NEW RESULTS ON REMOTALITY IN BANACH SPACES

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Abstract. A set E in a Banach space X is called remotal if for each $x \in X$, there exists an $e \in E$ such that $\|x - e\| = \sup\{\|x - e\| : e \in E\}$. If e is unique, E is called uniquely remotal. One of the main results of this paper is: a weakly closed bounded set E in a reflexive Banach space is uniquely remotal if and only if the weak closed convex hull of E is uniquely remotal.

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1. Introduction

Let X be any normed space and let E be a closed bounded subset of X. For $x \in X$ we define $D(x, E) = \sup\{\|x - e\| : e \in E\}$. We say that E is remotal in X if, for every $x \in X$ there exists $e \in E$ such that $\|x - e\| = D(x, E)$. It is clear that if X is finite dimensional then any closed bounded (compact) subset E is remotal. In fact, it is proved in [8] that "Every closed bounded subset of the normed space X is remotal if and only if X is finite dimensional". Many questions regarding remotality have been raised and some partial results are available, we refer the reader to [3], [8] and [7].

For $x \in X$, we shall denote by $F(x, E)$ the set of all $e \in E$ such that $\|x-e\| = D(x, E)$. For a set $E \subset X$ we denote the convex hull of E by $[E]$, the closed convex hull by $\overline{[E]}$ and the weak closed convex hull of E by $\overline{[E]}^w$. For $x, y \in X$ we denote the set $\{tx + (1 - t)y : t \in (0, 1)\}$ by $L(x, y)$ and the set $\{tx + (1 - t)y : t \in [0, 1]\}$

by $L[x,y]$. For $x \in X$ and $r > 0$, $S(x,r) = \{y \in X : \|y - x\| = r\}$ and $B(x,r) = \{y \in X : \|y-x\| \leq r\}$. A point $e \in E$ is called an extreme point of E if whenever $e \in L[x,y]$ for some $x, y \in E$ we have $e = x = y$. The set of all extreme points of E is denoted by $Ext(E)$. Finally, for a closed bounded set $E \subset X$ we denote the set $\{e \in E :$ there exists $x \in X$ such that $e \in F(x,E)\}$ by $F(E)$.

A strongly related concept is the proximinality concept; a set $E \subset X$ is said to be proximinal if for each $x \in X$ there exists $e \in E$ such that $\|x - e\| = \inf\{\|x - y\| : y \in E\}$.

2. Basic results

The proof of the following Lemma can be found in [8].

Lemma 2.1 *Let X be a Banach space, $x \in X$ and let E be a closed bounded subset of X. Then $D(x,E) = D(x, \overline{[E]})$.*

A similar result for $\overline{[E]}^w$ is valid, namely we have the following result:

Lemma 2.2 *Let E be a closed bounded subset of the Banach space X, then $D(x,E) = D(x, \overline{[E]}^w)$ for all $x \in X$.*

Proof. Since $E \subseteq \overline{[E]}^w$ we clearly have $D(x,E) \leq D(x, \overline{[E]}^w)$. On the other hand, let $z \in \overline{[E]}^w$ and let $(z_n) \subset [E]$ be such that $z_n \overset{w}{\to} z$, then $\|x - z\| \leq \liminf \|x - z_n| \leq D(x,[E])$. Since this is true for any $z \in \overline{[E]}^w$, we infer that $D(x, \overline{[E]}^w) \leq D(x,[E])$. But, by Lemma 2.1 we get that $D(x, \overline{[E]}^w) \leq D(x,E)$. This completes the proof of the lemma. ∎

The proof of the following Lemma can be found in [3].

Lemma 2.3 *Let $x \in X$ and suppose that $D(x,E)$ is attained by a unique element $e \in E$, then e is an extreme point of E.*

The proof of the following Corollary can be found in [8].

Corollary 2.4 *Let E be a closed bounded subset of the normed space X and let $x \in X$ be such that $F(x,E) \neq \phi$. If $e \in F(x,E)$ is such that $e \in L(y,z)$ for some $y, z \in E$, then $y, z \in F(x,E)$.*

The following proposition gives us more.

Proposition 2.5 *Let E be a closed bounded convex subset of the normed space X and let $e \in F(x,E)$ for some $x \in X$. If $e \in L(y,z)$ for some $y, z \in E$ then $L[y,z] \subset F(x,E)$.*

Proof. By Corollary 2.4 we know that $y, z \in F(x,E)$. So, let $e' \in L(y,z)$ and suppose that $e' \neq e$. Let $t, t' \in (0,1)$ be such that

$$e' = y + t'(z - y) \text{ and } e = y + t(z - y). \tag{2.1}$$

We assert that $e' \in F(x, E)$. For, we have three cases: either $t = t'$ or $t < t'$ or $t > t'$.

If $t = t'$ then $e = e'$ and we are done. If $t < t'$, then (2.1) implies that

$$e = y + \frac{t}{t'}(e' - y) \Rightarrow e \in L(e', y).$$

But according to Corollary 2.4 we have $e' \in F(x, E)$. As for the third case, if $t > t'$ then (2.1) implies

$$e = z + \frac{1-t}{1-t'}(e' - z) \Rightarrow e \in L(e', z).$$

Again Corollary 2.4 gives $e' \in F(x, E)$. This completes the discussion of the three cases and finishes the proof. ∎

Observe that the convexity condition in proposition 2.5 can be removed to get the following Corollary.

Corollary 2.6 *Let E be a closed bounded subset of the normed space X and let $e \in F(x, E)$ for some $x \in X$. If $e \in L(y, z)$ for some $y, z \in E$, then $L[y, z] \cap E \subset F(x, E)$.*

It is trivial to show that if $e \in F(x, E)$ then e is a boundary point of E. This observation together with Corollary 2.4 gives the following result.

Proposition 2.7 *Let X be a normed space, $x \in X$ and E be a closed bounded convex subset of X. If $e \in F(x, E)$ is such that $e \in L(x_1, x_2)$ for some $x_1, x_2 \in E$ then $L[x_1, x_2] \subset \partial E$; the boundary of E.*

Proposition 2.8 *Let X be a normed space, $x \in X$ and let E be a closed bounded subset of X. If $e \in F(x, E)$ then $e \in F(x + t(x - e), E)$ for all $t \geq 0$.*

Proof. Denote $x + t(x - e)$ by z and Observe that for any $e' \in E$ we have

$$\begin{aligned} \|z - e'\| &\leq \|x - e'\| + t\|x - e\| \leq \|x - e\| + t\|x - e\| \\ &= (1 + t)\|x - e\| = \|x + t(x - e) - e\| = \|z - e\|. \end{aligned}$$

This completes the proof. ∎

The following proposition gives us an equivalent condition that a point $e \in E$ is a farthest point from other points of X. We leave the easy proof to the reader.

Proposition 2.9 *Let E be a closed bounded subset of a normed space X and let $e \in E$. Then $e \in F(E)$ if and only if there exists $x_0 \in X$ and $r > 0$ such that $e \in S(x_0, r)$ and $E \subset B(x_0, r)$. In this case we have $e \in F(x_0, E)$ and $D(x_0, E) = r$.*

Observe that Corollary 2.6 and propositions 2.7 and 2.9 give us a clear image of farthest points.

The proof of the following proposition is left to the reader.

Proposition 2.10 *For $x \in X$, a normed space, let*

$$A_x = \bigcap_{e \in E} \{z \in E : \|z - x\| \geq \|x - e\|\} .$$

Then

i) $F(x, E) = A_x$*; including the case when they are both empty.*

ii) $F(E) = \bigcup_{x \in X} A_x$.

3. Convexity results

Convex sets play an important rule in the subject specially when X is a reflexive space. The following result gives us the relationship between remotality of the set and of its convex hull. We remark that the result of the Proposition is known in the literature, see [1].

Proposition 3.1 *Let E be a closed bounded subset of the normed space X. Then E is remotal if and only if $[E]$ is remotal.*

Proof. Suppose that E is remotal, then $[E]$ is remotal by virtue of lemma 2.1. For the converse suppose that $[E]$ is remotal and let $x \in X$ and $y \in F(x, [E])$. If $y \in E$ then $y \in F(x.E)$. If $y \notin E$ then there exist $\alpha_1, ..., \alpha_n \in (0, 1)$ and $y_1, ..., y_n \in E$ such that $y = \sum_{k=1}^{n} \alpha_k y_k$. But then

$$D(x, E) \;=\; \|x - y\| = \|\sum_{k=1}^{n} \alpha_k (x - y_k)\| \leq \sum_{k=1}^{n} \alpha_k \|x - y_k\|.$$

If $\|x - y_k\| < D(x, E)$ for any k then we would have $D(x, E) < D(x, E)$. Thus, $\|x - y_k\| = D(x, E)$ for all $k = 1, ..., n$. This proves that $y_k \in F(x, E)$. Since this is valid for all $x \in X$ we infer that E is remotal. ∎

Since $[E]$ is not necessarily closed even if E is closed, a natural question that comes to mind is: Is it true that E is remotal if and only if $\overline{[E]}$ is remotal?
In [8], this question was answered affirmatively when X is a reflexive space. Unfortunately, we do not have an answer for the general case.

Uniquely remotal sets are of special interest because of their connection to the geometry of Banach spaces. One conjecture concerning the problem is the following.

Conjecture 3.2 *Let X be a Banach space and let E be a closed bounded subset of X. If E is uniquely remotal in X then E is a singleton.*

Many attempts were made to solve this conjecture and many partial results were obtained. We refer the reader to [7] and [4] for some results.

The following two results discuss the relation between unique remotality of the set and the convex hull of it. The importance of these two results is a natural consequence of the easiness of studying convex sets rather than studying general sets. For example, closed bounded convex sets are weakly compact in reflexive spaces. In fact this observation was the key in proving that every closed bounded convex set in a reflexive space is proximinal.

Proposition 3.3 *Let E be a closed bounded subset of the Banach space X. Then E is uniquely remotal if and only if $[E]$ is uniquely remotal.*

Proof. If $[E]$ is uniquely remotal then trivially E is uniquely remotal by virtue of lemma 2.1 and proposition 3.1 and the fact that $E \subset [E]$. For the converse, suppose that E is uniquely remotal and let $\hat{x} \in F(x, E)$ for given $x \in X$. Then surely $[E]$ is remotal and $\hat{x} \in F(x, [E])$. Let $y \in F(x, [E])$. We assert that $y \in F(x, E)$ and this will finish the proof. If $y \in E$ then we are done. If $y \in [E] \backslash E$ then there exist $\alpha_1, ..., \alpha_n \in (0, 1)$ with the property that $\sum_{i=1}^{n} \alpha_i = 1$ and $e_1, ..., e_n \in E$ such that $y = \sum_{i=1}^{n} \alpha_i e_i$. Now,

$$D(x, E) \;=\; \|x - y\| = \left\| \sum_{i=1}^{n} \alpha_i (x - e_i) \right\| \leq \sum_{i=1}^{n} \alpha_i \|x - e_i\|.$$

We know that $\|x-e_i\| \leq D(x, E)$ for each i. If for any i we have $\|x-e_i\| < D(x, E)$ then we would have $D(x, E) = \|x - y\| < D(x, E)$ which is a contradiction. This proves that $e_i \in F(x, E), \forall i$. Since E is uniquely remotal we have $e_i = \hat{x}$ for all i. That is $y = \hat{x}$. This completes the proof. ∎

We remark that the result of this result in known in the literature, see [6].

Observe that this result says nothing about the relationship between unique remotality of E and of $\overline{[E]}$, the closed convex hull of E! In the following result we prove this relation but under the conditions that X is reflexive and that E is weakly closed.

Theorem 3.4 *Let X be a reflexive Banach space and let E be a weakly closed bounded subset of X. Then, E is uniquely remotal if, and only if $\overline{[E]}^w$ is uniquely remotal.*

Proof. Suppose that $\overline{[E]}^w$ is uniquely remotal and let $x \in X$. We assert that $F(x, E)$ is attained by a unique element $\hat{x} \in E$. Indeed, let $\hat{x} \in \overline{[E]}^w$ be the unique element of $\overline{[E]}^w$ such that $D(x, \overline{[E]}^w) = \|x - \hat{x}\|$. But according to lemma 2.3, $\hat{x}$ is an extreme point of $\overline{[E]}^w$. Since X is reflexive and E is weakly closed, we infer that all extreme points of $\overline{[E]}^w$ are elements of E, see [5]. Hence $\hat{x} \in E$. By lemma 2.2 we get $\hat{x} \in F(x, E)$.

It is clear that $F(x, E)$ is a singleton or otherwise $F(x, \overline{[E]}^w)$ would have more than one element which is a contradiction.

For the converse, suppose that E is uniquely remotal and let $x \in X$. Since $D(x,E) = D(x, \overline{[E]}^w)$, by Lemma 2.2, we have $F(x, \overline{[E]}^w) \neq \phi$. Let $y \in F(x,E) \cap F(x, \overline{[E]}^w)$ and note that this intersection is a singleton because E is uniquely remotal. Let r denote $D(x,E)$ and let $\hat{y} \in F(x, \overline{[E]}^w)$. We assert that $\hat{y} = y$. If $\hat{y}$ is an extreme point of $\overline{[E]}^w$ then $\hat{y} \in E$, hence $\hat{y} = y$. Thus, it suffices to show that $\hat{y}$ is an extreme point of $\overline{[E]}^w$.

Let H be the hyperplane that supports $S(x,r)$ at $\hat{y}$ and let $f \in X^*$ be such that $H = \{w \in X : f(w) = \lambda\}$ where $\lambda \in \mathbb{R}$. Without loss of generality, assume that $S(x,r) \subset \{z \in X : f(z) \leq \lambda\}$. We assert that $M := F(x, \overline{[E]}^w) \cap H$ is a weakly closed convex subset of X. Indeed, let $(m_n) \subset M$ be such that $m_n \to m$ weakly. We claim that $m \in M$: Since $f \in X^*$ and $m_n \xrightarrow{w} m$ we infer that $f(m_n) \longrightarrow f(m)$. But $f(m_n) = \lambda$ because $m_n \in H$. Hence $f(m) = \lambda$ and $m \in H$. Further, $\|m_n - x\| = r, \ \forall n$. But since $m_n \xrightarrow{w} m$ we have $\|m - x\| \leq r$. If $\|m - x\| < r$ then we have a contradiction to the facts that H supports $S(x,r)$ and $m \in H$. Hence, $\|m - x\| = r$. This completes the proof that $m \in M$ and M is weakly closed. For convexity, suppose that $w_1, w_2 \in M$ and let $\alpha \in (0,1)$ and let $w = \alpha w_1 + (1-\alpha) w_2$. Since $w_1, w_2 \in H$ we have $f(w_1) = f(w_2) = \lambda$ and hence $f(w) = \lambda$. That is $w \in H$. Moreover, since $w \in (w_1, w_2)$ we have necessarily $w \in F(x, \overline{[E]}^w)$ by virtue of Corollary 2.4. This proves our claim that M is convex.

In fact, M is an extremal subset of $F(x, \overline{[E]}^w)$. Indeed, if $w \in M$ is such that $w \in (w_1, w_2)$ for some $w_1, w_2 \in F(x, \overline{[E]}^w)$ then $w_1, w_2 \in S(x,r)$ and $f(w) = \alpha f(w_1) + (1-\alpha) w_2$ for some $\alpha \in (0,1)$. But $f(w) = \lambda$ and $f(w_i) \leq \lambda$ for $i = 1,2$. Hence $f(w_1) = f(w_2) = \lambda$. That is, $w_1, w_2 \in H$ hence they are in M. This proves that M is extremal.

Now being weakly closed and bounded in a reflexive space, M is the closed convex hull of its extreme points. But the extreme points of M (an extremal set) are extreme points of $\overline{[E]}^w$, hence of E because X is reflexive and E is weakly closed.

So, if M contains more than one element then the set of extreme points of M contains more than one element, say y_1 and y_2. Then $y_1, y_2 \in E$ and $y_1, y_2 \in S(x,r)$. This contradicts the fact that E is uniquely remotal. This shows that M is a singleton. In fact, $M = \{\hat{y}\}$. Consequently, $\hat{y}$ is an extreme point of M, hence of $\overline{[E]}^w$. This completes the proof of the Theorem. ∎

Question: If E is a closed bounded subset of the normed space X, is it true that E is uniquely remotal if and only if $\overline{[E]}$ is uniquely remotal?

4. Further results

Definition 4.1 If E is a subset of the Banach space X, we say that E is sectionally uniquely remotal (s.u.r.) if $E \cap Y$ is uniquely remotal in Y for every finite dimensional subspace $Y \subseteq X$. Here we use the convention that ϕ is uniquely remotal in any space.

Proposition 4.2 *E is s.u.r. if and only if E is a singleton.*

Proof. If E is a singleton then the result is clear. For the converse, suppose that $E \cap Y$ is uniquely remotal in Y for every finite dimensional space $Y \subseteq X$. Suppose n the way of contrary that E contains two elements, say $e_1, e_2 \in E$ and let $Y = span\{e_1, e_2\}$. By assumption, $E \cap Y$ is uniquely remotal in Y. But $E \cap Y$ is compact in Y, hence $E \cap Y$ is a singleton [4]. But $E \cap Y \supset \{e_1, e_2\}$ which is a contradiction. This proves that E must be a singleton. ∎

Proposition 4.3 *If $E \subset X$ is such that $span(E)$ is finite dimensional and if E is uniquely remotal in X, then E is a singleton.*

Closed bounded subsets E of Banach spaces need not be remotal, however elements $x \in X$ which attains their farthest distance from E always exist. In the sequel, we give some criteria for elements $x \in X$ which attains their farthest distance.

Definition 4.4 Let X be a Banach space and let E be a closed bounded subset of X. If $x \in X$, we say that E is $x-$compact if $\operatorname{diam}(E \backslash B_n(x)) \longrightarrow 0$ as $n \longrightarrow \infty$. Here $B_n(x) = \left\{ y \in X : \|y - x\| \leq D(x, E) - \frac{1}{n} \right\}$.

Proposition 4.5 *Let E be a closed bounded subset of a Banach space X. If E is $x-$compact, for some $x \in X$, then $D(x, E)$ is attained.*

Proof. Let $(e_n) \subset E$ be such that $\|x - e_n\| \longrightarrow r := D(x, E)$. The sequence (e_n) can be selected so that $\|x - e_n\| \uparrow r$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\operatorname{diam}(E \backslash B_N(x)) < \epsilon$. Such N exists because E is $x-$compact. Now let $M \in \mathbb{N}$ be such that $\|x - e_n\| > r - \frac{1}{N}$. Then for $n, m > M$ we have $\|e_n - e_m\| < \epsilon$ because $e_n, e_m \in E \backslash B_N(x)$. This shows that (e_n) is a Cauchy sequence is a Banach space E. Hence $e_n \longrightarrow e \in E$ and $\|x - e\| = \lim \|x - e_n\| = r$. In other words, $D(x, E)$ is attained by e. ∎

Corollary 4.6 *Let E be a closed bounded subset of a Banach space X. If E is $x-$compact for every $x \in X$, then E is remotal.*

We should remark that the converse of this Corollary is not necessarily true. That is, there are remotal sets which are not $x-$compact for some $x \in X$. The unit sphere provides a good and easy example on this observation. In fact, the unit sphere in any any Banach space is remotal but is not $0-$compact.

Proposition 4.7 *If the closed bounded set E is $x-$compact for some x in the Banach space X, then $D(x, E)$ is attained by a unique element $e \in E$.*

Proof. Let $x \in X$ and suppose that E is $x-$compact. We know that $D(x, E)$ is attained by some $e \in E$, see proposition 4.5. Suppose that $D(x, E)$ is attained by e and e' and we are to show that $e = e'$. We now that $\|x - e\| = \|x - e'\| > D(x, E) - \frac{1}{N}$ for all $N \in \mathbb{N}$. But this implies that $\operatorname{diam}(E \backslash B_N(x)) \geq \|e - e'\|$. But since E is $x-$compact, $\operatorname{diam}(E \backslash B_N(x)) \longrightarrow 0$. This implies that $\|e - e'\| = 0$ as claimed. ∎

The converse of this proposition is not necessarily true as can be seen from the following example: Let $X = \ell^2$ and let

$$E = \overline{\left\{ \frac{n}{n+1} \delta_n \right\} \cup \{\delta_1\}}.$$

It can be easily seen that E is not $0-$compact. However, $D(0, E)$ is attained by the unique element $\delta_1 \in E$.

Definition 4.8 Let X be a Banach space, E be a closed bounded subset of X and let $x \in X$. We say that E is partially $x-$compact if there exists $F \subset E$ such that $D(x, F) = D(x, E) := r$ and such that F is $x-$compact.

Theorem 4.9 *The closed bounded set E is partially $x-$compact for some x in the Banach space X if, and only if $D(x, E)$ is attained.*

Proof. Suppose that $D(x, E)$ is attained by some element $e \in E$ and take $F = \{e\}$. Then $D(x, E) = D(x, F)$ and, clearly, F is $x-$compact. For the converse, suppose that E is partially $x-$compact and let F be as in Definition 4.8. Then since F is $x-$compact, $D(x, F)$ is attained by some $e \in F \subset E$. This completes the proof. ∎

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