LOCALIZED NEARLY m-EMBEDDED PROPERTY OF SOME SUBGROUPS OF FINITE GROUPS[1]

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Abstract. Let A be a subgroup of a finite group G and $\Sigma : G_0 \leq G_1 \leq \cdots \leq G_n$ some subgroup series of G. Suppose that for each pair (K, H) such that K is a maximal subgroup of H and $G_{i-1} \leq K < H \leq G_i$, for some i, either $A \cap H = A \cap K$ or $AH = AK$. Then A is said to be Σ-embedded in G; A is said to be nearly m-embedded in G if G has a subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$. In this paper, we localize the above conditions in the G-normalizer of Sylow subgroups of the group G. Some new characterizations of some classes of finite groups are given.

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1. Introduction

All groups considered in this paper will be finite and G stands for a finite group. Let $\pi(G)$ stand for the set of all prime divisors of the order of G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ denote the class of supersolvable groups. The other notation and terminology are standard(see [6]).

Let A be a subgroup of G, $K \leq H \leq G$ and p a prime. Then we say:

(i) A covers the pair (K, H) if $AH = AK$;

(ii) A avoids (K, H) if $A \cap H = A \cap K$.

(K, H) is said to be maximal pair of G if K is a maximal subgroup of H. Now, the authors in [4] introduced the following concepts.

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Definition 1.1 Let A be a subgroup of G and $\Sigma = \{G_0 \leq G_1 \leq \cdots \leq G_n\}$ some subgroup series of G. Then we say that A is Σ-embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$, for some i.

Definition 1.2 Let A be a subgroup of G. We say that

(1) A is m-embedded in G if G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$.

(2) A is nearly m-embedded in G if G has a subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$.

In finite groups, the localized subgroups of a group play an important role in the structure of groups. A question of particular interest is the influence of the properties of the normalizers of the Sylow subgroups on the structure of the group. A nice example is Burnside′s Theorem.

Theorem 1.1 (Burnside) *Let P be a Sylow p-subgroup of G. If $N_G(P) = C_G(P)$, then G is p-nilpotent.*

The following extension of Burnside′s Theorem due to Hall is also interesting.

Theorem 1.2 ([5]) *Let P be a Sylow p-subgroup of G. If p'-elements of $N_G(P)$ commute with the elements of P and the nilpotency class of P is less than p, then G is p-nilpotent.*

Wielandt, Ballester-Bolinches and Esteban-Romero proved the following results respectively.

Theorem 1.3 ([7]) *A group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Theorem 1.4 ([1]) *A group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

The main purpose of the present paper is to investigate the structure of the group G that maximal subgroups of Sylow subgroups are nearly m-embedded in G-normalizer.

2. Preliminary results

Lemma 2.1 ([4, Lemma 2.3]) *Let $M \leq G$, N and R be normal subgroups of G. Then*

(a) *If $E \leq V$ and M is $\{E \leq G\}$-embedded in G, then $M \cap V$ is $\{E \leq V\}$-embedded in V.*

(b) *If $R \leq N$ and M is $\{R \leq G\}$-embedded in G, then NM is $\{R \leq G\}$-embedded in G and NM/N is $\{1 \leq G/N\}$-embedded in G/N.*

Lemma 2.2 ([4, Lemma 2.13]) *Let U be a nearly m-embedded subgroup of G and N a normal subgroup of G. Then:*

(1) *If $U \leq H \leq G$, then U is nearly m-embedded in H.*

(2) *If $N \leq U$, then U/N is nearly m-embedded in G/N.*

(3) *Let π be a set of primes, U a π-subgroup and N a π'-subgroup. Then UN/N is nearly m-embedded in G/N.*

Lemma 2.3 ([9, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.*

Lemma 2.4 ([4, Theorem 4.2]) *Let G be a group and p a prime dividing of G such that $(|G|, p-1) = 1$. If every maximal subgroup of a Sylow p-subgroup P of G is nearly m-embedded in G, then G is p-nilpotent.*

Lemma 2.5 ([4, Lemma 2.14]) *Let P be a normal non-identity p-subgroup of G with $|P| = p^n$ and $P \cap \Phi(G) = 1$. Suppose that either every maximal subgroup of P is nearly m-embedded in G or there is an integer k such that $1 \leq k < n$ and the subgroups of P of order p^k are m-embedded in G. Then some maximal subgroup of P is normal in G.*

3. Main results

Theorem 3.1 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is nearly m-embedded in $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G, then G is p-nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

(1) Let L be a normal subgroup of G contained in P. Then G/L satisfies the hypothesis.

It is clear that $(|G/L|, p-1) = 1$. For any maximal subgroup P_1/L of P/L, $p = |P/L : P_1/L| = |P : P_1|$, so P_1 is a maximal subgroup of P. By the hypothesis, P_1 is nearly m-embedded in $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G. By Lemma 2.2 (2), P_1/L is nearly m-embedded in $N_G(P)/L = N_{G/L}(P/L)$, and $(P/L)' = P'L/L$ is $\{1 \leq G/L\}$-embedded in G/L by Lemma 2.1 (b). The choice of G implies that (1) is true.

(2) $1 \neq P' \leq O_p(G)$ and G is solvable.

For any $Q \in Syl_q(N_G(P))$, where $q \neq p$. It is easy to see that every maximal subgroup of P is nearly m-embedded in PQ by Lemma 2.2 (1). Thus PQ satisfies the hypothesis of Lemma 2.4, so PQ is p-nilpotent, hence $Q \leq C_G(P)$. Assume that P is abelian, then $N_G(P) = C_G(P)$, hence G is p-nilpotent by Burnside′s Theorem, a contradiction. So $P' \neq 1$. By the hypothesis, P' is $\{1 \leq G\}$-embedded in G. By [4, Lemma 2.3], P' is subnormal in G. Hence $1 \neq P' \leq O_p(G)$. By (1), $G/O_p(G)$ is p-nilpotent. Since $(|G|, p-1) = 1$, we conclude that $G/O_p(G)$ is solvable, thus G is solvable.

(3) $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$; $L = O_p(G)$ is a unique minimal normal subgroup of G, and $\Phi(G) = 1$.

Let $\{G_r \mid r \in \pi(G)\}$ be a Sylow system of G and $H = G_pG_r$ for any $r \in \pi(G)$ with $r \neq p$. By Lemma 2.1(a) and Lemma 2.2 (1), the hypothesis is still true for H. If $|\pi(G)| > 2$, then $G_r \trianglelefteq H$, which implies that G_p normalizes G_r for any $r \in \pi(G)$, hence G is p-nilpotent, a contradiction. Thus we may assume that $|G| = p^aq^b$.

Let L be a minimal normal subgroup of G. By Lemma 2.1(b), $P'L/L$ is $\{1 \leq G/L\}$-embedded in G/L. If L is a q-group, then we consider the quotient group G/L. Evidently, $PL/L \in Syl_p(G/L)$. For any maximal subgroup T/L of PL/L, we have $p = |(PL/L) : (T/L)|$, and $T = PL \cap T = (P \cap T)L$. Let $P_1 = P \cap T$. Then $P_1 \cap L = P \cap T \cap L = P \cap L$, so

$$p = |PL : T| = |PL : (P \cap T)L| = |P : P \cap T| = |P : P_1|.$$

Thus P_1 is a maximal subgroup of P. By the hypothesis, P_1 is nearly m-embedded in $N_G(P)$. Then by Lemma 2.2 (3), it is easy to see P_1L/L is also nearly m-embedded in $N_G(P)L/L = N_{G/L}(PL/L)$. By the minimality of G, G/L is p-nilpotent and so is G, a contradiction. Hence L is a p-group and $L \leq P$. By (1), G/L is p-nilpotent. Similarly, if N is another minimal normal subgroup of G, then $N \leq P$ and so G/N is also p-nilpotent. Now it follows that $G \cong G/N \cap L$ is p-nilpotent, a contradiction. Thus L must be the unique minimal normal subgroup of G. Since the class of p-nilpotent groups is a saturated formation, $L \nleq \Phi(G)$, and $\Phi(G) = 1$. By [8, Thorem 5.3], we get $O_p(G) = F(G) = L$.

(4) The final contradiction.

By (3), $\Phi(G) = 1$. Then G has a maximal subgroup M such that $G = ML$ and $M \cap L = 1$. So $M \cong G/L$ is p-nilpotent. Let $M_{p'}$ be a normal p-complement of M, then $M_{p'} \lhd M$. Thus $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that either $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, a contradiction. Hence $M = N_G(M_{p'})$. Clearly, $P = L(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By the hypothesis, there are a subgroup T of G and a $\{1 \leq G\}$-embedded subgroup C of G such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. Thus C either covers or avoids (M, G). But $CM \leq P_1M \neq G$, hence $C \cap M = C$, that is, $C \leq M$. By [4, Lemma 2.3], C is subnormal in G. Then $C \leq O_p(G) = L$. Hence, $C \leq M \cap L = 1$, and then $|T|_p = p$, where $T_p \in Syl_p(T)$. Since $N_T(T_p)/C_T(T_p)$ is isomorphic to a subgroup of $Aut(T_p)$ and $|Aut(T_p)| = p-1$. By $(|G|, p-1) = 1$, we get $|N_T(T_p)/C_T(T_p)| = 1$, that is, $N_T(T_p) = C_T(T_p)$. So T is p-nilpotent by Burnside′s Theorem. Let $T_{p'}$ be a normal p-complement of T, then $T_{p'} \lhd T$. Clearly, both $T_{p'}$ and $M_{p'}$ are Hall p'-subgroup of G with odd order. By applying Groos's result in [3, main Theorem], there exists $g \in G$ such that $T_{p'}^{g} = M_{p'}$, and $T_{p'}$ is normalized by T, then we may assume that $g \in P_1$. Thus $T^g \leq N_G(T_{p'}^{g}) \leq N_G(M_{p'}) = M$, hence $G = P_1T^g = P_1M$ and $P = P \cap P_1M = P_1(P \cap M) = P_1$, a contradiction. This contradiction completes the proof of this theorem. ∎

Theorem 3.2 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is nearly m-embedded in $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

By Lemma 2.1 (a) and Lemma 2.2 (1), it is easy to see that every maximal subgroup of P is nearly m-embedded in $N_H(P)$ and P' is $\{1 \leq H\}$-embedded in H, then H is p-nilpotent by Theorem 3.1. Let M be a normal p-complement of H. Then $M \trianglelefteq G$. Assume that $M \neq 1$. We consider the quotient group G/M. Similar to the proof of (3) in Theorem 3.1, it is easy to see that the hypothesis is still true for $(G/M, H/M)$, hence G/M is p-nilpotent and so is G, a contradiction. Thus we conclude that $M = 1$. Now $H = P$ is a p-subgroup. Let T/P be a normal p-complement of G/P. It is clear that every maximal subgroup of P is nearly m-embedded in $N_T(P)$ and P' is $\{1 \leq T\}$-embedded in T by Lemma 2.1 (a) and Lemma 2.2 (1), then T is p-nilpotent by Theorem 3.1, so $T_{p'} \trianglelefteq T \trianglelefteq G$ and $T_{p'}$ is also a Hall p'-subgroup of G, thus $T_{p'} \trianglelefteq G$, hence G is p-nilpotent, a contradiction.

This completes the proof. ∎

In [2], the authors introduced the c-supplement subgroup. A subgroup A of G is said to be c-supplement in G if G has a subgroup T such that $TA = G$ and $T \cap A \leq A_G$. Since a normal subgroup is always $\{1 \leq G\}$-embedded in G, every c-supplement subgroup is nearly m-embedded. Then the following corollary is clear by Theorem 3.2.

Corollary 3.1 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is a c-supplemented subgroup of $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G, then G is p-nilpotent.*

Theorem 3.3 *Let G be a group. For any prime factor p of $|G|$, there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is nearly m-embedded in $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G, then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

By Theorem 3.1, G is a Sylow tower group of supersolvable type, so G is solvable. Let L be a minimal normal subgroup of G. Then L is an elementary r-subgroup, where $r \in \pi(G)$. By Lemma 2.1 and Lemma 2.2, G/L satisfies the hypothesis, thus G/L is supersolvable by the minimal choice of G. Since the class of supersolvable subgroups is a saturated formation, we may assume that L is a unique minimal normal subgroup of G and $L \nleq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$, thus $L \leq Q$ by the unique minimal normality of L. Since $Q = O_q(G) \leq F(G) \leq C_G(L)$, L and M normalize $Q \cap M$, thus $Q \cap M \lhd G$. So $Q \cap M = 1$ or $L \leq Q \cap M$. If the later happens, then $L \leq M$, that is, $G = LM = M$, a contradiction. So $Q \cap M = 1$. This implies that $|Q| = |G : M| = |L|$, hence $L = Q$. By the hypothesis, every maximal subgroup of Q is nearly m-embedded in $N_G(Q) = G$. Then by Lemma 2.5, there is a maximal subgroup of Q is normal in G. The minimal normality of Q in G implies that $|Q| = |L| = q$. By Lemma 2.3, G is supersolvable, a contradiction. This final contradiction completes our proof. ∎

Theorem 3.4 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. For any prime factor p of $|H|$, there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is nearly m-embedded in $N_G(P)$ and P' is $\{1 \leq G\}$-embedded in G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

By Lemma 2.1 (a) and Lemma 2.2 (1), H satisfies the hypothesis of Theorem 3.3, hence H is supersolvable. Let $p = max\pi(H)$ and $P \in Syl_p(H)$. Then $P \trianglelefteq G$. We consider the quotient group G/P, then $G/H \cong (G/P)/(H/P) \in \mathcal{F}$. By Lemma 2.1 and Lemma 2.2, $(G/P, H/P)$ satisfies the hypothesis, thus $G/P \in \mathcal{F}$. Hence we may assume that $H = P$.

Let N be a minimal normal subgroup of G contained in P. By Lemma 2.1 (b) and Lemma 2.2 (2), the hypothesis is still true for $(G/N, P/N)$, then $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, $N \nleq \Phi(G)$. So there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. On the other hand, we can conclude that $\Phi(P) = 1$. Otherwise, we have $G/\Phi(P) \in \mathcal{F}$, then $G/\Phi(G) \cong (G/\Phi(P))/(\Phi(G)/\Phi(P)) \in \mathcal{F}$, so $G \in \mathcal{F}$, a contradiction. Hence P is an elementary abelian subgroup. By $N \leq P$, we get $G = NM = PM$ and $P \cap M \lhd G$. If $P \cap M \neq 1$, then $N \leq P \cap M$, $N \leq M$, so $G = NM = M$, a contradiction. Thus $P \cap M = 1$, hence $P = N$ is a minimal normal subgroup of G. By the hypothesis, every maximal subgroup of P is nearly m-embedded in $N_G(P) = G$. Then by Lemma 2.5, there is a maximal subgroup of P is normal in G. The minimal normality of P in G implies that $|P| = p$. By Lemma 2.3, $G \in \mathcal{F}$, a contradiction. This final contradiction completes our proof. ∎

Recall that a group G was called an A-group if all of its Sylow subgroups are abelian. Let G be an A-group. Then for any $P \in Syl_p(G)$, we have $P' = 1$, of course, it is $\{1 \leq G\}$-embedded in G, so we have the following corollary.

Corollary 3.2 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal A-subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup P of H is a c-supplemented subgroup of $N_G(P)$, then $G \in \mathcal{F}$.*

4. Remarks

Remark 4.1 The following example illustrates that the hypothesis in Theorem 3.1 that "P' is $\{1 \leq G\}$-embedded in G" can not be removed.

Example. Let $G = PSL_2(q)$, where $q \equiv \pm 1 \pmod 8$. Let P be a Sylow 2-subgroup of G. By [6, II, Theorem 8.27], we have the Sylow 2-subgroup of $PSL_2(q)$ is selfnormalizing in $PSL_2(q)$. Evidently, every maximal subgroup of P is normal in $N_G(P) = P$, so every maximal subgroup of P is nearly m-embedded in $N_G(P)$. However, G is not 2-nilpotent.

Remark 4.2 Even if G is a solvable group and p is an odd prime, the hypothesis in Theorem 3.1 that "P' is $\{1 \leq G\}$-embedded in G" could not be omitted, either.

Example. Let $H = Z_3 \times Z_3 \times Z_3$ be an elementary abelian 3-group. It is clear that $Aut(H)$ has a subgroup $Z_{13} \rtimes Z_3$. Now suppose that

$$G = (Z_3 \times Z_3 \times Z_3) \rtimes (Z_{13} \rtimes Z_3)$$

Let P_3 be a Sylow 3-subgroup of G. It is clear that $N_G(P_3) = P_3$, so every maximal subgroup of P_3 is nearly m-embedded in $N_G(P_3) = P_3$. However, G is not 3-nilpotent.

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