FUZZY SUBNEXUSES

A. Saeidi Rashkolia

A. Hasankhani

Faculty of Mathematics
Islamic Azad
University-Kerman Branch
Kerman
Iran
e-mail: alisaeidi@iauk.ac.ir
abhasan@mail.uk.ac.ir

Abstract. The notion of fuzzy prime subnexus of a nexus is defined, and some related results are obtained. In particular, by considering the concept of homomorphism, some theorems about the coimage and preimage are proved. Finally, the notion of Quotient nexus induced by a fuzzy subnexus is introduced and some its properties are investigated.

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1. Introduction and preliminaries

The space Structure Research Center of the University of Surrey was founded by Z.S. Makoswski as part of Department of Civil Engineering in 1963.

The aim of the Center is to carry out research into the design and analysis of space structures. Space structures include structural forms such as single and double layer girds, barrel vaults, shells and various forms of tension structures.

The basic idea of a nexus has been further developed as a mathematical object for general use. The aim of recent study has been to evolve a mathematical object that allows complex processes on groups of mathematical objects to be formulated with ease of elegance. This notion is very useful for study of space structure.

In this paper the notion of a fuzzy subnexus of a nexus is defined and some related results are obtained.

Definition 1.1. [1]

(i) An address is a sequence of $\aleph^* = \aleph \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$ for all $i \geq k$. The sequence of zero is called the empty address and is denoted by (). In other word, every nonempty address is of the form

$$(a_1, a_2, ..., a_n, 0, 0, ...),$$

where n, belongs to $\aleph$. Hereafter, this address will be denoted by

$$(a_1, a_2, ..., a_n).$$

(ii) A nexus N is the set of address with the following properties:

$$(a_1, a_2, ..., a_n) \in N \implies (a_1, a_2, ..., a_{n-1}, t) \in N,\ \forall 0 \leq t \leq a_n.$$

$$\{a_i\}_{i=1}^{\infty} \in N, a_i \in \aleph \implies \forall n \in \aleph, (a_1, a_2, ..., a_n) \in N.$$

For example the set:

$$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3)\}$$

is a nexus.

Remark 1.1. Condition (ii) in Definition 1.1 implies that

$$(a_1, a_2, ..., a_n) \in N \Rightarrow (a_1, a_2, ..., a_{n-1}) \in N,\ \ \forall\, n \geq 2.$$

Definition 1.2. [1] Let $w \in N$. The level of w is said to be:

(i) n, if $w = (a_1, a_2, ..., a_n)$, for some $a_n \in \aleph$,

(ii) ∞, if w is an infinite sequence of $\aleph$,

(iii) 0, if $w = ()$.

The level of w is denoted by $l(w)$.

Definition 1.3. [1] Let $w = \{a_i\},\ i \in \aleph$ and $v = \{b_i\}\ i \in \aleph$ be addresses. Then $w \leq v$ if $l(w) = 0$ or one of the following cases are satisfied:

Case 1. If $l(w) = 1$, that is $w = (a_1)$, for all $a_1 \in \aleph$ and $a_1 \leq b_1$.

Case 2. If $1 < l(w) < \infty$, then $l(w) \leq l(v)$ and $a_{l(w)} \leq b_{l(v)}$ and for any $1 \leq i \leq l(w)$, $a_i = b_i$.

Case 3. If $l(w) = \infty$, then $w = v$. For example, consider the nexus:

$$N = \{(), (1), (2), (1,1), (1,2), (1,3), (1,3,1), (1,3,2)\}.$$

Therefore

$$(1) \leq (2), (1,2) \leq (1,3,1), (1,3,1) \leq (1,3,2).$$

Proposition 1.1. [1] $(N, \leq)$ *is a lower semi-lattice.*

Proof. Clearly, $\leq$ is reflexive. Let $w, v \in N, w \leq v$ and $v \leq w$. Now one can show that $\leq$ is anti-symmetric. For this consider three cases:

Case 1. If $l(w) = 0$, then $l(v) \leq l(w) = 0$, hence $v = () = w$.

Case 2. If $l(w) = 1$, then $1 \leq l(v) \leq l(w)$ and hence, $l(v) = 1$. suppose that $v = (a)$ and $w = (b)$, for some $a, b \in \aleph$. Thus, $a \leq b$ and $b \leq a$. Therefore $a = b$ and $v = w$.

Case 3. Let $n = l(w) > 1$. That is, $w = (a_1, a_2, ..., a_n)$. Then $l(v) = n$. Assume that $v = (b_1, b_2, ..., b_n)$. Therefore, $a_n \leq b_n$, $b_n \leq a_n$ and $a_i = b_i$, for all $1 \leq i \leq n$. Hence $a_j = b_j$ for all $1 \leq j \leq n$. In other words, $w = v$.

Clearly, if $l(w) = \infty$, then $w = v$.

Now, the transitive property of $\leq$ must be proved. To this aim, let $v, w, t \in N$, $w \leq v$ and $v \leq t$.

If $l(t) = 0$, then $l(v) = l(w) = 0$ and hence $w = t = ()$.

If $l(t) = 1$, then $l(v) = 0$ or $l(v) = 1$. Firstly, assume that $l(v) = 0$. It is easy to see that $l(w) = 0$, therefore $w = () \leq t$.

Now suppose that $l(v) = 1$ and $l(w) \neq 0$. Then $w = (a), v = (b)$, and $t = (c)$ for some $a, b, c \in \aleph$, and $a \leq b \leq c$. Therefore, $w \leq t$. At last, suppose that $1 < l(t) = k < \infty$ and $t = (c_1, c_2, ..., c_k)$ for some $c_i \in \aleph$. Then, $v = (b_1, b_2, ..., b_m)$ and $w = (a_1, a_2, ..., a_n)$ where that $b_i, a_i \in \aleph, n \leq m \leq k, a_n \leq b_n, b_m \leq c_m, a_i = b_i$ and $b_j = c_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore, $a_n \leq c_n$ and $a_r \leq c_r$ for all $1 \leq r \leq n$. Consequently, $w \leq t$. It is easy to see that, if $l(t) = \infty$, then $t = v = w$.

Now, for the last part of the proof of the theorem, it must be shown that the greatest lower bounded of v and w that $v \wedge w$ belongs to N. Suppose that $v = \{a_i\}_{i \in \aleph}$ and $w = \{b_i\}_{i \in \aleph}$ are two addresses. Now, consider three cases:

Case 1. If v or w is empty address, then $v \wedge w = ()$.

Case 2. If v and w are not empty addresses and $a_1 \neq b_1$, then $v \wedge w = (a_1 \wedge b_1)$.

Case 3. Assume that v and w are not empty addresses and $a_1 = b_1$. In this case, suppose that n is the greatest element of $\aleph$ such that $a_n = b_n$. Put $s = (a_1, a_2, ..., a_n, a_{n+1} \wedge b_{n+1})$. It is easy to see that $v \wedge w = s$ and s belongs to N.

Proposition 1.2. [1] *Suppose that N is a set of addresses. Then N is a nexus if and only if, $v \in N$ and $w \leq v$ implies that, $w \in N$.*

Definition 1.4. [1] A nonempty subset S of N is called a subnexus of N provided that S itself is a nexus. The set of all subnexus of N is denoted by $SUB(N)$.

Definition 1.5. [1] Let $\emptyset \neq X \subseteq N$. Then the smallest subnexus of N containing X is called the subnexus of N generated by X and denoted $\langle X \rangle$. If $\mid X \mid = 1$, then $\langle X \rangle$ is called a cyclic subnexus of N.

Definition 1.6. [5] A proper subnexus P of a nexus N is said to be a prime subnexus of N if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$ for any $a, b \in N$.

Definition 1.7. [7] A fuzzy subset of a set S is a function μ from S into $[0, 1]$.

2. Fuzzy subnexus

Definition 2.1. Let $\widetilde{P}$ be fuzzy subset of nexus N. Then $\widetilde{P}$ is called a fuzzy subnexus of N, if $w \leq v$ implies that $\widetilde{P}(v) \leq \widetilde{P}(w)$, for all $v, w \in N$.

The set of all fuzzy subnexus of N is denoted by $FSUB(N)$.

Remark 2.1. If $\widetilde{P} \in FSUB(N)$, then $\widetilde{P}(0) \geq \widetilde{P}(v)$, for all $v \in N$.

Example 2.1. Consider the nexus $N = \{(), (1), (2), (2, 1), (2, 2)\}$ define the fuzzy subset $\widetilde{P}$ of N as follows,

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}((1)) = \alpha_2, \widetilde{P}((2, 1)) = \alpha_4, \widetilde{P}((2, 2)) = \alpha_5,$$

where that $\alpha_5 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1$. Therefore, $\widetilde{P} \in FSUB(N)$.

Example 2.2. Let N be an arbitrary nexus. If $0 \neq w = (a_1, a_2, ..., a_n) \in N$, then $a_1a_2...a_n$ we mean a number with n, digits. Define the fuzzy subset $\widetilde{P}$ of N as follows if $w = (a_1, a_2, ..., a_n) \in N$ is of level n, then

$$\widetilde{P}(w) = \frac{1}{a_1a_2...a_n}.$$

Moreover,

$$\widetilde{P}(v) = \begin{cases} 1, & v = 0, \\ 0, & l(v) = \infty. \end{cases}$$

Clearly, $\widetilde{P} \in FSUB(N)$.

Definition 2.2. Let A is a subset of N. A function $\chi_A : N \longrightarrow [0, 1]$ is called a characteristic function of A if

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin a. \end{cases}$$

Theorem 2.1. *Let A be a nonempty subset of nexus N, $A \in N$, if and only if*

$$\chi_A \in FSUB(N),$$

where that χ_A is the characteristic function of A.

Proof. Let A be a subnexus of N and $v \leq w$, for some $v, w \in N$. If $w \in A$, then $v \in A$. Hence $\chi_A(w) = \chi_A(v)$ and if $w \notin A$, then $\chi_A(w) \leq \chi_A(v)$.

Conversely, suppose $w \in N$, $v \in A$ and $w \leq v$. Then, $\chi_A(v) \leq \chi_A(w)$. Since $\chi_A(v) = 1$, we have $\chi_A(w) = 1$ and $w \in A$. Hence, $A \in SUB(N)$.

Definition 2.3. [3] Let $\widetilde{P}$ be a fuzzy subset of a set S.

For $t \in [0, 1]$, the set $\widetilde{P}_t = \{s \in S \mid \widetilde{P}(s) \geq t\}$ is called a level subset of $\widetilde{P}$.

Theorem 2.2. *$\widetilde{P} \in FSUB(N)$ if and only if $\widetilde{P}_t \in SUB(N)$, for all $t \in [0, 1]$, where $\widetilde{P}_t \neq \emptyset$.*

Proof. Suppose $\widetilde{P} \in SUB(N)$ and $\widetilde{P}_t \neq \emptyset$, for $t \in [0, 1]$ and let $v \in N, w \in \widetilde{P}_t$ such that, $v \leq w$. Then, $t \leq \widetilde{P}(w) \leq \widetilde{P}(v)$. Hence, $v \in \widetilde{P}_t$.

Conversely, suppose $v, w \in N$ and $v \leq w$ and suppose $\widetilde{P}(w) = t$, for $t \in [0, 1]$.

Since $\widetilde{P}_t \in SUB(N)$, then $v \in \widetilde{P}_t$ i.e $\widetilde{P}(v) \geq \widetilde{P}(w) = t$.

Theorem 2.3. *Let N be a nexus and $\mathcal{A} = \{\widetilde{P}_\alpha \mid \widetilde{P}_\alpha \in FSUB(N)\}$. Then*

(i) $\bigcap_{\alpha \in I} \widetilde{P}_\alpha \in FSUB(N)$.

(ii) $\bigcup_{\alpha \in I} \widetilde{P}_\alpha \in FSUB(N)$.

Proof. Suppose $v, w \in N$ and $w \leq v$. Now we have:

$$\left(\bigcap_{\alpha \in I} \widetilde{P}_\alpha\right)(v) = \inf_{\alpha \in I}(\widetilde{P}_\alpha(v)) \leq \inf_{\alpha \in I}(\widetilde{P}_\alpha(w)) = \left(\bigcap_{\alpha \in I} \widetilde{P}_\alpha(v)\right).$$

And, similarly,

$$\left(\bigcup_{\alpha \in I} \widetilde{P}_\alpha\right)(v) = \sup_{\alpha \in I}(\widetilde{P}_\alpha(v)) \leq \sup_{\alpha \in I}(\widetilde{P}_\alpha(w)) = \left(\bigcup_{\alpha \in I} \widetilde{P}_\alpha\right)(w).$$

3. Prime fuzzy subnexus

Definition 3.1. Let N be a nontrivial nexus, (i.e., $N \neq \{()\}$). A fuzzy subnexus $\widetilde{P}$ of N is called a *prime fuzzy subnexus* if

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\},$$

for all $a, b \in N$.

The set of all prime fuzzy subnexus of N is denoted by $FPSUB(N)$.

Remark 3.1. If N is a nontrivial nexus and $\widetilde{P}$ a prime fuzzy subnexus of N, then

$$\widetilde{P}(a \wedge b) = \widetilde{P}(a) \text{ or } \widetilde{P}(b).$$

Example 3.1. Let $N = \{(), (1), (2), (2,1), (2,2)\}$.

Consider the fuzzy subnexus $\widetilde{P}$ of as follows,

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}(1) = \alpha_2, \widetilde{P}(2) = \alpha_3, \widetilde{P}(2,1) = \alpha_4, \widetilde{P}(2,2) = \alpha_4,$$

such that,

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4.$$

The $\widetilde{P}$ is a prime fuzzy subnexus of N.

Example 3.2. Let

$N=\{(), (1), (2), (3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,1,1), (3,1,2), (3,2,1), (3,2,2)\}$.

Now define the fuzzy subnexus $\widetilde{P}$ of N as follows, $\widetilde{P}(()) = 1$ and, if

$$w = (a_1, a_2, ..., a_n) \in N,$$

$$\widetilde{P}(w) = \frac{1}{a_1 a_2 ... a_n},$$

then $\widetilde{P}$ is a fuzzy subnexus of N, but $\widetilde{P}$ is not prime, because

$$\widetilde{P}((3,2) \wedge (3,1,1)) = \widetilde{P}(3,1) = \frac{1}{31} > \max\{\widetilde{P}(3,2), \widetilde{P}(3,1,1)\} = \frac{1}{32}.$$

Example 3.3. Let $N = \{(), (1), (2), (1,1), (1,2), (1,3), (1,3,1), (1,3,2)\}$.

Define

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}(1) = \widetilde{P}(2) = \alpha_2,\ \widetilde{P}(1,1) = \alpha_3,$$

$$\widetilde{P}(1,2) = \alpha_4, \widetilde{P}(1,3) = \alpha_5,\ \widetilde{P}(1,3,1) = \alpha_6,\ \widetilde{P}(1,3,2) = \alpha_7,$$

such that,

$$\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_7.$$

Then, $\widetilde{P}$ is a prime fuzzy subnexus of N.

Proposition 3.1. *Let N be a cyclic subnexus and $\widetilde{P}$ is a fuzzy subnexus of N. Then, $\widetilde{P} \in FPSUB(N)$.*

Proof. Since N is cyclic, then for all $a, b \in N$, $a \leq b$ or $b \leq a$ without loss of generality suppose $a \leq b$. Then, we have

$$\widetilde{P}(b) \leq \widetilde{P}(a)$$

and it is obvious that

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}.$$

Definition 3.2. Let F be a mapping from M onto N, if $\widetilde{P}$ is a fuzzy subset of M. Then, the fuzzy subset $\widetilde{Q}$ of N defined by

$$\widetilde{Q}(y) = \inf_{x \in F^{-1}(y)} \widetilde{P}(x),$$

for all $y \in N$, is called the coimage of M under F.

Similarly, if $\widetilde{Q}$ is a fuzzy subset of N, then the fuzzy subset $\widetilde{P} = \widetilde{Q}F$ of M (i.e., the fuzzy subset defined by

$$\widetilde{P}(x) = \widetilde{Q}(F(x))$$

for all $x \in X$) is called the preimage of v under F.

Definition 3.3. A fuzzy subnexus $\widetilde{P}$ of N has inf-property if, for any subset T of N, there exists $t_0 \in T$ such that

$$\widetilde{P}(t_0) = \inf_{t \in T} \widetilde{P}(t).$$

Example 3.4. In Example 2.2, if N is infinite, then $\widetilde{P}$ has inf-property.

Definition 3.4. Let M, N be two nexus and $F : M \longrightarrow N$ be a function. Then

(i) F is called homomorphism of nexuses if

$$w \leq v \text{ implies } F(w) \leq F(v), \text{ for all } w, v \in M.$$

(ii) F is called a semi-lattice homomorphism of nexuses if

$$F(w \wedge v) = F(w) \wedge F(v), \text{ for all } w, v \in M.$$

Remark 3.2. Clearly every semi-lattice homomorphism of nexuses is a homomorphism of nexuses.

Example 3.5. Let $M = \{(), (1), (2), (1,1)\}$, $N = \{(), (1), (2), (1,1), (2,1)\}$ and $F : M \longrightarrow N$ be a function such that,

$$F(()) = F((1)) = (1),\ F((1,1)) = F((2)) = (2).$$

Then

(i) F is a homomorphism of nexuses.

(ii) F is not a semi-lattice homomorphism of nexuses.

(iii) $F(()) \neq ()$.

Theorem 3.1. *Let $F : M \longrightarrow N$ be an onto homomorphism of nexuses.*

(i) $F(()) = ()$.

(ii) *If $F((v)) = ()$, then $F((w)) = ()$, for all $w \leq v$.*

Proof. It is obvious.

Remark 3.3. Let $F : M \longrightarrow N$ be an onto homomorphism of nexuses and $M \neq ()$. If $F((1)) \neq ()$, then $F((v)) \neq ()$, for all $() \neq v \in M$.

Theorem 3.2. *An onto semi-lattice homomorphism coimage of a one to one prime fuzzy subnexus with inf-property is a prime fuzzy subnexus.*

Proof. Let $F : M \longrightarrow N$ be an onto semi-lattice homomorphism of nexuses and $\widetilde{P}$ a one to one prime fuzzy subnexus of M with inf-property and so $\widetilde{Q}$ the coimage of $\widetilde{P}$ under F and suppose $w', v' \in N$ are arbitrary. Consider the following cases.

Case 1. If $w' \leq v'$ and $w' \neq v'$.

Since F is onto there exists $w \in F^{-1}(w')$, $v \in F^{-1}(v')$ such that,

$$\widetilde{P}(w) = \inf_{t \in F^{-1}(w')} \widetilde{P}(t),$$

$$\widetilde{P}(v) = \inf_{t \in F^{-1}(v')} \widetilde{P}(t).$$

If $\widetilde{P}(w) < \widetilde{P}(v)$, since $\widetilde{P}$ is prime, then $\widetilde{P}(w \wedge v) \leq \widetilde{P}(v)$. Hence, $v = w \wedge v$, i.e., $v \leq w$ implies that $F(v) \leq F(w)$. That is a contradiction.

Now, we have

$$\widetilde{Q}(v') = \inf_{t \in F^{-1}(v')} \widetilde{P}(t) = \widetilde{P}(v) \leq \widetilde{P}(w) = \inf_{t \in F^{-1}(w')} \widetilde{P}(t) = \widetilde{Q}(w').$$

Case 2. If $w' = v'$, then $\widetilde{Q}(v') = \widetilde{Q}(w')$. Therefore, $\widetilde{Q} \in FSUB(N)$.

Finally,

$$\begin{aligned}
\widetilde{Q}(v' \wedge w') &= \inf_{t \in F^{-1}(v' \wedge w')} \widetilde{P}(t) \\
&\leq \widetilde{P}(v \wedge w) \\
&\leq \max\{\widetilde{P}(v), \widetilde{P}(w)\} \\
&= \max\left\{\inf_{t \in F^{-1}(v')} \widetilde{P}(t), \inf_{t \in F^{-1}(w')} \widetilde{P}(t)\right\} \\
&= \max\{\widetilde{Q}(v'), \widetilde{Q}(w')\}.
\end{aligned}$$

Theorem 3.3. *Any onto homomorphism of nexuses preimage of a fuzzy subnexus is also a fuzzy subnexus.*

Proof. Suppose that $F : M \longrightarrow N$ is a homomorphism of nexuses and $\widetilde{Q} \in FSUB(N)$ and $\widetilde{P}$ the preimage of $\widetilde{Q}$ under F.

Now, for all $v, w \in M$, if $v \leq w$, then $F(v) \leq F(w)$. Hence,

$$\widetilde{Q}(F(w)) \leq \widetilde{Q}(F(v)),$$

i.e., $\widetilde{P}(w) \leq \widetilde{P}(v)$ implies that $\widetilde{P} \in FSUB(N)$.

Theorem 3.4. *Any onto semi-lattice homomorphism of nexuses preimage of a prime fuzzy subnexus is also a prime fuzzy subnexus.*

Proof. Let $F : M \longrightarrow N$ be an onto semi-lattice homomorphism of nexus and $\widetilde{Q} \in FSUB(N)$ and $\widetilde{P}$ the preimage of $\widetilde{Q}$ under F.

Now for all $a, b \in M$, $\widetilde{Q}(F(a) \wedge F(b)) \leq \max\{\widetilde{Q}(F(a)), \widetilde{Q}(F(b))\}$ implies

$$\widetilde{Q}(F(a \wedge b)) \leq \max\{(\widetilde{Q}F)(a), (\widetilde{Q}F)(b)\}.$$

Hence, by the previous theorem, $\widetilde{P} \in FPSUB(M)$.

Proposition 3.2. *Suppose N be a nexus and $\widetilde{P}$ a fuzzy subnexus arbitrary of N.*

(i) *If N is a chain, then $\widetilde{P}$ is a prime fuzzy subnexus.*

(ii) *If $\widetilde{P}$ is a prime fuzzy subnexus and one to one, then N is a chain.*

Proof. (i) Let $v, w \in N$ and $v \leq w$ implies $\widetilde{P}(v) \geq \widetilde{P}(w)$. Then

$$\widetilde{P}(w \wedge v) = \max\{\widetilde{P}(w), \widetilde{P}(v)\}.$$

(ii) Let $v, w \in N$ and $v \wedge w = u$ such that $u \neq v$, $u \neq w$. Hence,

$$\widetilde{P}(w) < \widetilde{P}(u), \ \ \widetilde{P}(v) < \widetilde{P}(u).$$

Then,

$$\widetilde{P}(u) > \max\{\widetilde{P}(w), \widetilde{P}(v)\}.$$

That is a contradiction.

Proposition 3.3. *Let B be a subnexus of nexus N. Then, $B \in PSUB(N)$ if and only if $\chi_B \in FPSUB(N)$, where that χ_B is a characteristic function of B.*

Proof. Let $a, b \in N$. If $a \wedge b \notin B$, then

$$\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}.$$

If $a \wedge b \in B$, then by assumption $a \in B$ or $b \in B$. Thus,

$$\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}.$$

Conversely, let $a \wedge b \in B$. Since $\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}$, then,

$$\max\{\chi_B(a), \chi_B(b)\} = 1.$$

Hence, $a \in B$ or $b \in B$.

Theorem 3.5. *Suppose that N is a nexus. Then, $\widetilde{P} \in FPSUB(N)$ if and only if $\widetilde{P}_t \in PSUB(N)$, for all $t \in [0,1]$, whenever $\widetilde{P}_t \neq \emptyset$.*

Proof. Let $\widetilde{P} \in FPSUB(N)$, $t \in [0,1]$, $\widetilde{P}_t \neq \emptyset$ and suppose $a \wedge b \in \widetilde{P}_t$.

Hence, $t \leq \widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}$ implies that $a \in \widetilde{P}_t$ or $b \in \widetilde{P}_t$.

Conversely, suppose $a, b \in N$ is arbitrary and suppose $\widetilde{P}(a \wedge b) = t$, for $t \in [0,1]$. Then, $a \wedge b \in \widetilde{P}_t$ implies that $a \in \widetilde{P}_t$ or $b \in \widetilde{P}_t$.

Hence,

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}.$$

Theorem 3.6. *Suppose that N is a nexus and $\mathbf{B} = \{\widetilde{P} \mid \widetilde{P} \in FPSUB(N)\}$.*

(i) $\bigcap_{i\in I} \widetilde{P}_i \in FPSUB(N)$.

(ii) $\bigcap_{i\in I} \widetilde{P}_i \in FPSUB(N)$.

Proof. (i) Let $a, b \in N$.

$$\begin{aligned}
\bigcap_{i\in I} \widetilde{P}_i(a \wedge b) &= \inf_{i\in I} \widetilde{P}_i(a \wedge b) \\
&\leq \inf_{i\in I} [\max\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}] \\
&= \max\left\{\inf_{i\in I}\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right\} \\
&= \max\left\{\inf_{i\in I} \widetilde{P}_i(a), \inf_{i\in I} \widetilde{P}_i(b)\right\} \\
&= \max\left\{\left(\bigcap_{i\in I} \widetilde{P}_i\right)(a), \left(\bigcap_{i\in I} \widetilde{P}_i\right)(b)\right\}.
\end{aligned}$$

(ii) Similarly, for all $a, b \in N$. We have:

$$\begin{aligned}
\bigcup_{i\in I} \widetilde{P}_i(a \wedge b) &= \sup_{i\in I} \widetilde{P}_i(a \wedge b) \\
&\leq \sup_{i\in I} [\max\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}] \\
&= \max\left\{\sup_{i\in I}\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right\} \\
&= \max\left\{\sup_{i\in I} \widetilde{P}_i(a), \sup_{i\in I} \widetilde{P}_i(b)\right\} \\
&= \max\left\{\left(\bigcup_{i\in I} \widetilde{P}_i\right)(a), \left(\bigcup_{i\in I} \widetilde{P}_i\right)(b)\right\}.
\end{aligned}$$

4. Quotient nexuses induced by a fuzzy subnexus

Definition 4.1. Let $N \neq \{()\}$ be a nexus and $\widetilde{P}$ a fuzzy subnexus of N such that, $\widetilde{P}(x) \neq 0$, for some $x \neq ()$. Define the relation " $\sim$ " as follow.

$$v \sim w \;\Leftrightarrow\; \exists\, t \in N - \{()\}$$

such that, $\min\{\widetilde{P}(v \wedge t), \widetilde{P}(w \wedge t)\} > 0$, for all $v, w \in N$.

Theorem 4.1. " $\sim$ " *is a an equivalence relation.*

Proof. First we show that " $\sim$ " is reflexive. To do this let v be an arbitrary element of N. If $v = ()$, then it is obvious that $v \sim v$. Now let $v \neq ()$. If $\widetilde{P}(v) > 0$, then put $t = v$ and hence $v \sim v$. If $\widetilde{P}(v) = 0$, then there exists $w \in N - \{()\}$ such that $\widetilde{P}(w) > 0$. Choose $t = v \wedge w$ and hence

$$\widetilde{P}(v \wedge w) \geq \widetilde{P}(w) > 0,$$

which implies that $v \sim v$. The symmetric proof of " $\sim$ " is easy.

To prove the transitivity of " $\sim$ ", we assume that $v, w, z \in N$ and $v \sim w$, $w \sim z$. Thus,

$$\min\{\widetilde{P}(v \wedge t), \widetilde{P}(w \wedge t)\} > 0 \text{ and } \min\{\widetilde{P}(w \wedge t'), \widetilde{P}(z \wedge t')\} > 0,$$

for some $t \in N - \{()\}$, $t' \in N - \{()\}$.

Choose $t'' = t \wedge t'$ and hence

$$\min\{\widetilde{P}(v \wedge t''), \widetilde{P}(z \wedge t'')\} > 0.$$

Remark 4.1. The following example shows that the condition

$$\widetilde{P}(x) \neq 0, \text{ for some } () \neq x \in N$$

is necessary.

Example 4.1. Let $N = \{(), (1)\}$ and $\widetilde{P}(()) = 1, \widetilde{P}(1) = 0$. Then $(1) \nsim (1)$, because $\min\{\widetilde{P}((1) \wedge (1)), \widetilde{P}((1) \wedge (1))\} = 0$.

Remark 4.2. Let $N \neq \{()\}$ be a nexus and $\widetilde{P}$ a fuzzy subnexus of N such that $\widetilde{P}(x) \neq 0$, for some $x \neq ()$. The equivalence class of v and the set of all equivalence classes of $\widetilde{P}$ are denoted by $\widetilde{P}_v$ and $\dfrac{N}{\widetilde{P}}$ respectively.

Definition 4.2. We define " $\leq$ " on $\dfrac{N}{\widetilde{P}}$ as follows:

$$\widetilde{P}_v \leq \widetilde{P}_w \Leftrightarrow v \leq w.$$

Lemma 4.1. $\left(\frac{N}{\widetilde{P}}, \leq\right)$ *is a* $\wedge$*-semi-lattice.*

Proof. Let $v, w \in N$. Since $v \wedge w \leq v$, $v \wedge w \leq w$. Then $\widetilde{P}_{v\wedge w} \leq \widetilde{P}_v$, $\widetilde{P}_{v\wedge w} \leq \widetilde{P}_w$. Now if $\widetilde{P}_s \leq \widetilde{P}_v$, $\widetilde{P}_s \leq \widetilde{P}_w$, for all $s \in N$, then it is obvious that $\widetilde{P}_s \leq \widetilde{P}_{v\wedge w}$. Hence

$$\inf\{\widetilde{P}_v, \widetilde{P}_w\} = \widetilde{P}_{v\wedge w}.$$

$\widetilde{P}_{v\wedge w}$ is denoted by $\widetilde{P}_v \widetilde{\wedge} \widetilde{p}_w$.

Theorem 4.2. *Let* $M \neq \emptyset$, $F : M \longrightarrow N$ *be an onto semi-lattices homomorphism of nexuses such that,* $F((1)) \neq ()$ *and* $\widetilde{Q}$ *a fuzzy subnexus of* N *such that* $\widetilde{Q}(x) \neq 0$ *for some* $x \neq ()$. *Then,*

$$\begin{array}{rcl} \Psi : \dfrac{M}{\widetilde{Q}F} & \longrightarrow & \dfrac{N}{\widetilde{Q}} \\ (\widetilde{Q}F)_w & \longmapsto & \widetilde{Q}_{F(w)} \end{array}$$

is an isomorphism of semi-lattices.

Proof. Ψ is well defined, because for all $w_1, w_2 \in M$, if $(\widetilde{Q}F)_{w_1} = (\widetilde{Q}F)_{w_2}$, then $w_1 \sim w_2$. Hence

$$\begin{aligned} &\min\{\widetilde{Q}F(w_1 \wedge t), \widetilde{Q}F(w_2 \wedge t)\} > 0, \text{ for some } t \in M - \{()\}. \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1 \wedge t)), \widetilde{Q}(F(w_2 \wedge t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1) \wedge F(t)), \widetilde{Q}(F(w_2) \wedge F(t))\} > 0 \\ &\Rightarrow F(w_1) \sim F(w_2) \Rightarrow \widetilde{Q}_{F(w_1)} = \widetilde{Q}_{F(w_2)}. \end{aligned}$$

To prove one to one of Ψ, let $w_1, w_2 \in M$ and $\widetilde{Q}_{F(w_1)} = \widetilde{Q}_{F(w_2)}$. Then

$$F(w_1) \sim F(w_2).$$

Hence,

$$\min\{\widetilde{Q}(F(w_1 \wedge t')), \widetilde{Q}(F(w_2) \wedge t')\} > 0, \text{ for some } t' \in N - \{()\}.$$

Then, since F is onto, there exists $t \in M - \{()\}$ such that

$$\begin{aligned} &\min\{\widetilde{Q}(F(w_1) \wedge F(t)), \widetilde{Q}(F(w_2) \wedge F(t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1 \wedge t)), \widetilde{Q}(F(w_2 \wedge t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}F(w_1 \wedge t), \widetilde{Q}F(w_2 \wedge t)\} > 0. \end{aligned}$$

Hence, $w_1 \sim w_2$ and $(\widetilde{Q}F)_{w_1} = (\widetilde{Q}F)_{w_2}$.

The proof onto of Ψ is obvious.

Now, for all $w_1, w_2 \in M$, we have:

$$\begin{aligned}\Psi((\widetilde{Q}F)_{w_1}\widetilde{\wedge}(\widetilde{Q}F)_{w_2}) &= \Psi((\widetilde{Q}F)_{w_1\wedge w_2}) \\ &= \widetilde{Q}_{F(w_1\wedge w_2)} = \widetilde{Q}_{F(w_1)\wedge F(w_2)} = \widetilde{Q}_{F(w_1)}\widetilde{\wedge}\widetilde{Q}_{F(w_2)} \\ &= \Psi((\widetilde{Q}F)_{w_1})\widetilde{\wedge}\Psi((\widetilde{Q}F)_{w_2}).\end{aligned}$$

Hence Ψ is isomorphism.

Remark 4.3. The following example shows that in the above theorem the condition $F((1)) \neq ()$ is necessary.

Example 4.2. Let $M = \{(), (1), (1,1)\}$, $N = \{(), (1)\}$ and $F : M \longrightarrow N$ be an function such that,

$$F(()) = F((1)) = () \ , \ F((1,1)) = (1).$$

Also, suppose $\widetilde{Q}$ to be a fuzzy subnexus of N such that,

$$\widetilde{Q}(()) = 1, \ \widetilde{Q}((1)) = 0.$$

Now, F is a semi-lattice homomorphism and

$$\begin{aligned}\frac{M}{\widetilde{Q}F} &= \{(), (1)\}, \\ \frac{N}{\widetilde{Q}} &= \{\{(), (1), (1,1)\}\}.\end{aligned}$$

Hence, $\frac{M}{\widetilde{Q}F}$ and $\frac{N}{\widetilde{Q}}$ are not isomorphic.

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