATORS

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Abstract. Let X be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X, T is said decomposably Fredholm if there exists an Fredholm operator S such that $TST = T$.

In this paper we consider the subset $\rho_{hF}(T)$ of $\mathbb{C}$: $\mu_0 \in \rho_{hF}(T)$ if and only if there is a neighbourhood U_0 of μ_0 and an analytic function $W : U_0 \to \mathcal{B}(X)$ such that $W(\lambda)$ is Fredholm and $(T-\lambda I)W(\lambda)(T-\lambda I) = T-\lambda I$. for all $\lambda \in U_0$, consequently we define a new spectrum $\sigma_{hF}(T) = \mathbb{C} \setminus \rho_{hF}(T)$ said holomorphically decomposable Fredholm spectrum of T. We prove that $\sigma_{hF}(T)$ is a non-empty compact subset of the classical spectrum $\sigma(T)$, and not satisfies the spectral mapping theorem.

Keywords: Fredholm operator, Kato spectrum, holomorphically Decomposable Fredholm operator.

1. Introduction and terminology

Throughout this paper, X will denote an infinite-dimensional complex Banach space. By $\mathcal{B}(X)$ we denote the Banach algebra of all bounded linear operators on X. Let $T \in \mathcal{B}(X)$. By $N(T)$, $T(X)$, $T^\infty(X)$, $\sigma(T)$, $\rho(T)$, $\rho_K(T)$, and $\sigma_K(T)$, we denote respectively the kernel, range, hyper-range, spectrum, resolvent set, Kato resolvent set, and Kato spectrum of T. Moreover, these sets are defined by:

$$\begin{aligned}
N(T) &= \{u \in X / Tu = 0\} \\
T(X) &= \{Tu / u \in X\} \\
T^\infty(X) &= \bigcap_{k\geq 1} T^k(X) \\
\sigma(T) &= \{\lambda \in \mathbb{C} / T-\lambda I \text{ is not invertible on } \mathcal{B}(X)\} \\
\rho(T) &= \mathbb{C} \setminus \sigma(T) \\
\rho_K(T) &= \{\lambda \in \mathbb{C} / (T-\lambda I)(X) \text{ is closed and } N(T-\lambda I) \subset (T-\lambda I)^\infty(X)\} \\
\sigma_K(T) &= \mathbb{C} \setminus \rho_K(T)
\end{aligned}$$

Recall that $T \in \mathcal{B}(X)$ is said to be relatively regular if there exists an operator $S \in \mathcal{B}(X)$ for which $TST = T$; S is called a pseudo-inverse of T, the necessary and sufficient condition for which T is relatively regular if and only if $N(T)$ and $T(X)$ are the complemented in X, see [1], [2], [5] and [8].

Denote by $\alpha(T)$ the dimension of $N(T)$, and by $\beta(T)$ the codimension of the range $T(X)$. Next define the following classes of operators by:

$$\Phi(X) = \{T \in \mathcal{B}(X) : \alpha(T) \text{ and } \beta(T) \text{ are finite}\}$$

the class of all Fredholm operators, where $\text{ind}(T) := \alpha(T) - \beta(T)\}$ is the index of $T \in \Phi(X)$;

$$\mathcal{R}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{B}(X) \text{ such that } TST = T\}$$

the class of all relatively regular operators;

$$\mathcal{J} = \{T \in \mathcal{B}(X) : T \text{ invertible}\}$$

the group of invertible operators;

$$\mathcal{J}\mathcal{R}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{J}, \text{ such that } TST = T\}$$

the class of all decomposably regular operators;

$$\Phi\mathcal{R}(X) = \{T \in \mathcal{B}(X) : TST = T \text{ for some } S \in \Phi(X)\}$$

the class of all decomposably Fredholm operators, (see Definition 2.1, below).

Note that $\mathcal{J}, \Phi(X)$ are open subsets of $\mathcal{B}(X)$ but $\mathcal{R}(X), \mathcal{J}\mathcal{R}(X)$ and $\Phi\mathcal{R}(X)$ are neither open nor closed, see [10].

Recently, Schmoeger [10] introduced and studied the holomorphically decomposably regular resolvent set of $T \in \mathcal{B}(X)$, $\rho_{gr}(T)$, where:

(∗) $\rho_{gr}(T) = \{\lambda \in \mathbb{C} : \exists U$, a neighbourhood of λ, $\exists F : U \to \mathcal{B}(X)$ analytic such that $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$, and $F(\lambda) \in \mathcal{J}$, $\forall \lambda \in U.\}$.

The holomorphically decomposably regular spectrum of T is defined by

$$\sigma_{gr}(T) = \mathbb{C} \setminus \rho_{g.r}(T).$$

In [10], the author proved that $\sigma_{gr}(T)$ is a non empty compact set and does not satisfy the spectral mapping theorem, [10, Proposition 3.3 and Theorem 3.6]. We do have, however, $\sigma_{gr}(f(T)) \subseteq f(\sigma_{gr}(T))$ for $f \in \mathcal{H}(\sigma(T))$, see Example 3.4 in [10], for $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ is the set of all complex-valued functions which are analytic in some neighbourhood of $\sigma(T)$, and $f(T) \in \mathcal{B}(X)$ defined by Riesz-Dunford functional calculus, see [1], [5] and [8]. Notice that $\mathcal{J} \subsetneq \Phi(X)$. Hence, it is natural to see what happened if we replace $\mathcal{J}$ by $\Phi(X)$ in (*). To this end, let

(∗) $\rho_{hF}(T) = \{\lambda \in \mathbb{C}/\exists U$, a neighbourhood of λ, $\exists F : U \to \mathcal{B}(X)$ analytic such that $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$, and $F(\lambda) \in \Phi(X)$ for all $\lambda \in U\}$.

It is clear that $\rho_{gr}(T) \subset \rho_{hF}(T)$; and setting $\sigma_{hF}(T) = \mathbb{C} \setminus \rho_{hF}(T)$, it follows that the new spectrum set satisfies $\sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T)$ (see Definition 2.2. below).

Our aim in this paper is to tackle the following problem:

Let $T \in \mathcal{B}(X)$ and consider the part $\sigma_{hF}(T)$ of the spectrum of T.
1. *Does $\sigma_{hF}(T)$ a non-empty compact subset of $\mathbb{C}$?*
2. *Does $\sigma_{hF}(T)$ satisfy the spectrum mapping theorem?*

We answer positively the first question and negatively the second.

In Section 2, we present the class of holomorphically decomposable Fredholm operators: we show that $\sigma_{hF}(T)$ is a non-empty compact subset of the spectrum, and we characterize the set $\rho_{hF}(T)$ (see Proposition 2.1).

In Section 3, we show that $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$ for every $f \in \mathcal{H}(\sigma(T))$ (Theorem 3.1.) and if f is univalent then this inclusion is indeed an equality (Theorem 3.2.).

Finally, Example 1 shows that the inclusion may be proper and conclude that $\sigma_{hF}(T)$ does not satisfy the spectral mapping theorem.

2. Class of holomorphically decomposable Fredholm operators

We first begin with some definitions and examples.

Definition 2.1 Let $T \in \mathcal{B}(X)$, T is said to be decomposably Fredholm if $TST = T$ for some $S \in \Phi(X)$.

Let the set of decomposably Fredholm operators be denoted by $\Phi\mathcal{R}(X)$.

Examples. We give some classes of $\Phi\mathcal{R}(X)$.

1. If $T \in \mathcal{JR}(X)$, then T is decomposably regular and there exists $S \in \mathcal{B}(X)$ invertible such that $TST = T$, clearly $S \in \Phi(X)$; in particular, $\mathcal{JR}(X) \subset \Phi\mathcal{R}(X)$.

2. For $n \in \mathbb{Z}$, denote by $\Phi_n(X) = \{T \in \Phi(X) : \operatorname{ind}(T) = \alpha(T) - \beta(T) = n\}$ and $\Phi_n\mathcal{R}(X) = \{T \in \mathcal{B}(X) : TST = T \text{ such that } S \in \Phi_n(X)\}$, then $\Phi\mathcal{R}(X) = \bigcup_{n\in\mathbb{Z}} \Phi_n\mathcal{R}(X)$ (see [10]).

3. Rakocevic in [9, Theorem 3] shows that $\Phi\mathcal{R}(X) = \mathcal{R}(X) \cap \overline{\Phi(X)}$.

4. Schmoeger in [10] proved that $\overset{\circ}{\Phi_n\mathcal{R}(X)} = \Phi_{-n}(X)$ and $\overset{\circ}{\Phi\mathcal{R}(X)} = \Phi(X)$.

5. If T is generalized Fredholm operator, i.e., if T is relatively regular and $(I - ST - TS) \in \Phi(X)$ for some pseudo-inverse S of T [15], we have $T \in \overline{\Phi(X)}$, and consequently $T \in \Phi\mathcal{R}(X)$.

Definition 2.2 Let $T \in \mathcal{B}(X)$, the holomorphically decomposable Fredholm resolvent set of T is defined by:

$\rho_{hF}(T) = \{\lambda \in \mathbb{C} : \exists U$, a neighbourhood of λ, $\exists F : U \to \mathcal{B}(X)$ analytic such that $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$ and $F(\lambda) \in \Phi(X)$ for all $\lambda \in U\}$.

The holomorphically decomposable Fredholm spectrum of T is defined by

$$\sigma_{hF}(T) := \mathbb{C} \setminus \rho_{hF}(T).$$

An operator $T \in \mathcal{B}(X)$ for which $0 \in \rho_{hF}(T)$ is called a holomorphically decomposable Fredholm operator.

Obviously,

$$\rho(T) \subset \rho_{gr}(T) \subset \rho_{hF}(T) \quad \text{and} \quad \sigma_{hF}(T) \subset \sigma_{gr}(T) \subset \sigma(T).$$

Remarks.

1. Let $0 \in \rho_K(T)$. Then T is said Kato operator (or semi-regular/singular operator) see [1], [4], [5], [7], [8]) and, if T is relatively regular, then T is called Saphar operator. We denote the Saphar resolvent set by $\rho_{rr}(T) = \{\lambda \in \rho_K(T)/T - \lambda I \in \mathcal{R}(X)\}$ and the Saphar spectrum $\sigma_{rr}(T) := \mathbb{C} \setminus \rho_{rr}(T)$.

   It is well known that (see [6], [8], [12]):

   $\lambda_0 \in \rho_{rr}(T) \Leftrightarrow \exists U$, a neighbourhood of λ, $\exists F : U \to \mathcal{B}(X)$ such that F is analytic and $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I, \forall \lambda \in U$.

   Consequently, if $\lambda_0 \in \rho_{hF}(T)$, then $\lambda_0 \in \rho_{rr}(T)$, we conclude that:

   $$\rho(T) \subseteq \rho_{gr}(T) \subseteq \rho_{hF}(T) \subseteq \rho_{rr}(T) \subseteq \rho_K(T) \text{ and}$$
   $$\sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T).$$

2. In [13] Schmoeger proved that $\partial\sigma(T) \subseteq \sigma_K(T)$, see also [1], [5], [7] and [8]. Hence, it follows that $\partial\sigma(T) \subseteq \sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T)$.

   In particular, $\sigma_{hF}(T)$ is a non empty compact subset of $\sigma(T)$.

3. If X is a Hilbert space, then T is relatively regular if and only if $T(X)$ is closed, in this special case $\rho_{rr}(T) = \rho_K(T)$ and $\sigma_K(T) = \sigma_{rr}(T) = \sigma_g(T)$ where $\sigma_g(T)$ is the generalized spectrum introduced by Mbekhta in [6], see also [1], [5], [7] and [8].

In general, $\sigma_K(T) \subsetneqq \sigma_{rr}(T)$ (see [14]) and, from the strict inclusion $\mathcal{J} \subsetneqq \Phi(X)$, we conclude that $\sigma_{hF}(T) \subsetneqq \sigma_{gr}(T)$.

Proposition 2.1 *For an operator $T \in \mathcal{B}(X)$ the following statements are equivalent:*

(i) $\mu_0 \in \rho_{hF}(T)$

(ii) $\mu_0 \in \rho_K(T)$ *and* $T - \mu_0 I \in \Phi\mathcal{R}(X)$.

Proof. Suppose first that $\mu_0 \in \rho_{hF}(T)$, then by definition there exists a neighborhood U_0 of μ_0 and analytic function $W : U_0 \to \mathcal{B}(X)$ satisfying the equation of pseudo-inverse of $T - \lambda I$ which $W(\lambda)$ is Fredholm. For all $\lambda \in U_0$, we obtain $\mu_0 \in \rho_{rr}(T) \subset \rho_K(T)$ and since $(T - \mu_0 I)W(\mu_0)(T - \mu_0 I) = T - \mu_0 I$, we conclude that $T - \mu_0 I \in \Phi\mathcal{R}(X)$.

Conversely, assume that $\mu_0 \in \rho_K(T)$ and $T - \mu_0 I \in \Phi\mathcal{R}(X)$. By replacing T with $T - \mu_0 I$ we may assume without loss of generality $\mu_0 = 0$ and T $\in \Phi\mathcal{R}(X)$. Since $T \in \Phi\mathcal{R}(X)$ then there is $S \in \mathcal{B}(X)$ that $TST = T$ and S is Fredholm. Next, we consider the function

$$\begin{array}{rcl} W : \mathbb{D}(0, \|S\|^{-1}) & \to & \mathcal{B}(X) \\ \lambda & \mapsto & W(\lambda) \end{array}$$

where $\mathbb{D}(0, \|S\|^{-1})$ is the open disc centred at 0 and radius $\|S\|^{-1} > 0$, defined by $W(\lambda) := (I - \lambda S)^{-1}S = S(I - \lambda S)^{-1}$ for all $\lambda \in \mathbb{D}(0, \|S\|^{-1})$, then, by Corollary 1.5 of [12], W is analytic and verifies the equation of pseudo-inverse $(T - \lambda I)W(\lambda)(T - \lambda I) = T - \lambda I$ for every $\lambda \in \mathbb{D}(0, \|S\|^{-1})$. On the other hand, since S is Fredholm and $(I - \lambda S)^{-1}$ is invertible, we have that $(I - \lambda I)^{-1}S$ is Fredholm, consequently, $0 \in \rho_{hF}(T)$; this completes the proof of the theorem. ∎

Let $T \in \mathcal{B}(X)$ and $g \in \mathcal{H}(\sigma(T))$, conditions on $g(T)$ to be element of $\Phi\mathcal{R}(X)$ are given by Schmoeger in Proposition 3.8. of [10].

Proposition 2.2 [10, Proposition 3.8] *Assume the following conditions:*

a. $T \in \mathcal{B}(X)$ *and* $g \in \mathcal{H}(\sigma(T))$ *has only a finite number of zeros in* $\sigma(T)$

b. $\mu_1, ..., \mu_m$ *are the zeros of* g *in* $\sigma(T)$ *with respective orders of multiplicity,* $n_1, ..., n_m$ ($\mu_i \neq \mu_j$ *for* $i \neq j$)

c. $(T - \mu_j I)^{n_j} \in \Phi\mathcal{R}(X)$ *for* $j = 1, ..., m$, *then* $g(T) \in \Phi\mathcal{R}(X)$. *To be more precise, if* $(T - \mu_j I)^{n_j} \in \Phi_{k_j}\mathcal{R}(X)$ *and* $k = \sum_{i=1}^{m} k_i$.

Then, $g(T) \in \Phi_k\mathcal{R}(X)$.

3. Spectral Mapping Theorem

We now discuss the spectral theorem for $\sigma_{hF}(T)$, of $T \in \mathcal{B}(X)$. Generally, the following theorem holds.

Theorem 3.1 *Let* $T \in \mathcal{B}(X)$, *and* $f \in \mathcal{H}(\sigma(T))$. *We have:* $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$.

Proof. Suppose $\lambda_0 \notin f(\sigma_{hF}(T))$, we have to show $\lambda_0 \in \rho_{hF}(f(T))$. Let us consider $g(z) = f(z) - \lambda_0$, g is analytic and $g \in \mathcal{H}(\sigma(T))$, since

$$(*) \qquad 0 \notin g(\sigma_{hF}(T)).$$

Since $\sigma_K(T) \subset \sigma_{hF}(T)$, it follows that $0 \notin g(\sigma_K(T))$; by the spectral mapping theorem for the Kato spectrum we have

$$(**) \qquad 0 \notin g(\sigma_K(T)) = \sigma_K(g(T)).$$

Case 1. Assume that g does not vanish in $\sigma(T)$; in this case $g(T) = f(T) - \lambda_0 I$ is invertible, we then obtain $\lambda_0 \in \rho(f(T)) \subset \rho_{hF}(f(T))$.
Case 2. Assume that g has zeros in $\sigma(T)$: Since $\sigma_K(T) \subset \sigma(T)$ and $(**)$, by Proposition 3 [13], g may have only a finite number of zeros in $\sigma(T)$, say $\mu_1, ..., \mu_m$, where $\mu_i \neq \mu_j$ for $i \neq j$, and $n_i \geq 0$ denote the order of multiplicity of μ_i.

By $(*)$ we have $\mu_i \notin \sigma_{hF}(T), \forall i = 1, ..., m$, and then $\mu_i \in \rho_{hF}(T)$ for all i. Consequently, there exists $W_i(\mu_i) \in \Phi_{k_i}(X)$ such that

$$(T - \mu_i I)W_i^{n_i}(T - \mu_i I) = T - \mu_i I, \text{ or } \rho_{hF}(T) \subset \rho_{rr}(T),$$

we also have $T - \mu_i I$ Saphar having $W_i(\mu_i)$ as a pseudo-inverse, and by Lemma 5-c [11], we have

$$(T - \mu_i I)^{n_i} W_i^{n_i}(\mu_i)(T - \mu_i I)^{n_i} = (T - \mu_i I)^{n_i},$$

since $W_i^{n_i}(\mu_i) \in \Phi_{n_i k_i}(X)$, we obtain $(T - \mu_i I)^{n_i} \in \Phi_{n_i k_i}\mathcal{R}(X) \subset \Phi\mathcal{R}(X)$.
We have:

(1) $g \in \mathcal{H}(\sigma(T))$ has only a finite number of zeros in $\sigma(T)$.

(2) $\mu_1, ...\mu_m$ are the zeros of g in $\sigma(T)$ with respective orders of multiplicity, $n_1, ..., n_m$ ($\mu_i \neq \mu_j$ for $i \neq j$)

(3) $(T - \mu_i I)^{n_i} \in \Phi\mathcal{R}(X)$ for $i = 1, ..., m$.

Then, by Proposition 2.2, $g(T) \in \Phi\mathcal{R}(X)$; on the other hand, by $(**)$, $0 \notin \sigma_K(g(T))$, we have $0 \in \rho_K(g(T))$.
We conclude that $0 \in \rho_K(g(T))$ and $g(T) \in \Phi\mathcal{R}(X)$.
Consequently, by Proposition 2.1, $0 \in \rho_{hF}(g(T))$ and $\lambda_0 \in \rho_{hF}(f(T))$, hence

$$\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T)).$$

∎

Theorem 3.2 *Let $T \in \mathcal{B}(X)$, then $sigma_{hF}(f(T)) = f(\sigma_{hF}(T))$, for every $f \in \mathcal{H}(\sigma(T))$, injective.*

Proof. By Theorem 3.1, we have $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$, and hence it suffices to prove that $f(\sigma_{hF}(T)) \subset \sigma_{hF}(f(T))$.
To see this, let $\lambda_0 \in f(\sigma_{hF}(T))$ and suppose that $\lambda_0 \notin \sigma_{hF}(f(T))$.
Hence, $\lambda_0 \in \rho_{hF}(f(T)) \subset \rho_K(f(T))$, by spectral theorem for $\sigma_K(T)$ we have $\lambda_0 \in f(\rho_K(T))$, and there exists $\mu_0 \in \rho_K(T)$ $(*)$ such that $f(\mu_0) = \lambda_0$.
Define $h(z) = f(z) - \lambda_0$, then $h(\lambda_0) = 0$ and $h \in \mathcal{H}(\sigma(T))$.

Consider

$$g(z) := \begin{cases} \dfrac{h(z)}{z-\mu_0} & \text{if } z \neq \mu_0, \\ f'(\mu_0) & \text{if } z = \mu_0. \end{cases}$$

Then $h(z) = (z-\mu_0)g(z)$ and $g \in \mathcal{H}(\sigma(T))$ such that $g(z) \neq 0$ *for all* $z \in \sigma(T)$ since f is injective.

We have $h(T) = (T-\mu_0 I)g(T) = g(T)(T-\mu_0 I)$ and $g(T)$ is invertible. By $\lambda_0 \in \rho_{hF}(f(T))$ we have $0 \in \rho_{hF}(h(T))$ and there exists a Fredholm operator $W(0)$ such that $h(T)W(0)h(T) = h(T)$.

Consequently, $(T-\mu_0 I)g(T)W(0)g(T)(T-\mu_0 I) = (T-\mu_0 I)g(T)$.

From invertibility of $g(T)$, we obtain that $(T-\mu_0 I)g(T)W(0)(T-\mu_0) = T-\mu_0$, and, since $(T-\mu_0 I) \in \Phi\mathcal{R}(X)$, $(g(T)W(0)$ is Fredholm, since $g(T)$ invertible and $W(0)$ Fredholm).

From $(*)$ and $(T-\mu_0 I) \in \Phi\mathcal{R}(X)$, we obtain, by Proposition 2.1, $\mu_0 \in \rho_{hF}(T)$; hence, $\mu_0 \notin \sigma_{hF}(T)$ and $\lambda_0 = f(\mu_0) \notin f(\sigma_{hF}(T))$, which contradicts the assumption $\lambda_0 \in f(\sigma_{hF}(T))$. Therefore, $f(\sigma_{hF}(T)) \subset \sigma_{hF}(f(T))$ is proved. This completes the proof of the theorem. ∎

In the following example, we find an operator T, and $f \in \mathcal{H}(\sigma(T))$ which f is not injective, and $\sigma_{hF}(f(T)) \neq f(\sigma_{hF}(T))$.

Example 1 Denote by $\mathcal{K}(X) = \{T \in \mathcal{B}(X) : T(X) \text{ is closed and } N(T) \subset T^{\infty}(X)\}$, then

$$T \in \mathcal{K}(X) \Leftrightarrow 0 \in \rho_K(T).$$

On the other hand, we consider

$$\Phi\mathcal{R}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{B}(X),\ TST = T \text{ such that } S \in \Phi(X)\}.$$

Let $T \in \mathcal{K}(X) \setminus \Phi\mathcal{R}(X)$ satisfying there exists $k_0 \geq 2$ such that $T^{k_0} \in \Phi\mathcal{R}(X)$ or, equivalently, $T^{k_0}ST^{k_0} = T^{k_0}$, $k_0 \geq 2$ for S Fredholm.

Notice that, by $T \in \mathcal{K}(X) \setminus \Phi\mathcal{R}(X)$ we have $0 \in \rho_K(T)$ and $T \notin \Phi\mathcal{R}(X)$.

This implies, by Proposition 2.1, that $0 \notin \rho_{hF}(T)$ i.e.,

$$(*) \qquad 0 \in \sigma_{hF}(T).$$

Let $\mu_0 \notin \sigma(T)$, then $\mu_0 \neq 0$ (since $\sigma_{hF}(T) \subset \sigma(T)$). Let us show that

$$T^{k_0}(T-\mu_0 I) \in \Phi\mathcal{R}(X).$$

$T^{k_0} \in \Phi\mathcal{R}(X)$ implies that there exists $S \in \Phi(X)$, such that $T^{k_0}ST^{k_0} = T^{k_0}$. Since T^{k_0} and $T-\mu_0 I$ commute, we have $T^{k_0}(T-\mu_0)ST^{k_0} = T^{k_0}(T-\mu_0 I)$. Put $Q = S(T-\mu_0 I)^{-1}$. Then Q is Fredholm and $T^{k_0}(T-\mu_0 I)QT^{k_0}(T-\mu_0 I) = T^{k_0}(T-\mu_0 I)$. Thus,

$$T^{k_0}(T-\mu_0 I) \in \Phi\mathcal{R}(X).$$

Let $P(z) = z^{k_0}(z-\mu_0)$, then P is not injective and $P(T) = T^{k_0}(T-\mu_0)$, $P(T) \in \Phi\mathcal{R}(X)$. Since $0 \in \rho_K(T)$, and $P(0) = 0$, by the spectral mapping theorem of $\sigma_K(T)$, we have $0 \in \rho_K(P(T))$. From $0 \in \rho_K(P(T))$ and since $P(T) \in \Phi\mathcal{R}(X)$. We have, by Proposition 2.1,

$$(**) \qquad 0 \in \rho_{hF}(P(T)).$$

Next, if we assume that $P(\sigma_{hF}(T)) \subset \sigma_{hF}(P(T))$, by $(*)$ we have $0 \in \sigma_{hF}(T)$, and, consequently, $0 \in \sigma_{hF}(P(T))$, which contradicts $(**)$. Thus, $P(\sigma_{hF}(T)) \not\subset \sigma_{hF}(P(T))$. Therefore, $\sigma_{hF}(T)$ does not satisfy the spectral mapping theorem.

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