HYPERACTION OF SEMIGROUPS AND MONOIDS

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Abstract. The purpose of this note is the study of hyper action as a generalization of action of a monoid on a set. In this regards, first we introduce the notion of hyperactions of type 1 and type 2 and then we study the basic properties of this notion. In particular, we investigate the relationship between hyperactions and non-deterministic automata.

Keywords: hyperaction, non-deterministic finite automata, congruence.

1. Introduction

Hyperstructure theory was born in 1934, when Marty defined hypergroups ([6]), began to analysis their properties and applied them to groups, rational algebraic functions. Now, they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied mathematics. Since then, many researchers have studied in this field and developed it, for example, see [2].

The notion of a fuzzy subset of a nonempty set was introduced by L.A. Zadeh in 1965 [10] as a function from a nonempty set X to interval $I = [0, 1]$. Rosenfeld defined the concept of a fuzzy subgroup of a given group G [8] and then many researchers has developed it in all subjects of algebra. In [7], Malik, Mordeson and Sen have studied some properties of S-set in the name poly transformation semigroup. In the literature, we find that Wee [2] first introduced the concept of fuzzy automata. In this note, we introduce the notion of a hyperactions of types 1 and 2 as a generalization of action of a monoid on a set and obtain basic results of them. In this regards, we study the relationship between non-deterministic automata and hyperactions. In addition to the above mentioned concept of hyperaction, some other directions have been pursued in the literature. A. Madanshekaf and A.R. Ashrafi have introduced a notion of generalized action of a hypergroup on a set in [5], which is different from the concept we present in this paper.

2. Preliminaries

Let H be a nonempty set and $P(H)$ the family of all nonempty subsets of H.

A map $\cdot : H \times H \longrightarrow P(H)$ is called *hyperoperation* or join operation (see [2]).

The join operation is extended to subsets of H in natural way, so that $A \cdot B$ or AB is given by

$$AB = \bigcup \{ab \mid a \in A \text{ and } b \in B\}.$$

The relational notation $A \approx B$ (read A meets B) is used to asserts that A and B have an element in common, that is, $A \cap B \neq \emptyset$. The notations aA and Aa are used for $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Throughout the paper, S denotes a monoid and Q denotes a nonempty set.

A (left) action of S on Q is a function $f : S \times Q \longrightarrow Q$ (usually denoted by $f(x, q) \longrightarrow xq$) for all $x \in X$ and $q \in Q$. Q is called an S-set [3] if there exists an action of S on Q such that

(i) $(xy)q = x(yq)$

(ii) $1q = q$, for all $x, y \in X$ and $q \in Q$.

This concept of S-set also plays an important role in the theory of Deterministic finite automata [1]. Considering the theory of Non-deterministic finite automata [1], one can introduce the concept of the hyperaction of S on Q. We denote the set of all nonempty subsets of Q by $P(Q)$.

3. Hyper action

Definition 3.1 A (left) hyper action of S on Q is a function $\circ : S \times Q \mapsto P(Q)$ (usually denoted by $\circ(x, q) \mapsto x \circ q$) for all $x \in S$ and $q \in Q$.

Let $A \in P(Q)$ and $x \in S$. We define $x \circ A \in P(Q)$ by

$$x \circ A = \begin{cases} \bigcup_{a \in A} (x \circ a), & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}$$

Q is called an S-hyper set of type 1 if there exists an hyper action $\circ$ of S on Q such that

(i) $(xy) \circ q = x \circ (y \circ q)$

(ii) $q \in 1 \circ q$, for all $x, y \in S$ and $q \in Q$.

Q is called an S-hyper set of type 2 if there exists an hyper action $\circ$ of S on Q such that

(i) $(xy) \circ q = x \circ (y \circ q)$

(ii) $1 \circ q = \{q\}$, for all $x, y \in S$ and $q \in Q$.

Example 3.2 Let Q be a non-empty set. Let B_Q denotes the set of all binary relations on Q. Let $\alpha, \beta \in B_Q$. Now B_Q is a semigroup where the binary operation o is defined by $\alpha o \beta = \{(x, y) \in Q \times Q : (x, z) \in \beta, \ (z, y) \in \alpha, \text{ for some } z \in Q\}$. One can show that B_Q is a semigroup with identity.

Theorem 3.3 *Q is a B_Q-hyper set of type* 2.

Proof. We define $\circ : B_Q \times Q \mapsto P(Q)$ by

$$\alpha \circ x = \{y \in Q : (x, y) \in \alpha \ \}, \text{ for all } \alpha \in B_Q \text{ and } x \in Q.$$

Clearly, this is a hyper action of B_Q on Q. $(x, x) \in i$ (identity relation) $\in B_Q$, for all $x \in Q$ i.e. $i \circ x = \{x\}$, for all $x \in Q$.

$$\alpha \circ (\beta \circ x) = \bigcup\{\alpha \circ y : y \in \beta \circ x\} = \bigcup\{\alpha \circ y : (x, y) \in \beta\} = \{z \in Q : (y, z) \in \alpha$$

and

$$(x, y) \in \beta\} = \{z \in Q : (x, z) \in \alpha \circ \beta\} = (\alpha \circ \beta) \circ x, \text{ for all } \alpha \in B_Q \text{ and } x \in Q.$$

Hence the result. ∎

Example 3.4 Let Q be a non-empty set. Let $f : Q \mapsto P(Q)$. Let $S = \{f : (f : Q \mapsto P(Q))\}$. Clearly, $i : Q \mapsto P(Q)$, where $i(q) = \{q\}$ for all $q \in Q$, is a member of S. Let $f, g \in S$. We define $f \bullet g : Q \mapsto P(Q)$ by

$$f \bullet g(q) = \begin{cases} \bigcup\{f(p) : p \in g(q)\}, & \text{if } g(q) \neq \emptyset \\ \emptyset, & \text{if } g(q) = \emptyset, \end{cases}$$

for all $q \in Q$.

Then, $f \bullet g \in S$.

Let $f, g, h \in S$. Let $h(q) = \emptyset$, then $(f \bullet g) \bullet h(q) = \emptyset$ and $f \bullet (g \bullet h)(q) = \bigcup\{f(p) : p \in g \bullet h(q)\} = \bigcup\{f(p) : p \in \emptyset\} = \emptyset$, i.e., $(f \bullet g) \bullet h(q) = f \bullet (g \bullet h)(q)$.

Let $h(q) \neq \emptyset$, then $f \bullet (g \bullet h)(q) = \bigcup\{p : p \in f(r)$, where $r \in g \bullet h(q)\} = \bigcup\{p : p \in f(r)$, where $r \in g(t)$ and $t \in \bullet h(q)\} = \bigcup\{p : p \in f(g(t))$, where $t \in h(q)\} = \bigcup\{p : p \in f \bullet g(t)$, where $t \in h(q)\} = (f \bullet g) \bullet h(q)$.

Therefore, S is a monoid.

Now, we define $\circ : S \times Q \mapsto P(Q)$ (described as $(f, q) \mapsto f \circ q$) by

$$f \circ q = f(q),$$

where

$$f \circ A = \begin{cases} \bigcup\{f \circ p : p \in A\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}$$

Here $i \circ q = i(q) = \{q\}$, for all $q \in Q$.

Now, it can be shown that $f \circ (g \circ q) = fg \circ q$, where fg stands for $f \bullet g$, in short.

Let $g \circ q = g(q) = \emptyset$. Then $f \circ (g \circ q) = \emptyset$ (by definition of $\circ$). In this case, $(fg) \circ q = \emptyset$ (by definition of $\bullet$ on S). Thus, $f \circ (g \circ q) = fg \circ q$, whenever $g \circ q = \emptyset$.

Now, let $g \circ q \neq \emptyset$, then we have

$$f \circ (g \circ q) = \bigcup\{f \circ p : p \in (g \circ q)\} = \bigcup\{f(p) : p \in g(q)\} = fg(q).$$

Therefore, Q is an S-hyper set of type 2.

Example 3.5 Let G be a group and H be a normal subgroup of G. Define a hyper action of G on G by

$$a \circ b = aHb, \text{ for all } a, b \in G.$$

We can show that G is a G-hyper set of type 1.

Proof. Let e be the identity element of G. Then $e \circ q = eHq = Hq$. This implies that $q \in Hq = e \circ q$, for all $q \in G$.

$$ab \circ q = abHq = abHHq = aHbHq = \bigcup_{c \in bHq} a \circ c = \bigcup_{c \in b \circ q} a \circ c = a \circ (b \circ q),$$

for all $a, b, q \in G$. Hence the result. ∎

Theorem 3.6 *S is an S-hyper set of type 2.*

Proof. We define $\circ : S \times S \mapsto P(S)$ by $x \circ q = \{xq\}$

$$\lambda \circ q = \{\lambda q\} = \{q\}, \quad \text{for all } q \in S.$$

$$x \circ (y \circ q) = x \circ yq = \{xyq\} = xy \circ q, \text{ for all } x, y, q \in S.$$

Therefore, S is an S-hyper set. ∎

Theorem 3.7 *Let Σ and Q be two non-empty finite sets. $\Sigma^\star$ denotes the free monoid generated by Σ. Suppose for all $x \in \Sigma^\star$, $q \in Q$, $x \circ q \in P(Q)$ such that $xa \circ q = x \circ (a \circ q)$ for all $x \in \Sigma^\star$, $a \in \Sigma$, $q \in Q$ and $\lambda \circ q = \{q\}$, for all $q \in Q$. Then Q is an $\Sigma^\star$-hyper set of type 2.*

Proof. We have to show that $xy \circ q = x \circ (y \circ q)$ for all $x, y \in \Sigma^\star$ and $q \in Q$.

We prove the result by induction on $|y| = n$.

If $n = 0$, then $y = \lambda$.

$x \circ (y \circ q) = x \circ (\lambda \circ q) = x \circ \{q\} = x \circ q = x\lambda \circ q = xy \circ q$. Given that the result is true for $|y| = 1$, assume the result is true for $|y| = n - 1$, $n > 1$.

Let $y = ua$, where $|u| = n - 1$, $n > 1$ and $a \in \Sigma$. Now,

$$\begin{aligned} x \circ (y \circ q) &= x \circ (ua \circ q) = x \circ (u \circ (a \circ q)) \\ &= \bigcup \{x \circ t : t \in u \circ (a \circ q)\} \\ &= \bigcup \{x \circ t : t \in u \circ r, \text{ for some } r \in a \circ q\} \\ &= \bigcup \{x \circ (u \circ r) : r \in a \circ q\} \\ &= \bigcup \{xu \circ r : r \in a \circ q\} \\ &= xu \circ (a \circ q) = xua \circ q = xy \circ q. \end{aligned}$$

Therefore, Q is a $\Sigma^\star$-hyper set of type 2. ■

Theorem 3.8 *Let $\mathcal{M} = (\Sigma, Q, \delta)$ be a non-deterministic finite automata. Then, Q is a $\Sigma^\star$-hyper set of type 2.*

Proof. We define $\circ : \Sigma^\star \times Q \mapsto P(Q)$ by

$$\begin{aligned} x \circ q &= \delta^\star(x, q) && \text{for all } x \in \Sigma, q \in Q. \\ \lambda \circ q &= \delta^\star(\lambda, q) = \{q\}, && \text{for all } q \in Q. \end{aligned}$$

By definition of $\delta^\star$, we have

$$\delta^\star(xa, q) = \bigcup_{r \in \delta(a,q)} \delta^\star(x, r).$$

We have to show that

$$x \circ (y \circ q) = xy \circ q, \text{ for all } x, y \in \Sigma^\star, q \in Q.$$

We prove the result by induction on $|y| = n$.

If $n = 0$, then $y = \lambda$.

$$x \circ (y \circ q) = x \circ (\lambda \circ q) = x \circ \delta^\star(\lambda, q) = x \circ \{q\} = x \circ q = x\lambda \circ q = xy \circ q.$$

Assume the result is true for $|y| = n - 1, n > 0$.

Let $y = ua$, where $|u| = n-1,\ n > 0,\ a \in \Sigma$.

$$\begin{aligned} x \circ (y \circ q) &= \bigcup\{x \circ r : r \in y \circ q\} = \bigcup\{x \circ r : r \in ua \circ q\} \\ &= \bigcup\{x \circ r : r \in u \circ (a \circ q)\} = \bigcup\{x \circ r : r \in u \circ p, \end{aligned}$$

where $p \in a \circ q\} = \bigcup\{x \circ (u \circ p) : p \in a \circ q\} = \bigcup\{xu \circ p : p \in a \circ q\} = xu \circ (a \circ q) = xua \circ q = xy \circ q$.

Hence the result. ∎

Theorem 3.9 *Let Σ and Q be two non-empty sets. Suppose that Q is $\Sigma^\star$-hyper set of type 2. Then, there exists a non-deterministic finite automata $\mathcal{M} = (\Sigma, Q, \delta)$ such that*

$$\delta^\star(\lambda, q) = \{q\},\ \textit{for all } q \in Q \textit{ and}$$

$$\delta^\star(xy, q) = \bigcup_{r \in \delta^\star(y,q)} \delta^\star(x, r).$$

Proof. We define $\delta^\star : \Sigma^\star \times Q \mapsto P(Q)$ by

$$\delta^\star(x, q) = x \circ q, \text{ for all } x \in \Sigma^\star, q \in Q.$$

If $x \in \Sigma$, then $\delta^\star(x, q) = \delta(x, q)$.

$$\delta^\star(\lambda, q) = \lambda \circ q = \{q\},$$

for all $q \in Q$.

$$\delta^\star(xy, q) = xy \circ q = x \circ (y \circ q) = x \circ \delta^\star(y, q) = \bigcup_{r \in \delta^\star(y,q)} x \circ r = \bigcup_{r \in \delta^\star(y,q)} \delta^\star(x, r),$$

for all $x \in \Sigma^\star, q \in Q$. ∎

Theorem 3.10 *Let Q_1 and Q_2 be two S-hyper sets of type 1 (or type 2), then $Q_1 \times Q_2$ is also an S-hyper set of type 1 (or type 2).*

Proof. We define $\circ : S \times (Q_1 \times Q_2) \mapsto P(Q_1 \times Q_2)$ by

$$x \circ (q_1, q_2) = (x \circ q_1) \times (x \circ q_2),$$

for all $x \in S, q_1 \in Q_1, q_2 \in Q_2$.

$$(q_1, q_2) \in (1 \circ q_1) \times (1 \circ q_2) = 1 \circ (q_1, q_2)\ \ [1 \circ (q_1, q_2) = (1 \circ q_1) \times (1 \circ q_2) = \{(q_1, q_2)\}],$$

for all $(q_1, q_2) \in (Q_1 \times Q_2)$.

$$\begin{aligned} x \circ (y \circ (q_1, q_2)) &= \bigcup_{(r_1, r_2) \in y \circ (q_1, q_2)} x \circ (r_1, r_2) = \bigcup_{r_1 \in y \circ q_1, r_2 \in y \circ q_2} (x \circ r_1) \times (x \circ r_2) \\ &= (x \circ (y \circ q_1)) \times (x \circ (y \circ q_2)) = xy \circ q_1 \times xy \circ q_2 = xy \circ (q_1, q_2), \end{aligned}$$

for all $x, y \in S$ and $(q_1, q_2) \in (Q_1 \times Q_2)$. ∎

Definition 3.11 Let S and T be two semigroups. Then, their cartesian product $S \times T$ will be a semigroup with respect to the binary operation $\bullet$ defined by

$$(x,y) \bullet (s,t) = (xs, yt), \text{ for all } x,s \in S \text{ and } y,t \in T.$$

The semigroup $(S \times T, \bullet)$ is called the direct product of S and T, written by $S \times T$.

Theorem 3.12 *Let P be an S-hyper set of type 1 (type 2) and Q be a T-hyper set of type 1 (type 2), then $P \times Q$ is $S \times T$-hyper set of type 1 (type 2).*

Proof. We define $\circ : (S \times T) \times (P \times Q) \mapsto P(P \times Q)$ by

$$(x,y) \circ (p,q) = \begin{cases} (x \circ p) \times (y \circ q), & \text{if } (x \circ p) \neq \emptyset \text{ and } (y \circ q) \neq \emptyset \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $(x,y) \in (S \times T)$ and $(p,q) \in (P \times Q)$.

Let $(x,y) \circ ((s,t) \circ (p,q)) = \emptyset$, then either $(s,t) \circ (p,q) = \emptyset$ or $(x,y) \circ (p_1,q_1) = \emptyset$, for all $(p_1,q_1) \in (s,t) \circ (p,q)$.

If $(s,t) \circ (p,q) = \emptyset$, then either $s \circ p = \emptyset$ or $t \circ q = \emptyset$, i.e., either $x \circ (s \circ p) = \emptyset$ or $y \circ (t \circ q) = \emptyset$, i.e., either $xs \circ p = \emptyset$ or $yt \circ q = \emptyset$, i.e., $(xs, yt) \circ (p,q) = \emptyset$, i.e., $((x,y)(s,t)) \circ (p,q) = \emptyset$.

If $(x,y) \circ (p_1,q_1) = \emptyset$, for all $(p_1,q_1) \in (s,t) \circ (p,q)$, then either $x \circ p_1 = \emptyset$ or $y \circ q_1 = \emptyset$, for all $p_1 \in s \circ p$ and for all $q_1 \in t \circ q$, i.e., either $x \circ (s \circ p) = \emptyset$ or $y \circ (t \circ q) = \emptyset$, i.e., either $xs \circ p = \emptyset$ or $yt \circ q = \emptyset$, i.e., $(xs, yt) \circ (p,q) = \emptyset$, i.e., $((x,y)(s,t)) \circ (p,q) = \emptyset$.

Let $(x,y) \circ ((s,t) \circ (p,q)) \neq \emptyset$, then

$$\begin{aligned} (x,y) \circ ((s,t) \circ (p,q)) &= \bigcup \{(x,y) \circ (p_1,q_1) : (p_1,q_1) \in (s,t) \circ (p,q)\} \\ &= \bigcup \{x \circ p_1 \times y \circ q_1 : p_1 \in s \circ p \end{aligned}$$

and

$$\begin{aligned} q_1 \in t \circ q\} &= x \circ (s \circ p) \times y \circ (t \circ q) = xs \circ p \times yt \circ q \\ &= (xs, yt) \circ (p,q) = ((x,y)(s,t)) \circ (p,q). \end{aligned}$$

$$(p,q) \in (1 \circ p) \times (1 \circ q) = 1 \circ (p,q) \ \ [1 \circ (p,q) = (1 \circ p) \times (1 \circ q) = \{(p,q)\}],$$

for all $(x,y), (s,t) \in (S \times T)$ and $(p,q) \in (P \times Q)$. Hence the result. ∎

Definition 3.13 Let Q_1 and Q_2 be two S-hyper sets. A mapping $f : Q_1 \mapsto Q_2$ is said to be an S-hyper homomorphism if $f(x \circ q) = x \circ f(q)$, for all $x \in S$ and $q \in Q_1$ where $f(x \circ q) = \{f(t) : t \in x \circ q\}$.

Definition 3.14 Let σ be an equivalence relation on Q(S-hyper set). Let $A, B \in P(Q)$. We say that $A \sigma B$ if for each $a \in A$ there exists $b \in B$ such that $a \sigma b$ and for each $y \in B$ there exists $x \in A$ such that $x \sigma y$.

The equivalence relation σ on Q is said to be S-hyper congruence if $q \sigma p$ implies $x \circ q \sigma x \circ p$, for all $x \in S$ and for all $q, p \in Q$.

If σ be an equivalence relation on Q then we define $Q/\sigma = \{q\sigma : q \in Q\}$.

Theorem 3.15 *If σ be an S-hyper congruence on an S-hyper set Q, then Q/σ is an S-hyper set.*

Proof. We define $\star : S \times Q/\sigma \mapsto P(Q/\sigma)$ by

$$x \star q\sigma = \begin{cases} \{p\sigma \in Q/\sigma : p \in x \circ q\}, & \text{if } x \circ q \neq \emptyset \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $x \in S$ and $q\sigma \in Q/\sigma$.

Let $q\sigma = r\sigma$, then $q\sigma r$ implies $x \circ q\sigma x \circ r$, for all $x \in S$.

Let $p\sigma \in x \star q\sigma$ implies $p \in x \circ q$ implies there exists $t \in x \circ r$ such that $p\sigma t$ implies $p\sigma = t\sigma$, for some $t \in x \circ r$ implies $p\sigma = t\sigma$, $t\sigma \in x \star r\sigma$ implies $p\sigma \in x \star r\sigma$.

This implies that $x \star q\sigma \subseteq x \star r\sigma$. Similarly we can show that $x \star r\sigma \subseteq x \star q\sigma$. Therefore $x \star q\sigma = x \star r\sigma$.

If $x \star q\sigma = \emptyset$, then $x \circ q = \emptyset$ and so $x \circ r = \emptyset$ and therefore $x \star r\sigma = \emptyset$, i.e., $x \star q\sigma = x \star r\sigma$.

Hence, $\star$ is well-defined.

$q \in 1 \circ q$ $[1 \circ q = \{q\}]$ implies $q\sigma \in 1 \star q\sigma$ $[1 \star q\sigma = \{q\sigma\}]$, for all $q\sigma \in Q/\sigma$.

Let $x \star (y \star q\sigma) = \emptyset$, then either $y \star q\sigma = \emptyset$ or $x \star p\sigma = \emptyset$, for all $p\sigma \in y \star q\sigma$. If $y \star q\sigma = \emptyset$, then $y \circ q = \emptyset$, implies $x \circ (y \circ q) = \emptyset$, implies $xy \circ q = \emptyset$, i.e., $xy \star q\sigma = \emptyset$. Again if $x \star p\sigma = \emptyset$, for all $p\sigma \in y \star q\sigma$, then $x \circ p = \emptyset$, for all $p \in y \circ q$ i.e. $x \circ (y \circ q) = \emptyset$, i.e., $xy \circ q = \emptyset$, i.e., $xy \star q\sigma = \emptyset$. Similarly if $xy \star q\sigma = \emptyset$, then we can prove $x \star (y \star q\sigma) = \emptyset$.

Now let $r\sigma \in x \star (y \star q\sigma)$ then $r\sigma \in x \star p\sigma$, for some $p \in y \star q\sigma$ implies $r \in x \circ p\sigma$ for some $p \in y \circ q$ implies $r \in x \circ (y \circ q)$ implies $r \in xy \circ q$ implies $r\sigma \in xy \star q\sigma$.

This implies that $x \star (y \star q\sigma) \subseteq xy \star q\sigma$.

Similarly, we can show that $xy \star q\sigma \subseteq x \star (y \star q\sigma)$.

Therefore, $x \star (y \star q\sigma) = xy \star q\sigma$. Hence, Q/σ is an S-hyper set. ∎

Theorem 3.16 *If $f : Q_1 \mapsto Q_2$ be an S-hyper homomorphism, then $\sigma = \{(q, p) : f(q) = f(p)\}$ is an S-hyper congruence on Q_1.*

Proof. Clearly, σ is an equivalence relation on Q_1. Let $q\sigma p$ then $f(q) = f(p)$ implies $x \circ f(q) = x \circ f(p)$ i.e. $f(x \circ q) = f(x \circ p)$, for all $x \in S$

Let $r \in x \circ q$ then $f(r) \in f(x \circ q) = f(x \circ p)$ implies there exists $t \in x \circ p$ such that $f(r) = f(t)$ implies there exists $t \in x \circ p$ such that $t\sigma r$. Similarly, we can show that if $t \in x \circ p$ then there exists $r \in x \circ q$ such that $t\sigma r$.

Therefore, $x \circ q\sigma x \circ p$. Hence, σ is an S-hyper congruence on Q_1. ∎

Theorem 3.17 *If σ be an S-hyper congruence on an S-hyper set Q then there is an S-hyper homomorphism $f : Q \mapsto Q/\sigma$ such that* $\ker f = \sigma$.

Proof. We define $f : Q \mapsto Q/\sigma$ by $f(q) = q\sigma$, for all $q \in Q$.

Now $f(x \circ q) = \{f(p) : p \in x \circ q\} = \{p\sigma : p \in x \circ q\} = x \star q\sigma = x \star f(q)$.

Therefore, f is S-hyper homomorphism.

Let $(q, p) \in \ker f$, then $f(q) = f(p)$, implies $q\sigma = p\sigma$, implies $q\sigma p$, i.e., $(q, p) \in \sigma$. Similarly, we can show that, if $(q, p) \in \sigma$, then $(q, p) \in \ker f$. Therefore, $\ker f = \sigma$. Hence the result. ∎

Theorem 3.18 *Let $f : Q_1 \mapsto Q_2$ be a surjective S-hyper homomorphism and σ be the S-hyper congruence on Q_1 induced by f. There exists a bijective S-hyper homomorphism $g : Q_1/\sigma \mapsto Q_2$ such that $g(q\sigma) = f(q)$, for all $q \in Q_1$.*

Proof. Let $q\sigma = p\sigma$ in Q_1/σ, then $q\sigma p$ implies $f(q) = f(p)$. Therefore, $g(q\sigma) = f(q)$ is well-defined as well as injective.

Let $q_2 \in Q_2$ then there exists $q_1 \in Q_1$ such that $f(q_1) = q_2$. Then there exists $q_1\sigma \in Q_1/\sigma$ such that $g(q_1\sigma) = f(q_1) = q_2$. Hence g is surjective.

$g(x \star q\sigma) = \{g(p\sigma) : p\sigma \in x \star q\sigma\} = \{f(p) : p \in x \circ q\} = f(x \circ q) = x \circ f(q) = x \star g(q\sigma)$.

Therefore, g is a bijective S-hyper homomorphism. ∎

Definition 3.19 A subset T of an S-hyper set Q is said to be an S-subhyper set if $x \circ T \subseteq T$ for all $x \in S$.

Corollary 3.20 *If T be an S-subhyper set of Q, then $1 \circ T = T$.*

Proof. T is an S-subhyper set of Q, then $x \circ T \subseteq T$, for all $x \in S$. In particular, $1 \circ T \subseteq T$. Again $q \in 1 \circ q$, for all $q \in Q$. In particular, $t \in 1 \circ t$, for all $t \in T$, implies $T \subseteq 1 \circ T$. Therefore, $T = 1 \circ T$. □

Corollary 3.21 *If $f : Q_1 \mapsto Q_2$ be an S-hyper homomorphism then $f(Q_1)$ is an S-subhyper set.*

Proof. $x \circ f(Q_1) = \bigcup_{f(p) \in f(Q_1)} x \circ f(p) = \bigcup_{f(p) \in f(Q_1)} f(x \circ p) = f\left(\bigcup_{f(p) \in f(Q_1)} x \circ p\right)$

$= f\left(\bigcup_{p \in Q_1} x \circ p\right) \subseteq f(Q_1)$, for all $x \in S$. ∎

Theorem 3.22 *Let Q be an S-hyper set. Then*

(i) *Q is itself an S-subhyper set.*

(ii) *$S \circ q$ is an S-subhyper set.*

(iii) *Union and Intersection of two S-subhyper sets are subhyper sets.*

Proof. (i) $x \circ Q \subseteq Q$, for all $x \in S$.

(ii) $x \circ (S \circ q) = \bigcup\{x \circ p : p \in S \circ q\} = \bigcup\{x \circ p : p \in y \circ q$, where $y \in S\} = \bigcup\{x \circ (y \circ q) : y \in S\} = \bigcup\{xy \circ q : y \in S\} = xS \circ q \subseteq S \circ q$, for all $x \in S$. Hence $S \circ q$ is an S-subhyper set.

(iii) Let Q_1 and Q_2 be two S-subhyper sets of Q.

$x \circ (Q_1 \cup Q_2) = \bigcup\{x \circ t : t \in Q_1 \cup Q_2\} = \bigcup\{x \circ t : t \in Q_1\} \cup \bigcup\{x \circ t : t \in Q_2\} = x \circ Q_1 \cup x \circ Q_2 \subseteq Q_1 \cup Q_2$, for all $x \in S$.

$x \circ (Q_1 \cap Q_2) = \bigcup\{x \circ t : t \in Q_1 \cap Q_2\} \subseteq x \circ Q_1 \cap x \circ Q_2 \subseteq Q_1 \cap Q_2$, for all $x \in S$. ∎

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