A NEW CHARACTERIZATION OF SPORADIC SIMPLE GROUPS[1]

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Abstract. Let G be a finite group, $k_1(G)$ denote the largest element order of G, and $k_2(G)$, the second largest element order. In this paper, we show that each sporadic simple group G can be uniquely determined by the order of G and $k_i(G)$, where $i \leq 2$.

Key words: finite group, the largest element order, the second largest element order, sporadic simple group, characterization.

AMS Subject Classfication: 20D05, 20D08, 20D60

1. Introduction

It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G. W.J. Shi characterized some finite simple groups by using $\pi_e(G)$ and $|G|$, for example, see [1]-[6]. Recently, this topic has been finished by V.D. Mazurov, et al. (See [7]). Now the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized simple K_3-groups, some K_4-groups by using three numbers: the order of a group, the largest and the second largest element orders. In this paper, we characterize sporadic simple groups via the order of a group and the largest and the second largest element orders.

Notations. The groups mentioned are all finite groups, the number in bracket "( )" behind a group is the order of the group, e.g., $L_2(7)(2^3 \cdot 3 \cdot 7)$ means that

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$L_2(7)$ is of order $2^3 \cdot 3 \cdot 7$. Let $\pi_e(G)$ denote the set of orders of elements in G, $\pi(G)$ denote the set of all prime divisors of $|G|$, $k_1(G)$, the largest element order of G, and $k_2(G)$, the second largest element order. S_p-subgroup is a Sylow p-subgroup of G. We denote by $\Gamma(G)$ the prime graph of G and $t(G)$ is the number of connected components of $\Gamma(G)$. And we also denote the sets of vertex of the connected components of the prime graph by $\{\pi_i,\ i = 1, ..., t(G)\}$, for convenience we call Pi_i $(1 \leq i \leq t(G))$ the connected components and if the order of G is even, denote the component containing 2 by π_1 (see [8]).

In this paper, we come to the following theorems:

Theorem 1. *Let G be a group and A be one of the following sporadic simple groups: M_{11}, M_{12}, J_1, M_{22}, M_{23}, HS, J_3, M_{24}, He, Ru, Suz, $O'N$, Co_3, Co_2, Ly, Th, J_4, B, M. Then $G \cong A$ if and only if*

(i) $k_1(G) = k_1(A)$;

(ii) $|G| = |A|$.

For other sporadic simple groups, we have

Theorem 2. *Let G be a group and A be one of the following sporadic simple groups: J_2, McL, Fi_{22}, HN, Fi_{23}, Co_1, Fi_{24}. Then $G \cong A$ if and only if*

(i) $k_i(G) = k_i(A)$, *where* $i = 1, 2$;

(ii) $|G| = |A|$.

2. Preliminary results

Lemma 1. *Suppose that G has more than one prime graph component. Then one of the following holds:*

(1) *G is a $Frobenius$ group or a $2-Frobenius$ group;*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.*

Proof. The lemma follows from theorem A and Lemma 3 in [8].

Remark. A group G will be called a $2-Frobenius$ group provided G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are $Frobenius$ groups with K/H and H as their $Frobenius$ kernels respectively.

Lemma 2. *If G is a Frobenius group of even order with K the Frobenius kernel and H the Frobenius complement, then $t(G) = 2$ and $\Gamma(G) = \{\pi(H),\ \pi(K)\}$ (see [9]).*

Lemma 3. *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2$, and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying that $|G/K| \mid |Aut(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$. Particularly, G is solvable* (*see* [9]).

Lemma 4. *Let A be a $\pi'-$group of automorphisms of the $\pi-$group G, and suppose G or A is solvable. Then for each prime p in π, A leaves invariant some S_p-subgroup of G* (*See* [10], *Theorem* 6.22).

Lemma 5. *Let k be a positive integer, and $\pi(k)$ denote the set of prime divisors of k. Suppose that t is a positive integer satisfying that $22 \leq t \leq 46$. Then $\pi(2^{2t} - 1) \nsubseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.*

Lemma 6. *Let G be a sporadic simple group. Then $|G|$, $k_1(G)$ and $k_2(G)$ are as in Table* 1:

Table 1

G	$\|G\|$	$k_1(G)$	$k_2(G)$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	11	8
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	11	10
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	11	8
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	15
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	21
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	15	12
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	24	21
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	20	15
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	30	15
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	30	24
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	30	28
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	60	42
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	28	21
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	30	24
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	60	42
Fi_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	60	45
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	40	35
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	39	36
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	70	66
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	119	110
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	19	15
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	31	28
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	19	17
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	67	42
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	29	26
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	66	44

Proof. The lemma follows from [11].

Lemma 7. *Let G be a finite group and A be one of the following sporadic simple groups: M_{11}, M_{12}, J_1, M_{22}, M_{23}, HS, J_3, M_{24}, He, Ru, Suz, $O'N$, Co_3, Co_2, Ly, Th, J_4, B, M. Suppose that*

(i) $k_1(G) = k_1(A)$;

(ii) $|G| = |A|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the cases $A = M_{11},\ M_{12}$. For the other cases, we can prove them similarly.

1. Assume that $A = M_{11}$ $(2^4 \cdot 3^2 \cdot 5 \cdot 11)$. In such case, $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Because $k_1(G) = 11$, it follows that 11 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

First, we suppose that G is a Frobenius group. Then by Lemma 2 we get that $t(G) = 2$ and $\Gamma(G) = \{\ \pi(H), \pi(K)\ \}$, where K is the Frobenius kernel and H the Frobenius complement. Since $k_1(G) = 11$, K is either a {2, 3, 5}-Hall subgroup or a Sylow 11-subgroup of G. Since K is nilpotent, let S be a Sylow subgroup of K, one has that $|H| \mid (|S| - 1)$. We can find an suitable Sylow subgroup of K such that $|H| \nmid (|S| - 1)$, then we get a contradiction. For this reason, K can't be a Sylow 11-subgroup of G. Hence K is a {2, 3, 5}-Hall subgroup. Consider the Sylow 5-subgroup of K, we get $11 \mid 4$, a contradiction. Therefore, G is not a Frobenius group.

Second, we suppose that G is a 2-Frobenius group. By Lemma 3, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 11 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{11\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 5\}$ and $|K/H| = 11$. Since $|G/K| \mid |Aut(K/H)| = 10$, we know that $3 \mid |H|$. Consider the action on H by the element of order 11. By Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Since $|L| = 3^2$, we have $11 \nmid |Aut(L)|$, which means that such action on L is trivial. Therefore, G has an element of order 33, a contradiction. So G is not a 2-Frobenius group.

Remark. This approach can be used to prove that G is not 2-Frobenius group for the most of other cases. For a few exceptions, we only need to consider $\Omega_1(Z(L))$ of some special Sylow subgroup L to lead to a contradiction. The process can be seen in the case $A = M_{12}$.

2. Assume that $A = M_{12}$ $(2^6 \cdot 3^3 \cdot 5 \cdot 11)$. In such case, $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Because $k_1(G) = 11$, it follows that 11 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

Using the similar arguments in case $A = M_{11}$, we can easily show that G is not a Frobenius group. Now we assert that G is not a 2-Frobenius group. Assume the contrary. Let G be a 2-Frobenius group. By Lemma 3, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 11 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{11\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 5\}$ and $|K/H| = 11$. Since $|G/K| \mid |Aut(K/H)| = 10$, we know that $3 \mid |H|$. Consider the action on H by the element y of order 11. Again by Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Obviously, $|L| = 3^3$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))| \mid 3^3$. Because $\Omega_1(Z(L))$ is characteristic in L, y fixes $\Omega_1(Z(L))$ too. As $11 \nmid |Aut(\Omega_1(Z(L)))|$, the action on $\Omega_1(Z(L))$ by y is trivial, which implies that G has an element of order 33, a contradiction. So G is not a 2-Frobenius group.

Lemma 8. *Let G be a finite group and A be one of the following sporadic simple groups: J_2, McL, Fi_{22}, HN, Fi_{23}, Co_1, Fi_{24}. Suppose that*

(i) $k_i(G) = k_i(A)$, *where* $i = 1, 2$;

(ii) $|G| = |A|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the case $A = J_2$, and for the other cases, they can be proved similarly.

Assume that $A = J_2$ $(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$. In such case, $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $k_1(G) = 15$ and $k_2(G) = 12$. Because $k_1(G) = 15$, $k_2(G) = 12$, we have 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar arguments in Lemma 7, we can prove that G is neither a Frobenius group nor a 2-Frobenius group. Therefore, the lemma follows from part 2 of Lemma 1.

From [11], we know all the simple groups of order less than 10^{25}, except that the $L_2(q)$, $L_3(q)$, $U_3(q)$, $L_4(q)$, $U_4(q)$, $S_4(q)$, $G_2(q)$ are stopped at orders 10^6 $(q \leq 125)$, 10^{12} $(q \leq 31)$, 10^{12} $(q \leq 32)$, 10^{16} $(q \leq 11)$, 10^{16} $(q \leq 11)$, 10^{16} $(q \leq 41)$, 10^{20} $(q \leq 25)$ respectively.

Let $B(q)$ be one of the following Lie-type simple groups: $L_2(q)$, $L_3(q)$, $U_3(q)$, $L_4(q)$, $U_4(q)$, $S_4(q)$, $G_2(q)$. For the need of discussion, we list the orders of $B(q)$ in Table 2.

Table 2

$B(q)$	$L_2(q)$	$L_3(q)$
$\|B(q)\|$	$q(q^2-1)/(q-1,2)$	$q^3(q^3-1)(q^2-1)/(q-1,3)$
$B(q)$	$L_4(q)$	$U_4(q)$
$\|B(q)\|$	$q^6(q^4-1)(q^3-1)(q^2-1)/(q-1,4)$	$q^6(q^4-1)(q^3+1)(q^2-1)/(q+1,4)$
$B(q)$	$G_2(q)$	$U_3(q)$
$\|B(q)\|$	$q^6(q^6-1)(q^2-1)$	$q^3(q^3+1)(q^2-1)/(q+1,3)$
$B(q)$	$S_4(q)$	
$\|B(q)\|$	$q^4(q^4-1)(q^2-1)/(q-1,2)$	

And for convenient discussion, we list the orders of some $L_2(q)$ in Table 3.

Table 3

$L_2(q)$	$\|L_2(q)\|$	q^2-1
$L_2(2^5)$	$2^5\cdot 3\cdot 11\cdot 31$	$3\cdot 11\cdot 31$
$L_2(2^6)$	$2^6\cdot 3^2\cdot 5\cdot 7\cdot 13$	$3^2\cdot 5\cdot 7\cdot 13$
$L_2(2^7)$	$2^7\cdot 3\cdot 43\cdot 127$	$3\cdot 43\cdot 127$
$L_2(2^8)$	$2^8\cdot 3\cdot 5\cdot 17\cdot 257$	$3\cdot 5\cdot 17\cdot 257$
$L_2(2^9)$	$2^9\cdot 3^3\cdot 7\cdot 19\cdot 73$	$3^3\cdot 7\cdot 19\cdot 73$
$L_2(2^{10})$	$2^{10}\cdot 3\cdot 5^2\cdot 11\cdot 31\cdot 41$	$3\cdot 5^2\cdot 11\cdot 31\cdot 41$
$L_2(2^{11})$	$2^{11}\cdot 3\cdot 23\cdot 89\cdot 683$	$3\cdot 23\cdot 89\cdot 683$
$L_2(2^{12})$	$2^{12}\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241$	$3^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241$
$L_2(2^{13})$	$2^{13}\cdot 3\cdot 2731\cdot 8191$	$3\cdot 2731\cdot 8191$
$L_2(2^{14})$	$2^{14}\cdot 3\cdot 5\cdot 29\cdot 43\cdot 113\cdot 127$	$3\cdot 5\cdot 29\cdot 43\cdot 113\cdot 127$
$L_2(2^{15})$	$2^{15}\cdot 3^2\cdot 7\cdot 11\cdot 31\cdot 151\cdot 331$	$3^2\cdot 7\cdot 11\cdot 31\cdot 151\cdot 331$
$L_2(2^{16})$	$2^{16}\cdot 3\cdot 5\cdot 17\cdot 257\cdot 65537$	$3\cdot 5\cdot 17\cdot 257\cdot 65537$
$L_2(2^{17})$	$2^{17}\cdot 3\cdot 43691\cdot 131071$	$3\cdot 43691\cdot 131071$
$L_2(2^{18})$	$2^{18}\cdot 3^3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73\cdot 109$	$3^3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73\cdot 109$
$L_2(2^{19})$	$2^{19}\cdot 3\cdot 174763\cdot 524287$	$3\cdot 174763\cdot 524287$
$L_2(2^{20})$	$2^{20}\cdot 3\cdot 5^2\cdot 11\cdot 17\cdot 31\cdot 41\cdot 61681$	$3\cdot 5^2\cdot 11\cdot 17\cdot 31\cdot 41\cdot 61681$
$L_2(2^{21})$	$2^{21}\cdot 3^2\cdot 7^2\cdot 43\cdot 127\cdot 337\cdot 5419$	$3^2\cdot 7^2\cdot 43\cdot 127\cdot 337\cdot 5419$
$L_2(3^5)$	$2^2\cdot 3^5\cdot 11^2\cdot 61$	$2^3\cdot 11^2\cdot 61$
$L_2(3^6)$	$2^3\cdot 3^6\cdot 5\cdot 7\cdot 13\cdot 73$	$2^4\cdot 5\cdot \cdot 7\cdot 13\cdot 73$
$L_2(3^7)$	$2^2\cdot 3^7\cdot 547\cdot 1093$	$2^3\cdot 547\cdot 1093$
$L_2(3^8)$	$2^5\cdot 3^8\cdot 5\cdot 17\cdot 41\cdot 193$	$2^6\cdot 5\cdot 17\cdot 41\cdot 193$
$L_2(3^9)$	$2^2\cdot 3^9\cdot 7\cdot 13\cdot 19\cdot 37\cdot 757$	$2^3\cdot 7\cdot 13\cdot 19\cdot 37\cdot 757$
$L_2(3^{10})$	$3^{10}\cdot 2^3\cdot 5^2\cdot 11^2\cdot 61\cdot 1181$	$2^4\cdot 5^2\cdot 11^2\cdot 61\cdot 1181$
$L_2(3^{11})$	$3^{11}\cdot 2^2\cdot 23\cdot 67\cdot 661\cdot 3851$	$2^3\cdot 23\cdot 67\cdot 661\cdot 3851$
$L_2(3^{12})$	$3^{12}\cdot 2^4\cdot 5\cdot 7\cdot 13\cdot 41\cdot 73\cdot 6481$	$2^5\cdot 5\cdot 7\cdot 13\cdot 41\cdot 73\cdot 6481$
$L_2(3^{13})$	$3^{13}\cdot 2^2\cdot 398581\cdot 797261$	$2^3\cdot 398581\cdot 797261$
$L_2(3^{14})$	$3^{14}\cdot 2^3\cdot 5\cdot 29\cdot 547\cdot 1093\cdot 16493$	$2^4\cdot 5\cdot 29\cdot 547\cdot 1093\cdot 16493$
$L_2(3^{15})$	$3^{15}\cdot 2^2\cdot 7\cdot 11^2\cdot 13\cdot 31\cdot 61\cdot 271\cdot 4561$	$2^3\cdot 7\cdot 11^2\cdot 13\cdot 31\cdot 61\cdot 271\cdot 4561$
$L_2(3^{16})$	$3^{16}\cdot 2^6\cdot 5\cdot 17\cdot 41\cdot 193\cdot 21523361$	$2^7\cdot 5\cdot 17\cdot 41\cdot 193\cdot 21523361$

Table 3

$L_2(q)$	$\lvert L_2(q)\rvert$	q^2-1
$L_2(3^{17})$	$3^{17}{\cdot}2^2{\cdot}32285041{\cdot}64570081$	$2^3{\cdot}32285041{\cdot}64570081$
$L_2(3^{18})$	$3^{18}{\cdot}2^3{\cdot}5{\cdot}7{\cdot}13{\cdot}19{\cdot}37{\cdot}73{\cdot}757{\cdot}530713$	$2^4{\cdot}5{\cdot}7{\cdot}13{\cdot}19{\cdot}37{\cdot}73{\cdot}757{\cdot}530713$
$L_2(3^{19})$	$3^{19}{\cdot}2^2{\cdot}290565367{\cdot}581130733$	$2^3{\cdot}290565367{\cdot}581130733$
$L_2(3^{20})$	$3^{20}{\cdot}2^4{\cdot}5^2{\cdot}11^2{\cdot}41{\cdot}61{\cdot}1182{\cdot}42521761$	$2^5{\cdot}5^2{\cdot}11^2{\cdot}41{\cdot}61{\cdot}1182\cdot 42521761$
$L_2(5^4)$	$5^4\cdot 2^4\cdot 3\cdot 13\cdot 313$	$2^5\cdot 3\cdot 13\cdot 313$
$L_2(5^5)$	$5^5\cdot 2^2\cdot 3\cdot 11\cdot 71\cdot 521$	$2^3\cdot 3\cdot 11\cdot 71\cdot 521$
$L_2(5^6)$	$5^6\cdot 2^3\cdot 3^2\cdot 7\cdot 13\cdot 31\cdot 601$	$2^4\cdot 3^2\cdot 7\cdot 13\cdot 31\cdot 601$
$L_2(5^7)$	$5^7\cdot 2^2\cdot 3\cdot 29\cdot 449\cdot 19531$	$2^3\cdot 3\cdot 29\cdot 449\cdot 19531$
$L_2(5^8)$	$5^8\cdot 2^5\cdot 3\cdot 13\cdot 17\cdot 313\cdot 11489$	$2^6\cdot 3\cdot 13\cdot 17\cdot 313\cdot 11489$
$L_2(5^9)$	$5^9\cdot 2^2\cdot 3^3\cdot 7\cdot 5176\cdot 488281$	$2^3\cdot 3^3\cdot 7\cdot 5176\cdot 488281$
$L_2(7^3)$	$7^3\cdot 2^3\cdot 3^2\cdot 19\cdot 43$	$2^4\cdot 3^2\cdot 19\cdot 43$
$L_2(7^4)$	$7^4\cdot 2^5\cdot 3\cdot 5^2\cdot 1201$	$2^6\cdot 3\cdot 5^2\cdot 1201$
$L_2(7^5)$	$7^5\cdot 2^3\cdot 3\cdot 11\cdot 191\cdot 2801$	$2^4\cdot 3\cdot 11\cdot 191\cdot 2801$
$L_2(7^6)$	$7^6\cdot 2^4\cdot 3^2\cdot 5^2\cdot 13\cdot 19\cdot 43\cdot 181$	$2^5\cdot 3^2\cdot 5^2\cdot 13\cdot 19\cdot 43\cdot 181$
$L_2(11^3)$	$11^3\cdot 2^2\cdot 3^2\cdot 5\cdot 7\cdot 19\cdot 37$	$2^3\cdot 3^2\cdot 5\cdot 7\cdot 19\cdot 37$
$L_2(13^2)$	$13^2\cdot 2^3\cdot 3\cdot 5\cdot 7\cdot 17$	$2^4\cdot 3\cdot 5\cdot 7\cdot 17$
$L_2(13^3)$	$13^3\cdot 2^2\cdot 3^2\cdot 7\cdot 61\cdot 157$	$2^3\cdot 3^2\cdot 7\cdot 61\cdot 157$

3. Proofs of theorems

Proof of Theorem 1. We only need to prove the sufficiency. And the proof will be made through a case by case analysis.

Case 1.1. Assume that $A = M_{11}$ $(2^4\cdot 3^2\cdot 5\cdot 11)$. In this case, $|G| = 2^4\cdot 3^2\cdot 5\cdot 11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 2$. By Lemma 7, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H)\cup\pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that $K/H \cong L_2(11)$ $(2^2\cdot 3\cdot 5\cdot 11)$ or M_{11} $(2^4\cdot 3^2\cdot 5\cdot 11)$.

Suppose that $K/H \cong L_2(11)$ $(2^2\cdot 3\cdot 5\cdot 11)$. From [11] we know that $|Out(L_2(11))| = 2$, so we can get that $3 \mid |H|$ by comparing the order of G. Let L be a Sylow 3-subgroup of H. Then $|L| = 3$. As H is a nilpotent group, we have L is characteristic in H, and thus $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial. It implies that G has an element of order 33, a contradiction.

Therefore, we have $K/H \cong M_{11}$ $(2^4\cdot 3^2\cdot 5\cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong M_{11}$.

Case 1.2. Assume that $A = M_{12}$ $(2^6\cdot 3^3\cdot 5\cdot 11)$. In this case, $|G| = 2^6\cdot 3^3\cdot 5\cdot 11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 2$. By Lemma 7,

G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that $K/H \cong L_2(11)$ $(2^2 \cdot 3 \cdot 5 \cdot 11)$, M_{11} $(2^4 \cdot 3^2 \cdot 5 \cdot 11)$ or M_{12} $(2^6 \cdot 3^3 \cdot 5 \cdot 11)$.

Suppose that $K/H \cong L_2(11)$ $(2^2 \cdot 3 \cdot 5 \cdot 11)$ or M_{11} $(2^4 \cdot 3^2 \cdot 5 \cdot 11)$. From [11] we get that $3 \nmid |Out(K/H))| = 2$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H. Then $|L| \mid 3^2$. As H is a nilpotent group, we have L is characteristic in H, and thus $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 33, a contradiction.

Therefore, we have $K/H \cong M_{12}$ $(2^6 \cdot 3^3 \cdot 5 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong M_{12}$.

Case 1.3. Assume that $A = J_1$ $(2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}19)$. In this case, $|G| = 2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}19$, and $k_1(G) = 19$. Since $k_1(G) = 19$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenius group. So by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $19 \in \pi(K/H)$. From [11] we can know that K/H is isomorphic only to J_1 $(2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$. So $H = 1$, $K = G$, and therefore $G \cong J_1$.

Case 1.4. Assume that $A = M_{22}$ $(2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11)$. In this case, $|G| = 2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenius group. So by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(11)$ $(2^2{\cdot}3{\cdot}5{\cdot}11)$, M_{11} $(2^4 \cdot 3^2 \cdot 5 \cdot 11)$ and M_{22} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$.

Suppose that $K/H \cong L_2(11)$ $(2^2 \cdot 3 \cdot 5 \cdot 11)$ or M_{11} $(2^4 \cdot 3^2 \cdot 5 \cdot 11)$. From [11] we have $7 \nmid |Out(K/H))| = 2$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 35, a contradiction.

Therefore, we have $K/H \cong M_{22}$ $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong M_{22}$.

Case 1.5. Assume that $A = M_{23}$ $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 23$. Since $k_1(G) = 23$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, $L_2(2^7)$ and M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$.

Suppose that $K/H \cong L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$. Then $7 \nmid |Out(K/H))|$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 77, a contradiction.

Suppose that $K/H \cong L_2(2^7)$. From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong M_{22}$ $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong M_{23}$.

Case 1.6. Assume that $A = HS$ $(2^9{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11)$. In this case, $|G| = 2^9{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11$, and $k_1(G) = 20$. Since $k_1(G) = 20$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(11)$ $(2^2{\cdot}3{\cdot}5{\cdot}11)$, M_{11} $(2^4{\cdot}3^2{\cdot}5{\cdot}11)$, M_{22} $(2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11)$, $L_2(2^t)$ $(7 \leq t \leq 9)$ and HS $(2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11)$.

Suppose that $K/H \cong L_2(11)$ $(2^2{\cdot}3{\cdot}5{\cdot}11)$, $M_{11}(2^4{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11)$ or M_{22} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$. Then $5 \nmid |Out(K/H))|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. Then $L \trianglelefteq G$ and $|L| \mid 5^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 55, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ $(7 \leq t \leq 9)$. From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong HS$ $(2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong HS$.

Case 1.7. Assume that $A = J_3$ $(2^7{\cdot}3^5{\cdot}5{\cdot}17{\cdot}19)$. In this case, $|G| = 2^7{\cdot}3^5{\cdot}5{\cdot}17{\cdot}19$, and $k_1(G) = 19$. Since $k_1(G) = 19$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenus group. Therefore, by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 17\}$ and $19 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(19)$ $(2^2{\cdot}3^2{\cdot}5{\cdot}19)$, $L_2(2^7)$, $L_2(3^5)$ and J_3 $(2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19)$.

Suppose that $K/H \cong L_2(19)$ $(2^2 \cdot 3^2 \cdot 5 \cdot 19)$. Then $17 \nmid |Out(K/H))|$, and thus $17 \mid |H|$. Let L be a Sylow 17-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action trivial, which implies that G has an element of order 51, a contradiction.

Suppose that $K/H \cong L_2(2^7)$ or $L_2(3^5)$. From Table 3 we can get that $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong J_3$ $(2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19)$. So $H = 1$, $K = G$, and therefore $G \cong J_3$.

Case 1.8. Assume that $A = M_{24}$ $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In this case, $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 23$. Since $k_1(G) = 23$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $L_2(2^t)(7 \leq t \leq 10)$ and M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$.

Suppose that $K/H \cong L_2(23)$ $(2^2 \cdot 3 \cdot 11 \cdot 23)$ or M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. Then $3 \nmid |Out(K/H))|$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H. Then $L \trianglelefteq G$ and $|L| \mid 3^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 33, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ $(7 \leq t \leq 10)$. From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong M_{24}$ $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong M_{24}$.

Case 1.9. Assume that $A = He$ $(2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17)$. In this case, $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, and $k_1(G) = 28$. Since $k_1(G) = 28$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $17 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(17)$ $(2^4 \cdot 3^2 \cdot 17)$, $L_2(16)$ $(2^4 \cdot 3 \cdot 5 \cdot 17)$, $S_4(4)$ $(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $L_2(2^t)$ $(7 \leq t \leq 10)$, $L_2(7^3)$ and He $(2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17)$.

Suppose that $K/H \cong L_2(17)$ $(2^4 \cdot 3^2 \cdot 17)$ or $L_2(16)$ $(2^4 \cdot 3 \cdot 5 \cdot 17)$. Then $5 \nmid |Out(K/H))|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. Then $L \trianglelefteq G$ and $|L| \mid 5^2$. Consider the action on L by the element of order 17. Clearly, this action is trivial, which implies that G has an element of order 85, a contradiction.

Suppose that $K/H \cong S_4(4)$ $(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$. Then $3 \nmid |Out(K/H))|$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H. Then $L \trianglelefteq G$ and $|L| = 3$. Consider the action on L by the element of order 17. Clearly, this action is also trivial, which implies that G has an element of order 51, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ $(7 \leq t \leq 10)$ or $L_2(7^3)$. From Table 3 we can know that $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong He$ $(2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17)$. So $H = 1$, $K = G$, and therefore $G \cong He$.

Case 1.10. Assume that $A = Ru$ $(2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29)$. In this case, $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, and $k_1(G) = 29$. Since $k_1(G) = 29$, we have $t(G) \geq 2$. By

Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 13\}$ and $29 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(29)$ $(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29)$, $L_2(2^t)$ $(7 \le t \le 14)$ and Ru $(2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29)$.

Suppose that $K/H \cong L_2(29)$ $(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29)$. From [11] we know that $13 \nmid |Out(K/H))|$, and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 65, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ $(7 \le t \le 14)$. From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Ru$ $(2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29)$. So $H = 1$, $K = G$, and therefore $G \cong Ru$.

Case 1.11. Assume that $A = Suz$ $(2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. In this case, $|G| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, and $k_1(G) = 24$. Since $k_1(G) = 24$, we have $t(G) \ge 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ $(2^2{\cdot}3{\cdot}7{\cdot}13)$, $L_3(3)$ $(2^4{\cdot}3^3{\cdot}13)$, $L_2(25)$ $(2^3{\cdot}3{\cdot}5^2{\cdot}13)$, $L_2(27)$ $(2^2{\cdot}3^3{\cdot}7{\cdot}13)$, $Sz(8)$ $(2^6 \cdot 5 \cdot 7 \cdot 13)$, $U_3(4)$ $(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$, $G_2(3)$ $(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $L_4(3)$ $(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)'$ $(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9)$ $(2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$, $G_2(4)(2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13)$, $A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$, $L_2(2^t)$ $(7 \le t \le 13)$, $L_2(3^s)$ $(5 \le s \le 7)$ and Suz $(2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$.

Suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ $(2^2{\cdot}3{\cdot}7{\cdot}13)$, $L_3(3)$ $(2^4{\cdot}3^3{\cdot}13)$, $L_2(25)$ $(2^3{\cdot}3{\cdot}5^2{\cdot}13)$, $L_2(27)$ $(2^2{\cdot}3^3{\cdot}7{\cdot}13)$, $Sz(8)$ $(2^6{\cdot}5{\cdot}7{\cdot}13)$, $U_3(4)$ $(2^6{\cdot}3{\cdot}5^2{\cdot}13)$, $L_2(64)$ $(2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13)$, $G_2(3)$ $(2^6{\cdot}3^6{\cdot}7{\cdot}13)$, $L_4(3)$ $(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)'$ $(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9)$ $(2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$ and $G_2(4)$ $(2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13)$. From [11] we get that $11 \dagger |Out(K/H))|$, and thus $11 \mid |H|$. Let L be a Sylow 11-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action trivial, which implies that G has an element of order 33, a contradiction.

Suppose that $K/H \cong A_{13}$ $(2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11{\cdot}13)$. In such case, $|Out(K/H)| = 2$. So $|G/K| \mid 2$ and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H. Then $L \trianglelefteq G$ and $|L| = 3^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 33, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ $(7 \le t \le 13)$ or $L_2(3^s)$ $(5 \le s \le 7)$. From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Suz$ $(2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong Suz$.

Case 1.12. Assume that $A = O'N$ $(2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31)$. In this case,

$|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$, and $k_1(G) = 31$. Since $k_1(G) = 31$, we have $t(G) \geq 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 19\}$ and $31 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(31)$ $(2^5 \cdot 3 \cdot 5 \cdot 31)$, $L_2(32)$ $(2^5 \cdot 3 \cdot 11 \cdot 31)$, $L_2(2^t)$ $(7 \leq t \leq 9)$, $L_2(7^3)$ and $O'N(2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31)$.

Suppose that $K/H \cong L_2(31)$ $(2^5 \cdot 3 \cdot 5 \cdot 31)$ or $L_2(32)$ $(2^5 \cdot 3 \cdot 11 \cdot 31)$. Then $19 \nmid |Out(K/H))|$, and thus $19 \mid |H|$. Let L be a Sylow 19-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 95, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ $(7 \leq t \leq 9)$ or $L_2(7^3)$. From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong O'N$ $(2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31)$. So $H = 1$, $K = G$, and therefore $G \cong O'N$.

Case 1.13. Assume that $A = Co_3$ $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. In this case, $|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 30$. Since $k_1(G) = 30$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $L_2(2^t)$ $(7 \leq t \leq 10)$, $L_2(3^s)$ $(5 \leq s \leq 7)$ and $Co_3(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$.

Suppose that $K/H \cong L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$. Then $7 \nmid |Out(K/H))|$, and thus $7 \mid |H|$. Let L be a Sylow 7- subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 35, a contradiction.

Suppose that $K/H \cong M_{23}$ $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ or M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In such case, $5 \nmid |Out(K/H)|$ and therefore $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. Then $L \trianglelefteq G$ and $|L| = 5^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 55, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ $(7 \leq t \leq 10)$ or $L_2(3^s)$ $(5 \leq s \leq 7)$. From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Co_3$ $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Co_3$.

Case 1.14. Assume that $A = Co_2$ $(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. In this case, $|G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 30$. Since $k_1(G) = 30$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

So, we have $\pi(H)\cup\pi(G/K) \subseteq \{2, 3, 5, 7, 11,\}$ and $23 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, Co_2 $(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 18)$, $L_2(3^s)$ $(5 \le s \le 6)$, $L_3(2^t)$ $(5 \le t \le 6)$ and $U_3(2^s)$ $(5 \le s \le 6)$.

Suppose that $K/H \cong L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$. Then $7 \nmid |Out(K/H))|$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 35, a contradiction.

Suppose that $K/H \cong M_{23}$ $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ or M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In such case, $5 \nmid |Out(K/H)|$, and therefore $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. Then $L \trianglelefteq G$ and $|L| = 5^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 55, a contradiction too.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which implies $(q^2 - 1) \mid |G|$. We can see it is impossible from Table 3.

Therefore, we have $K/H \cong Co_2$ $(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Co_2$.

Case 1.15. Assume that $A = Ly$ $(2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67)$. In this case, $|G| = 2^8{\cdot}3^7{\cdot}5^6{\cdot}7{\cdot}11{\cdot}31{\cdot}37{\cdot}67$, and $k_1(G) = 67$. Since $k_1(G) = 67$, we have $t(G) \ge 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H)\cup\pi(G/K) \subseteq \{2, 3, 5, 7, 11, 31, 37\}$ and $67 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to Ly $(2^8{\cdot}3^7{\cdot}5^6{\cdot}7{\cdot}11{\cdot}31{\cdot}37{\cdot}67)$ or $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 8)$, $L_2(3^s)$ $(5 \le s \le 7)$ and $L_2(5^r)$ $(4 \le r \le 6)$.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. From Table 3, we can see it is impossible.

Therefore, we have $K/H \cong Ly$ $(2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67)$. So $H = 1$, $K = G$, and therefore $G \cong Ly$.

Case 1.16. Assume that $A = Th$ $(2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31)$. In this case, $|G| = 2^{15}{\cdot}3^{10}{\cdot}5^3{\cdot}7^2{\cdot}13{\cdot}19{\cdot}31$, and $k_1(G) = 39$. Since $k_1(G) = 39$, we have $t(G) \ge 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 13, 19\}$ and $31 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(31)$ $(2^5{\cdot}3{\cdot}5{\cdot}31)$, $L_2(32)$ $(2^5{\cdot}3{\cdot}11{\cdot}31)$, $L_3(5)$ $(2^5{\cdot}3{\cdot}5^3{\cdot}31)$, $L_2(125)$ $(2^2{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}31)$, $L_5(2)$ $(2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31)$, $L_6(2)$ $(2^{15}{\cdot}3^4{\cdot}5{\cdot}7^2{\cdot}31)$, Th $(2^{15}{\cdot}3^{10}{\cdot}5^3{\cdot}7^2{\cdot}13{\cdot}19{\cdot}31)$

and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 15)$, $L_2(3^s)$ $(5 \le s \le 10)$, $L_3(2^5)$ and $U_3(2^5)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(31)$ $(2^5 \cdot 3 \cdot 5 \cdot 31)$, $L_2(32)$ $(2^5 \cdot 3 \cdot 11 \cdot 31)$, $L_3(5)$ $(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_2(125)$ $(2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31)$, $L_5(2)$ $(2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31)$, $L_6(2)$ $(2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31)$. Then $19 \nmid |Out(K/H))|$, and thus $19 \mid |H|$. Let L be a Sylow 19-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 95, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. We can see it is impossible from Table 3.

Therefore, we have $K/H \cong Th$ $(2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31)$. So $H = 1$, $K = G$, and therefore $G \cong Th$.

Case 1.17. Assume that $A = J_4$ $(2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43)$. In this case, $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, and $k_1(G) = 66$. Since $k_1(G) = 66$, we have $t(G) \ge 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 23, 29, 31, 37\}$ and $43 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(43)$ $(2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43)$, J_4 $(2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 21)$, $L_2(11^3)$, $L_3(2^s)$ $(5 \le s \le 7)$ and $U_3(2^r)$ $(5 \le r \le 7)$.

Suppose that $K/H \cong L_2(43)$ $(2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43)$. Then $37 \nmid |Out(K/H))|$, and thus $37 \mid |H|$. Let L be a Sylow 37-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 185, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. If $B(q)$ is $L_2(2^t)$ $(7 \le t \le 21)$, $L_2(11^3)$, $L_3(2^s)$ $(6 \le s \le 7)$ or $U_3(2^r)$ $(6 \le r \le 7)$, then $(q^2 - 1) \nmid |G|$ by Table 3, which is a contradiction. If $B(q)$ is $L_3(2^5)$ or $U_3(2^5)$, we can easily know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong J_4$ $(2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43)$. So $H = 1$, $K = G$, and therefore $G \cong J_4$.

Case 1.18. Assume that $A = B$ $(2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47)$. In this case, $|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, and $k_1(G) = 70$. Since $k_1(G) = 70$, we have $t(G) \ge 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31\}$ and $47 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(47)$ $(2^4 \cdot 3 \cdot 23 \cdot 47)$, B $(2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47)$ and $B(q)$, where $B(q)$ is one of the following simple groups:

$L_2(2^t)$ $(7 \le t \le 41)$, $L_2(3^s)$ $(5 \le s \le 13)$, $L_2(5^r)$ $(4 \le r \le 6)$, $L_3(2^t)$ $(5 \le t \le 13)$, $L_3(3^4)$, $U_3(2^t)$ $(5 \le t \le 13)$, $U_3(3^4)$, $L_4(2^t)$ $(4 \le t \le 6)$, $U_4(2^t)$ $(4 \le t \le 6)$, $S_4(2^t)$ $(6 \le t \le 10)$ and $G_4(2^t)$ $(5 \le t \le 6)$.

Suppose that $K/H \cong L_2(47)$ $(2^4 \cdot 3 \cdot 23 \cdot 47)$. Then $31 \nmid |Out(K/H))|$, and thus $31 \mid |H|$. Let L be a Sylow 31-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is also trivial, which implies that G has an element of order 217, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2-1) \mid |B(q)|$, which means $(q^2-1) \mid |G|$. We can easily know that $|B(q)| \nmid |G|$ by Lemma 5, Table 2 and Table 3, also a contradiction.

Therefore, we have $K/H \cong B$ $(2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47)$. So $H = 1$, $K = G$, and therefore $G \cong B$.

Case 1.19. Assume that $A = M$ $(2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71)$. In this case, $|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71)$, and $k_1(G) = 119$. Since $k_1(G) = 119$, we have $t(G) \ge 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59\}$ and $71 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(71)$ $(2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71)$, $M(2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71)$ and $B(q)$, where $B(q)$ is one of the following simple groups:

$L_2(2^t)$ $(7 \le t \le 46)$, $L_2(3^s)$ $(5 \le s \le 20)$, $L_2(5^r)$ $(4 \le r \le 9)$, $L_2(7^x)$ $(3 \le x \le 6)$, $L_2(13^y)$ $(2 \le y \le 3)$, $L_3(2^t)$ $(5 \le t \le 15)$, $L_3(3^s)$ $(4 \le s \le 6)$, $L_3(5^3)$, $L_3(7^2)$, $U_3(2^t)$ $(5 \le t \le 15)$, $U_3(3^s)$ $(4 \le t \le 6)$, $U_3(5^3)$, $U_3(7^2)$, $L_4(2^t)$ $(4 \le t \le 7)$, $L_4(3^3)$, $U_4(2^t)$ $(4 \le t \le 7)$, $U_4(3^3)$, $S_4(2^t)$ $(6 \le t \le 11)$, $S_4(3^s)$ $(4 \le s \le 5)$, $G_4(2^t)$ $(5 \le t \le 7)$ and $G_2(3^3)$.

Suppose that $K/H \cong L_2(71)$ $(2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71)$. Then $31 \nmid |Out(K/H))|$, and thus $31 \mid |H|$. Let L be a Sylow 31-subgroup of H. Then $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is also trivial, which implies that G has an element of order 217, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2-1) \mid |B(q)|$, and therefore $(q^2-1) \mid |G|$. If $B(q)$ is not $L_2(2^{10})$ and $L_2(13^2)$, then we can easily know that $|B(q)| \nmid |G|$ by Lemma 5, Table 2 and Table 3, a contradiction. We now assume that $B(q)$ is $L_2(2^{10})$ or $L_2(13^2)$. From Table 3, we have $71 \nmid |B(q)|$, also a contradiction.

Therefore, we have $K/H \cong M(2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71)$. So $H = 1$, $K = G$, and therefore $G \cong M$.

Now Theorem 1 follows from Case 1.1 to Case 1.19.

Proof of Theorem 2. It is enough to prove the sufficiency.

Case 2.1. Assume that $A = J_2$ $(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$. In this case, $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $k_1(G) = 15$ and $k_2(G) = 12$. Since $k_1(G) = 15$, $k_2(G) = 12$, we have $t(G) \ge 2$.

By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5,\}$ and $7 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ $(2^3 \cdot 3 \cdot 7)$, $L_2(8)$ $(2^3 \cdot 3^2 \cdot 7)$, A_7 $(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $U_3(3)$ $(2^5 \cdot 3^3 \cdot 7)$, $L_4(2)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(4)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7)$ and $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ $(2^3{\cdot}3{\cdot}7)$, $L_2(8)$ $(2^3{\cdot}3^2{\cdot}7)$, A_7 $(2^3{\cdot}3^2{\cdot}5{\cdot}7)$, $U_3(3)$ $(2^5{\cdot}3^3{\cdot}7)$, $L_4(2)$ $(2^6{\cdot}3^2{\cdot}5{\cdot}7)$ and $L_3(4)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7)$. By [11] we know that $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. We know that $L \trianglelefteq G$, and $|L| \mid 5^2$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 35, a contradiction.

Therefore, we have $K/H \cong J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$. So $H = 1$, $K = G$, and therefore $G \cong J_2$.

Case 2.2. Assume that $A = McL$ $(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11)$. In this case, $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$, $k_1(G) = 30$ and $k_2(G) = 15$. Since $k_1(G) = 30$ and $k_2(G) = 15$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(11)$ $(2^2{\cdot}3{\cdot}5{\cdot}11)$, M_{11} $(2^4{\cdot}3^2{\cdot}5{\cdot}11)$, M_{12} $(2^6{\cdot}3^3{\cdot}5{\cdot}11)$, M_{22} $(2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11)$, A_{11} $(2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11)$, McL $(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11)$, $L_2(2^7)$ and $L_2(3^s)$ $(5 \leq s \leq 6)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(11)$ $(2^2{\cdot}3{\cdot}5{\cdot}11)$, M_{11} $(2^4{\cdot}3^2{\cdot}5{\cdot}11)$, M_{12} $(2^6{\cdot}3^3{\cdot}5{\cdot}11)$, M_{22} $(2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11)$ and A_{11} $(2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11)$. By [11] we know that $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H. We know that $L \trianglelefteq G$, and $|L| \mid 5^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial. It implies that G has an element of order 55, which is a contradiction.

Suppose that $K/H \cong L_2(2^7)$ or $L_2(3^s)$ $(5 \leq s \leq 6)$. From Table 3, we have $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong McL$ $(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong McL$.

Case 2.3. Assume that $A = Fi_{22}$ $(2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. In this case, $|G| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $k_1(G) = 30$ and $k_2(G) = 24$. Since $k_1(G) = 30$ and $k_2(G) = 24$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ $(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13)$, $L_3(3)$ $(2^4 \cdot 3^3 \cdot 13)$, $L_2(25)$ $(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(27)$ $(2^2 \cdot 3^3 \cdot 7 \cdot 13)$, $Sz(8)$ $(2^6 \cdot 5 \cdot 7 \cdot 13)$, $U_3(4)$ $(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$,

$G_2(3)$ $(2^6{\cdot}3^6{\cdot}7{\cdot}13)$, $L_4(3)$ $(2^7{\cdot}3^6{\cdot}5{\cdot}13)$, ${}^2F_4(2)'$ $(2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13)$, $L_3(9)$ $(2^7{\cdot}3^6{\cdot}5{\cdot}7{\cdot}13)$ $G_2(4)$ $(2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13)$, A_{13} $(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13)$, $S_6(3)$ $(2^9\cdot 3^9\cdot 5\cdot 7\cdot 13)$, $O_7(3)$ $(2^9\cdot 3^9\cdot 5\cdot 7\cdot 13)$, Suz $(2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13)$, Fi_{22} $(2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7\le t\le 17)$, $L_2(3^s)$ $(5\le s\le 9)$, $L_3(2^t)$ $(5\le t\le 6)$ and $U_3(2^5)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(13)$ $(2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}13)$, $L_3(3)$ $(2^4{\cdot}3^3{\cdot}13)$, $L_2(25)$ $(2^3{\cdot}3{\cdot}5^2{\cdot}13)$, $L_2(27)$ $(2^2{\cdot}3^3{\cdot}7{\cdot}13)$, $Sz(8)$ $(2^6{\cdot}5{\cdot}7{\cdot}13)$, $U_3(4)$ $(2^6{\cdot}3{\cdot}5^2{\cdot}13)$, $L_2(64)$ $(2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13)$, $G_2(3)$ $(2^6{\cdot}3^6{\cdot}7{\cdot}13)$, $L_4(3)$ $(2^7\cdot 3^6\cdot 5\cdot 13)$, ${}^2F_4(2)'$ $(2^{11}\cdot 3^3\cdot 5^2\cdot 13)$, $L_3(9)$ $(2^7\cdot 3^6\cdot 5\cdot 7\cdot 13)$, $G_2(4)$ $(2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13)$, $S_6(3)$ $(2^9\cdot 3^9\cdot 5\cdot 7\cdot 13)$ and $O_7(3)$ $(2^9\cdot 3^9\cdot 5\cdot 7\cdot 13)$, By [11] we know that $11\nmid |Out(K/H)|$, and thus $11 \mid |H|$. Let L be a Sylow 11-subgroup of H. We know that $L\trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 77, which is a contradiction.

Suppose that $K/H\cong A_{13}$ $(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13)$. By [11] we know that $|Out(K/H)|=2$, and thus $3\mid |H|$. Let L be a Sylow 3-subgroup of H. We know that $L\trianglelefteq G$ and $|L|\mid 3^4$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))|\mid 3^4$. Since $\Omega_1(Z(L))$ is characteristic in L, we have $\Omega_1(Z(L))\trianglelefteq G$ for $L\trianglelefteq G$. Consider the action on $\Omega_1(Z(L))$ by the element of order 11. Since $11\nmid |Aut(\Omega_1(Z(L)))|$, this action is trivial. It implies that G has an element of order 33, also a contradiction.

Suppose that $K/H\cong Suz$ $(2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13)$. By [11] we know that $|Out(K/H)|=2$, and therefore $3\mid |H|$. Let L be a Sylow 3-subgroup of H. We know that $L\trianglelefteq G$, and $|L|\mid 3^2$. Consider the action on L by the element of order 13. Clearly, this action is trivial, which implies that G has an element of order 39, a contradiction too.

Suppose that $K/H\cong B(q)$. From Table 2, we know that $(q^2-1)\mid |B(q)|$ and therefore $(q^2-1)\mid |G|$. But Table 2 and Table 3 show that $|B(q)|\nmid |G|$, still a contradiction.

Therefore, we have $K/H\cong Fi_{22}(2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13)$. So $H=1$, $K=G$, and therefore $G\cong Fi_{22}$.

Case 2.4. Assume that $A=HN$ $(2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19)$. In this case, $|G|=2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$, $k_1(G)=40$ and $k_2(G)=35$. Since $k_1(G)=40$ and $k_2(G)=35$, we have $t(G)\ge 2$. By Lemma 8, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K|\mid |Out(K/H)|$. So, we have $\pi(H)\cup\pi(G/K)\subseteq\{2,3,5,7,11\}$ and $19\in\pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(19)$ $(2^2\cdot 3^2\cdot\cdot 5\cdot 19)$, J_1 $(2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19)$, $U_3(8)$ $(2^9\cdot 3^4\cdot 7\cdot 19)$, HN $(2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19)$ and B(q), where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7\le t\le 14)$, $L_2(3^s)$ $(5\le s\le 6)$, $L_2(5^r)$ $(4\le s\le 6)$ and $U_3(2^5)$.

Suppose that $K/H\cong L_2(19)$ $(2^2\cdot 3^2\cdot 5\cdot 19)$ or $U_3(8)$ $(2^9\cdot 3^4\cdot 7\cdot 19)$, By [11] we know that $11\nmid |Out(K/H)|$, and thus $11\mid |H|$. Let L be a Sylow 11-subgroup of H. We know that $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 77, a contradiction.

Suppose that $K/H \cong J_1$ $(2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$. By [11] we know that $|Out(K/H)| = 1$, and $3 \mid |H|$. Let L be a Sylow 3-subgroup of H. We know that $L \trianglelefteq G$ and $|L| \mid 3^5$. Consider $\Omega_1(Z(L))$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group and $\Omega_1(Z(L)) \trianglelefteq G$. Because $|\Omega_1(Z(L))| \mid 3^5$, we have $19 \nmid |Aut(\Omega_1(Z(L)))|$. Consider the action on $\Omega_1(Z(L))$ by the element of order 19. One can see that this action is trivial. It implies that G has an element of order 57, also a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we can know that $|B(q)| \nmid |G|$, which is a contradiction.

Therefore, we have $K/H \cong HN$ $(2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19)$. So $H = 1$, $K = G$, and therefore $G \cong HN$.

Case 2.5. Assume that $A = Fi_{23}$ $(2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$. In this case, $|G| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $k_1(G) = 60$ and $k_2(G) = 42$. Since $k_1(G) = 60$ and $k_2(G) = 42$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17\}$ and $23 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, Co_3 $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, Fi_{23} $(2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \leq t \leq 18)$, $L_2(3^s)$ $(5 \leq s \leq 13)$, $L_3(2^t)$ $(5 \leq t \leq 6)$, $L_3(3^4)$, $U_3(2^t)$ $(5 \leq t \leq 6)$ and $U_3(3^4)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ and Co_3 $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, By [11] we know that $17 \nmid |Out(K/H)|$, and thus $17 \mid |H|$. Let L be a Sylow 17-subgroup of H. We know that $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 85, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Fi_{23}$ $(2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Fi_{23}$.

Case 2.6. Assume that $A = Co_1$ $(2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23)$. In this case, $|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$, $k_1(G) = 60$ and $k_2(G) = 42$. Since $k_1(G) = 60$ and $k_2(G) = 42$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13\}$ and $23 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, Co_3 $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, Co_2 $(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, $Co_1(2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 21)$, $L_2(3^s)$ $(5 \le s \le 9)$, $L_2(5^4)$, $L_3(2^t)$ $(5 \le t \le 7)$ and $U_3(2^t)$ $(5 \le t \le 7)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(23)$ $(2^3 \cdot 3 \cdot 11 \cdot 23)$, M_{23} $(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, M_{24} $(2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, Co_3 $(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$ and Co_2 $(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, By [11] we know that $13 \nmid |Out(K/H)|$, and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H. We know that $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 65, which is a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Co_1$ $(2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Co_1$.

Case 2.7. Assume that $A = Fi_{24}$ $(2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29)$. In this case, $|G| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, $k_1(G) = 60$ and $k_2(G) = 45$. Since $k_1(G) = 60$ and $k_2(G) = 45$, we have $t(G) \ge 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 23\}$ and $29 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(29)$ $(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29)$, Fi_{24} $(2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ $(7 \le t \le 21)$, $L_2(3^s)$ $(5 \le s \le 16)$, $L_2(7^3)$, $L_3(2^t)$ $(5 \le t \le 7)$, $L_3(3^s)$ $(4 \le s \le 5)$, $U_3(2^t)$ $(5 \le t \le 7)$ and $U_3(3^s)$ $(4 \le s \le 5)$.

Suppose that $K/H \cong L_2(29)$ $(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29)$. By [11] we know that $23 \nmid |Out(K/H)|$, and thus $23 \mid |H|$. Let L be a Sylow 23-subgroup of H. We know that $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action is trivial. It implies that G has an element of order 69, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, a contradiction too.

Therefore, we have $K/H \cong Fi_{24}$ $(2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29)$. So $H = 1$, $K = G$, and therefore $G \cong Fi_{24}$.

Now Theorem 2 follows from Case2.1 to Case 2.7.

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