SOME PROPERTIES OF n-ISOCLINISM IN LIE ALGEBRAS

Foroud Parvaneh

Mohammad Reza R. Moghaddam

Department of Mathematics
Islamic Azad University
Mashhad-Branch, Mashhad
Iran
e-mail: foroudparvane@gmail.com
mrrm5@yahoo.ca

A. Khaksar

Faculty of Mathematical Sciences
and
Centre of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
Mashhad
Iran

Abstract. In 1940, P. Hall introduced the concept of isoclinism of groups and it was generalized to n-isoclinism and isologism with respect to a given variety of groups.

In the present article this notion is studied in Lie algebras and give some results similar to N.S. Hekster in 1986. In particular, it is shown that every family of n-isoclinism of Lie algebras contains an n-stem Lie algebra of minimal dimension.

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1. Introduction and Preliminaries

Let L be a Lie algebra, then the *lower* and *upper central series* of L are defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^{n+1} \supseteq \cdots ,$$

and

$$(0) = Z_0(L) \subseteq Z_1(L) = Z(L) \subseteq Z_2(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots ,$$

respectively, where $L^{n+1} = [L, L^n]$ and

$$\frac{Z_n(L)}{Z_{n-1}(L)} = Z\left(\frac{L}{Z_{n-1}(L)}\right).$$

The following definition is vital in our investigation and it is similar to the case for groups (see [1] or [2]).

Definition 1.1. Let L and K be two Lie algebras, $\alpha : \dfrac{L}{Z_n(L)} \longrightarrow \dfrac{K}{Z_n(K)}$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ be Lie algebra homomorphisms such that the following diagram is commutative

$$\begin{array}{ccc}
\dfrac{L}{Z_n(L)} \times \cdots \times \dfrac{L}{Z_n(L)} & \xrightarrow{\varphi} & L^{n+1} \\
\Big\downarrow \underbrace{\alpha \times \cdots \times \alpha}_{(n+1)-times} & & \Big\downarrow \beta \\
\underbrace{\dfrac{K}{Z_n(K)} \times \cdots \times \dfrac{K}{Z_n(K)}}_{(n+1)-times} & \xrightarrow{\psi} & K^{n+1}
\end{array}$$

where $\varphi : (\bar{l_1}, \bar{l_2}, ..., \bar{l}_{n+1}) \mapsto [l_1, l_2, ..., l_{n+1}]$, for all $\bar{l_i} \in \dfrac{L}{Z_n(L)}$, $i = 1, ..., n+1$ and similarly for ψ. In fact, α and β are defined in such a way that they are compatible, i.e., for all $l_i \in L$, $\beta([l_1, l_2, \cdots, l_{n+1}]) = [k_1, k_2, ..., k_{n+1}]$, in which $k_i \in \alpha(l_i + Z_n(L)), 1 \leq i \leq n+1$.

The pair (α, β) is called *n-homoclinism* and if they are both isomorphisms, then (α, β) is called *n-isoclinism*. In this case, L and K are said to be *n-isoclinic*, which is denoted by $L \underset{n}{\sim} K$. If $n = 1$, then it will be the notion of isoclinism, which was first introduced by P. Hall [1] in 1940. The kernel and the image of (α, β) are defined as follows:

$$Ker(\alpha, \beta) = Ker\beta \text{ and } Im(\alpha, \beta) = I \subseteq K,$$

where $\alpha\left(\dfrac{L}{Z_n(L)}\right) = \dfrac{I}{Z_n(K)}$.

Now, the above definition gives the following

Theorem 1.2. *Let (α, β) be an n-homoclinism of Lie algebras L into K, then*

(i) $Ker(\alpha, \beta) \trianglelefteq L$;

(ii) $\dfrac{L}{Ker(\alpha, \beta)} \underset{n}{\sim} Im(\alpha, \beta)$.

Proof. It is obvious that $Ker(\alpha, \beta) = Ker\beta$ is an ideal of L. To prove the second part, assume that

$$\alpha\left(\frac{L}{Z_n(L)}\right) = \frac{I}{Z_n(K)},$$
$$Z_n\left(\frac{L}{Ker\beta}\right) = \frac{J}{Ker\beta}.$$

Clearly, $J \trianglelefteq L$, $Z_n(L) + Ker\beta \trianglelefteq J$, $Z_n(K) \trianglelefteq Z_n(I)$ and $Im\beta = I^{n+1}$. Thus

$$\frac{L/Ker\beta}{Z_n(L/Ker\beta)} \cong \frac{L}{J} \cong \frac{L/Z_n(L)}{J/Z_n(L)}.$$

Now, the homomorphism α induces the homomorphism

$$\bar{\alpha} : \frac{L}{Z_n(L)} \longrightarrow \frac{I/Z_n(K)}{Z_n(I)/Z_n(K)}$$
$$l + Z_n(L) \longmapsto \alpha(l + Z_n(L)) + \frac{Z_n(I)}{Z_n(K)}$$

Clearly, $Ker\bar{\alpha} = \dfrac{J}{Z_n(L)}$ and $\bar{\alpha}$ is surjective. We also have $\dfrac{L}{J} \cong \dfrac{I}{Z_n(I)}$, which implies that

$$\frac{L/Ker\beta}{Z_n(L/Ker\beta)} \cong \frac{I}{Z_n(I)}.$$

Since $(L/Ker\beta)^{n+1} = L^{n+1}/Ker\beta$, the homomorphism β induces the homomorphism $\bar{\beta} : L^{n+1}/Ker\beta \longrightarrow I^{n+1}$. Therefore, a pair $(\bar{\alpha}, \bar{\beta})$ is an n-isoclinism from $L/Ker(\alpha, \beta)$ onto $Im(\alpha, \beta)$ and the proof is complete. ∎

2. Some properties of n-isoclinism of Lie algebras

In this section, some of the basic properties of n-isoclinism in Lie algebras are discussed.

The following lemma is useful in proving the next result and its proof is straightforward.

Lemma 2.1. *Let L_1 and L_2 be two Lie algebras and $L = L_1 \oplus L_2$, then, for all $n \geq 1$,*

$$L^n = L_1^n \oplus L_2^n \text{ and } Z_n(L) = Z_n(L_1) \oplus Z_n(L_2).$$

Theorem 2.2. *Let L be a Lie algebra and M an abelian Lie algebra. Then, for all $n > 1$,*

$$L \underset{n}{\sim} L \oplus M.$$

Proof. Clearly, since M is abelian we have $M^{n+1} = 0$ and $Z_n(M) = M$, for some $n \geq 1$. Hence

$$Z_n(L \oplus M) = Z_n(L) \oplus M, \ (L \oplus M)^{n+1} = L^{n+1}.$$

Now, we define

$$\begin{aligned} \alpha : \frac{L}{Z_n(L)} &\longrightarrow \frac{L \oplus M}{Z_n(L \oplus M)} \\ x + Z_n(L) &\longmapsto x + Z_n(L \oplus M) \end{aligned}$$

and β is assumed to be the identity on L^{n+1}. Then α and β are both isomorphisms and the following diagram is commutative

$$\begin{array}{ccc} (x_1 + Z_n(L), ..., x_{n+1} + Z_n(L)) & \longmapsto & [x_1, ..., x_{n+1}] \\ \downarrow{\scriptstyle \alpha \times \cdots \times \alpha} & & \downarrow{\scriptstyle \beta} \\ (x_1 + Z_n(L \oplus M), ..., x_{n+1} + Z_n(L \oplus M) & \longmapsto & [x_1, ..., x_{n+1}], \end{array}$$

where $x_i \in \alpha(x_i + Z_n(L))$, for $i = 1, \cdots, n+1$. Thus

$$L \underset{n}{\sim} L \oplus M.$$

■

The following result is a criterion for two Lie algebras being n-isoclinic.

Theorem 2.3. *Two Lie algebras L and K are n-isoclinic if and only if there exist ideals L_1 and K_1 of L and K contained in $Z_n(L)$ and $Z_n(K)$, respectively, and the isomorphisms $\alpha : L/L_1 \longrightarrow K/K_1$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ such that α induces β.*

Proof. If $L \underset{n}{\sim} K$, then choosing $L_1 = Z_n(L)$ and $K_1 = Z_n(K)$ will do the job.

Conversely, consider the pair of isomorphisms (α, β) satisfying the assumption. We show that $\alpha\left(\frac{Z_n(L)}{L_1}\right) = \frac{Z_n(K)}{K_1}$. To do this, let the elements $x \in Z_n(L)$, $y \in \alpha(x + L_1)$, $l_1, ..., l_n \in L$ and $k_1, ..., k_n \in \alpha(l_i + L_1)$ such that

$$[y, k_1, ..., k_n] = \beta[x, l_1, ..., l_n] = \beta(0) = 0,$$

which implies that $y \in Z_n(K)$ and hence

$$\alpha\left(\frac{Z_n(L)}{L_1}\right) \subseteq \frac{Z_n(K)}{K_1}.$$

The reverse inclusion follows similarly, by using the property of β. Thus

$$\alpha\left(\frac{Z_n(L)}{L_1}\right) = \frac{Z_n(K)}{K_1},$$

which gives the following isomorphism

$$\bar{\alpha} : \frac{L}{Z_n(L)} \longrightarrow \frac{K}{Z_n(K)}.$$

Now, the pair of isomorphisms $(\bar{\alpha}, \beta)$ implies that $L \underset{n}{\sim} K$. ∎

The following corollary is an immediate consequence of the above theorem.

Corollary 2.4. *Let L and K be Lie algebras. If $L \underset{n}{\sim} K$, then $L \underset{m}{\sim} K$, for all $m \geq n$.*

Theorem 2.5. *Let (α, β) be the pair of n-isoclinism between two Lie algebras L_1 and L_2. Then, for all $n \geq 1$*

(a) *if M_1 is a subalgebra of L_1 containing $Z_n(L_1)$ such that $\alpha(M_1/Z_n(L_1)) = M_2/Z_n(L_2)$, then $M_1 \underset{n}{\sim} M_2$.*

(b) *if M_1 is an ideal of L_1 and $M_1 \subseteq L_1^{n+1}$, then $M_2 = \beta(M_1) \lhd L_2$ and $\dfrac{L_1}{M_1} \underset{n}{\sim} \dfrac{L_2}{M_2}$.*

Proof. (a) Clearly, $Z_n(L_1) \subseteq Z_n(M_1)$ and similarly $Z_n(L_2) \subseteq Z_n(M_2)$.

Now, as in Theorem 2.3, by choosing $L_1 = Z_n(L_1)$, $K_1 = Z_n(L_2)$, $\bar{\alpha} = \alpha\big|_{\frac{M_1}{Z_n(L_1)}}$ and $\bar{\beta} = \beta\big|_{M_1^{n+1}}$, then $(\bar{\alpha}, \bar{\beta})$ is the required n-isoclinim pair for $M_1 \underset{n}{\sim} M_2$.

(b) We first show that $M_2 = \beta(M_1)$ is an ideal of L_2, i.e., $[\beta(m_1), y] \in M_2$ for all $m_1 \in M_1$ and $y \in L_2$. By the assumption $M_1 \subseteq L_1^{n+1}$, and hence

$$\alpha(m_1 + Z_n(L_1)) = \beta(m_1) + Z_n(L_2).$$

Thus there exists $x \in L_1$ so that for any $y \in \alpha(x + Z_n(L_1))$,

$$\beta[m_1, x] = [\beta(m_1), y] \in M_2,$$

which implies that $M_2 \lhd L_2$.

Clearly, $\left(\dfrac{L_i}{M_i}\right)^{n+1} = \dfrac{L_i^{n+1}}{M_i}$, $i = 1, 2$ and $\beta(M_1) = M_2$. Therefore, β induces

$$\bar{\beta} : \left(\frac{L_1}{M_1}\right)^{n+1} \longrightarrow \left(\frac{L_2}{M_2}\right)^{n+1}.$$

Now, since $\alpha\left(\dfrac{Z_n(L_1) + M_1}{Z_n(L_1)}\right) = \dfrac{Z_n(L_2) + M_2}{Z_n(L_2)}$, the following isomorphism

$$\frac{L_1/Z_n(L_1)}{(Z_n(L_1)+M_1)/Z_n(L_1)} \cong \frac{L_2/Z_n(L_2)}{(Z_n(L_2)+M_2)/Z_n(L_2)},$$

implies $\dfrac{L_1}{Z_n(L_1)+M_1} \cong \dfrac{L_2}{Z_n(L_2)+M_2}$. So they give the following isomorphisms

$$\bar{\alpha} : \frac{L_1/M_1}{(Z_n(L_1)+M_1)/M_1} \cong \frac{L_1}{Z_n(L_1)+M_1} \cong \frac{L_2}{Z_n(L_2)+M_2} \cong \frac{L_2/M_2}{(Z_n(L_2)+M_2)/M_2}$$

One can easily see that, using Theorem 2.3, $\dfrac{L_1}{M_1} \underset{n}{\sim} \dfrac{L_2}{M_2}$, via the n-isoclinism pair $(\bar{\alpha}, \bar{\beta})$. ∎

Theorem 2.6. *Let K and M be subalgebra and ideal of a Lie algebra L, respectively. Then for all $n \geq 0$,*

(a) $K \underset{n}{\sim} K + Z_n(L)$. *In particular, if $L = K + Z_n(L)$ then $L \underset{n}{\sim} K$.*

Conversely, if $\dfrac{L}{Z_n(L)}$ is finite dimensional and $L \underset{n}{\sim} K$, then $L = K + Z_n(L)$.

(b) $\dfrac{L}{M} \underset{n}{\sim} \dfrac{L}{M \cap L^{n+1}}$. *In particular, if $L \cap L^{n+1} = 0$, then $\dfrac{L}{M} \underset{n}{\sim} L$.*

Conversely, if L^{n+1} is of finite dimension and $L \underset{n}{\sim} \dfrac{L}{M}$, then $L^{n+1} \cap M = 0$.

Proof. (a) Clearly, $Z_n(K+Z_n(L)) = Z_n(K)+Z_n(L)$, for all $n \geq 0$. Now, consider

$$\alpha : \frac{K}{Z_n(K)} \longrightarrow \frac{K+Z_n(L)}{Z_n(K+Z_n(L))}$$

given by $\alpha : k + Z_n(K) \mapsto k + Z_n(K + Z_n(L))$, for all $k \in K$ and $\beta = id_{K^{n+1}}$. Then the pair of (α, β), guaranties the property $K \underset{n}{\sim} K + Z_n(L)$. In particular, if $L = K + Z_n(L)$ then $L \underset{n}{\sim} K$.

Conversely, suppose that $dim\left(\dfrac{L}{Z_n(L)}\right) < \infty$ and $J = K + Z_n(L)$. Then, we show that $L = J$. Clearly, $L \underset{n}{\sim} K \underset{n}{\sim} J$, which imply that $L \underset{n}{\sim} J$ and hence $\dfrac{L}{Z_n(L)} \cong \dfrac{J}{Z_n(J)}$ and $Z_n(L) \subseteq Z_n(J)$. Thus, $dim\left(\dfrac{J}{Z_n(J)}\right)$ is finite and it follows that

$$dim\left(\frac{J}{Z_n(J)}\right) \leq dim\left(\frac{J}{Z_n(L)}\right).$$

Hence, $dim\left(\dfrac{L}{Z_n(L)}\right) \leq dim\left(\dfrac{J}{Z_n(L)}\right) \leq dim\left(\dfrac{L}{Z_n(L)}\right)$, which follow that $J = L$.

(b) Put $\bar{L} = L/M$ and $\tilde{L} = \dfrac{L}{M \cap L^{n+1}}$. Then one can easily check that $\bar{l} \in Z_n(\bar{L})$ if and only if $\tilde{l} \in Z_n(\tilde{L})$. Therefore, the following maps

$$\begin{aligned} \alpha : \frac{\bar{L}}{Z_n(\bar{L})} &\longrightarrow \frac{\tilde{L}}{Z_n(\tilde{L})} \\ \bar{l} + Z_n(\bar{L}) &\longmapsto \tilde{l} + Z_n(\tilde{L}) \end{aligned}$$

and

$$\begin{aligned} \beta : \bar{L}^{n+1} &\longrightarrow \tilde{L}^{n+1} \\ \bar{l} &\longmapsto \tilde{l} \end{aligned}$$

are the required n-isoclinism pair.

Now, if $M \cap L^{n+1} = 0$, then $\dfrac{L}{M} \underset{n}{\sim} L$.

To prove the converse, since $L \underset{n}{\sim} \dfrac{L}{M} \underset{n}{\sim} \dfrac{L}{M \cap L^{n+1}}$, we have

$$L^{n+1} \cong \left(\frac{L}{M \cap L^{n+1}} \right)^{n+1} = \frac{L^{n+1}}{M \cap L^{n+1}},$$

which implies that

$$dimL^{n+1} = dimL^{n+1} + dim(M \cap L^{n+1})$$

and so $dim(M \cap L^{n+1}) = 0$. Hence $M \cap L^{n+1} = 0$. ■

Theorem 2.7. *Let $f : L \longrightarrow K$ be a homomorphism of Lie algebras such that $f(Z_n(L)) \subseteq Z_n(K)$, for some $n \geq 1$. If*

$$\begin{aligned} \alpha : \frac{L}{Z_n(L)} &\longrightarrow \frac{K}{Z_n(K)} \\ l + Z_n(L) &\longmapsto f(l) + Z_n(K) \end{aligned}$$

and

$$\begin{aligned} \beta : L^{n+1} &\longrightarrow K^{n+1} \\ [l_1, ..., l_{n+1}] &\longmapsto [f(l_1), ..., f(l_{n+1})], \end{aligned}$$

for all $l, l_1, ..., l_{n+1} \in L$. Then, the following statements hold:

(a) *(α, β) is an n-homoclinism of L into K;*

(b) *$L/Kerf \underset{n}{\sim} L/Ker(\alpha, \beta)$;*

(c) *$Imf \underset{n}{\sim} Im(\alpha, \beta)$.*

Proof. (a) is obvious.

(b) Clearly, $Ker(\alpha,\beta) = Ker\beta = Kerf \cap L^{n+1}$. Then, using Theorem 2.6(b), it implies that

$$\frac{L}{Kerf} \underset{n}{\sim} \frac{L}{Kerf \cap L^{n+1}} = \frac{L}{Ker(\alpha,\beta)}.$$

(c) By Theorem 1.2,

$$Imf \cong \frac{L}{Kerf} \underset{n}{\sim} \frac{L}{Ker(\alpha,\beta)} \underset{n}{\sim} Im(\alpha,\beta). \qquad \blacksquare$$

The following corollary gives a sufficient condition that two Lie algebras are n-isoclinic.

Corollary 2.8. *If $f : L \longrightarrow K$ is an epimorphism of Lie algebras such that $Kerf \cap L^{n+1} = 0$, then $L \underset{n}{\sim} K$.*

Proof . Since f is surjective, it follows that $f(Z_n(L)) \subseteq Z_n(K)$. Hence the pair (α,β) is defined in the above theorem is an n-homoclinism such that $Im(\alpha,\beta) = K$. Thus

$$L \cong \frac{L}{Kerf \cap L^{n+1}} \underset{n}{\sim} \frac{L}{Kerf} \cong K. \qquad \blacksquare$$

Theorem 2.9. *Let K be a subalgebra of a Lie algebra L and $\frac{L}{Z_n(L)}$ satisfies the descending chain condition on subalgebras. Then the following are equivalent:*

(a) $L = K + Z_n(L)$;

(b) $L \underset{n}{\sim} K$;

(c) $\dfrac{L}{Z_n(L)} \cong \dfrac{K}{Z_n(K)}$.

Proof. The conclusions (a)⇒ (b) and (b)⇒ (c) follow by Theorem 2.6 and Definition 1.1, respectively.

(c)⇒(a). Put $J = K + Z_n(L)$ and so $K \underset{n}{\sim} J$. Hence $\alpha : \dfrac{L}{Z_n(L)} \longrightarrow \dfrac{J}{Z_n(J)}$ is an isomorphism, which implies that

$$\frac{L}{Z_n(L)} \cong \frac{K}{Z_n(K)} \cong \frac{J}{Z_n(J)}.$$

Clearly, $\dfrac{J}{Z_n(L)} \subseteq \dfrac{L}{Z_n(L)}$ and so assume $\alpha\left(\dfrac{J}{Z_n(L)}\right) = \dfrac{J_1}{Z_n(J)}$, for some $J_1 \subseteq J$.

Now, if $J_1 = J$, then

$$\alpha\left(\frac{J}{Z_n(L)}\right) = \frac{J}{Z_n(J)} \cong \frac{L}{Z_n(L)},$$

which gives $J = L$.

Conversely, we have $Z_n(L) \subseteq Z_n(J) \subseteq J_1$ and hence $\frac{J_1}{Z_n(L)} \subseteq \frac{J}{Z_n(L)}$. So, there exists a subalgebra J_2 of J contained in J_1 such that

$$\alpha\left(\frac{J_1}{Z_n(L)}\right) = \frac{J_2}{Z_n(J)}.$$

Thus $J_2 = J_1 \Leftrightarrow J_1 = J \Leftrightarrow J = L$. Clearly, using this process we obtain a descending chain of subalgebras of $\frac{J}{Z_n(J)}$ as follows:

$$\frac{J}{Z_n(J)} \supseteq \frac{J_1}{Z_n(J)} \supseteq \frac{J_2}{Z_n(J)} \supseteq \cdots.$$

By the assumption, $\frac{J}{Z_n(J)}$ satisfies the descending chain condition, and hence there exists a natural number n such that for all $m > n$,

$$\frac{J_n}{Z_n(L)} = \frac{J_m}{Z_n(L)} \Rightarrow J_n = J_m, \quad \forall m > n.$$

Thus $J = K$, which completes the proof. ∎

The following result gives some equivalence conditions on n-isoclinism of Lie algebras, which can be proved using Theorem 2.6(b).

Theorem 2.10. *Let M be an ideal of a Lie algebra L. If L^{n+1} satisfies the ascending chain condition on ideals, for all $n \geq 1$, then the following conditions are equivalent:*

(a) $M \cap L^{n+1} = 0$;

(b) $L \underset{n}{\sim} \frac{L}{M}$;

(c) $L^{n+1} \cong \left(\frac{L}{M}\right)^{n+1}$.

The above theorems have the following corollary.

Corollary 2.11. *Let K and M be a subalgebra and an ideal of a Lie algebra L, respectively.*

(i) *If N is an ideal of L and $M \cap L^{n+1} = 0$, then $L \underset{n}{\sim} \frac{L}{M \cap N}$.*

(ii) *If* $M \cap L^{n+1} = 0$, *then* $\dfrac{K+M}{M} \underset{n}{\sim} K$.

(iii) *If* $L = K + Z_n(L)$, *then* $\dfrac{K+M}{M} \underset{n}{\sim} \dfrac{L}{M}$.

(iv) *If* $J \leq L$ *and* $L = K + Z_n(L)$, *then* $L \underset{n}{\sim} K + J$.

Proof. (i) Clearly, $(M \cap N) \cap L^{n+1} = 0$. Now, the result follows by Theorem 2.8.

(ii) By the assumption, we have $(M \cap K) \cap K^{n+1} = 0$ and so

$$\frac{K+M}{M} \cong \frac{K}{M \cap K} \underset{n}{\sim} K.$$

(iii) One observes that

$$\frac{L}{M} = \frac{K + Z_n(L)}{M} = \frac{K+M}{M} + \frac{Z_n(L)+M}{M} \text{ and } \frac{Z_n(L)+M}{M} \leq Z_n\left(\frac{L}{M}\right).$$

Thus

$$\frac{L}{M} = \frac{K+M}{M} + Z_n\left(\frac{L}{M}\right).$$

Now, the result follows, using Theorem 2.7.

(iv) Clearly, $L = (K+L) + Z_n(L)$ and hence

$$L \underset{n}{\sim} K + J$$

■

In the next section, we study the concept of n-stem Lie algebras.

3. The structure of n-stem Lie algebras

In this section, we introduce the concept of n-stem Lie algebras and similar to group theory case (see [1], [2]), it is shown that every family of n-isoclinic Lie algebras contains an n-stem Lie algebra of minimum dimension.

Definition 3.1. A Lie algebra L is said to be an *n-stem Lie algebra*, if $Z_n(L) \leq L^{n+1}$, for some $n \geq 1$.

Theorem 3.2. *Let $\mathcal{C}$ be a family of n-isoclinic Lie algebras, then*

(a) *$\mathcal{C}$ contains an n-stem Lie algebra;*

(b) *Suppose T is a finite dimensional Lie algebra in $\mathcal{C}$, then T is an n-stem Lie algebra if and only if*

$$dimT = \min\{dimL \mid L \in \mathcal{C}\}.$$

Proof. Let $L \in \mathcal{C}$ and S be a vector space complement of $Z_n(L) \cap L^{n+1}$ in $Z_n(L)$, i.e., $Z_n(L) = S \oplus (Z_n(L) \cap L^{n+1})$. It follows that $S \cap L^{n+1} = 0$ and $S \lhd L$. By Theorem 2.6, $T = \dfrac{L}{S} \underset{n}{\sim} L$ and so $T \in \mathcal{C}$. Thus, we have

$$\begin{aligned} Z_n(T) = Z_n\left(\frac{L}{S}\right) &= \frac{Z_n(L)+S}{S} = \frac{S + Z_n(L) \cap L^{n+1}}{S} \subseteq \frac{S + L^{n+1}}{S} \\ &= \left(\frac{L}{S}\right)^{n+1} = T^{n+1}, \end{aligned}$$

which implies that T is an n-stem Lie algebra in $\mathcal{C}$.

(b) Let L be a finite dimension Lie algebra in $\mathcal{C}$ and T be the n-stem Lie algebra of $\mathcal{C}$. Then

$$\begin{aligned} \frac{L^{n+1}}{L^{n+1} \cap Z_n(L)} \cong \frac{L^{n+1} + Z_n(L)}{Z_n(L)} &= \left(\frac{L}{Z_n(L)}\right)^{n+1} \cong \left(\frac{T}{Z_n(T)}\right)^{n+1} \\ &= \frac{T^{n+1} + Z_n(T)}{Z_n(T)} = \frac{T^{n+1}}{Z_n(T)}. \end{aligned}$$

Now, since $L^{n+1} \cong T^{n+1}$, it follows that

$$dim(Z_n(T)) = dim(Z_n(L) \cap L^{n+1}) \leq dim Z_n(L).$$

Clearly, $\dfrac{L}{Z_n(L)} \cong \dfrac{T}{Z_n(T)}$ and hence $dim T \leq dim L$.

Conversely, Let T be a Lie algebra in the family $\mathcal{C}$ of minimum dimension. As in the first part, there exists an ideal S of T such that $\dfrac{T}{S} \underset{n}{\sim} T$. Now, since T is of minimum dimension, it implies that

$$Z_n(T) = Z_n(T) \cap T^{n+1},$$

i.e., $S = 0$. Therefore $Z_n(T) \subseteq T^{n+1}$ and hence T is an n-stem Lie algebra. ∎

The following lemma shortens the proof of our final result.

Lemma 3.3. *Let L and M be two Lie algebras such that $L \underset{n}{\sim} M$ with the isomorphisms pair (α, β). Then, for all $x \in L^{n+1}$*

(a) $\alpha(x + Z_n(L)) = \beta(x) + Z_n(M)$;

(b) $\beta([x, y]) = [\beta(x), m], \quad \forall y \in L, \ m \in \alpha(y + Z_n(L))$.

Theorem 3.4. *If L and M are n-stem Lie algebras. Then $Z_n(L) \cong Z_n(M)$, for $n \geq 1$.*

Proof . Suppose $L \underset{n}{\sim} M$ with the isomorphisms pair (α, β). By the assumption, we have $Z_n(L) \subseteq L^{n+1}$. Then using Lemma 3.3(a), for all $z \in Z_n(L)$

$$\alpha(z + Z_n(L)) = \beta(z) + Z_n(M),$$

which implies that $\beta(z) \in Z_n(M)$ and hence $\beta(Z_n(L)) \subseteq Z_n(M)$.

Now, take an arbitrary element $x \in Z_n(M)$. Then, there exists $l \in L$ with $\alpha(l + Z_n(L)) = x + Z_n(M) = Z_n(M)$. Clearly, using Lemma 3.3(b),

$$0 = [x, m_1, ..., m_n] = [\beta(l), l_1, ..., l_n],$$

where $\beta(l) = x \in Z_n(M)$ and $m_i \in \alpha(l_i + Z_n(L))$, for all $1 \leq i \leq n$. Thus $Z_n(M) = \beta(Z_n(L))$, which implies that $Z_n(L) \cong Z_n(M)$. ∎

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