A RADICAL PROPERTY OF HYPERRINGS

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Abstract. In this paper we prove that Von Neumann regularity is a radical property on hyperrings.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [11] at the 8^{th} Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini [5], [6], Mittas [12], [13], Stratigopoulos [17], Vougiouklis [20] and by various authors. Basic definitions and propositions about the hyperstructures are found in [5], [6] and [20]. Krasner [10] has studied the notion of hyperfields, hyperrings and then many researchers like Davvaz [7], Massouros [14] and others followed him.

There are different notions of hyperrings $(R, +, \cdot)$. If in a hyperring the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called a Krasner (additive) hyperring [10]. The monograph [8] of Davvaz and Leoreanu-Fotea contains many results about various hyperrings. Asokkumar and Velrajan [1], [4] have studied Von Neumann regularity in Krasner hyperrings. Rota [16] introduced multiplicative hyperrings, where the additions are binary operations and multiplications are hyperoperations. De Salvo [9] introduced hyperrings in which the additions and the multiplications are hyperoperations. These hyperrings are studied by Rahnamani Barghi [15] and also by Asokkumar and Velrajan [2], [3], [19].

In this paper we prove that regularity (Von Neumann) is a radical property on hyperrings, where the additions and the multiplications are hyperoperations. We also prove that if a hyperring R is regular, then for a hyperideal I of R both I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.

2. Basic definitions and notations

This section explains some basic definitions that have been used in the sequel. A *hyperoperation* $\circ$ on a nonempty set H is a mapping of $H \times H$ into the family of nonempty subsets of H (i.e., $x \circ y \subseteq H$ for every $x, y \in H$). A *hypergroupoid* is a nonempty set H equipped with a hyperoperation $\circ$. For any two subsets A, B of a hypergroupoid H, the set $A \circ B$ means $\bigcup_{\substack{a\in A\\ b\in B}} (a \circ b)$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x\circ(y\circ z) = (x\circ y)\circ z$ for every $x, y, z \in H$(the associative axiom). A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if $x \circ H = H \circ x = H$ for every $x \in H$(the reproductive axiom). A reproductive semihypergroup is called a *hypergroup* (Marty). A comprehensive review of the theory of hypergroups appears in [5].

A nonempty set H with a hyperoperation $+$ is said to be a *canonical hypergroup* if the following conditions hold:

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$,

(ii) for every $x, y \in H$, $x + y = y + x$,

(iii) there exists $0 \in H$ such that $0 + x = x$ for all $x \in H$,

(iv) for every $x \in H$ there exists an unique element denoted by $-x \in H$ such that $0 \in x + (-x)$,

(v) for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

A nonempty subset N of a canonical hypergroup of H is called a *subcanonical hypergroup* of H if N itself is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, for every $x, y \in N$, $x - y \subseteq N$. Moreover, for any subset A of H, $-A$ denotes the set $\{-a : a \in A\}$.

The following elementary facts in a canonical hypergroup easily follow from the axioms.

(i) $-(-a) = a$ for every $a \in R$;

(ii) 0 is the unique element such that for every $a \in R$, there is an element $-a \in R$ with the property $0 \in a + (-a)$;

(iii) $-0 = 0$;

(iv) $-(a + b) = -b - a$ for all $a, b \in R$.

Theorem 2.1 [19] *Let H be a canonical hypergroup and N be a subcanonical hypergroup of H. For any two elements $a, b \in H$, if we define a relation $a \sim b$ if and only if $a \in b + N$, then $\sim$ is an equivalence relation on H.*

Let $\overline{x}$ be the equivalence class determined by the element $x \in H$ and H/N be the collection of all equivalence classes.

Theorem 2.2 [19] *Let H be a canonical hypergroup and N be a subcanonical hypergroup of H. Then $\overline{x} = x + N$ for any $x \in H$.*

Theorem 2.3 [19] *Let H be a canonical hypergroup, N be a subcanonical hypergroup of H. If we define $\overline{x} \oplus \overline{y} = \{\overline{z} : z \in x + y\}$ for all $\overline{x}, \overline{y} \in H/N$, then H/N is a canonical hypergroup.*

A nonempty set R with two hyperoperations $+$ and $\cdot$ is said to be a *hyperring* if $(R, +)$ is a canonical hypergroup, $(R, \cdot)$ is a semihypergroup with $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$ (0 as a bilaterally absorbing element) and the hyperoperation $\cdot$ is *distributive* over $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. The hyperoperation $+$ is usually called *hyperaddition* and the hyperoperation $\cdot$ is called *hypermultiplication*.

Definition 2.4 Let R be a hyperring and I be a nonempty subset of R. Then I is called a *left* (resp. *right*) *hyperideal* of R if $(I, +)$ is a canonical subhypergroup of R and for every $a \in I$and $r \in R$, $ra \subseteq I$ (resp. $ar \subseteq I$). A *hyperideal* of R is one which is a left as well as a right hyperideal of R.

If I, J are left (resp. right) hyperideals of a hyperring R, then $I + J$ is a left (resp. right) hyperideal of R. If I, J are hyperideals of a hyperring R, then $I + J$ is a hyperideal of R. Let R be a hyperring, I a hyperideal of R and R/I be the set of all distinct equivalence classes of I in R obtained by considering I as a subcanonical hypergroup of R. Then R/I is a canonical hypergroup under the hyperaddition defined in the Theorem 2.3.

Theorem 2.5 [19] *If we define $\overline{x} \otimes \overline{y} = \{\overline{z} : z \in xy\}$ for all $\overline{x}, \overline{y} \in R/I$, then R/I is a hyperring.*

Definition 2.6 Let R_1 and R_2 be two hyperrings. A mapping ϕ from R_1 into R_2 is called a *homomorphism* if the following conditions hold for all $a, b \in R_1$:

(i) $\phi(a + b) \subseteq \phi(a) + \phi(b)$;

(ii) $\phi(ab) \subseteq \phi(a)\phi(b)$, and

(iii) $\phi(0) = 0$.

The mapping ϕ is called a *good homomorphism* or *a strong homomorphism* if

(i) $\phi(a + b) = \phi(a) + \phi(b)$;

(ii) $\phi(ab) = \phi(a)\phi(b)$, and

(iii) $\phi(0) = 0$ for all $a, b \in R_1$.

Definition 2.7 A homomorphism (resp. strong homomorphism)ϕ from a hyperring R_1 into a hyperring R_2 is said to be an *isomorphism* (*resp. strong isomorphism*) if ϕ is one to one and onto. In this case we say R_1 is *isomorphic* (resp. *strongly isomorphic*) to R_2 and is denoted by $R_1 \cong R_2$.

Definition 2.8 Let ϕ be a homomorphism from a hyperring R_1 into another hyperring R_2. Then the set $\{x \in R_1 : \phi(x) = 0\}$ is called the *kernel* of ϕ and is denoted by $Ker\phi$ and the set $\{\phi(x) : x \in R_1\}$ is called *Image* of ϕ and is denoted by $Im\phi$.

It is clear that $Ker\phi$ is a hyperideal of R_1 and $Im\phi$ is a subcanonical hypergroup of R_2 and $R_1/Ker\phi$ is a hyperring.

Theorem 2.9 [19] (First Isomorphism Theorem) *Let ϕ be a strong homomorphism from a hyperring R_1 onto a hyperring R_2 with kernel K. Then R_1/K is strongly isomorphic to R_2.*

Theorem 2.10 [19] (Second Isomorphism Theorem) *If I and J are hyperideals of a hyperring R then $J/(I \cap J) \cong (I + J)/I$.*

3. Regular hyperring

First, let us recall the definition of a regular ring. An element a in a ring R is said to be regular if $a \in aRa$. A ring R is called regular if every element of R is regular. We define a regular hyperring as follows.

Definition 3.1 [2] An element $a \in R$ is said to be regular if $a \in aRa$. That is, there exists an element $b \in R$ such that $a \in aba$. A hyperring R is said to be regular if every element of R is regular.

Proposition 3.2 [2] *Strong homomorphic image of a regular hyperring is a regular hyperring.*

Proposition 3.3 *If I is a hyperideal of a regular hyperring R, then I is regular.*

Proof. Consider a hyperideal I of R. Let $a \in I$. Since R is regular, there exists $x \in R$ such that $a \in axa$. Then $a \in a(xa) \subseteq (axa)(xa) = a(xax)a$ where $xax \subseteq I$. Thus I is regular. ∎

Theorem 3.4 *If I, J are regular hyperideals of a hyperring R, then $I \cap J$ is also a regular hyperideal of R.*

Proof. It is clear that $I \cap J$ is a hyperideal of R. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $a \in axa$ and $a \in aya$. Now,

$$a \in axa \subseteq (axa)x(aya) = a(xaxay)a.$$

Since I, J are hyperideals of R, $xaxay \subseteq I \cap J$. Thus $I \cap J$ is regular. ∎

4. Regularity is a radical property on hyperrings

In this section, we show that regularity is a radical property on hyperrings. We also prove that if a hyperring R is regular, then for a hyperideal I of R both I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.

Definition 4.1 Let P be a property of hyperrings. A hyperring with the property P is called a *P-hyperring*. A hyperideal I of a hyperring R is called a *P-hyperideal* if the hyperideal I, as a hyperring, is a P-hyperring.

Definition 4.2 A P-hyperideal $P(R)$ of a hyperring R which contains every P-hyperideal of R is called the *P-hyperradical* of R.

Definition 4.3 A property P of a hyperring is called a *radical property* (in the sense of Amitsur and Kurosh [18]) if P satisfies the following conditions:

(i) Strong homomorphic image of a P-hyperring is a P-hyperring.

(ii) Every hyperring R has a P-hyperradical $P(R)$.

(iii) The hyperring $R/P(R)$ has no non-zero P-hyperideals.

Lemma 4.4 *Let R be a hyperring and $a \in R$. If there exists $x \in R$ and $c \in axa-a$ such that c is regular, then a is regular.*

Proof. Since $c \in axa - a$ is regular, there exists $d \in R$ such that $c \in cdc$. This means that

$$\begin{aligned} c &\in (axa-a)d(axa-a) \\ &= (axad-ad)(axa-a) \\ &\subseteq axadaxa - axada - adaxa + ada \\ &= a(xadaxa - xada - daxa + da) \\ &= a(xadax - xad - dax + d)a \end{aligned}$$

Hence $c \in aba$ for some $b \in xadax - xad - dax + d$. Since $c \in (axa-a)$, we get $a \in (axa - c) \subseteq axa - aba = a(x-b)a$. So $a \in aya$ for some $y \in x - b$. That is, a is regular. ∎

Theorem 4.5 *Let R be a regular hyperring and I be a hyperideal of R. Then I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.*

Proof. Let R be a regular hyperring and I be a hyperideal of R. Then by the Proposition 3.3, I is a regular hyperideal. Let $x+I \in R/I$. Since R is regular, there exists $y \in R$ such that $x \in xyx$. Consider $\overline{y} = y + I$. Now, $\overline{x}\,\overline{y}\,\overline{x} = \{\overline{z} : z \in xyx\}$. Since $x \in xyx$ we have $\overline{x} \in \{\overline{z} : z \in xyx\}$. That is, $\overline{x} \in \overline{x}\,\overline{y}\,\overline{x}$. So $x + I$ is regular in R/I. Hence R/I is regular.

Conversely, suppose R is a hyperring and there exists a hyperideal I of R such that both I and R/I are regular. Let $a \in R$. Then $\overline{a} \in R/I$. Since R/I is regular, there exists an element $\overline{b} \in R/I$ such that $\overline{a} \in \overline{a}\,\overline{b}\,\overline{a} = \{\overline{z} : z \in aba\}$. This means that $\overline{a} = \overline{z}$ for some $z \in aba$. That is, $a + I = z + I$ for some $z \in aba$. Since $z \in a+I$, we get $z \in a+i$ for some $i \in I$. Therefore, $i \in -a+z = z-a \subseteq aba-a$. Thus $i \in aba - a$. Since I is regular, i is a regular element of I and therefore i is a regular element of R. Thus the set $aba - a$ contains a regular element i of R. Then by the Lemma 4.4, the element a is regular in R. Hence R is regular. ∎

Theorem 4.6 *Let R be a hyperring. If I and J are regular hyperideals of R, then $I + J$ is regular.*

Proof. Since $J/(I \cap J)$ is a homomorphic image of a regular hyperideal J, it is regular. By the Theorem 2.10, $J/(I \cap J)$ is isomorphic to $(I + J)/I$. Therefore, $(I + J)/I$ is regular. Since both I and $(I + J)/I$ are regular, by the Theorem 4.5, the hyperideal $I + J$ is regular. ∎

Theorem 4.7 *Any hyperring has a regular hyperradical.*

Proof. Let R be a hyperring. Consider the hyperideal (0) of R. Clearly, (0) is a regular hyperideal of R. If (0) is the only regular hyperideal of R, then this is the regular hyperradical.

Otherwise, let $\{I_i\}$ be the collection of all regular hyperideals in a hyperring R. Their sum is given by $M = \bigcup\{\sum_{finite} a_i : a_i \in I_i\}$. Clearly, M is a hyperideal of R. If $x \in M$, then $x \in a_i + a_j + a_k + \cdots + a_l$, where $a_i \in I_i$. By Theorem 4.6, $I_i + I_j + I_k + \cdots + I_l$ is a regular hyperideal. Therefore, x is regular. Hence, M is regular. Since M contains all regular hyperideals of R, we have M is the regular hyperradical of R. ∎

Theorem 4.8 *Let R be a hyperring and M be the regular hyperradical of R. Then the hyperring R/M has no non-zero regular hyperideals.*

Proof. Let J be a regular hyperideal of R/M. Then $J = I/M$ for some hyperideal I of R containing M. Since M and I/M are regular, by the Theorem 4.5, I is regular. By the definition of M, we have $I \subseteq M$. Hence $I = M$. Therefore, J is a zero hyperideal of R/M. ∎

Theorem 4.9 *The regularity is a radical property on hyperrings.*

Proof. The proof follows from the Proposition 3.2, and the Theorems 4.7, 4.8. ∎

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