MULTIDIMENSIONAL GENERATING RELATIONS SUGGESTED BY A GENERATING RELATION FOR HYPER-BESSEL FUNCTIONS

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Abstract. The authors derive a general theorem on multidimensional generating functions involving arbitrary coefficients. By appropriately specializing these coefficients a number of (known and new) results are shown to follow as applications of the theorem.

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1. Introduction and notations

We define the generalized hypergeometric series ${}_AF_B$ [6] with A numerator and B denominator parameters by

$$(1.1)\qquad {}_AF_B\left[(a_A);(b_B);x\right] = \sum_{m=0}^{\infty} \frac{(a_A)_m}{(b_B)_m}\frac{x^m}{m!}, \quad x \in C,$$

where $(a_A) = a_1, a_2, ..., a_A$; $b_j \neq 0, -1, -2, ...$; $(j = 1, 2, .., B)$ and $(a)_n$ denotes the Pochhammmer symbol given by

$$(a)_0 = 1 \text{ and } (a)_n = a(a+1)...(a+n-1), \ (n = 1, 2, 3, ...).$$

A generating function for hyper-Bessel function $J_{m,n,p}(x)$ due to Pathan [3, p. 40-41, Eq. (2.1)] is recalled here in the following form:

$$exp\left[x\left(y+s+t-\frac{1}{yst}\right)\right]=\sum_{m,n,p=-\infty}^{\infty}J_{m,n,p}(x)y^m s^n t^p, \tag{1.2}$$

where

$$J_{m,n,p}(x)=\frac{x^{m+n+p}}{\Gamma(m+1)\Gamma(n+1)\Gamma(p+1)}{}_0F_3\left[--;m+1,n+1,p+1;-x^4\right]. \tag{1.3}$$

Motivated by the aforementioned work of Pathan [3], we aim here at presenting a new general theorem on multidimensional generating relations with essentially arbitrary coefficients. Many earlier (known) results given by [3], [4], [7] and [8] are shown to be special cases of our main result.

For convenience, a few conventions and some notations are introduced here:

(1) Boldface letters $(\mathbf{p_r})$ and $(\mathbf{k_s})$ denote vectors of dimension r and s, respectively; for instance, we have

$$\mathbf{p_r}=(p_1,...,p_r) \text{ and } \mathbf{k_s}=(k_1,...,k_s). \tag{1.4}$$

(2) The symbol $\Omega(p,k)$ denotes a double sequence and the symbol $\Omega(\mathbf{p_r},\mathbf{k_s})$ denotes a multiple $(r+s)$-dimensional sequence as follows:

$$\Omega(\mathbf{p_r},\mathbf{k_s})=\Omega(p_1,...,p_r;k_1,...,k_s). \tag{1.5}$$

(3) Throughout our present investigation, sufficient conditions to ensure absolute convergence of the series involved are understood to hold true for the essentially arbitrary double sequences and the essentially arbitrary multiple sequences of the types described above.

(4) Quite frequently, multiple series are written in simplified notation. Thus

$$\begin{aligned}&\sum_{\mathbf{p_r}=\mathbf{0}}^{\infty} \text{ means } \sum_{p_1,..,p_r=0}^{\infty},\\ &\sum_{\mathbf{p_r}=\mathbf{0}}^{\mathbf{n}} \text{ means } \sum_{p_1=0}^{n_1}...\sum_{p_r=0}^{n_r} \text{ and}\\ &\sum_{\mathbf{p_r},\mathbf{k_s}=\mathbf{0}}^{\infty} \text{ means } \sum_{p_1,..,p_r=0}^{\infty}\times\sum_{k_1,..,k_s=0}^{\infty}.\end{aligned} \tag{1.6}$$

(5) The symbols M_k^s , $N_k^s$and R_k^s are defined by

$$M_k^s=m_k+..+m_s, N_k^s=n_k+..+n_s \text{ and } R_k^s=r_k+..+r_s \tag{1.7}$$

respectively. In particular, we have

$$M_1^s=m_1+...+m_s, N_1^s=n_1+...+n_s \text{ and } R_1^s=r_1+...+r_s. \tag{1.8}$$

2. A class of multidimensional generating relations

Our main results on generating functions involving bilateral series are given by the following theorem:

Theorem. *Let $\Omega(p,k)$ be a suitably bounded double sequence of complex numbers then*

$$\sum_{p,k=0}^{\infty}\Omega(p,k)\frac{u^p}{p!}\frac{\left(x\left(y+s+t+\frac{z}{yst}\right)\right)^k}{k!}$$

$$(2.1)\qquad =\sum_{m,n,r=-\infty}^{\infty}\frac{(xy)^m\,(xs)^n\,(xt)^r}{\Gamma(m+1)\,\Gamma(n+1)\Gamma(r+1)}$$

$$\sum_{p,k=0}^{\infty}\Omega(p,4k+m+n+r)\frac{u^p}{p!}\,\frac{x^{4k}z^k}{(m+1)_k(n+1)_k(r+1)_k k!},$$

provided that each member of (2.1) *exists.*

More generally, let $\Omega(\mathbf{p_r},\mathbf{k_s})$ be a suitably bounded multiple $(r+s)$ *dimensional* sequence of complex numbers, then

$$\sum_{\mathbf{p_r},\mathbf{k_s}=\mathbf{0}}^{\infty}\Omega(\mathbf{p_r},\mathbf{k_s})\prod_{j=1}^{r}\left\{\frac{u_j^{p_j}}{p_j!}\right\}\prod_{j=1}^{s}\left\{\frac{x_j^{k_j}\left(y_j+s_j+t_j+\frac{z_j}{y_js_jt_j}\right)^{k_j}}{k_j!}\right\}$$

$$(2.2)\qquad =\sum_{\mathbf{m_s},\mathbf{n_s},\mathbf{r_s}=-\infty}^{\infty}\prod_{j=1}^{s}\left\{\frac{(x_jy_j)^{m_j}\,(x_js_j)^{n_j}(x_jt_j)^{r_j}}{\Gamma(m_j+1)\,\Gamma(n_j+1)\Gamma(r_j+1)}\right\}$$

$$\cdot\sum_{\mathbf{p_r},\mathbf{k_s}=\mathbf{0}}^{\infty}\Omega(\mathbf{p_r},\mathbf{4k_s}+\mathbf{m_s}+\mathbf{n_s}+\mathbf{r_s})$$

$$\cdot\prod_{j=1}^{r}\left\{\frac{u_j^{p_j}}{p_j!}\right\}\cdot\prod_{j=1}^{s}\left\{\frac{x_j^{4k_j}z_j^{k_j}}{(m_j+1)_{k_j}(n_j+1)_{k_j}(r_j+1)_{k_j}k_j!}\right\},$$

provided that each member of (2.2) exists.

Proof. Denote, for convenience, the first member of (2.1) by $f(u,x,y,s,t,z)$. Then, as a consequence of the binomial theorem, it is easily seen that:

$$(2.3)\quad f(u,x,y,s,t,z)=\sum_{p,k,m,n,r=0}^{\infty}\Omega(p,k+m+n+r)\frac{u^px^{k+m+n+r}y^{m-k}s^{n-k}t^{r-k}z^k}{p!k!m!n!r!}.$$

Upon replacing the summation indices m, n and r in (2.3) by $m+k$, $n+k$ and $r+k$, respectively, if we rearrange the resulting series (which can be justified by absolute convergence of the series involved), we are led finally to assertion (2.1). The derivation of assertion (2.2) runs parallel to that of (2.1). We skip the details involved.

3. Applications of the theorem

First, in its special case when

$$u \mapsto 0,\ z = -1 \text{ and } \Omega(p, k) = 1,$$

assertion (2.1) would obviously correspond to the generating function (1.2).

Secondly, upon setting

$$u \mapsto 0,\ z = -1 \quad \text{and} \quad \Omega(p, k) = \Omega_1(p)\ \Omega_2(k),$$

where

$$\Omega_1(p) = 1 \quad \text{and} \quad \Omega_2(k) = \frac{1}{(b_B)_k},$$

we shall obtain an extension of the result of Pathan (1.2) in the following form:

$$(3.1)\quad {}_0F_B\left[--; b_1, b_2, ..., b_B; x\left(y + s + t - \frac{1}{yst}\right)\right] = \sum_{m,n,r=-\infty}^{\infty} J_{m,n,r}\left(b_B; x\right) y^m s^n t^r,$$

where

$$(3.2)\quad J_{m,n,r}(b_B; x) = {}_0F_{4B+3}\left[--; \frac{D_B}{4}, \frac{D_B+1}{4}, \frac{D_B+2}{4}, \frac{D_B+3}{4}, m+1, n+1, r+1; -x^4\right],$$

$$(D_B = (b_B) + m + n + r\ ).$$

Now, we consider some applications of assertion (2.2). By setting

$$z_j \mapsto -1,\quad (j = 1, 2, ..., s),\ u_i \mapsto 0\ , (i = 1, 2, ..., r) \text{ and } \Omega(\mathbf{p_r}, \mathbf{k_s}) = 1,$$

(2.2) immediately yields the following simple consequence of Pathan's generating function (1.2):

$$(3.3)\quad \begin{aligned} &\exp\left[\sum_{j=1}^{s} x_j\left(y_j + s_j + t_j - \frac{1}{y_j s_j t_j}\right)\right] \\ &= \sum_{\mathbf{m_s},\mathbf{n_s},\mathbf{r_s}=-\infty}^{\infty} \prod_{j=1}^{s} \left\{J_{m_j,n_j,r_j}(x_j) y_j^{m_j} s_j^{n_j} t_j^{r_j}\right\}, \end{aligned}$$

where

$$J_{m_j,n_j,r_j}(x_j) = \frac{x_j^{m_j+n_j+r_j}}{\Gamma(m_j + 1)\Gamma(n_j + 1)\Gamma(r_j + 1)}$$

$${}_0F_3\left[--; m_j + 1, n_j + 1, r_j + 1; -x_j^4\right],\ (j = 1, 2, ..., s).$$

Next, if we let

$$z_j \mapsto -z_j s_j^2 t_j \quad , \quad x_j \mapsto 1 \quad \text{and} \quad t_j \mapsto 0, \; (j = 1, 2, ..., s),$$

in (2.2) and making use of the series rearrangement technique illustrated fairly fully in [10, Ch. 2], then we shall obtain a generating relation in the following form

$$\sum_{\mathbf{p_r}, \mathbf{k_s} = \mathbf{0}}^{\infty} \Omega(\mathbf{p_r}, \mathbf{k_s}) \cdot \prod_{j=1}^{r} \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \prod_{j=1}^{s} \left\{ \frac{(y_j + s_j - \frac{z_j s_j}{y_j})^{k_j}}{k_j!} \right\}$$

$$(3.4) \qquad = \sum_{\mathbf{m_s} = -\infty}^{\infty} \sum_{\mathbf{n_s} = \mathbf{0}}^{\infty} \prod_{j=1}^{s} \left\{ \frac{y_j^{m_j} \; s_j^{n_j}}{\Gamma(m_j + 1) \; \Gamma(n_j + 1)} \right\} \sum_{\mathbf{p_r} = \mathbf{0}}^{\infty} \sum_{\mathbf{k_s} = \mathbf{0}}^{\mathbf{n}} \Omega(\mathbf{p_r}, \mathbf{k_s} + \mathbf{m_s} + \mathbf{n_s})$$

$$\cdot \prod_{j=1}^{r} \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \cdot \prod_{j=1}^{s} \left\{ \binom{n_j}{k_j} \frac{(-z_j)^{k_j}}{(m_j + 1)_{k_j}} \right\}.$$

Indeed, except for some obvious notional variations, it is the main generating relation of Srivastava et al. [7, p. 10, Equation (2.2)], which is a generalization of another known result of Srivastava et al. [8, p.477, Eq. (2,2)]. Further, according to the multinomial expansion [9, p. 329, Eq. 9.4(220)]:

$$(1 - x_1 - ... - x_r)^{-\nu} = \sum_{k_1, .., k_r = 0}^{\infty} (\nu)_{k_1 + ... + k_r} \frac{x_1^{k_1}}{k_1!} ... \frac{x_r^{k_r}}{k_r!}, \quad (|x_1 + ... + x_r| < 1);$$

(2.2) with $z_j \mapsto z_j s_j t_j$, $(j = 1, 2, ..., s)$, and

$$\Omega(\mathbf{p_r}, \mathbf{k_s}) = \frac{(a)_{k_1 + ... + k_s + p_1 + ... + p_r} (b_1)_{p_1} ... (b_r)_{p_r}}{(c_1)_{p_1} ... (c_r)_{p_r}},$$

would yield a generating relation involving Lauricella function $F_A^{(n)}$ of n-variables (see [10, p. 35, Eq. 1.4(1)]) and Exton function of n-variables ${}^{(p)}H_4^{(n)}$ (see [1, p.97, Eq. 3.4(3.5.2)])

$$(1 - X)^{-a} \; F_A^{(r)} \left[a, b_1, .., b_r; c_1, ..., c_r; \frac{u_1}{(1 - X)}, .., \frac{u_r}{(1 - X)} \right]$$

$$(3.5) \qquad = \sum_{\mathbf{m_s} = -\infty}^{\infty} \sum_{\mathbf{n_s}, \mathbf{r_s} = 0}^{\infty} \prod_{j=1}^{s} \left\{ \frac{y_j^{m_j} s_j^{n_j} t_j^{r_j} x_j^{m_j + n_j + r_j}}{\Gamma(m_j + 1) \Gamma(n_j + 1) \Gamma(r_j + 1)} \right\} (a)_{M_1^s + N_1^s + R_1^s}$$

$${}^{(r)}H_4^{(r+s)} \left[a_{M_1^s + N_1^s + R_1^s}, b_1, ..., b_r, ; m_1 + 1, ..., m_s + 1, c_1, ..., c_r \; ; \right.$$
$$\left. z_1 x_1^2, ..., z_s x_s^2, u_1, ..., u_r \right],$$

where

$$(3.6) \qquad X = \sum_{j=1}^{s} x_j \left(y_j + s_j + t_j + \frac{z_j}{y_j} \right)$$

and M_1^s, N_1^s and R_1^s are given by (1.8). Furthermore, if in assertion (2.2) we set

$$s = r, \quad z_j \mapsto z_j y_j^2 s_j t_j, \; (s_j, t_j) \mapsto (0,0), \; u_j \mapsto \frac{u_j}{y_j}, \quad (j = 1, 2, ..., r),$$

and

$$\Omega(\mathbf{p_r}, \mathbf{k_r}) = (a)_{k_1+...+k_r}(b_1)_{p_1}...(b_r)_{p_r}, \tag{3.7}$$

we obtain a generating relation for Lauricella function $F_A^{(n)}$ in the form

$$\left(1-\sum_{j=1}^{r} x_j y_j(1+z_j)\right)^{-a} \prod_{j=1}^{r}\left\{\left(1-\frac{u_j}{y_j}\right)^{-b_j}\right\} = \sum_{\mathbf{m_r}=-\infty}^{\infty} \prod_{j=1}^{r}\left\{\frac{[x_j y_j(1+z_j)]^{m_j}}{\Gamma(m_j+1)}\right\}(a)_{M_1^r}$$

$$F_A^{(r)}\left[a_{M_1^r}, b_1, .., b_r; m_1+1, ..., m_r+1; u_1x_1(1+z_1), ..., u_rx_r(1+z_r)\right]. \tag{3.8}$$

Equation (3.8) provides a generalization of a known result [9, p. 325, Eq. 6.5(91)].

In order to derive another consequence of assertion (2.2), we now set

$$s = r, \; z_j \mapsto z_j s_j t_j, \; (s_j, t_j) \mapsto (0,0) \;\text{ and }\; u_j \mapsto u_j y_j, \; (j = 1, 2, 3, ..., r).$$

We thus obtain the following elegant result

$$\begin{aligned} &\sum_{\mathbf{P_r},\mathbf{k_r}=\mathbf{0}}^{\infty} \Omega(\mathbf{p_r}, \mathbf{k_r}) \prod_{j=1}^{r}\left\{\frac{(u_j y_j)^{p_j} x_j^{k_j}(y_j+z_j)^{k_j}}{p_j! k_j!}\right\} \\ &= \sum_{\mathbf{m_r}=\mathbf{0}}^{\infty} \prod_{j=1}^{r} \frac{(y_j x_j)^{m_j}}{m_j!} \sum_{\mathbf{k_r}=\mathbf{0}}^{\infty} \sum_{\mathbf{P_r}=\mathbf{0}}^{\mathbf{m}} \Omega(\mathbf{p_r}, \mathbf{k_r}+\mathbf{m_r}-\mathbf{p_r}) \\ &\quad \prod_{j=1}^{r}\left\{\binom{m_j}{p_j}\left(\frac{u_j}{x_j}\right)^{p_j} \frac{(z_j x_j)^{k_j}}{k_j!}\right\}. \end{aligned} \tag{3.9}$$

Now, if in (3.9), we set $\Omega(\mathbf{p_r}, \mathbf{k_r}) = \dfrac{(a)_{k_1+...+k_r}(b_1)_{p_1+k_1}...(b_r)_{p_r+k_r}(d_1)_{p_1}...(d_r)_{p_r}}{(c_1)_{p_1+k_1}...(c_r)_{p_r+k_r}}$, we find for the Erdélyi series $H_{n,p}$ (see [9, p. 36, Eq. 1.4(19)]; see also [6]) that

$$\begin{aligned} &\sum_{\mathbf{k_r}=0}^{\infty} \prod_{j=1}^{r}\left\{\frac{(b_j)_{k_j}[x_j(y_j+z_j)]^{k_j}}{(c_j)_{k_j} k_j!} {}_2F_1[b_j+k_j, d_j; c_j+k_j]\right\} (a)_{K_1^r} \\ &= \sum_{\mathbf{m_r}=0}^{\infty} \prod_{j=1}^{r}\left\{\frac{(x_j y_j)^{m_j}(b_j)_{m_j}}{(c_j)_{m_j} m_j!}\right\}(a)_{M_1^r} \\ &\quad H_{2r,r}\left[a_{M_1^r}, b_1+m_1, ..., b_r+m_r, -m_1, ..., -m_r, d_1, ..., d_r;\right. \\ &\quad \left. c_1+m_1, ..., c_r+m_r; z_1x_1, ..., z_rx_r, -u_1/x_1, ..., -u_r/x_r\right]. \end{aligned} \tag{3.10}$$

For Extonś functions ${}_{(1)}^{(k)}E_D^{(n)}$ and ${}_{(2)}^{(k)}E_D^{(n)}$ (see [1, p. 104, Eq. (3.6.1) and p. 89, Eq. 3.4(3.4.2)]), formula (3.9) similarly yields the following generating relations:

$$
{}_0F_1\left[\begin{matrix} --; \\ \qquad \sum_{j=1}^{r} x_j(y_j+z_j) \\ a; \end{matrix}\right] {}_0F_1\left[\begin{matrix} --; \\ \qquad \sum_{j=1}^{q} u_j y_j \\ c; \end{matrix}\right] {}_0F_1\left[\begin{matrix} --; \\ \qquad \sum_{j=q+1}^{r} u_j y_j \\ d; \end{matrix}\right]
$$

$$
= \sum_{\mathbf{m_r},\mathbf{k_r}=0}^{\infty} \prod_{j=1}^{r} \left\{ \frac{(x_j y_j)^{m_j}(x_j z_j)^{k_j}}{m_j! k_j!} \right\} \frac{1}{(a) - M_1^r + K_1^r}
$$

$$
(3.11) \qquad {}^{(q)}_{(1)}E^{(r)}_{D}\left[1-a-M_1^r-K_1^r, -m_1, ..., -m_r; c, d; \frac{u_1}{x_1}, ..., \frac{u_r}{x_r}\right],
$$

$$
(\ q\ <\ r\ , q = 1, 2, 3, ...),
$$

$$
\left[1-\sum_{j=1}^{q} u_j y_j\right]^{-a} \left[1-\sum_{j=q+1}^{r} u_j y_j\right]^{-b} \left[1-\sum_{j=1}^{r} x_j(y_j+z_j)\right]^{-c}
$$

$$
= \sum_{\mathbf{m_r},\mathbf{k_r}=0}^{\infty} \prod_{j=1}^{r} \left\{ \frac{(x_j y_j)^{m_j}(x_j z_j)^{k_j}}{m_j! k_j!} \right\} (c) - M_1^r + k_1^r
$$

$$
(3.12) \qquad {}^{(q)}_{(2)}E^{(r)}_{D}\left[a, b, , -m_1, ..., m_r; 1-c-M_1^r+K_1^r; \frac{u_1}{x_1}, ..., \frac{u_r}{x_r}\right],
$$

$$
(\ q\ <\ r\ , q = 1, 2, 3, ...),
$$

Moreover, if in the formula (3.9), we let

$$
(3.13) \qquad \Omega(\mathbf{p_r}, \mathbf{k_r}) = \Omega_1(\mathbf{p_r})\Omega_2(\mathbf{k_r})
$$

the left-hand side of (3.10) would reduces at once to a product of two multiple series with essentially arbitrary coefficients. Thus, by assigning suitable special values to the coefficients $\Omega_1(\mathbf{p_r})$ and $\Omega_2(\mathbf{k_r})$, we can derive a number of generating functions involving the product of multiple functions and polynomials.

For instance, in view of the following definition of the multiple Bessel polynomials introduced by Pathan and Bin-Saad [4, p. 91, Equation (2.1)]:

$$
(3.14) \quad y^{(\alpha_1,..,\alpha_n;\beta)}_{m_1,..,m_n}(x_1,..,x_n) = \sum_{k_1=0}^{m_1} ... \sum_{k_n=0}^{m_n} (1+\beta+\nu)_{k_1+..+k_n} \prod_{j=1}^{n} \left\{ \binom{m_j l}{k_j} x_j^{k_j} \right\},
$$

where $\nu = \alpha_1 m_1 + ... + \alpha_n m_n$ by appropriately choosing the multiple sequences $\Omega(\mathbf{p_r}, \mathbf{k_r})$ in accordance with the definition (3.14), we shall find

$$
y^{(\alpha_1,..,\alpha_r;\beta)}_{s_1,..,s_r}(x_1(y_1+z_1), .., x_r(y_r+z_r)) \times y^{(\gamma_1,..,\gamma_r;\delta)}_{l_1,..,l_r}(u_1 y_1, .., u_r y_r)
$$

$$
= \sum_{\mathbf{m_r},\mathbf{k_r}=0}^{\infty} \prod_{j=1}^{r} \left\{ \frac{(x_j y_j)^{m_j}(x_j z_j)^{k_j}(-s_j)_{k_j+m_j}}{m_j! k_j!} \right\} (1+\beta+\nu_1)_{M_1^r+K_1^r}
$$

(3.15)

$$F^{1:2;..;2}_{1:1;..;1}\left[\begin{array}{l} 1+\delta+\nu_2 : -l_1, -m_1; ...; -l_r, -m_r; \\ \\ -\beta-\nu_1-M_1^r-K_1^r : 1+s_1-m_1-k_1, ..., 1+s_1-m_r-k_r; \end{array} \frac{u_1}{x_1}, ..., \frac{u_r}{x_r}\right],$$

$\nu_1 = \alpha_1 s_1 + ... + \alpha_r s_r,\ \nu_2 = \gamma_1 l_1 + ... + \gamma_r l_r,\ (s_j, l_j) = 1, 2, 3, ...,\ (j = 1, 2, 3, .., r),$

where $F^{l:m_1;...;m_n}_{p:q_1;...;q_{1n}}[x_1, ..., x_n]$ is the generalized Kampé de Fériet series of n-variables [9, p. 38, (24) and (25)].

We find it worthwhile to conclude by mentioning that the general assertion (2.2) can be applied further fairly easily to derive a considerable wide variety of generating functions for double, triple and several multivariable hypergeometric polynomials including, for example, Hermite polynomials, Whittaker's function, Humbert's function, the generalized Kampé de Fériet function of two variables, the general triple hypergeometric series (cf. [5] and [9]) and Exton's triple series (see [2]).

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