# NON PERIODIC SOLUTIONS OF FOURTH ORDER NONLINEAR EQUATIONS

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**Abstract.** In this paper we obtain a result of existence of solutions for a 4th order nonlinear equation subject to non periodic boundary conditions.

**Subject Classification:** 34B15.

**Keywords and phrases:** boundary value problems, fourth order ordinary differential equations.

## 1. Introduction

The purpose of this paper is to provide sufficient and/or necessary conditions on $f(x) \in C^o(\overline{I})$ ($\overline{I} = [0,1]$) for the existence of a solution to equations of the type:

$$\text{(i)} \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + g(x, u(x)) = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

where $g(x,\xi) \in C^o(\overline{I} \times \Re)$.

The conditions on $g$ will be conditions of sign and not conditions of growth. More exactly, we assume $g(x,\xi) \geq 0$ for all $\xi \geq 0$ and for every $x \in \overline{I}$ and that $g(x,\xi) \leq 0$ for all $\xi \leq 0$ and for every $x \in \overline{I}$. An important example is the jumping nonlinearities (i.e. $g(x,\xi) = \mu\xi^+ - \nu\xi^-$ for all $x \in \overline{I}$, where: $\mu \geq 0$ and $\nu \geq 0$)

Boundary value problems associated to fourth order equations raised a lot of interest in these years.

The assumptions that we are going to introduce on the nonlinearity do not prevent the linear or the sub-linear cases. Therefore they seem not directly to be tackled by Mountain Pass Theorem (MPL) (recall that in [1] the authors considered the superlinear situation).

The fourth derivative makes methods such sub or supersolutions not easily applicable. Indeed, these methods need a maximum principle for the following linear inequality [2]:

$$\text{(ii)} \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + \omega.u(x) \geq 0 \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

or for some related inequalities. That's true for a second order equation, if $\omega \geq 0$, and if the second derivative coefficient is negative but is in general false for a fourth order differential equation.

The linear term has a bidimensional kernel. The non invertibility (resonance) of this term raises a lot of problems [13].

Abstract methods that use the surjectivity of a nonlinear operator $T\colon V \to V^*$, where $V$ is a Banach space, are applicable with difficulty in our situation. For instance, the results in [3] require a one-dimensional kernel for a linear application $L$ associated to $T$. Again, this is usually true for a second order differential equation, whereas, in general, it is false for a fourth order equation. It seems also hard to generalize the use of these theorems to bidimensional kernels. With respect to some methods relying on the application of the Schauder fixed point theorem, we observe that the usual approach based on the consideration of the Green function and the evaluation of the norm of the inverse, appears rather complicated, due to the presence of a fourth derivative and the difficulty to estimate the norm of the inverse operator [4].

In this work we use an abstract Theorem of Hess [5] which is very suitable for ours boundary conditions.

Several articles deal with similar problems, even if the case of the *periodic* boundary conditions is more often considered. In some cases, the authors considered also more general nonlinearities by taking into account the presence of lower order derivatives or higher order systems. However, the nonlinearities and boundary conditions treated in this work, appear not yet fully investigated.

We present now a brief discussion of some related results appeared in the literature.

Ward [7], using some a priori bounds, proved that *periodic* solutions exist for a second order differential equation with a nonnull linear term in the first derivative.

Besides, Bates and Ward [8] studied *periodic* solutions for a $n$-order system and allowed nonlinearities in the derivative until the $n-1$-th degree, too.

By means of a continuation theorem of Leray-Schauder [9], they showed the existence of periodic solutions under assumptions of sub-linear growth (at infinity) or super-linear growth (at zero) for the function. He also considered the possibility of nonlinearities in lower derivative which depend from nonlinear components of the vector $g$. If the nonlinearity in the function is supposed to be superlinear at infinity (i.e. $\frac{\|g(\xi)\|}{\xi} \to +\infty$ if $\xi \to +\infty$), these results don't apply.

Further general results were obtained by Ward [10] using topological degree. He studied *periodic* solutions of $2n$-order differential equations where nonlinearities until the n-order derivative are allowed. Ward requires that the nonlinearity satisfies: $g(t,\xi)\text{sign}\xi \geq \alpha(t)$ if $|\xi| \to +\infty$ for every $t \in \overline{I}$, where $\alpha(t) \not\equiv 0$ and $\alpha(t) \geq 0$. Note that, the function $g(t,\xi) = \xi^+$ for every $t \in \overline{I}$ doesn't satisfy the above condition but it satisfies ours (see Theorems 4 and 4A in the cap. 4).

Finally, we recall that Ward in [11] proved the existence of periodic solutions for a class of $n-$th order scalar differential equations, using only sign conditions and one-sided growth restrictions on the nonlinearity.

Fučík[12] has got other results about periodic solutions in two variables for fourth order equations. In his conference [12] the author, by using Leray-Schauder topological degree, treats fourth order nonlinear problems (only in the function), when the function depends also from a time-variable; he obtains results if $g$ is bounded or at most linear. Our work gives a solution to the *Open problem* at page 335 if the function $u$ doesn't depend from $t$ (stationary case). Moreover it gives a solution to the problems for the case of jumping nonlinearities (cap. 7) with $\nu = 0$ that isn't studied in Fučík's article ("TERRA INCOGNITA" for the author).

In conclusion, we also mention the papers of V. Khoi Le and K. Schmitt in [14] and [15], where the authors have used a method based on elliptic regularization in order to find several results of existence of solutions for partial and ordinary differential equations.

We recall the following nonlinear boundary value problem:

$$\begin{cases} \dfrac{d^4u(x)}{dx^4} = \alpha \left(k|u'|^{\alpha-2}u'\right)' \quad \text{in } (0,T) \\ \dfrac{d^2u(0)}{dx^2} = \dfrac{d^2u(T)}{dx^2} = 0 \\ \alpha k(0)|u'(0)|^{\alpha-2}u'(0) = \dfrac{d^3u(0)}{dx^3} \\ \alpha k(T)|u'(T)|^{\alpha-2}u'(T) = \dfrac{d^3u(T)}{dx^3} \end{cases}$$

where $k \in C^1[0,T]$, which was considered by Khoi Le and Schmitt and has some relations with our problem.

In this case, it is clear that constants are trivial solutions. To obtain nontrivial solutions the authors use a method based on elliptic regularization. They treat with polynomial nonlinearities on the first derivative. The boundary conditions are the same as those of (i) if $k(0) = k(1) = 0$ ($T = 1$). Also, in this case, our results are not contained in that work.

## 2. The weak problem

Before stating our main results, we put together some notations and definitions we employ. $I = (0,1)$. The Hilbert space $W^{2,2}(I)$ $(W^{2,2})$ is endowed with the inner product

$$\langle u,v\rangle = \int_I \left(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + u(x)v(x)\right) dx$$

and the norm $\|w\| = \langle w,w\rangle^{\frac{1}{2}}$, while $(u,v)$ indicates the inner product in $L^2(I)$, i.e., $(u,v) = \int_I uvdx$.

We will consider the problem:

$$\text{(1)} \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + g(x,u(x)) = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

where $f(x) \in C^0(\overline{I})$ and $g(x,\xi) \in C^0(\overline{I} \times \Re)$.

We will impose suitable conditions on $g(x,\xi)$ which will allow us to find necessary and sufficient conditions on $f(x)$ so that there exists at least one solution for equation (1).

We make the following hypothesis:

(A) $f(x) \in L^1(I)$.

(B) $g(x,\xi)$ of Carathéodory.

(C) $\forall\, \tau > 0,\ \int_I \sup_{|\xi|\le\tau} g(x,\xi)dx < \infty$.

**Remark A.** The assumption (C) doesn't put conditions on the "functional" growth of $g(x,\xi)$, but restricts only to the $x$ dependence, thanks to the validity of the following Sobolev (compact) embedding in one-dimension:

(C1) $W^{2,2}(I) \overset{\hookrightarrow}{\hookrightarrow} C^0(\overline{I})$.

The conditions (B), (C) and (C1) ensure that

(D) $g(x,u(x)) \in L^1(I),\ \forall u \in W^{2,2}(I)$.

Moreover, from (D)+(C1) we have:

(E) $g(x,u(x))v(x) \in L^1(I),\ \forall v \in W^{2,2}$.

Therefore, if $u$ satisfies (1), then:

$$\int_I \Big(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + g(x,u(x))\Big)v(x)dx = \int_I f(x)v(x)dx,\ \forall v \in W^{2,2} \tag{2}$$

for (E)+(C1), equation (2) is well defined in $W^{2,2}$.

## 3. Abstract results

For the proof of our result, we will rely on an abstract theorem of P. Hess [4].

**Theorem 1.** *Let $T : \Re^n \to \Re^n$ be a continuous map. We suppose $\exists R > 0$:*

$$\langle Tw, w\rangle > 0,\ \forall w : \|w\| = R,$$

*then $\exists x_0$ with $\|x_0\| < R$ :*

$$Tx_0 = 0.$$

To generalize this theorem to arbitrary Banach (possibly infinite dimensional) spaces, we must impose some conditions on the operator $T$ [5].

Let $V$ be a reflexive separable Banach space. We impose on $T\colon V \to V^*$ the following conditions:

(i) $T$ is continuous from each finite-dimensional subspace $F$ of $V$.

(ii) to the weak topology on $V^*$, if $v_n \rightharpoonup v$ and $Tv_n \rightharpoonup g$, then $Tv = g$.

(iii) $T$ maps bounded sets of $V$ to bounded sets of $V^*$

The following theorem has been proved by Hess under conditions lightly more general than ours. Our proof simplifies that in [4].

**Theorem 2.** *If $T : V \to V^*$, satisfies* (i), (ii) *and* (iii) *and $\exists R > 0$ :*

(iv) $\langle Tw, w\rangle > 0,\ \forall\ w : \|w\| = R,$

*then $\exists\ x_0 \in V : \|x_0\| < R$ and:*

$$Tx_0 = 0.$$

**Proof.** Let $V_n$ be spaces of finite dimension such that $V_1 \subset V_2 \subset V_3 \subset ...$ and $\bigcup_{n=1}^{\infty} V_n$ is dense in $V$.

Let $j_n : V_n \to V$ be the canonical immersion and $j_n^* : V^* \to V_n^*$ denotes the adjoint. We put:

$$T_n = j_n^* T j_n\colon V_n \to V_n^*.$$

Let $w \in V_n$ be such that $\|w\| = R$. Since:

$$\langle T_n w, w\rangle = \langle Tw, w\rangle > 0$$

for the theorem (1) $\exists\, v_n : \|v_n\| < R$ and $T_n v_n = 0$.

By reflexivity, we can suppose that $\{v_n\}_{n\in\mathbf{N}}$ converges weakly to $v$.

Therefore, $v_n$ is bounded in $V$. For (iii), $\{Tv_n\}_{n\in\mathbf{N}}$ is bounded in $V^*$.

If $w \in \bigcup_{n=1}^{\infty} V_n$, then there exists $n_0$ such that $w \in V_n$, $\forall n \geq n_0$. For these $n$, we have:

$$\langle Tv_n, w\rangle = \langle T_n v_n, w\rangle = 0. \tag{3}$$

Since $\bigcup_{n=1}^{\infty} V_n$ is dense in $V$ and $\{Tv_n\}_{n\in\mathbf{N}}$ is bounded from (3) we have $Tv_n \rightharpoonup 0$ in $V^*$. From (ii), we conclude that:

$$Tv = 0.$$

## 4. Existence of solutions for the weak problem

In addition to (B) and (C), we suppose that $g(x,\xi)$ satisfies:

$$g(x,\xi)\ \mathrm{sign}\ \xi \geq 0, \quad \forall \text{ a.e. } x \in I,\ \forall\, \xi \in \Re. \tag{F}$$

We study the following problem:

$$\begin{cases} \displaystyle\int_I \Big(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + u(x)v(x)\Big) - \lambda_1 \int_I u(x)v(x) \\ \displaystyle\qquad + \int_I g(x,u(x))v(x) = \int f(x)v(x), \quad \forall\, v \in W^{2,2}(I), \end{cases} \tag{4}$$

where $\lambda_1$ is the lower eigenvalue of the compact linear operator $B\colon W^{2,2} \to W^{2,2}$ defined as follows:

The application

$$v \to \int_I uv$$

is, for each *fixed* $u \in W^{2,2}$, a linear and continuous functional on $W^{2,2}$, therefore by the Riesz theorem, there exists a unique $Bu \in W^{2,2}$ such that:

$$(u,v) = \int_I uv = \langle Bu, v\rangle\,,\ \forall\, v \in W^{2,2}.$$

It is easy to prove that $B \in \mathcal{L}\,(W^{2,2}, W^{2,2})$.

For the immersion (C1), we have $B \in \mathcal{K}(W^{2,2}, W^{2,2})$ and $B$ is self-adjoint.

Using Freedholm's theory for compact operators, we have the existence of a first eigenvalue $\lambda_1 > 0$, and there exists a diverging sequence of eigenvalues of $B$ such that: $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 ... \to \infty$, dim $V_{\lambda_i} < \infty$ for every $i \in \mathbf{N}$, where $V_{\lambda_i}$ is the eigenspace associated to the eigenvalue $\lambda_i$. We define:

$$L := I - \lambda_1 B$$

so that $\mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

Since $v \to \int_I fv$ is a linear and continuous functional on $W^{2,2}$, by the Riesz theorem there exists a unique $\mathcal{F} = F(u) \in W^{2,2}$ such that:

$$(f, v) = \int_I fv = \langle \mathcal{F}, v \rangle = \langle F(u), v \rangle \,, \ \forall \ v \in W^{2,2}.$$

In this case, $F : W^{2,2} \longrightarrow W^{2,2}$ is the (affine and continuous) constant application

$$F(u) = \mathcal{F}, \ \forall \ u \in W^{2,2}.$$

With respect to the nonlinearity, it is easy to prove that the linear form $v \to \int_I g(x, u(x))v(x)$ is, for each *fixed* $u \in W^{2,2}$, continuous (by using (C1), the hypothesis (B) and the Lebesgue theorem). So that, using again the Riesz theorem, there exists a non linear application $G : W^{2,2} \to W^{2,2}$ such that:

$$\int_I g\,(x, u(x))\, v(x) = \langle G(u), v \rangle \quad \forall \ v \in W^{2,2}.$$

It is standard to prove that such application is continuous.

Therefore, our problem becomes a problem formulated in a Hilbert space. More precisely, we define the operator:

$$T = L + G - F : W^{2,2} \to W^{2,2}. \tag{5}$$

The zeroes of the operator $T$ defined in (5) are the solutions of (4).

Now, we must prove that $T$ satisfies (i), (ii), (iii) and (iv).

Hypothesis (i) is immediate ($T$ is continuous from $W^{2,2} \to W^{2,2}$).

Hypothesis (ii) is easily proved if we decompose $T$ into its own linear part $L$ and into its nonlinear part $G$ ((ii) is clearly true for the constant part $F$).

For the linear part we use the following [6]:

**Theorem.** *Let $E$ and $F$ be two Banach spaces. If $L$ is a linear operator, continuous from $(E, \|\cdot\|) \to (F, \|\cdot\|)$, then $L$ is continuous from $(E, \mathrm{weak}) \to (F, \mathrm{weak})$.*

For the non linear part, we use the compactness of the immersion (C1) and hypothesis (B).

The hypothesis (iii) is verified by use of (C) and by following the equivalent definition of the norm $\|Tu\| = \sup\limits_{\|v\| \leq 1} \langle Tu, v \rangle$.

The (iv) is verified by imposing on constant part $F$ a condition that will be formulated later on.

Therefore, we can apply Theorem 2 and obtain:

**Theorem 3.** *If $g$ satisfies* (B) *and* (C) *and there exists $R > 0$ such that:*

$$(5A)\qquad \begin{cases} \displaystyle\int_I \left(\frac{d^2w(x)}{dx^2}\frac{d^2w(x)}{dx^2} + w^2(x)\right) - \lambda_1 \int_I w^2(x) \\ \displaystyle\qquad + \int_I g\,(x, w(x))\, w(x) - \int_I fw > 0, \\ \qquad\qquad \forall w \in W^{2,2} : \|w\| = R, \end{cases}$$

*then equation* (4) *has at least one solution $u_o \in W^{2,2}$ such that $\|u_o\| < R$.*

We note that, in Theorem 3, the hypothesis (F) has not been used.

From the equality:

$$\int_I \left(\frac{d^2u(x)}{dx^2}\right)^2 = (\lambda_1 - 1)\int_I u^2, \quad \forall u \in \mathrm{KerL} = \mathrm{V}_{\lambda_1},$$

we have that all eigenvalues of $B$ are $\geq 1$. Indeed, $\lambda_1 = 1$ as:

$$\mathrm{dim}V_{\lambda_1} = \mathrm{dimKer}L = 2$$

and the eigenvectors of $\lambda_1$ are given from $ax + b$ where $a$ and $b \in \Re$. Therefore, the solutions of equation (4) are the same of the equation (2).

Since $B$ is self-adjoint and compact, it can be represented as:

$$Bv = \sum_{i=1}^{+\infty} \mu_i < w_i, v > w_i, \ \forall\ v \in W^{2,2},$$

where $w_i$ is the eigenvector of $B$, which corresponds to the characteristics numbers $\mu_i = \dfrac{1}{\lambda_i}$.

Such representation allows to obtain the following valuation:

if $v \in Im(L) = Ker(L)^{\perp}$ then:

$$\begin{aligned} \langle v, (I - \lambda_1 B)v\rangle = &\ \|v\|^2 - \lambda_1 \langle v, Bv\rangle = \|v\|^2 - \lambda_1 \sum_{i=1}^{+\infty} \mu_i \langle w_i, v\rangle\, w_i \\ \geq &\ \|v\|^2 - \lambda_1\mu_2\|v\|^2 = \|v\|^2(1 - \lambda_1\mu_2) > 0. \end{aligned}$$

Hence, we have proved that $\exists m > 0$ such that:

$$(6)\qquad a(v, v) \geq m\|v\|^2, \ \forall\ v \in Ker(L)^{\perp},$$

where $a(v, v) = \langle v, Lv\rangle = (\ddot{v}, \ddot{v}) + (v, v) - \lambda_1(v, v)\ \forall\ \ v \in W^{2,2}$.

Therefore, on the subspace $Ker(L)^{\perp}$ the norm of $W^{2,2}$ is equivalent to the one induced from the bilinear form $a$.

Decomposing every vector of $W^{2,2}$ in the component in KerL $=$ $V_{\lambda_1}$ and in the normal component ($\in KerL^{\perp} = ImL$), we see that the bilinear form $a$ is semidefinite positive and is, therefore, l.s.c. for the weak topology, i.e., if $v_n \rightharpoonup v$, then:

$$\liminf_{n\to\infty} a(v_n, v_n) \geq a(v, v). \tag{7}$$

Now, we search a condition on $f(x)$ more "applicable" of the $(5A)$.

We put:

(I) $\liminf_{\xi\to+\infty} g(x,\xi) = d(x), \quad \forall$ a.e. $x \in I$

(II) $\limsup_{\xi\to-\infty} g(x,\xi) = c(x) \quad \forall$ a.e. $x \in I.$

We prove the following (cfr. Fučík [4]):

**Theorem 4.** *Suppose that $g$ satisfies* (B), (C) *and:*

(F) $g(x,\xi)$ sign $\xi \geq 0, \quad \forall$ a.e. $x \in I, \ \forall\, \xi \in \Re.$

*Suppose further that $f(x) \in L^1(I)$ satisfies:*

$$\int_I f\phi < \int_I (d\phi_+ - c\phi_-), \ \forall\ \phi \not\equiv 0 \in Ker(L), \tag{8}$$

*then problem* (2) *has at least one solution.*

**Proof.** By contradiction, assume (see Theorem 3) that there exists a sequence $\{w_n\}_{n\in\mathbf{N}}$ with $\|w_n\| \to \infty$ such that:

$$\text{(8A)}\qquad \begin{cases} \displaystyle\int_I \Big(\frac{d^2w_n(x)}{dx^2}\frac{d^2w_n(x)}{dx^2} + {w_n}^2(x)\Big) - \lambda_1 \int_I {w_n}^2(x) \\ \displaystyle\qquad + \int_I g\left(x, w_n(x)\right) w_n(x) - \int_I f(x) w_n(x) \leq 0 \end{cases}$$

If we put $v_n = \dfrac{w_n}{\|w_n\|}$, we may suppose that $v_n \rightharpoonup v$. We wish to show that $v \not\equiv 0$ and $v \in$ KerL $= V_{\lambda_1}$, i.e., $a(v,w) = 0, \ \forall w \in W^{2,2}$. We can suppose that $v_n \xrightarrow{L^2} v$ and $v_n \longrightarrow v$ a.e.

From (8A), we obtain:

$$a(v_n, v_n) + \frac{1}{\|w_n\|}\Big(\int_I g(x, w_n(x)) v_n(x) - \int_I f v_n\Big) \leq 0. \tag{8B}$$

Hence, the following relations are true:

$$\frac{1}{\|w_n\|}\int_I f v_n \overset{n\to\infty}{\longrightarrow} 0$$
$$\frac{1}{\|w_n\|}\Big(\int_I g(x, w_n(x))v_n(x)\Big) \geq 0,\ \forall\, n \in \mathbf{N}.$$

The last, by relation (F). We have, therefore, obtained:

$$\limsup_{n\to\infty} a(v_n, v_n) \leq 0. \tag{9}$$

From the weak l.s.c. of the form $a(\cdot,\cdot)$, we obtain:

$$a(v, v) \leq \liminf_{n\to\infty} a(v_n, v_n) \leq \limsup_{n\to\infty} a(v_n, v_n) \leq 0$$

and, since $a$ is positive semidefinite, we get:

$$a(v, v) = 0.$$

This relation implies that $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

To see this, put $v = (v - \phi) + \phi$, where $\phi \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$. Since $a(\phi, w) = 0$ for all $w \in W^{2,2}$, we obtain:

$$a(v - \phi, v - \phi) = 0.$$

From (6), we get:

$$\|v - \phi\| = 0.$$

Therefore, $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

Now, we calculate $\|v_n - v\|$.

If $P$ is the projection $P : W^{2,2}(I) \to \mathrm{KerL} = \mathrm{V}_{\lambda_1}$and $P^c$ is the projection $P^c : W^{2,2} \to ImL$, we have that $P^c + P = I$ and then:

$$\|v_n - v\| \leq \|P(v_n - v)\| + \|P^c(v_n - v)\|.$$

Then, since $\|P(v_n - v)\|^2 = \lambda_1(P(v_n - v), P(v_n - v))$ and $v_n \overset{L^2}{\longrightarrow} v$, one has:

$$\|P(v_n - v)\| \to 0.$$

Moreover, for (6) , (9) and since $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$, we have:

$$\|P^c(v_n - v)\|^2 \leq \frac{1}{m}a(v_n - v, v_n - v) = \frac{1}{m}a(v_n, v_n) \leq 0.$$

Then $v_n \overset{W^{2,2}}{\longrightarrow} v$ and, therefore, $v \not\equiv 0$.

By passing to limit in (8B), since $\lim_{n\to\infty} a(v_n, v_n) = 0$, we have:

$$\limsup_{n\to\infty} \frac{1}{\|w_n\|}\int_I g(x, w_n(x))v_n(x) \leq \liminf_{n\to\infty} \frac{1}{\|w_n\|}\int_I f v_n$$

i.e.

$$\limsup_{n\to\infty}\int_I g(x,w_n(x))v_n(x) \le \liminf_{n\to\infty}\int_I fv_n. \tag{10}$$

By Fatou's lemma we have:

$$\liminf_{n\to\infty}\int_I g(x,w_n(x))v_n(x) \ge \int_I d(x)v(x) - c(x)v(x). \tag{11}$$

Since $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$ and $v \not\equiv 0$, (10) and (11) contradict hypothesis (8). ∎

If (I) becomes:

$$\liminf_{\xi\to+\infty} g(x,\xi) = +\infty, \quad \forall \text{ a.e. } x \in I, \tag{M}$$

then (8) is not trivial only in the case $\phi_+ = 0$, i.e., $\phi = -\phi_- = ax + b$, where $a$ and $b$ are two real numbers such that $ax + b \le 0$ for every $x \in \overline{I}$.

If $\phi_{-b} = -b(1-x)$, where $b < 0$, then we have: $\phi_{-b} \le \phi_-$ for every $\phi_-$ such that $\phi_-(0) = -b = \phi_{-b}(0)$. Since, for $(F)$, $c(x) \le 0$ for every $x \in \overline{I}$, we have that $\phi_{-b}c \ge \phi_-c$ for every $\phi_-$ con $\phi_-(0) = -b$. In (8), we replace therefore only cases with: 1) $a < 0$ and $b = 0$; 2) $a = 0$ e $b < 0$; 3) and the worst case $a = -b > 0$ i.e. $\phi_{-b} = \phi_-$ and we get:

$$\int_I xf(x) > \int_I xc(x) \tag{12}$$

$$\int_I f(x) > \int_I c(x) \tag{13}$$

$$\int_I f(x).(1-x) > \int_I c(x).(1-x). \tag{14}$$

Therefore, we have proved the following:

**Theorem 4A.** *Let $g(x,\xi)$ be a function of Carathéodory that satisfies the conditions* (C), (M) *and* (F). *If $f \in L^1(I)$ satisfies the hypothesis* (12), (13) *and* (14), *then equation* (2) *has at least one solution in $W^{2,2}$.*

## 5. Existence of one solution for the classical problem

The following theorem will be basic in our further consideration:

**Theorem 5.** *If:*

$$\begin{aligned} &f(x) \in C^0(\overline{I}) \\ &g(x,\xi) \in C^0(\overline{I} \times \Re) \end{aligned} \tag{N}$$

*then every weak solution of* (4) (*or of* (2)) *is in $C^4(\overline{I})$ and satisfies the classical equation* (1).

**Proof.** See the appendix. ∎

**Remark 1.** If $g(x, u(x))$ satisfies hypothesis (N), then it verifies automatically hypothesis (C). Therefore, in this case, the only non trivial hypothesis is (F).

**Corollary 6.** *If $g(x,\xi) \in C^0(\overline{I} \times \Re)$ satisfies $(F)$ and $(M)$ (or $(I)$), while $f(x) \in C^0(\overline{I})$ satisfies* (12), (13) *and* (14) *(or* (8)*), then equation* (1) *has at least one solution.*

**Proof.** We need only to apply Theorem 4A, Remark 1 and Theorem 5. ∎

**Corollary 7.** *If $c(x) \equiv -\infty$, $\forall x \in \overline{I}$ (for example, the linear case),* (M) *is true and $g(x,\xi) \in C^0(\overline{I} \times \Re)$, then the condition* (8) *is trivially verified, and therefore equation* (1) *has at least a solution for every $f \in C^0(\overline{I})$ (well known result in the linear case).*

We assume, now, the following hypothesis in the study of the necessary conditions on $f(x) \in C^0(\overline{I})$:

$$\text{(O)} \qquad c(x) \le g(x,\xi), \;\; \forall x \in \overline{I} \;\text{ and }\; \forall \xi \in \Re.$$

We note that for hypothesis (F) we have that: $c(x) \le 0$ for every $x \in \overline{I}$.

If $u$ satisfies equation (2), then, choosing $v = ax + b$ so that segment $ax + b$ belongs to the first quadrant for every $x \in \overline{I}$, using hypothesis (O) we obtain that the following conditions are necessary for the existence of solution of equation (2)

$$(15) \qquad \int_I x f(x) \ge \int_I x c(x)$$

$$(16) \qquad \int_I f(x) \ge \int_I c(x)$$

$$(17) \qquad \int_I f(x).(1-x) \ge \int_I c(x).(1-x)$$

We summarize:

**Theorem 8.** *If $g(x,\xi) \in C^0(\overline{I} \times \Re)$ satisfies* (F), (M) *and* (O), *and $f(x) \in C^0(\overline{I})$, then conditions* (12), (13) *and* (14) *are sufficient, while conditions* (15), (16) *and* (17) *are necessary for the existence of at least one solution for the equation* (1).

**Remark 1A.** This theorem is true for the weak solutions of (2) if $f \in L^1(I)$ and if $g\,(x,\xi)$ is of Carathéodory and satisfies (C), (M), (O) and (F), where condition (O) is for a.e. $x \in I$ and $\forall\, \xi \in \Re$.

Now, what happens when there is an equality in at least one of the conditions (15), (16) or (17)?

Do they suffice for the existence of at least one solution for equation (1)?

If $g(x,\xi) \in C^0(\overline{I} \times \Re)$, instead of the hypothesis (O), satisfies the following (more restrictive) condition:

(P) $g(x,\xi) \equiv 0, \quad \forall x \in I$ and $\forall \xi \leq 0$,

we shall show that (P) implies the next result:

**Theorem 9.** *Let* $g(x,\xi) \in C^0(\overline{I} \times \Re)$ *be a function that verifies hypotheses* (F), (M) *and* (P), *and* $f(x) \in C^0(\overline{I})$.

*If*

$$\int_I f(x)dx = 0, \tag{18}$$

*then the further condition:*

$$\int_I xf(x)dx = 0 \tag{18A}$$

*is necessary and sufficient so that equation* (1) *had infinitely many non-positive solutions.*

*If*

$$\int_I f(x)dx > 0, \tag{19}$$

*then the further conditions:*

(19A) $\int_I xf(x)dx > 0$

(19B) $\int f(x)dx > \int_I xf(x)dx$

*are necessary and sufficient so that equation* (1) *has at least one solution.*

**Proof.** Suppose first that (1) has at least one solution. Then, for Theorem 8, one has that conditions (18) and (18A) are necessary.

Now, let us assume (18) and (18A). It follows that this linear

$$\begin{cases} \dfrac{d^4u(x)}{dx^4} = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases} \tag{20}$$

has a solution given by:

$$u(x) = \alpha + \beta x + \int_0^x \left( \int_0^s \left( \int_0^t \left( \int_0^r f(m)dm \right) dr \right) dt \right) ds$$

where $\alpha$ and $\beta \in \Re$.

Now, we choose $\alpha$ and $\beta$ so that:

$$\max_{x\in\overline{I}} u(x) \leq 0.$$

For the hypothesis (P), we have found infinitely many not positive solutions of equation (1).

Now, suppose that (19), (19A) and (19B) are true. Then, for Theorem 8, equation (1) has at least one solution.

In view of Theorem 8, the last step of the proof concerns with the fact that there exist no solution of equation (1) in the following cases below:

$1^o$ Case:

$$\int_I xf = 0$$

$$\int_I f > 0.$$

$2^o$ Case:

$$\int_I xf = \int_I f > 0. \tag{21}$$

About the first case, we assume to the contrary that there exists one solution $u(x)$ of problem (1). If we integrate (1) on $I$, we have:

$$\int_I g\,(t, u(t))\,dt > 0. \tag{22}$$

If we multiply the same equation for $x$ and we integrate on $I$, we get:

$$\int_I tg\,(t, u(t))\,dt = 0 \tag{23}$$

since $g(x,\xi) \geq 0$ for all $x \in \overline{I}$ and for every $\xi \in \Re$ relation (22) contradicts (23).

About the second case, we suppose by contradiction that there exists one solution $u(x)$ of problem (1). Hypothesis (21) is equivalent to:

$$\int_I \left(\int_0^x f(t)dt\right) dx = 0. \tag{24}$$

If we integrate first equation (1) from 0 to $x$ and then on $I$, we get:

$$\int_I \left(\int_0^x g\,(t, u(t))\,dt\right) dx = \int_I \left(\int_0^x f(t)dt\right) dx = 0. \tag{25}$$

Since the function:

$$q(x) = \int_0^x g(t, u(t))\,dt$$

is monotone nondecreasing with $q(0) = 0$ and $q(1) = \int_I g(t, u(t))\, dt = \int_I f > 0$, the relation (25) is the needed contradiction. ∎

**Remark 1B.** If the equality $(P)$ means for a.e. $x \in I$ and $\forall\ \xi \leq 0$, then this theorem is true for the weak solutions of (2) if $f(x) \in L^1(I)$ and if $g\,(x, \xi)$ is of Carathéodory and satisfies hypothesis (C), (M), (P) and (F).

**Remark 2.** As it was already observed in the introduction, an example particularly significant to which it is possible to apply Theorem 9 is the following:

$$g(x, \xi) = q(x)\xi^+$$

where $q(x) \in C^0(\overline{I})$ and $q(x) \geq 0$.

## Appendix

There is a proof of Theorem 5 much simpler and more direct than the following version which, nevertheless, can be applied to more general situations. (See Fučík[4].)

We start with the proof of the following:

**Lemma A.** *Let $R(x) \in L^2(I)$ be such that:*

$$\text{(a)} \qquad \int_I R(x) \frac{d^2 v(x)}{dx^2} = 0, \quad \forall\ v \in W_0^{2,2}(I),$$

*then there exists two numbers $\alpha_1$ and $\alpha_2$ such that:*

$$\text{(b)} \qquad R(x) = \alpha_1 + \alpha_2 x, \quad \text{for a.e. } \ x \in I.$$

**Proof.** The relation (a) is equivalent to the following:

$$\text{(c)} \qquad \frac{d^2 R(x)}{dx^2} = 0,$$

where the equality in (c) is in $\mathcal{D}'(\mathcal{I})$.

We obtain relation (b) if we remark that also in $\mathcal{D}'(\mathcal{I})$ the solutions of (c) are only first grade polynomial. ∎

**Proof of Theorem 5.** If $u(x) \in W^{2,2}(I)$ is a weak solution of equation (2), then we get:

$$\int_I \left( \frac{d^2 u(x)}{dx^2} \cdot \frac{d^2 v(x)}{dx^2} + g\,(x, u(x)) - f(x) \right) v(x) = 0, \qquad \forall\ v \in W_0^{2,2}(I).$$

Integrating by parts the above relation and remembering that $v \in W_0^{2,2}(I)$, we obtain:

$$\int_I \left( \frac{d^2u(x)}{dx^2} + \int_0^x \left( \int_0^t \left(g\left(m, u(m)\right) - f(m)\right) dm \right) dt \right) .v(x)dx = 0, \quad \forall\ v \in W_0^{2,2}(I).$$

For Lemma A, we have that there exists two constants $\alpha_1$ and $\alpha_2$ such that:

$$\frac{d^2u(x)}{dx^2} + \int_0^x \left( \int_0^t \left(g\left(m, u(m)\right) - f(m)\right) dm \right) dt + \alpha_1 + \alpha_2 x = 0, \quad \text{for a.e. } x \in I.$$

Now, we put:

$$F(x,z) = z + \int_0^x \left( \int_0^t \left(g\left(m, u(m)\right) - f(m)\right) dm \right) dt + \alpha_1 + \alpha_2 x.$$

Of course, $F(x,z) \in C^2(\overline{I})$ and $\dfrac{\partial F(x,z)}{\partial z} = 1$ for all $x \in \overline{I}$. Applying the implicit function theorem we get that there exists a unique function $z(x) \in C^2(\overline{I})$ such that:

$$F(x, z(x)) = 0.$$

For the uniqueness of that function, we have:

$$\frac{d^2u(x)}{dx^2} = z(x)$$

and, therefore, $u \in C^4(\overline{I})$. We can, therefore, integrate by parts equation (2) and obtain:

$$(d) \qquad \begin{cases} \dfrac{d^2u(1)}{dx^2}\dfrac{dv(1)}{dx} - \dfrac{d^2u(0)}{dx^2}\dfrac{dv(0)}{dx} - \dfrac{d^3u(1)}{dx^3}v(1) + \dfrac{d^3u(0)}{dx^3}v(0) \\ \qquad + \displaystyle\int_I \left( \frac{d^4u(x)}{dx^4} + g\left(x, u(x)\right) - f(x) \right) .v(x) = 0, \\ \qquad\qquad \forall\ v \in W^{2,2}(I). \end{cases}$$

In the relation (d), we can choose in particular $v \in W_0^{2,2}(I)$. Since this space is dense in $W^{2,2}$, we have:

$$\frac{d^4u(x)}{dx^4} + g\left(x, u(x)\right) = f(x), \quad \forall\ x \in \overline{I}.$$

Therefore, relation (d) becomes:

$$\frac{d^2u(1)}{dx^2}\frac{dv(1)}{dx} - \frac{d^2u(0)}{dx^2}\frac{dv(0)}{dx} - \frac{d^3u(1)}{dx^3}v(1) + \frac{d^3u(0)}{dx^3}v(0) = 0, \quad \forall\ v \in W^{2,2}.$$

For the arbitrary choice of $v \in W^{2,2}$, we get:

$$\begin{cases} \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0. \end{cases}$$

Therefore, every weak solution of equation (2) is a classical solution of equation (1). ∎

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Accepted: 05.04.2005