GENERALIZED QUASI-COINCIDENCE IN FUZZY SUB-HYPERMODULES

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Abstract. We consider a general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set. We introduce the notions of $(\in, q_k)$-fuzzy sub-hypermodule and $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of a given hypermodule, and investigate several properties of these notions.

Keywords: $(\in, q_k)$-fuzzy sub-hypermodule, $(\in, \in \vee q_k)$-fuzzy sub-hypermodule, $(\in \vee q_k)$-level sub-hypermodule.

1. Introduction

The concept of hyperstructure was first introduced by Marty at the eighth Congress of Scandinavian Mathematicians in 1934 when he defined hypergroups and started to analyse their properties ([14]). Because there are extensive applications in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures (or hypersystems) has nowadays become a well-established branch in algebraic theory. Later on, people have developed the semi-hypergroups, which are the simplest algebraic hyperstructures having closure and associativity properties. A comprehensive review of the theory of hyperstructures can be found in [6], [7] and [20].

The theory of fuzzy sets, proposed by Zadeh [22] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Murali [17] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [18], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [3,4] initiated the concept of (α, β)-fuzzy subgroup by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In fact, these notions were originally introduced by Pu and Liu in [18]. The $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of the fuzzy subgroups defined by Rosenfeld [19] and their structure was described by Bhakat and Das in [4].

In recent years, there has been considerable interest in the relationships between the fuzzy sets and the algebraic hyperstructures. Fuzzy hyperstructures are a direct generalization of the concept of fuzzy algebras (fuzzy groups, fuzzy rings, fuzzy modules etc). This approach can be extended to fuzzy hypergroups. For example, given a crisp hypergroup $(H, \circ)$ and a fuzzy subset μ in H, then we say that μ is a fuzzy subhypergroup of H if every level set of μ say μ_t, is a (crisp) subhypergroup of H. This was initiated by Zahedi et al. [24] and continued by Ameri and Hedayati [2], Davvaz and Corsini [8], Davvaz et al. [9], Zhan et al. [25], [26] and so on.

In this paper, we consider more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set. Thus, this is a natural generalization of the fuzzy sub-hypermodules. As a generalization of $(\in, \in \vee q)$-fuzzy sub-hypermodules, we introduce the notions of $(\in, q_k)$-fuzzy sub-hypermodules and $(\in, \in \vee q_k)$-fuzzy sub-hypermodules and investigate some of the interesting properties of $(\in, \in \vee q_k)$-fuzzy sub-hypermodules. Finally, we consider $(\in \vee q_k)$-level sub-hypermodule of a fuzzy set, and prove some related results.

2. Preliminaries

A *hyperstructure* is a non-empty set H together with a mapping "$\circ$" $: H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H. If $x \in H$ and $A, B \in P^*(H)$, then by $A \circ B$, $A \circ x$ and $x \circ B$, we mean $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively (For more details see [6], [7]).

Now, we call hyperstructure $(H, +)$ a *canonical hypergroup* ([16]) if the following axioms are satisfied:

(1) For every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;

(2) For every $x, y \in H$, $x + y = y + x$;

(3) There exists an element $0 \in H$, such that $0 + x = x$, for all $x \in H$;

(4) For every $x \in H$, there exists a unique element $x' \in H$, such that $0 \in x + x'$ (we call the element x' the *opposite* of x and it is denoted by $-x$).

Definition 2.1. [26] A *hyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(i) $(R, +)$ is a canonical hypergroup;

(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element;

(iii) The multiplication is distributive with respect to the hyperoperation " $+$ ".

Let $(R, +, \cdot)$ be a hyperring and A a non-empty subset of R. Then A is called a *sub-hyperring* of R if $(A, +, \cdot)$ itself is a hyperring.

Definition 2.2. [1] A non-empty set M is called a *left hypermodule* over a hyperring R (R-hypermodule) if $(M, +)$ is a canonical hypergroup and there exists the map " $\cdot$ " : $R \times M \longrightarrow P^*(M)$ by $(r, m) \mapsto r \cdot m$ such that, for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, we have

(i) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$;

(ii) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$;

(iii) $(r_1r_2)m_1 = r_1(r_2m_1)$.

Example 2.3. Let M be an R-module over a unitary ring R and G a subgroup of the multiplicative semigroup of R satisfying the condition $aGbG = abG$, for every $a, b \in R$. Note that this condition is equivalent to the normality of G only if $R\backslash\{0\}$ is a group which appears in the case of division rings. Now, we introduce in M an equivalence relation " $\sim$ " defined as follows:

$$x \sim y \Longleftrightarrow x = ty, \quad t \in G.$$

Let $\overline{M}$ be the set of the equivalence classes of M with respect to $\sim$. Then, a hyperoperation " $\oplus$ " can be endowed in $\overline{M}$ by

$$\overline{x} \oplus \overline{y} = \{\overline{w} \in \overline{M} |\ \overline{w} \subseteq \overline{x} + \overline{y}\},$$

i.e., $\overline{x} \oplus \overline{y}$ consists of all the classes $\overline{w} \in \overline{M}$ which are contained in the set-wise sum of $\overline{x}$ and $\overline{y}$. Thus, $(\overline{M}, \oplus)$ becomes a canonical hypergroup. Now, we suppose that $\overline{R}$ is the quotient hyperring of R by G. Consider an external composition from $\overline{R} \times \overline{M}$ to $\overline{M}$ defined by

$$\overline{a} \cdot \overline{x} = \overline{ax} \ \text{ for all } \ \overline{a} \in \overline{R}, \overline{x} \in \overline{M}.$$

Then the above composition satisfies the axioms of the hypermodule and so $\overline{M}$ becomes a hypermodule over hyperring $\overline{R}$. It was proved by Massouros ([15]) that this hypermodule is strongly related with the analytic projective geometries as well as with the Euclidean spherical geometries (for details see [15]).

In what follows, all hypermodules are left hypermodules. Recall that a non-empty subset A of a hypermodule M is a *sub-hypermodule* if $(A, +, \cdot)$ is a hypermodule.

Definition 2.4. [26] A fuzzy subset μ of a hypermodule M over a hyperring R is a *fuzzy sub-hypermodule* of M if

(1) $\inf\limits_{z\in x+y} \mu(z) \geq \min\{\mu(x), \mu(y)\ \}$, for all $x, y \in M$;

(2) $\mu(-x) \geq \mu(x)$, for all $x \in M$;

(3) $\mu(ax) \geq \mu(x)$, for all $a \in R$ and $x \in M$.

If μ is a fuzzy sub-hypermodule of M, clearly we have

$$\mu(-x) = \mu(x), \quad \inf_{z\in x-y} \mu(z) \geq \min\{\mu(x), \mu(y)\}, \text{ for all } x, y \in M.$$

Let M be an R-hypermodule. Then, for a fuzzy subset μ of M, the level subset μ_t is defined by

$$\mu_t = \{x \in M \mid \mu(x) \geq t\}; \quad t \in [0, 1].$$

Theorem 2.5. [26] *Let μ be a fuzzy subset of an* R*-hypermodule* M*. Then the following are equivalent:*

(1) *μ is a fuzzy sub-hypermodule of* M*;*

(2) *Each non-empty level subset of μ is a sub-hypermodule of* M*.*

A fuzzy set μ of a hypermodule M of the form

$$\mu(y) = \begin{cases} t\ (\neq 0), & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

is said to be a *fuzzy point* with *support* x and *value* t and is denoted by $(x)_t$. A fuzzy point $(x)_t$ is said to "*belong to*" (resp. be *quasi-coincident with*) a fuzzy set μ, written as $(x)_t \in \mu$ (resp. $(x)_t\ q\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $(x)_t \in \mu$ or $(x)_t\ q\mu$, then we write $(x)_t \in \vee q\mu$. If $(x)_t \in \mu$ and $(x)_t\ q\mu$, then we write $(x)_t \in \wedge q\mu$. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

3. Generalization of $(\in, \in \vee q)$-fuzzy sub-hypermodules

Let k denote an arbitrary element of $(0, 1]$ unless otherwise specified. To say that $(x)_t\ q_k\mu$, we mean $\mu(x) + t + k > 1$. To say that $(x)_t \in \vee q_k\mu$, we mean $(x)_t \in \mu$ or $(x)_t\ q_k\mu$. The symbol $\overline{\in \vee q_k}$ means that $\in \vee q_k$ does not hold.

Definition 3.1. A fuzzy set μ of M is called an $(\in, \in \vee q_k)-$*fuzzy sub-hypermodule* of M if, for all $t_1, t_2 \in [0,1)$, $a \in R$ and $x, y \in M$

(1) $(x)_{t_1} \in \mu$, $(y)_{t_2} \in \mu \Longrightarrow (z)_{min(t_1,t_2)} \in \vee q_k \mu$, for all $z \in x+y$;

(2) $(x)_{t_1} \in \mu \Longrightarrow (-x)_{t_1} \in \vee q_k \mu$;

(3) $(x)_{t_1} \in \mu \Longrightarrow (ax)_{t_1} \in \vee q_k \mu$.

Similarly, $(\in, \in \vee q)$-fuzzy sub-hypermodule, $(\in, q)$-fuzzy sub-hypermodule and $(\in, \in)$-fuzzy sub-hypermodule are defined (see [26]).

Theorem 3.2. *A fuzzy set μ of hypermodule* M *over hyperring* R *is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of* M *if only if, for all $x, y \in$ M and $a \in$ R the following conditions hold:*

(1) $\mu(z) \geq \min\left\{\mu(x), \mu(y), \dfrac{1-k}{2}\right\}$, *for all $z \in x+y$;*

(2) $\mu(-x) \geq \min\left\{\mu(x), \dfrac{1-k}{2}\right\}$;

(3) $\mu(ax) \geq \min\left\{\mu(x), \dfrac{1-k}{2}\right\}$.

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. Assume that (1) is not valid. Then there exist $a, b \in M$ and $z \in a+b$ such that

$$\mu(z) < min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

If $\min\{\mu(a), \mu(b)\} < \dfrac{1-k}{2}$, then

$$\mu(z) < \min\{\mu(a), \mu(b)\}.$$

Hence $\mu(z) < t \leq \min\{\mu(a), \mu(b)\}$ for some $t \in (0,1)$. It follows that $(a)_t \in \mu$ and $(b)_t \in \mu$, but $(z)_t \overline{\in} \mu$. Moreover, $\mu(z) + t < 2t < \dfrac{1-k}{2} + \dfrac{1-k}{2} = 1-k$, and so $(z)_t\ \overline{q_k} \mu$. Consequently, $(z)_t\ \overline{\in \vee q_k} \mu$, this is a contradiction.

If $\min\{\mu(a), \mu(b)\} \geq \dfrac{1-k}{2}$, then

$$\mu(a) \geq \frac{1-k}{2},\ \mu(b) \geq \frac{1-k}{2} \text{ and } \mu(z) < \frac{1-k}{2}.$$

Thus $(a)_{\frac{1-k}{2}} \in \mu$ and $(b)_{\frac{1-k}{2}} \in \mu$, but $(z)_{\frac{1-k}{2}} \overline{\in} \mu$. Also,

$$\mu(z) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k,$$

i.e., $(z)_{\frac{1-k}{2}}\ \overline{q_k}\mu$. Hence $(z)_{\frac{1-k}{2}}\ \overline{\in \vee q_k}\mu$, again, a contradiction. Therefore (1) is valid.

Assume that (2) is not valid. Then there exists $a \in M$, such that

$$\mu(-a) < \min\left\{\mu(a), \frac{1-k}{2}\right\}.$$

If $\mu(a) < \dfrac{1-k}{2}$, then $\mu(-a) < \mu(a)$. Hence $\mu(-a) < t \leq \mu(a)$ for some $t \in (0,1)$. It follows that $(a)_t \in \mu$, but $(-a)_t\ \overline{\in}\mu$. Moreover, $\mu(-a)+t < 2t < 1-k$ and so $(-a)_t\ \overline{q_k}\mu$. Consequently $(-a)_t\ \overline{\in \vee q_k}\mu$, this is a contradiction. If $\mu(a) \geq \dfrac{1-k}{2}$, then $\mu(-a) < \frac{1-k}{2}$. Thus $(a)_{\frac{1-k}{2}} \in \mu$, but $(-a)_{\frac{1-k}{2}}\ \overline{\in}\mu$. Also,

$$\mu(-a) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k,$$

i.e., $(-a)_{\frac{1-k}{2}}\ \overline{q_k}\mu$. Hence $(-a)_{\frac{1-k}{2}}\ \overline{\in \vee q_k}\mu$, again, a contradiction. Therefore, (2) is valid. Similarly, (3) is valid.

Conversely, suppose that (1), (2) and (3) are valid. Let $x, y \in M$ and $t_1, t_2 \in (0,1]$ be such that $(x)_{t_1} \in \mu$ and $(y)_{t_2} \in \mu$. Then, for all $z \in x+y$,

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{t_1, t_2, \frac{1-k}{2}\right\}.$$

Assume that $t_1 \leq \dfrac{1-k}{2}$ or $t_2 \leq \dfrac{1-k}{2}$. Then $\mu(z) \geq \min(t_1, t_2)$, which implies that $(z)_{\min(t_1,t_2)} \in \mu$. Now, suppose that $t_1 > \dfrac{1-k}{2}$ and $t_2 > \dfrac{1-k}{2}$. Then $\mu(z) \geq \dfrac{1-k}{2}$ and thus

$$\mu(z) + min(t_1, t_2) > \frac{1-k}{2} + \frac{1-k}{2} = 1-k,$$

i.e., $(z)_{\min(t_1,t_2)}\ q_k\mu$. Hence $(z)_{\min(t_1,t_2)} \in \vee q_k\mu$.

Let $x \in M$ and $t \in (0,1]$ be such that $(x)_t \in \mu$. Then

$$\mu(-x) \geq \min\left\{t, \frac{1-k}{2}\right\}.$$

Assume that $t \leq \dfrac{1-k}{2}$. Then $\mu(-x) \geq t$, which implies that $(-x)_t \in \mu$. Now, suppose that $t > \dfrac{1-k}{2}$. Then $\mu(-x) \geq \dfrac{1-k}{2}$, and so

$$\mu(-x) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1-k,$$

i.e., $(-x)_t\ q_k\mu$. Hence $(-x)_t \in \vee q_k\mu$. Similarly, $(x)_t \in \mu$ implies $(ax)_t \in \vee q_k\mu$ for all $a \in R$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. ∎

Corollary 3.3. *A fuzzy set μ of* M *is an* $(\in, \in \vee q)$*-fuzzy sub-hypermodule of* M *if only if for all* $x, y \in$ M *and* $a \in$ R *the following conditions hold:*

(1) $\mu(z) \geq min\{\ \mu(x), \mu(y), 0.5\ \}$, *for all* $z \in x + y$;

(2) $\mu(-x) \geq min\{\ \mu(x), 0.5\ \}$;

(3) $\mu(ax) \geq min\{\ \mu(x), 0.5\ \}$.

Proof. It follows taking $k = 0$ in Theorem 3.2. ∎

Theorem 3.4. *Let μ be a fuzzy set of* M*. Then μ is an* $(\in, \in \vee q_k)$*-fuzzy sub-hypermodule of* M *if only if the level subset* $\mu_t = \{\ x \in \mathrm{M} \mid \mu(x) \geq t\ \}$ *is a sub-hypermodule of* M *for all* $t \in \left(0, \frac{1-k}{2}\right]$.

Proof. Assume that μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. Let $t \in \left(0, \frac{1-k}{2}\right]$ and $x, y \in \mu_t$. Then $(x)_t \in \mu$ and $(y)_t \in \mu$. Let $z \notin \mu_t$ for some $z \in x+y$. It follows that $(z)_t \overline{\in} \mu$. Moreover, $\mu(z)+t < 2t \leq \frac{1-k}{2}+\frac{1-k}{2} = 1-k$, and so $(z)_t\ \overline{q_k}\mu$. Consequently, $(z)_t\ \overline{\in \vee q_k}\mu$, this is a contradiction. Hence, $z \in \mu_t$ for all $z \in x + y$. Therefore $x + y \subseteq \mu_t$.

Let $x \in \mu_t$. Then $\mu(x) \geq t$. It follows from Theorem 3.2 that

$$\mu(-x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t,$$

so that $-x \in \mu_t$. Consequently $(\mu_t, +)$ is a sub-hypergroup of $(M, +)$. Similarly, for all $x \in \mu_t$ and $a \in R$, we have $ax \in \mu_t$, by Theorem 3.2. Therefore μ_t is a sub-hypermodule of M.

Conversely, suppose that μ_t is a sub-hypermodule of M for all $t \in \left(0, \frac{1-k}{2}\right]$. Let (1) of Theorem 3.2 be not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Then $t \in \left(0, \frac{1-k}{2}\right]$ and $a, b \in \mu_t$. Since μ_t is a sub-hypermodule of M, it follows that $z \in \mu_t$, so that $\mu(z) \geq t$. This is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently, μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.2. ∎

Corollary 3.5. *Let μ be a fuzzy set of* M. *Then μ is an $(\in, \in \vee q)$-fuzzy sub-hypermodule of* M *if only if the level subset $\mu_t = \{x \in \mathrm{M} \mid \mu(x) \geq t\}$ is a sub-hypermodule of* M *for all $t \in (0, 0.5]$.*

Proof. In Theorem 3.4, taking $k = 0$. ∎

Example 3.6. Consider the set $M = \{0, a, b, c\}$ with the following table:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Hence $(M, +)$ is a commutative group. Since every commutative group is a $\mathbb{Z}$-module therefore M is a $\mathbb{Z}$-module ($\mathbb{Z}$ is the set of all integers). By Example 2.3, consider $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ with the following table:

$\oplus$	$\overline{0}$	$\overline{a}$	$\overline{b}$	$\overline{c}$
$\overline{0}$	$\overline{0}$	$\overline{a}$	$\overline{b}$	$\overline{c}$
$\overline{a}$	$\overline{a}$	$\overline{0}$	$\overline{c}$	$\overline{b}$
$\overline{b}$	$\overline{b}$	$\overline{c}$	$\overline{0}$	$\overline{a}$
$\overline{c}$	$\overline{c}$	$\overline{b}$	$\overline{a}$	$\overline{0}$

Thus $(\overline{M}, \oplus)$ is a canonical hypergroup. Therefore, $(\overline{M}, \oplus, \cdot)$ is a hypermodule over hyperring $\overline{\mathbb{Z}}$. Let μ be a fuzzy set of hypermodule $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ defined by $\mu(\overline{0}) = 0.5$, $\mu(\overline{a}) = 0.8$, and $\mu(\overline{b}) = \mu(\overline{c}) = 0.3$.

(1) If $k = 0.4$, then $\mu_t = \overline{M}$ for all $t \in (0, 0.3]$. Hence μ is an $(\in, \in \vee q_{0.4})$-fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.

(2) If $k = 0.2$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{a}\}, & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since $\overline{M}$ and $\{\overline{0}, \overline{a}\}$ are sub-hypermodule of $\overline{M}$, μ is an $(\in, \in \vee q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.

Example 3.7. Let $\overline{M}$ be the hypermodule given in Example 3.6. Let μ be a fuzzy set of hypermodule $\overline{M}$ defined by $\mu(\overline{0}) = 0.45$, $\mu(\overline{a}) = \mu(\overline{c}) = 0.4$, and $\mu(\overline{b}) = 0.48$. If $k = 0.04$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.4] \\ \{\overline{0}, \overline{b}\}, & \text{if } t \in (0.4, 0.45] \\ \{\overline{b}\}, & \text{if } t \in (0.45, 0.48]. \end{cases}$$

Note that μ_t is not a sub-hypermodule of $\overline{M}$ for $t \in (0.45, 0.48]$. Hence μ is not an $(\in, \in \vee q_{0.04})$-fuzzy sub-hypermodule of $\overline{M}$.

Theorem 3.8. *Every* $(\in, \in)$*-fuzzy sub-hypermodule of* M *is an* $(\in, \in \vee q_k)$*-fuzzy sub-hypermodule of* M.

Proof. Let μ be an $(\in, \in)$-fuzzy sub-hypermodule of M. Let (1) of Theorem 3.2 is not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Thus $(a)_t \in \mu$ and $(b)_t \in \mu$, then $\mu(z) \geq t$. This is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.2. ∎

The next corollary immediately follow from theorem 3.8, by taking $k = 0$.

Corollary 3.9. *Every* $(\in, \in)$*-fuzzy sub-hypermodule of* M *is an* $(\in, \in \vee q)$*-fuzzy sub-hypermodule of* M.

The converse of Theorem 3.8 is not true as seen in the following example.

Example 3.10. Consider the $(\in, \in \vee q_{0.4})$-fuzzy sub-hypermodule of $\overline{M}$ given in Example 3.6. Then μ is not an $(\in, \in)$-fuzzy sub-hypermodule of $\overline{M}$ since, $(\overline{a})_{0.62} \in \mu$ and $(\overline{a})_{0.66} \in \mu$, but $(\overline{0})_{min(0.62,0.66)} = (\overline{a} \oplus \overline{a})_{min(0.62,0.66)} \overline{\in} \mu$, because $\mu(\overline{0}) = 0.5 < 0.62$.

4. Properties of $(\in, \in \vee q_k)$-fuzzy sub-hypermodules

Definition 4.1. A fuzzy set μ of M is called an $(\in, q_k)$-*fuzzy sub-hypermodule* of M if, for all $t_1, t_2 \in (0, 1]$, $a \in R$ and $x, y \in M$

(1) $(x)_{t_1} \in \mu,\ (y)_{t_2} \in \mu \Longrightarrow (z)_{min(t_1,t_2)}\ q_k\mu$, for all $z \in x + y$;

(2) $(x)_t \in \mu \Longrightarrow (-x)_t\ q_k\mu$;

(3) $(x)_t \in \mu \Longrightarrow (ax)_t\ q_k\mu$.

Theorem 4.2. *Every* $(\in, q_k)$*-fuzzy sub-hypermodule of* M *is an* $(\in, \in \vee q_k)$*-fuzzy sub-hypermodule of* M.

Proof. Let μ be an $(\in, q_k)$-fuzzy sub-hypermodule of M. If (1) of Theorem 3.2 is not valid, then there exists $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Thus $(a)_t \in \mu$ and $(b)_t \in \mu$, then $(z)_t\ q_k\mu$, but $\mu(z) + t < 2t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, this is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.2. ∎

Taking $k = 0$ in Theorem 4.2, we have the following corollary.

Corollary 4.3. *Every $(\in, q)$-fuzzy sub-hypermodule of M is an $(\in, \in \vee q)$-fuzzy sub-hypermodule of M.*

The next example shows that the converse of Theorem 4.2 does not hold.

Example 4.4. Consider the $(\in, \in \vee q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$ given in Example 3.6. Note that $(\overline{a})_{0.4} \in \mu$ and $(\overline{b})_{0.25} \in \mu$, but $(\overline{a} \oplus \overline{b})_{min(0.4,0.25)} = (\overline{c})_{0.25}\ \overline{q_{0.2}}\mu$, because $\mu(\overline{c}) + 0.25 + 0.2 < 1$. Therefore, μ is not an $(\in, q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$.

Theorem 4.5. *Let M be a hypermodule. If $0 \leq k < r < 1$, then every $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M is an $(\in, \in \vee q_r)$-fuzzy sub-hypermodule of M.*

Proof. Straightforward. ∎

The following example shows that if $0 \leq k < r < 1$, then an $(\in, \in \vee q_r)$-fuzzy sub-hypermodule of M may not be an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M.

Example 4.6. Let $\overline{M}$ and μ be as in Example 3.7. If $r = 0.16$ and $k = 0.04$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } \ t \in (0, 0.4] \\ \{\overline{0}, \overline{b}\}, & \text{if } \ t \in (0.4, 0.42]. \end{cases}$$

Since $\overline{M}$ and $\{\ \overline{0}, \overline{b}\ \}$ are sub-hypermodules of $\overline{M}$, then μ is an $(\in, \in \vee q_{0.16})$-fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4. But μ is not an $(\in, \in \vee q_{0.04})$-fuzzy sub-hypermodule of $\overline{M}$ (see Example 3.7).

Let S be a subset of hypermodule M. Consider a fuzzy set μ_s in M where for all $x \in M$ defined by

$$\mu_s(x) = \begin{cases} 1, & \text{if } \ x \in S \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.7. *A non-empty subset S of hypermodule M is a sub-hypermodule of M if and only if the fuzzy set μ_s in M is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M.*

Proof. Let S be a sub-hypermodule of M. Then $(\mu_s)_t$ is clearly a sub-hypermodule of M for all $t \in \left(0, \dfrac{1-k}{2}\right]$. Hence μ_s is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.4.

Conversely, assume that μ_s is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. Let $x, y \in S$. Then for all $z \in x + y$

$$\mu_s(z) \geq \min\left\{\mu_s(x), \mu_s(y), \frac{1-k}{2}\right\} = \min\left(1, \frac{1-k}{2}\right) = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, $\mu_s(z) = 1$ for all $z \in x + y$ and so $z \in S$. Hence $x + y \subseteq S$.

Let $x \in S$. Then

$$\mu_s(-x) \geq min\left\{\mu_s(x), \frac{1-k}{2}\right\} = \min\left(1, \frac{1-k}{2}\right) = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, $\mu_s(-x) = 1$ and so $-x \in S$. Similarly, $ax \in S$ for all $a \in R$. Therefore S is a sub-hypermodule of M. ∎

Theorem 4.8. *Let S be a sub-hypermodule of M. Then for every $t \in \left(0, \dfrac{1-k}{2}\right]$, there exists an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule μ of M such that $\mu_t = S$.*

Proof. Let μ be a fuzzy set of hypermodule M defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in M$, where $t \in \left(0, \dfrac{1-k}{2}\right]$. Obviously, $\mu_t = S$.

If (1) of Theorem 3.2 is not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Since $\#Im(\mu) = 2$, it follows that $\mu(z) = 0$ and $\min\left\{\mu(a), \mu(b), \dfrac{1-k}{2}\right\} = t$. Hence $\mu(a) = t = \mu(b)$, and so $a, b \in S$. Since S is sub-hypermodule of M, $z \in S$. Thus $\mu(z) = t$, which is a contradiction. Therefore (1) of Theorem 3.2

is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.2. ∎

Taking $k = 0$ in Theorem 4.8, we have the following corollary.

Corollary 4.9. *Let S be a sub-hypermodule of M. Then for every $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy sub-hypermodule μ of M such that $\mu_t = S$.*

Theorem 4.10. *Let μ be an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M such that $\mu(x) < \dfrac{1-k}{2}$ for all $x \in M$. Then μ is an $(\in, \in)$-fuzzy sub-hypermodule of M.*

Proof. Let $(x)_{t_1} \in \mu$, $(y)_{t_2} \in \mu$ and $z \in x + y$. By assumption, we have

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \min\{\mu(x), \mu(y)\} \geq \min(t_1, t_2).$$

Thus $(z)_{\min(t_1,t_2)} \in \mu$.

Let $(x)_t \in \mu$. By assumption, we have

$$\mu(-x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} = \mu(x) \geq t.$$

Thus $(-x)_t \in \mu$. Similarly, for all $(x)_t \in \mu$ and $r \in R$ we have $(rx)_t \in \mu$. Therefore μ is an $(\in, \in)$-fuzzy sub-hypermodule of M. ∎

Taking $k = 0$ in Theorem 4.10, we have the following corollary.

Corollary 4.11. *Let μ be an $(\in, \in \vee q)$-fuzzy sub-hypermodule of M such that $\mu(x) < 0.5$ for all $x \in M$. Then μ is an $(\in, \in)$-fuzzy sub-hypermodule of M.*

Theorem 4.12. *Let $\{ \mu_i \mid i \in A \}$ be a family of $(\in, \in \vee q_k)$-fuzzy sub-hypermodules of M. Then $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M.*

Proof. Let $x, y \in M$ and $t_1, t_2 \in (0, 1]$ be such that $(x)_{t_1} \in \mu$ and $(y)_{t_2} \in \mu$. Assume that $(z)_{\min(t_1,t_2)} \overline{\in \vee q_k} \mu$ for some $z \in x + y$. Then $\mu(z) < \min(t_1, t_2)$ and $\mu(z) + \min(t_1, t_2) \leq 1 - k$, which imply that

$$\mu(z) < \frac{1-k}{2}. \qquad (*)$$

Let

$$\Psi_1 = \{i \in A \mid (z)_{\min(t_1,t_2)} \in \mu_i \} \text{ and}$$

$$\Psi_2 = \{i \in A \mid (z)_{\min(t_1,t_2)}\ q_k \mu_i \} \cap \{j \in A \mid (z)_{\min(t_1,t_2)}\ \overline{\in} \mu_j \}.$$

Then, $A = \Psi_1 \cup \Psi_2$ and $\Psi_1 \cap \Psi_2 = \emptyset$. If $\Psi_2 = \emptyset$, then $(z)_{\min(t_1,t_2)} \in \mu_i$ for all $i \in A$, that is, $\mu_i(z) \geq \min(t_1, t_2)$ for all $i \in A$, which yields $\mu(z) \geq \min(t_1, t_2)$. This is a contradiction. Hence, $\Psi_2 \neq \emptyset$, and so for every $i \in \Psi_2$ we have $\mu_i(z) < \min(t_1, t_2)$

and $\mu_i(z) + min(t_1, t_2) > 1 - k$. It follows that $\min(t_1, t_2) > \frac{1-k}{2}$. Now, $(x)_{t_1} \in \mu$ implies $\mu(x) \geq t_1$ and thus $\mu_i(x) \geq \mu(x) \geq t_1 \geq \min(t_1, t_2) > \frac{1-k}{2}$ for all $i \in A$. Similarly, $\mu_i(y) \geq \frac{1-k}{2}$ for all $i \in A$.

Next, suppose that $t = \mu_i(z) < \frac{1-k}{2}$. Taking $t < r < \frac{1-k}{2}$, we get $(x)_r \in \mu_i$ and $(y)_r \in \mu_i$, but $(z)_r \overline{\in \vee q_k} \mu_i$. This contradicts that μ_i is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. Hence, $\mu_i(z) \geq \frac{1-k}{2}$ for all $i \in A$, and so $\mu(z) \geq \frac{1-k}{2}$ which contradicts $(*)$. Therefore, $(z)_{\min(t_1,t_2)} \in \vee q_k \mu$ for all $z \in x + y$. Similarly, $(-x)_t \in \vee q_k \mu$ and $(ax)_t \in \vee q_k \mu$ for all $x \in M$ and $a \in R$. Consequently, $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M. ∎

Taking $k = 0$ in Theorem 4.12, we have the following corollary.

Corollary 4.13. *Let $\{\mu_i \mid i \in A\}$ be a family of $(\in, \in \vee q)$-fuzzy sub-hypermodules of M. Then $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \vee q)$-fuzzy sub-hypermodule of M.*

The following example shows that there exists $k \in [0, 1)$ such that the union of two $(\in, \in \vee q_k)$-fuzzy sub-hypermodules of M may not be an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M.

Example 4.14. Let $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ be a hypermodule given in Example 3.6 and μ an $(\in, \in \vee q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$ described in Example 3.6. Let ν be a fuzzy set in $\overline{M}$ defined by $\nu(\overline{0}) = 0.4$, $\nu(\overline{a}) = \nu(\overline{c}) = 0.3$, and $\nu(\overline{b}) = 0.5$. Then

$$\nu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{b}\}, & \text{if } t \in (0.3, 0.4] \end{cases}$$

Since $\overline{M}$ and $\{\overline{0}, \overline{b}\}$ are sub-hypermodules of $\overline{M}$, so ν is an $(\in, \in \vee q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4. The union $\mu \cup \nu$ of μ and ν is given by $\mu \cup \nu(\overline{0}) = 0.5$, $\mu \cup \nu(\overline{a}) = 0.8$, $\mu \cup \nu(\overline{b}) = 0.5$, and $\mu \cup \nu(\overline{c}) = 0.3$. Hence

$$(\mu \cup \nu)_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{a}, \overline{b}\}, & \text{if } t \in (0.3, 0.4] \end{cases}$$

Since $\{\overline{0}, \overline{a}, \overline{b}\}$ is not a sub-hypermodule of M, it follows that $\mu \cup \nu$ is not an $(\in, \in \vee q_{0.2})$-fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.

For any fuzzy set μ in M and $t \in (0, 1]$, we denote

$$\langle \mu \rangle_t = \{x \in M \mid (x)_t \; q_k \mu\} \quad \text{and} \quad [\mu]_t = \{x \in M \mid (x)_t \in \vee q_k \mu\}.$$

Obviously, $[\mu]_t = \mu_t \cup \langle \mu \rangle_t$.

Theorem 4.15. *Let μ be a fuzzy set in hypermodule of M. Then μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M if and only if $[\mu]_t$ is a sub-hypermodule of M for all $t \in (0, 1]$.*

We call $[\mu]_t$ an $(\in \vee q_k)$-*level* sub-hypermodule of μ.

Proof. Assume that hen μ is an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M and let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $(x)_t \in \vee q_k \mu$ and $(y)_t \in \vee q_k \mu$, that is, $\mu(x) \geq t$ or $\mu(x) + t > 1 - k$, and $\mu(y) \geq t$ or $\mu(y) + t > 1 - k$. Using Theorem 3.2, for all $x, y \in M$, and $a \in R$ the following axioms are satisfied:

(1) $\mu(z) \geq \min\left\{\mu(x), \mu(y), \dfrac{1-k}{2}\right\}; \quad \forall z \in x + y;$

(2) $\mu(-x) \geq \min\left\{\mu(x), \dfrac{1-k}{2}\right\};$

(3) $\mu(ax) \geq \min\left\{\mu(x), \dfrac{1-k}{2}\right\}.$

Case 1. $\mu(x) \geq t$ and $\mu(y) \geq t$. If $t > \dfrac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Hence $\mu(z) + t > \dfrac{1-k}{2} + \dfrac{1-k}{2} = 1 - k$, and so $(z)_t \; q_k \mu$. Moreover,

$$\mu(-x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Thus $\mu(-x) + t > \dfrac{1-k}{2} + \dfrac{1-k}{2} = 1 - k$, and so $(-x)_t \; q_k \mu$. Similarly, $(ax)_t \; q_k \mu$ for all $a \in R$. If $t \leq \dfrac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq t,$$

and thus $(z)_t \in \mu$. Similarly, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Therefore $(z)_t \in \vee q_k \mu$, $(-x)_t \in \vee q_k \mu$, and $(ax)_t \in \vee q_k \mu$ for all $a \in R$. Hence $x + y \subseteq [\mu]_t$, $-x \in [\mu]_t$, and $ax \in [\mu]_t$ for all $a \in R$.

Case 2. $\mu(x) \geq t$ and $\mu(y) + t > 1 - k$. If $t > \dfrac{1-k}{2}$, then for all $z \in x + y$

$$\begin{aligned}\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} &\geq \min\left\{\mu(y), \frac{1-k}{2}\right\} \\ &> \min\left(1 - k - t, \frac{1-k}{2}\right) = 1 - k - t\end{aligned}$$

and so $(z)_t\ q_k\mu$. Moreover,

$$\mu(-x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \geq \min\left(t, \frac{1-k}{2}\right) = \frac{1-k}{2},$$

and thus $(-x)_t\ q_k\mu$. Similarly, $(ax)_t\ q_k\mu$ for all $a \in R$. If $t \leq \dfrac{1-k}{2}$, then for all $z \in x+y$

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{t, 1-k-t, \frac{1-k}{2}\right\} = t.$$

Hence $(z)_t \in \mu$. Furthermore, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Thus $(z)_t \in \vee q_k\mu$, $(-x)_t \in \vee q_k\mu$, and $(ax)_t \in \vee q_k\mu$ for all $a \in R$. Then $x+y \subseteq [\mu]_t$, $-x \in [\mu]_t$, and $ax \in [\mu]_t$ for all $a \in R$.

Case 3. $\mu(x)+t > 1-k$ and $\mu(y) \geq t$. Similar to the case 2.

Case 4. $\mu(x)+t > 1-k$ and $\mu(y)+t > 1-k$. If $t > \dfrac{1-k}{2}$, then for all $z \in x+y$

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} > \min\left(1-k-t, \frac{1-k}{2}\right) = 1-k-t.$$

Thus $(z)_t\ q_k\mu$. Moreover, $(-x)_t\ q_k\mu$ and $(ax)_t\ q_k\mu$ for all $a \in R$. If $t \leq \dfrac{1-k}{2}$, then for all $z \in x+y$

$$\mu(z) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left(1-k-t, \frac{1-k}{2}\right) = \frac{1-k}{2} \geq t,$$

and so $(z)_t \in \mu$. Similarly, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Hence $(z)_t \in \vee q_k\mu$, $(-x)_t \in \vee q_k\mu$, and $(ax)_t \in \vee q_k\mu$ for all $a \in R$. Then $x+y \subseteq [\mu]_t$, $-x \in [\mu]_t$ and $ax \in [\mu]_t$ for all $a \in R$. Consequently, $[\mu]_t$ is a sub-hypermodule of M.

Conversely, let μ be a fuzzy set in hypermodule of M and $t \in (0,1]$ be such that $[\mu]_t$ is a sub-hypermodule of M. Let there exists $a, b \in M$ and $z \in a+b$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}$$

for some $t \in (0,1)$. Then $a, b \in \mu_t \subseteq [\mu]_t$, which implies that $z \in [\mu]_t$ (since $a+b \subseteq [\mu]_t$). Hence $\mu(z) \geq t$ or $\mu(z)+t+k > 1$, a contradiction. Thus $\mu(z) \geq \min\left\{\mu(a), \mu(b), \dfrac{1-k}{2}\right\}$ for all $a, b \in M$ and $z \in a+b$. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in\vee q_k)$-fuzzy sub-hypermodule of M by Theorem 3.2. ∎

A fuzzy set μ of hypermodule M is said to be *proper* if $Im(\mu)$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 4.16. *Let μ be an $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M such that* $\#\left\{\mu(x) \mid \mu(x) < \dfrac{1-k}{2}\right\} \geq 2$. *Then there exist two proper non-equivalent $(\in, \in \vee q_k)$-fuzzy sub-hypermodule of M such that μ can be expressed as the union of them.*

Proof. Let $\left\{\mu(x) \mid \mu(x) < \dfrac{1-k}{2}\right\} = \{t_1, t_2, ..., t_r\}$, where $t_1 > t_2 > ... > t_r$ and $r \geq 2$. Then the chain of $(\in \vee q_k)$-level sub-hypermodules of μ is

$$[\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq ... \subseteq [\mu]_{t_r} = M.$$

Define two fuzzy sets ν and γ of M by

$$\nu(x) = \begin{cases} t_1, & \text{if } x \in [\mu]_{t_1} \\ t_2, & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{t_1} \\ ... \\ t_r, & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}} \end{cases} \qquad \gamma(x) = \begin{cases} \mu(x), & \text{if } x \in [\mu]_{\frac{1-k}{2}} \\ k, & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{\frac{1-k}{2}} \\ t_3, & \text{if } x \in [\mu]_{t_3} \setminus [\mu]_{t_2} \\ ... \\ t_r, & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}} \end{cases}$$

respectively, where $t_3 < k < t_2$. Then ν and γ are $(\in, \in \vee q_k)$-fuzzy sub-hypermodules of M, and $\nu, \gamma \leq \mu$. The chain of $(\in \vee q_k)$-level sub-hypermodules of ν and γ are, respectively, given by

$$[\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq ... \subseteq [\mu]_{t_r} \quad \text{and} \quad [\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_2} \subseteq ... \subseteq [\mu]_{t_r}.$$

Therefore ν and γ are non-equivalent and clearly $\mu = \nu \cup \gamma$. This completes the proof. ■

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