BLOCKWISE REPEATED LOW-DENSITY BURST ERROR CORRECTING LINEAR CODES

Bal Kishan Dass

Department of Mathematics
University of Delhi
Delhi - 110 007
India
e-mail: dassbk@rediffmail.com

Surbhi Madan[1]

Department of Mathematics
Shivaji College (University of Delhi)
Raja Garden
New Delhi - 110 027
India
e-mail: surbhimadan@gmail.com

Abstract. The paper presents necessary and sufficient condition on the number of parity-check digits required for the existence of a linear code capable of correcting errors in the form of 2-repeated low-density bursts occurring within a sub-block. An illustration of a code of length 24 correcting all 2-repeated low-density bursts of length 3 or less with weight 2 or less occurring within a sub-block of length 12 has also been provided.

Keywords: error locating codes, error correction, burst errors, repeated burst errors, low-density repeated burst errors.

AMS Subject Classification: 94B20, 94B65, 94B25.

1. Introduction

In the theory of error control coding, codes have been developed to detect, correct and/or to locate various kinds of errors. Amongst these, burst errors have played a dominant role and have been studied extensively by many authors. Most of the earlier studies in this direction have been made with respect to the following definition of a burst:

Definition 1. A burst of length b is a vector whose all non-zero components are among some b consecutive components, the first and the last of which is non-zero.

[1]Corresponding author.

Depending upon the type of channel used during the process of transmission the nature of burst errors differ. It has been observed that in very busy communication channels, errors repeat themselves. Recently, repeated bursts have been introduced and studied by Berardi, Dass and Verma [1]. An m-repeated burst of length b is defined as follows:

Definition 2. An m-repeated burst of length b is a vector of length n whose only non-zero components are confined to m distinct sets of b consecutive components, the first and the last component of each set being non-zero.

Certain situations like lightening or other disturbances which induce burst errors usually operate in a way that over a given length some digits are received correctly whereas others are corrupted. Such situations led to the development of codes dealing with errors that are bursts of length b or less with weight w or less ($w \leq b$), known as low-density bursts (refer Wyner [14]). A study of low-density burst error detecting and correcting linear codes has been made by Sharma and Dass [12] and Dass [2]. Different situations demanded the development of codes which correct those errors that are repeated low-density burst errors of length b or less with weight w or less. A study of such codes was initiated by Dass and Verma [6]. A 2-repeated low-density burst of length b with weight w ($w \leq b$) is defined as follows:

Definition 3. A 2-repeated low-density burst of length b with weight w is a vector of length n whose only non-zero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being non-zero, with w ($w \leq b$) non-zero components within each set of such b consecutive components.

For example, (01023000132400) is a 2-repeated low-density burst of length 4 with weight 3 over $GF(5)$.

Wolf and Elspas [13] introduced the coding technique called *error-locating codes* (EL Codes). The concept of error location coding lies midway between error detection and error correction. Error location technique provides an attractive alternative to the conventional error detection in decision feedback communications. Wolf and Elspas [13] obtained results in the form of bounds over the number of parity-check digits required for binary codes capable of detecting and locating a single sub-block containing random errors. Further, Dass [3], [4] studied codes locating burst errors and low-density burst errors. In our earlier papers [7], [9] the authors have obtained bounds for codes locating 2-repeated burst errors and 2-repeated low-density burst errors occurring within a single sub-block. This paper extends the study further to the correction of 2-repeated low-density burst errors occurring within a sub-block. The development of codes correcting repeated low-density burst errors within a sub-block economizes in the number of parity-check digits in comparison to the usual low-density burst error locating codes.

In this paper lower and upper bounds on the number of parity check digits required for the existence of such a code are obtained. The paper concludes with an illustration of such a code. Throughout the paper, we consider a block of n

digits, consisting of r check digits and $k = n - r$ information digits, subdivided into s mutually exclusive sub-blocks, each sub-block contains $t = n/s$ digits.

2. Bounds for linear codes correcting 2-repeated low-density bursts

An (n, k) linear EL code over $GF(q)$ capable of detecting and locating a single sub-block containing 2-repeated low-density burst of length b or less with weight w or less must satisfy the following two conditions:

(i) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within any one sub-block must be non-zero.

(ii) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated low-density burst of length b or less with weight w or less within *any other* sub-block.

Further, an (n, k) linear code over $GF(q)$ capable of correcting an error requires the syndromes of any two vectors to be distinct irrespective of whether they belong to the same sub-block or to different sub-blocks. So, in order to correct 2-repeated low-density bursts of length b or less with weight w or less lying within a sub-block the following conditions need to be satisfied:

(iii) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting from any other 2-repeated low-density burst of length b or less with weight w or less within the same sub-block.

(iv) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated low-density burst of length b or less with weight w or less within *any other* sub-block.

Remark 1. We observe that condition (iv) is the same as condition (ii). Also, for computational purposes condition (i) is taken care of by condition (iii). So we need to consider conditions (iii) and (iv) or equivalently conditions (ii) and (iii) for correction of the said type of errors.

We first obtain a lower bound over the number of parity check digits required for such a code.

Theorem 1. *The number of check digits r required for an (n, k) linear code over $GF(q)$, subdivided into s sub-blocks of length t each, that corrects 2-repeated low-density bursts of length b or less with weight w or less lying within a single corrupted sub-block is bounded from below by*

$$
\begin{aligned}
r \geq log_q \Bigg\{ 1+s \Bigg[& (q-1)[1+(q-1)]^{(b-1,w-1)} \Bigg(\binom{t-2b+2}{2} (q-1)[1+(q-1)]^{(b-1,w-1)} \\
& + \binom{t-2b+1}{1} [1+(q-1)]^{(b-1,\min(w,b-1))} \\
& + \sum_{i=t-2b+2}^{t-b-w+1} [1+(q-1)]^{(t-i-b+1,w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Bigg) \\
& + \Bigg(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4,r_5,r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4,r_5,r_6} \Bigg) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \cdot \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))} \\
& + [1+(q-1)]^{(b-1,\min(2w,b-1))} - 1 \Bigg] \Bigg\},
\end{aligned} \tag{1}
$$

where $0 \leq r_4 \leq w-1$, $1 \leq r_5 \leq 2w-2$, $0 \leq r_6 \leq w-2$, $r_4+r_5 \geq w$, $r_4+r_5+r_6 \leq 2w-2$.

Proof. Let V be an (n, k) linear code over $GF(q)$ that corrects 2-repeated low-density bursts of length b or less with weight w or less within a single corrupted sub-block. The maximum number of distinct syndromes available using r check digits is q^r. The proof proceeds by first counting the number of syndromes that are required to be distinct by the two conditions and then setting this number less than or equal to q^r.

Since the code is capable of correcting all errors which are 2-repeated low-density bursts of length b or less with weight w or less within any single sub-block, the syndrome produced by a 2-repeated low-density burst of length b or less with weight w or less in a given sub-block must be distinct from any such syndrome likewise resulting from another 2-repeated low-density burst of length b or less with weight w or less in the same sub-block(refer to condition (iii)). Moreover, syndromes produced by 2-repeated low-density bursts of length b or less with weight w or less in different sub-blocks must also be distinct by condition (iv).
Thus, the syndromes of vectors which are 2-repeated low-density bursts of length b or less with weight w or less, whether in the same sub-block or in different sub-blocks, must be distinct. Since there are

$$
\begin{aligned}
(q-1)[1+(q-1)]^{(b-1,w-1)} \Bigg(& \tbinom{t-2b+2}{2} (q-1)[1+(q-1)]^{(b-1,w-1)} \\
& + \binom{t-2b+1}{1} [1+(q-1)]^{(b-1,\min(w,b-1))}
\end{aligned}
$$

$$+\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\Bigg)$$
$$+\Bigg(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r_4,r_5,r_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r_4,r_5,r_6}\Bigg)\binom{k_1}{r_4}\binom{b-k_1-1}{r_5}\binom{k_1}{r_6}\cdot$$
$$\cdot(q-1)^{r_4+r_5+r_6+2}$$
$$+\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))}$$
$$+[1+(q-1)]^{(b-1,\min(2w,b-1))}-1$$

2-repeated low-density burst of length b or less with weight w or less within one sub-block of length t excluding the vector of all zeros [5], where $0 \le r_4 \le w-1$, $1 \le r_5 \le 2w-2$, $0 \le r_6 \le w-2$, $r_4+r_5 \ge w$, $r_4+r_5+r_6 \le 2w-2$, and as there are s sub-blocks in all, we must have at least

$$1+s\Bigg[(q-1)[1+(q-1)]^{(b-1,w-1)}\Bigg(\binom{t-2b+2}{2}(q-1)[1+(q-1)]^{(b-1,w-1)}$$
$$+\binom{t-2b+1}{1}[1+(q-1)]^{(b-1,\min(w,b-1))}$$
$$+\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\Bigg)$$
$$+\Bigg(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r_4,r_5,r_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r_4,r_5,r_6}\Bigg)\binom{k_1}{r_4}\binom{b-k_1-1}{r_5}\binom{k_1}{r_6}\cdot$$
$$\cdot(q-1)^{r_4+r_5+r_6+2}$$
$$+\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))}$$
$$+[1+(q-1)]^{(b-1,\min(2w,b-1))}-1\Bigg]$$

where $0 \le r_4 \le w-1$, $1 \le r_5 \le 2w-2$, $0 \le r_6 \le w-2$, $r_4+r_5 \ge w$, $r_4+r_5+r_6 \le 2w-2$, distinct syndromes including the all zeros syndrome. Therefore, we must have

$$q^r \ge 1+s\Bigg[(q-1)[1+(q-1)]^{(b-1,w-1)}\Bigg(\binom{t-2b+2}{2}(q-1)[1+(q-1)]^{(b-1,w-1)}$$
$$+\binom{t-2b+1}{1}[1+(q-1)]^{(b-1,\min(w,b-1))}$$
$$+\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\Bigg)$$

$$+\left(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r_4,r_5,r_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r_4,r_5,r_6}\right)\binom{k_1}{r_4}\binom{b-k_1-1}{r_5}\binom{k_1}{r_6}\cdot$$
$$\cdot(q-1)^{r_4+r_5+r_6+2}$$
$$+\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))}$$
$$+[1+(q-1)]^{(b-1,\min(2w,b-1))}-1\Bigg]$$

Taking logarithm on both the sides we get the result as stated in (1). ■

Remark 2. By taking $s = 1$ the bound obtained in (1) reduces to

$$\log_q\Bigg((q-1)[1+(q-1)]^{(b-1,w-1)}\left(\binom{t-2b+2}{2}(q-1)[1+(q-1)]^{(b-1,w-1)}\right.$$
$$+\binom{t-2b+1}{1}[1+(q-1)]^{(b-1,\min(w,b-1))}$$
$$\left.+\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\right)$$
$$+\left(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r_4,r_5,r_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r_4,r_5,r_6}\right)\binom{k_1}{r_4}\binom{b-k_1-1}{r_5}\binom{k_1}{r_6}\cdot$$
$$\cdot(q-1)^{r_4+r_5+r_6+2}$$
$$+\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))}$$
$$+[1+(q-1)]^{(b-1,\min(2w,b-1))}-1\Bigg)$$

which coincides with the result for correction of 2-repeated low-density bursts obtained by Dass and Verma [5].

Remark 3. For $w = b$ the bound obtained in (1) reduces to

$$log_q\left\{1+s\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]\right\}.$$

which coincides with the lower bound on the number of parity check digits required for the blockwise correction of 2-repeated bursts [8].

Several other particular cases by fixing up the parameters may also be deduced which would result into known results obtained earlier by various authors.

In the following result, we derive another bound on the number of check digits required for the existence of such a code. The proof is based on the technique used to establish Varshamov-Gilbert-Sacks bound by constructing a parity check matrix for such a code (refer Sacks [11], also Theorem 4.7, Peterson and Weldon [10]). This technique not only ensures the existence of such a code but also gives a method for the construction of the code.

Theorem 2. *An (n, k) linear code over $GF(q)$ capable of correcting 2-repeated low-density burst of length b or less with weight w or less or less occurring within a single sub-block of length t $(4b < t)$ can always be constructed using r check digits, where r is the smallest integer satisfying the inequality*

$$
\begin{aligned}
q^r \geq \Bigg\{ & [1+(q-1)]^{(b-1,w-1)} \pi^{b,w}_{3,t-b} \\
& + \Bigg(\sum_{k_1=1}^{b-1} \sum_{r_1,r_2,r_3} \binom{b-k_1-1}{r_1} \binom{k_1}{r_2} \binom{b-k_1-1}{r_3} (q-1)^{r_1+r_2+r_3+1} \Bigg) \pi^{b,w}_{2,t-2b+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_4,r_5,r_6,r_7,r_8} \binom{b-k_1-1}{r_4} \binom{k_1}{r_5} \binom{b-k_1-1}{r_6} \binom{k_1}{r_7} \binom{b-k_1-1}{r_8} \\
& \times (q-1)^{r_4+r_5+r_6+r_7+r_8+1} \pi^{b,w}_{1,t-3b+2} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_9,r_{10},\ldots,r_{15}} \binom{b-k_1-1}{r_9} \binom{k_1}{r_{10}} \binom{b-k_1-1}{r_{11}} \binom{k_1}{r_{12}} \binom{b-k_1-1}{r_{13}} \\
& \times \binom{k_1}{r_{14}} \binom{b-k_1-1}{r_{15}} (q-1)^{r_9+r_{10}+\ldots+r_{15}+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{16},r_{17},\ldots,r_{20}} \binom{b-k_1-1}{r_{16}} \binom{k_1}{r_{17}} \binom{b-k_1-1}{r_{18}} \binom{k_1}{r_{19}} \binom{b-k_1-1}{r_{20}} \\
& \times (q-1)^{r_{16}+r_{17}+r_{18}+r_{19}+r_{20}+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{21},r_{22},r_{23}} \binom{b-k_1-1}{r_{21}} \binom{k_1}{r_{22}} \binom{b-k_1-1}{r_{23}} (q-1)^{r_{21}+r_{22}+r_{23}+1} \pi^{b,w}_{1,t-3b+2} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{24},r_{25},r_{26}} \binom{b-k_1-1}{r_{24}} \binom{k_1}{r_{25}} \binom{b-k_1-1}{r_{26}} (q-1)^{r_{24}+r_{25}+r_{26}+1} \pi^{b,w}_{1,t-2b+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{27},r_{28},\ldots,r_{31}} \binom{b-k_1-1}{r_{27}} \binom{k_1}{r_{28}} \binom{b-k_1-1}{r_{29}} \binom{k_1}{r_{30}} \binom{b-k_1-1}{r_{31}} \\
& \times (q-1)^{r_{27}+r_{28}+r_{29}+r_{30}+r_{31}+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{32},r_{33},r_{34}} \binom{b-k_1-1}{r_{32}} \binom{k_1}{r_{33}} \binom{b-k_1-1}{r_{34}} (q-1)^{r_{32}+r_{33}+r_{34}+1} \\
& + [1+(q-1)]^{(b-1;w,2w-2)} \pi^{b,w}_{2,t-2b+1} + \binom{b-1}{2w-1} (q-1)^{2w-1} \pi^{b,w}_{2,t-b}
\end{aligned}
\tag{2}
$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{35},r_{36},r_{37}}\binom{b-k_1-1}{r_{35}}\binom{k_1}{r_{36}}\binom{b-k_1-1}{r_{37}}(q-1)^{r_{35}+r_{36}+r_{37}+1}\pi^{b,w}_{1,t-2b+1}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{38},r_{39},\dots,r_{42}}\binom{b-k_1-1}{r_{38}}\binom{k_1}{r_{39}}\binom{b-k_1-1}{r_{40}}\binom{k_1}{r_{41}}\binom{b-k_1-1}{r_{42}}$$

$$\times\ (q-1)^{r_{38}+r_{39}+r_{40}+r_{41}+r_{42}+1}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{43},r_{44},r_{45}}\binom{b-k_1-1}{r_{43}}\binom{k_1}{r_{44}}\binom{b-k_1-1}{r_{45}}(q-1)^{r_{43}+r_{44}+r_{45}+1}$$

$$+\ [1+(q-1)]^{(b-1;2w,3w-2)}\pi^{b,w}_{1,t-2b+1}+\binom{b-1}{3w-1}(q-1)^{3w-1}\pi^{b,w}_{1,t-b}$$

$$+\ \sum_{k_1=1}^{b-1}\sum_{r_{46},r_{47},r_{48}}\binom{b-k_1-1}{r_{46}}\binom{k_1}{r_{47}}\binom{b-k_1-1}{r_{48}}(q-1)^{r_{46}+r_{47}+r_{48}+1}$$

$$+\ [1+(q-1)]^{(b-1;3w,\min(4w-1,b-1))}\Bigg\}$$

$$+\Bigg\{\ \Bigg([1+(q-1)]^{(b-1,w-1)}\Big\{q^{w-1}((q-1)(t-b-w+1)+1)$$

$$+(q-1)^2\sum_{i=w+1}^{b}(t-b-i+1)[1+(q-1)]^{(i-2,w-2)}\Big\}+\sum_{i=w}^{2w-1}\binom{b-1}{i}(q-1)^i$$

$$+\sum_{k=1}^{b-1}\sum_{r'_1,r'_2,r'_3}\binom{b-k-1}{r'_1}\binom{k}{r'_2}\binom{b-k-1}{r'_3}(q-1)^{r'_1+r'_2+r'_3+1}\Bigg)$$

$$\times(s-1)\cdot\Bigg((q-1)[1+(q-1)]^{(b-1,w-1)}\Bigg(\binom{t-2b+2}{2}(q-1)[1+(q-1)]^{(b-1,w-1)}$$

$$+\binom{t-2b+1}{1}[1+(q-1)]^{(b-1,\min(w,b-1))}+\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}$$

$$+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\Bigg)$$

$$+\Bigg(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r'_4,r'_5,r'_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r'_4,r'_5,r'_6}\Bigg)\binom{k_1}{r'_4}\binom{b-k_1-1}{r'_5}\binom{k_1}{r'_6}\cdot$$

$$\cdot\,(q-1)^{r'_4+r'_5+r'_6+2}$$

$$+\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))}$$

$$+[1+(q-1)]^{(b-1,\min(2w,b-1))}-1\Bigg)\Bigg\},$$

where $0 \le r_1 \le w-2$, $1 \le r_2 \le 2w-2$, $0 \le r_3 \le w-1$, $r_2+r_3 \ge w$, $r_1+r_2+r_3 \le 2w-2$; $0 \le r_4 \le w-2$, $1 \le r_5 \le 2w-2$, $0 \le r_6 \le 2w-3$, $0 \le r_7 \le 2w-2$, $0 \le r_8 \le 2w-2$, $2 \le r_5+r_6 \le 2w-2$, $w \le r_7+r_8 \le 2w-2$, $2w \le r_5+r_6+r_7+r_8 \le 3w-2$, $2w \le r_4+r_5+\ldots+r_8 \le 3w-2$; $0 \le r_9 \le w-2$, $1 \le r_{10} \le 2w-2$, $0 \le r_{11} \le 2w-3$, $0 \le r_{12} \le 2w-2$, $0 \le r_{13} \le 2w-2$, $0 \le r_{14} \le 2w-2$, $0 \le r_{15} \le 2w-2$, $2 \le r_{10}+r_{11} \le 2w-2$, $2 \le r_{12}+r_{13} \le 2w-2$, $w \le r_{14}+r_{15} \le 2w-2$, $2w \le r_{12}+r_{13}+r_{14}+r_{15} \le 3w-2$, $3w \le r_{10}+r_{11}+\ldots+r_{15} \le 4w-2$, $3w \le r_9+r_{10}+\ldots+r_{15} \le 4w-2$; $1 \le r_{16} \le 2w-2$, $0 \le r_{17} \le 2w-2$, $0 \le r_{18} \le 2w-2$, $0 \le r_{19} \le 2w-2$, $0 \le r_{20} \le 2w-2$, $2 \le r_{17}+r_{18} \le 2w-2$, $w \le r_{19}+r_{20} \le 2w-2$, $2w \le r_{17}+r_{18}+r_{19}+r_{20} \le 3w-2$, $3w-1 \le r_{16}+r_{17}+\ldots+r_{20} \le 4w-2$; $1 \le r_{21} \le 2w-3$, $0 \le r_{22} \le 2w-2$, $0 \le r_{23} \le 2w-2$, $w \le r_{22}+r_{23} \le 2w-2$, $2w-1 \le r_{21}+r_{22}+r_{23} \le 3w-3$; $w \le r_{24} \le 2w-2$, $0 \le r_{25} \le 2w-2$, $0 \le r_{26} \le 2w-2$, $w \le r_{25}+r_{26} \le 2w-2$, $r_{24}+r_{25}+r_{26} = 3w-2$; $0 \le r_{27} \le w-2$, $1 \le r_{28} \le 3w-2$, $0 \le r_{29} \le 3w-3$, $0 \le r_{30} \le 2w-2$, $0 \le r_{31} \le 2w-2$, $w+2 \le r_{28}+r_{29} \le 3w-2$, $w \le r_{30}+r_{31} \le 2w-2$, $3w \le r_{28}+r_{29}+r_{30}+r_{31} \le 4w-2$, $3w \le r_{27}+r_{28}+\ldots+r_{31} \le 4w-2$; $w+1 \le r_{32} \le 3w-2$, $0 \le r_{33} \le 2w-2$, $0 \le r_{34} \le 2w-2$, $w \le r_{33}+r_{34} \le 2w-2$, $3w-1 \le r_{32}+r_{33}+r_{34} \le 4w-2$; $0 \le r_{35} \le w-2$, $1 \le r_{36} \le 3w-2$, $0 \le r_{37} \le 3w-3$, $r_{36}+r_{37} \ge 2w$, $r_{35}+r_{36}+r_{37} \le 3w-2$; $0 \le r_{38} \le w-2$, $1 \le r_{39} \le 2w-2$, $0 \le r_{40} \le 2w-3$, $0 \le r_{41} \le 3w-2$, $0 \le r_{42} \le 3w-2$, $2 \le r_{39}+r_{40} \le 2w-2$, $2w \le r_{41}+r_{42} \le 3w-2$, $3w \le r_{39}+r_{40}+r_{41}+r_{42} \le 4w-2$, $3w \le r_{38}+r_{39}+\ldots+r_{42} \le 4w-2$; $1 \le r_{43} \le 2w-2$, $0 \le r_{44} \le 3w-2$, $0 \le r_{45} \le 3w-2$, $2w \le r_{44}+r_{45} \le 3w-2$, $3w-1 \le r_{43}+r_{44}+r_{45} \le 4w-2$; $0 \le r_{46} \le w-2$, $1 \le r_{47} \le 4w-2$, $0 \le r_{48} \le 3w-1$, $r_{47}+r_{48} \ge 3w$, $r_{46}+r_{47}+r_{48} \le 4w-2$;
$0 \le r'_1 \le w-2$, $1 \le r'_2 \le 2w-2$, $0 \le r'_3 \le w-1$, $r'_2+r'_3 \ge w$, $r'_1+r'_2+r'_3 \le 2w-2$; $0 \le r'_4 \le w-1$, $1 \le r'_5 \le 2w-2$, $0 \le r'_6 \le w-2$, $r'_4+r'_5 \ge w$, $r'_4+r'_5+r'_6 \le 2w-2$ *and* $\pi^{b,w}_{m,n}$ *denotes the number of m-repeated low-density bursts of length b or less with weight w or less* ($w \le b$) *in a vector of length n*, $[1+x]^{(m,r)}$ *denotes the incomplete binomial expansion of* $(1+x)^m$ *upto the term* x^r *in ascending powers of* x *and* $[1+x]^{(m;r_1,r_2)}$ *denotes the incomplete binomial expansion of* $(1+x)^m$ *from the term* x^{r_1} *to the term* x^{r_2} ($r_1 < r_2$).

Proof. We shall prove the result by constructing an appropriate $(n-k)\times n$ parity check matrix H for the desired code. Suppose that the columns of the first $s-1$ sub-blocks of H and the first $j-1$ columns $h_1, h_2, \cdots, h_{j-1}$ of the s^{th} sub-block have been appropriately added. We lay down conditions to add the j^{th} column h_j to the s^{th} sub-block as follows:

Since the code is to correct 2-repeated low-density bursts of length b or less with weight w or less within a single sub-block, therefore, by condition (iii), the syndrome of any 2-repeated low-density burst in any sub-block must be different from the syndrome resulting from any other such burst within the same sub-block. Therefore the j^{th} column h_j can be added provided that h_j is not a linear combination of $w-1$ or fewer columns from the immediately preceding $b-1$ or fewer columns of H together with any w or fewer columns chosen from three sets of b or fewer consecutive columns each amongst the first $j-1$ columns. In other words,

$$\begin{aligned} h_j \neq\ & (\alpha_1 h_i + \alpha_2 h_{i+1} + \cdots + \alpha_{w-1} h_{i+w-2}) \\ & + (\beta_1 h_{i_1} + \beta_2 h_{i_1+1} + \cdots + \beta_w h_{i_1+w-1}) \\ & + (\gamma_1 h_{i_2} + \gamma_2 h_{i_2+1} + \cdots + \gamma_w h_{i_2+w-1}) \\ & + (\delta_1 h_{i_3} + \delta_2 h_{i_3+1} + \cdots + \delta_w h_{i_3+w-1}), \end{aligned} \tag{3}$$

where $\alpha_i, \beta_i, \gamma_i$ and $\delta_i \in GF(q)$ and the h_i are any $w-1$ or less columns amongst $h_{j-b+1}, \cdots, h_{j-1}$ and h_{i_1}, h_{i_2} and h_{i_3} are any w or less columns each from three sets of b or less consecutive columns amongst all the preceding $j-1$ columns.

The number of linear combinations corresponding to the right hand side of (3) is (refer Dass and Verma [5])

$$\begin{aligned} & [1+(q-1)]^{(b-1,w-1)} \pi^{b,w}_{3,j-b} \\ & + \Bigg(\sum_{k_1=1}^{b-1} \sum_{r_1,r_2,r_3} \binom{b-k_1-1}{r_1} \binom{k_1}{r_2} \binom{b-k_1-1}{r_3} (q-1)^{r_1+r_2+r_3+1} \Bigg) \pi^{b,w}_{2,j-2b+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_4,r_5,r_6,r_7,r_8} \binom{b-k_1-1}{r_4} \binom{k_1}{r_5} \binom{b-k_1-1}{r_6} \binom{k_1}{r_7} \binom{b-k_1-1}{r_8} \\ & \times (q-1)^{r_4+r_5+r_6+r_7+r_8+1} \pi^{b,w}_{1,j-3b+2} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_9,r_{10},\ldots,r_{15}} \binom{b-k_1-1}{r_9} \binom{k_1}{r_{10}} \binom{b-k_1-1}{r_{11}} \binom{k_1}{r_{12}} \binom{b-k_1-1}{r_{13}} \\ & \times \binom{k_1}{r_{14}} \binom{b-k_1-1}{r_{15}} (q-1)^{r_9+r_{10}+\ldots+r_{15}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{16},r_{17},\ldots,r_{20}} \binom{b-k_1-1}{r_{16}} \binom{k_1}{r_{17}} \binom{b-k_1-1}{r_{18}} \binom{k_1}{r_{19}} \binom{b-k_1-1}{r_{20}} \\ & \times (q-1)^{r_{16}+r_{17}+r_{18}+r_{19}+r_{20}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{21},r_{22},r_{23}} \binom{b-k_1-1}{r_{21}} \binom{k_1}{r_{22}} \binom{b-k_1-1}{r_{23}} (q-1)^{r_{21}+r_{22}+r_{23}+1} \pi^{b,w}_{1,j-3b+2} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{24},r_{25},r_{26}} \binom{b-k_1-1}{r_{24}} \binom{k_1}{r_{25}} \binom{b-k_1-1}{r_{26}} (q-1)^{r_{24}+r_{25}+r_{26}+1} \pi^{b,w}_{1,j-2b+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{27},r_{28},\ldots,r_{31}} \binom{b-k_1-1}{r_{27}} \binom{k_1}{r_{28}} \binom{b-k_1-1}{r_{29}} \binom{k_1}{r_{30}} \binom{b-k_1-1}{r_{31}} \\ & \times (q-1)^{r_{27}+r_{28}+r_{29}+r_{30}+r_{31}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{32},r_{33},r_{34}} \binom{b-k_1-1}{r_{32}} \binom{k_1}{r_{33}} \binom{b-k_1-1}{r_{34}} (q-1)^{r_{32}+r_{33}+r_{34}+1} \end{aligned} \tag{4}$$

$$+ [1+(q-1)]^{(b-1;w,2w-2)}\pi^{b,w}_{2,j-2b+1} + \binom{b-1}{2w-1}(q-1)^{2w-1}\pi^{b,w}_{2,j-b}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{35},r_{36},r_{37}}\binom{b-k_1-1}{r_{35}}\binom{k_1}{r_{36}}\binom{b-k_1-1}{r_{37}}(q-1)^{r_{35}+r_{36}+r_{37}+1}\pi^{b,w}_{1,j-2b+1}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{38},r_{39},\ldots,r_{42}}\binom{b-k_1-1}{r_{38}}\binom{k_1}{r_{39}}\binom{b-k_1-1}{r_{40}}\binom{k_1}{r_{41}}\binom{b-k_1-1}{r_{42}}$$

$$\times (q-1)^{r_{38}+r_{39}+r_{40}+r_{41}+r_{42}+1}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{43},r_{44},r_{45}}\binom{b-k_1-1}{r_{43}}\binom{k_1}{r_{44}}\binom{b-k_1-1}{r_{45}}(q-1)^{r_{43}+r_{44}+r_{45}+1}$$

$$+ [1+(q-1)]^{(b-1;2w,3w-2)}\pi^{b,w}_{1,j-2b+1} + \binom{b-1}{3w-1}(q-1)^{3w-1}\pi^{b,w}_{1,j-b}$$

$$+\sum_{k_1=1}^{b-1}\sum_{r_{46},r_{47},r_{48}}\binom{b-k_1-1}{r_{46}}\binom{k_1}{r_{47}}\binom{b-k_1-1}{r_{48}}(q-1)^{r_{46}+r_{47}+r_{48}+1}$$

$$+ [1+(q-1)]^{(b-1;3w,\min(4w-1,b-1))}$$

where conditions on $r_1 \cdots r_{48}$ are as stated in (2).

Further, by condition (iv), h_j can be added to the s^{th} sub-block provided thath_j is not a linear combination of w or fewer columns out of the immediately preceding $b-1$ or fewer columns together with w or fewer columns out of one set of b or fewer consecutive columns from amongst the first $j-1$ columns together with linear combinations of w or fewer columns out of any two sets of b or fewer consecutive columns each within *any other* sub-block, i.e.,

$$\begin{aligned} h_j \neq\ & (\alpha'_1 h_i + \alpha'_2 h_{i+1} + \cdots + \alpha'_{w-1} h_{i+w-2}) \\ +\ & (\beta'_1 h_{i_1} + \beta'_2 h_{i_1+1} + \cdots + \beta'_w h_{i_w}) \\ +\ & (\gamma'_1 h_{p_1} + \gamma'_2 h_{p_1+1} + \cdots + \gamma'_w h_{p_1+w-1}) \\ +\ & (\delta'_1 h_{p_2} + \delta'_2 h_{p_2+1} + \cdots + \delta'_w h_{p_2+w-1}), \end{aligned}$$

where $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i \in GF(q)$, not all γ'_i, δ'_i zero and h_i are any $w-1$ columns amongst $h_{j-b+1}, h_{j-b+2}, \cdots, h_{j-1}$ and $h'_{i_1}s$ are any w columns from a set of b consecutive columns from the previously chosen $j-1$ columns of s^{th} sub-block and both h_{p_1}'s and h_{p_2}'s are sets of w columns from any b consecutive columns each from *any other* sub-block.

The number of ways in which the coefficients α'_i and β'_i can be chosen is [2, 6]

$$
\begin{aligned}
(5)\quad & \Bigg([1+(q-1)]^{(b-1,w-1)}\Big\{q^{w-1}((q-1)(j-b-w+1)+1)+ \\
& (q-1)^2\sum_{i=w+1}^{b}(j-b-i+1)[1+(q-1)]^{(i-2,w-2)}\Big\}+\sum_{i=w}^{2w-1}\binom{b-1}{i}(q-1)^i \\
& +\sum_{k=1}^{b-1}\sum_{r'_1,r'_2,r'_3}\binom{b-k-1}{r'_1}\binom{k}{r'_2}\binom{b-k-1}{r'_3}(q-1)^{r'_1+r'_2+r'_3+1}\Bigg),
\end{aligned}
$$

where conditions on r'_1, r'_2, r'_3 are as stated in (2).

Also, the number of linear combinations corresponding to the last two terms on the right hand side of (5) is the same as the number of 2-repeated low-density bursts of length b or less with weight w or less within a sub-block of length t, excluding the vector of all zeros and this number in a sub-block of length t, is [5]

$$
\begin{aligned}
& (q-1)[1+(q-1)]^{(b-1,w-1)}\Bigg(\binom{t-2b+2}{2}(q-1)[1+(q-1)]^{(b-1,w-1)} \\
& +\binom{t-2b+1}{1}[1+(q-1)]^{(b-1,\min(w,b-1))} \\
& +\sum_{i=t-2b+2}^{t-b-w+1}[1+(q-1)]^{(t-i-b+1,w)}+\sum_{i=t-b-w+2}^{t-b+1}q^{t-i-b+1}\Bigg) \\
(6)\quad & +\Bigg(\binom{t-2b+2}{1}\sum_{k_1=0}^{b-2}\sum_{r_4,r_5,r_6}+\sum_{i=t-2b+3}^{t-b}\sum_{k_1=0}^{t-i-b}\sum_{r_4,r_5,r_6}\Bigg)\binom{k_1}{r_4}\binom{b-k_1-1}{r_5}\binom{k_1}{r_6}\cdot \\
& \cdot(q-1)^{r_4+r_5+r_6+2} \\
& +\binom{t-b+1}{1}(q-1)[1+(q-1)]^{(b-1;w,\min(2w-1,b-1))} \\
& +[1+(q-1)]^{(b-1,\min(2w,b-1))}-1,
\end{aligned}
$$

where conditions on r'_4, r'_5, r'_6 are as stated in (2).

Since there are $s-1$ previously chosen sub-blocks, therefore number of such linear combinations becomes

$$(s-1)\cdot expr(6). \tag{7}$$

Thus, according to condition (iv), the number of linear combinations to which h_j can not be equal to is the product computed in expr (5) and expr (7). i.e.

$$expr(5)\cdot expr(7). \tag{8}$$

Thus, for blockwise correction of 2-repeated low-density burst errors, the total number of linear combinations that h_j can not be equal to is the sum of linear combinations in (4) and (8). At worst, all these combinations might yield a distinct sum. Therefore, h_j can be added to the s^{th} sub- block of H provided that

$$q^r > expr(4) + expr.$$

For completing the s^{th} sub-block to length t, replacing j by t gives the result as stated in (2). ∎

Remark 4. By taking $s = 1$ in (2), the bound obtained in (2)coincides with the condition for existence of a code correcting 2-repeated low-density bursts of length b or less with weight w or less [5].

Remark 5. For $w = b$ the bound in (2) reduces to the upper bound on the number of parity-check digits for the existence of a code correcting 2-repeated bursts of length b or less occurring within a sub-block [8].

We conclude this section with an example.

Example 1. Consider a $(24, 10)$ binary code with a 14×24 parity-check matrix H given by

$$H = \begin{bmatrix}
1 0 0 0 0 0 0 0 0 0 0 0 \quad 0 0 1 0 0 0 0 0 0 0 0 0 \\
0 1 0 0 0 0 0 0 0 0 0 0 \quad 0 0 1 0 0 0 0 0 0 0 0 0 \\
0 0 1 0 0 0 0 0 0 0 0 0 \quad 0 0 1 0 0 0 0 0 0 0 1 0 \\
0 0 0 1 0 0 0 0 0 0 0 0 \quad 0 0 0 0 0 0 0 1 1 0 0 1 \\
0 0 0 0 1 0 0 0 0 0 0 0 \quad 0 0 0 0 0 1 1 0 1 1 0 0 \\
0 0 0 0 0 1 0 0 0 0 0 0 \quad 0 0 0 0 1 0 1 0 0 1 0 0 \\
0 0 0 0 0 0 1 0 0 0 0 0 \quad 0 0 0 0 1 0 1 0 1 0 0 1 \\
0 0 0 0 0 0 0 1 0 0 0 0 \quad 0 0 1 1 0 0 1 0 0 1 0 0 \\
0 0 0 0 0 0 0 0 1 0 0 0 \quad 0 0 0 1 1 1 0 0 1 0 1 1 \\
0 0 0 0 0 0 0 0 0 1 0 0 \quad 0 0 0 1 0 1 0 1 1 0 1 0 \\
0 0 0 0 0 0 0 0 0 0 1 0 \quad 0 0 0 1 0 1 1 1 1 1 0 0 \\
0 0 0 0 0 0 0 0 0 0 0 1 \quad 0 0 0 1 1 0 0 1 0 0 1 1 \\
0 0 0 0 0 0 0 0 0 0 0 0 \quad 1 0 0 0 1 1 0 1 1 1 0 1 \\
0 0 0 0 0 0 0 0 0 0 0 0 \quad 0 1 0 0 1 1 1 1 0 0 1 0
\end{bmatrix}$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 2 by taking $b = 3$, $w = 2$, $s = 2$, $t = 12$ over $GF(2)$. It has been verified though MS Excel program that the syndromes of all distinct 2-repeated bursts of length 3 or less with weight 2 or less whether in the same sub-block or in different sub-blocks are different, thereby ensuring that the code that is the null space of this matrix corrects all 2-repeated bursts of length 3 or less with weight 2 or less occurring within a sub-block. It should be noted that this code may not correct 2-repeated bursts of length 3 or less which are *not* 2-repeated low-density bursts of length 3 or less and weight 2 or less, e.g. the code does not correct the error (000000011111 000000000000) as its syndrome is the same as that of (000000000000 000100000000).

References

[1] Berardi L., Dass B.K., Verma Rashmi, *On 2-repeated Burst Error Detecting codes*, Journal of Statistical Theory and Practice, 3 (2) (2009), 381-391.

[2] Dass B.K., *A sufficient bound for codes correcting bursts with weight constraints*, Journal of the Association for Computing Machinery, 22 (4) (1975), 501-503.

[3] Dass B.K., *Burst Error Locating Codes,* J. Inf. and Optimization Sciences, 3 (1) (1982), 77-80.

[4] Dass B.K., *Low-Density Burst Error Locating Linear Codes,* IEE Proc., 129 (E) (4) (1984), 145-146.

[5] Dass B.K., Verma Rashmi, *Bounds for 2-Repeated Low-density Burst Error Correcting Linear Codes*, 2010 *communicated.*

[6] Dass B.K., Verma Rashmi, *Repeated Low-density Burst Error Detecting Codes,* 2010, Accepted for publication in the *Journal of the Korean Mathematical Society.*

[7] Dass B.K., Madan Surbhi, *Repeated Burst Error Locating Linear codes*, 2010, *Communicated.*

[8] Dass B.K., Madan Surbhi, *Blockwise Repeated Burst Error Correcting Linear Codes*, 2010, *Communicated*

[9] Dass B.K., Madan Surbhi, *Repeated Low-Density Burst Error Locating Linear Codes*, 2010, *Communicated.*

[10] Peterson W.W., Weldon, E.J.Jr., *Error-Correcting Codes*, Second Edition, The MIT Press, Mass, 1972.

[11] Sacks G.E., *Multiple Error Correction by Means of Parity-checks,* IRE Trans. Inform. Theory IT, 4 (1958), 145-147.

[12] Sharma B.D., Dass B.K., *Extended Varsharmov-Gilbert and sphere-packing bounds for burst correcting codes*, IEEE Trans. Inform. Theory, IT 20 (1974), 291-292.

[13] Wolf J., Elspas B., *Error-locating Codes-A New Concept in Error Control,* IEEE Transactions on Information Theory, 9 (2) (1963), 113-117.

[14] Wyner A.D., *Low-density-burst-correcting codes,* IEEE Trans. Information Theory, IT-9 (1963), 124.

Accepted: 22.07.2010