CHARACTERIZATION OF HYPER BCI-ALGEBRA OF ORDER 3

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Abstract. In this paper, first we introduce the concepts of weak hyper BCI-algebras and strong hyper BCI-algebras. Then by using that concepts, we characterize all of the hyper BCI-algebras of order 3 up to isomorphism.

Keywords: hyper BCI-algebra, weak hyper BCI-algebra, strong hyper BCI-algebra.
2000 Mathematics Subject Classification: 06D99, 08A30.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [4] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [7] at the 8th Congress of Scandinavian Mathematiciens. Since then many researchers have worked on algebraic hyperstructures and developed it. A recent book [3] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [1], Y.B. Jun et al. applied the hyper structures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra and investigated some related properties. In [6], X.X. Long applied the hyper structure to BCI-algebras and introduce the concepts of hyper BCI-algebras which is a generalization of BCI-algebras. Now, in this note we define the notions of weak hyper BCI-algebras and strong hyper BCI-algebras and we obtain some related results which have been mentioned in this paper.

2. Preliminary

Definition 2.1. [6] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(BCI-1): $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2): $x * 0 = x$,
(BCI-3): $x * y = 0$ and $y * x = 0$ imply $x = y$,

for any $x, y, z \in X$. A BCI-algebra X is called *p-semisimple BCI-algebra* if $0 * (0 * x) = x$, for all $x \in X$.

Definition 2.2. [6] Let H be a nonempty set and " $\circ$ " be a hyper operation on H. Then H is called a *hyper BCI-algebra*, if it contains a constant 0 and satisfies the following conditions:

(B_1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(B_2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(B_3) $x \ll x$,
(B_4) $x \ll y$ and $y \ll x \;\Rightarrow\; x = y$,
(B_5) $0 \circ (0 \circ x) \ll x, \quad x \neq 0$,

for all $x, y, z \in H$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 which satisfy axioms (B_1), (B_2), (B_4) and $x \circ H \ll \{x\}$, for all $x \in H$. It is easy to see that every hyper BCK-algebra is a hyper BCI-algebra.

Let $(H, \circ, 0)$ be a hyper BCI-algebra. By H^+ we mean

$$H^+ = \{x \in H \mid 0 \in 0 \circ x\}.$$

We note that $0 \in H^+$, thus $H^+ \neq \emptyset$.

Definition 2.3. [5] Let I be a nonempty subset of hyper BCI-algebra H and $0 \in I$. Then I is called a

(i) weak hyper BCI-ideal of H if $xoy \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,
(ii) hyper BCI-ideal of H if $xoy \ll I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,
(iii) strong hyper BCI-ideal of H if $xoy \approx I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$, where $xoy \approx I$ means $xoy \cap I \neq \emptyset$.

By $W(H)$ we means the class of all weak hyper BCI-ideals, by $I(H)$ we mean the class of all hyper BCI-ideals and by $S(H)$ we means the class of all strong hyper BCI-ideals of H.

Theorem 2.4. [2] *There are* 19 *hyper* BCK*-algebras of order* 3 *up to isomorphism.*

Proposition 2.5. [6] *In any hyper* BCI*-algebra, the following hold:*

(i) $x \ll x \circ o$,

(ii) $A \ll A$,

(iii) $y \ll z$ *implies* $x \circ z \ll x \circ y$,

for all $x, y, z \in H$ *and for all nonempty subsets* A *and* B *of* H.

Definition 2.6. [2] Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper BCI-algebras and $f : H_1 \longrightarrow H_2$ be a function. Then f is said to be a homomorphism if and only if

$$f(x \circ_1 y) = f(x) \circ_2 f(y), \text{ for all } x, y \in H_1.$$

If f is one to one (onto) we say that f is a monomorphism (epimorphism) and if f is both one to one and onto, we say that f is an isomorphism and $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ are isomorphic.

3. Some properties of hyper BCI-algebras

Remark 3.1. We note that if $(H, *, 0)$ is a BCI-algebra and we define $x \circ y = \{x * y\}$, then $(H, \circ, 0)$ is a hyper BCI-algebra. Hence hyper BCI-algebras are a generalization of BCI-algebras

Theorem 3.2. *In any hyper* BCI*-algebra, the following hold:*

(i) *if* $x \ll 0$, *then* $x = 0$,

(ii) *if* $A \ll \{0\}$, *then* $A = \{0\}$,

(iii) *if* $A \circ A = \{0\}$, *then* A *is singleton,*

(iv) $x \in x \circ 0$,

(v) $x \in 0 \circ 0$ *imply that* $x \in H^+$,

(vi) *if* $H^+ = \{0\}$, *then* $0 \circ 0 = \{0\}$,

(vii) *if* $0 \circ 0 = \{0\}$, *then* $0 \circ x = \{0\}$, *for all* $x \in H^+$,

for all $x \in H$ *and for all nonempty subset* A *of* H.

Proof. (i) Let $x \ll 0$. Thus $0 \in x \circ 0$. By (B_1), $0 \in 0 \circ 0(s)(0 \circ 0) \circ (x \circ 0) \ll 0 \circ x$. Hence there is $t \in 0 \circ x$ such that $0 \ll t$ and so $0 \in 0 \circ t$. By (B_5), $0 \in 0 \circ t(s)0 \circ (0 \circ x) \ll \{x\}$. Hence $0 \ll x$. By (B_4), we imply that $x = 0$.

(ii) Let $A \ll \{0\}$. Then for all $a \in A$, $a \ll 0$. By (i), we imply that $a = 0$. Hence $A = \{0\}$.

(iii) Let $A \circ A = \{0\}$ and $x, y \in A$. Then $x \circ y = \{0\}$ and $y \circ x = \{0\}$. By (B_4), $x = y$. Therefore, A is singleton.

(iv) By (B_2) and (B_3), $0 \in 0 \circ 0(s)(x \circ x) \circ 0 = (x \circ 0) \circ x$. Thus there is $t \in x \circ 0$ such that $0 \in t \circ x$ and so $t \ll x$. On the other hands $0 \in (x \circ 0) \circ t = (x \circ t) \circ 0$. Hence there is an element $a \in x \circ t$ such that $0 \in a \circ 0$. Thus $a \ll 0$. By (i), $a = 0$ and so $0 \in x \circ t$. Hence $x \ll t$. By (B_4), $x = t$. Therefore, $x \in x \circ 0$.

(v) Let $x \in 0 \circ 0$. By (B_3) and (B_2), $0 \in (0 \circ 0) \circ x = (0 \circ x) \circ 0$. Thus there exist $a \in 0 \circ x$ such that $0 \in a \circ 0$ and so $a \ll 0$. By (i), $a = 0$. Hence $0 \in 0 \circ x$. Therefore, $x \in H^+$.

(vi) By (v), the proof is easy.

(vii) Let $0 \circ 0 = \{0\}$ and $x \in H^+$. By (B_1), $(0 \circ x) \circ (0 \circ x) \ll 0 \circ 0 = \{0\}$. By (ii), $(0 \circ x) \circ (0 \circ x) = \{0\}$. By (iii), $0 \circ x$ is singleton. Since $x \in H^+$, $0 \in 0 \circ x$. Thus $0 \circ x = \{0\}$. ∎

Theorem 3.3. *Let H be a hyper BCI-algebra and $x \circ x = \{0\}$, for all $x \in H$. Then H is a BCI-algebra.*

Proof. By Remark 3.1, it is sufficient to prove that $x \circ y$ is singleton, for all $x, y \in H$. By (B_1),

$$(x \circ y) \circ (x \circ y) \ll x \circ x = \{0\}.$$

By Theorem 3.2(iii), $x \circ y$ is singleton. Therefore, H is a BCI-algebra. ∎

Lemma 3.4. *Let H be a hyper BCI-algebra and $H^+ = \{0\}$. Then the following hold:*

(i) *if $x \ll y$, then $x = y$,*

(ii) *if $A \ll B$, then $A(s)B$,*

(iii) $0 \circ (0 \circ x) = \{x\}$,

(iv) $x \circ x = \{0\}$,

for all $x, y \in H$ and for all nonempty subsets A, B of H.

Proof. (i) Let H be a hyper BCI-algebra, $H^+ = \{0\}$ and $x \ll y$. By Theorem 3.2(vi), $0 \circ 0 = \{0\}$. By $0 \in x \circ y$, (B_1) and (B_3), $\{0\} = 0 \circ 0(s)(y \circ y) \circ (x \circ y) \ll y \circ x$. Hence there exist $a \in y \circ x$ such that $0 \ll a$ and so $0 \in 0 \circ a$. Thus $a \in H^+ = \{0\}$. Hence $a = 0$ and so $y \ll x$. By (B_4), $x = y$.

(ii) Let $A \ll B$. Then, for all $a \in A$ there exist $b \in B$ such that $a \ll b$. By (i), $a = b$. Therefore, $A(s)B$.

(iii) By (B_5) and (ii), the proof is easy.

(iv) By (iii), $0 \circ (0 \circ x) = \{x\}$, for all $x \in H$. Thus $0 \circ y = \{x\}$, for all $y \in 0 \circ x$. By (iii), $0 \circ x = 0 \circ (0 \circ y) = \{y\}$. Thus $0 \circ x = \{y\}$ and $0 \circ y = \{x\}$. By $0 \circ 0 = \{0\}$ and by (B_1), $x \circ x = (0 \circ y) \circ (0 \circ y) \ll 0 \circ 0 = \{0\}$.

Now, by Theorem 3.2(ii), $x \circ x = \{0\}$. ■

Theorem 3.5. *Let H be a hyper BCI-algebra and $H^+ = \{0\}$. Then H is a p-semisimple BCI-algebra.*

Proof. Let H be a hyper BCI-algebra and $H^+ = \{0\}$. By Lemma 3.4(iv), $x \circ x = \{0\}$ and so by Theorem 3.3, H is a BCI-algebra. By Lemma 3.4(iii), $0 \circ (0 \circ x) = \{x\}$. Therefore, H is a p-semisimple BCI-algebra. ■

Theorem 3.6. *If $f : (H_1, \circ_1, 0_1) \longrightarrow (H_2, \circ_2, 0_2)$ is an isomorphism of hyper BCI-algebras, then*

(i) *if $0_1 \circ_1 0_1 = \{0_1\}$, then $f(0_1) = 0_2$,*

(ii) *if $x \in 0_1 \circ_1 x$, for all $x \in 0_1 \circ_1 0_1$, then $f(0_1) = 0_2$.*

Proof. (i) Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper BCI-algebras. Then $f(0_1) = f(0_1 \circ_1 0_1) = f(0_1) \circ_2 f(0_1)$ and $0_2 \in f(0_1) \circ_2 f(0_1)$. Thus $f(0_1) = 0_2$.

(ii) Let $f(y) = 0_2$ and $f(0_1) = x$. By $0_2 \in f(0_1) \circ_2 f(0_1) = f(0_1 \circ_1 0_1)$, we imply that $y \in 0_1 \circ_1 0_1$. By hypothesis, $y \in 0_1 \circ y$ and by Theorem 3.2(iv), $y \in y \circ_1 0_1$. Thus $f(y) \in f(0_1) \circ_2 f(y)$ and $f(y) \in f(y) \circ_2 f(0_1)$. Hence $0_2 \in x \circ_2 0_2$ and $0_2 \in 0_2 \circ_1 x$. By (B_4), $x = 0_2$. Therefore, $f(0_1) = 0_2$. ■

In what follows, first we introduce the concepts of weak hyper BCI-algebras and strong hyper BCI-algebras. Then, we find some results on (strong) weak hyper BCI-algebras of order 3. Finally, we characterize the hyper BCI-algebras of order 3.

Definition 3.7. Let H be a hyper BCI-algebra. Then the set $S_k = \{x \in H : x \circ H \ll \{x\}\}$ is defined as hyper BCK-part of H. If $H \neq S_k$, then H is known as a proper hyper BCI-algebra.

A hyper BCI-algebra H is called a

(i) weak proper hyper BCI-algebra if H is proper and $H^+ = H$. In other word if 0 is the smallest element of H,

(ii) strong proper hyper BCI-algebra if $H^+ \neq H$. We note that if $x \notin H^+$, then $0 \notin 0 \circ x$. Thus $0 \circ x \not\ll \{0\}$. Therefore, $0 \circ H \not\ll \{0\}$ and $(H, \circ, 0)$ is proper.

4. Characterization of weak proper hyper *BCI*-algebra of order 3

Lemma 4.1. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then, the following hold:*

(i) $0 \circ 0 \neq \{0, a, b\}$,

(ii) *if $0 \circ 0 = \{0, a\}$, then $a \ll b$,*

(iii) *if $0 \circ a = \{0\}$, then $0 \circ 0 = \{0\}$.*

Proof. (i) Let $0 \circ 0 = \{0, a, b\}$. Since H is a weak proper hyper BCI-algebra, then $0 \in 0 \circ b$. By (B_5),

$$\{0, a, b\} = 0 \circ 0 (s) 0 \circ (0 \circ b) \ll \{b\}.$$

Thus $a \ll b$. By the similar way, $b \ll a$. By (B_4), $a = b$, which is a contradiction.

(ii) Let $0 \circ 0 = \{0, a\}$. By $b \in H^+$ and (B_5), $\{0, a\} = 0 \circ 0(s) 0 \circ (0 \circ b) \ll \{b\}$. Therefore, $a \ll b$.

(iii) Let $0 \circ a = \{0\}$. By (B_1), $0 \circ 0(s)(0 \circ a) \circ (0 \circ a) \ll a \circ a = \{0\}$. Now, by Theorem 3.2(ii), $0 \circ 0 = \{0\}$. ∎

Lemma 4.2. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then the following hold:*

(i) *if $0 \circ 0 \not\ll \{0\}$, then $(H, \circ, 0)$ is a chain,*

(ii) *if $0 \circ a \not\ll \{0\}$, then $0 \circ 0 \not\ll \{0\}$,*

(iii) *if $a \circ 0 \not\ll \{a\}$, then $a \ll b$,*

(iv) *if $a \circ a \not\ll \{a\}$, then $a \circ 0 \not\ll \{a\}$,*

(v) *if $a \circ b \not\ll \{a\}$, then $a \circ 0 \not\ll \{a\}$.*

Proof. (i) Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then $0 \ll a$ and $0 \ll b$. It is remind to prove that a and b are comparable. If $0 \circ 0 \not\ll 0$, then $0 \circ 0 = \{0, a\}$ or $\{0, b\}$ or $\{0, a, b\}$. By Lemma 4.1(i), $0 \circ 0 \neq \{0, a, b\}$. Thus $0 \circ 0 = \{0, a\}$ or $\{0, b\}$. If $0 \circ 0 = \{0, a\}$, then by Lemma 4.1(ii), $a \ll b$ and if $0 \circ 0 = \{0, b\}$, $b \ll a$. Therefore, H is a chain.

(ii) If $0 \circ a \not\ll \{0\}$, then $0 \circ a \neq \{0\}$. By Theorem 3.2(vii), $0 \circ 0 \neq \{0\}$ and so $0 \circ 0 \not\ll \{0\}$.

(iii) If $a \circ 0 \not\ll \{a\}$, then $b \in a \circ 0$ and $b \not\ll a$. By Theorems 3.2(i),(iv), $a \in a \circ 0$ and $0 \notin a \circ 0$. Thus $a \circ 0 = \{a, b\}$. By (B_2), $0 \in b \circ b(s)(a \circ 0) \circ b = (a \circ b) \circ 0$. Hence $0 \in (a \circ b) \circ 0$. It means that there is an element $x \in a \circ b$ such that $0 \in x \circ 0$ and so $x \ll 0$. By Theorem 3.2(i), $x = 0$ and so $0 \in a \circ b$. Therefore, $a \ll b$.

(iv) If $a \circ a \not\ll \{a\}$, then $b \in a \circ a$ and $b \not\ll a$. By Theorem 3.2(iv) and (B_1),

$$b \in b \circ 0(s)(a \circ a) \circ (0 \circ a) \ll a \circ 0.$$

Thus there is an element $t \in a \circ 0$ such that $b \ll t$. From $b \not\ll a$ and $b \not\ll 0$ we imply that $t = b$. Hence $b \in a \circ 0$ and $b \not\ll a$. Therefore, $a \circ 0 \not\ll \{a\}$.

(v) If $a \circ b \not\ll a$, then $b \in a \circ b$ and $b \not\ll a$. We have $b \in b \circ 0(s)(a \circ b) \circ (0 \circ b) \ll a \circ 0$. Thus there is an element $t \in a \circ 0$ such that $b \ll t$. From $b \not\ll a$ and $b \not\ll 0$ we imply that $t = b$. Hence $b \in a \circ 0$ and $b \not\ll a$. Therefore, $a \circ 0 \not\ll a$. ∎

Theorem 4.3. *Every weak proper hyper BCI-algebra of order three is a chain.*

Proof. Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. It is clear that $0 \ll a$ and $0 \ll b$. It is remind to prove that a and b are comparable. Since H is proper, then

$$0 \circ H \not\ll \{0\} \text{ or } a \circ H \not\ll \{a\} \text{ or } b \circ H \not\ll \{b\}.$$

So we will investigate three cases.

Case 1. If $0 \circ H \not\ll \{0\}$, then $0 \circ 0 \not\ll \{0\}$ or $0 \circ a \not\ll \{0\}$ or $0 \circ b \not\ll \{0\}$. By Lemma 4.2(i) and (ii), we imply that H is a chain.

Case 2. If $a \circ H \not\ll \{a\}$, then $a \circ o \not\ll \{a\}$ or $a \circ a \not\ll \{a\}$ or $a \circ b \not\ll \{a\}$. By Lemma 4.2(iii),(iv) and (v), we imply that H is a chain.

Case 3. $b \circ H \not\ll \{b\}$ is similar to the case two.

Therefore, in every case H is a chain. ∎

Proposition 4.4. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then, the following hold:*

(i) *if* $0 \circ 0 = \{0, a\}$, *then* $0 \circ a = 0 \circ b = \{0, a\}$,

(ii) $0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}$ *or* $\{0, a\}$ *or* $\{0, b\}$.

Proof. (i) Let $0 \circ 0 = \{0, a\}$. By Lemma 4.1(ii), $a \ll b$. If $0 \circ a = \{0\}$, then by Lemma 4.1, $0 \circ 0 = \{0\}$ which is a contradiction. By the similar way $0 \circ b \neq \{0\}$. Thus $a \in 0 \circ a$ or $b \in 0 \circ a$ and $a \in 0 \circ b$ or $b \in 0 \circ b$. If $b \in 0 \circ a$, then $0 \circ b(s) 0 \circ (0 \circ a) \ll \{a\}$. From $a \ll b$ we imply that $a \in 0 \circ b$ and $b \notin 0 \circ b$. Hence $0 \circ b = \{0, a\}$. Thus $b \in 0 \circ a(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\}$. It means that $b \ll 0$ or $b \ll a$ which is a contradiction. Thus $b \notin 0 \circ a$. By $0 \circ a \neq \{0\}$ we conclude that $0 \circ a = \{0, a\}$. If $b \in 0 \circ b$, then $b \in 0 \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\}$. Hence $b \ll 0$ or $b \ll a$ which both of them are contradiction. Therefore, $0 \circ b = \{0, a\}$.

(ii) By (i) and Lemma 4.1(i),(iii), the proof is easy. ∎

Theorem 4.5. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then the following hold:*

(i) *if* $0 \circ 0 = \{0\}$, *then* $a \circ 0 = \{a, b\}$ *and* $b \circ 0 = \{b\}$ *or* $a \circ 0 = \{a\}$ *and* $b \circ 0 = \{a, b\}$,

(ii) *if* $0 \circ 0 = \{0, a\}$, *then* $a \circ 0 = \{a\}$ *and* $b \circ 0 = \{b\}$.

Proof. (i) Let $0 \circ 0 = \{0\}$. Then, by Proposition 4.4, $0 \circ a = 0 \circ b = 0 \circ 0 = \{0\}$. Thus $0 \circ H \ll \{0\}$. Since H is proper, then $a \circ H \not\ll \{a\}$ or $b \circ H \not\ll \{b\}$. Without loss of generality, let $a \circ H \not\ll \{a\}$. By Lemma 4.2(iii),(iv),(v), we imply that $a \circ 0 \not\ll \{a\}$ and $a \ll b$ and so $a \circ 0 = \{a, b\}$. By Theorem 3.2, $b \in b \circ 0$ and from $0 \ll b$ we conclude that $0 \notin b \circ 0$. If $a \in b \circ 0$, then $0 \in a \circ a(s)(b \circ 0) \circ a = (b \circ a) \circ 0$. Hence $0 \in (b \circ a) \circ 0$. Since $0 \notin a \circ 0$ and $0 \notin b \circ 0$, $0 \in b \circ a$ and so $b \ll a$ which is a contradiction. Thus $a \notin b \circ 0$ and so $b \circ 0 = \{b\}$. By the similar way if $b \circ H \not\ll \{b\}$, then $a \circ 0 = \{a\}$ and $b \circ 0 = \{a, b\}$.

(ii) Let $0 \circ 0 = \{0, a\}$. By Lemma 4.4, $0 \circ b = 0 \circ a = \{0, a\}$ and by Lemma 4.1, $a \ll b$. By Theorem 3.2, $a \in a \circ 0$ and $b \in b \circ 0$. If $b \in a \circ 0$, then

$$b \in a \circ 0(s)(0 \circ a) \circ (0 \circ a) \ll 0 \circ 0 = \{0, a\}.$$

Hence $b \ll 0$ or $b \ll a$ which is a contradiction. Thus $b \notin a \circ 0$ and so $a \circ 0 = \{a\}$.

If $a \in b \circ 0$, then

$$0 \in a \circ a(s)(b \circ 0) \circ a = (b \circ a) \circ 0.$$

Thus there is an element $x \in b \circ a$ such that $0 \in x \circ 0$ and so $x \ll 0$. By Theorem 3.2, $x = 0$ and so $0 \in b \circ a$. Hence $b \ll a$ which is a contradiction. Therefore, $0 \circ b = \{b\}$. ∎

Theorem 4.6. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra, $0 \circ 0 = \{0\}$ and $a \circ 0 = \{a, b\}$. Then $b \in a \circ a$, $b \in b \circ a$, $a \circ b \neq \{0, a\}$ and $b \circ b \neq \{0, a\}$.*

Proof. By Propositions 4.5 and 4.4,

$$0 \circ a = 0 \circ b = 0 \circ 0 = \{0\}, \quad a \circ 0 = \{a, b\}, \quad b \circ 0 = \{b\} \text{ and } a \ll b.$$

Thus $b \in a \circ 0(s)(b \circ a) \circ (a \circ a) \ll b \circ a$. From $a \ll b$ and $0 \ll b$ we imply that $b \in b \circ a$. By (B_1), $b \in b \circ a(s)(a \circ 0) \circ (a \circ 0) \ll a \circ a$. Therefore, $b \in a \circ a$. To prove $a \circ b \neq \{0, a\}$ and $b \circ b \neq \{0, a\}$ we will prove that if $a \in a \circ b$, then $b \in a \circ b$ and if $a \in b \circ b$, then $b \in b \circ b$. Let $a \in a \circ b$, then

$$(1) \qquad b \in a \circ 0(s)(a \circ b) \circ (b \circ b) \ll a \circ b.$$

Since $a \ll b$ and $0 \ll b$, then by (1), $b \in a \circ b$.

Let $a \in b \circ b$, then $b \in a \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b$. Hence $b \in b \circ b$. ∎

Theorem 4.7. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra and $0 \circ 0 = \{0, a\}$. Then*

(i) $a \circ a = a \circ b = \{0, a\}$,

(ii) $a \in b \circ b$,

(iii) *if $b \in b \circ b$, then $b \in b \circ a$.*

Proof. (i) By Propositions 4.5 and 4.4 we conclude that

$$0 \circ a = 0 \circ b = 0 \circ 0 = \{0, a\}, \quad a \circ 0 = \{a\}, \quad b \circ 0 = \{b\} \text{ and } a \ll b.$$

By (B_1), $a \in 0 \circ 0(s)(a \circ b) \circ (a \circ b) \ll a \circ a$. Hence $a \ll a \circ a$ and so $a \in a \circ a$ or $b \in a \circ a$. If $b \in a \circ a$, then

$$b \in a \circ a(s)(0 \circ 0) \circ (0 \circ 0) \ll 0 \circ 0 = \{0, a\}.$$

Thus $b \ll 0$ or $b \ll a$ which both of them are contradiction. Hence $b \notin a \circ a$ and so $a \circ a = \{0, a\}$. Also by (B_1), $\{0, a\} = 0 \circ (b \circ a)(s)(a \circ a) \circ (b \circ a) \ll a \circ b$.

Hence $a \ll a \circ b$. Therefore, $a \in a \circ b$ or $b \in a \circ b$. If $b \in a \circ b$, then $b \in b \circ 0(s)(a \circ b) \circ (0 \circ b) \ll a \circ 0 = \{a\}$. It means that $b \ll a$ which is a contradiction. Thus $b \notin a \circ b$. Since $a \in a \circ b$ and $a \ll b$, $a \circ b = \{0, a\}$.

(ii) By (B_2), $a \in 0 \circ 0(s)(b \circ b) \circ 0 = (b \circ 0) \circ b = b \circ b$. Thus $a \in b \circ b$.

(iii) If $b \in b \circ b$, then $b \in b \circ b(s)(b \circ 0) \circ (a \circ 0) \ll b \circ a$. Therefore, $b \in b \circ a$. ■

Theorem 4.8. *Let $H = \{0, a, b\}$ and $f : (H, \circ_1, 0) \longrightarrow (H, \circ_2, 0)$ be a non identity isomorphism of weak proper hyper BCI-algebras, then*

(i) $f(0) = 0$,

(ii) *if $a \ll_1 b$, then $b \ll_2 a$.*

Proof. (i) By Lemma 4.4 and Theorem 3.6, the proof is clear.

(ii) Let $a \ll_1 b$. Since $f(0) = 0$ and f is not identity, then $f(a) = b$ and $f(b) = a$. By $0 \in a \circ_1 b$, $f(0) \in f(a) \circ_2 f(b)$ and so $0 \in b \circ_2 a$. Therefore, $b \ll_2 a$. ■

By Theorems 4.8 and 4.3 any weak proper hyper BCI-algebra is a chain and any chain $0 \ll a \ll b$ is isomorph by a chain $0 \ll b \ll a$. So if we let $H = \{0, a, b\}$ and $0 \ll a \ll b$, then we can find all weak proper hyper BCI-algebra of order 3 up to isomorphism.

According Theorems 4.4, 4.5, 4.6 and 4.7 we have two following structures:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$ or $\{0, a, b\}$	$\{0\}, \{0, b\}$ or $\{0, a, b\}$
b	$\{b\}$	$\{b\}$ or $\{a, b\}$	$\{0\}, \{0, b\}$ or $\{0, a, b\}$

$\circ$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a\}, \{b\}$ or $\{a, b\}$	$\{0, a\}$ or $\{0, a, b\}$

Case 1. Let $0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}$, $a \circ 0 = \{a, b\}$, $b \circ 0 = \{b\}$.

Let $a \circ a = \{0, b\}$. If $a \in b \circ a$, then

$$a \in b \circ a(s)(a \circ 0) \circ a = (a \circ a) \circ 0 = 0 \circ 0 \cup b \circ 0 = \{0, b\}.$$

which is a contradiction. Hence $b \circ a = \{b\}$. If $a \circ b = \{0\}$, then

$$b \circ b(s)(a \circ a) \circ (b \circ a) \ll a \circ b = \{0\}.$$

By Theorem 3.2, $b \circ b = \{0\}$ and we get the following Cayley table which is a weak proper hyper BCI-algebra.

$\circ_1$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, b\}$, then by $\{0\} = 0 \circ b(s)b \circ b$ and (B_3),

$$b \circ b = 0 \circ b \cup b \circ b = \{0, b\} \circ b = (a \circ a) \circ b = (a \circ b) \circ a = \{0, b\} \circ a = \{0, b\}.$$

Thus $b \circ b = \{0, b\}$ and we get the following weak proper hyper BCI-algebra.

$\circ_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, a, b\}$, then

$$b \circ b = 0 \circ b \cup b \circ b = \{0, b\} \circ b = (a \circ a) \circ b = (a \circ b) \circ a = \{0, a, b\} \circ a = \{0, b\}.$$

Thus $b \circ b = \{0, b\}$ and we get the following weak proper hyper BCI-algebra.

$\circ_3$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$I(H) = S(H) = \{\{0\}, \{0, a\}, H\} \text{ and } W(H) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

Let $a \circ a = \{0, a, b\}$ and $b \circ a = \{b\}$. If $a \circ b = \{0\}$, then $b \circ b(s)(a \circ 0) \circ (b \circ 0) \ll a \circ b = \{0\}$. By Theorem 3.2, $b \circ b = \{0\}$ and we get the next weak proper hyper BCI-algebra.

$\circ_4$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, b\}$, then $a \circ b \cup b \circ b = \{a, b\} \circ b = (a \circ 0) \circ b = (a \circ b) \circ 0 = \{0, b\}$. Hence $\{0, b\} \cup b \circ b = \{0, b\}$. which means that $b \circ b = \{0\}$ or $b \circ b = \{0, b\}$ which both of them are weak proper hyper BCI-algebras.

$\circ_5$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$\circ_6$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$W(H,\circ_5)=I(H,\circ_5)=S(H,\circ_5)=W(H,\circ_6)=I(H,\circ_6)=S(H,\circ_6)=\{\{0\},\{0,a\},H\}.$$

If $a\circ b=\{0,a,b\}$, then $b\circ b=\{0\},\{0,b\}$or $\{0,a,b\}$ which all of them are weak proper hyper BCI-algebras.

$\circ_7$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$\circ_8$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

$$I(H,\circ_7)=S(H,\circ_7)=I(H,\circ_8)=S(H,\circ_8)=\{\{0\},\{0,a\},H\},$$
$$W(H,\circ_7)=W(H,\circ_8)=\{\{0\},\{0,b\},\{0,a\},H\}.$$

$\circ_9$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

$$I(H)=S(H)=\{\{0\},\{0,a\},H\} \text{ and } W(H)=\{\{0\},\{0,b\},\{0,a\},H\}.$$

Let $a\circ a=\{0,a,b\}$ and $b\circ a=\{a,b\}$. By (B_1), $(a\circ a)\circ(b\circ a)\ll a\circ b$. Hence $\{0,a,b\}\ll a\circ b$. We imply that $b\in a\circ b$. Thus $a\circ b=\{0,b\}$ or $\{0,a,b\}$. If $a\circ b=\{0,b\}$, then by $(a\circ a)\circ b=(a\circ b)\circ a$ we conclude that $\{0,b\}\cup b\circ b=\{0,a,b\}$. Hence $a\in b\circ b$. By Theorem 4.7(iii), $b\circ b=\{0,a,b\}$. But in this case $(a\circ 0)\circ b\neq(a\circ b)\circ 0$. Hence $a\circ b\neq\{0,b\}$.

If $a\circ b=\{0,a,b\}$, then $b\circ b=\{0\}$, $\{0,b\}$ or $\{0,a,b\}$. If $b\circ b=\{0\}$, then $(b\circ a)\circ b\neq(b\circ b)\circ a$. If $b\circ b=\{0,b\}$ or $\{0,a,b\}$, then we get two following weak proper hyper BCI-algebras.

$\circ_{10}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

$\circ_{11}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

$$I(H,\circ_{10})=S(H,\circ_{10})=I(H,\circ_{11})=S(H,\circ_{11})=\{\{0\},\{0,a\},H\},$$
$$W(H,\circ_{10})=W(H,\circ_{11})=\{\{0\},\{0,b\},\{0,a\},H\}.$$

Case 2. Let $0\circ 0=0\circ a=0\circ b=\{0,a\}$, $a\circ 0=\{a\}$, $b\circ 0=\{b\}$ and $a\circ a=a\circ b=\{0,a\}$.

If $b\circ b=\{0,a\}$, then $b\circ a=\{b\}$, $\{a\}$ or $\{a,b\}$. If $b\circ a=\{a,b\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. The other cases are weak proper hyper BCI-algebras with the following Cayley tables.

$\circ_{12}$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

$\circ_{13}$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

$I(H,\circ_{12}) = S(H,\circ_{12}) = \{\{0\},\{0,a\},H\}$, $I(H,\circ_{13}) = S(H,\circ_{13}) = \{\{0\},H\}$, $W(H,\circ_{12}) = \{\{0\},\{0,b\},\{0,a\},H\}$ and $W(H,\circ_{13}) = \{\{0\},\{0,b\},H\}$.

If $b \circ b$=$\{0,a,b\}$, then by Theorem 4.7, $b \in b \circ a$. Thus $b \circ a$=$\{b\}$ or $\{a,b\}$, which both of them are weak proper hyper BCI-algebra with the following Cyley tables:

$\circ_{14}$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

$\circ_{15}$	0	a	b
0	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

$I(H,\circ_{14}) = S(H,\circ_{14}) = I(H,\circ_{15}) = \{\{0\},\{0,a\},H\}$, $S(H,\circ_{15}) = \{\{0\},H\}$, $W(H,\circ_{14}) = W(H,\circ_{15}) = \{\{0\},\{0,b\},\{0,a\},H\}$.

5. Characterization of strong proper hyper BCI-algebra of order 3

Let $H = \{0,a,b\}$ and $(H,\circ,0)$ be a strong proper hyper BCI-algebra. By definition of strong proper hyper BCI-algebra, $H^+ \neq H$. Thus $H^+ = \{0\}$ or $H^+ = \{0,a\}$.

Theorem 5.1. *Let $H = \{0,a,b\}$ be a strong proper hyper BCI-algebra and $H^+ = \{0,a\}$. Then, the following hold:*

(i) $0 \circ b = \{b\}$,

(ii) $0 \circ 0 = \{0\}$ *or* $\{0,a\}$,

(iii) *if* $0 \circ 0 = \{0\}$, *then* $0 \circ a = \{0\}$,

(iv) *if* $0 \circ 0 = \{0,a\}$, *then* $0 \circ a = \{0,a\}$,

(v) $0 \circ a = 0 \circ 0 = \{0\}$ *or* $\{0,a\}$,

(vi) *if* $0 \circ 0 = \{0,a\}$, *then* $a \in b \circ b$ *and* $a \in a \circ a$.

Proof. (i) Since $b \notin H^+$, $0 \notin 0 \circ b$. If $a \in 0 \circ b$, then by $a \in H^+$ and (B_5), $0 \in 0 \circ a(s)0 \circ (0 \circ b) \ll \{b\}$, which is a contradiction. Hence $a \notin 0 \circ b$. Since $b \notin H^+$, $0 \notin 0 \circ b$. Therefore, $0 \circ b = \{b\}$.

(ii) If $b \in 0 \circ 0$, then by Theorem 3.2(i) and (B$_2$),

$$0 \notin b \circ 0 = (0 \circ b) \circ 0 = (0 \circ 0) \circ b \supseteq b \circ b \ni 0,$$

which is a contradiction. Thus $b \notin 0 \circ 0$ and so $0 \circ 0 = \{0\}$ or $0 \circ 0 = \{0,a\}$.

(iii) By Theorem 3.2(vii), the proof is easy.

(iv) Let $0 \circ 0 = \{0,a\}$. If $0 \circ a = \{0\}$, then by (iv), $0 \circ 0 = \{0\}$ which is a contradiction. Since $a \in H^+$, $0 \in 0 \circ a$. It is sufficient to prove that $b \notin 0 \circ a$. On the contrary let $b \in 0 \circ a$. By (i) and (B$_2$),

$$b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b \supseteq 0 \circ b \quad \cup \quad b \circ b \ni \{0,b\}.$$

Thus $b \in b \circ a$ and $0 \in b \circ a$. Hence $b \ll a$ and so $a \not\ll b$.

By (B_1) and (B_3),

$$a \in 0 \circ 0(s)(a \circ a) \circ (b \circ a) \ll a \circ b.$$

Thus $a \in a \circ b$. On the other hand,

$$\{0, a\} = 0 \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b.$$

Hence $a \in b \circ b$. By (B_1),

$$a \in a \circ b(s)(b \circ b) \circ (0 \circ b) \ll b \circ 0.$$

Since $b \ll a$, then $a \in b \circ 0$. By Theorem 3.2, $b \circ 0 = \{a, b\}$. By (B_1) and (i),

$$0 \in a \circ a(s)(0 \circ 0) \circ (b \circ 0) \ll 0 \circ b = \{b\}.$$

Thus $0 \ll b$, which is a contradiction. Hence $b \notin 0 \circ a$. Therefore, $0 \circ a = \{0, a\}$.

(v) By (iii) and (iv), the proof is easy.

(vi) Let $0 \circ 0 = \{0, a\}$. By (v), $0 \circ a = \{0, a\}$. By (B_1),

$$\{0, a\} = 0 \circ 0(s)(a \circ a) \circ (a \circ a) \ll a \circ a.$$

Hence there is $t \in a \circ a$ such that $a \ll t$.

Let $t = b$ and so $b \in a \circ a$ and $a \ll b$. Thus $b \not\ll a$. By (B_2),

$$0 \notin b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b \supseteq a \circ b \ni 0,$$

which is a contradiction. Thus $t = a$ and so $a \in a \circ a$. By similar way from

$$\{0, a\} = 0 \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b,$$

we imply that $a \in b \circ b$. ∎

Theorem 5.2. *Let $H = \{0, a, b\}$ be a strong proper hyper BCI-algebra and $H^+ = \{0, a\}$. If $0 \circ 0 = \{0, a\}$, then the following hold:*

(i) $b \circ a = \{b\}$,

(ii) $a \circ b = \{b\}$,

(iii) $b \circ 0 = \{b\}$,

(iv) $a \circ 0 = \{a\}$,

(v) $a \circ a = \{0, a\}$,

(vi) $b \circ b = \{0, a\}$.

Proof. If $0 \circ 0 = \{0, a\}$, then by Theorem 5.1, $0 \circ a = \{0, a\}$, $0 \circ b = \{b\}$, $a \in b \circ b$ and $a \in a \circ a$.

(i) By (B_2),

$$b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b = \{0, a\} \circ b = \{b\} \quad \cup a \circ b. \tag{1}$$

Thus $b \in b \circ a$. If $0 \in a \circ b$, then by (1), $0 \in b \circ a$. It means that $a \ll b$ and $b \ll a$. Therefore, $a = b$ which is a contradiction. Hence $0 \notin a \circ b$ and $0 \notin b \circ a$. If $b \circ a = \{a, b\}$, then $a \circ a(s)(0 \circ a) \circ (b \circ a) \ll 0 \circ b = \{b\}$. Hence $a \circ a = \{b\}$ which is a contradiction. Therefore, $b \circ a = \{b\}$.

(ii) By $b \circ a = \{b\}$ and (1), $a \circ b = \{b\}$.

(iii) By Theorem 3.2(i),(iii), $b \in b \circ 0$ and $0 \notin b \circ 0$. By (B_2),

$$b \circ 0 = (b \circ a) \circ 0 = (b \circ 0) \circ a.$$

If $a \in b \circ 0$, then we conclude that $0 \in b \circ 0$ which is a contradiction. Hence $b \circ 0 = \{b\}$.

(iv) By Theorem 3.2, $a \in a \circ 0$ and $0 \notin a \circ 0$. If $b \in a \circ 0$, then

$$b = 0 \circ b(s)(0 \circ 0) \circ (a \circ 0) \ll 0 \circ a = \{0, a\}.$$

Thus $b \ll a$. But in (1) we proved that $a \not\ll b$ and $b \not\ll a$. Thus $b \notin a \circ 0$ and $a \circ 0 = \{a\}$.

(v) By Theorem 5.1(vii), $a \in a \circ a$. If $b \in a \circ a$, then

$$b \in a \circ a(s)(0 \circ a) \circ (0 \circ a) \ll 0 \circ 0 = \{0, a\},$$

which is a contradiction. Thus $a \circ a = \{0, a\}$.

(vi) By Theorem 5.1(vii), $a \in b \circ b$. If $b \in b \circ b$, then

$$b \in b \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\},$$

which is a contradiction. Thus $b \circ b = \{0, a\}$. ∎

By Theorem 5.2, if $0 \circ 0 = \{0, a\}$ we just have one weak proper hyper *BCI*-algebra with the following Cayley table:

$\circ_{16}$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

Theorem 5.3. *Let $H = \{0, a, b\}$ and $(H, \circ, 0)$ be a strong proper hyper BCI-algebra and $H^+ = \{0, a\}$. If $0 \circ 0 = \{0\}$, then the following hold:*

(i) $b \circ a = \{b\}$,

(ii) $b \circ b = \{0\}$,

(iii) $b \circ 0 = \{b\}$,

(iv) $a \circ 0 = \{a\}$,

(v) $a \circ b = \{b\}$,

(vi) $a \circ a = \{0, a\}$ *or* $\{0\}$.

Proof. If $0 \circ 0 = \{0\}$, then by Theorem 5.1, $0 \circ a = \{0\}$ and $0 \circ b = \{b\}$.

(i) By (B_2), $b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b = 0 \circ b = \{b\}$. Thus $b \circ a = \{b\}$.

(ii) By (B_1), $b \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0\}$. By Theorem 3.2(ii), $b \circ b = \{0\}$.

(iii) By Theorem 3.2, $b \in b \circ 0$ and we have $(b \circ 0) \circ (b \circ 0) \ll b \circ b = \{0\}$. By Theorem 3.2(ii), $(b \circ 0) \circ (b \circ 0) = \{0\}$ and so by Theorem 3.2(iii), $b \circ 0$ is singleton. By Theorem 3.2(iv), we imply that $b \circ 0 = \{b\}$.

(iv) By Theorem 3.2, $a \in a \circ 0$. If $b \in a \circ 0$, then $b = 0 \circ b(s)(0 \circ 0) \circ (a \circ 0) \ll 0 \circ a = \{0\}$. It means that $b \ll 0$ which is a contradiction. Hence $a \circ 0 = \{a\}$.

(v) By (B_1), (i) and (ii), $0 \circ (a \circ b) = (b \circ b) \circ (a \circ b) \ll b \circ a = \{b\}$. Thus $0 \circ (a \circ b) = \{b\}$ or $\{a\}$. From $0 \circ 0 = 0 \circ a = \{0\}$ and $0 \circ b = \{b\}$ we conclude that $a \circ b = \{b\}$.

(vi) By (B_1), *(ii)* and *(v)*, $0 = b \circ b = (a \circ b) \circ (a \circ b) \ll a \circ a$. Since $b \notin H^+$, $a \circ a = \{0\}$ or $\{0, a\}$. ∎

By Theorem 5.3, if $0 \circ 0 = \{0\}$ we can have two strong proper hyper *BCI*-algebras and we get two following strong proper hyper *BCI*-algebras.

$\circ_{17}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{b\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$\circ_{18}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$W(H, \circ_{17}) = I(H, \circ_{17}) = S(H, \circ_{17}) = W(H, \circ_{18}) = I(H, \circ_{18}) = S(H, \circ_{18}) = \{\{0\}, \{0, a\}, H\}$.

Theorem 5.4. *Let* $H = \{0, a, b\}$ *and* $f : (H, \circ_1, 0) \longrightarrow (H, \circ_2, 0)$ *be a non identity isomorphism of strong proper hyper BCI-algebras, then*

(i) $f(0) = 0$,

(ii) *if* $(H, \circ_1)^+ = \{0, a\}$, *then* $(H, \circ_2)^+ = \{0, b\}$.

Proof. (i) By Theorems 5.2, 5.3, 3.5 and 3.6, the proof is routine and we are omitted.

(ii) Let $(H, \circ_1)^+ = \{0, a\}$. Since $f(0) = 0$ and f is not identity, $f(a) = b$ and $f(b) = a$. By $0 \in 0 \circ_1 a$, $f(0) \in f(0) \circ_2 f(a)$ and so $0 \in 0 \circ_2 b$. Therefore, $(H, \circ_2)^+ = \{0, b\}$. ∎

Let $H = \{0, a, b\}$ and H be a strong proper hyper BCI-algebra. Then $H^+ = \{0\}$ or $\{0, a\}$ or $\{0, b\}$. If $(H, \circ_1)^+ = \{0, a\}$, then by Theorem 5.4, $(H, \circ_1)$ is isomorph by a strong proper hyper BCI-algebra $(H, \circ_2)$ $(H, \circ_2)^+ = \{0, b\}$.

Let $(H, \circ, 0)$ be a strong proper hyper BCI-algebra and $H^+ = \{0\}$. Then by Theorem 3.5, H is p-semisimple BCI-algebra and we get the only p-semisimple BCI-algebra of order 3 with the following Cayley table:

$\circ_{19}$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, H\}.$$

6. Characterization of hyper BCI-algebra of order 3

Theorem 6.1. *The number of proper hyper BCI-algebra of order* 3 *is* 19.

Theorem 6.2. *The number of hyper BCI-algebra of order* 3 *is* 38.

Proof. By Theorem 2.4, there are 19 hyper BCK-algebra of order 3 up to isomorphism. In this note we proved that there are 19 proper hyper BCI-algebra of order 3 up to isomorphism. Also every hyper BCK-algebra is hyper BCI-algebra. Thus there are 38 hyper BCI-algebra of order 3. ∎

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Accepted: 21.10.2011