CAUCHY'S METHOD AND BILATERAL BASIC HYPERGEOMETRIC SERIES

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Abstract. In this paper, we find bilateral basic hypergeometric series adapting Cauchy's method used by Bailey, Slater, Fredric Jouhet and Michael Schlosser.

Key words: bilateral basic hypergeometric series, q-series.

1. Introduction

Schlosser [10] gave a new proof of Ramanujans ${}_1\psi_1$ summation formula [6].

$$ {}_1\psi_1\left[\begin{array}{c} a;q;z \\ b \end{array}\right] = \frac{(q,b/a,az,q/az)_\infty}{(b,q/a,z,b/az)_\infty} \tag{1.1}$$

valid for $|q| < 1$, $|b/a| < |z| < 1$. This proof was given by a famous method already utilized by Cauchy [5] in the second proof of Jacobi's [8] triple product identity. The same method known as Cauchys method had also been used by Bailey [3] and Slater [11]. Cauchy's method is used to obtain any bilateral sum from an approximately chosen terminating identity.

Jouhet and Schlosser [9] used Cauchy's method in Jackson's [7] q-Ptaff–Saalschutz summation

$$ {}_3\phi_2\left[\begin{array}{c} a,b,q^{-n};q;q \\ c,abq^{1-n/c} \end{array}\right] = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n} \tag{1.2}$$

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to prove Ramanujan's ${}_1\psi_1$ summation. Also, by considering Bailey's [2] transformation formula, Jouhet and Schlosser proved Bailey's ${}_6\psi_6$ summation formula.

It is necessary to give some standard notations for q-series and basic hypergeometric series. Let q be a complex number, i.e., base with $0 < |q| < 1$. The q-shifted factorial is defined for any complex parameter a by

$$(a)_\infty = (a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$$

and

$$(a)_k = (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer.

Also,

$$(a_1, a_2, ..., a_m)_k = (a_1)_k, ..., (a_m)_k,$$

where k is an integer or infinity.

The basic hypergeometic series is given by

$$ {}_s\phi_{s-1}\left[\begin{matrix} a_1, ..., a_s; q; z \\ b_1, ..., b_{s-1} \end{matrix}\right] = \sum_{k=0}^{\infty}\frac{(a_1, ..., a_s; q)_k}{(q, b_1, ..., b_{s-1}; q)_k}\, z^k \tag{1.3}$$

and of bilateral basic hypergeometric series,

$$ {}_s\psi_s\left[\begin{matrix} a_1, ..., a_s; q; z \\ b_1, ..., b_s \end{matrix}\right] = \sum_{k=-\infty}^{\infty}\frac{(a_1, ..., a_s; q)_k}{(b_1, ..., b_s; q)_k}\, z^k \tag{1.4}$$

Also this paper depends on some elementary identities for q-shifted factorials listed by Gasper and Rahman [6]. In this paper, we use Cauchy's method to bilateralize well-known terminating identities of Carlitz [4], Verma and Jain [12] and Agarwal [1].

2. Main results

First, let us consider Carlitz [4] transformation formula, to bilateralize it by Cauchy's method;

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, c, d, \frac{q^{\frac{1}{2}-n}}{cd}; q; q^2 \\ \frac{1}{c}\, q^{1-n}, \frac{1}{d}\, q^{1-n}, cd\sqrt{q} \end{matrix}\right] = \frac{(cd;q)_n(c,d,-\sqrt{q};\sqrt{q})_n}{(cd;\sqrt{q})_n(c,d;q)_n}. \tag{2.1}$$

First replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n,

$$\sum_{k=0}^{2n}\frac{\left(q^{-2n},c,d,\frac{1}{cd}\,q^{\frac{1}{2}-2n};q\right)_k q^2}{\left(q,\frac{1}{c}\,q^{1-2n},\frac{1}{d}\,q^{1-2n},cd\sqrt{q};q\right)_k}=\frac{(cd;q)_{2n}(c,d,-\sqrt{q};\sqrt{q})_{2n}}{(cd;\sqrt{q})_{2n}(c,d;q)_{2n}}$$

$$\Longrightarrow \sum_{k=-n}^{n}\frac{\left(q^{-2n},c,d,\frac{1}{cd}\,q^{\frac{1}{2}-2n};q\right)_k q^{2(n+k)}}{\left(q,\frac{1}{c}\,q^{1-2n},\frac{1}{d}\,q^{1-2n},cd\sqrt{q};q\right)_k}=\frac{(cd;q)_{2n}(c,d,-\sqrt{q};\sqrt{q})_{2n}}{(cd;\sqrt{q})_{2n}(c,d;q)_{2n}}$$

$$\Longrightarrow \frac{\left(q^{-2n},c,d,\frac{1}{cd}\,q^{\frac{1}{2}-2n};q\right)_n q^{2n}}{\left(q,\frac{1}{c}\,q^{1-2n},\frac{1}{d}\,q^{1-2n},cd\sqrt{q};q\right)_k}\sum_{k=-n}^{n}\frac{\left(q^{-n},cq^n,dq^n,\frac{1}{cd}\,q^{\frac{1}{2}-n};q\right)_k q^{2k}}{\left(q^{1+n},\frac{1}{c}\,q^{1-n},\frac{1}{d}\,q^{1-n},cdq^{\frac{1}{2}+n};q\right)_k}$$

$$=\frac{(cd;q)_{2n}(c,d,-\sqrt{q};\sqrt{q})_{2n}}{(cd;\sqrt{q})_{2n}(c,d;q)_{2n}}\,.$$

Take $c=cq^{-n}$ and $d=dq^{-n}$, we get

$$\sum_{k=-n}^{n}\frac{\left(q^{-n},c,d,\frac{1}{cd}\,q^{\frac{1}{2}+n};q\right)_k q^{2k}}{\left(q^{1+n},\frac{q}{c},\frac{q}{d},cdq^{\frac{1}{2}-n};q\right)_k}$$
$$=\frac{(cdq^{-2n};q)_{2n}(cq^{-n},dq^{-n},-\sqrt{q};\sqrt{q})_{2n}\left(q,\frac{1}{c}\,q^{1-n},\frac{1}{d}\,q^{1-n},cdq^{\frac{1}{2}-2n};q\right)_n q^{-2n}}{(cdq^{-2n};\sqrt{q})_{2n}(cq^{-n},dq^{-n};q)_{2n}\left(q^{-2n},cq^{-n},dq^{-n},\frac{\sqrt{q}}{cd};q\right)_n}$$

$$\sum_{k=-n}^{n}\frac{\left(q^{-n},c,d,\frac{1}{cd}\,q^{\frac{1}{2}+n};q\right)_k q^{2k}}{\left(q^{1+n},\frac{q}{c},\frac{q}{d},cdq^{\frac{1}{2}-n};q\right)_k}$$
$$=\frac{(-\sqrt{q},-q,q,q;q)_n(q/cd;q)_n(\sqrt{q}/c,\sqrt{q}/d;q)_n}{(q/c,q/d,\sqrt{q}/cd;q)_n(q;q)_{2n}}\,.$$

Now, let $n\to\infty$, and assuming $|q^{3/2}/cd|<1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$\sum_{k=-\infty}^{\infty}\frac{(c,d;q)_k}{\left(\frac{q}{c},\frac{q}{d};q\right)_k}\left(\frac{q^{3/2}}{cd}\right)^k=\frac{(-\sqrt{q},-q,q,q,q/cd;q)_\infty(\sqrt{q}/c,\sqrt{q}/d;q)_\infty}{(q/c,q/d,\sqrt{q}/cd;q)_\infty(q;q)_\infty}$$

(2.2)

$$\Longrightarrow {}_2\psi_2\left[\begin{matrix}c,d;q;q^{3/2}/cd\\ q/c,q/d\end{matrix}\right]=\frac{(-\sqrt{q},-q,q,q/cd,\sqrt{q}/c,\sqrt{q}/d;q)_\infty}{(q/c,q/d,\sqrt{q}/cd;q)_\infty}\,.$$

Next, consider Verma and Jain's summation [12]

(2.3)

$${}_4\phi_3\left[\begin{matrix}q^{-n},c,d,\frac{1}{cd}\,q^{\frac{3}{2}-n};q;q\\ \frac{1}{c}\,q^{1-n},\frac{1}{d}\,q^{1-n},cd\sqrt{q}\end{matrix}\right]$$
$$=\frac{(cd/\sqrt{q};\sqrt{q})_{2n}(c,d;\sqrt{q})_n(q;q)_n}{(cd/\sqrt{q};\sqrt{q})_n(cd\sqrt{q};q)_n(c,d;q)_n(\sqrt{q};\sqrt{q})_n}\,.$$

First replace n by $2n$ and then shift the index of summation by n, such that the new sum runs from $-n$ to n,

$$\begin{aligned}
&\frac{(cd/\sqrt{q}, cd; q)_{2n}(c, d; \sqrt{q})_{2n}(q; q)_{2n}}{(cd/\sqrt{q}; \sqrt{q})_{2n}(cd\sqrt{q}; q)_{2n}(c, d; q)_{2n}(\sqrt{q}; \sqrt{q})_{2n}} \\
&= \sum_{k=0}^{2n} \frac{\left(q^{-2n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}-2n}; q\right)_k \cdot q^k}{\left(q, \frac{1}{c}\, q^{1-2n}, \frac{1}{d}\, q^{1-2n}, cd\sqrt{q}; q\right)_k} \\
\Longrightarrow \quad & \sum_{k=-n}^{n} \frac{\left(q^{-2n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}-2n}; q\right)_k \cdot q^{n+k}}{\left(q, \frac{1}{c}\, q^{1-2n}, \frac{1}{d}\, q^{1-2n}, cd\sqrt{q}; q\right)_{n+k}} \\
&= \frac{\left(q^{-2n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}-2n}; q\right)_n \cdot q^n}{\left(q, \frac{1}{c}\, q^{1-2n}, \frac{1}{d}\, q^{1-2n}, cd\sqrt{q}; q\right)_n} \sum_{k=-n}^{n} \frac{\left(q^{-n}, cq^n, dq^n, \frac{1}{cd}\, q^{\frac{3}{2}-n}; q\right)_k \cdot q^k}{\left(q^{1+n}, \frac{1}{c}\, q^{1-n}, \frac{1}{d}\, q^{1-n}, cd\sqrt{q}q^n; q\right)_k}
\end{aligned}$$

Take $c = cq^{-n}$ and $d = dq^{-n}$, we get

$$\begin{aligned}
&\frac{\left(cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q\right)_{2n} (cq^{-n}, dq^{-n}; \sqrt{q})_{2n}(q; q)_{2n}}{\left(cdq^{-2n-\frac{1}{2}}; \sqrt{q}\right)_{2n} \left(cdq^{-2n+\frac{1}{2}}; q\right)_{2n} (cq^{-n}, dq^{-n}; q)_{2n}(\sqrt{q}; \sqrt{q})_{2n}} \\
&= \frac{(q^{-2n}, cq^{-n}, dq^{-n}, q^{3/2}/cd; q)_n \cdot q^n}{\left(q, \frac{1}{c}\, q^{1-n}, \frac{1}{d}\, q^{1-n}, cdq^{-2n+\frac{1}{2}}; q\right)_n} \sum_{k=-n}^{n} \frac{\left(q^{-n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}+n}; q\right)_k \cdot q^k}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q\right)_k} \\
\Longrightarrow \quad & \sum_{k=-n}^{n} \frac{\left(q^{-n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}+n}; q\right)_k \cdot q^k}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q\right)_k} \\
&= \frac{\left(cdq^{-2n-\frac{1}{2}}, cdq^{-n-\frac{1}{2}}, cdq^{-2n}, cdq^{-n}; q\right)_n}{\left(cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q\right)_n \left(cdq^{-2n+\frac{1}{2}}, cdq^{-n+\frac{1}{2}}; q\right)_n} \\
&\qquad \times \frac{\left(cq^{-n}, dq^{-n}, cq^{-n+\frac{1}{2}}, dq^{-n+\frac{1}{2}}; q\right)_n (q; q)_{2n}(q; q)_n}{(cq^{-n}, dq^{-n}, c, d; q)_n(\sqrt{q}, q; q)_n(q^{3/2}/cd; q)_n} \\
&\qquad \times \frac{\left(\frac{1}{c}\, q^{1-n}, \frac{1}{d}\, q^{1-n}, cdq^{-2n+\frac{1}{2}}; q\right)_n \cdot q^{-n}}{(q^{-2n}, cq^{-n}, dq^{-n}; q)_n} \\
\Longrightarrow \quad & \sum_{k=-n}^{n} \frac{\left(q^{-n}, c, d, \frac{1}{cd}\, q^{\frac{3}{2}+n}; q\right)_k \cdot q^k}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q\right)_k} = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d, q; q)_n}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_n(q; q)_{2n}}.
\end{aligned}$$

Let $n \to \infty$, and assuming $|\sqrt{q}/cd| < 1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

(2.4)
$$\sum_{k=-\infty}^{\infty} \frac{(c,d;q)_k}{\left(\frac{q}{c},\frac{q}{d};q\right)_k} \left(\frac{\sqrt{q}}{cd}\right)^k = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d; q)_\infty}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_\infty}$$
$$\Longrightarrow {}_2\psi_2\left[\begin{matrix} c, d; q, \sqrt{q}/cd \\ q/c, q/d \end{matrix}\right] = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d; q)_\infty}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_\infty}.$$

Again consider another summation of Verma and Jain [12];

(2.5)
$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, c, d, q^{\frac{1}{2}-n}/cd; q; q \\ q^{1-n}/c, q^{1-n}/d, cd\sqrt{q} \end{matrix}\right] = \frac{(cd;q)_n (c,d;\sqrt{q})_n (q;q)_n}{(c,d;q)_n(\sqrt{q};\sqrt{q})_n(cd/\sqrt{q};\sqrt{q})_n} \cdot q^{\frac{-n}{2}}.$$

First replace n by $2n$ and then shift the index of summation by n, such that the new sum runs from $-n$ to n,

$$\frac{(cd;q)_{2n}(c,d;\sqrt{q})_{2n}(q;q)_{2n}\cdot q^{-n}}{(c,d;q)_{2n}(\sqrt{q};\sqrt{q})_{2n}(cd/\sqrt{q};\sqrt{q})_{2n}} = \sum_{k=0}^{2n} \frac{\left(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q\right)_k \cdot q^k}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_k}$$
$$\Longrightarrow \sum_{k=-n}^{n} \frac{\left(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q\right)_k \cdot q^{n+k}}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_{n+k}}$$
$$= \frac{\left(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q\right)_n \cdot q^n}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_n} \sum_{k=-n}^{n} \frac{\left(q^{-n}, cq^n, dq^n, q^{\frac{1}{2}-n}/cd; q\right)_k \cdot q^k}{\left(q^{1+n}, q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}+n}; q\right)_k}.$$

Take $c = cq^{-n}$ and $d = dq^{-n}$, we get

$$\sum_{k=-n}^{n} \frac{\left(q^{-n}, c, d, \frac{1}{cd}\, q^{\frac{1}{2}+n}; q\right)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q)_k}$$
$$= \frac{(cdq^{-2n};q)_{2n}(cq^{-n}, dq^{-n}; \sqrt{q})_{2n}(q;q)_{2n}}{(cq^{-n}, dq^{-n}; q)_{2n}(\sqrt{q};\sqrt{q})_{2n}\left(cdq^{-2n-\frac{1}{2}};\sqrt{q}\right)_{2n}} \cdot q^{-n}$$
$$\times \frac{\left(q, q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}-2n}; q\right)_n \cdot q^{-n}}{(q^{-2n}, cq^{-n}, dq^{-n}, \sqrt{q}/cd; q)_n}$$
$$= \frac{(cdq^{-2n}, cdq^{-n}; q)_n \left(cq^{-n}, dq^{-n}, cq^{-n+\frac{1}{2}}, dq^{-n+\frac{1}{2}}; q\right)_n}{(cq^{-n}, dq^{-n}, c, d; q)_n(\sqrt{q}, q; q)_n\left(cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q\right)_n}$$
$$\times \frac{(q;q)_{2n}(q;q)_n\left(q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}-2n}; q\right)_n \cdot q^{-2n}}{(q^{-2n}, cq^{-n}, dq^{-n}, \sqrt{q}/cd; q)_n}$$
$$\Longrightarrow \sum_{k=-n}^{n} \frac{\left(q^{-n}, c, d, \frac{1}{cd}\, q^{\frac{1}{2}+n}; q\right)_k \cdot q^k}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q\right)_k} = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d, q; q)_n}{(\sqrt{q}, \sqrt{q}/cd, q/c, q/d; q)_n}.$$

Let $n \to \infty$, and assuming $|\sqrt{q}/cd| < 1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{(c,d;q)_\infty}{\left(\frac{q}{c},\frac{q}{d};q\right)_\infty} &= \frac{(q/cd,\sqrt{q}/c,\sqrt{q}/d;q;q)_\infty}{(\sqrt{q},\sqrt{q}/cd,q/c,q/d;q)_\infty} \\ {}_2\psi_2\left[\begin{matrix} c,d;q;\sqrt{q}/cd \\ q/c,q/d \end{matrix}\right] &= \frac{(q/cd,\sqrt{q}/c,\sqrt{q}/d,q;q)_\infty}{(\sqrt{q},\sqrt{q}/cd,q/c,q/d;q)_\infty}. \end{aligned} \tag{2.6}$$

Consider Agarwal's summation [1];

$$ {}_2\phi_1\left[\begin{matrix} a,b;q,q \\ abq \end{matrix}\right] = \frac{(aq,bq;q)_n}{(q,abq;q)_n}. \tag{2.7}$$

First replace n by $2n$ and then shift the index of summation by n, such that the new sum runs from $-n$ to n,

$$\begin{aligned} &\sum_{k=0}^{2n} \frac{(a,b;q)_k}{(q,abq;q)_k}\cdot q^k = \frac{(aq,bq;q)_{2n}}{(q,abq;q)_{2n}} \\ \Longrightarrow &\sum_{k=-n}^{n} \frac{(a,b;q)_{n+k}}{(q,abq;q)_{n+k}}\cdot q^{n+k} = \frac{(aq,bq;a)_{2n}}{(q,abq;q)_{2n}} \\ \Longrightarrow &\frac{(q,b;q)_n}{(q,abq;q)_n}\cdot q^n \cdot \sum_{k=-n}^{n} \frac{(aq^n,bq^n;q)_k}{(q^{1+n},abq^{1+n};q)_k}\cdot q^k = \frac{(aq,bq;q)_{2n}}{(q,abq;q)_{2n}} \\ \Longrightarrow &\sum_{k=-n}^{n} \frac{(aq^n,bq^n;q)_k}{(q^{1+n},abq^{1+n};q)_k}\cdot q^k = \frac{(aq,bq;q)_{2n}(q,abq;q)_n}{(q,abq;q)_{2n}(a,b;q)_n}\cdot q^{-n}. \end{aligned}$$

Take $a = aq^{-n}$, we get

$$\begin{aligned} \sum_{k=-n}^{n} \frac{(a,bq^n;q)_k}{(q^{1+n},abq;q)_k}\cdot q^k &= \frac{(aq^{1-n},bq;q)_{2n}(q,abq^{1-n};q)_n}{(q,abq^{1-n};q)_{2n}(aq^{-n},b;q)_n}\cdot q^{-n} \\ &= \frac{(bq;q)_{2n}(q,aq,1/a;q)_n}{(q;q)_{2n}(b,abq,q/a;q)_n}. \end{aligned}$$

Let $n \to \infty$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(abq;q)_k}\cdot q^k &= \frac{(bq,aq,q/a;q)_\infty}{(b,abq,q/a,q)_\infty}, \\ {}_1\psi_1\left[\begin{matrix} a;q;q \\ abq \end{matrix}\right] &= \frac{(bq,aq,1/a;q)_\infty}{(b,abq,q/a;q)_\infty}. \end{aligned} \tag{2.8}$$

3. Conclusion

We have established bilateral basic hypergeometric series adapting Cauchy's method used by Bailey, Slater, Fredric Jouhet and Michael Schlosser. It could also be very interesting to find bilateral series for any other terminating basic hypergeometric functions by Cauchy's method.

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