$\mathcal{M}$-INJECTIVITY IN THE CATEGORY Act-S

Leila Shahbaz

Department of Mathematics
University of Maragheh
Maragheh 55181-83111
Iran
e-mail: leilashahbaz@yahoo.com

Abstract. Injectivity is one of the central notions in many branches of mathematics. Different kinds of injectivity with respect to the class of all monomorphisms and with respect to some special subclasses of monomorphisms in the category **Act-S** of acts over a semigroup S have been studied before. In this paper, we take the category **Act-S** of acts over a semigroup S, and $\mathcal{M}$ as an arbitrary subclass of monomorphisms, and study some kinds of injectivity with respect to $\mathcal{M}$. Also, the behaviour of these notions of injectivity with respect to products, coproducts, and direct sums is studied. As a result we give some characterizations of semigroups.

Keywords: $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, ideal $\mathcal{M}$-injective.

2000 Mathematics Subject Classification: 08A60, 18A20, 20M30, 20M50.

1. Introduction and preliminaries

One of the very useful notions in many branches of mathematics as well as in computer science is the notion of acts of a semigroup or a monoid on a set. In the following we first recall some facts about the category **Act-S** needed in this paper.

Let S be a semigroup, A be a set, and

$$\begin{array}{rcl} \mu : A \times S & \longrightarrow & A \\ (a, s) & \longmapsto & as := \mu(a, s), \end{array}$$

be a map. The set A is called a (*right*) S-*act* or a (*right*) *act* over S, if the map μ satisfies $a(st) = (as)t$ for $a \in A$ and $s, t \in S$. In this case, μ is called the action of S on A.

If S is a monoid with 1 as its identity, we usually also require that $a1 = a$ for $a \in A$.

A subset A' of an S-act A is said to be a *subact* of A if $a's \in A'$ for all $s \in S$ and $a' \in A'$; and in this case we write $A' \leq A$.

A *homomorphism* (also called an *equivariant* map or an S-*map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A, s \in S$, $f(as) = f(a)s$.

Since id_A and the composition of two S-maps are S-maps, we have the category **Act-S** of all right S-acts and S-maps between them.

Note that, the class of S-acts is an equational class, and so the category **Act-S** is complete and cocomplete (has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets and equipped with a natural action. Also, monomorphisms (epimorphisms) in **Act-S** are exactly one-one (onto) S-maps. Therefore, we do not distinguish between monomorphisms of acts and inclusions, and call an S-act B containing (an isomorphic copy of) an S-act A an *extension* of A.

An element z of S is called a right zero element if for each $s \in S, sz = z$. Also, an element z of S is called a left zero element if for each $s \in S, zs = z$. An element 0 or θ of S is a zero element if it is right and left zero.

An S-act A is said to be finitely generated if $A = \bigcup_{i=1}^{n} a_i S^1$, for some $a_1, ...,$ $a_n \in A$ and $n \in N$, where S^1 is the semigroup S with an adjoined identity 1. We say that a semigroup S is finitely generated if it is finitely generated as an S-act with its operation as the action.

A semigroup S is said to be *Noetherian* if it satisfies the ascending chain condition on its right ideals.

A semigroup S is said to be *left reversible* if every two right ideals of S have a nonempty intersection. For more information about semigroups and acts see [6], [8] and [9].

An S-act A is said to be *decomposable* if there exist subacts $B, C \subsetneq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$. Otherwise, A is called *indecomposable*.

An act A is called torsion free if for any $x, y \in A$ and for any element $s \in S$ the equality $xs = ys$ implies $x = y$.

Recall that for a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathcal{A}$, we say that $\mathcal{A}$ satisfies the $\mathcal{M}$-*transferability property* if any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

with $f \in \mathcal{M}$ can be completed to a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow{v} & D \end{array}$$

with $v \in \mathcal{M}$.

Note that, since pushouts exist in the category **Act-S**, the above condition is equivalent to "pushout transfer monomorphism"; that is, the pushout map corresponding to a morphism in $\mathcal{M}$ again belongs to $\mathcal{M}$.

Recall that for a family $\{A_i : i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is defined to be the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

2. $\mathcal{M}$-injectivity and some kinds of weak $\mathcal{M}$-injectivity

In this section, we study $\mathcal{M}$-injectivity and some kinds of weak $\mathcal{M}$-injectivity of acts for a subclass $\mathcal{M}$ of monomorphisms. Any map with prefix $\mathcal{M}$ means being in $\mathcal{M}$.

Definition 2.1 We call an S-act E:

(1) $\mathcal{M}$-*injective* if it is injective with respect to $\mathcal{M}$-morphisms. An injective act with respect to all act monomorphisms is simply called *injective.*

(2) $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is for every $\mathcal{M}$-morphism $h : A \to B$ there exists an S-map $g : B \to A$ such that $gh = id_A$.

Clearly, if A is an injective S-act, then it is $\mathcal{M}$-injective for any subclass $\mathcal{M}$ of monomorphisms, but the converse is not necessarily true. For example, take $\mathcal{M}$ as the class of sequentially dense or sequentially pure monomorphisms and see [10] and [2].

The following theorem is one of the important theorems about injectivity of S-acts with respect to any subclass of monomorphisms. This theorem was first proved by P. Berthiaume in [4] for injective acts. H. Barzegar [3] and B. Banaschewski [1] proved it for $\mathcal{M}$-injective acts for any subclass $\mathcal{M}$ of monomorphisms.

Theorem 2.2 *Let S be a semigroup. Then, if pushouts transfer $\mathcal{M}$-morphisms, the following are equivalent for an S-act A:*

(i) *A is $\mathcal{M}$-injective.*

(ii) *A is an $\mathcal{M}$-absolute retract.*

Here we give a criterion like the Skornjakov-Baer criterion [13], for $\mathcal{M}$-injectivity of acts over a semigroup.

Theorem 2.3 *Let for each $\mathcal{M}$-extension A of B, D be an $\mathcal{M}$-extension of B with $D \subseteq A$. Then, for an S-act E, the following are equivalent:*

(i) *E is $\mathcal{M}$-injective.*

(ii) *For every $\mathcal{M}$-morphism $h : B \rightarrowtail B \cup aS^1$ to a singly generated extension of B and every S-map $f : B \to E$ there exists an S-map $g : B \cup aS^1 \to E$ such that $gh = f$.*

Proof. It is clear that (i)⇒(ii).

(ii)⇒(i) Let $h : B \rightarrowtail A$ be an $\mathcal{M}$-morphism and $f : B \to E$ be an S-map. Applying Zorn's Lemma on the poset of all subacts (D_α, g_α) of A which are $\mathcal{M}$-extensions of B, with $h(B) \subseteq D_\alpha$ and such that there exists an S-map $g_\alpha : D_\alpha \to E$ with $g_\alpha h = f$ with the order

$$(D_\alpha, g_\alpha) \leq (D_\beta, g_\beta) \Leftrightarrow D_\alpha \subseteq D_\beta, g_\beta \mid_{D_\alpha} = g_\alpha$$

we get a maximal such subact, say (D, g). If $D = A$ then the proof is complete. Otherwise, there exists $a \in A - D$. Now, since by the hypothesis $D \cup aS^1$ is an $\mathcal{M}$-extension of B we get that $\overline{g} : B \to D \cup aS^1$ is an $\mathcal{M}$-morphism, and by (ii) there is an S-map $\overline{g}$ which extends g. This contradicts the maximality of D, so $A = D$. ∎

Now, we define some kinds of weak $\mathcal{M}$-injectivity and then compare them with $\mathcal{M}$-injectivity.

Definition 2.4 A right ideal I of S is called a right $\mathcal{M}$-ideal if the inclusion map from I into S belongs to $\mathcal{M}$.

Definition 2.5 An S-act A is said to be

(1) *ideal $\mathcal{M}$-injective*, if every S-map $f : I \to A$ from a right $\mathcal{M}$-ideal I of S can be represented as λ_a, for some $a \in A$.

(2) *weakly $\mathcal{M}$-injective* if for each right $\mathcal{M}$-ideal I of S, any S-map $f : I \to A$ can be extended to an S-map $g : S \to A$.

(3) *finitely $\mathcal{M}$-injective* (*c$\mathcal{M}$-injective*) if for each $\mathcal{M}$-morphism $h : F \to B$ from a finitely generated (cyclic) act F and for any S-map $f : F \to A$ there exits an S-map $g : B \to A$ such that $gh = f$.

(4) *$F\mathcal{M}$-injective* (*$P\mathcal{M}$-injective)* if every S-map $f : I \to A$ from a finitely generated (principal) right $\mathcal{M}$-ideal I of S can be extended to an S-map $\overline{f} : S \to A$.

(5) A semigroup S is called *completely $\mathcal{M}$-injective* if all right S-acts are $\mathcal{M}$-injective. Similarly for the other types of $\mathcal{M}$-injectivity.

Remark 2.6

(1) Ideal (weakly) $\mathcal{M}$-injective acts need not be $\mathcal{M}$-injective in the usual sense (let $\mathcal{M} = \mathcal{M}$ono and see [9]).

(2) Ideal $\mathcal{M}$-injectivity implies weak $\mathcal{M}$-injectivity but weakly $\mathcal{M}$-injective acts need not be ideal $\mathcal{M}$-injective. For the case where S is a monoid, ideal $\mathcal{M}$-injectivity coincides with weak $\mathcal{M}$-injectivity.

Lemma 2.7 *A retract of any kind of $\mathcal{M}$-injective act is $\mathcal{M}$-injective of that type.*

The following results show when all acts are finitely $\mathcal{M}$-injective.

Lemma 2.8 *If pushouts transfer $\mathcal{M}$-morphisms then every finitely generated finitely $\mathcal{M}$-injective S-act is $\mathcal{M}$-injective.*

Proof. Let A be a finitely generated finitely $\mathcal{M}$-injective S-act. Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ {\scriptstyle id_A}\downarrow & & \\ A & & \end{array}$$

in which B is an $\mathcal{M}$-extension of A. Using that A is finitely generated, there exists $\pi : B \to A$ such that $\pi \circ i = id_A$. This implies that A is an $\mathcal{M}$-absolute retract. Now, by Theorem 2.2, A is $\mathcal{M}$-injective. ∎

Theorem 2.9 *If pushouts transfer $\mathcal{M}$-morphisms then a semigroup S is completely finitely $\mathcal{M}$-injective if and only if all finitely generated S-acts are $\mathcal{M}$-injective.*

Proof. ($\Rightarrow$) The proof is similar to the proof of the above lemma.

($\Leftarrow$) Let A be any S-act and $h : F \to B$ be an $\mathcal{M}$-morphism from a finitely generated act F, and $f : F \to A$ be any S-map. Then, by hypothesis, F is $\mathcal{M}$-injective and so an $\mathcal{M}$-absolute retract, by Theorem 2.2. Thus there exists an S-map $g : B \to F$ such that $gh = id_F$. Then the composite $fg : B \to A$ is an S-map with $(fg)h = f$. So, A is finitely $\mathcal{M}$-injective. ∎

Proposition 2.10 *An S-act A with a zero element is $c\mathcal{M}$-injective if and only if for any $\mathcal{M}$-morphism $h : P \rightarrowtail D$ from a cyclic act P into any indecomposable act D, any S-map $f : P \to A$ can be extended to an S-map $g : D \to A$.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) Let A be an S-act with a zero element and $h : P \to B$ be an $\mathcal{M}$-morphism from a cyclic act P, and $f : P \to A$ be any S-map. Consider the decomposition of $B = \coprod_{i\in I} B_i$ into its indecomposable subacts B_i which exists by Theorem I.5.10 of [9]. Since P is cyclic, there exists $i \in I$ such that $h(P) \subseteq B_i$. Thus by the hypothesis there exists an S-map $g : B_i \to A$ which extends f. Define $\overline{f} : B = \coprod_{i\in I} B_i \to A$ by

$$\overline{f}(b) = \begin{cases} g(b) & \text{if } b \in B_i \\ \theta & \text{if } b \notin B_i \end{cases}$$

where θ is the zero element of A. Then $\overline{f}$ is an S-map which extends f. ∎

The following results show when all acts are $c\mathcal{M}$-injective.

Lemma 2.11 *If pushouts transfer $\mathcal{M}$-morphisms then every cyclic $c\mathcal{M}$-injective S-act is $\mathcal{M}$-injective.*

Proof. It is similar to the proof of Theorem 2.9 by replacing finitely generated acts with cyclic acts. ∎

Theorem 2.12 *If pushouts transfer $\mathcal{M}$-morphisms then a semigroup S is completely $c\mathcal{M}$-injective if and only if all cyclic S-acts are $\mathcal{M}$-injective.*

Proof. The proof is similar to the proof of the above lemma. ∎

The following theorem characterizes semigroups over which all acts are ideal $\mathcal{M}$-injective.

Theorem 2.13 *Every S-act is ideal $\mathcal{M}$-injective if and only if every right $\mathcal{M}$-ideal of the semigroup S is generated by an idempotent.*

Proof. Consider the identity map id_I from a right $\mathcal{M}$-ideal I of S which is of the form λ_a for some element a in I, by hypothesis. Thus, $a = id_I(a) = \lambda_a(a) = aa = a^2$ and so a is an idempotent element. For the converse, let $I = eS$ be a right $\mathcal{M}$-ideal of S, where e is an idempotent element. Consider an S-map $f : I = eS \to A$. Thus $f = \lambda_a$ for $a = f(e)$. Thus A is ideal $\mathcal{M}$-injective. ∎

In the following we characterize semigroups over which all acts are weakly $\mathcal{M}$-injective.

Theorem 2.14 *The following conditions are equivalent:*

(1) *Each right $\mathcal{M}$-ideal of S is a retract of S.*

(2) *S is completely weakly $\mathcal{M}$-injective.*

(3) *Each right $\mathcal{M}$-ideal of S is weakly $\mathcal{M}$-injective.*

Proof. (1) $\Rightarrow$ (2) Let A be an S-act, and $f : I \to A$ be an S-map from a right $\mathcal{M}$-ideal I of S. By the hypothesis, there is a retraction $g : S \to I$. Then $fg : S \to A$ is an S-map which extends f. So, A is weakly $\mathcal{M}$-injective.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let I be a right $\mathcal{M}$-ideal of S. Then there exists an S-map π from I to S with $\pi\mid_I = id_I$, since I is weakly $\mathcal{M}$-injective by (2). So I is a retract of S. ∎

In the following we characterize semigroups over which all acts are $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).

Theorem 2.15 *The following conditions are equivalent:*

(1) *Each finitely generated (principal) right $\mathcal{M}$-ideal of S is a retract of S.*

(2) *S is completely $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).*

(3) *Each finitely generated (principal) right $\mathcal{M}$-ideal of S is $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).*

Proof. (1) $\Rightarrow$ (2) Let A be an S-act, and $f : I \to A$ be an S-map from a finitely generated (principal) right $\mathcal{M}$-ideal I of S. By the hypothesis, there is a retraction $g : S \to I$. Then $fg : S \to A$ is an S-map which extends f. So, A is $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let I be a finitely generated (principal) right $\mathcal{M}$-ideal of S. Then there exists an S-map π from I to S with $\pi \mid_I = id_I$, since I is $F\mathcal{M}$-injective ($P\mathcal{M}$-injective) by (2). So I is a retract of S. ∎

Definition 2.16 A right $\mathcal{M}$-ideal I of S is called $\mathcal{M}$-*intersection large* in S if the intersection of I with any nonempty right $\mathcal{M}$-ideal of S is always nonempty.

Theorem 2.17 *Suppose that S is a semigroup with a zero element and for each right $\mathcal{M}$-ideal I of S, J is a right $\mathcal{M}$-ideal of S with $I \subseteq J$. Then a right S-act A is weakly $\mathcal{M}$-injective if and only if for any $\mathcal{M}$-intersection large right $\mathcal{M}$-ideal I of S, every S-map from I into A can be extended to an S-map from S into A.*

Proof. The only if part is obvious. To prove the converse, let I be any right $\mathcal{M}$-ideal of S and f be an S-map from I into A. Let P be the set of pairs (J, g) where J is a right $\mathcal{M}$-ideal of S which contains I and g is an S-map from J into A which extends f. Consider P as the ordered set with the order

$$(J_1, g_1) \leq (J_2, g_2) \Leftrightarrow J_1 \subseteq J_2, g_2 \mid_{J_1} = g_1.$$

Applying Zorn's lemma, P has a maximal element (K, g_K). If K is not an $\mathcal{M}$-intersection large ideal, there exists a right $\mathcal{M}$-ideal $L \neq \emptyset$ of S such that $L \cap K = \emptyset$. Define a mapping $g^* : K \cup L \to A$ by

$$g^*(s) = \begin{cases} g_K(s) & \text{if } s \in K \\ \theta \text{ (the zero element of A)} & \text{if } s \in L \end{cases}$$

It is easy to see that g^* is an S-map from $K \cup L$ into A, which extends g_K and hence f. Thus $(K \cup L, g^*) \in P$, which contradicts the fact that (K, g_K) is maximal in P. Hence K must be an $\mathcal{M}$-intersection large ideal. Hence by the hypothesis, g_K can be extended to an S-map from S into A, which is therefore an extension of f. ∎

Definition 2.18 A semigroup S is called $\mathcal{M}$-Noetherian if it satisfies the ascending chain condition on its right $\mathcal{M}$-ideals.

Remark 2.19 If a semigroup S is $\mathcal{M}$-Noetherian then every right $\mathcal{M}$-ideal I of S is finitely generated. For if, let I be a right $\mathcal{M}$-ideal of S, and assume that $I \neq \emptyset$ is not finitely generated. Thus there exists $x_1 \in I$. Let $\{x_1, x_2, x_3, \ldots\} \subseteq I$ be a countable subset of generating elements of I which are different. Thus

$$\langle x_1 \rangle = I_1 \subseteq \langle x_1, x_2 \rangle = I_2 \subseteq \langle x_1, x_2, x_3 \rangle = I_3 \subseteq ... \subseteq \langle x_1, x_2, ..., x_n \rangle = I_n \subseteq ...$$

is an ascending chain of right $\mathcal{M}$-ideals of S which does not stop and this is a contradiction. Thus I is finitely generated.

Theorem 2.20 *Let S be a left reversible and right $\mathcal{M}$-Noetherian semigroup. Suppose A is a torsion free S-act such that every cyclic subact of A is ideal $\mathcal{M}$-injective. Then A is itself ideal $\mathcal{M}$-injective.*

Proof. Let $I = \bigcup_{i=1}^{n} s_i S^1$ be a right $\mathcal{M}$-ideal of S, and $f : I \to A$ be an S-map. If $f(s_i) = x_i$, then $f\,|_{s_i S^1} \colon s_i S^1 \to x_i S^1$ is an S-map. Since $x_i S^1$ is ideal $\mathcal{M}$-injective by the assumption, there exists $a_i \in x_i S^1$ such that $f(s_i) = a_i s_i, i = 1, 2, \ldots, n$. Since by left reversiblity of S the intersection of all right ideals of S are nonempty, there exists $c_i \in S$ such that $s_1 c_1 = s_2 c_2 = \ldots = s_n c_n$. Then $f(s_1)c_1 = f(s_2)c_2 = \ldots = f(s_n)c_n$ and so $a_1 s_1 c_1 = a_2 s_2 c_2 = \ldots = a_n s_n c_n$. Since A is torsion free we have $a_1 = a_2 = \ldots = a_n = a$. Thus $f(s) = as, s \in I$ and hence A is ideal $\mathcal{M}$-injective. ∎

Remark 2.21 The condition that S is left reversible is necessary in the above theorem. For, take $\mathcal{M}$ as the class of all monomorphisms and see [11], Example 1.3.

Theorem 2.22 *Let S be an $\mathcal{M}$-Noetherian semigroup, A be an S-act which is not finitely generated and each proper subact of A is ideal $\mathcal{M}$-injective. Then A is ideal $\mathcal{M}$-injective.*

Proof. Let $I = \bigcup_{i=1}^{n} s_i S^1$ be a right $\mathcal{M}$-ideal of S, and $f : I \to A$ be an S-map. If $f(I) = A$, then $A = \bigcup_{i=1}^{n} a_i S^1$ where $f(s_i) = a_i \in A$, which implies that A is finitely generated. Hence $f(I) \neq A$. Then $f(I)$ is ideal $\mathcal{M}$-injective by the hypothesis and so $f : I \to f(I)$ is given by $f(s) = as$ for every $s \in I$. Hence A is ideal $\mathcal{M}$-injective. ∎

Remark 2.23 The condition that A is not finitely generated is necessary in the above theorem. For, take $\mathcal{M}$ as the class of all monomorphisms and see [11], Example 1.5.

3. Products, coproducts, and direct sums of different kinds of $\mathcal{M}$-injective acts

In this section we consider the behaviour of different kinds of $\mathcal{M}$-injective acts with respect to products, coproducts, and direct sums.

In the following theorems, it is shown that as usual, the above types of $\mathcal{M}$-injectivity well-behaves with respect to products similar to the case of injectivity using the universal property of products, but not as well with coproducts and direct sums.

Theorem 3.24 *Let $\{A_i : i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is $\mathcal{M}$-injective (finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) if each A_i is $\mathcal{M}$-injective (finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective). The converse is true if each A_i has a zero element.*

In the case of weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective) acts we have the following.

Theorem 3.25 *Let $\{A_i : i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective) if and only if each A_i is weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective).*

Proof. The fact that the product of weakly $\mathcal{M}$-injective acts is weakly $\mathcal{M}$-injective is proved similar to the case of injectivity using the universal property of products. To prove the converse, let $A = \prod_{i\in I} A_i$ be weakly $\mathcal{M}$-injective, $k \in I$, and define an S-map $f_i : J \to A_i$ by $f_k = f$, and for $i \neq k, f = \lambda_{a_i}$, where a_i is any element of A_i and J is an $\mathcal{M}$-right ideal of S. Then we get an S-map $\overline{f}$ using the universal property of products which extends to an S-map $\overline{\overline{f}} : S \to A$ by weak $\mathcal{M}$-injectivity of A. Now, $p_k\overline{\overline{f}} : S \to A_k$ extends f, where $p_k : A \to A_k$ is the kth projection map. So A is weakly $\mathcal{M}$-injective. ∎

In regard with coproducts, first note that the following is trivially true.

Proposition 3.26 *Let $\{A_i : i \in I\}$ be a family of S-acts with a zero element. If the coproduct $\coprod_{i\in I} A_i$ is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) then each A_i is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

The converse of the above theorem is not necessarily true in general (see [10], Theorem 3.4). But in the case of $c\mathcal{M}$-injective ($P\mathcal{M}$-injective) acts the converse is also true.

Proposition 3.27 *Let $\{A_i : i \in I\}$ be a family of $c\mathcal{M}$-injective ($P\mathcal{M}$-injective) S-acts. Then the coproduct $\coprod_{i\in I} A_i$ is $c\mathcal{M}$-injective ($P\mathcal{M}$-injective).*

Proof. Let $\{A_i : i \in I\}$ be a family of $c\mathcal{M}$-injective acts. Notice that for any $\mathcal{M}$-morphism $g : F \to B$ from a cyclic act F and any S-map $f : F \to \coprod A_i$ we have $Imf \subseteq A_i$ for some $i \in I$. Hence f can be extended to an S-map $\overline{f}$, since A_i is $c\mathcal{M}$-injective. ∎

In the case of ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective) acts we have the following.

Theorem 3.28 *Let a semigroup S be left reversible. Then the coproduct $\coprod_{i\in I} A_i$ of each family of ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective) acts is ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective).*

Proof. Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts. Let $f : J \to \coprod_{i\in I} A_i$, which J is an $\mathcal{M}$-right ideal of S, be an S-map. Suppose there exist $i, j \in I, i \neq j$, with $Imf \cap A_i \neq \emptyset$ and $Imf \cap A_j \neq \emptyset$. Then J is a disjoint union of two ideals in contradiction with left reversibility of S. This implies the existence of $i \in I$ such that $Imf \subseteq A_i$. Since A_i is ideal $\mathcal{M}$-injective, f can be written in the form of λ_a for some a in A_i. Thus $\coprod_{i\in I} A_i$ is ideal $\mathcal{M}$-injective. ∎

Theorem 3.29 *If the coproduct $\coprod_{i\in I} A_i$ of each family of ideal $\mathcal{M}$-injective acts is ideal $\mathcal{M}$-injective then any two right $\mathcal{M}$-ideals of S have a nonempty intersection.*

Proof. Let I, J be two $\mathcal{M}$-right ideals of S such that $I \cap J = \emptyset$. By assumption $\Theta \sqcup \Theta = \{b_1, b_2\}$, where Θ is the one element act, is ideal $\mathcal{M}$-injective. The S-map $f : I \sqcup J \to \Theta \sqcup \Theta$ given by

$$f(s) = \begin{cases} b_1 & \text{if } s \in I \\ b_2 & \text{if } s \in J \end{cases}$$

can not be written in the form of λ_a for some $a \in \Theta \sqcup \Theta$ which is a contradiction. Thus each pair of $\mathcal{M}$-right ideals of S have a nonempty intersection. ∎

For the direct sum of different kinds of $\mathcal{M}$-injectivity we first trivially have:

Theorem 3.30 *Let $\{A_i : i \in I\}$ be a family of S-acts with zero such that the direct sum $\bigoplus_{i\in I} A_i$ is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective). Then each A_i is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

Remark 3.31 The converse of the above theorem is true in the case of finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, and $P\mathcal{M}$-injective acts. But it is not true in general, for example take $\mathcal{M}$ as the class of sequentially dense monomorphisms and see [10].

Proposition 3.32 *Each direct sum of finitely $\mathcal{M}$-injective ($c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) acts is finitely $\mathcal{M}$-injective ($c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

Proof. Let $\{A_i\}_{i\in I}$ be a family of finitely $\mathcal{M}$-injective acts. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & Y \\ {\scriptstyle f}\downarrow & & \\ \bigoplus_{i\in I} A_i & & \end{array}$$

in which λ is an $\mathcal{M}$-morphism, f is a homomorphism, and X is a finitely generated act. Let $\{x_1, x_2, ..., x_n\}$ be the generating set of X. Then for each j, the element $f(x_j)$ in $\bigoplus_{i\in I} A_i$ has only finitely many nonzero coordinates. Since there are only finitely many x_j, the set $\{f(x_1), f(x_2), ..., f(x_n)\}$ collectively involves only finitely many A_i, say $A_{i_1}, ..., A_{i_n}$. Hence $Imf \subseteq A_{i_1} \bigoplus ... \bigoplus A_{i_n}$ which, being a finite direct sum of finitely $\mathcal{M}$-injective acts (which is in fact a product of finitely $\mathcal{M}$-injective acts), is finitely $\mathcal{M}$-injective, by Theorem 3.25. Hence there is a homomorphism $\bar{f} : Y \to A_{i_1} \bigoplus ... \bigoplus A_{i_n}$ which extends f. We may regard $\bar{f}$ as a homomorphism whose image is in the larger S-act $\bigoplus_{i\in I} A_i$. ∎

Theorem 3.33 *Let a semigroup S be ($\mathcal{M}$-)Noetherian. Then the direct sum $\bigoplus_{i\in I} A_i$ of each family of weakly $\mathcal{M}$-injective acts is weakly $\mathcal{M}$-injective.*

Proof. The proof is similar to the proof of the above theorem by replacing X with every right $\mathcal{M}$-ideal I of S. ∎

About direct sum of ideal $\mathcal{M}$-injective acts we have

Theorem 3.34 *Let S be a semigroup with the zero 0 and for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. Then each direct sum of (ideal $\mathcal{M}$-) injective acts is ideal $\mathcal{M}$-injective if and only if S is $\mathcal{M}$-Noetherian.*

Proof. ($\Rightarrow$) Let $\{0\} = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_i \subseteq \ldots$ be an ascending chain of right $\mathcal{M}$-ideals of S, and $I = \bigcup_{i\in I} I_i$. By the hypothesis, I is a right $\mathcal{M}$-ideal of S. Consider the Rees factor acts I/I_i for each i, and let E_i be the injective hull of

I/I_i. Then $E = \bigoplus_i E_i$ is ideal $\mathcal{M}$-injective by the hypothesis. Consider natural epimorphisms $f_i : I \to I/I_i$ and define an S-map $f : I \to E$ by $f(s) = (f_i(s))_i$. Notice that for each $s \in I$ only finitely many components of $f(s)$ are nonzero, because $s \in I_k$ for some k, and so $f_i(s) = 0$ for all $i \geq k$. Now, since E is ideal $\mathcal{M}$-injective by assumption, there exists an element $a \in E, f = \lambda_a$. Since $a \in E = \bigoplus_i E_i, a = (a_1, a_2, \ldots, a_k, \ldots)$, so there is t such that $a_k = 0$ for all $k \geq t$. Then since for any $x \in I, f(x) = ax$, and since $(ax)_t = a_t x = 0$, it follows that $I \subseteq I_t$. Hence $I_{t+1} = I_{t+2} = \ldots = I$. Thus S satisfies the ascending chain condition on its $\mathcal{M}$-right ideals and so is $\mathcal{M}$-Noetherian.

($\Leftarrow$) Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts with zero elements. Let $f : I \to \bigoplus_{i\in I} A_i$ be an S-map from a right $\mathcal{M}$-ideal I of S and assume that I is generated by $\{s_1, s_2, \ldots, s_n\}$, since S is $\mathcal{M}$-Noetherian. Then, since only finitely many components of each $f(s_i)$ are nonzero, we get that Imf is contained in a direct sum of finitely many A_i, say $i_1, i_2, \ldots, i_m$. Then, $A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m}$ which is in fact a product, is ideal $\mathcal{M}$-injective, $f = \lambda_a$, for some $a \in A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m} \subseteq \bigoplus_{i\in I} A_i$. ■

Proposition 3.35 *Let for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of a semigroup S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. If every finitely $\mathcal{M}$-injective act is ideal $\mathcal{M}$-injective then S is $\mathcal{M}$-Noetherian. The converse is true if S is a monoid.*

Proof. Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts. Since each ideal $\mathcal{M}$-injective act is finitely $\mathcal{M}$-injective thus each A_i is finitely $\mathcal{M}$-injective. We know that each direct sum of finitely $\mathcal{M}$-injective acts is finitely $\mathcal{M}$-injective and so by the hypothesis it is ideal $\mathcal{M}$-injective. So, by Theorem 3.34, S is $\mathcal{M}$-Noetherian.

For the converse, let S be a monoid, A be a finitely $\mathcal{M}$-injective act, and $f : I \to A$ be an S-map from a right $\mathcal{M}$-ideal I of S. Since S is $\mathcal{M}$-Noetherian, I is finitely generated. Now, since A is finitely $\mathcal{M}$-injective there exists an S-map $g : S \to A$ which extends f. Then g is of the form λ_a for $a = g(1)$ where 1 is the identity element of the monoid S. Thus f is also of the form λ_a and hence A is ideal $\mathcal{M}$-injective. ■

We recall the following Theorem from [12].

Theorem 3.36 *Each direct sum of injective S-acts is injective if and only if S is Noetherian.*

Corollary 3.37 *If S is an $\mathcal{M}$-Noetherian semigroup and ideal $\mathcal{M}$-injectivity (weak $\mathcal{M}$-injectivity) for S-acts implies injectivity, then S must be Noetherian.*

Proof. By Theorem 3.36, S is Noetherian if and only if every direct sum of injective S-acts is injective. Now, if $\{A_i : i \in I\}$ is any family of injective acts then they are also ideal $\mathcal{M}$-injective, and so, by Theorem 3.34, their direct sum is ideal $\mathcal{M}$-injective and hence injective by the hypothesis, and so the result. ■

Definition 3.38 An S-act A is called *countably* $\sum$*-ideal* $\mathcal{M}$*-injective* if any countable direct sum of A with itself is ideal $\mathcal{M}$-injective.

Theorem 3.39 *Let for each family* $\{I_i : i \in I\}$ *of right* $\mathcal{M}$*-ideals of* S, $I = \bigcup_{i\in I} I_i$ *be a right* $\mathcal{M}$*-ideal of* S*. Then the following are equivalent:*

(1) *Each direct sum of injective acts is ideal* $\mathcal{M}$*-injective.*

(2) *Each injective act is countably* $\sum$*-ideal* $\mathcal{M}$*-injective.*

(3) S *is* $\mathcal{M}$*-Noetherian.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) Applying the notations of Theorem 3.34, put $A = \prod_{n\in\mathbb{N}} E_n$. Then A is injective, and so by (2), $\bigoplus_{n\in\mathbb{N}} A$ is ideal $\mathcal{M}$-injective. But, $A = \prod_{n\in\mathbb{N}} E_n$ $= E_m \oplus \prod_{n\neq m} E_n$, for each $m \in \mathbb{N}$. Thus $\bigoplus_{m\in\mathbb{N}} A = \bigoplus_{m\in\mathbb{N}} E_m \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n =$ $E \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n$, which means that E is a direct summand of an ideal $\mathcal{M}$-injective act and hence is ideal $\mathcal{M}$-injective, by Theorem 3.30. The rest of the proof is similar to Theorem 3.34.

(3)⇒(1) is proved similar to Theorem 3.34. ■

4. Some Baer conditions

The condition that weak injectivity implies injectivity is known as the *Baer Criterion* for injectivity. In this section we give some Baer conditions.

Definition 4.40 An S-act A is called

(i) *quasi injective* if any S-map $f : B \to A$ from a subact B of A can be extended to A.

(ii) $\sum$*-injective* ($\sum$*-quasi injective)* if every direct sum of A with itself is injective (quasi injective).

Theorem 4.41 *The following conditions are equivalent:*

(1) *Each weakly* $\mathcal{M}$*-injective act is injective and* S *is* $\mathcal{M}$*-Noetherian.*

(2) *Each* $F\mathcal{M}$*-injective act is injective.*

(3) *Each $F\mathcal{M}$-injective act is quasi injective with a zero element.*

Proof. (1)⇒(2) Since S is $\mathcal{M}$-Noetherian, each right $\mathcal{M}$-ideal is finitely generated, so an act A is weakly $\mathcal{M}$-injective if and only if it is $F\mathcal{M}$-injective.

(2)⇒(3) is clear.

(3)⇒(2) Let A be $F\mathcal{M}$-injective and $E(A)$ be the injective hull of A. Then, by Theorem 3.32, $A \oplus E(A)$ is $F\mathcal{M}$-injective and hence by (3) quasi injective. The rest of the proof is quite similar to the proof of (4)⇒(1) of Theorem 4.42.

(2)⇒(1) Since every weakly $\mathcal{M}$-injective act is $F\mathcal{M}$-injective, we get the first part. To see that S is $\mathcal{M}$-Noetherian, using Theorem 3.36 we show that any direct sum of injective acts is injective. This is true because of (2) and since ,by Proposition 3.32, any direct sum of $F\mathcal{M}$-injective acts is $F\mathcal{M}$-injective. ∎

Theorem 4.42 *If S is an $\mathcal{M}$-Noetherian semigroup, then the following are equivalent:*

(1) *Each weakly (ideal) $\mathcal{M}$-injective act is injective.*

(2) *Each weakly (ideal) $\mathcal{M}$-injective act is $\sum$-injective and has a fixed element.*

(3) *Each weakly (ideal) $\mathcal{M}$-injective act is $\sum$-quasi injective and has a fixed element.*

(4) *Each weakly (ideal) $\mathcal{M}$-injective act is quasi injective and has a fixed element.*

Proof. (1)⇒(2) is true using Theorem 3.33. Also recall that each injective act has a fixed element.

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(1) Let A be weakly $\mathcal{M}$-injective and $E(A)$ be the injective hull of A. Then $A \oplus E(A)$ is weakly $\mathcal{M}$-injective and hence quasi injective by (4). So, considering the monomorphism $A \hookrightarrow E(A) \stackrel{\tau_{E(A)}}{\longrightarrow} A \oplus E(A)$, where $\tau_{E(A)}$ is the injection $x \mapsto (0, x)$, and the S-map $\tau_A : A \to A \oplus E(A)$, there exists an S-map $g : A \oplus E(A) \to A \oplus E(A)$ such that $g\tau_{E(A)} |_A = \tau_A$. This implies that $p_A g \tau_{E(A)} |_A = id_A$ and so A is a retract of the injective act $E(A)$ and hence is injective. ∎

Theorem 4.43 *Let for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. Then the following are equivalent:*

(1) *The direct sum of each family of weakly $\mathcal{M}$-injective acts is ideal $\mathcal{M}$-injective.*

(2) *S is $\mathcal{M}$-Noetherian and weak $\mathcal{M}$-injectivity implies ideal $\mathcal{M}$-injectivity.*

(3) *Each $F\mathcal{M}$-injective act is ideal $\mathcal{M}$-injective.*

Proof. (1)⇒(2) Similar to the proof of Theorem 3.34 it is shown that S is $\mathcal{M}$-Noetherian, the rest is clear.

(2)⇒(1) It is concluded from Theorem 3.34.

(2) ⇒ (3) Since S is $\mathcal{M}$-Noetherian, each right $\mathcal{M}$-ideal of S is finitely generated and so weak $\mathcal{M}$-injectivity coincides with $F\mathcal{M}$-injectivity. So (3) holds.

(3) ⇒ (2) Since every weakly $\mathcal{M}$-injective act is $F\mathcal{M}$-injective we get the first part. For the second part, let $\{A_i : i \in I\}$ be a family of injective acts. Since each injective act is $F\mathcal{M}$-injective and since each direct sum of $F\mathcal{M}$-injective acts is $F\mathcal{M}$-injective, thus the direct sum $\bigoplus_{i\in I} A_i$ is $F\mathcal{M}$-injective and so ideal $\mathcal{M}$-injective by the hypothesis. Thus S is $\mathcal{M}$-Noetherian, by Theorem 3.34. ∎

Acknowledgments. The author would like to thank Professor M. Mehdi Ebrahimi and Professor Mojgan Mahmoudi for their very useful comments and helpful conversations during this research.

References

[1] Banaschewski, B., *Injectivity and essential extensions in equational classes of algebras*, Queen's Papers in Pure and Applied Mathematics, 25 (1970), 131-147.

[2] Barzegar, H. and Ebrahimi, M.M, *Essentiality and injectivity relative to sequential purity of acts*, Semigroup Forum, 79 (2009), 128 144.

[3] Barzegar, H., Ebrahimi, M. M., Mahmoudi, M., *Essentiality and injectivity*, Appl. Categor. Struct., 18 (2010), 73-83.

[4] Berthiaume, P., *The injective envelope of S-sets*, Canad. Math. Bull. 10 (2) (1967), 261-273.

[5] Ebrahimi, M.M., *Algebra in a Grothendieck topos: injectivity in quasi-equational classes*, J. Pure Appl. Algebra, 26 (3) (1982), 269-280.

[6] Ebrahimi, M.M. and Mahmoudi, M., *The category of M-sets*, Ital. J. Pure Appl. Math., 9 (2001), 123-132.

[7] Herrlich H. and Ehrig, H., *The construct* **PRO** *of projection spaces: its internal structure*, Lecture Notes in Computer Science, 393 (1988), 286-293.

[8] Howie, J.M., *Fundamentals of Semigroup Theory*, Oxford Science Publications, Oxford, 1995.

[9] Kilp, M., Knauer U., and Mikhalev, A., *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[10] Mahmoudi M. and Shahbaz, L., *Characterizing semigroups by sequentially dense injective acts*, Semigroup Forum, 75 (1) (2007), 116-128.

[11] Satyanarayana, M., *Quasi and weakly injective S-systems*, Mathematische Nachrichten, 71 (1) (1973), 183-190.

[12] L.A. Skornjakov, L.A., *Axiomatizability of the class of injective M-sets*, Trudy Seminara im. I.G. Petrovskogo, 4 (1978), 233-239 (in Russian).

[13] Skornjakov, L.A., *On homological classification of monoids*, Sib. Math. J., 10 (1969), 1139-1143 (in Russian). Correction ibid., 12 (1971), 689.

Accepted: 12.03.2010