HAAR WAVELET METHOD FOR NUMERICAL SOLUTION OF TELEGRAPH EQUATIONS

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Abstract. In this paper we modified the result given by Hariharan [18] on the solution of Fisher's equation. We are giving the solution of second -order linear hyperbolic telegraph equation in one - space dimension. The telegraph equation is solved numerically by Haar wavelet method. Two numerical examples show the accuracy of the method. The present method is very simple, small computation costs and flexible.

Keywords: Haar wavelet, second-order hyperbolic telegraph equation, Matlab.

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1. Introduction

Wavelets have been applied extensively in many engineering field. In this paper we use Haar wavelet method to solve the telegraph equations. The telegraph equations appeared in many engineering field, such as modeling of anomalous diffusive and wave propagation phenomenon, modeling of anomalous diffusion and sub-diffusive systems, Continuous-time random walks.

In this paper we use second-order linear hyperbolic telegraph equation in one-space dimension, given by

$$\frac{\delta^2 u}{\delta t^2} + 2\alpha\frac{\delta u}{\delta t} + \beta^2 u = \frac{\delta^2 u}{\delta x^2} + f(x,t), \; a \leq x \leq b, \quad t \geq 0. \tag{1.1}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= f(x) \quad a \leq x \leq b, \\ \dot{u}(x,0) &= f_1(x) \quad a \leq x \leq b \end{aligned}$$

and the Dirichlet boundary conditions

$$U(a,t) = g_0(t), \; U(b,t) = g_1(t), \quad t \geq 0,$$

where α and β are known constant coefficients, for $\alpha > 0, \; \beta = 0$ equation (1.1) represents a damped wave equation and for $\alpha > \; \beta > 0$, it is called telegraph equation. We assume that $f(x), f_1(x)$ and their derivatives are continuous functions of x, and $g_0(t), \; g_1(t)$ and their derivatives are continuous function of t. both the electric voltage and the current in a double conductor satisfy the telegraph equation, where x is distance and t is time.

The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beam and machines) and are basis for fundamental equations of atomic physics. Equations of the form equation (1.1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [8], [9], [15], [18]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. For example biologists encounter these equation in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge [16]. Also the propagation of acoustic waves in Darcy-type porous media [17], and parallel floes of viscous Maxwell fluids [1] are just some of the phenomena governed [3], [10] by equation (1.1).

Haar wavelet are makeup of pairs of piecewise constant functions and mathematically the simplest orthonormal wavelets with a constant support. Due to the mathematical simplicity the Haar wavelet method has turned out to be an effective tool for solving differential and integral equations. Lipik [11]-[14] use Haar wavelet to solve differential and integral equations. Fasal-I Hak, Imram Aziz and Siraj-ul-islam [4] have used Haar wavelet numerical method for eight-order boundary value problem. Hariharan, Kannan and Sharma [5], [6] present a method for solving Fisher's and Fitzhugh - Nagmo equations.

In this paper we modified the result given by Hariharan [5[, [6] on the solution of Fisher's equation. This method consists of reducing the problem to a set of algebraic equations by first expanding the terms, which has maximum derivative, given in the equation as Haar functions with unknown coefficients. The operational matrix of integration and product operational matrix are utilized to evaluate the coefficients of the Haar functions. Since the differentiation of the Haar wavelet results into impulse functions, this approach is avoided and instead, method of integration is preferred. One main advantage of this method is that, we don't need to solve it manually it is fully computer supported.

2. Haar wavelet

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [7]. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions on the real line that can take only three values 1, -1 and 0. Haar wavelets, like the well-known Walsh functions, form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions.

Haar wavelet is defined for $x \in [0\ 1]$

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Haar wavelet family for $x \in [0\ 1]$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\eta_1, \eta_2) \\ -1 & \text{for } x \in [\eta_2, \eta_3) \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where $\eta_1 = \frac{K}{m}$, $\eta_2 = \frac{K+0.5}{m}$, $\eta_3 = \frac{K+1}{m}$. The integer $m = 2^j$ $(j = 0, 1, ..., J)$ indicates the level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter. The maximal level of relation is J. The index i is calculated according to the formula $i = m + k + 1$; In the case of minimal values $m = 1, k = 0$, we have $i = 2$. The maximum value of i is $i = 2^{J+1} = M$ It is assume that the value $i = 1$ corresponding to the scaling function for which $h_1 = 1$ for $x \in [0\ 1]$. The interval $[A, B]$ will be divided into M subintervals, hence $\Delta x = \frac{B-A}{M}$ and the matrices are in the dimension of $M \times M$.

We introduce the following notations

$$p_{i,1}(x) = \int_0^x h_i(x)\, dx\ , \tag{2.3}$$

$$p_{i,v}(x) = \int_0^x p_{i,v-1}(x)\, dx, \quad v = 2, 3, ... \tag{2.4}$$

These integrals can be evaluated by using equation (2.2) and the first two of them are given by

$$p_{i,1}(x) = \begin{cases} x-\eta_1 & \text{for } x \in [\eta_1, \eta_2) \\ \eta_3 - x & \text{for } x \in [\eta_2, \eta_3] \\ 0 & \text{elsewhere} \end{cases} \tag{2.5}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\eta_1)^2 & \text{for } x \in [\eta_1, \eta_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - x)^2 & \text{for } x \in [\eta_2, \eta_3), \\ \frac{1}{4m^2} & \text{for } x \in [\eta_3, 1], \\ 0 & \text{elsewhere.} \end{cases} \tag{2.6}$$

Similarly, we can find other integrals $p_{i,n}(x)$, $n = 1, 2, ...$

3. Method for solving telegraph equation

We consider telegraph equation (1.1) with the initial conditions $u(x,0) = f(x)$, $\dot{u}(x,0) = f_1(x)$, $0 < x < 1$ and the boundary conditions $u(0,t) = g_0(t)$ and $u(1,t) = g_1(t)$, $t > 0$.

In terms of the Haar wavelet, $\ddot{u}''(x,t)$ can be expanded as

$$\ddot{u}''(x, t) = \sum_{i=1}^{M} a_i h_i(x) \tag{3.1}$$

where ".." and "″" means differentiation with respect to t and x, respectively, Haar wavelet coefficient is constant in the subinterval $t \in [t_n, t_{n+1}]$.

On twice integration of equation (3.1) with respect to t from t_n to t and with respect to x from 0 to x, the following equations are obtained

$$\dot{u}''(x,t) = (t - t_n) \sum_{i=1}^{M} a_i h_i(x) + \dot{u}''(x, t_n) \tag{3.2}$$

$$\begin{aligned} u''(x,t) = \; & \frac{1}{2}\left(t^2 - 2tt_n + t_n^2\right) \sum_{i=1}^{M} a_i h_i(x) \\ & + (t - t_n)\, \dot{u}''(x, t_n) + u''(x, t_n) \end{aligned} \tag{3.3}$$

$$\begin{aligned} u'(x,t) = \; & \frac{1}{2}\left(t^2 - 2\,tt_n + t_n^2\right) \sum_{i=1}^{M} a_i P_{i,1}(x) \\ & + (t - t_n)\left[\dot{u}'(x, t_n) - \dot{u}'(0, t_n)\right] \\ & + u'(x, t_n) - u'(0, t_n) + u'(0, t) \end{aligned} \tag{3.4}$$

$$(3.5)\quad \begin{aligned} u(x,t) &= \frac{1}{2}\left(t^2 - 2\ tt_n + {t_n}^2\right)\sum_{i=1}^{M} a_i P_{i,2}\left(x\right) \\ &+ \left(t - t_n\right)\left[\dot{u}\left(x, t_n\right) - \dot{u}\left(0, t_n\right) - x\dot{u}^{'}\left(0, t_n\right)\right] + u\left(x, t_n\right) - u\left(0\ , t_n\right) \\ &- x\left[u^{'}\left(0\ , t_n\right) - u^{'}\left(0, t\right)\right] + u(0,t) \end{aligned}$$

$$(3.6)\quad \begin{aligned} \dot{u}\left(x,t\right) = &\ \left(t - t_n\right)\sum_{i=1}^{M} a_i P_{i,2}\left(x\right) \\ &+ \left[\dot{u}\left(x, t_n\right) - \dot{u}\left(0, t_n\right) - x\dot{u}^{'}\left(0, t_n\right)\right] + \ x\dot{u}^{'}\left(0, t\right) + \dot{u}\left(0, t\right) \end{aligned}$$

$$(3.7)\quad \ddot{u}\left(x,t\right) = \sum_{i=1}^{M} a_i P_{i,2}\left(x\right) + x\ \ddot{u}^{'}\left(0,t\right) + \ddot{u}\left(0,\ t\right)$$

From the initial and boundary conditions, we have the following equations as

$$\begin{aligned} &u(x,0) = f(x), && u(0,t) = g_0(t), && u(1,t) = g_1(t), \\ &u(0,t_n) = g_0(t_n), && u(1,t_n) = g_1(t_n), && \dot{u}\left(0,t_n\right) = g_0^{'}\left(t_n\right), \\ &\dot{u}\left(1,t_n\right) = g_1^{'}\left(t_n\right), && \ddot{u}\left(0,t_n\right) = g_0^{''}(t_n), && \ddot{u}\left(1,t_n\right) = g_1^{''}(t_n). \end{aligned}$$

At $x = 1$ in the formula (3.5) and (3.7) and by using condition, we have

$$(3.8)\quad \begin{aligned} u^{'}\left(0,t\right) - u^{'}\left(0,t_n\right) = &\ -\frac{1}{2}\left(t^2 - \ 2tt_n + {t_n}^2\right)\sum_{i=1}^{M} a_i P_{i,2}\left(1\right) \\ &- \left(t - t_n\right)\left[g_1^{'}\left(t_n\right) - g_0^{'}\left(t_n\right) - \dot{u}^{'}\left(0, t_n\right)\right] \\ &+ g_1\left(t\right) - g_1\left(t_n\right) + g_0\left(t_n\right) - g_0(t) \end{aligned}$$

$$(3.9)\quad \ddot{u}^{'}\left(0,t\right) = -\sum_{i=1}^{M} a_i P_{i,2}\left(1\right) - g_0^{''}\left(t\right) + g_1^{''}\left(t\right)$$

If equations (3.8) and (3.9) are substituted into equations (3.3)-(3.5) and the results are discriticised by assuming $x \to x_l$, $t \to t_{n+1}$, we obtain

$$(3.10)\quad \begin{aligned} u^{''}\left(x_l\ , t_{n+1}\right) \ = &\ \frac{1}{2}\left(t_{n+1}^2 - 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i h_i\left(x_l\right) \\ &+ \left(t_{n+1} - t_n\right)\dot{u}^{''}\left(x_l, t_n\right) + \ u^{''}\left(x_l, t_n\right) \end{aligned}$$

$$u'(x_l,t_{n+1}) = \frac{1}{2}\left(t_{n+1}^2-2t_{n+1}t_n+t_n^2\right)\sum_{i=1}^{M}a_iP_{i,1}(x_l) + (t_{n+1}-t_n)\dot{u}'(x_l,t_n) + u'(x_l,t_n) - \frac{1}{2}\left(t_{n+1}^2-2t_{n+1}t_n+t_n^2\right)\sum_{i=1}^{M}a_iP_{i,2}(1) - (t_{n+1}-t_n)\left[g_1'(t_n)-g_0'(t_n)\right] + g_1(t_{n+1})-g_1(t_n)+g_0(t_n)-g_0(t_{n+1}) \tag{3.11}$$

$$u(x_l,t_{n+1}) = \frac{1}{2}\left(t_{n+1}^2-2t_{n+1}t_n+t_n^2\right)\sum_{i=1}^{M}a_iP_{i,2}(x_l) + (t_{n+1}-t_n)\left[\dot{u}(x_l,t_n)-\dot{u}(0,t_n)\right] + u(x_l,t_n)-u(0,t_n) - \frac{x_l}{2}\left(t_{n+1}^2-2t_{n+1}t_n+t_n^2\right)\sum_{i=1}^{M}a_iP_{i,2}(1) - x_l(t_{n+1}-t_n)\left[g_1'(t_n)-g_0'(t_n)\right] - x_l[g_1(t_n)-g_0(t_n)+g_0(t_{n+1})-g_1(t_{n+1})]+g_0(t_{n+1}) \tag{3.12}$$

$$\dot{u}(x_l,\ t_{n+1}) = (t_{n+1}-t_n)\sum_{i=1}^{M}a_iP_{i,2}(x_l) + \left[\dot{u}(x_l,t_n)-\dot{u}(0,t_n)\right] - x_l(t_{n+1}-t_n)\sum_{i=1}^{M}a_iP_{i,2}(1) - x_l[g_1'(t_n)-g_0'(t_n)] - x_l[g_0'(t_{n+1})-g_1'(t_{n+1})]+g_0'(t_{n+1}) \tag{3.13}$$

$$\ddot{u}(x_l,t_{n+1}) = \sum_{i=1}^{M}a_i[P_{i,2}(x_l)-x_l\ P_{i,2}(1)] - x_l\left[g_0''(t_{n+1})-g_1''(t_{n+1})\right]+g_0''(t_{n+1}) \tag{3.14}$$

From equation (2.6), we obtain

$$p_{i,2}(1) = \begin{cases} 0.5 & \text{if} \quad i=1 \\ \dfrac{1}{4m^2} & \text{if} \quad i>1 \end{cases} \tag{3.15}$$

4. Numerical example

Example1. Consider equation (1.1) with $\alpha = 6, \beta = 2, 0 \leq x \leq 1$ and the following conditions:

$$\begin{aligned} f(x) &= \sin x, \\ f_1(x) &= 0, \\ g_0(x) &= 0, \\ g_1(x) &= \cos(t)\sin(1), \\ f(x,t) &= -2\alpha \sin(t)\sin(x) + \beta^2 \cos(t) \sin(x). \end{aligned}$$

The exact solution of this example is $u(x,t) = \cos(t)\sin(x)$.

After substituting values from equations (3.12)-(3.14) in equation (1.1) and using conditions, we have

$$\begin{aligned} &\sum_{i=1}^{M} a_i [P_{i,2}(x_l) - x_l \, P_{i,2}(1) + 12(t_{n+1} - t_n) P_{i,2}(x_l) \\ &-12x_l (t_{n+1} - t_n) P_{i,2}(1) + 2\left(t_{n+1}^2 - 2t_{n+1}t_n + t_n^2\right) P_{i,2}(x_l) \\ &- 2x_l \left(t_{n+1}^2 - 2t_{n+1}t_n + t_n^2\right) P_{i,2}(1)] \\ &= -sinx_l - 12sintsinx_l + 4costsinx_l - 12x_l sint_n sin1 + 12sint_{n+1}sin1 \\ &\quad 4u(x_l, t_n) - 4x_l (t_{n+1} - t_n) sint_n sin1 - 4x_l [cost_n sin1 - cost_{n+1} sin1] \\ &\quad + x_l cost_{n+1} sin1 \end{aligned} \tag{4.1}$$

Equation (4.1) is the algebraic form of the telegraph equation (1.1). After solving these algebraic equations, we can compute the Haar coefficients $a_i^{'}s$. Then, from equation (3.12), we obtain the value of u, which is very near to the exact solution. This solution process is started with

$$\begin{aligned} u(x_l, 0) &= f(x_l) \\ u^{'}(x_l, 0) &= f^{'}(x_l) \\ u^{''}(x_l, 0) &= f^{''}(x_l) \end{aligned}$$

In all the results given in the following, J is taken as 3.

Table 1
The absolute error for different values of x & t is

x /32	At t=0.1	At t=0.2	At t=0.3
1	2.66106E-06	2.15080 E-04	4.271600 E-04
3	7.98322E-06	6.44370 E-04	1.279750 E-03
5	1.33055E-05	1.07105 E-03	2.127200 E-03
7	1.86280E-05	1.49340 E-03	2.966080 E-03
9	2.39509E-05	1.90969 E-03	3.793030 E-03
11	2.92741E-05	2.31826 E-03	4.604710 E-03
13	3.45979E-05	2.71743 E-03	5.397840 E-03
15	3.99222E-05	3.10559 E-03	6.169230 E-03
17	4.52472E-05	3.48116 E-03	6.915780 E-03
19	5.05729E-05	3.84261 E-03	7.634440 E-03
21	5.58994E-05	4.18847 E-03	8.322330 E-03
23	6.12269E-05	4.51733 E-03	8.976660 E-03
25	6.65553E-05	4.82783 E-03	9.594750 E-03
27	7.18847E-05	5.11872 E-03	1.017411 E-02
29	7.72152E-05	5.38878 E-03	1.071237 E-02
31	8.25469E-05	5.63691 E-03	1.120731 E-02

The plot of the numerical solution of $u(x,t)$ at different value of t is shown in following figures (1)-(3):

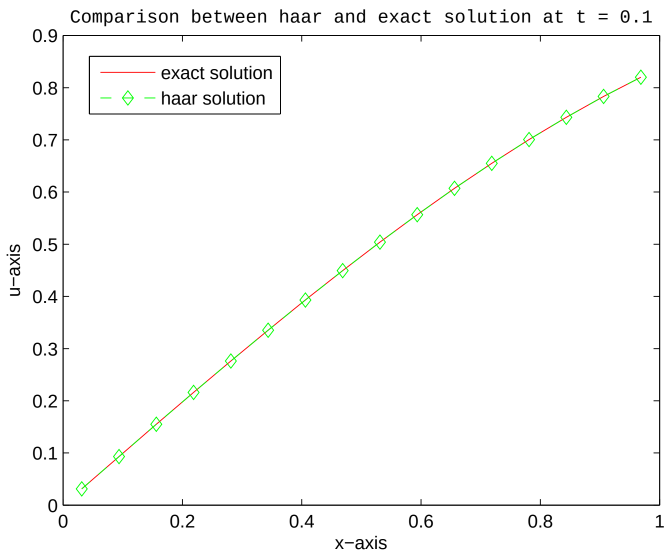

Figure-(1)

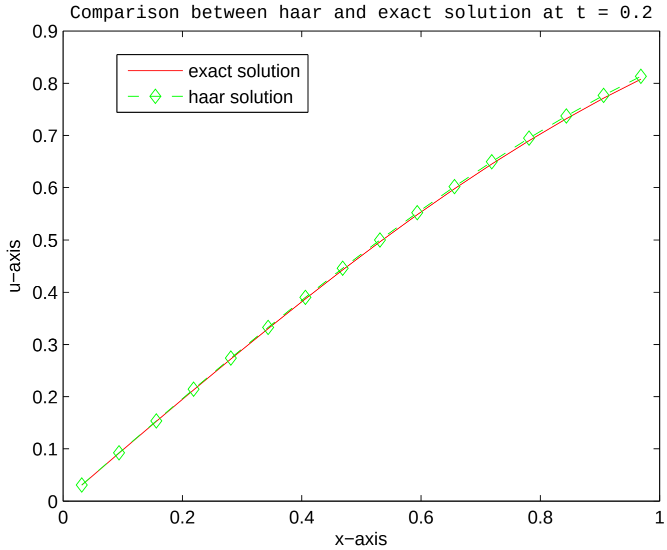

Figure-(2)

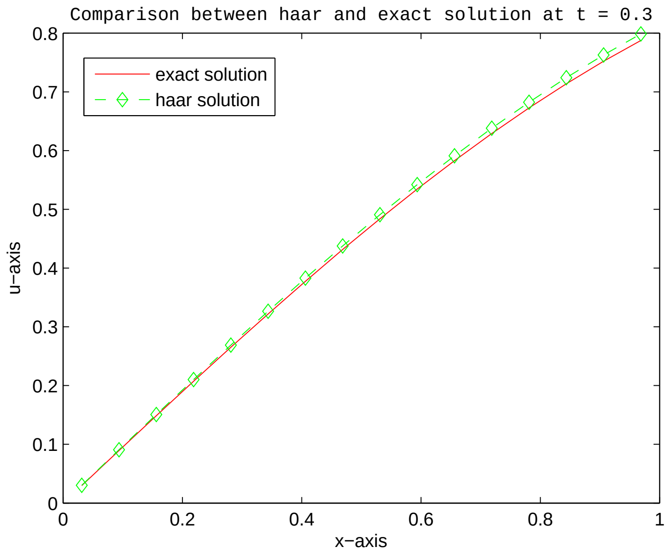

Figure-(3)

Example 2. Consider equation (1.1) with $\alpha = 1,\ \beta = 1,\ 0 \le x \le 1$ and the following conditions

$$\begin{aligned} f(x) &= x^2, \\ f_1(x) &= 1, \\ g_0(x) &= t, \\ g_1(x) &= 1 + t, \\ f\,(x,t) &= \ x^2 + \ t - 1. \end{aligned}$$

The exact solution of this example is $u(x,t) = x^2 + \ t$.

Table 2
The absolute error for different values of x & t is

x /32	At t=0.1	At t=0.2
1	4.589840 E-04	9.135300 E-04
3	4.944960 E-04	9.490410 E-04
5	5.655180 E-04	1.020064 E-03
7	6.720530 E-04	1.126598 E-03
9	8.140980 E-04	1.268643 E-03
11	9.916550 E-04	1.446200 E-03
13	1.204723 E-03	1.659268 E-03
15	1.453303 E-03	1.907848 E-03
17	1.737393 E-03	2.191939 E-03
19	2.056996 E-03	2.511541 E-03
21	2.412109 E-03	2.866655 E-03
23	2.802734 E-03	3.257280 E-03
25	3.228871 E-03	3.683416 E-03
27	3.690518 E-03	4.145064 E-03
29	4.187678 E-03	4.642223 E-03
31	4.589840 E-04	9.135300 E-04

The plot of the numerical solution of $u(x, t)$ at different value of t is shown in following figures (4) & (5) :

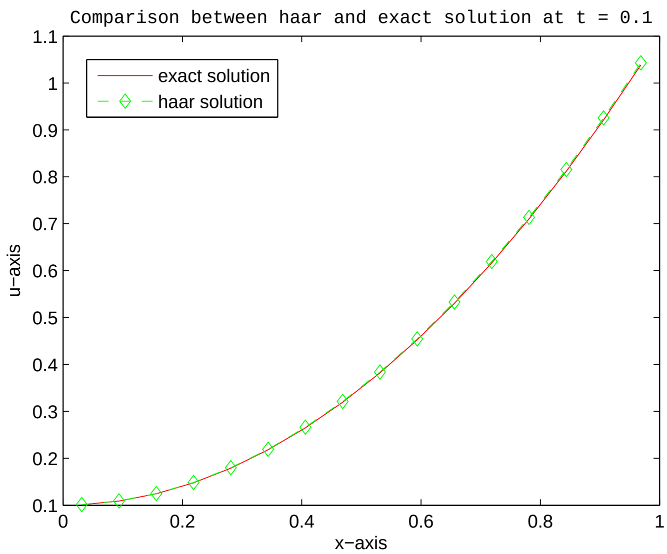

Figure-(4)

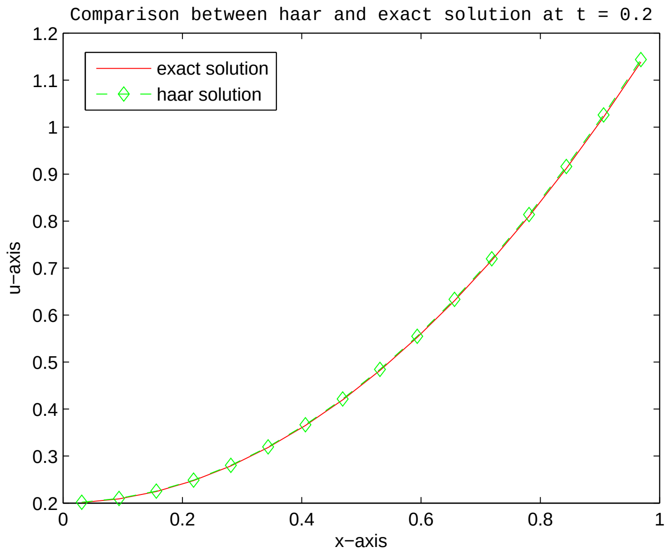

Figure-(5)

Conclusion

In this paper, Haar wavelet method is proposed for the numerical solution for the second-order hyperbolic telegraph equation. Approximate solution of the telegraph equation, obtain by Matlab, are compared with exact solution. This calculation demonstrates the accuracy of the Haar wavelet solution. The main advantage of this method is its simplicity and small computation costs, which is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficient. It is worth mentioning that Haar solution provides excellent results even for small values of M ($M = 16$). For larger values of M, we can obtain the results closer to the real values.

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