PSEUDO-D-LATTICES AND TOPOLOGIES GENERATED BY MEASURES

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Abstract. We prove that every modular measure on a pseudo-D-lattice L generates on L a lattice uniformity which makes uniformly continuous the pseudo-D-lattice operations. As an application, we obtain a uniqueness theorem for modular measures on pseudo D-lattices.

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1. Introduction

Effect algebras (alias D-posets) have been independently introduced in 1994 by D.J. Foulis and M.K. Bennett in [9] and by F. Chovanek and F. Kopka in [11] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics (see [14]) and in Mathematical Economics (see [18], [22], [10]), in particular of orthomodular posets and MV-algebras. After 1994, a great number of papers concerning effect algebras have been published.

In 2001, G. Georgescu and A. Jogulescu in [20] introduced the concept of a pseudo-MV-algebra, which is a non-commutative generalization of an MV-algebra, and A. Dvurecenskij and T. Vetterlein in [15] introduced the more general structure of a pseudo-effect algebra, which is a non-commutative generalization of effect algebra. The study of these structures is motivated by the non-commutative nature of certain psychological processes and quantum mechanical experiments (see [13]) and there even exists a programming language based on a non-commutative logic (see [7]). For a study, see for example [15], [16], [12], [13], [23], [26] and many others.

In the study of modular measures on lattice ordered effect algebras, essential tools are topological methods based on the theory of uniform lattices introduced by H. Weber in [24] (see for example [6], [4], [2], [5], [3]). In particular, a starting point for these topological methods was the result that the lattice uniformity generated by a modular measure on a lattice-ordered effect algebra E makes uniformly continuous the effect algebra operations of E (see [4]).

The aim of this paper is to set up the basis for topological methods in the more general study of modular measures on pseudo-D-lattices (i.e. lattice-ordered pseudo-effect algebras), for future development of a measure theory in pseudo-D-lattices.

Thus, in the first part of this paper, we prove that every modular measure on a pseudo-D-lattice L generates on L a D-uniformity, i.e. a lattice uniformity which makes uniformly continuous the pseudo-effect algebra operations, and we study D-uniformities on L.

In the second part, we first prove a uniqueness theorem for measures on pseudo-effect algebras which extends previous results of [21] for a particular case of effect algebra and of [4] in arbitrary effect algebras; then we give a first example of application of the results of the first part, proving that, for modular measures on pseudo-D-lattices, the above uniqueness theorem holds without completeness assumptions on L.

1. Preliminaries

A partial algebra $(E, +, 0, 1)$, where $+$ is a partial binary operation and $0, 1$ are constants, is called a *pseudo-effect algebra* if, for all $a, b, c \in E$, the following properties hold:

(1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist and in this case $(a + b) + c = a + (b + c)$.

(2) For any $a \in E$, there exist exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.

(3) If $a + b$ exists, there are $d, e \in E$ such that $a + b = d + a = b + e$.

(4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

We note that, if $+$ is commutative, then E becomes an effect algebra.

If we define $a \leq b$ if and only if there exists $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. If E is a lattice with respect to this order, then we say that E is a *lattice pseudo-effect algebra* or a *pseudo-D-lattice*.

If E is a pseudo-effect algebra, we can define two partial binary operations on E such that, for $a, b \in E$, a/b is defined if and only if $b\backslash a$ is defined if and only if $a \le b$ and in this case we have $(b\backslash a) + a = a + (a/b) = b$. In particular, we set ${}^{\perp}a = 1\backslash a$ and $a^{\perp} = a/1$.

In the sequel E is a *pseudo-effect algebra*, L is a *pseudo-D-lattice* and $(G, +)$ is a *topological Abelian group.*

For basic properties of pseudo-effect algebras we refer to [15], [13] and [26]. In particular, we need the following (see 1.4 and 1.6 of [15], 2.7, 2.9, 2.10 and 2.11 of [26] and [13], pp. 32 and 33).

Proposition 1.1. *Let $a, b, c \in E$. Then:*

(1) $a + 0 = 0 + a = a$.
(2) ${}^{\perp}(a^{\perp}) = ({}^{\perp}a)^{\perp} = a$.
(3) $a + b = c$ *if and only if* $a = {}^{\perp}(b + c^{\perp})$ *if and only if* $b = ({}^{\perp}c + a)^{\perp}$.
(4) $a + b = a + c$ *implies* $b = c$; $b + a = c + a$ *implies* $b = c$.
(5) $a + b$ *exists if and only if* $a \le {}^{\perp}b$ *if and only if* $b \le a^{\perp}$.
(6) $a \le b$ *if and only if* ${}^{\perp}b \le {}^{\perp}a$ *if and only if* $b^{\perp} \le a^{\perp}$.
(7) *If* $b + c$ *exists, then* $a \le b$ *if and only if (*$a + c$ *exists and)* $a + c \le b + c$.
(8) *If* $c + b$ *exists, then* $a \le b$ *if and only if (*$c + a$ *exists and)* $c + a \le c + b$.
(9) *If* $a \le b \le c$, *then* $c\backslash b \le c\backslash a$ *and* $b/c \le a/c$.
(10) *If* $a \le b \le c$, *then* $b\backslash a \le c\backslash a$ *and* $a/b \le a/c$.
(11) *If* $a \le b \le c$, *then* $(c\backslash b)/(c\backslash a) = b\backslash a$ *and* $(a/c)\backslash(b/c) = a/b$.
(12) *If* $a \le b \le c$, *then* $(c\backslash a)\backslash(b\backslash a) = c\backslash b$ *and* $(a/b)/(a/c) = b/c$.

Proposition 1.2. *Let $a, b, c \in L$. Then:*

(1) *If* $a \le c$ *and* $b \le c$, *then* $c\backslash(a\wedge b) = (c\backslash a)\vee(c\backslash b)$ *and* $(a\wedge b)/c = (a/c)\vee(b/c)$.
(2) *If* $a \le c$ *and* $b \le c$, *then* $c\backslash(a\vee b) = (c\backslash a)\wedge(c\backslash b)$ *and* $(a\vee b)/c = (a/c)\wedge(b/c)$.
(3) *If* $c \le a$ *and* $c \le b$, *then* $(a\wedge b)\backslash c = (a\backslash c)\wedge(b\backslash c)$ *and* $c/(a\wedge b) = (c/a)\wedge(b/c)$.
(4) *If* $c \le a$ *and* $c \le b$, *then* $(a\vee b)\backslash c = (a\backslash c)\vee(b\backslash c)$ *and* $c/(a\vee b) = (c/a)\vee(c/b)$.
(5) $((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) = 0$ *and* $(a/(a \vee b)) \wedge (b/(a \vee b)) = 0$.

In the sequel, we set $\Delta = \{(a, b) \in E \times E : a = b\}$.

If $a \le b$, we set $[a, b] = \{c \in E : a \le c \le b\}$.

If $(a_n)_{n\in\mathbb{N}}$ is a sequence in E and $a \in E$, we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) if $(a_n)_{n\in\mathbb{N}}$ is increasing and $a = \sup_n a_n$ (respectively, $(a_n)_{n\in\mathbb{N}}$ is decreasing and $a = \inf_n a_n$).

If $a_1, \ldots, a_n \in E$, we inductively define $a_1 + \cdots + a_n = (a_1 + \cdots + a_{n-1}) + a_n$, provided that the right hand side exists. We say that the finite sequence $(a_1, ..., a_n)$ of E is *orthogonal* if $a_1 + \cdots + a_n$ exists. Given an infinite sequence $(a_n)_{n\in\mathbb{N}}$, we say that it is *orthogonal* if, for every positive integer n, $a_1 + \cdots + a_n$ exists. If, moreover, $\sup_{n\in\mathbb{N}}(a_1 + \cdots + a_n)$ exists, we set $\sum_{n=1}^{\infty} a_n = \sup_{n\in\mathbb{N}}(a_1 + \cdots + a_n)$.

We say that E is *Archimedean* if, for every $a \in E$ with $a \ne 0$, there exists an integer $k > 0$ such that ka exists and $(k+1)a$ does not exist, where $ka = a+\cdots+a$ k times.

We say that E is *σ-complete* if, for every orthogonal sequence (a_n), $\sum_{n=1}^{\infty} a_n$ exists.

If E is a pseudo-D-lattice, we set $a \,{}^*\!\triangle\, b = (a \vee b)\backslash(a \wedge b)$ and $a \triangle^* b = (a \wedge b)/(a \vee b)$.

A function $\mu : E \to G$ is said to be a *measure* if, for every $a, b \in E$ with $a \leq b$, $\mu(b) - \mu(a) = \mu(b\backslash a) = \mu(a/b)$. It is easy to see that μ is a measure if and only if, for every $a, b \in E$ such that $a + b$ exists, $\mu(a + b) = \mu(a) + \mu(b)$. We say that μ is *σ-additive* if, for every orthogonal sequence (a_n) in E such that $a = \sum_{n=1}^{\infty} a_n$ exists, $\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$. If $\mu : L \to G$, we say that μ is *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

A uniformity $\mathcal{U}$ on L is said to be a *lattice uniformity* if the lattice operations $\vee$ and $\wedge$ are uniformly continuous with respect to $\mathcal{U}$. For a study, see [24].

As proved in [19], if L_1 is a lattice, every modular function $\mu : L_1 \to G$ generates on L_1 a lattice uniformity $\mathcal{U}(\mu)$, called *μ-uniformity,* which is the weakest lattice uniformity which makes μ uniformly continuous and a basis of $\mathcal{U}(\mu)$ is the family consisting of the sets

$$U_W = \{(a, b) \in L_1 \times L_1 : \mu(c) - \mu(d) \in W \ \forall\ c, d \in [a \wedge b, a \vee b], d \leq c\},$$

where W is a neighbourhood of 0 in G.

A lattice uniformity $\mathcal{U}$ on L_1 is said to be *exhaustive* if every monotone sequence in L_1 is a Cauchy sequence in $\mathcal{U}$, *σ-order-continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L_1 implies that $\{a_n\}$ converges to a in $\mathcal{U}$, and *order-continuous* (o.c.) if the same condition holds for nets.

If $\mu : L_1 \to G$ is a modular function, μ is said to be *exhaustive* (respectively, *σ-o.c.* or *o.c.*) if $\mathcal{U}(\mu)$ is exhaustive (respectively, σ-o.c. or o.c.).

By 3.5 and 3.6 of [25], we have that a modular function μ is exhaustive if and only if $\mu(a_{n+1}) - \mu(a_n)$ converges to 0 for every monotone sequence $(a_n)_{n\in\mathbb{N}}$ in L_1, μ is σ-o.c. if and only if $a_n \uparrow a$ or $a_n \downarrow a$ imply that $(\mu(a_n))$ converges to $\mu(a)$, and μ is o.c. if and only if, for every monotone net $(a_\alpha)_{\alpha\in J}$ order convergent to a, $(\mu(a_\alpha))$ converges to $\mu(a)$.

2. D-uniformities and modular measures

In this section we introduce the concept of D-uniformity on L, which arises in a natural way from the study of modular measures since, as we will see in Theorem 2.9, every G-valued modular measure μ on L generates on L a D-uniformity.

First we need some preliminaries.

Lemma 2.1. *Let $a, b, c \in E$.*

(1) *If $a+b$ exists and $a+b \leq c$, then $c\backslash(a+b) = (c\backslash b)\backslash a$ and $(a+b)/c = b/(a/c)$.*

(2) *If $a + b$ exists, then $a + b = ({}^{\perp}b\backslash a)^{\perp} = {}^{\perp}(b/a^{\perp})$.*

(3) *If $a \leq b$, then $b\backslash a = {}^{\perp}(a + b^{\perp})$ and $a/b = ({}^{\perp}b + a)^{\perp}$.*

Proof. (1) Set $d = c\backslash(a+b)$. Then $c = d+(a+b) = (d+a)+b$, whence $d+a = c\backslash b$. Therefore $d = (c\backslash b)\backslash a$.

In a similar way, setting $e = (a+b)/c$, we have $b+e = a/c$ and therefore $e = b/(a/c)$.

(2) Setting $c = 1$ in (1), we have ${}^\perp(a+b) = {}^\perp b\backslash a$ and $(a+b)^\perp = b/a^\perp$. Therefore, by Proposition 1.1-(2), we obtain $a+b = ({}^\perp b\backslash a)^\perp$ and $a+b = {}^\perp(b/a^\perp)$.

(3) By Proposition 1.1-(5), $a \leq b$ implies that $a + b^\perp$ exists. Then (3) follows from (2) and Proposition 1.1-(2). ∎

Lemma 2.2. *Let $a, b \in E$, with $a \leq b$. Then*

$$a/b = a^\perp\backslash b^\perp \text{ and } b\backslash a = {}^\perp b/{}^\perp a.$$

Proof. It is sufficient to set $c = 1$ in Proposition 1.1-(11). ∎

Lemma 2.3. *If $a, b \in L$, then*

$$a \triangle^* b = a^\perp \,{}^*\!\triangle\, b^\perp \text{ (and } a \,{}^*\!\triangle\, b = {}^\perp a \triangle^* {}^\perp b).$$

Proof. By Lemma 2.2, we have $a \triangle^* b = (a \wedge b)/(a \vee b) = (a \wedge b)^\perp\backslash(a \vee b)^\perp = (a^\perp \vee b^\perp)\backslash(a^\perp \wedge b^\perp) = a^\perp \,{}^*\!\triangle\, b^\perp$. The other equality can be proved in a similar way. ∎

Lemma 2.4. *Let $c, d \in E$ be such that $c \leq d$. Set*

$$I_{c,d} = \{a \in E : \exists\, r, s \in [c, d] : r \leq s, a = s\backslash r\}$$

and

$$J_{c,d} = \{a \in E : \exists\, r, s \in [c, d] : r \leq s, a = r/s\}.$$

Then $I_{c,d} = [0, d\backslash c]$ and $J_{c,d} = [0, c/d]$.

Proof. We prove the first equality. The other equality can be proved in a similar way. Let $a \in I_{c,d}$ and choose $r, s \in E$ such that $c \leq r \leq s \leq d$ and $a = s\backslash r$. By Proposition 1.1-(9) and (10), we obtain $a \leq d\backslash r \leq d\backslash c$. Conversely, let $a \in [0, d\backslash c]$. Then, by Proposition 1.1-(8), $a + c$ exists and $a + c \leq d$. Set $s = a + c$ and $r = c$. Then $c = r \leq s \leq d$ and $a = (a+c)\backslash c = s\backslash r$. ∎

Proposition 2.5. *Let $\mathcal{U}$ be a uniformity on E. Set $E_1 = \{(a, b) \in E \times E : b \leq a\}$ and $E_2 = \{(a, b) \in E \times E : a + b$ exists $\}$ $(= \{(a, b) \in E \times E : b \leq a^\perp\})$. Then the following conditions are equivalent:*

1. *The operations $(a, b) \in E_2 \to a + b \in E$, $a \in E \to {}^\perp a \in E$ and $a \in E \to a^\perp \in E$ are uniformly continuous with respect to $\mathcal{U}$.*
2. *The operations $(a, b) \in E_1 \to a\backslash b \in E$ and $(a, b) \in E_1 \to b/a \in E$ are uniformly continuous with respect to $\mathcal{U}$.*

(3) *The operations* $(a,b) \in E_1 \to a \backslash b \in E$ *and* $a \in E \to a^{\perp} \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(4) *The operations* $(a,b) \in E_1 \to b/a \in E$ *and* $a \in E \to {}^{\perp}a \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(5) *The operation* $(a,b) \in E_2 \to a^{\perp} \backslash b \ (= a/{}^{\perp}b) \in E$ *is uniformly continuous with respect to* $\mathcal{U}$.

Moreover, if E *is a pseudo D-lattice and* $\mathcal{U}$ *is a lattice uniformity on* E, *each of the above conditions is equivalent to each of the following:*

(6) *The operations* $(a,b) \in E \times E \to a \,{}^{*}\!\triangle\, b \in E$ *and* $a \in E \to a^{\perp} \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(7) *The operations* $(a,b) \in E \times E \to a \triangle^{*} b \in E$ *and* $a \in E \to {}^{\perp}a \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(8) *The operation* $(a,b) \in E \times E \to a^{\perp} \,{}^{*}\!\triangle\, b \ (= a \triangle^{*} {}^{\perp}b) \in E$ *is uniformly continuous with respect to* $\mathcal{U}$.

Proof. (1) ⇒ (2) The uniform continuity of \ and / follows from Lemma 2.1-(3).

(2) ⇒ (3) it is trivial.

(2) ⇒ (4) it is trivial.

(3) ⇒ (1) It is clear that the operation $a \in E \to {}^{\perp}a$ is uniformly continuous. The uniform continuity of + follows from Lemma 2.1-(2).

(4) ⇒ (1) is similar to the proof of (3) ⇒ (1).

(3) ⇒ (5) is trivial.

(5) ⇒ (3) Set $a * b = a^{\perp} \backslash b$. Then, since $a * 0 = a^{\perp}$ and $0 * a = {}^{\perp}a$, we obtain $a \backslash b = ({}^{\perp}a)^{\perp} \backslash b = {}^{\perp}a * b = (0 * a) * b$ and therefore \ is uniformly continuous.

(6) ⇒ (3), (7) ⇒ (4), (8) ⇒ (5) and (6) ⇒ (8) are trivial.

(7) ⇒ (6) follows from Lemma 2.3 and the equality $a^{\perp} = a \triangle^{*} 1$.

(4) ⇒ (7) follows from the definition of $a \triangle^{*} b$. ∎

Definition 2.6. We say that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if it satisfies one of the conditions in the above proposition (and hence all).

Therefore, if L is a lattice-ordered effect algebra, a D-uniformity in the sense of Definition 2.6 is a D-uniformity according to [4].

If we set, for subsets U and V of $L \times L$,

$$U \backslash V = \{(a \backslash c, b \backslash d) : c \leq a, d \leq b, (a,b) \in U, (c,d) \in V\},$$

$$U/V = \{(c/a, d/b) : c \leq a, d \leq b, (a,b) \in U, (c,d) \in V\},$$

$$U^{\perp} = \{(a^{\perp}, b^{\perp}) \in L \times L : (a,b) \in U\},$$

$${}^{\perp}U = \{({}^{\perp}a, {}^{\perp}b) \in L \times L : (a,b) \in U\}$$

it is clear that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if and only if, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $V \backslash V \subseteq U$ and $W^{\perp} \subseteq U$ if and only if, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $V/V \subseteq U$ and ${}^{\perp}W \subseteq U$.

Moreover the following result holds.

Proposition 2.7. *Let $\mathcal{U}$ be a lattice uniformity on L. Then $\mathcal{U}$ is a D-uniformity if and only if, for every $U \in \mathcal{U}$, there exists $V, W \in \mathcal{U}$ such that $V \backslash \Delta \subseteq U$, $\Delta \backslash V \subseteq U$ and $W^\perp \subseteq U$.*

Proof. Suppose that the above conditions are satisfied and we prove that $\mathcal{U}$ is a lattice uniformity.

Since it follows by assumption that $\perp$ is uniformly continuous, we have only to prove that, for every $U \in \mathcal{U}$, there exists $V_2 \in \mathcal{U}$ such that $V_2 \backslash V_2 \subseteq U$.

Let $U \in \mathcal{U}$ and choose $V, V_1, V_2 \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$, $V_1 \backslash \Delta \subseteq V$, $\Delta / V_1 \subseteq V$ and $V_2 \wedge V_2 \subseteq V_1$.

We prove that $V_2 \backslash V_2 \subseteq U$. Let $(a, b), (c, d)$ be in V_2 such that $c \leq a$ and $d \leq b$. We prove that $(a \backslash c, b \backslash d) \in U$. Indeed, since $(c, c \wedge d) \in \Delta \wedge V_2 \subseteq V_2 \wedge V_2 \subseteq V_1$, we have

$$(*) \qquad (a \backslash c, a \backslash (c \wedge d)) \in \Delta \backslash V_1 \subseteq V.$$

Moreover, from $V_2 \subseteq V_1$, we get

$$(**) \qquad (a \backslash (c \wedge d), b \backslash (c \wedge d)) \in V_1 \backslash \Delta \subseteq V.$$

Finally, since $(c \wedge d, d) \in V_2 \wedge V_2 \subseteq V_1$, we have

$$(***) \qquad (b \backslash (c \wedge d), b \backslash d) \in \Delta \backslash V_1 \subseteq V.$$

From $(*)$, $(**)$ and $(***)$, we obtain $(a \backslash c, b \backslash d) \in V \circ V \circ V \subseteq U$. ∎

In the sequel, if $\mu : L \to G$ is a function and W is a neighbourhood of 0 in G, we set

$$\begin{aligned} U_W &= \{(a, b) \in L \times L : \mu(s) - \mu(r) \in W \ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s\}, \\ A_W &= \{(a, b) \in L \times L : \mu(c) \in W \ \forall\, c \leq a \,{}^*\!\triangle\, b\}, \\ B_W &= \{(a, b) \in L \times L : \mu(c) \in W \ \forall\, c \leq a \triangle^* b\}. \end{aligned}$$

Lemma 2.8. *Let $\mu : L \to G$ be a measure and W a neighbourhood of 0 in G. Then $U_W = A_W = B_W$.*

Proof. We use the notations of Lemma 2.4. Applying Lemma 2.4, we have

$$\begin{aligned} A_W &= \{(a, b) \in L \times L :\ \forall\, c \in [0, a \,{}^*\!\triangle\, b],\ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in I_{a \wedge b, a \vee b},\ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s,\ \mu(s \backslash r) \in W\} = U_W \\ &= \{(a, b) \in L \times L :\ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s,\ \mu(r / s) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in J_{a \wedge b, a \vee b},\ \ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in [0, a \triangle^* b],\ \ \mu(c) \in W\} = B_W. \end{aligned}$$

∎

Theorem 2.9. *Let $\mu : L \to G$ be a modular measure. Then the μ-uniformity $\mathcal{U}(\mu)$ is a D-uniformity on L and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets A_W, where W is a neighbourhood of* 0 *in G.*

Proof. By Lemma 2.8, a base of $\mathcal{U}(\mu)$ is the family consisting of the sets A_W, where W is a neighbourhood of 0 in G. Then, by Proposition (2.7), it is sufficient to prove the following conditions:

(1) $A_W^{\perp} = A_W$.

(2) $A_W \backslash \Delta \subseteq A_W$.

(3) $\Delta / A_W \subseteq A_W$.

Proof of (1). By Lemma 2.3 and 2.8, we have

$$\begin{aligned} A_W^{\perp} &= B_W^{\perp} = \{(a^{\perp}, b^{\perp}) : (a,b) \in B_W\} = \{(a^{\perp}, b^{\perp}) :\ \forall\, c \le a \mathbin{\triangle^*} b,\ \mu(c) \in W\} \\ &= \{(a^{\perp}, b^{\perp}) :\ \forall\, c \le a^{\perp} \mathbin{{}^*\!\triangle} b^{\perp},\ \mu(c) \in W\} = A_W. \end{aligned}$$

Proof of (2). It is sufficient to prove that, for every a, b, c in L with $c \le a$ and $c \le b$, $(a\backslash c) \mathbin{{}^*\!\triangle} (b\backslash c) = a \mathbin{{}^*\!\triangle} b$.

Set $d = a\backslash c$ and $e = b\backslash c$. By Proposition 1.2-(3) and (4), we have $d \vee e = (a \vee b)\backslash c$ and $d \wedge e = (a \wedge b)\backslash c$. Therefore, by Proposition 1.1-(12), we obtain $d \mathbin{{}^*\!\triangle} e = ((a \vee b)\backslash c)\backslash((a \wedge b)\backslash c) = (a \vee b)\backslash(a \wedge b) = a \mathbin{{}^*\!\triangle} b$.

Proof of (3). Since $A_W = B_W$ by Lemma 2.8, it is sufficient to prove that, for every a, b, c in L with $a \le c$ and $b \le c$, $(c\backslash a) \mathbin{\triangle^*} (c\backslash b) = a \mathbin{{}^*\!\triangle} b$.

Set $d = c\backslash a$ and $e = c\backslash b$. By Proposition 1.2-(1) and (2), we obtain $d \vee e = c\backslash(a \wedge b)$ and $d \wedge e = c\backslash(a \vee b)$. By Proposition 1.1-(11), we get $d \mathbin{\triangle^*} e = (d \wedge e)/(d \vee e) = (c\backslash(a \vee b))/(c\backslash(a \wedge b)) = (a \vee b)\backslash(a \wedge b) = a \mathbin{{}^*\!\triangle} b$. ∎

Definition 2.10. A DV-congruence on L (after A. Dvurecenskij and T. Vetterlein) is an equivalence relation N which satisfies the following conditions:

(a) For every $a, b \in L$, if $(a, c) \in N$, $(b, d) \in N$, $a + b$ and $c + d$ exist, then $(a + b, c + d) \in N$.

(b) If $a + b$ exists, then, for every $c \in L$ such that $(c, a) \in N$, there exists $d \in L$ such that $(d, b) \in N$ and $c + d$ exists; and, for every $h \in L$ such that $(h, b) \in N$, there exists $k \in L$ such that $(k, a) \in N$ and $k + h$ exists.

Proposition 2.11. *Let N be a DV-congruence on E. Define the operation $+$ on the quotient E/N in the following way: For every $\hat{a}, \hat{b} \in E/N$, $\hat{a} + \hat{b} = \hat{c}$ if and only if there exist $a' \in \hat{a}$, $b' \in \hat{b}$ and $c' \in \hat{c}$ such that $a' + b' = c'$ in E. Then:*

(1) *$+$ is well defined on E/N and $(E/N, +, \hat{0}, \hat{1})$ is a pseudo-effect algebra.*

(2) *If $c \ge b$, then $\widehat{c\backslash b} = \hat{c}\backslash\hat{b}$ and $\widehat{b/c} = \hat{b}/\hat{c}$.*

Proof. (1) has been proved in 3.3 of [16].

(2) follows from the definition of $+$ in E/N, since, if we set $a = c\backslash b$, we have $\hat{a} + \hat{b} = \hat{c}$, whence $\hat{a} = \hat{c}\backslash\hat{b}$. In a similar way we obtain the other equality. ∎

The aim of the next two theorems is to obtain also in pseudo D-lattices a technique based on the "completion method" of H. Weber (see [24] and [25]) which allowed in many cases to reduce the study of exhaustive modular measures on D-lattices to the study of o.c. modular measures on complete D-lattices (see for example [1]–[5]).

Theorem 2.12. *Let $\mathcal{U}$ be a D-uniformity on L. Then the following properties hold:*

(1) $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\}$ *is a DV-congruence and a lattice congruence.*

(2) *The quotient* $\hat{L} = L/N(\mathcal{U})$ *is a pseudo-D-lattice.*

(3) *Setting, for* $U \in \mathcal{U}$, $\widehat{U} = \{(\hat{a}, \hat{b}) \in \hat{L} \times \hat{L} : (a,b) \in U\}$, *the quotient uniformity* $\widehat{\mathcal{U}} = \{\widehat{U} : U \in \mathcal{U}\}$ *is a Hausdorff D-uniformity on* $\hat{L}$.

(4) *If G is Hausdorff and $\mu : L \to G$ is a modular measure which is uniformly continuous with respect to $\mathcal{U}$, then the function $\hat{\mu} : \hat{L} \to G$ defined as $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ is a well defined modular measure on $\hat{L}$ and the D-uniformity generated by $\hat{\mu}$ coincides with $\widehat{\mathcal{U}}$.*

Proof. (1) $N(\mathcal{U})$ is a lattice congruence by 1.2.2 of [24]. Moreover, it is clear that $N(\mathcal{U})$ satisfies condition (a) of Definition 2.10. Indeed, it is sufficient to observe that, since $\mathcal{U}$ is a D-uniformity, then, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$, where

$$V + V = \{(a+c, b+d) : (a,b) \in V, (c,d) \in V, a+c \text{ and } b+d \text{ exist}\}.$$

Now we prove condition (b) of Definition 2.10. Since $\mathcal{U}$ is a D-uniformity, it is clear that $(a,b) \in N(\mathcal{U})$ implies $(a^{\perp}, b^{\perp}) \in N(\mathcal{U})$ and $({}^{\perp}a, {}^{\perp}b) \in N(\mathcal{U})$.

Now, suppose that $a+b$ exists and let $c \in L$ be such that $(c,a) \in N(\mathcal{U})$. We prove that there exists $d \in L$ such that $(d,b) \in N(\mathcal{U})$ and $c+d$ exists. Set $d = c^{\perp} \wedge b$. By Proposition 1.1-(5), $c+d$ exists since $d \le c^{\perp}$. Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $V \wedge \Delta \subseteq U$. Since $a+b$ exists, by Proposition 1.1-(5) we have that $b \le a^{\perp}$. Therefore, we get $(d,b) = (c^{\perp} \wedge b, a^{\perp} \wedge b) = (c^{\perp}, a^{\perp}) \wedge (b,b) \in V \wedge \Delta \subseteq U$. Hence, $(d,b) \in N(\mathcal{U})$.

In a similar way, let $h \in L$ be such that $(h,b) \in N(\mathcal{U})$. We prove that there exists $k \in L$ such that $(k,a) \in N(\mathcal{U})$ and $k+h$ exists. Set $k = {}^{\perp}h \wedge a$. Since $k \le {}^{\perp}h$, by Proposition 1.1-(5) we have that $h+k$ exists. Moreover, since $a+b$ exists, we have $a \le {}^{\perp}b$. Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $\Delta \wedge V \subseteq U$. Then we obtain $(k,a) = (a \wedge {}^{\perp}h, a \wedge {}^{\perp}b) = (a,a) \wedge ({}^{\perp}h, {}^{\perp}b) \in \Delta \wedge V \subseteq U$.

(2) By Proposition 2.11, $\hat{L}$ is a pseudo-effect-algebra. It remains to prove that $\hat{L}$ is a pseudo-D-lattice. By (3) of [16] (page 7), we have that $\hat{a} \le \hat{b}$ if and only if there exists $h \in L$ such that $\hat{a} + \hat{h} = \hat{b}$ if and only if there exist $c, d \in L$ such that $(c,a) \in N(\mathcal{U}), (d,h) \in N(\mathcal{U}), c+d$ exists and $(c+d, b) \in N(\mathcal{U})$.

Moreover, by 1.2.3 of [24], $\hat{L}$ is a lattice with respect to the following order: $\hat{a} \le' \hat{b}$ if and only if there exist $c, k \in L$ such that $(c,a) \in N(\mathcal{U}), (k,b) \in N(\mathcal{U})$ and $c \le k$. Therefore, it is sufficient to observe that $\le$ and $\le'$ coincide.

(3) It is known by Proposition 1.2.4 of [24] that $\widehat{\mathcal{U}}$ is a Hausdorff lattice uniformity. To prove that $\widehat{\mathcal{U}}$ is a D-uniformity, we apply Proposition 2.7. Let $\widehat{U} \in \widehat{\mathcal{U}}$ and choose $W \in \mathcal{U}$ such that $W^{\perp} \subseteq U$. Applying Proposition 2.11-(3), we obtain that $\widehat{W}^{\perp} = \widehat{W^{\perp}}$ and therefore $\widehat{W}^{\perp} \subseteq \widehat{U}$.

Now, choose $V \in \mathcal{U}$ closed such that $V \backslash V \subseteq U$. We prove that $\widehat{V} \backslash \widehat{\Delta} \subseteq \widehat{U}$. Let $a, b, c \in L$ be such that $(\hat{a}, \hat{b}) \in \widehat{V}$, $\hat{c} \leq \hat{a}$ and $\hat{c} \leq \hat{b}$. We prove that $(\hat{a} \backslash \hat{c}, \hat{b} \backslash \hat{c}) \in \widehat{U}$. Since $\hat{c} \leq \hat{a}$ and $\hat{c} \leq \hat{b}$, by 1.2.3 of [24] we can find $d, e, r, s \in L$ such that $\hat{d} = \hat{c}$, $\hat{e} = \hat{a}$, $\hat{r} = \hat{c}$, $\hat{s} = \hat{b}$, $d \leq e$ and $r \leq s$. Since $(\hat{e}, \hat{s}) = (\hat{a}, \hat{b}) \in \widehat{V}$, by Proposition 1.2.4 of [24] we obtain that $(e, s) \in V$. Moreover, since $\hat{d} = \hat{r}$, we have that $(d, r) \in N(\mathcal{U})$. Therefore, we get $(e \backslash d, s \backslash r) \in V \backslash V \subseteq U$. Set $h = e \backslash d$ and $k = s \backslash r$. Hence we have $(\hat{h}, \hat{k}) \in \widehat{U}$. Now it is sufficient to observe that $(\hat{h}, \hat{k}) = (\hat{a} \backslash \hat{c}, \hat{b} \backslash \hat{c})$ by Proposition 2.11-(2).

The other condition of Proposition 2.7 can be proved in a similar way.

(4) is known by the theory of uniform lattices (see [25]). Indeed, since $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous, we have $\mathcal{U}(\mu) \leq \mathcal{U}$, from which $N(\mathcal{U}) \subseteq N(\mathcal{U}(\mu))$. By Propositions 2.5 and 3.1 of [25], $(a, b) \in N(\mathcal{U}(\mu))$ if and only if μ is constant on the interval $[a \wedge b, a \vee b]$. Then, if $\hat{a} = \hat{b}$, we have $\mu(a) = \mu(b)$ and therefore $\hat{\mu}$ is well defined on $\hat{L}$. It is also known that $\hat{\mu}$ is a modular function, too, and $\mathcal{U}(\hat{\mu}) = \widehat{\mathcal{U}}$. Here we have only to observe that, because of the definition of $+$ in $\hat{L}$ (see Proposition 2.11), $\hat{\mu}$ is a measure, too. ∎

Theorem 2.13. *Let $\mathcal{U}$ be a Hausdorff D-uniformity on L and let $(\tilde{L}, \widetilde{\mathcal{U}})$ be the uniform completion of $(L, \mathcal{U})$. Then the following properties hold:*

(1) *The lattice operations $\vee$ and $\wedge$ and the pseudo-D-lattice operations $\backslash$ and $/$ can be extended in a unique way such that $\tilde{L}$ becomes a pseudo-D-lattice.*
(2) *$\widetilde{\mathcal{U}}$ is a D-uniformity on $\tilde{L}$.*
(3) *If $\mathcal{U}$ is exhaustive, then $\tilde{L}$ is complete as lattice and $\widetilde{\mathcal{U}}$ is o.c.*
(4) *If G is complete and Hausdorff and $\mu : L \to G$ is a modular measure which is uniformly continuous with respect to $\mathcal{U}$, then μ can be extended in a unique way to a modular measure $\tilde{\mu} : \tilde{L} \to G$ which is uniformly continuous with respect to $\widetilde{\mathcal{U}}$ and o.c. and $\mu(L)$ is dense in $\tilde{\mu}(\tilde{L})$.*

Proof. (1) By Proposition 1.3.1 of [24], it is known that the lattice operations $\vee$ and $\wedge$ can be extended in a unique way such that $\tilde{L}$ becomes a lattice and $\widetilde{\mathcal{U}}$ is a lattice uniformity.

Then the set $\tilde{L}' = \{(a, b) \in \tilde{L} \times \tilde{L} : b \leq a\}$ coincides with the closure in $(\tilde{L}, \widetilde{\mathcal{U}})$ of the set $\{(a, b) \in L \times L : b \leq a\}$.

Denote again by $/$ and $\backslash$ the uniformly continuous extensions, respectively, of $/$ and $\backslash$ to $\tilde{L}'$.

To prove that $\tilde{L}$ is a pseudo-D-lattice, it is sufficient, by Theorem 2.7 of [26], to prove that $\backslash$ and $/$ have the following properties:

(a) If $a \le b \le c$, then $b/c \le a/c$, $c\backslash b \le c\backslash a$, $(a/c)\backslash(b/c) = a/b$, $(c\backslash b)/(c\backslash a) = b\backslash a$.

(b) For every $a \in L$, $a\backslash 0 = 0/a = a$.

(a) Let a, b, c in $\tilde{L}$ such that $a \le b \le c$. Choose nets (a_α), (b_α) and (c_α) in L convergent, respectively, to a, b and c in $(\tilde{L}, \widetilde{\mathcal{U}})$. Without loss of generality, we may assume that they are indexed in the same way. Moreover, we may suppose that $a_\alpha \le b_\alpha \le c_\alpha$ for each α, since (a_α) can be replaced by $(a_\alpha \wedge b_\alpha)$ and (b_α) by $(b_\alpha \wedge c_\alpha)$. Therefore, by the definition of $\backslash$ and $/$, we obtain $b/c = \lim_\alpha(b_\alpha/c_\alpha) \le \lim_\alpha(a_\alpha/c_\alpha) = a/c$ and $c\backslash b = \lim_\alpha(c_\alpha\backslash b_\alpha) \le \lim_\alpha(c_\alpha\backslash a_\alpha) = c\backslash a$. Moreover, since $b_\alpha/c_\alpha \le a_\alpha/c_\alpha$, we have $(a/c)\backslash(b/c) = \lim_\alpha((a_\alpha/c_\alpha)\backslash(b_\alpha/c_\alpha)) = \lim_\alpha(a_\alpha/b_\alpha) = a/b$. In a similar way, since $c_\alpha\backslash b_\alpha \le c_\alpha\backslash a_\alpha$, we obtain that $(c\backslash b)/(c\backslash a) = b\backslash a$.

In the same way we obtain (b).

(2) It is known that a base of $\widetilde{\mathcal{U}}$ consists of the sets $\{\overline{U} : U \in \mathcal{U}\}$, where $\overline{U}$ is the closure of U in $\widetilde{\mathcal{U}}$. Then, to prove that $\widetilde{\mathcal{U}}$ is a D-uniformity, it is sufficient to prove that, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $\overline{V}\backslash\overline{V} \subseteq U$ and $\overline{W}/\overline{W} \subseteq U$.

Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $V\backslash V \subseteq U$. Let $(a, b) \in \overline{V}$ and $(c, d) \in \overline{V}$ be such that $c \le a$ and $d \le b$. Choose nets $((a_\alpha, b_\alpha))$ and $((c_\alpha, d_\alpha))$ in V convergent, respectively, to (a, b) and (c, d) in $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}$. We may suppose that, for each α, $c_\alpha \le a_\alpha$ and $d_\alpha \le b_\alpha$. Then $((a_\alpha\backslash c_\alpha, b_\alpha\backslash d_\alpha)) \in V\backslash V \subseteq U$ and, by the definition of $\backslash$, converges to $(a\backslash c, b\backslash d)$. Therefore we get $(a\backslash c, b\backslash d) \in \overline{U}$.

In a similar way we prove that there exists $W \in \mathcal{U}$ such that $\overline{W}/\overline{W} \subseteq \overline{U}$.

(3) and (4) have been proved in Proposition 3.7 of [25]. We have only to observe that the continuity of $\tilde{\mu}$ and the definition of $\backslash$ and $/$ on $\tilde{L}$ imply that $\tilde{\mu}$ is a measure. ■

Remark. In Theorem 4.6 of [17], it is proved that every Archimedean (and, therefore, every σ-complete) pseudo-MV-algebra is commutative. This is not true if L is an Archimedean pseudo-effect algebra, as the next examples prove.

Let $E = \{0, 1, a, b, c\}$, where a, b, and c are not comparable. Define $a + b = b + c = c + a = 1$, while $b + a$, $c + b$ and $a + c$ are undefined. Then, E is a complete modular pseudo-D-lattice which is not commutative.

Moreover let $\mu : E \to [0, 1]$ be defined as $\mu(a) = \mu(b) = \mu(c) = 1/2$, $\mu(0) = 0$ and $\mu(1) = 1$. Then μ is a modular measure on E with $N(\mathcal{U}(\mu)) = \Delta$ and therefore $\tilde{E} = E/N(\mathcal{U}(\mu)) = E$.

Now we obtain an infinite example considering the set F of all sequences with values in E, in which we define $(a_n) + (b_n)$ if and only if, for each $n \in \mathbb{N}$, $a_n + b_n$ exists and in this case $(a_n) + (b_n) = (a_n + b_n)$.

It is easy to see that, since E is finite, F is a complete pseudo-D-lattice. Moreover, if we define $\lambda : F \to [0, 1]$ as $\lambda(a) = \sum_{n=1}^{\infty} \mu(a_n)/2^n$ for $a = (a_n) \in F$, we obtain a modular measure on F with $\lambda(a) > 0$ for every $a \in F$ with $a \ne 0$. Then $\tilde{F} = F/N(\mathcal{U}(\lambda)) = F$.

3. Uniqueness theorems

In this section we prove a uniqueness theorem for measures on pseudo-effect-algebras and we apply the results of the previous section to prove that, for modular measures on pseudo D-lattices, the uniqueness theorem holds without completeness assumptions.

We say that E *has the interpolation property* if, for every sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in E, with $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for each n, there exists $a \in E$ such that, for each n, $a_n \le a \le b_n$.

It is clear that, if E is σ-complete, then E has the interpolation property.

If $\mu : E \to G$ is a measure, we say that:

- E is *μ-chained* if, for every neighbourhood W of 0 in G and every $a \in E$, there exist $a_0, a_1, \ldots, a_r$ in E such that $0 = a_0 \le a_1 \le \ldots \le a_r = a$ and $\mu(c) - \mu(d) \in W$ whenever $c, d \in [a_{i-1}, a_i]$ for some $i \in \{1, \ldots, r\}$.
- μ is *strongly continuous* if, for every neighbourhood W of 0 in G and every $a \in E$, there exists an orthogonal finite family $(b_1, \ldots, b_r)$ in E such that $b_1 + \ldots + b_r = a$ and $\mu(b) \in W$ whenever $b \le b_i$ for some $i \le r$.
- If G is a linear space, μ is *convex-ranged* if, for every $a \in E$, $\mu([0, a])$ is convex.

Lemma 3.1. *Let a, b, c in E.*

(1) *If $c \le a$ and $a + b$ exists, then $(c/a) + b$ exists and $c/(a + b) = (c/a) + b$.*

(2) *If $c \le a$ and $b + a$ exists, then $b + (a\backslash c)$ exists and $(b + a)\backslash c = b + (a\backslash c)$.*

(3) *If $a \le b \le c$, then $(a/b) + (b/c)$ exists and $(a/b) + (b/c) = a/c$.*

(4) *If $a \le b \le c$, then $(c\backslash b) + (b\backslash a)$ exists and $(c\backslash b) + (b\backslash a) = c\backslash a$.*

Proof. (1) Since $a + b$ and $c + (c/a) = a$ exist and $+$ is associative, then $d = (c/a) + b$ and $c + d$ exist and we have

$$c/(a + b) = c/((c + (c/a)) + b) = c/(c + d) = d = (c/a) + b.$$

(2) can be proved as (1).

(3) Since $b + (b/c) = c$ exists, by (1) we obtain that $(a/b) + (b/c)$ exists and $(a/b) + (b/c) = a/(b + (b/c)) = a/c$.

(4) In a similar way as (3), we obtain (4) applying (2). ∎

Lemma 3.2. *Let h, k, r, s and a, b, c, d be in E. Then:*

(1) *If $h+k$ and $r+s$ exist and $h+k \le r+s$, then $k \le h/(r+s)$ and $h \le (r+s)\backslash k$.*

(2) *If $b \le a$ and $c \le a\backslash b$, then $c + b$ exists, $c + b \le a$ and $b \le c/a$.*

(3) *If $b \le a$ and $c \le b/a$, then $b + c$ exists, $b + c \le a$ and $b \le a\backslash c$.*

Proof. (1) We first apply Proposition 1.1-(8) with $a = k$, $b = h/(r + s)$ and $c = h$. Indeed, by assumption, $r + s = h + (h/(r + s)) = c + b$ and $h + k = c + a$ exist, and $c + a \leq c + b$. Therefore, we have $k = a \leq b = h/(r + s)$.

Now, we apply Proposition 1.1-(7) with $a = h$, $c = k$ and $b = (r + s)\backslash k$. By assumption, we have that $r + s = ((r + s)\backslash k) + k = b + c$ and $h + k = a + c$ exist, and $a + c \leq b + c$. Therefore, $h = a \leq b = (r + s)\backslash k$.

(2) Since $(a\backslash b) + b = a$ exists and $c \leq a\backslash b$ by assumption, then by Proposition 1.1-(7) we have that $c + b$ exists and $c + b \leq a$. By (1), we get $b \leq c/a$.

(3) Since $b + (b/a) = a$ exists and $c \leq b/a$ by assumption, by Proposition 1.1-(8) we have that $b + c$ exists and $b + c \leq a$. By (1), we get $b \leq a\backslash c$. ■

Lemma 3.3. *Let $a_0, a_1, ..., a_n$ be in E such that $a_0 \leq a_1 \leq \ldots \leq a_n$ and, for every $i \in \{1, ..., n\}$, let $b_i = a_{i-1}/a_i$. Then $(b_1, ..., b_n)$ is orthogonal and $b_1 + \cdots + b_n = a_0/a_n$.*

Proof. By Lemma 3.1-(3), we get that $b_1 + b_2 = (a_0/a_1) + (a_1/a_2)$ exists and it is equal to a_0/a_2. By induction, suppose that $b_1 + \cdots b_{n-1}$ exists and it is equal to a_0/a_{n-1}. Then, by Lemma 3.1-(3), we obtain that $b_1 + \cdots b_{n-1} + b_n = (a_0/a_{n-1}) + (a_{n-1}/a_n)$ exists and it is equal to a_0/a_n. ■

Proposition 3.4. *The following conditions are equivalent*

(1) *E is σ-complete.*

(2) *For every increasing sequence $(a_n)_{n\in\mathbb{N}}$ in E, $\sup_n a_n$ exists.*

(3) *For every decreasing sequence $(a_n)_{n\in\mathbb{N}}$ in E, $\inf_n a_n$ exists.*

Proof. The equivalence of (2) and (3) is trivial by Proposition 1.1-(6).

(1) $\Rightarrow$ (2) Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence in E and set $b_n = a_{n-1}/a_n$ (where $a_0 = 0$). By Lemma 3.3, (b_n) is an orthogonal sequence and $b_1 + \cdots + b_n = a_n$. Hence, by (1), $a = \sup_n \sum_{i\leq n} b_i = \sup_n a_n$ exists.

(2) $\Rightarrow$ (1) Let $(a_n)_{n\in\mathbb{N}}$ be an orthogonal sequence in E. Set $b_n = a_1 + \ldots + a_n$. Since (b_n) is an increasing sequence, we have that $\sum_{n\in\mathbb{N}} a_n = \sup_n b_n$ exists. ■

Proposition 3.5. *Let $\mu : E \to G$ be a measure. Then the following conditions are equivalent:*

(1) *μ is σ-additive.*

(2) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \uparrow a \Rightarrow \mu(a) = \lim_n \mu(a_n)$.*

(3) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \downarrow a \Rightarrow \mu(a) = \lim_n \mu(a_n)$.*

(4) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \downarrow 0 \Rightarrow \lim_n \mu(a_n) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let (a_n) be such that $a_n \uparrow a$. For each $n \in \mathbb{N}$, set $b_n = a_{n-1}/a_n$, where $a_0 = 0$. By Lemma 3.3, (b_n) is orthogonal and, for each $n \in \mathbb{N}$, $b_1 + \cdots + b_n = 0/a_n = a_n$. Therefore, $a = \sup_n a_n = \sum_{n\in\mathbb{N}} b_n$. Since μ is σ-additive, we obtain $\mu(a) = \sum_{n=1}^{\infty} \mu(b_n) = \lim_n \sum_{k=1}^{n} \mu(b_k) = \lim_n \mu(b_1 + \cdots + b_n) = \lim_\mu(a_n)$.

(2) $\Rightarrow$ (3) Let $(a_n)_{n\in\mathbb{N}}$ be such that $a_n \downarrow a$. By Proposition 1.1-(6), we get that $a_n^\perp \uparrow a^\perp$. By (2), we obtain $\mu(a^\perp) = \lim_n \mu(a_n^\perp)$, from which $\mu(a) = \lim_n \mu(a_n)$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Let (a_n) be an orthogonal sequence in E such that $a = \sum_{n\in\mathbb{N}} a_n$ exists. Since $a_1 + \cdots + a_n \leq a$, then, for each $n \in \mathbb{N}$, $b_n = a\backslash(a_1 + \cdots + a_n)$ exists. By Proposition (1.1)-9, (b_n) is a decreasing sequence. Moreover we have that $\inf_n b_n = 0$. Indeed, if $c \leq b_n$ for each n, by Lemma 3.2-(2) we obtain that, for each n, $a_1 + \ldots + a_n \leq c/a$, from which we get $a = \sup_n(a_1 + \ldots + a_n) \leq c/a$. By Lemma 3.2-(3), we have $c \leq a\backslash a = 0$. Now, since $b_n \downarrow 0$, by (4) we have $\lim_n \mu(b_n) = 0$. Since $\lim_n \mu(b_n) = \lim_n(\mu(a) - \mu(a_1 + \cdots + a_n)) = \mu(a) - \sum_{n=1}^{\infty} \mu(a_n)$, we obtain $\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$. ∎

Proposition 3.6. *Let $\mu : E \to G$ be a measure. Then E is μ-chained if and only if μ is strongly continuous.*

Proof. Suppose that E is μ-chained. Let W be a neighbourhood of 0 in G and $a \in E$. Choose $a_0, a_1, \cdots, a_r$ in E such that $0 = a_0 \leq a_1 \leq \cdots \leq a_r = a$ and $\mu(h) - \mu(k) \in W$ whenever $h, k \in [a_{i-1}, a_i]$ for some $i \in \{1, ..., r\}$. Set $b_i = a_{i-1}/a_i$ for each $i \in \{1, ..., r\}$. By Lemma 3.3, $(b_1, ..., b_r)$ is orthogonal and $b_1 + \cdots + b_r = a$. Let $i \leq r$ and choose $b \leq b_i$. Since $a_{i-1} + b_i$ exists, by Proposition 1.1-(8) $a_{i-1} + b$ exists and $a_{i-1} \leq a_{i-1} + b \leq a_{i-1} + b_i = a_i$. Therefore, we obtain $\mu(b) = \mu(a_{i-1} + b) - \mu(a_{i-1}) \in W$.

Now, suppose that μ is strongly continuous. Let W and V be neighbourhoods of 0 in G with $V - V \subseteq W$ and $a \in E$. Choose an orthogonal family $(b_1, \ldots, b_r)$ in E such that $b_1 + \cdots + b_r = a$ and $\mu(b) \in V$ whenever $b \leq b_i$ for some $i \leq r$. Set $a_0 = 0$ and $a_i = b_1 + \ldots + b_i$ for every $i \leq r$. Then, we have $0 = a_0 \leq a_1 \leq \cdots \leq a_r = a$. Let $i \leq r$ and choose $h, k \in [a_{i-1}, a_i]$. By Proposition 1.1-(10), we have $a_{i-1}/h \leq a_{i-1}/a_i = b_i$ and $a_{i-1}/k \leq a_{i-1}/a_i = b_i$. Therefore, $\mu(h) - \mu(k) = \mu(a_{i-1}/h) - \mu(a_{i-1}/k) \in V - V \subseteq W$. ∎

The following result can be derived by Theorems 4.2 and 4.4 of [8].

Theorem 3.7. *Suppose that E has the interpolation property and let $\mu : E \to R^n$ be a strongly continuous measure. Then, if μ has nonnegative components, $\mu(E)$ is star-shaped with respect to 0. Moreover, if E is a lattice and μ is modular, then μ is convex-ranged.*

Proof. By Theorem 4.2 of [8], $\mu(E')$ is star-shaped with respect to 0 if E' is a μ-chained poset with smallest element 0 and greatest element 1, with a binary relation $\perp$ and a partially defined binary operation $\oplus$ satisfying the following properties:

(a) $a \perp b$ if and only if $a \oplus b$ exists.

(b) $a \oplus 0 = 0 \oplus a = a$.

(c) If $a \leq b$, then there exists c in E' with $a \perp c$ and $a \oplus c = b$.

(d) If $a \perp b$, $c \leq a$ and $d \leq b$, then $c \perp d$ and $c \oplus d \leq a \oplus b$.

Moreover, by Theorem 4.4 of [8], $\mu(E')$ is convex if E' satisfies the additional condition:

(e) If $a \leq c \leq a \oplus b$, then there exists $d \leq b$ such that $a \oplus d = c$.

Now, observe that a pseudo-effect algebra satisfies all the above conditions if we define $a \oplus b = a + b$ and $a \perp b$ if and only if $a + b$ exists. Indeed (a), (b) and (c) are trivial, (d) follows from Proposition 1.1-(7) and (8), and (e) follows from Proposition 1.1-(8). Moreover it is easy to see that, if $d \in E$, the interval $[0, d]$ is a pseudo-effect-algebra if we define, for every $a, b \in E$, $a + b = c$ if and only if $a + b = c$ in E and $c \leq d$. The assumptions on E imply that $[0, d]$ has the interpolation property and it is μ-chained by Proposition 3.6. Therefore, we can apply to $[0, d]$ Theorems 4.2 and 4.4 of [8]. ∎

Now, using the results of Section 3 instead of the corresponding results of [4], it is possible to prove the following Uniqueness theorem for measures on pseudo-effect algebras, proved in [21] for measures on particular effect algebras and in [4] for measures on arbitrary effect algebras.

Theorem 3.8. *Let μ and ν be $[0, +\infty[$-valued measures on E which satisfy the following conditions:*

(a) *μ is convex-ranged.*

(b) *There exist $\alpha \in]0, \mu(1)[$ and $\beta \in]0, \nu(1)[$ such that, for every $a \in E$, $\mu(a) = \alpha$ implies $\nu(a) = \beta$.*

Moreover suppose that one of the following conditions is satisfied:

(1) *E is σ-complete and ν is σ-additive.*

(2) *E has the interpolation property and, for every $a \in E$, $\nu(a) = 0$ implies $\mu(a) = 0$.*

(3) *E has the interpolation property and, for every $a \in E$, $\mu(a) = \alpha$ if and only if $\nu(a) = \beta$.*

Then $\mu = \lambda\nu$, where $\lambda = \frac{\mu(1)}{\nu(1)}$.

Proof. The proof is similar to the proof of Theorem 3.1 of [4]. ∎

Now, we apply the results of Section 2 to prove that, if μ is a modular measure on L, then the Uniqueness theorem holds without completeness assumptions on L.

First, we need the following result.

Proposition 3.9. *Let $\mu : L \to [0, +\infty[$ be a modular measure. Then the following conditions are equivalent:*

(1) *μ is strongly continuous.*

(2) *For every $\varepsilon > 0$, there exists an orthogonal family $(a_1, ..., a_r)$ in L such that $a_1 + \cdots + a_r = 1$ and $\mu(a_i) < \varepsilon$ for every $i \leq n$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $a \in L$ and $\varepsilon > 0$. Choose an orthogonal family $(a_1, ..., a_r)$ in L such that $a_1 + \cdots + a_r = 1$ and $\mu(a_i) < \varepsilon$ for every $i \leq r$. Set $b_0 = 0$ and $b_i = a_1 + \cdots + a_i$ for every $i \leq r$. Then we have $0 = b_0 \leq b_1 \leq ... \leq b_n = 1$. Since $b_i = b_{i-1} + a_i$, we have $a_i = b_{i-1}/b_i$, from which we obtain $\mu(b_i) - \mu(b_{i-1}) = \mu(a_i) < \varepsilon$ for every $i \leq n$. Setting $c_i = b_i \wedge a$, we can see as in (2.3) of [1], that $0 = c_0 \leq c_1 \leq \cdots \leq c_r = a$ and $\mu(c_i) - \mu(c_{i-1}) < \varepsilon$ for each $i \leq r$. Set $d_0 = 0$ and $d_i = c_{i-1}/c_i$ for $i \leq r$. Then $\mu(d_i) < \varepsilon$ for each $i \leq r$ and, by Lemma 3.3, $d_1 + \ldots + d_r = a$. ∎

Theorem 3.10. *Let $\mu, \nu : L \to [0, +\infty[$ be modular measures with the following properties:*

(1) *μ is strongly continuous.*

(2) *There exist $\alpha \in]0, \mu(1)[$ and $\beta \in]0, \nu(1)[$ such that, if (a_n) is a sequence in L with $\lim_n \mu(a_n) = \alpha$, then $\lim_n \nu(a_n) = \beta$.*

Then $\mu = \lambda\nu$, where $\lambda = \frac{\mu(1)}{\nu(1)}$.

Proof. Denote by $\mathcal{U}$ the supremum of the D-uniformities generated by μ and ν (see Theorem 2.9). It is clear that $\mathcal{U}$ is a D-uniformity. Moreover μ and ν are obviously exhaustive, since they are monotone real-valued. Then $\mathcal{U}$ is exhaustive, too.

Set $\hat{L} = L/N(\mathcal{U})$, $\hat{\mu}(a) = \mu(a)$ and $\hat{\nu}(a) = \nu(a)$ for $a \in \hat{a} \in \hat{L}$, and denote by $\tilde{\mu}$ and $\tilde{\nu}$ the uniformly continuous extensions, respectively, of $\hat{\mu}$ and $\hat{\nu}$ to the uniform completion $(\tilde{L}, \tilde{\mathcal{U}})$ of $\hat{L}$ (see Theorems 2.12 and 2.13).

By Theorem 2.13, $\tilde{L}$ is a complete D-lattice and $\tilde{\mu}$, $\tilde{\nu}$ are o.c. modular measures and therefore σ-additive by Proposition 3.5.

By Proposition 3.9, it is clear that $\tilde{\mu}$ is strongly continuous, too, since $\widehat{1_L} = 1_{\tilde{L}}$. Therefore, by Theorem 3.7, $\tilde{\mu}$ is convex-ranged.

Now let $a \in \tilde{L}$ such that $\tilde{\mu}(a) = \alpha$. Choose (a_n) in $\hat{L}$ which converges to a in $(\tilde{L}, \tilde{\mathcal{U}})$. By the continuity of $\tilde{\mu}$ and $\tilde{\nu}$, we get $\lim_n \hat{\mu}(a_n) = \tilde{\mu}(a) = \alpha$ and $\lim_n \hat{\nu}(a_n) = \tilde{\nu}(a)$. By (2), we get $\tilde{\nu}(a) = \beta$. Then $\tilde{\mu}$ and $\tilde{\nu}$ verify the assumptions of Theorem 3.8.

By Theorem 3.8, we get $\tilde{\mu} = \lambda\tilde{\nu}$, from which $\mu = \lambda\nu$. ∎

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