RELATED FIXED POINT THEOREM FOR SIX MAPPINGS ON THREE MODIFIED INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. Related fixed point theorems on two or three metric spaces have been proved in different ways. Sharma, Deshpande and Thakur were the first who have established related fixed point theorem for four mappings on two complete fuzzy metric spaces. Their work was maiden in this line. In this paper we obtain a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces. Of course this is a new result on this line.

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1. Introduction

Motivated by the potential applicability of fuzzy topology to quantum particle physics particularly in connection with both string and $e^{(\infty)}$ theory developed by El Naschie [10], [11], Park introduced and discussed in [24] a notion of intuitionistic fuzzy metric spaces which is based on the idea of intuitionistic fuzzy sets due to Atanassov [3] and the concept of fuzzy metric space given by George and Veeramani [18]. Actually, Park's notion is useful in modelling some phenomena where it is necessary to study the relationship between two probability functions. It has direct physics motivation in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by El Naschie [12], [13].

Alaca et al. [2] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [24] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [22]. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well known fixed point theorems of Banach [4] and Edelstein [9] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [13]. Turkoglu et al. [30] introduced the concept of compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces and gave some relations between the concepts of compatible maps and compatible maps of types (α) and (β).

Since the intuitionistic fuzzy metric space has extra conditions, Saadati, Sedghi and Shobe [28] modified the idea of intuitionistic fuzzy metric spaces and gave the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable.

Related fixed point theorems on two or three metric spaces were proved by Fisher[14],[15], Nung[23], Popa [24], Jain, Sahu and Fisher [19], Jain, Shrivastava and Fisher [20], Cho, Kang and Kim [5], Fisher and Murthy [16] and many others. Sharma, Deshpande and Thakur [29] established a related fixed point theorem for four mappings on two complete fuzzy metric spaces. Deshpande and Pathak [8] intuitionistically fuzzified the results of Sharma, Deshpande and Thakur [29] and proved a related fixed point theorem for two pairs of mappings on two intuitionistic fuzzy metric spaces. In this paper, we extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

2. Preliminaries

Definition 2.1. ([26]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Definition 2.2. ([26]) A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,

(iv) $a \diamond b = c \diamond d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Lemma 2.1. ([7]) *Consider the set* L^* *and operation* $\leq_{L^*}$ *defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \ \leq_{L^*})$ *is a complete lattice.*

Definition 2.3. ([3]) An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universe U is an object $A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) | \ u \in U\}$, where, for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$, and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$, if $c_i \in [0,1]$ such that $\sum\limits_{j=1}^{n} c_j = 1$, then it is easy that

$$c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) = \sum_{j=1}^{n} c_j \ (x_j, \ y_j) = \left(\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} c_j y_j \right) \in L^*. \tag{2.1}$$

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, \ x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $\mathrm{S} = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0, \ x) = 0 \diamond x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \ \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 2.4. ([6]) A triangular norm (t-norm) on L^* is a mapping $\tau : (L^*)^2 \to L^*$ satisfying the following conditions:

$(\forall \ x \in L^*)(\tau(x, 1_{L^*}) = x)$ (boundary condition),
$(\forall (x,y) \in (L^*)^2)(\tau(x,y) = \tau(y,x))$ (commutativity),
$(\forall (x,y,z) \in (L^*)^3)(\tau(x, \tau(y,z)) = \tau(\tau(x,y), z))$ (associativity),
$(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x')$ and $(y \leq_{L^*} y' \to \tau(x,y) \leq_{L^*} \tau(x',y'))$ (monotonicity).

Definition 2.5. ([6], [7]) A continuous t-norm τ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on [0, 1] such that, for all $x = (x_1, x_2), \ y = (y_1, y_2) \in L^*$,

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now, define a sequence τ^n recursively by $\tau^1 = \tau$ and

$$\tau^n(x^{(1)}, \ ..., x^{(n+1)}) \ = \tau(\tau^{n-1}(x^{(1)}, \ ..., \ x^{(n)}, \ x^{(n+1)}) \text{ for } n \geq 2 \text{ and } x^{(i)} \in L^*.$$

Definition 2.6. ([28]) Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{M}_{M,N}, \tau)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, τ is a continuous t-representable and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 2.3) satisfying the following conditions for every $x, \ y \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x, y, t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{M}_{M,N}(x, y, t) = 1_{L^*}$ if and only if $x = y$;

(c) $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$;

(d) $\mathcal{M}_{M,N}(x, y, t + s) \geq_{L^*} \tau(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, s))$;

(e) $\mathcal{M}_{M,N}(x, y, \cdot) : (0, \infty) \to L^*$ is continuous.

In this case, $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy metric.*

Here,

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)).$$

Example 2.1. ([28]) Let (X, d) be a metric space. Denote

$$\tau(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,\ N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{ht^n}{ht^n\ +\ md(x,\ y)}, \frac{md(x,\ y)}{ht^n\ +\ md(x,\ y)} \right)$$

for all $t, h, m, n \in R^+$.

Then, $(X, \mathcal{M}_{M,N}, \tau)$ is an intuitionistic fuzzy metric space.

Example 2.2. ([28]) Let $X = N$. Define

$$\tau(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{Bmatrix} \left(\frac{x}{y}, \frac{y - x}{y} \right) & \text{if } x \leq y, \\ \left(\frac{y}{x}, \frac{x - y}{x} \right) & \text{if } y \leq x, \end{Bmatrix}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,\ N}, \tau)$ is an intuitionistic fuzzy metric space.

Definition 2.7. ([28]) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\ N}, \tau)$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$$

and for each $n, m \geq n_0$, here N_s is the standard negator. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \tau)$ and denoted by $x_n \overset{\mathcal{M}_{M,N}}{\longrightarrow} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \to 1_{L^*}$ whenever $n \to \infty$ for every $t > 0$. An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 2.2. ([27]) *Let $\mathcal{M}_{M,\ N}$ be an intuitionistic fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x, y, t)$ is non-decreasing with respect to t, in $(L^*, \leq_{L^*})$, for all x, y in X.*

Lemma 2.3. ([1]) *Let $(X, \mathcal{M}_{M,N}, \tau)$ be a modified intuitionistic fuzzy metric space. For each $\lambda \in (0, 1)$, define the map $E_\lambda : X^2 \to R^+ \cup \{0\}$ by*

$$E_\lambda(x, y) = \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) >_{L^*} (1 - \lambda, \lambda)\},$$

then

(a) *For each $\lambda \in (0, 1)$, we have a $\mu \in (0, 1)$ such that*

$$E_\lambda(x_1, x_n) \leq E_\mu(x_1, x_2) + E_\mu(x_2, x_3) + \cdots + E_\mu(x_{n-1}, x_n),$$

for any $x_1, x_2, x_3, ..., x_n \in X$.

(b) *The sequence $\{x_n\}_{n\in N}$ in X is convergent to x if and only if $E_\lambda(x_n, x) \to 0$.*

Also, the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence in X if and only if it is a Cauchy sequence with respect to E_λ.

Lemma 2.4. ([21]) *Let $(X, \mathcal{M}_{M,\ N}, \tau)$ be an intuitionistic fuzzy metric space. If for a sequence $\{x_n\}$ in X, there exists $k \in (0, 1)$ such that*

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, kt) \geq_{L^*} \mathcal{M}_{M,N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and for all } t,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $(X, \mathcal{M}_{M,\ N}, \tau)$ be an intuitionistic fuzzy metric space. Let for a sequence $\{x_n\}$ in X, there exists $k \in (0, 1)$ such that

$$\mathcal{M}_{M,\ N}(x_n, x_{n+1}, kt) \geq_{L^*} \mathcal{M}_{M,\ N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and } t,$$

then

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, t) \geq_{L^*} \mathcal{M}_{M,N}\left(x_{n-1}, x_n, \frac{t}{k}\right) \geq_{L^*} \mathcal{M}_{M,N}\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right)$$
$$\ldots \geq_{L^*} \mathcal{M}_{M,N}(x_0, x_1, \frac{t}{k^n}), \text{ for all } n.$$

Now

$$\begin{aligned}
E_\lambda(x_{n+1}, x_n) &= \inf\{t > 0 : \mathcal{M}_{M,N}(x_{n+1}, x_n, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&\leq \inf\{t > 0 : \mathcal{M}_{M,N}\left(x_1, x_0, \frac{t}{k^n}\right) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= \inf\{k^n t > 0 : \mathcal{M}_{M,N}(x_1, x_0, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k^n \inf\{t > 0 : \mathcal{M}_{M,N}(x_1, x_0, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k^n \ E_\lambda(x_0, x_1). \\
E_\lambda(x_{n+1}, x_n) &\leq k^n E_\lambda(x_0, x_1) \ldots (A)
\end{aligned}$$

Again from Lemma 2.3, for $\lambda \in (0,1)$, there exists $\mu \in (0,1)$ such that

$$\begin{aligned}
E_\lambda(x_n, x_{n+p}) &\leq E_\mu(x_n, x_{n+1}) + E_\mu(x_{n+1}, x_{n+2}) + \cdots + E_\mu(x_{n+p-1}, x_{n+p}) \\
&\leq k^n \ E_\mu(x_0, x_1) + k^{n+1} \ E_\mu(x_0, x_1) + \cdots + k^{n+p-1} \ E_\mu(x_0, x_1), \\
&\qquad \text{using (A)} \\
&= (k^n + k^{n+1} + \cdots + k^{n+p-1}) E_\mu(x_0, x_1), \\
&= \frac{k^n}{1-k} \ E_\mu(x_0, x_1), \text{ as } 0 < k < 1,
\end{aligned}$$

which tends to 0, as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X. ∎

Lemma 2.5. ([21]) *In an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\ N}, \tau)$, if for some x, y in X there exists $k \in (0,1)$ such that*

$$\mathcal{M}_{M,\ N}(x, y, kt) \geq_{L^*} \mathcal{M}_{M,\ N}(x, y, t), \text{for all } t,$$

then $x = y$.

Proof. Let for $\lambda \in (0,1)$

$$\begin{aligned}
E_\lambda(x, y) &= \inf\{t > \ 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&\leq \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t/k) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= \inf\{kt > 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k \, E_\lambda(x, y).
\end{aligned}$$

Therefore, $E_\lambda(x, y) = 0$. Hence $x = y$. ∎

Sharma, Deshpande and Thakur [29] established the following related fixed point theorem for four mappings on two complete fuzzy metric spaces.

Theorem A. *Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete fuzzy metric spaces. Let A, B be mappings from X into Y and S, T be mappings from Y into X satisfying the inequalities:*

(i) $M_1(SAx, TBx', kt) \geq M_1(x, x', t) * M_1(x, SAx, t) * M_1(x', TBx', t)$
$* M_1(SAx, TBx', t)$

(ii) $M_2(BSy, ATy', kt) \geq M_2(y, y', t) * M_2(y, BSy, t) * M_2(y', ATy', t)$
$* M_2(BSy, ATy', kt)$

for all x, x' in X and y, y' in Y. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Deshpande and Pathak [8] intuitionistically fuzzify the results of Sharma, Deshpande and Thakur [29] and proved the following:

Theorem B. *$(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two complete intuitionistic fuzzy metric spaces. Let A, B be mappings from X into Y and let S, T be mappings from Y into X satisfying the inequalities:*

(i) $M_1(SAx, TBx\prime, kt) \geq M_1(x, x\prime, t) * M_1(x, SAx, t) * M_1(x\prime, TBx\prime, t)$
$*M_1(SAx, TBx\prime, t)$

and

$N_1(SAx, TBx\prime, kt) \leq N_1(x, x\prime, t) \diamond N_1(x, SAx, t) \diamond N_1(x\prime, TBx\prime, t)$
$\diamond N_1(SAx, TBx\prime, t)$

(ii) $M_2(BSy, ATy\prime, kt) \geq M_2(y, y\prime, t) * M_2(y, BSy, t) * M_2(y\prime, ATy\prime, t)$
$*M_2(BSy, ATy\prime, t)$

and

$N_2(BSy, ATy\prime, kt) \leq N_2(y, y\prime, t) \diamond N_2(y, BSy, t) \diamond N_2(y\prime, ATy\prime, t)$
$\diamond N_2(BSy, ATy\prime, t)$

for all $x, x\prime$ in X and $y, y\prime$ in Y. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

We extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

3. Main result

Theorem 3.1. *Let $(X, \mathcal{M}_{M_1,\ N_1}, \tau)$, $(Y, \mathcal{M}_{M_2,\ N_2}, \tau)$ and $(Z, \mathcal{M}_{M_3,\ N_3}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A, B be continuous mappings from X into Y, let S, T be continuous mappings from Y into Z and let P, Q be continuous mappings from Z into X satisfying the inequalities:*

(3.1) $$\mathcal{M}_{M_1,N_1}(PSAx, QTBx', kt) \geq_{L^*} \mathcal{M}_{M_1,\ N_1}(x, x', t) * \mathcal{M}_{M_1,N_1}(x, PSAx, t) * \mathcal{M}_{M_1,N_1}(x', QTBx', t) * \mathcal{M}_{M_1,N_1}(PSAx, QTBx', t)$$

(3.2) $$\mathcal{M}_{M_2,N_2}(APSy, BQTy', kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(y, y', t) * \mathcal{M}_{M_2,N_2}(y, APSy, t) * \mathcal{M}_{M_2,N_2}(y', BQTy', t) * \mathcal{M}_{M_2,N_2}(APSy, BQTy', t)$$

(3.3) $$\mathcal{M}_{M_3,N_3}(SAPz, TBQz', kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(z, z', t) * \mathcal{M}_{M_3,N_3}(z, SAPz, t) * \mathcal{M}_{M_3,N_3}(z', TBQz', t) * \mathcal{M}_{M_3,N_3}(SAPz, TBQz', t)$$

for all x, x' in X, y, y' in Y and z, z' in Z, $t > 0$ and $k \in (0, 1)$, then PSA and QTB have a unique common fixed point u in X, APS and BQT have a unique common fixed point v in Y and SAP and TBQ have a unique common fixed point w in Z. Further, $Au = Bu = v, Sv = Tv = w$ and $Pw = Qw = u$.

Proof. Let $x = x_0$ be an arbitrary point in X and define sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively as follows:

Choose a point $z_1 = Sy_1$, a point $y_1 = Ax_0$, a point $x_1 = Pz_1$, a point $z_2 = Ty_2$, a point $y_2 = Bx_1$ and a point $x_2 = Qz_2$. In general, having chosen x_{2n-2} in X, choose a point $y_{2n-1} = Ax_{2n-2}$, a point $y_{2n} = Bx_{2n-1}$, a point $z_{2n-1} = Sy_{2n-1}$, a point $z_{2n} = Ty_{2n}$, a point $x_{2n-1} = Pz_{2n-1}$ and a point $x_{2n} = Qz_{2n}$ for all $n = 1, 2, ...$

Applying inequality (3.1), we have

(3.4) $$\begin{aligned} \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, kt) &= \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, kt) \\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, t) \\ &* \mathcal{M}_{M_1,\ N_1}(x_{2n-1}, QTBx_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, t) \\ &= \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, t) \\ &* \mathcal{M}_{M_1,N_1}(x_{2n-1}, x_{2n}, t) * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, t) \\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, t) \end{aligned}$$

Similarly, we have

(3.5) $$\mathcal{M}_{M_1,N_1}(x_{2n+2}, x_{2n+1}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, t) * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n+2}, t).$$

Thus, from (3.4) and (3.5), it follows that

$$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t) * \mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, t),$$

for $n = 1, 2, ...$

Consequently, for positive integers n, p we have

$$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t) * \mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, t/k^p).$$

Thus, since $\mathcal{M}_{M_1,\ N_1}(x_{n+1}, x_{n+2}, kt) \to 1_{L^*}$ as $p \to \infty$, we have

(3.6) $$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t)$$

Similarly, applying inequality (3.2) and (3.3), we have

$$\mathcal{M}_{M_2,N_2}(y_{n+1}, y_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(y_n, y_{n+1}, t) \tag{3.7}$$

$$\mathcal{M}_{M_3,N_3}(z_{n+1}, z_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(z_n, z_{n+1}, t) \tag{3.8}$$

By Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in a complete intuitionistic fuzzy metric space X and so has a limit u in X. It follows similarly that the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in complete intuitionistic fuzzy metric space Y and Z and so have limits v in Y and w in Z.

Using (3.1), we have

$$\begin{aligned}\mathcal{M}_{M_1,N_1}(PSAx_{2n}, u, kt) &\geq_{L^*} \mathcal{M}_{M_1,N_1}(PSAx_{2n}, x_{2n}, \tfrac{kt}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2})\\ &= \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, \tfrac{kt}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2})\\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, \tfrac{t}{2})\\ &\qquad * \mathcal{M}_{M_1,N_1}(x_{2n-1}, QTBx_{2n-1}, \tfrac{t}{2})\\ &* \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2})\\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, \tfrac{t}{2})\\ &* \mathcal{M}_{M_1,N_1}(x_{2n-1}, x_{2n}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2})\end{aligned}$$

Taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \mathcal{M}_{M_1,\ N_1}(PSAx_{2n}, u, kt) \to 1_{L^*}.$$

Thus, we have

$$\lim_{n\to\infty} PSAx_{2n} = u = \lim_{n\to\infty} PSy_{2n+1} \tag{3.9}$$

Similarly, we can prove that

$$\lim_{n\to\infty} QTBx_{2n-1} = u = \lim_{n\to\infty} QTy_{2n} \tag{3.10}$$

$$\lim_{n\to\infty} APSy_{2n-1} = v = \lim_{n\to\infty} APz_{2n-1} \tag{3.11}$$

$$\lim_{n\to\infty} BQTy_{2n} = v = \lim_{n\to\infty} BQz_{2n} \tag{3.12}$$

$$\lim_{n\to\infty} SAPz_{2n} = w = \lim_{n\to\infty} SAx_{2n} \tag{3.13}$$

$$\lim_{n\to\infty} TBQz_{2n-1} = w = \lim_{n\to\infty} TBx_{2n-1} \tag{3.14}$$

Since A and B are continuous, we have

$$\lim_{n\to\infty} Ax_{2n} = Au = v, \quad \lim_{n\to\infty} Bx_{2n-1} = Bu = v. \tag{3.15}$$

Using inequality (3.1), we have

$$\mathcal{M}_{M_1,N_1}(PSAu, QTBx_{2n-1}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(u, PSAu, t)$$
$$* \mathcal{M}_{M_1,N_1}(x_{2n-1}, QTBx_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(PSAu, QTBx_{2n-1}, t).$$

Letting $n \to \infty$ and using (3.10), we have

$$\mathcal{M}_{M_1,\ N_1}(PSAu, u, kt) \geq_{L^*} \mathcal{M}_{M_1,\ N_1}(u, PSAu, t).$$

Therefore, by Lemma 2.5 and using (3.15), we have $PSAu = u = PSv$.

Using inequality (3.1), we have

$$\mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBu, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, u, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, t)$$
$$* \mathcal{M}_{M_1,N_1}(u, QTBu, t) * \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBu, t).$$

Letting $n \to \infty$ and using (3.9), we have

$$\mathcal{M}_{M_1,N_1}(u, QTBu, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, QTBu, t).$$

Therefore, by Lemma 2.5 and using (3.15), we have $QTBu = u = QTv$.

Since S and T are continuous, we have

$$\lim_{n\to\infty} Sy_{2n-1} = Sv = w, \quad \lim_{n\to\infty} Ty_{2n} = Tv = w. \tag{3.16}$$

Using inequality (3.2), we have

$$\mathcal{M}_{M_2,\ N_2}(APSv, BQTy_{2n}, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, y_{2n}, t) * \mathcal{M}_{M_2,\ N_2}(v, APSv, t)$$
$$* \mathcal{M}_{M_2,\ N_2}(y_{2n}, BQTy_{2n}, t) * \mathcal{M}_{M_2,\ N_2}(APSv, BQTy_{2n}, t).$$

Letting $n \to \infty$ and using (3.12), we have

$$\mathcal{M}_{M_2,\ N_2}(APSv, v, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, APSv, t).$$

Therefore, by Lemma 2.5 and using (3.16), we have $APSv = v = APw$.

Using inequality (3.2), we have

$$\mathcal{M}_{M_2,N_2}(APSy_{2n-1}, BQTv, kt)$$
$$\geq_{L^*} \mathcal{M}_{M_2,N_2}(y_{2n-1}, v, t) * \mathcal{M}_{M_2,N_2}(y_{2n-1}, APSy_{2n-1}, t)$$
$$* \mathcal{M}_{M_2,N_2}(v, BQTv, t) * \mathcal{M}_{M_2,N_2}(APSy_{2n-1}, BQTv, t).$$

Letting $n \to \infty$ and using (3.11), we have

$$\mathcal{M}_{M_2,\ N_2}(v, BQTv, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, BQTv, t).$$

Therefore, by Lemma 2.5 and using (3.16), we have $BQTv = v = BQw$.

Since P and S are continuous, we have

$$\lim_{n\to\infty} Pz_{2n} = Pw = u, \quad \lim_{n\to\infty} Qz_{2n-1} = Qw = u. \tag{3.17}$$

Using inequality (3.3), we have

$$\mathcal{M}_{M_3,N_3}(SAPw, TBQz_{2n-1}, kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(w, z_{2n-1}, t) * \mathcal{M}_{M_3,N_3}(w, SAPw, t) \\ *\mathcal{M}_{M_3,N_3}(z_{2n-1}, TBQz_{2n-1}, t) * \mathcal{M}_{M_3,N_3}(SAPw, TBQz_{2n-1}, t).$$

Letting $n \to \infty$ and using (3.14), we have

$$\mathcal{M}_{M_3,\ N_3}(SAPw, w, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(w, SAPw, t).$$

Therefore, by Lemma 2.5 and using (3.17), we have $SAPw = w = SAu$.

Using inequality (3.3), we have

$$\mathcal{M}_{M_3,\ N_3}(SAPz_{2n}, TBQw, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(z_{2n}, w, t) * \mathcal{M}_{M_3,\ N_3}(z_{2n}, SAPz_{2n}, t) \\ * \mathcal{M}_{M_3,\ N_3}(w, TBQw, t) * \mathcal{M}_{M_3,\ N_3}(SAPz_{2n}, TBQw, t).$$

Letting $n \to \infty$ and using (3.13), we have

$$\mathcal{M}_{M_3,\ N_3}(w, TBQw, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(w, TBQw, t).$$

Therefore, by Lemma 2.5 and using (3.17), we have $TBQw = w = TBu$.

Thus, we have

$$\left\{ \begin{array}{l} PSAu = QTBu = PSv = QTv = Pw = Qw = u, \\ APSv = BQTv = APw = BQw = Au = Bu = v, \\ SAPw = TBQw = SAu = TBu = Sv = Tv = w. \end{array} \right\} \tag{3.18}$$

To prove the uniqueness of the fixed point, suppose that PSA and QTB have a common fixed point u' also.

Using inequality (3.1), we have

$$\mathcal{M}_{M_1,N_1}(PSAu, QTBu', kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, u', t) * \mathcal{M}_{M_1,N_1}(u, PSAu, t) \\ *\mathcal{M}_{M_1,N_1}(u', QTBu', t) * \mathcal{M}_{M_1,N_1}(PSAu, QTBu', t).$$

Therefore, we have

$$\mathcal{M}_{M_1,\ N_1}(u, u', kt) \geq_{L^*} \mathcal{M}_{M_1,\ N_1}(u, u', t).$$

By Lemma 2.5, we have $u = u'$. Similarly we can prove that v and w are unique common fixed point of APS and BQT and of SAP and TBQ. This completes the proof. ∎

If we put $M_1 = M_2 = M_3 = M$ and $N_1 = N_2 = N_3 = N$ in Theorem 3.1, we get the following:

Corollary 1. *Let $(X, \mathcal{M}_{M,\ N}, \tau)$, $(Y, \mathcal{M}_{M,\ N}, \tau)$ and $(Z, \mathcal{M}_{M,\ N}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A, B be continuous mappings from X into Y, let S, T be continuous mappings from Y into Z and let P, Q be continuous mappings from Z into X satisfying the inequalities:*

$$\begin{aligned}(3.1)\quad \mathcal{M}_{M,N}(PSAx, QTBx', kt) \geq_{L^*} & \mathcal{M}_{M,N}(x, x', t) * \mathcal{M}_{M,N}(x, PSAx, t) \\ & * \mathcal{M}_{M,N}(x', QTBx', t) * \mathcal{M}_{M,N}(PSAx, QTBx', t)\end{aligned}$$

$$\begin{aligned}(3.2)\quad \mathcal{M}_{M,N}(APSy, BQTy', kt) \geq_{L^*} & \mathcal{M}_{M,N}(y, y', t) * \mathcal{M}_{M,N}(y, APSy, t) \\ & * \mathcal{M}_{M,N}(y', BQTy', t) * \mathcal{M}_{M,N}(APSy, BQTy', t)\end{aligned}$$

$$\begin{aligned}(3.3)\quad \mathcal{M}_{M,N}(SAPz, TBQz', kt) \geq_{L^*} & \mathcal{M}_{M,\ N}(z, z', t) * \mathcal{M}_{M,\ N}(z, SAPz, t) \\ & * \mathcal{M}_{M,\ N}(z', TBQz', t) * \mathcal{M}_{M,\ N}(SAPz, TBQz', t)\end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in $Z, t > 0$ and $k \in (0, 1)$, then PSA and QTB have a unique common fixed point u in X, APS and BQT have a unique common fixed point v in Y and SAP and TBQ have a unique common fixed point w in Z. Further, $Au = Bu = v, Sv = Tv = w$ and $Pw = Qw = u$. ∎

If we put $A = B, S = T$ and $P = Q$ in Theorem 3.1, we get the following:

Corollary 2. *Let $(X, \mathcal{M}_{M_1,N_1}, \tau)$, $(Y, \mathcal{M}_{M_2,N_2}, \tau)$ and $(Z, \mathcal{M}_{M_3,N_3}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A be continuous mapping from X into Y, let S be continuous mapping from Y into Z and let P be continuous mapping from Z into X satisfying the inequalities:*

$$\begin{aligned}(3.4)\quad \mathcal{M}_{M_1,N_1}(PSAx, PSAx', kt) \geq_{L^*} & \mathcal{M}_{M_1,N_1}(x, x', t) * \mathcal{M}_{M_1,N_1}(x, PSAx, t) \\ & * \mathcal{M}_{M_1,N_1}(x', PSAx', t) * \mathcal{M}_{M_1,N_1}(PSAx, PSAx', t)\end{aligned}$$

$$\begin{aligned}(3.5)\quad \mathcal{M}_{M_2,N_2}(APSy, APSy', kt) \geq_{L^*} & \mathcal{M}_{M_2,N_2}(y, y', t) * \mathcal{M}_{M_2,N_2}(y, APSy, t) \\ & * \mathcal{M}_{M_2,N_2}(y', APSy', t) * \mathcal{M}_{M_2,N_2}(APSy, APSy', t)\end{aligned}$$

$$\begin{aligned}(3.6)\quad \mathcal{M}_{M_3,N_3}(SAPz, SAPz', kt) \geq_{L^*} & \mathcal{M}_{M_3,N_3}(z, z', t) * \mathcal{M}_{M_3,N_3}(z, SAPz, t) \\ & * \mathcal{M}_{M_3,N_3}(z', SAPz', t) * \mathcal{M}_{M_3,N_3}(SAPz, SAPz', t)\end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z, $t > 0$ and $k \in (0, 1)$, then PSA has a unique common fixed point u in X, APS has a unique common fixed point v in Y and SAP has a unique common fixed point w in Z. Further, $Au = v, Sv = w$ and $Pw = u$. ∎

Remark 3.1. From Theorem 3.1, with $P = Q = Ix$ (the identity mapping on X), we obtain modified intuitionistic version of the results of Sharma, Deshpande and Thakur [29] and Deshpande and Pathak [8].

References

[1] ABIDI, H., CHO, Y.J., REGAN, D.O. and SAADATI, R., *Common fixed point in L-fuzzy metric spaces*, Applied Mathematics and Computation, 182 (2006), 820-828.

[2] Alaca, C., Turkoglu, D. and Yildiz, C., Fixed points in intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals, 29 (2006), 1073-1078.

[3] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87-96.

[4] Banach, S., *Theoriles Operations Linearies Manograie Mathematyezne*, Warsaw, Poland, 1932.

[5] Cho, Y. J., Kang, S.M. and Kim, S.S., *Fixed points in two metric spaces*, Novi Sad J. Math., 29 (1) (1999), 47-53.

[6] Deschrilver, G., Cornelis, C. and Kerre, E.E., *On representation of Intuitionistic fuzzy t-norms and t-conorms*, IEEE Trans Fuzzy Sets and Systems, 12 (2001), 45-61.

[7] Deschrilver, G. and Kerre, E.E., *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems, 133 (2003), 517-529.

[8] Deshpande, B. and Pathak, R., *Related fixed point theorem on two intuitionistic fuzzy metric spaces*, J. Korean Soc. Math. Educ. Ser. B.: Pure Appl. Math., 16 (4) (2009), 345-357.

[9] Edelstein, M., *On fixed and periodic points under contractive mappings*, J. London Math. Soc., 37 (1962), 74-79.

[10] El Naschie, *On the uncertainty of Cantorian geometry and two-slit experiment*, Chaos, Solitons & Fractals, 9 (1998), 517-529.

[11] El Naschie, *On the verifications of heterotic string theory and* $e^{(\infty)}$ *theory*, Chaos, Solitons & Fractals, 11 (2000), 397-407.

[12] El Naschie, *A review of E-infinity theory and the mass spectrum of high energy particle physics*, Chaos, Solitons & Fractals, 19 (2004), 209-36.

[13] El Naschie, *Fuzzy dodecahedron topology and E-infinity spacetime as a model for quantum physics*, Chaos, Solitons & Fractals, 30 (2006), 1025-33.

[14] Fisher, B., *Fixed points on two metric spaces*, Glasnik Mate, 16 (36) (1981), 333-337.

[15] Fisher, B., *Related fixed points on two metric spaces*, Math. Seminar Notes, Kobe Univ., 10 (1982), 17-26.

[16] Fisher, B. and Murthy, P.P., *Related fixed point theorems for two pairs of mappings on two metric spaces*, Kyungpook Math. J., 37 (1997), 343-347.

[17] Grabiec, M., *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27 (1988), 385-389.

[18] George, A. and Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395-399.

[19] Jain, R.K., Sahu, H.K. and Fisher, B., *Related fixed point theorems for three metric spaces*, Novi Sad J. Math., 26 (1) (1996), 11-17.

[20] Jain, R.K., Shrivastava, A.K. and Fisher, B., *Fixed points on three complete metric spaces*, Novi Sad J. Math., 27 (1) (1997), 27-35.

[21] Jain, S., Jain, S. and Jain, L.B., *Compatibility of type (P) in intuitionistic fuzzy metric space*, J. Nonlinear Sci Appl., 3, no. 2 (2010), 96-109.

[22] Karmosil, O. and Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (1975), 326-334.

[23] Nung, N.P., *A fixed point theorem in three metric spaces,* Math. Seminar Notes, Kobe Univ., 11 (1983), 77-79.

[24] Park, J.H., *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, 22 (2004), 1039-1046.

[25] Popa, V., *Fixed points on two complete metric spaces*, Review of Research, Faculty of Sci. Math. Series Univ. of Novi Sad, 21 (1) (1991), 83-93.

[26] Schweizer, B. and Sklar, A., *Statistical metric spaces*, Pacific Journal Math., 10 (1960), 314-334.

[27] Saadati, R. and Park, J.H., *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons & Fractals, 27 (2006), 331-344.

[28] Saadati, R., Sedghi, S. and Shobe, N., *Modified intuitionistic fuzzy metric spaces and some fixed point theorems*, Chaos, Solitons & Fractals, 38 (2008), 36-47.

[29] Sushil Sharma, Deshpande, B. and Thakur, B., *Related fixed point theorem for four mappings on two fuzzy metric spaces*, Internet. J. Pure and Applied Mathematics, 41 (2) (2007), 241-250.

[30] Turkoglu, D., Alaca, C. and Yildiz, C., *Compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces*, Demonstratio Math., 39 (2006), 671-684.

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