ON THE FINITE GROUPS WITH AVERAGE LENGTH 3 OF CONJUGACY CLASSES[1]

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Abstract. This article studies the problem of average length of conjugacy classes of finite groups, and classifies all finite groups with the average length 3 of conjugacy classes.

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1. Introduction

Let G be a finite group, $k(G)$ be the conjugacy number of G. Let $\mu(G) = |G|/k(G)$. By classes equation: $|G| = c_1 + c_2 + ... + c_k$, where c_i are length of conjugacy classes of elements of $G, i = 1, 2, ..., k$. In fact $\mu(G) = |G|/k(G)$ is average length of conjugacy classes of the finite group G. We know that conjugacy classes length can show some character of the group. Further more, the average length of a group has strong restriction to the group. Shi ([1]) proved that if $Z(G) = 1$, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$. Du ([2]) generalized this result that: if $|Z(G)|$ is a odd, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$. Du and Qian ([3]) proved that if $G' \not\cong Z_6$, then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4$, D_{18}, G_{18} and for any $x, y \in G$, $[x, y] \notin Z^*(G)$ (where $G_{18} \cong \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$, G_{18} is a group of order 18, it contains six conjugacy classes, there are $\{1\}$, $\{a, a^2\}$, $\{b, b^2\}$, $\{ab, a^2b^2\}$, $\{ab^2, a^2b\}$, $\{c, ca, ca^2, cb, cb^2, cab, ca^2b, cab^2, ca^2b^2\}$)

This paper generalizes the results of [3], we will get rid of the condition of $G' \not\cong Z_6$ in paper [3] and have the same results as paper [3].

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For the sake of convenience, let $G_{18} = \langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ c^{-1}ac = a^{-1},\ c^{-1}bc = b^{-1} \rangle$, $G^* = G - \{1\}$, $k(G)$ be the number of conjugacy classes of elements of G, a^G be the conjugacy class containing a, $\mathrm{Irr}(G)$ be the set of all irreducible characters of G, $\mathrm{Irr}^*(G)$ be the set of nonlinear irreducible characters of G. Let χ_P be the character of P by χ restricting on P, θ^G be the induced character of G, where $\chi \in \mathrm{Irr}(G)$, $\theta \in \mathrm{Irr}(G)$. Throughout this paper, all groups are finite.

2. Preliminaries

We need the following lemmas in this paper.

Lemma 2.1. [1]

(1) *Let G be a finite nonAbelian group, then $\mu(G) \geq 8/5$.*

(2) *Let $G = A \times B$, where A and B are finite groups. Then $\mu(G) = \mu(A)\mu(B)$.*

Lemma 2.2. [3] *Let H be a subgroup of a finite group G. Then*

(1) *$\mu(H) \leq \mu(G)$, and the equality holds if and only if for any $\chi \in Irr(G)$, $\chi_H \in Irr(H)$, and $H' = G'$.*

(2) *If $H \lhd G$, then $\mu(G/H) \leq \mu(G)$. $\mu(G/H) = \mu(G)$ if and only if $H \leq Z(G)$, and moreover, for any $x, y \in G$, $[x, y] \notin H^*$.*

Lemma 2.3. [4] *Let G be a finite group. Then $|G| = \sum_{\chi \in Irr(G)} \chi^2(1)$.*

Lemma 2.4. *Let G is a finite group. Then*

(1) *$\mu(G) = 2$ if and only if $G/Z(G) \simeq S_3$.*

(2) *Suppose $G' \not\simeq Z_6$. Then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4,\ D_{18},\ G_{18}$, and for any $x, y \in G,\ [x, y] \notin Z(G) - \{1\}$.*

Proof. See [3], Theorem 3.3.

3. Main theorem

Theorem 3.1. *Suppose that G is a finite group, if $\mu(G) = 3$, then $G' \not\simeq Z_6$.*

Proof. If $G' \cong Z_6 = Z_2 \times Z_3$, then $Z_2 \leq Z(G)$. Since $G' \cong Z_6$, G is solvable. Let $G = HL$, where H is a $\{2, 3\}$-Hall subgroup, L is a $\{2, 3\}'$-Hall subgroup. It is easy to see that $H \lhd G$ and L is an Abelian group. Moreover, we prove that $L \leq Z(G)$.

Let $Q \in Sly_3(G)$ and let $Z_3 = Q \cap G' = \langle u \rangle$, then $\langle u \rangle$ char $G' \lhd G$. So $\langle u \rangle \lhd G$.

Firstly, we prove that $L = C_L(u) \lhd G$. Because $\langle u \rangle \lhd G$, we have $C_L(u) \lhd G$. By n-c theorem, $L/C_L(u) \leq Aut(\langle u \rangle) = Z_2$, it follows that $C_L(u) = L$ since L is a $\{2,3\}'$-group.

Secondly, we prove that $L \leq Z(G)$.

For any element $a \in L$, $g \in G$, Since $a^{-1}a^g \in G' = Z_2 \times \langle u \rangle$, we have $a^g = ab$, where $b \in Z_2 \times \langle u \rangle$. Since $Z_2 \leq Z(G)$ and $L = C_L(u)$, we have $ab = ba$. If $b \neq 1$, then a^g is a $\{2,3\}'$- element, and the order of ab can be divided by 2 or 3, this is a contradiction. So $b = 1$, $a^g = a$, and $L \leq Z(G)$ follows.

Therefore, $G = H \times L$. By Lemma 2.1, we have $\mu(G) = \mu(H)\mu(L) = \mu(H)$.

In the following proof, we only consider $G = H = PQ$, where $P \in Syl_2(G)$, $Q \in Syl_3(G)$, clearly, $Q \lhd G$. Let $|G| = 2^n.3^m$. We will complete the proof of Theorem 3.1 in four steps.

Step 1. $\langle u \rangle \nleq Z(G)$.

Proof. Because $G' = Z_2 \times Z_3$, we have $P' = Z_2 = \langle z \rangle \leq Z(G)$. If $\langle u \rangle \leq Z(G)$, for any $a \in P$, $g \in G$, $g^{-1}aga^{-1} \in G' = Z_2 \times \langle u \rangle$. It implies that $g^{-1}ag = axy$, where $x \in Z_2$, $y \in \langle u \rangle$ and $xy \in Z(G)$. Since $\langle u \rangle$ is a subgroup of order 3, we have $g^{-1}a^3g = (ax)^3 \in P$. P is Sylow-2 subgroup of G implies that $g^{-1}ag \in P$. So $P \lhd G$, and $G = P \times Q$. In this case P and Q are not Abelian since $G' = Z_2 \times Z_3$. Furthermore, we have $P' = Z_2$, $Q' = Z_3$. Now we compute $k(Q)$, the number of conjugacy classes of Q. Since $|Q/Q'| = 3^{m-1}$ we have $k(Q) > 3^{m-1}$. Let $k(Q) = k$, clearly, for any $a \in Q$, $|a^Q| = 1$, or 3, and the number of conjugacy classes of elements with length 1 of Q is $|Z(Q)|$. Therefore,

$$3^m = |Q| = |Z(Q)| + 3(k - |Z(Q)|) = 3k - 2|Z(Q)|.$$

So we can conclude that $\mu(Q) = |Q|/k = 3 - 2|Z(Q)|/k$. Since $|Z(Q)| \leq 3^{m-2}$ and $k > |Q/Q'| = 3^{m-1}$, we have

$$\mu(Q) = |Q|/k > 3 - 2.3^{m-2}/3^{m-1} = 3 - 2/3 > 2.$$

We know from Lemma 2.1 (1) that $\mu(P) \geq 8/5$, therefore

$$3 = \mu(G) = \mu(P \times Q) = \mu(P).\mu(Q) > \mu(P) \geq (8/5).2 = 16/5 > 3,$$

a contradiction.

Step 2. Q is an Abelian subgroup.

Proof. We have know that $\langle u \rangle \lhd G$, so for any $g \in G$, $u^g = u^i \in \langle u \rangle$, $i = 1, 2$. It follows that $|G : C_G(u)| = |u^G| = 2$. Let $C_G(u) = P_1Q$, where $|P : P_1| = 2$, which implies that $P_1 \lhd P$. Let $P = \langle a, P_1 \rangle$. For any $x \in P_1$, $y \in Q$, we have $x^{-1}x^y \in Q \cap G' = \langle u \rangle$. Thus $x^y = u^i$. If $i \neq 0$, then the left is an elements of order 2 and the right is an element of order 6. Therefore, $i = 0$ and $xy = yx$. That means that $C_G(u) = P_1Q = P_1 \times Q$. Since $P = \langle a, P_1 \rangle$ and $a \notin C_G(u)$,

$|a^G| = 3$, or 6. Therefore $3^{m-1} \mid |C_G(a)|$. Let Q_1 is a Sylow-3 subgroup of $C_G(a)$, then $|Q : Q_1| = 3$. We prove Q_1 is an Abelian subgroup. For any x, $y \in Q_1$, $x^{-1}x^y = u^i \in Q_1 \cap G' \leq \langle u\rangle$. $i \neq 0$ means that $u \in Q_1 \leq C_G(a)$, and $a \in C_G(u)$ follows, it is a contradiction. So $i = 0$, that is $xy = yx$. Therefore Q_1 is an Abelian. Clearly, $u \in Z(Q)$. So $Q = \langle u, Q_1\rangle$, and Q is an Abelian subgroup.

Step 3. $G = (P\langle u\rangle) \times Q_1$.

Proof. Since $Q_1 \leq C_G(a)$ and $P_1Q = P_1 \times Q$, $Q_1 \leq C_G(P)$. Therefore, $Q_1 \lhd G$. Clearly, $Q_1 \cap \langle u\rangle = 1$, so $Q = \langle u\rangle \times Q_1$ and $Q_1 \lhd Z(G)$. Therefore $G = (P\langle u\rangle) \times Q_1$.

Step 4. The finial contradiction.

Proof. By Step 3, $G = (P\langle u\rangle) \times Q_1$, by Lemma 2.1, $\mu(G) = \mu(P\langle u\rangle).\mu(Q_1) = \mu(P\langle u\rangle)$. So in the following proof, we can think $G = P\langle u\rangle$. Because $\langle u\rangle$ is an abelian normal subgroup of G, for any $\chi \in Irr(G)$, we have $\chi(1) \mid |G : \langle u\rangle| = 2^n$. (See [4].)

Remember that

$$Z_2 = \langle z\rangle \leq Z(G).$$

Now, we compute the number of irreducible characters of G.

The irreducible characters of $\bar{G} = G/Z_2$ may be identified as irreducible characters of G. For any $\chi \in Irr(G)$, if $z \in ker\chi$, then $\chi \in Irr(G/Z_2)$. Therefore, $Irr(G) = Irr(G/Z_2) \cup \{\chi \in Irr(G) \mid z \notin ker\chi\}$. If $\chi(1) = 1$, since $z \in G'$, $\chi(z) = 1$ and $z \in ker\chi$, it follows that $\chi \in Irr(G/Z_2)$. So for any $\chi \in Irr(G)$, if $z \notin ker\chi$, then $\chi(1) > 1$.

Let $\bar{G} = G/Z_2$, $\bar{A} = \bar{P}_1\bar{Q}$, then $\bar{A}$ is an Abelian normal subgroup of $\bar{G}$. Hence $\chi(1) \mid |\bar{G} : \bar{A}| = |P : P_1| = 2$ ([4]). Therefore any $\chi \in Irr(\bar{G})$, $\chi(1) = 1$, or 2. Thus

$$2^{n-1}.3 = |\bar{G}| = |\bar{G}/\bar{G}'| + |Irr^*(\bar{G})|.2^2.$$

Since $|\bar{G}/\bar{G}'| = |\bar{P}| = 2^{n-1}$, it is easy to see that

$$|Irr^*(\bar{G})| = 2^{n-2}.$$

Therefore $\bar{G}$ has exactly $2^{n-1}+2^{n-2} = 3.2^{n-2}$ irreducible characters. We know that G has exactly $|G|/3 = 2^n$ irreducible characters, therefore G has $2^n - 3.2^{n-2} = 2^{n-2}$ other irreducible characters χ with $z \notin ker\chi$.

Let $\chi \in Irr(G)$ with $z \notin ker\chi$, ρ be the irreducible C-representation affording χ.

Since $P' = Z_2 = \langle z\rangle \leq Z(G)$, for $a \notin Z(P)$, there exists $g \in P$ such that $a^g = az$. $z \in Z(G)$ implies that $\rho(z) = \omega I$, where I is an unit matrix with $\chi(1)$ degree, ω is a root of unity. So

$$\chi(a) = \mathrm{tr}\rho(a) = \mathrm{tr}\rho(a^g) = \mathrm{tr}\rho(az) = \mathrm{tr}(\rho(z)\rho(a)) = \mathrm{tr}(\omega\rho(a)) = \omega\chi(a).$$

$z \notin ker\chi$ implies that $\omega \neq 1$. Therefore $\chi(a) = 0$.

We restrict χ in subgroup P. Then we have

$$(*)\quad (\chi,\chi)_P = \frac{1}{|P|}\sum_{a\in P}\chi(a)\chi(a^{-1}) = \frac{|Z(P)|\chi^2(1)}{|P|}$$

Hence

$$\chi^2(1) = |P|/|Z(P)|(\chi,\chi)_P.$$

Since P is non-Abelian,

$$|P:Z(P)| = 2^{n-s} \geq 4,$$

where $|Z(P)| = 2^s$. If $(\chi,\chi)_P > 1$, then $\chi(1) \geq 2^2$. If $\chi(1) = 2$, then $(\chi,\chi)_P = 1$ and $\chi \in Irr(P)$.

If any $\chi \in Irr(G)$ with $z \notin ker\chi$, $\chi(1) \geq 2^2$. Then we have

$$3.2^n = |G| = |G/Z_2| + \sum_{z\notin ker\chi} \chi^2(1) \geq 3.2^{n-1} + 2^{n-2}.2^4$$

Thus $3.2^{n-1} \geq 8.2^{n-1}$, a contradiction.

Therefore there exists at least one $\chi \in Irr(G)$ with $z \notin ker\chi$, $\chi(1) = 2$.

Now we prove, for any $\chi \in Irr(G)$ with $z \notin ker\chi$, $\chi(1) = 2$ or $\chi(1) = 4$. Let $\theta \in Irr(G)$ with $\theta(1) = 2$, $z \notin ker\theta$, we have known that $\theta \in Irr(P)$, and by (*), $|P:Z(P)| = 4$. Also by (*), for any $\theta \in Irr^*(P)$, we have $\theta(1) = 2$.

Since $|P|/|Z(P)| = 4$, by (*) for any $\chi \in Irr(G)$ with $\chi(1) > 2$, $z \notin ker\chi$, we have $(\chi,\chi)_P > 1$. Restricting χ on subgroup P, let $\chi_P = n_1\theta_1 + ... + n_t\theta_t$, where $\theta_i \in Irr(P)$. Then there is a $\theta_i(1) = 2$ since $z \notin ker\chi$. Let it be $\theta = \theta_1$, by Frobenius Reciprocity theorem, the induced character $\theta_1^G = n_1\chi + ..., +m_t\chi_t$. We know that $\theta_1^G(1) = |G:P|\theta_1(1) = 3.2 = 6$, $\chi(1) > 2$ and $\chi(1)$ is power of 2, therefore the only possible is $\chi(1) = 4$.

Suppose that there exists n_1 irreducible characters $\chi \in Irr(G)$ with $\chi(1) = 2$, $z \notin ker\chi$, n_2 irreducible characters $\chi \in Irr(G)$ with $\chi(1) = 4$, $z \notin ker\chi$, then $n_1 + n_2 = 2^{n-2}$. Therefore we have

$$3.2^n = |G| = |G/Z_2| + \sum_{z\notin ker\chi} \chi^2(1) = 3.2^{n-1} + \sum_{z\notin ker\chi} \chi^2(1) = 3.2^{n-1} + n_1.2^2 + n_2.2^4.$$

From here we have $2^{n-3} = 3.n_2$. It is a final contradiction.

Theorem 3.2. *Suppose that G is a finite group, Then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4, D_{18}, G_{18}$, and for any $x, y \in G, [x,y] \notin Z^*(G)$.*

Proof. By Theorem 3.1, $G' \ncong Z_6$. So by Lemma 2.4(2), the Theorem is true. The proof is completed.

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