THE GROUPS OF TWO CLASSES OF CERTAIN CYCLICALLY PRESENTED GROUPS ARE ESSENTIALLY 3-GENERATED

Dedicated to Dr. D.L. Johnson

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Abstract. Two classes of cyclically presented groups were introduced in [3] and proven infinite for $n \succeq 3$ in [2]. I show that the groups of these classes of certain cyclically presented groups are essentially 3-generated. The groups G_n and H_n for $n = 3$ and 4 were shown to be 2-generated in [9] and [1], while the abelianized groups G_n^{ab} of G_n were dealt with in [8]. Naturally, the groups G_n and H_n for $n = 1$ and 2 are trivial. Showing that the groups of these two classes are essentially 3–generated has been the most difficult to solve thus far.

Keywords and phrases: Cyclically presented groups; 2-generated, 3-generated.

1. Introduction

The cyclically presented groups

$$H_n = \left\langle x_i | x_{i+1}^{-1} x_{i+2} x_{i+1}^{-1} x_{i+2} x_{i+1}^{-2} x_i x_{i+1}^{-1} x_i \right\rangle_n ,$$

where subscripts are reduced mod n to lie in the set $\{1, 2, ..., n\}$ belong to a second class of groups introduced in [3]. They have the Alexander polynomial $f(t) = 2t^2 - 5t + 2$, which is equal to the polynomial associated with the cyclically presented groups, of knot with 6 crossings denoted by 6_1 and is equivalent to the 2-bridge knot $b(9, 4)$ in [7].

The other class of groups

$$G_n = \left\langle x_i | x_{i+1}^{-1} x_{i+2} x_{i+1}^{-1} x_{i+2} x_i x_{i+1}^{-1} x_i \right\rangle_n$$

has the Alexander polynomial $f(t) = 2t^2 - 3t + 2$, which is equal to the polynomial associated with the cyclically presented groups, of knot with 5 crossings denoted by 5_2 in [7] were previously dealt with in [8] and [9].

A detailed study of the connections between these group presentations and closed 3-dimensional manifolds can be found in [4] and [5].

2. A general relation of each class of groups

Before we can approach the matter of showing that the groups of these classes of groups are essentially 3-generated, we will need to compute an essential formula for each class.

Lemma 2.1. *For any H_n,*

$$x_1x_2...x_{n-2}x_{n-1}x_n = 1,$$

and for any G_n,

$$x_nx_{n-1}x_{n-2}...x_2x_1 = 1,$$

for $n \in N$.

Proof. From the relations of

$$H_n = \left\langle x_i | x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_{i+1}^{-2}x_ix_{i+1}^{-1}x_i \right\rangle_n,$$

starting with the first relation derived from $i = n-1$, and post-multiplying the next one derived from $i = n-2$, successively, until you have multiplied the last relation derived from $i = n$, gives:

$$\begin{aligned}(1)\qquad x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1}x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-2}x_{n-2}x_{n-1}^{-1}x_{n-2}...\\ x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1,\end{aligned}$$

which means,

$$x_n^{-1}x_{n-1}^{-1}x_{n-2}^{-1}...x_2^{-1}x_1^{-1} = 1,$$

or

$$(2)\qquad x_1x_2...x_{n-2}x_{n-1}x_n = 1,$$

for $n \in N$.

Similarly, for any G_n,

$$(3)\qquad x_nx_{n-1}x_{n-2}...x_2x_1 = 1,$$

for $n \in N$. So all the $x_i's$ within each relation above are related.

3. The groups $H_n's$ are essentially 3-generated

Looking at patterns in the relations of the groups $H_n's$, we are choosing new generators and then reducing the number of generators to the least possible. We then simplify the presentations of the groups $H_n's$.

Theorem 3.1. *The groups $H_n's$ are essentially 3-generated.*

The relations of H_n are shown below:

$$x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1 = 1, \tag{4}$$
$$x_3^{-1}x_4x_3^{-1}x_4x_3^{-2}x_2x_3^{-1}x_2 = 1, \tag{5}$$
$$x_4^{-1}x_5x_4^{-1}x_5x_4^{-2}x_3x_4^{-1}x_3 = 1, \tag{6}$$
$$|, \tag{7}$$
$$|, \tag{8}$$
$$|, \tag{9}$$
$$x_{n-2}^{-1}x_{n-1}x_{n-2}^{-1}x_{n-1}x_{n-2}^{-2}x_{n-3}x_{n-2}^{-1}x_{n-3} = 1, \tag{10}$$
$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-2}x_{n-2}x_{n-1}^{-1}x_{n-2} = 1, \tag{11}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{12}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1. \tag{13}$$

Now, pre-multiplying these relations, starting with the first one to the 3^{th} to last one, we get

$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}...x_4^{-1}x_3^{-1}x_2^{-2}x_1x_2^{-1}x_1 = 1, \tag{14}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{15}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1. \tag{16}$$

However, from equation (2)

$$x_{n-1}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}...x_4^{-1}x_3^{-1}x_2^{-1} = x_nx_1,$$

and therefore H_n can be re-written as

$$(x_{n-1}^{-1}x_n)^2x_nx_1(x_2^{-1}x_1)^2 = 1, \tag{17}$$
$$(x_n^{-1}x_1)^2x_n^{-1}(x_n^{-1}x_{n-1})^2 = 1, \tag{18}$$
$$(x_1^{-1}x_2)^2x_1^{-1}(x_1^{-1}x_n)^2 = 1. \tag{19}$$

Having looked at patterns in the relators, we set

$$z = x_1^{-1}x_n^{-1}(x_n^{-1}x_{n-1})^2,$$
$$u = (x_{n-1}^{-1}x_n)^2x_nx_1x_2^{-1}x_1$$
$$t = (x_n^{-1}x_1)^2x_n^{-2}x_{n-1},$$
$$s = x_1^{-2}x_nx_1^{-1}x_n,$$
$$r = x_n^{-1}x_1.$$

Therefore,

$$x_1 = r^{-2}s^{-1}, \tag{20}$$
$$x_n = r^{-2}s^{-1}r^{-1}, \tag{21}$$
$$x_n^{-2}x_{n-1} = r^{-2}t, \tag{22}$$
$$x_2^{-1}x_1 = zu, \tag{23}$$
$$x_2 = r^{-2}s^{-1}u^{-1}z^{-1}, \tag{24}$$
$$x_{n-1} = (r^{-2}s^{-1}r^{-1})^2r^{-2}t. \tag{25}$$

We now simplify the presentation in terms of u, t and r, but we start by using the above equations to re-write the above relations in terms of r, z, u, s and t. Using relator (17), we get:

$$uzu = 1 \Rightarrow z = u^{-2}, \tag{26}$$

while from relator (18), we get:

$$s^{-1}z = 1 \Rightarrow z = s, \tag{27}$$

and from relator (19), we get:

$$(u^{-1}z^{-1})^2s = 1 \Rightarrow s = (zu)^2. \tag{28}$$

Now the relations (26) and (27) imply that $s = z = u^{-2}$. So z and s can be eliminated from the set of generators, as they can be expressed in terms of u. Hence the groups $H_n's$ are generated by t, u and r, thus the groups $H_n's$ are 3-generated. We know, however from [1], that the groups H_1 and H_2 are trivial, while H_3 and H_4 are 2-generated.

Theorem 3.2. *The groups $H_n's$ can be re-written as:*

$$\langle r, t, u | r^{-2}u^2r^{-3}t^2 \rangle .$$

Clearly,

$$z = x_1^{-1}x_n^{-1}(x_n^{-1}x_{n-1})^2, \tag{29}$$
$$z = sr^3sr^2(rsr^2(r^{-2}s^{-1}r^{-1})^2r^{-2}t)^2, \tag{30}$$
$$z = str^{-2}s^{-1}r^{-3}t, \tag{31}$$
$$tr^{-2}s^{-1}r^{-3}t = 1, \tag{32}$$

since $z = s$. Now, replacing s from the above relation (32), we get:

$$tr^{-2}u^2r^{-3}t = 1,$$

which gives

$$r^{-2}u^2r^{-3}t^2 = 1. \tag{33}$$

All the other manipulations of relations give this same relation, so

$$\left\langle r, t, u | r^{-2} u^2 r^{-3} t^2 \right\rangle.$$

4. The groups $G_n's$ are essentially 3-generated

Looking at patterns in the relations of the groups $G_n's$, we are choosing new generators and then reducing the number of generators to the least possible. We then simplify the presentations of the groups $G_n's$.

Theorem 4.1. *The groups $G_n's$ are essentially 3-generated.*

The relations of G_n are shown below:

$$x_2^{-1}x_3x_2^{-1}x_3x_1x_2^{-1}x_1 = 1, \tag{34}$$
$$x_3^{-1}x_4x_3^{-1}x_4x_2x_3^{-1}x_2 = 1, \tag{35}$$
$$x_4^{-1}x_5x_4^{-1}x_5x_3x_4^{-1}x_3 = 1, \tag{36}$$
$$|, \tag{37}$$
$$|, \tag{38}$$
$$|, \tag{39}$$
$$x_{n-2}^{-1}x_{n-1}x_{n-2}^{-1}x_{n-1}x_{n-3}x_{n-2}^{-1}x_{n-3} = 1, \tag{40}$$
$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-2}x_{n-1}^{-1}x_{n-2} = 1, \tag{41}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{42}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_nx_1^{-1}x_n = 1. \tag{43}$$

Now, pre-multiplying these relations, starting with the first one to the 3^{th} to the last one, we get

$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}x_{n-2}...x_4x_3x_1x_2^{-1}x_1 = 1, \tag{44}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{45}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_nx_1^{-1}x_n = 1. \tag{46}$$

However, from equation (3)

$$x_nx_{n-1}x_{n-2}...x_4x_3 = x_1^{-1}x_2^{-1},$$

and therefore G_n can be re-written as

$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1 = 1, \tag{47}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{48}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_nx_1^{-1}x_n = 1. \tag{49}$$

Having looked at patterns in the relators, we set

$$u = x_{n-1}^{-1} x_n x_{n-1}^{-1} x_1^{-1} x_2^{-1} x_1,$$
$$t = (x_n^{-1} x_1)^2 x_{n-1},$$
$$z = x_1 x_{n-1} x_n^{-1} x_{n-1},$$
$$s = x_n x_1^{-1} x_n$$
$$r = x_1^{-1} x_n.$$

Therefore,

$$x_2^{-1} x_1 = zu, \tag{50}$$
$$x_n = sr^{-1}, \tag{51}$$
$$x_2 = sr^{-2} u^{-1} z^{-1}, \tag{52}$$
$$x_{n-1} = r^2 t, \tag{53}$$
$$x_1 = sr^{-2}. \tag{54}$$

We now simplify the presentation in terms of u, r and t, but we start by using the above equations to re-write the above relations in terms of r, s, u, z and t. Using relator (47), we get:

$$uzu = 1 \Rightarrow z = u^{-2}, \tag{55}$$

while from relator (48), we get:

$$s^{-1} z = 1 \Rightarrow z = s, \tag{56}$$

and from relator (49), we get:

$$(u^{-1} z^{-1})^2 s = 1 \Rightarrow s = (zu)^2, \tag{57}$$

and so from equations (55) and (56), $s = z = u^{-2}$. This means z and s can be eliminated from the set of generators. Hence the groups $G_n's$ are generated by t, u and r. Thus the groups $G_n's$ are 3-generated. We know, however, that the groups G_1 and G_2 are trivial, while G_3 and G_4 are 2-generated as proven in the paper [9] derived from my 2000 thesis and also in [1]. It was also proven that G_5 is 3-generated in the latter.

Theorem 4.2. *The groups $G_n's$ can be re-written as:*

$$\left\langle t, r, u | r u^2 r^2 t^2 \right\rangle.$$

Clearly,

$$z = x_1 x_{n-1} x_n^{-1} x_{n-1}, \tag{58}$$
$$z = sr^{-2} r^2 t r s^{-1} r^2 t, \tag{59}$$
$$z = s t r s^{-1} r^2 t, \tag{60}$$
$$t r s^{-1} r^2 t = 1, \tag{61}$$

since $z = s$. Now, replacing s from relation (61), we get:

$$tru^2r^2t = 1, \tag{62}$$

which gives

$$ru^2r^2t^2 = 1. \tag{63}$$

All other manipulations of relations give this same relation, so

$$\left\langle t, r, u | ru^2r^2t^2 \right\rangle .$$

5. Remark

These groups, actually, have 'small' generating sets as purported by Dr. D.L. Johnson – Remark 4.4 in [3], which was questioned by Professor M.F. Newman in [6].

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