CERTAIN PROPERTIES OF MITTAG-LEFFLER FUNCTION WITH ARGUMENT x^α, $\alpha > 0$

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Abstract. In this paper, author discusses some interesting properties such as Composition property, Power series expansion, Inverse property, Increasing property, Positivity and Limiting case of Mittag-Leffler function with argument x^α, $\alpha > 0$.

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1. Introduction

In 1903, a Swedish mathematician Gösta Mittag-Leffler [6] introduced the function $E_\alpha(z)$ in the form:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where z is a complex variable, $\alpha > 0$ and $\Gamma(\cdot)$ is the well-known gamma function.

The Mittag-Leffler function (1.1) is an entire function of order $(\operatorname{Re}\alpha)^{-1}$ and is also direct generalization of the exponential function to which it reduces when $\alpha = 1$, or in other words, the Mittag-Leffler function is the parameterized exponential function. If $0 < \alpha < 1$, then it interpolates between the pure exponential $\exp(z)$ and a hypergeometric function $\frac{1}{1-z} = {}_1F_0(1; -; z)$. In recent years, the Mittag-Leffler function has caused extensive interest among scientist, engineers and applied mathematicians. The Mittag-Leffler functions naturally occur as the solution of fractional order differential equation or fractional order integral equations. Some applications of the function (1.1) have already been discussed in [4], [5], [7] and [8].

Few interesting special cases of $E_\alpha(z)$ are as listed below.

$$E_0(z) = \frac{1}{1-z};\ |z| < 1, \tag{1.2}$$

$$E_{\frac{1}{2}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{2}+1\right)} = \exp(z^2)\mathrm{erf}_c(-z) \tag{1.3}$$

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = e^z \tag{1.4}$$

$$E_2(z) = \cos h(\sqrt{z}) \tag{1.5}$$

$$E_3(z) = \frac{1}{3}\left[\cos\left(z^{\frac{1}{4}}\right) + 2\exp\left(-\frac{z^{\frac{1}{3}}}{2}\right)\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right] \tag{1.6}$$

$$E_4(z) = \frac{1}{2}\left[\cos\left(z^{\frac{1}{4}}\right) + \cos h\left(z^{\frac{1}{4}}\right)\right] \tag{1.7}$$

2. Mittag-Leffler function with argument x^α and its properties

In this section, the author establishes some interesting properties of the Mittag-Leffler function x^α.

The Mittag-Leffler function does not satisfy the composition property, $E_\alpha(x)E_\alpha(y) \neq E_\alpha(x+y)$, but it can be observed that (Jumarie [1], [2], [3]) the function

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n+1)};\ \alpha > 0, \tag{2.1}$$

does satisfy the composition property

$$E_\alpha(x^\alpha)E_\alpha(y^\alpha) = E_\alpha\{(x+y)^\alpha\},\quad \forall\, x \in \mathbb{R}. \tag{2.2}$$

The function $E_\alpha(x^\alpha)$ defined in (2.1) converges absolutely for

$$|x| < \left(\frac{\Gamma(\alpha n+\alpha+1)}{\Gamma(\alpha n+1)}\right)^{\frac{1}{\alpha}}$$

is a Mittag-Leffler function with argument x^α, $\alpha > 0$, and this also can be reduced in the exponential function for $\alpha = 1$.

(i) Power series expansion of $E_\alpha(x^\alpha)$

$$
\begin{aligned}
E_\alpha(x^\alpha) &= \sum_{n=0}^{\infty}\frac{x^{\alpha n}}{\Gamma(\alpha n+1)};\ \alpha>0,\ \forall\, x\in \mathbb{R} \\
&= 1+\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\gamma(3\alpha+1)}+\cdots
\end{aligned}
\tag{2.3}
$$

Taking $x=0$ in (2.3), we get

$$E_\alpha(0)=1. \tag{2.4}$$

(ii) Inverse Property and its particular cases:

Putting $y=-x$ in (2.2), yields

$$E_\alpha(0)=E_\alpha(x^\alpha)E_\alpha\{(-x)^\alpha\}.$$

Using (2.4), the above equation yields

$$E_\alpha\{(-x)^\alpha\}=\frac{1}{E_\alpha(x^\alpha)}. \tag{2.5}$$

If $\alpha=1$, then (2.2) and (2.5) becomes

$$\exp(x+y)=\exp(x)\exp(y)\quad\text{and}\quad \exp(-x)=\frac{1}{\exp(x)}\,\cdot$$

(iii) Increasing property:

If $x>y>0 \Longrightarrow -y>-x$ for odd positive integer α, we can write

$$(-y)^\alpha>(-x)^\alpha;$$

this gives

$$E_\alpha\{(-y)^\alpha\}>E_\alpha\{(-x)^\alpha\};$$

using (2.5), this leads to

$$\frac{1}{E_\alpha(y^\alpha)}>\frac{1}{E_\alpha(x^\alpha)}\quad\text{i.e.}\quad E_\alpha(x^\alpha)>E_\alpha(y^\alpha). \tag{2.6}$$

Now, again if $x>y>0$, then $x^\alpha>y^\alpha>0$ implies that

$$E_\alpha(x^\alpha)>E_\alpha(y^\alpha). \tag{2.7}$$

Equations (2.6) and (2.7) imply that $E_\alpha(x^\alpha)$ is strictly increasing function for odd positive integer α.

(iv) Positivity:

For $\alpha \in \mathbb{N}$ and $x \geq 0$, we have

$$E_\alpha(x^\alpha) > 0 \tag{2.8}$$

again for

$$x < 0 \Longrightarrow -x > 0.$$

Therefore, for $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} &(-x)^\alpha > 0 \\ &\Longrightarrow E_\alpha\{(-x^\alpha)\} > E_\alpha(0) = 1 > 0; \end{aligned}$$

using (2.5), this leads to

$$\frac{1}{E_\alpha(x^\alpha)} > 0$$

and hence

$$E_\alpha(x^\alpha) > 0. \tag{2.9}$$

Equations (2.8) and (2.9) show that

$$E_\alpha(x^\alpha) > 0;\ \alpha \in \mathbb{N} \text{ and } \forall\, x \in \mathbb{R}.$$

(v) Limiting case:

Equation (2.3), gives

$$E_\alpha(x^\alpha) \to \infty \quad \text{as} \quad x \to \infty \quad \text{for} \quad \alpha > 0. \tag{2.10}$$

Now, consider

$$\begin{aligned} \lim_{x \to -\infty} E_\alpha(x^\alpha) &= \lim_{y \to \infty} E_\alpha\{(-y)^\alpha\} \\ &= \lim_{y \to \infty} \frac{1}{E_\alpha(y^\alpha)} = 0. \end{aligned}$$

Therefore,

$$E_\alpha(x^\alpha) \to 0 \quad \text{as} \quad x \to -\infty \quad \text{for} \quad \alpha > 0. \tag{2.11}$$

From (2.5),

$$\begin{aligned} E_\alpha\{(-x)^\alpha\} &= \frac{1}{E_\alpha(x^\alpha)} \\ &= \frac{1}{1+\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{2\alpha}}{G(2\alpha+1)}+\frac{x^{3a}}{\Gamma(3\alpha+1)}+\cdots+\frac{x^{\alpha(n+1)}}{\Gamma(\alpha n+\alpha+1)}+\cdots} \\ &< \frac{\Gamma(\alpha n+\alpha+1)}{x^{\alpha n+\alpha}}. \end{aligned}$$

Therefore,

$$\lim_{x\to\infty} x^{\alpha n} E_a\{(-x)^\alpha\} < \lim_{x\to\infty} \frac{\Gamma(\alpha n+\alpha+1)}{x^\alpha} = 0, \quad \alpha > 0,$$

hence

$$x^{\alpha n} E_\alpha\{(-x)^\alpha\} \to 0 \text{ as } x \to \infty \text{ for } \alpha > 0. \tag{2.12}$$

3. Concluding remarks

The results established in this paper seem to be new and stimulate the scope of further research and other computational aspects.

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