WEAK LATTICES[1]

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Abstract. The ordered set induced by a BCK-algebra $\mathcal{A}$ can be equipped with a binary term operation on $\mathcal{A}$ such that the resulting structure is a so-called weak semilattice. If this structure is endowed with an antitone involution we can introduce a second binary operation and the structure arising this way is called a weak lattice. Properties of weak lattices and weak semilattices are investigated and connections to directoids and semilattices are established. Moreover, a derived structure similar to basic algebras is introduced and called a skew basic algebra. An axiomatization of these algebras is presented. It is shown that every bounded poset can be organized into a weak lattice and the number of non-isomorphic weak lattices of cardinality less than five is determined.

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1. Introduction

Let $(L, \leq)$ be a poset. For $x, y \in L$ let $U(x, y)$ respectively $L(x, y)$ denote the set of all upper respectively lower bounds of $\{x, y\}$. The poset $(L, \leq)$ is called *up-directed* respectively *down-directed* if $U(x, y) \neq \emptyset$ respectively $L(x, y) \neq \emptyset$ for all $x, y \in L$. A lattice may be regarded as an up- and down-directed poset where for each $x, y \in L$ the set $U(x, y)$ has a smallest and the set $L(x, y)$ a greatest element.

Several generalizations of lattices can be found in literature, see e. g. [5] and [6]. Let $(L, \leq)$ be a poset being both up-directed and down-directed. For each $x, y \in L$ we can pick up an arbitrary element $x \sqcup y$ of $U(x, y)$ and $x \sqcap y$ of $L(x, y)$ with the only constraint that $x \sqcup y = y$ and $x \sqcap y = x$ whenever $x \leq y$. The resulting algebra is the so-called λ-lattice introduced by V. Snášel in [6]. λ-lattices can be axiomatized as follows:

Definition 1.1 (cf. [6]) A *λ-lattice* is an algebra $(L, \sqcup, \sqcap)$ of type $(2, 2)$ satisfying the following axioms:

(L1)	$x \sqcup x = x$	(L1′)	$x \sqcap x = x$
(L2)	$x \sqcup y = y \sqcup x$	(L2′)	$x \sqcap y = y \sqcap x$
(L3)	$x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$	(L3′)	$x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$
(L4)	$x \sqcup (x \sqcap y) = x$	(L4′)	$x \sqcap (x \sqcup y) = x$

λ-lattices are applied e. g. in the theory of so-called non-associative MV-algebras (cf. e. g. [3]) where it is shown that every non-associative MV-algebra is a λ-lattice with respect to the induced order.

Another interesting structure on an up-directed poset (or, alternatively, on a down-directed poset) was introduced by J. Ježek and R. Quackenbush (cf. [5]), namely a so-called *directoid* (more precisely, a *join-directoid*) which is defined as follows:

Definition 1.2 (cf. [5]) A *directoid* (more precisely, a *join-directoid*) is a groupoid $(L, \sqcup)$ satisfying the following axioms:

(D1) $x \sqcup x = x$

(D2) $(x \sqcup y) \sqcup x = x \sqcup y$

(D3) $y \sqcup (x \sqcup y) = x \sqcup y$

(D4) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$

Dually, one can define so-called *meet-directoids*. Directoids can be characterized by the existence of a certain partial order relation.

Lemma 1.1 *A groupoid $(L, \sqcup)$ is a (join-)directoid if and only if there exists a partial ordering $\leq$ on L satisfying*

(JD1) $x, y \leq x \sqcup y$

(JD2) *If $x \leq y$ then $x \sqcup y = y \sqcup x = y$.*

Proof. First assume $(L, \sqcup)$ to be a join-directoid. Define $x \leq y$ by $x \sqcup y = y$. (D1) implies reflexivity of $\leq$. If $x \leq y \leq x$ then $x \sqcup y = y$ and $y \sqcup x = x$ and hence $x = y \sqcup x = (y \sqcup x) \sqcup y = x \sqcup y = y$ according to (D2). If, finally, $x \leq y \leq z$ then $x \sqcup y = y$ and $y \sqcup z = z$ and hence $x \sqcup z = x \sqcup (y \sqcup z) = x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z = z$ according to (D4) showing $x \leq z$. Since $x \sqcup (x \sqcup y) = x \sqcup ((x \sqcup x) \sqcup y) = (x \sqcup x) \sqcup y = x \sqcup y$ according to (D1) and (D4) we have $x \leq x \sqcup y$. Moreover, $y \sqcup (x \sqcup y) = x \sqcup y$ according to (D3) and hence $y \leq x \sqcup y$. Finally, $x \leq y$ implies $x \sqcup y = y$ according to the definition of $\leq$ and $y \sqcup x = (x \sqcup y) \sqcup x = x \sqcup y = y$ according to (D2) and the definition of $\leq$. The rest of the proof is clear. ∎

Remark 1.1 Obviously, the partial order relation $\leq$ on a (join-)directoid mentioned in Lemma 1.1 is uniquely determined by $(L, \sqcup)$ via $x \leq y$ if $x \sqcup y = y$. $(L, \leq)$ is called the *poset corresponding to* $(L, \sqcup)$.

It is immediately clear that in a directoid, $x \sqcup y$ is an arbitrarily chosen element of $U(x, y)$ having the property that $x \sqcup y = y \sqcup x = y$ whenever $x \leq y$. But, contrary to the case of λ-lattices, $x \sqcup y$ need not coincide with $y \sqcup x$ in general. In other words, for any $x, y \in L$ there are picked up some elements $x \sqcup y$ and $y \sqcup x$ of $U(x, y)$ such that $x \sqcup y = y \sqcup x = y$ in case $x \leq y$.

2. Weak semilattices

It turns out that also the concept of a directoid is not general enough for some investigations of algebras of non-classical logic (cf. [2]). Let us recall the necessary concepts. In logics, BCK-algebras play an important role. They reflect certain properties of the implication operation and are defined as follows:

Definition 2.1 (cf. [4]) A BCK-*algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ of type $(2, 0)$ satisfying the following axioms:

$$(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$$
$$x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$$
$$x \rightarrow x = 1$$
$$x \rightarrow y = 1 \text{ and } y \rightarrow x = 1 \text{ together imply } x = y.$$

In [2] the following was proved:

Proposition 2.1 (*cf.* [2]) *If* $(A, \rightarrow, 1)$ *is a* BCK-*algebra and* $x \sqcup y := (x \rightarrow y) \rightarrow y$ *then*

(W1) $x \sqcup x = x$.

(W2) $x \sqcup y = y$ *and* $y \sqcup x = x$ *together imply* $x = y$.

(W3) $x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y) = (x \sqcup y) \sqcup y = x \sqcup y$.

(W4) $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

This motivates the following

Definition 2.2 A *weak join-semilattice* is a groupoid $(L, \sqcup)$ satisfying (W1)-(W4).

Now we characterize weak join-semilattices by the existence of a certain partial order relation.

Theorem 2.1 *A groupoid* $(L, \sqcup)$ *is a weak join-semilattice if and only if* (P1) *holds:*

(P1) $(x \sqcup y) \sqcup y = x \sqcup y$

and if there exists a partial ordering $\leq$ *on* L *satisfying* (P2)–(P4)*:*

(P2) $x, y \leq x \sqcup y$

(P3) $x \leq y$ *implies* $x \sqcup y = y$.

(P4) $x \leq y$ *implies* $x \sqcup z \leq y \sqcup z$.

Proof. Let $a, b, c \in L$.

First, assume $(L, \sqcup)$ to be a weak join-semilattice.

(P1): follows from (W3).

Now, define $x \leq y$ by $x \sqcup y = y$. Then $\leq$ is reflexive and because of (W2) it is antisymmetric. If $a \leq b \leq c$ then $a \sqcup b = b$ and $b \sqcup c = c$ and hence

$$\begin{aligned} a \sqcup c &= (a \sqcup c) \sqcup c = (a \sqcup c) \sqcup (b \sqcup c) = (a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) \\ &= (a \sqcup b) \sqcup c = b \sqcup c = c \end{aligned}$$

according to (W3) and (W4), i. e. $a \leq c$ showing transitivity of $\leq$. Hence $(L, \leq)$ is a poset.

(P2): follows from (W3).

(P3): follows from the definition of $\leq$.

(P4): If $a \leq b$ then $a \sqcup b = b$ and hence

$$(a \sqcup c) \sqcup (b \sqcup c) = (a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c = b \sqcup c$$

according to (W4), i. e. $a \sqcup c \leq b \sqcup c$.

Conversely, assume $(L, \sqcup)$ to satisfy (P1) and assume there exists a partial ordering $\leq$ on L satisfying (P2)-(P4).

(W1): follows from reflexivity of $\leq$ and (P3).

(W2): If $a \sqcup b = b$ and $b \sqcup a = a$ then, according to (P2), $a \leq a \sqcup b = b$ and $b \leq b \sqcup a = a$ and hence by antisymmetry of $\leq$, $a = b$.

(W3): Because of (P2), $a \leq a \sqcup b$ which according to (P3) implies $a \sqcup (a \sqcup b) = a \sqcup b$ showing the first part of (W3). Analogously, the second part of (W3) follows. The last part of (W3) follows from (P1).

(W4): $a \leq a \sqcup b$ according to (P2) whence by (P4) $a \sqcup c \leq (a \sqcup b) \sqcup c$ showing $(a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c$ according to (P3). ∎

Remark 2.1 Obviously, the partial order relation $\leq$ on a weak join-semilattice mentioned in Theorem 2.1 is uniquely determined by $(L, \sqcup)$ via $x \leq y$ if $x \sqcup y = y$. $(L, \leq)$ is called the *poset corresponding to* $(L, \sqcup)$.

Next, we define a bounded weak join-semilattice with sectionally antitone involutions.

Definition 2.3 A *bounded weak join-semilattice with sectionally antitone involutions* is an algebra of the form $(L, \sqcup, 0, 1, ({}^a; a \in L))$ where $(L, \sqcup)$ is a weak join-semilattice with corresponding partial ordering $\leq$, 0 is the smallest and 1 the greatest element of $(L, \leq)$ and for every $a \in L$, $x \mapsto x^a$ is an antitone involution of $([a, 1], \leq)$, i. e. $x \leq y$ implies $y^a \leq x^a$ and $(x^a)^a = x$ for all $x, y \in [a, 1]$.

3. Skew basic algebras

Now we introduce total algebras which will turn out to correspond to bounded weak join-semilattices with sectionally antitone involutions in a natural bijective way.

Definition 3.1 A *skew basic algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying (A1) – (A8):

(A1) $\neg\neg x = x$

(A2) $\neg x \oplus x = 1$

(A3) $x \oplus 0 = 0 \oplus x = x$

(A4) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$

(A5) $\neg(\neg x \oplus y) \oplus y = x$ and $\neg(\neg y \oplus x) \oplus x = y$ together imply $x = y$.

(A6) $\neg(x \oplus (\neg(x \oplus y) \oplus y)) \oplus (\neg(x \oplus y) \oplus y) = \neg(x \oplus y) \oplus y$

(A7) $\neg(\neg x \oplus y) \oplus y = x$ and $\neg(z \oplus x) \oplus x = \neg z$ together imply
$\neg(\neg(\neg x \oplus y) \oplus (z \oplus y)) \oplus (z \oplus y) = \neg x \oplus y$

(A8) $\neg(\neg(\neg(x \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(x \oplus z) \oplus z)) \oplus (\neg(x \oplus z) \oplus z) =$
$\neg(x \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(y \oplus z) \oplus z)$

Here and in the following, 1 is an abbreviation for $\neg 0$.

Now, one can formulate and prove the mentioned natural bijective correspondence.

Theorem 3.1 *If $\mathcal{L} = (L, \sqcup, 0, 1, ({}^a; a \in L))$ is a bounded weak join-semilattice with sectionally antitone involutions and one defines*

$$\begin{aligned} x \oplus y &:= (y \sqcup x^0)^y \\ \neg x &:= x^0 \end{aligned}$$

then $\mathbf{S}(\mathcal{L}) := (L, \oplus, \neg, 0)$ is a skew basic algebra. If, conversely, $\mathcal{A} = (A, \oplus, \neg, 0)$ is a skew basic algebra and one defines

$$\begin{aligned} x \sqcup y &:= \neg(\neg y \oplus x) \oplus x \\ 1 &:= \neg 0 \\ x^y &:= \neg x \oplus y \end{aligned}$$

then $\mathbf{L}(\mathcal{A}) := (A, \sqcup, 0, 1, ({}^a; a \in A))$ is a bounded weak join-semilattice with sectionally antitone involutions. Moreover, $\mathbf{L}(\mathbf{S}(\mathcal{L})) = \mathcal{L}$ for every bounded weak join-semilattice $\mathcal{L}$ with sectionally antitone involutions and $\mathbf{S}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$ for every skew basic algebra $\mathcal{A}$.

Proof. Let $a, b, c \in L$.

First, assume $\mathcal{L} = (L, \sqcup, 0, 1, ({}^a; a \in L))$ to be a bounded weak join-semilattice with sectionally antitone involutions and define

$$\begin{aligned} x \oplus y &:= (y \sqcup x^0)^y \\ \neg x &:= x^0 \end{aligned}$$

Let $(L, \leq)$ denote the poset corresponding to $(L, \sqcup)$. Then

(1) $$a \oplus b \geq b.$$

Moreover, $\neg(\neg b \oplus a) \oplus a = (a \sqcup (a \sqcup b)^a)^a = (a \sqcup b)^{aa} = a \sqcup b$ and if $a \geq b$ then $\neg a \oplus b = (b \sqcup a)^b = a^b$. We check axioms (A1)-(A8):

(A1) $\neg\neg a = a^{00} = a$

(A2) $\neg a \oplus a = a^a = 1$

(A3) $a \oplus 0 = (0 \sqcup a^0)^0 = a^{00} = a$ and $0 \oplus a = (a \sqcup 0^0)^a = (a \sqcup 1)^a = 1^a = a$

(A4) $\neg(\neg(a \oplus b) \oplus b) \oplus b = b \sqcup (a \oplus b) = a \oplus b$ according to (1)

(A5) $\neg(\neg a \oplus b) \oplus b = a$ and $\neg(\neg b \oplus a) \oplus a = b$ together imply $b \sqcup a = a$ and $a \sqcup b = b$ whence $a = b$ according to (W2)

(A6) $\neg(a \oplus (\neg(a \oplus b) \oplus b)) \oplus (\neg(a \oplus b) \oplus b) = \neg(a \oplus (b \sqcup a^0)) \oplus (b \sqcup a^0) = (b \sqcup a^0) \sqcup a^0 = b \sqcup a^0 = \neg(a \oplus b) \oplus b$ according to (W3)

(A7) $\neg(\neg a \oplus b) \oplus b = a$ and $\neg(c \oplus a) \oplus a = \neg c$ together imply $b \sqcup a = a$ and $a \sqcup c^0 = c^0$ whence $b \leq a \leq c^0$ which implies $(c^0)^b \leq a^b$. Hence $\neg(\neg(\neg a \oplus b) \oplus (c \oplus b)) \oplus (c \oplus b) = (c^0)^b \sqcup a^b = a^b = \neg a \oplus b$.

(A8) $\neg(\neg(\neg(a\oplus(\neg(b\oplus c)\oplus c))\oplus(\neg(b\oplus c)\oplus c))\oplus(\neg(a\oplus c)\oplus c))\oplus(\neg(a\oplus c)\oplus c) = \neg(\neg(\neg(a \oplus (c \sqcup b^0)) \oplus (c \sqcup b^0)) \oplus (c \sqcup a^0)) \oplus (c \sqcup a^0) = \neg(\neg((c \sqcup b^0) \sqcup a^0) \oplus (c \sqcup a^0)) \oplus (c \sqcup a^0) = (c \sqcup a^0) \sqcup ((c \sqcup b^0) \sqcup a^0) = (c \sqcup b^0) \sqcup a^0 = \neg(a \oplus (\neg(b \oplus c) \oplus c)) \oplus (\neg(b \oplus c) \oplus c)$ according to (W4)

Hence, $\mathbf{S}(\mathcal{L})$ is a skew basic algebra.

Conversely, assume $\mathcal{A} = (A, \oplus, \neg, 0)$ to be a skew basic algebra and define

$$x \sqcup y := \neg(\neg y \oplus x) \oplus x$$
$$1 := \neg 0$$
$$x^y := \neg x \oplus y$$

Then $a^0 = \neg a\oplus 0 = \neg a$ according to (A3) and $(b\sqcup a^0)^b = \neg(\neg(a\oplus b)\oplus b)\oplus b = a\oplus b$ according to (A4). We check axioms (W1)-(W4):

(W1) $a \sqcup a = \neg(\neg a \oplus a) \oplus a = \neg 1 \oplus a = 0 \oplus a = a$ according to (A2), (A1) and (A3)

(W2) $a\sqcup b = b$ and $b\sqcup a = a$ together imply $\neg(\neg b\oplus a)\oplus a = b$ and $\neg(\neg a\oplus b)\oplus b = a$ whence according to (A5) $b = a$, i. e. $a = b$.

(W3) $a \sqcup (a \sqcup b) = \neg(\neg(\neg(\neg b \oplus a) \oplus a) \oplus a) \oplus a = \neg(\neg b \oplus a) \oplus a = a \sqcup b$ according to (A4). Now put $z = 0$ in (A8). Then one obtains $\neg(\neg(\neg(x \oplus \neg y) \oplus \neg y) \oplus \neg x)\oplus\neg x = \neg(x\oplus\neg y)\oplus\neg y$. Hence $b\sqcup(a\sqcup b) = \neg(\neg(\neg(\neg b\oplus a)\oplus a)\oplus b)\oplus b = \neg(\neg b \oplus a) \oplus a = a \sqcup b$. Finally, $(a \sqcup b) \sqcup b = \neg(\neg b \oplus (\neg(\neg b \oplus a) \oplus a)) \oplus (\neg(\neg b \oplus a) \oplus a) = \neg(\neg b \oplus a) \oplus a = a \sqcup b$ according to (A6).

(W4) $(a\sqcup c)\sqcup((a\sqcup b)\sqcup c) = \neg(\neg(\neg(\neg c\oplus(\neg(\neg b\oplus a)\oplus a))\oplus(\neg(\neg b\oplus a)\oplus a))\oplus(\neg(\neg c\oplus a)\oplus a))\oplus(\neg(\neg c\oplus a)\oplus a) = \neg(\neg c\oplus(\neg(\neg b\oplus a)\oplus a))\oplus(\neg(\neg b\oplus a)\oplus a) = (a\sqcup b)\sqcup c$ according to (A8)

Hence $(L, \sqcup)$ is a weak join-semilattice. Let $(L, \leq)$ denote its corresponding poset. Then
$a \geq b$ implies $b \sqcup a^b = \neg(\neg(\neg a \oplus b) \oplus b) \oplus b = \neg a \oplus b = a^b$ according to (A4) which implies $b \leq a^b$.
$c \leq a \leq b$ implies $c \sqcup a = a$ and $a \sqcup b = b$ whence $\neg(\neg a \oplus c) \oplus c = a$ and $\neg(\neg b \oplus a) \oplus a = b = \neg\neg b$ according to (A1) and hence $b^c \sqcup a^c = \neg(\neg(\neg a \oplus c) \oplus (\neg b \oplus c)) \oplus (\neg b \oplus c) = \neg a \oplus c = a^c$ according to (A7) whence $b^c \leq a^c$.
$0 \sqcup a = \neg(\neg a \oplus 0) \oplus 0 = \neg\neg a \oplus 0 = \neg\neg a = a$ according to (A3) and (A1) whence $0 \leq a$.
$a \sqcup 1 = \neg(\neg 1 \oplus a) \oplus a = \neg(0 \oplus a) \oplus a = \neg a \oplus a = 1$ according to (A1), (A3) and (A2) whence $a \leq 1$.

Finally, $b \leq a$ implies $a^{bb} = \neg(\neg a \oplus b) \oplus b = b \sqcup a = a$.

Hence, $\mathbf{L}(\mathcal{A})$ is a bounded weak semilattice with sectionally antitone involutions.

Conversely, let $\mathcal{L} = (L, \sqcup, 0, 1, (^a; a \in L))$ be a bounded weak semilattice with sectionally antitone involutions and put $\mathbf{L}(\mathbf{S}(\mathcal{L})) = (L, \cup, 0, \bar{1}, (_a; a \in L))$. Then

$$a \cup b = \neg(\neg b \oplus a) \oplus a = (a \sqcup (a \sqcup b)^a)^a = (a \sqcup b)^{aa} = a \sqcup b.$$

Hence the posets corresponding to $(L, \sqcup)$ respectively $(L, \cup)$ coincide.
Let $(L, \leq)$ denote this poset.
Moreover, $\bar{1} = \neg 0 = 0^0 = 1$.
If $a \leq b$ then $a \sqcup b = b$ whence $b_a = \neg b \oplus a = (a \sqcup b)^a = b^a$.
Hence $\mathbf{L}(\mathbf{S}(\mathcal{L})) = \mathcal{L}$.
Now let, finally, $\mathcal{A} = (A, \oplus, \neg, 0)$ be a skew basic algebra and put $\mathbf{S}(\mathbf{L}(\mathcal{A})) = (A, +, ', 0)$. Then
$a + b = (b \sqcup a^0)^b = \neg(\neg(\neg(\neg a \oplus 0) \oplus b) \oplus b) \oplus b = \neg(\neg(\neg\neg a \oplus b) \oplus b) \oplus b = \neg(\neg(a \oplus b) \oplus b) \oplus b = a \oplus b$ according to (A3), (A1) and (A4).
Finally, $a' = a^0 = \neg a \oplus 0 = \neg a$ according to (A3).
Hence $\mathbf{S}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$. ∎

Dually to the notion of a weak join-semilattice we define the notion of a weak meet-semilattice:

Definition 3.2 A *weak meet-semilattice* is a groupoid $(L, \sqcap)$ satisfying (W1′)-(W4′):

(W1′) $x \sqcap x = x$

(W2′) $x \sqcap y = x$ and $y \sqcap x = y$ together imply $x = y$.

(W3′) $(x \sqcap y) \sqcap x = (x \sqcap y) \sqcap y = x \sqcap (x \sqcap y) = x \sqcap y$

(W4′) $(z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y)$

Corollary 3.1 *Dually to the case of weak join-semilattices it can be proved that a groupoid $(L, \sqcap)$ is a weak meet-semilattice if and only if* (P1′) *holds:*

(P1′) $x \sqcap (x \sqcap y) = x \sqcap y$

and if there exists a partial ordering $\leq$ on L satisfying (P2′) – (P4′)*:*

(P2′) $x \sqcap y \leq x, y$.

(P3′) $x \leq y$ *implies* $x \sqcap y = x$.

(P4′) $x \leq y$ *implies* $z \sqcap x \leq z \sqcap y$.

$\leq$ *is uniquely determined by $(L, \sqcap)$ via $x \leq y$ if $x \sqcap y = x$. $(L, \leq)$ is called the* poset corresponding to $(L, \sqcap)$.

4. Weak lattices

Now, we define the notion of a weak lattice.

Definition 4.1 A *weak lattice* is an algebra $(L, \sqcup, \sqcap)$ of type $(2, 2)$ satisfying (W1)-(W5) and (W1')-(W5') where

(W5) $x \sqcap (x \sqcup y) = x$.

(W5′) $(x \sqcap y) \sqcup y = y$.

Weak lattices can also be characterized by the existence of a certain partial order relation.

Lemma 4.1 *An algebra* $(L, \sqcup, \sqcap)$ *of type* $(2, 2)$ *is a weak lattice if and only if* (P1) *and* (P1′) *hold and if there exists a partial ordering* $\leq$ *on* L *satisfying* (P2)-(P4) *and* (P2′)-(P4′).

Proof. Let $a, b \in L$.

First assume $(L, \sqcup, \sqcap)$ to be a weak lattice and define $x \leq y$ by $x \sqcup y = y$ and $x \sqsubseteq y$ by $x \sqcap y = x$. Then $a \leq b$ implies $a \sqcup b = b$ and hence $a \sqcap b = a \sqcap (a \sqcup b) = a$ according to (W5), i.e., $a \sqsubseteq b$, and, conversely, $a \sqsubseteq b$ implies $a \sqcap b = a$ and hence $a \sqcup b = (a \sqcap b) \sqcup b = b$ according to (W5′), i.e., $a \leq b$. This shows that both partial orderings coincide. According to Theorem 2.1 and Corollary 3.1, (P1) and (P1′) hold and there exists a partial ordering $\leq$ on L satisfying (P2)-(P4) and (P2′)-(P4′). The rest of the proof follows from Theorem 2.1 and Corollary 3.1. ∎

Remark 4.1 Obviously, the partial order relation $\leq$ mentioned in Lemma 4.1 is uniquely determined by $(L, \sqcup, \sqcap)$ via $x \leq y$ if and only if $x \sqcup y = y$ if and only if $x \sqcap y = x$. $(L, \leq)$ is called the *poset corresponding to* $(L, \sqcup, \sqcap)$.

Example 4.1 Consider the four-element chain $0 < a < b < 1$ and define the operations $\sqcup$ and $\sqcap$ on $L := \{0, a, b, 1\}$ as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	0	b	b
1	0	a	b	1

It is easy to check that $(L, \sqcup, \sqcap)$ is a weak lattice which is not a directoid (neither w. r. t. $\sqcup$ nor w. r. t. $\sqcap$) since e. g. $a \leq b$ and $a \sqcup b = b$, but $b \sqcup a = 1 \neq b$. Of course, the order of the original chain coincides with the order induced by $(L, \sqcup, \sqcap)$.

Example 4.2 If $(L, \leq, 0, 1)$ is a bounded poset and one defines

$$x \sqcup y := \begin{cases} y & \text{if } x \leq y \\ 1 & \text{otherwise} \end{cases}$$

$$x \sqcap y := \begin{cases} x & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

then $(L, \sqcup, \sqcap)$ is a weak lattice. This can be seen as follows:

Let $a, b, c \in L$. We check (P1)-(P4):

(P1): If $a \leq b$ then $(a \sqcup b) \sqcup b = b \sqcup b = b = a \sqcup b$ and if $a \not\leq b$ then $(a \sqcup b) \sqcup b = 1 \sqcup b = 1 = a \sqcup b$.

(P2): If $a \leq b$ then $a \leq b = a \sqcup b$ and $b = a \sqcup b$ and if $a \not\leq b$ then $a \leq 1 = a \sqcup b$ and $b \leq 1 = a \sqcup b$.

(P3): If $a \leq b$ then $a \sqcup b = b$.

(P4): If $a \leq b \leq c$ then $a \leq c$ and hence $a \sqcup c = c = b \sqcup c$ and if $a \leq b \not\leq c$ then $a \sqcup c \leq 1 = b \sqcup c$.

The dual conditions follow analogously.

According to Lemma 4.1, $(L, \sqcup, \sqcap)$ is a weak lattice.

Now, we introduce a unary operation on weak lattices.

Definition 4.2 A *weak lattice with an antitone involution* is an algebra $(L, \sqcup, \sqcap, ')$ of type $(2, 2, 1)$ such that $(L, \sqcup, \sqcap)$ is a weak lattice and $'$ is an antitone involution. If, in addition, $(L, \sqcup, \sqcap, ')$ satisfies the de Morgan laws $(x \sqcup y)' = y' \sqcap x'$ and $(x \sqcap y)' = y' \sqcup x'$ then it is called a *weak lattice with an antitone involution satisfying the de Morgan laws.*

Example 4.3 Consider the weak lattice $(L, \sqcup, \sqcap)$ of Example 4.1 and define a unary operation $'$ on L by $0' := 1$, $a' := b$, $b' := a$ and $1' := 0$. One can easily verify that $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution.

Remark 4.2 That $'$ is antitone follows from the de Morgan laws: If $x \leq y$ then $x \sqcup y = y$ which implies $y' \sqcap x' = (x \sqcup y)' = y'$ whence $y' \leq x'$.

That there exist weak lattices with an antitone involution not satisfying the de Morgan laws is shown by the following example:

Example 4.4 Let L and $\sqcup$ be defined as in Example 4.1, $(L, \sqcap)$ a meet-semilattice, i.e.,

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

and $'$ be defined as in Example 4.3. Then $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution not satisfying the de Morgan laws since $(b \sqcup a)' = 1' = 0 \neq a = b \sqcap a = a' \sqcap b'$.

Now, it can be proved that every weak join-semilattice together with an antitone involution induces in a natural way a weak lattice with an antitone involution satisfying the de Morgan laws:

Theorem 4.1 *Let $(L, \sqcup)$ be a weak join-semilattice and $'$ an antitone involution of the corresponding poset $(L, \leq)$. Define $x \sqcap y := (y' \sqcup x')'$. Then $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution satisfying the de Morgan laws.*

Proof. There hold (W1)-(W4). (W1')-(W4') can be easily proved, e.g.:

(W1′): $x \sqcap x = (x' \sqcup x')' = x'' = x$ according to (W1).

Now

(W5): $x \sqcap (x \sqcup y) = ((x \sqcup y)' \sqcup x')' = x'' = x$ since $x \leq x \sqcup y$ and hence $(x \sqcup y)' \leq x'$.

(W5′): $(x \sqcap y) \sqcup y = (y' \sqcup x')' \sqcup y'' = y'' = y$ since $y' \leq y' \sqcup x'$ and hence $(y' \sqcup x')' \leq y''$.

$(x \sqcup y)' = (x'' \sqcup y'')' = y' \sqcap x'$

$(x \sqcap y)' = (y' \sqcup x')'' = y' \sqcup x'$. ■

Remark 4.3 It was proved in [2] that in a BCK-algebra $(A, \rightarrow, 1)$ one can define a partial ordering $\leq$ by $x \leq y$ if $x \rightarrow y = 1$ and it holds $x \leq y$ if and only if $x \sqcup y = y$ where $x \sqcup y = (x \rightarrow y) \rightarrow y$. Moreover, 1 is the greatest element of $(A, \leq)$. If $(A, \leq)$ has a smallest element 0 then the term operation $'$ defined by $x' := x \rightarrow 0$ is antitone.

Now, we introduce BCK-algebras with 0 satisfying the double negation law.

Definition 4.3 (cf. [1]) A BCK-algebra $(A, \rightarrow, 1)$ with 0 is said to satisfy the *double negation law* if $(x \rightarrow 0) \rightarrow 0 = x$.

In fact, in logics represented by BCK-algebras the unary operation $\neg x = x \rightarrow 0$ is considered as the connective negation . Thus the double negation law expresses the fact that $\neg\neg x = x$ (contrary to the intuitionistic case where only $\neg\neg\neg x = \neg x$ and where it can happen that $\neg\neg x \neq x$). BCK-algebras satisfying the double negation law were treated in [1]. Here we can state

Theorem 4.2 *If* $(A, \rightarrow, 1)$ *is a* BCK-*algebra with* 0 *satisfying the double negation law and*

$$x \sqcup y := (x \rightarrow y) \rightarrow y$$
$$x \sqcap y := (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow (x \rightarrow 0)) \rightarrow 0$$
$$x' := x \rightarrow 0$$

then $(A, \sqcup, \sqcap, ')$ *is a weak lattice with an antitone involution satisfying the de Morgan laws.*

Proof. This follows from Proposition 2.1. ■

Theorem 4.3 *Up to isomorphism there exist exactly one one-element weak lattice, one two-element weak lattice, four three-element weak lattices and* 74 *four-element weak lattices.*

Proof. The case concerning weak lattices of cardinality less than three is trivial.

Let us first assume $L = \{0, a, 1\}$ with $0 < a < 1$.

(P2), (P3), (P2') and (P3'): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$\sqcup$	0	a	1
0	0	a	1
a	b	a	1
1	1	1	1

$\sqcap$	0	a	1
0	0	0	0
a	0	a	a
1	0	c	1

where $b \in \{a, 1\}$ and $c \in \{0, a\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

$$\text{If } (x, y) = (a, 0) \text{ then } (x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = b \sqcup 0 = b = a \sqcup 0 = x \sqcup y.$$

Hence (P1) holds.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

$$\text{If } (x, y) = (1, a) \text{ then } x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap c = c = 1 \sqcap a = x \sqcap y.$$

Hence (P1') holds.

(P4) and (P4'): These conditions can be easily checked.

Since all four possibilities for b, c yield pairwise non-isomorphic weak lattices, up to isomorphism there exist exactly four three-element weak lattices.

Now we turn to the four-element case. Let $L = \{0, a, b, 1\}$.

First we consider the case $0 < a, b < 1$ with a and b being incomparable.

(P2), (P3), (P2′) and (P3′): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	c	a	1	1
b	d	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	e	f	1

where $c \in \{a, 1\}$, $d \in \{b, 1\}$, $e \in \{0, a\}$ and $f \in \{0, b\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

$$\text{If } (x, y) = (a, 0) \text{ then } (x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = c \sqcup 0 = c = a \sqcup 0 = x \sqcup y.$$
$$\text{If } (x, y) = (a, b) \text{ then } (x \sqcup y) \sqcup y = (a \sqcup b) \sqcup b = 1 \sqcup b = 1 = a \sqcup b = x \sqcup y.$$
$$\text{If } (x, y) = (b, 0) \text{ then } (x \sqcup y) \sqcup y = (b \sqcup 0) \sqcup 0 = d \sqcup 0 = d = b \sqcup 0 = x \sqcup y.$$
$$\text{If } (x, y) = (b, a) \text{ then } (x \sqcup y) \sqcup y = (b \sqcup a) \sqcup a = 1 \sqcup a = 1 = b \sqcup a = x \sqcup y.$$

Hence (P1) holds.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

If $(x, y) = (a, b)$ then $x \sqcap (x \sqcap y) = a \sqcap (a \sqcap b) = a \sqcap 0 = 0 = a \sqcap b = x \sqcap y$.
If $(x, y) = (b, a)$ then $x \sqcap (x \sqcap y) = b \sqcap (b \sqcap a) = b \sqcap 0 = 0 = b \sqcap a = x \sqcap y$.
If $(x, y) = (1, a)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap e = e = 1 \sqcap a = x \sqcap y$.
If $(x, y) = (1, b)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap b) = 1 \sqcap f = f = 1 \sqcap b = x \sqcap y$.

Hence (P1') holds.

Since every homomorphism of a weak lattice is order-preserving, the only non-trivial isomorphism between our algebras could be the interchanging of a and b. We have the following cases: $(c, d, e, f) \in \{a, 1\} \times \{b, 1\} \times \{0, a\} \times \{0, b\}$. Now the weak lattices corresponding to the following quadruples are isomorphic:

$$(a, b, 0, b), (a, b, a, 0)$$
$$(a, 1, 0, 0), (1, b, 0, 0)$$
$$(a, 1, 0, b), (1, b, a, 0)$$
$$(a, 1, a, 0), (1, b, 0, b)$$
$$(a, 1, a, b), (1, b, a, b)$$
$$(1, 1, 0, b), (1, 1, a, 0)$$

So we have 10 pairwise non-isomorphic four-element weak lattices whose corresponding order is not a chain.

Finally, consider the case $0 < a < b < 1$.

(P2), (P3), (P2') and (P3'): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	c	a	b	1
b	d	e	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	f	b	b
1	0	g	h	1

where $c \in \{a, b, 1\}$, $d, e \in \{b, 1\}$, $f, g \in \{0, a\}$ and $h \in \{0, a, b\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

If $(x, y) = (a, 0)$ then $(x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = c \sqcup 0$ and $x \sqcup y = a \sqcup 0 = c$.
If $(x, y) = (b, 0)$ then $(x \sqcup y) \sqcup y = (b \sqcup 0) \sqcup 0 = d \sqcup 0 = d = b \sqcup 0 = x \sqcup y$.
If $(x, y) = (b, a)$ then $(x \sqcup y) \sqcup y = (b \sqcup a) \sqcup a = e \sqcup a = e = b \sqcup a = x \sqcup y$.

Hence (P1) holds if and only if $(c, d) \neq (b, 1)$.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

If $(x, y) = (b, a)$ then $x \sqcap (x \sqcap y) = b \sqcap (b \sqcap a) = b \sqcap f = f = b \sqcap a = x \sqcap y$.
If $(x, y) = (1, a)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap g = g = 1 \sqcap a = x \sqcap y$.
If $(x, y) = (1, b)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap b) = 1 \sqcap h$ and $x \sqcap y = 1 \sqcap b = h$.

Hence (P1') holds if and only if $(g,h) \neq (0,a)$.

(P4): This condition holds if and only if $(c,d) \neq (1,b)$.

(P4'): This condition holds if and only if $(g,h) \neq (a,0)$.

Summing up, we have that the algebra with the above operation tables is a weak lattice if and only if

$$((c,d),e,f,(g,h)) \in \{(a,b),(a,1),(b,b),(1,1)\} \times \{b,1\} \\ \times \{0,a\} \times \{(0,0),(a,a),(0,b),(a,b)\}.$$

Hence we have $4 \cdot 2 \cdot 2 \cdot 4 = 64$ possibilities. The corresponding algebras are pairwise non-isomorphic. Together this yields $10 + 64 = 74$ pairwise non-isomorphic four-element weak lattices. ∎

Finally, we introduce some axioms.

Definition 4.4 For an algebra $(L, \sqcup, \sqcap)$ we define (W6), (W6'), (W7) and (W7') as follows:

(W6) $(x \sqcup y) \sqcup x = x \sqcup y$

(W6') $y \sqcap (x \sqcap y) = x \sqcap y$

(W7) $y \sqcap (x \sqcup y) = y$

(W7') $(x \sqcap y) \sqcup x = x$

Next, we show some implications between some axioms:

Lemma 4.2 *There hold* (i)-(v)*:*

(i) (W3) *and* (W5) *together imply* (W7).

(ii) (W3') *and* (W5') *together imply* (W7').

(iii) (W1) *and* (W5) *together imply* (W1').

(iv) (W2) *and* (W5') *together imply* (W2').

(v) (W6) *implies* (W2).

Proof. (i): $y \sqcap (x \sqcup y) = y \sqcap (y \sqcup (x \sqcup y)) = y$.

(ii): $(x \sqcap y) \sqcup x = ((x \sqcap y) \sqcap x) \sqcup x = x$. This follows analogously.

(iii): $x \sqcap x = x \sqcap (x \sqcup x) = x$.

(iv): $x \sqcap y = x$ and $y \sqcap x = y$ together imply $x \sqcup y = (x \sqcap y) \sqcup y = y$ and $y \sqcup x = (y \sqcap x) \sqcup x = x$ whence $x = y$.

(v): $x \sqcup y = y$ and $y \sqcup x = x$ together imply $x = y \sqcup x = (x \sqcup y) \sqcup x = x \sqcup y = y$. ∎

Lemma 4.3 *There hold* (i) *and* (ii)*:*

(i) *Any join-directoid satisfying* (W4) *is a join-semilattice.*

(ii) *Any weak join-semilattice satisfying* (W6) *is a join-semilattice.*

Proof. (i) If $(L, \sqcup)$ is a join-directoid, $(L, \leq)$ denotes the corresponding poset, $a, b, c \in L$ and $a, b \leq c$ then

$$\begin{aligned}(a \sqcup b) \sqcup c &= (a \sqcup b) \sqcup (c \sqcup b) \\ &= (a \sqcup b) \sqcup ((a \sqcup c) \sqcup b) \\ &= (a \sqcup c) \sqcup b = c \sqcup b = c,\end{aligned}$$

i. e. $a \sqcup b \leq c$.

(ii) If $(L, \sqcup)$ is a weak join-semilattice, $(L, \leq)$ denotes the corresponding poset, $a, b \in L$ and $a \leq b$ then $b \sqcup a = (a \sqcup b) \sqcup a = a \sqcup b = b$ and according to Lemma 1.1 and Theorem 2.1, $(L, \sqcup)$ is a join-directoid and according to Lemma 4.3 (i) a join-semilattice. ∎

Corollary 4.1 *There hold* (i) *and* (ii)*:*

(i) *If* $(L, \sqcup)$ *is a weak join-semilattice satisfying* (W5) *and* (W6)*,* $(L, \leq)$ *denotes the corresponding poset,* $'$ *is an antitone involution of* $(L, \leq)$ *and one defines* $x \sqcap y := (x' \sqcup y')'$ *for all* $x, y \in L$ *then* $(L, \sqcup, \sqcap)$ *is a lattice.*

(ii) *Any weak lattice satisfying* (W6) *and* (W6') *is a lattice.*

Proof. (i): According to Lemma 4.3 (ii), $(L, \sqcup)$ is a join-semilattice and hence $(L, \sqcap)$ a meet-semilattice. Because of (W5) one absorption law holds. The other one follows by duality.

(ii): Let $(L, \sqcup, \sqcap)$ be a weak lattice. According to Lemma 4.3 (ii) and its dual $(L, \sqcup)$ and $(L, \sqcap)$ are semilattices. Because of (W5) and (W5') the absorption laws hold. ∎

Remark 4.4 Since (W3) and $x \sqcup y = y \sqcup x$ together imply (W6), according to Lemma 4.3 (ii) it follows that any commutative weak join-semilattice is a join-semilattice and according to Corollary 4.1 (ii) that every commutative weak lattice is a lattice. If in Definition 2.2 (W2) is replaced by the stronger axiom $(x \sqcup y) \sqcup x = x \sqcup y$ or if the axiom $(x \sqcup y) \sqcup x = x \sqcup (x \sqcup y)$ is included, then the weak join-semilattice becomes a join-semilattice.

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