REDEFINED GENERALIZED FUZZY R-SUBGROUPS OF NEAR-RINGS

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Abstract. By means of a kind of new idea, we redefine generalized fuzzy R-subgroups of a near-ring and investigate some of its related properties. Some new characterizations are also given. In particular, we introduce the concepts of strong prime (semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings, and discuss the relationship between strong prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups and prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings.

Keywords: near-ring; prime (semiprime) R-subgroup; $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup; $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup.

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1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting.

A near-ring satisfying all axioms of an associative ring, expect for commutativity of addition and one of the two distributive laws. Abou-Zaid [1] introduced the concept of fuzzy subnear-ring and studied fuzzy ideals of near-rings. The concept was discussed further by many researchers, for example [3]-[7], [10], [11]. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee \mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat

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and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, the $(\in, \in \vee \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures, see [3], [4], [8], [9].

By means of a kind of new idea, we redefine generalized fuzzy R-subgroups of a near-ring and investigate some of its related properties. In particular, we introduce the concepts of strong prime (semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings, and discuss the relationship between strong prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups and prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings.

2. Preliminaries

A non-empty set R with two binary operation " $+$ " and " $\cdot$ " is called a *near-ring* if it satisfies:

(1) $(R, +)$ is a group,

(2) $(R, \cdot)$ is a semigroup,

(3) $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in R$.

We will use the word "near-ring" to mean " left near-ring" and denote xy instead of $x \cdot y$.

An R-subgroup H of a near-ring R is a subset of R such that

(i) $(H, +)$ is a subgroup of $(R, +)$,

(ii) $RH \subseteq H$,

(iii) $HR \subseteq H$.

If H satisfies (i) and (ii), then it is called a left R-subgroup of R. If H satisfies (i) and (iii), then it is called a right R-subgroup of R.

If I and J are R-subgroups of near-ring R. An R-subgroup P of R is called prime if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all R-subgroups I and J of R. An R-subgroup P of R is called semiprime if $I^2 \subseteq P$ implies $I \subseteq P$ for all R-subgroups I of R.

We next state some fuzzy logic concepts. Recall that a fuzzy set is a function $\mu : R \to [0, 1]$. For any $A \subseteq R$, the characteristic function of A is denoted by χ_A. We define μ^{-1} by $\mu^{-1}(x) = \mu(-x)$, for all $x \in R$.

A fuzzy set μ of S of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t and is denoted by x_t*. A fuzzy point x_t is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t\ q\ \mu$, then we write $x_t \in\vee\,\mathrm{q}\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we call $x_t\ \overline{\in}\mu$ (resp., $x_t\ \overline{q}\ \mu$). We note that the symbol $\overline{\in\vee\,\mathrm{q}}$ means that $\in\vee\,\mathrm{q}$ does not hold.

Definition 2.1. [1] A fuzzy set μ of R is called a *fuzzy right* (resp., *left*) *R-subgroup* of R if

(F1a) $\mu(x + y) \geq \mu \wedge \mu(y), \forall x, y \in R,$
(F1a') $\mu(-x) \geq \mu(x),\ \forall x \in R,$
(F1b) $\mu(xy) \geq \mu(x)$ (resp.,$\mu(yx) \geq \mu(x)$), $\forall x, y \in R.$

In what follows, a (fuzzy) R-subgroup means a (fuzzy) right R-subgroup and R is a near-ring unless otherwise specified.

Definition 2.2. [4] A fuzzy set μ of R is called an $(\in, \in\vee\,\mathrm{q})$-*fuzzy R-subgroup* of R if for all $t, r \in (0, 1]$ and $x, y \in R$,

(F2a) $x_t \in \mu$ and $y_r \in \mu$ imply $(x + y)_{t\wedge r} \in\vee\,\mathrm{q}\mu$,
(F2a') $x_t \in \mu$ implies $(-x)_t \in\vee\,\mathrm{q}\mu$,
(F2b) $x_t \in \mu$ implies $(xy)_t \in\vee\,\mathrm{q}\mu$.

Theorem 2.3. [4] *A fuzzy set μ of R is an $(\in, \in\vee\,\mathrm{q})$-fuzzy R-subgroup of R if and only if for any $x, y, a \in R$,*

(F3a) $\mu(x + y) \geq \mu(x) \wedge \mu(y) \wedge 0.5,$
(F3a') $\mu(-x) \geq \mu(x) \wedge 0.5,$
(F3b) $\mu(xy) \geq \mu(x) \wedge 0.5.$

Naturally, we consider the concept of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R by means of Davvaz's way.

Definition 2.4. A fuzzy set μ of R called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R if for all $t, r \in (0, 1]$ and for all $x, y \in R$,

(F4a) $(x + y)_{t\wedge r}\overline{\in}\mu$ implies $x_t\overline{\in} \vee \overline{q}\mu$ or $y_r\overline{\in} \vee \overline{q}\mu$,
(F4a') $(-x)_t\overline{\in}\mu$ implies $(-x)_t\overline{\in} \vee \overline{q}\mu$,
(F4b) $(xy)_{t\wedge r}\overline{\in}\mu$ implies $x_t\overline{\in} \vee \overline{q}\mu$.

Example 2.5. Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	b	a

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then $(R, +, \cdot)$ is a near-ring. Define a fuzzy set μ of R by $\mu(a) = 0.9$, $\mu(b) = 0.8, \mu(c) = 0.4$ and $\mu(d) = 0.6$. Thus, μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R.

Theorem 2.6. *A fuzzy set μ of R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R if and only if for any $x, y \in R$,*

(F5a) $\mu(x + y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$,

(F5a') $\mu(-x) \vee 0.5 \geq \mu(x)$,

(F5b) $\mu(xy) \vee 0.5 \geq \mu(x)$.

Proof. We only prove (F4a) $\Leftrightarrow$ (F5a). The others are similar.

(F4a1)$\Rightarrow$ (F5a) If there exist $x, y \in R$ such that $\mu(x + y) \vee 0.5 < t = \mu(x) \wedge \mu(y)$, then $0.5 < t \leq 1, (x + y)_t \overline{\in} \mu$, but $x_t \in \mu, y_t \in \mu$. By (F1), we have $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$. Then, ($t \leq \mu(x)$ and $t + \mu(x) \leq 1$) or ($t \leq \mu(y)$ and $t + \mu(y) \leq 1$). Thus, $t \leq 0.5$, contradiction.

(F5a)$\Rightarrow$ (F4a) Let $(x + y)_{t \wedge r} \overline{\in} \mu$, then $\mu(x + y) < t \wedge r$.

(1) If $\mu(x + y) \geq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \mu(y) < t \wedge r$, and consequently, $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t \overline{\in} \mu$ or $y_r \overline{\in} \mu$. Thus, $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$.

(2) If $\mu(x+y) < \mu(x) \wedge \mu(y)$ then by (F4), we have $0.5 \geq \mu(x) \wedge \mu(y)$. Putting $x_t \overline{\in} \mu$ or, $y_r \overline{\in} \mu$, then $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t \overline{q} \mu$ or $y_r \overline{q} \mu$, and thus, $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$. This completes the proof. ∎

3. Main results

In this Section, we introduce the concepts of generalized fuzzy R-subgroups of near-rings by means of a new way, which is different with the related topic.

Remark 3.1. Let μ and ν be any two fuzzy sets of R. Then

(i) If $x_t \in \mu$ implies $x_t \in \vee \mathrm{q}\ \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \subseteq \vee q\ \nu$.

(ii) If $x_t \overline{\in}\ \mu$ implies $x_t \overline{\in} \vee \overline{q}\ \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \supseteq \vee \overline{q}\ \nu$.

Proposition 3.2. *For any two fuzzy sets μ and ν of R.*

(i) *$\mu \subseteq \vee\ q\ \nu$ if and only if $\nu(x) \geq \min\{\mu(x), 0.5\}, \forall x \in R$;*

(ii) *$\mu \supseteq \vee\ \overline{q}\ \nu$ if and only if $\max\{\mu(x), 0.5\} \geq \nu(x), \forall x \in R$.*

Proof. (i) Let $\mu \subseteq \vee q\ \nu$. If there exists $x \in R$ such that $\nu(x) < t = \mu(x) \wedge 0.5$, then $x_t \in \mu$, but $x_t \overline{\in \vee \mathrm{q}} \nu$, contradiction.

Conversely, let $\nu(x) \geq \mu(x) \wedge 0.5, \forall x \in R$. If $\mu \overline{\subseteq \vee q} \nu$, then there exists $x_t \in \mu$, but $x_t \overline{\in \vee \mathrm{q}} \nu$, and so $\mu(x) \geq t$ and $\nu(x) < t < 0.5$, contradiction.

(ii) Let $\mu \supseteq \vee \overline{q}\, \nu$. If there exists $x \in R$ such that $\mu(x) \vee 0.5 < t = \nu(x)$, then $x_t \in \nu$, but $x_t \overline{\in} \mu$ and $t > 0.5$. Hence $x_t \overline{\in} \vee\ \overline{q}\nu$, and so $x_t \overline{q} \nu$, that is, $G(x) + t \leq 1$, and so $t \leq 0.5$, contradiction.

Conversely, let $\mu(x) \vee 0.5 \geq \nu(x), \forall x \in R$. If $\mu \overline{\supseteq \vee\ \overline{q}} \nu$, then there exists $x_t \overline{\in} \mu$, but $x_t \overline{\overline{\in} \vee\ \overline{q}} \nu$. Hence $\mu(x) < t, \nu(x) \geq t$ and $\nu(x) + t > 1$.

Case (1). $\mu(x) > 0.5$. Then $\mu(x) \geq \nu(x)$, contradiction.

Case (2). $\mu(x) \leq 0.5$. Then $0.5 \geq \nu(x)$. By $\nu(x) \geq t, 0.5 \geq \nu(x) \geq t$. But $2\nu(x) \geq \nu(x) + t > 1$, and so $\nu(x) > 0.5$, contradiction. ■

Now, we give the concepts of the product and sum of two fuzzy sets of R.

Definition 3.3. Let μ and ν be fuzzy sets of R. Then the sum of μ and ν is defined by

$$(\mu \circ \nu)(x) = \bigvee_{x=ab} (\mu(a) \wedge \nu(b))$$

and $(\mu \circ \nu)(x) = 0$ if x cannot be expressed as $x = ab$.

Definition 3.4. Let μ and ν be fuzzy sets of R. Then the sum of μ and ν is defined by

$$(\mu + \nu)(x) = \bigvee_{x=a+b} (\mu(a) \wedge \nu(b))$$

and $(\mu + \nu)(x) = 0$ if x cannot be expressed as $x = a + b$.

Now, by means of a new way, we consider another generalized fuzzy R-subgroup of near-rings, which is called a new $(\in, \in \vee\, \mathrm{q})$-fuzzy R-subgroups.

Definition 3.5. A fuzzy set μ of R is called a *new* $(\in, \in \vee\, \mathrm{q})$*-fuzzy R-subgroup* of R if it satisfies:

(F6a) $(\mu + \mu) \subseteq \vee q\mu$,

(F6a') $\mu^{-1} \subseteq \vee q\mu$,

(F6b) $(\mu \circ \chi_R) \subseteq \vee q\mu$.

Theorem 3.6. *A fuzzy set μ of R is a new* $(\in, \in \vee\, \mathrm{q})$*-fuzzy R-subgroup of R if and only if it satisfies* (F3a), (F3a') and (F3b).

Proof. Let μ be a new $(\in, \in \vee\, \mathrm{q})$-fuzzy R-subgroup of R. If there exist $x, y \in R$ such that $\mu(x + y) < t < \mu(x) \wedge \mu(y) \wedge 0.5$, then $t < 0.5, x_t \in \mu, y_t \in \mu$, but $(x + y)_t \overline{\in} \mu$, and so $(x + y)_t \overline{\in \vee\, \mathrm{q}} \mu$. But, $(\mu + \mu)(x + y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b)) \geq \mu(x) \wedge \mu(y) \geq t$, and so $(x + y)_t \in (\mu + \mu)$. Thus, $(x + y)_t \in \vee\, \mathrm{q}\mu$, contradiction. This proves (F3a) holds.

Now, if there exists $x \in R$ such that $\mu(-x) < t < \mu(x) \wedge 0.5$, then $\mu(x) \geq t$ and $t < 0.5$, but $(-x) \overline{\in} \mu$. Thus, $(-x)_t \overline{\in \vee\, \mathrm{q}} \mu$. But $\mu^{-1}(-x) = \mu(x) \geq t$, and so $(-x)_t \in \mu^{-1}$, which implies, $(-x)_t \in\vee\,\mathrm{q}\mu$, contradiction. This proves (F3a') holds.

The proof of (F3b) is similar to the proof of (F3a).

Conversely, μ satisfies (F3a), (F3a') and (F3b). Let $x_t \in (\mu+\mu)$, but $x_t\overline{\in\vee\,\mathrm{q}}\mu$. Then $\mu(x) < t$ and $\mu(x) < 0.5$. By definition, $(\mu+\mu)(x+y) = \bigvee_{x+y=a+b} (\mu(a)\wedge\mu(b))$. Since $0.5 > \mu(x) = \mu(a+b) \geq \mu(a) \wedge \mu(b) \wedge 0.5$, and so $\mu(x) \geq \mu(a) \wedge \mu(b)$. Thus, $t \leq (\mu+\mu)(x) \leq \bigvee_{x=a+b} \mu(x) = \mu(x)$, that is, $\mu(x) \geq t$, contradiction. This proves that (F6a)holds.

Now, let $x_t \in \mu^{-1}$, but $x_t\overline{\in\vee\,\mathrm{q}}\mu$, then $\mu(x) < t$ and $\mu(x) < 0.5$. Thus $\mu(-x) \geq \mu(x) \wedge 0.5 = \mu(x)$, and so $\mu(-x) = \mu(x)$. Hence $t \leq \mu^{-1}(x) = \mu(-x) = \mu(x)$, contradiction.

The proof of (F6b) is similar to the proof of (F6a).

Therefore, μ is a new $(\in, \in\vee\,\mathrm{q})$-fuzzy R-subgroup of R. ∎

The following is a consequence of Theorem 2.3 and 3.6.

Corollary 3.7. *The concepts of new $(\in, \in\vee\,\mathrm{q})$-fuzzy R-subgroups of R and $(\in, \in\vee\,\mathrm{q})$-fuzzy R-subgroups are equivalent, respectively.*

Next, by means of a new way, we consider another a generalized fuzzy R-subgroup of near-rings, which is called a new $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup.

Definition 3.8. A fuzzy set μ of R is called a *new $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy R-subgroup* of R if it satisfies:

(F7a) $\mu \supseteq \vee\overline{q}(\mu+\mu)$,

(F7a') $\mu \supseteq \vee\overline{q}\mu^{-1}$,

(F7b) $\mu \supseteq \vee\overline{q}(\mu \circ \chi_R)$.

Theorem 3.9. *A fuzzy set μ of R is a new $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R if and only if it satisfies* (F5a), (F5a') *and* (F5b).

Proof. Let μ be a new $(\in, \in\vee\,\mathrm{q})$-fuzzy R-subgroup of R. If there exist $x, y \in R$ such that $\mu(x+y) \vee 0.5 < t < \mu(x) \wedge \mu(y)\wedge$, then $t > 0.5, x_t \in \mu, y_t \in \mu$, but $(x+y)_t\overline{\in}\mu$, and so $(x+y)_t\overline{\in} \vee \overline{q}(\mu+\mu)$. Thus,

$$(*) \qquad (\mu+\mu)(x+y) < t$$

and

$$(**) \qquad (\mu+\mu)(x+y) + t \leq 1$$

But, $(\mu+\mu)(x+y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b)) \geq \mu(x) \wedge \mu(y) \geq t$, which implies, $(\mu+\mu)(x+y) \geq t$. By $(*)$ and $(**)$, we have $(\mu+\mu)(x+y) + t \leq 1$, and so $t \leq 0.5$, contradiction. This proves (F5a) holds. Similarly, we can prove (F5a') and (F5b).

Conversely, μ satisfies (F5a), (F5a') and (F5b). Let $x_t \in \mu$, but $x_t \overline{\overline{\in} \vee \overline{q}}(\mu+\mu)$. Then $\mu(x) < t$, but $(\mu+\mu)(x) \geq t$ and $(\mu+\mu)(x)+t > 1$, and so $(\mu+\mu)(x) > 0.5$. By definition, $(\mu+\mu)(x) = \bigvee_{x=a+b} (\mu(a) \wedge \mu(b))$. Since $0.5 \vee \mu(x) = \mu(a+b) \vee 0.5 \geq \mu(a) \wedge \mu(b)$, and so $0.5 \vee \mu(x) \geq \mu(a) \wedge \mu(b)$. Thus, $t \leq (\mu+\mu)(x) \leq \bigvee_{x=a+b} \mu(x) \vee 0.5$. Since $(\mu+\mu)(x) > 0.5$, then $\mu(x) \geq 0.5$, and so $\mu(x) \geq t$, contradiction. This proves that (F7a) holds. Similarly, we can prove (F7a') and (F7b) hold. Therefore, μ is a new $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroup of R. ∎

The following is a consequence of Theorem 2.5 and 3.9.

Corollary 3.10. *The concepts of new $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroups of R and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy R-subgroups are equivalent, respectively.*

4. Strong prime (semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups

In this Section, we introduce the concepts of strong prime (semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings. In particular, we discuss the relationship between strong prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups and prime (resp., semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings.

Definition 4.1. [8]

(i) An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R is called prime if for all $x, y \in R$ and $t \in (0, 1]$, we have

(P) $(xy)_t \in \mu \Rightarrow x_t \in \vee \mathrm{q}\mu$ or $y_t \in \vee \mathrm{q}\mu$.

(ii) An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R is called semiprime if for all $x \in R, t \in (0, 1]$, we have

(SP) $(x^2)_t \in \mu \Rightarrow x_t \in \vee \mathrm{q}\mu$.

Theorem 4.2. [8]

(i) *An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R is prime if for all $x, y \in R$, it satisfies:*

(P') $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge 0.5$.

(ii) *An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R is semiprime if for all $x \in R$, it satisfies:*

(SP') $\mu(x) \geq \mu(x^2) \wedge 0.5$.

For a fuzzy set μ of R and $t \in (0,1]$, the crisp set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called the *level subset* of μ.

Theorem 4.3. [8] *An* $(\in, \in \vee \mathrm{q})$*-fuzzy* R*-subgroup* μ *of* R *is prime (semiprime) if and only if* $\mu_t (\neq \emptyset)$ *is a prime (semiprime)* R*-subgroup of* R *for all* $t \in (0, 0.5]$*, respectively.*

Now, we give the concept of strong prime (semiprime) $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of near-rings.

Definition 4.4.

(i) An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup ρ of R is called strong prime if for every $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups μ and ν of R, it satisfies:

(P") $\mu \circ \nu \subseteq \rho$ implies $\mu \subseteq \rho$ or $\nu \subseteq \rho$.

(ii) An $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R is called strong semiprime if for every $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup μ of R, it satisfies:

(SP") $\mu \circ \mu \subseteq \rho$ implies $\mu \subseteq \rho$.

Theorem 4.5. *Let* μ *be a strong prime (semiprime)* $(\in, \in \vee \mathrm{q})$*-fuzzy* R*-subgroup of* R*. Then* $\mu_t (\neq \emptyset)$ *is a prime (semiprime)* R*-subgroup of* R *for all* $t \in (0, 0.5]$*, respectively.*

Proof. We only consider strong prime $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups. The case for strong semiprimeness is similar.

Let $t \in (0, 0.5]$ be such that μ_t is non-empty. Then μ_t is an R-subgroup of R. Now we show that μ_t is prime. Let I and J be two R-subgroups of R such that $IJ \subseteq \mu_t$. Then it is easy to see that t_I and t_J are two $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of R and that $t_I \circ t_J \subseteq \mu$. In fact, let $x \in R$. If $(t_I \circ t_J)(x) = 0$, then $(t_I \circ t_J)(x) = 0 \leq \mu(x)$. Otherwise, there exist $a, b \in R$ such that $x = ab$ and $t_I(a) \wedge t_J(b) \neq 0$. This implies $a \in I$ and $b \in J$, hence $x \in IJ \subseteq \mu_t$, that is, $\mu(x) \geq t$. Hence $(t_I \circ t_J)(x) = \bigvee_{x=yz} t_I(y) \wedge t_J(z) \leq t \leq \mu(x)$. Therefore, $t_I \circ t_J \subseteq \mu$. Since μ is a strong prime $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup of R, we have $t_I \subseteq \mu$ or $t_J \subseteq \mu$, this implies $I \subseteq \mu_t$ or $J \subseteq \mu_t$. This completes the proof. ∎

The following is a consequence of Theorems 4.3 and 4.5.

Theorem 4.6. *Every strong prime (semiprime)* $(\in, \in \vee \mathrm{q})$*-fuzzy* R*-subgroup of a near-ring is a prime (semiprime)* $(\in, \in \vee \mathrm{q})$*-fuzzy* R*-subgroup, respectively.*

Remark 4.7. The converse of Theorem 4.6 is not true in general as shown in the following example.

Example 4.8. Let $(\mathbb{Z}, +, \cdot)$ be the near-ring (it is also a ring) of all integers. Then $0.4_{(2)}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup of $\mathbb{Z}$ and non-empty subset $(0.4_{(2)})_t$

is a prime R-subgroup of R for all $t \in (0, 0.5]$. By Theorem 4.3, we know that $0.4_{(2)}$ is a prime $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroup of $\mathbb{Z}$, but it is not strong prime.

In fact, $0.4_{(3)} \circ 0.5_{(4)} \subseteq 0.4_{(2)}$, but $0.4_{(3)} \nsubseteq 0.4_{(2)}$ and $0.5_{(4)} \nsubseteq 0.4_{(2)}$, where both $0.4_{(3)}$ and $0.5_{(4)}$ are $(\in, \in \vee \mathrm{q})$-fuzzy R-subgroups of $\mathbb{Z}$.

5. Conclusions

In study the structure of a fuzzy algebraic system, we notice that the (fuzzy) R-subgroups with special properties always play an important role. In this paper, by means of a kind of new idea, we redefine generalized fuzzy R-subgroups of a near-ring and investigate some of its related properties.

We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of near-rings. In our future study of fuzzy structure of near-rings, may be the following topics should be considered:

(1) To describe soft near-rings and its applications;

(2) To establish an $(\in, \in \vee \mathrm{q})$-fuzzy spectrum of near-rings.

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