INEQUALITIES FOR MARKS IN MULTIDIGRAPHS

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Abstract. An r-digraph (multidigraph) D is an orientation of a multigraph that is without loops and contains at most two edges between any pair of distinct vertices. So 1-digraph is an oriented graph, and complete 1-digraph is a tournament. Define $p_v = r(n-1) + d_v^+ - d_v^-$, the mark ($r$-score) of a vertex v in an r-digraph D, where d_v^+ and d_v^-, respectively denote the outdegree and indegree of v and n is the number of vertices in D. In this paper, we obtain some stronger inequalities for marks in r-digraphs.

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1. Introduction

A tournament is an orientation of a complete simple graph. The score s_{v_i} (or simply, s_i) of a vertex v_i in a tournament is the outdegree of v_i. The score sequence of a tournament is formed by listing the vertex scores in non-decreasing order. The following result of Landau [3] gives necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

Theorem 1.1. *A sequence $[s_i]_1^n$ of nonnegative integers in non-decreasing order is the score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \text{ for } 1 \leq k \leq n,$$

with equality when $k = n$.

With the marking system, the mark p_v of a vertex v in a tournament is given by $p_v = 2s_v + n - 1$, and Landaus conditions become

$$\sum_{i=1}^{k} p_i \geq k(n + k - 2), \text{ for } 1 \leq k \leq n,$$

with equality for $k = n$.

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Avery [1] defined a_{v_i} (or, simply, a_i) $= n - 1 + d_v^+ - d_v^-$, the score of a vertex v_i in an oriented graph, where d_v^+ and d_v^-, respectively denote the outdegree and indegree of v_i and n is the number of vertices. The score sequence of an oriented graph is formed by listing the vertex scores in non-decreasing order. The following result is due to Avery [1].

Theorem 1.2. *A sequence $[a_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of some oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geq k(k-1), \text{ for } 1 \leq k \leq n,$$

with equality when $k = n$.

Once again, with the marking system, the mark p_v of a vertex v in an oriented graph is given by $p_v = a_v + n - 1$, and Averys conditions become

$$\sum_{i=1}^{k} p_i \geq k(n + k - 2), \text{ for } 1 \leq k \leq n,$$

with equality for $k = n$.

A digraph D is semicomplete if for any pair of vertices $u \neq v$ in D, there is an arc from u to v or an arc v to u (or both). The following necessary and sufficient conditions for a non-decreasing sequence of integers to be the score sequence for a semicomplete digraph is given by Reid and Zhang [8].

Theorem 1.3. *A sequence $[s_i]_1^n$ of integers in non-decreasing order is the score sequence of some semicomplete digraph if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \quad \text{and} \quad s_k \leq n - 1, \quad \text{for all } k, \ 1 \leq k \leq n.$$

An r-digraph (multidigraph) D is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. Let D be an r-digraph with vertex set $V = \{v_1, v_2, ..., v_n\}$, and let d_v^+ and d_v^-, respectively denote the outdegree and indegree of a vertex v. Define $p_v = r(n-1) + d_v^+ - d_v^-$, the mark ($r$-score) of v, so that $0 \leq p_v < 2r(n-1)$. The sequence $P = [p_i]_1^n$, where $p_i = p_{v_i}$, in nondecreasing order is the mark sequence of D. An r-digraph can be interpreted as the result of a competition in which the participants play each other at most r times, with an arc from u to v if and only u defeats v. A player receives two points for each win, and one point for each tie (draw). With this marking system, player v obtains a total of p_v points. A sequence P of non-negative integers in nondecreasing order is said to be realizable if there exists an r-digraph with mark sequence P. The following existence criteria for realizability is due to Pirzada and Samee [5]. Various results on marks in digraphs can be found in [6], [7].

Theorem 1.4. *A sequence $[p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of some r-digraph if and only if*

$$\sum_{i=1}^{k} p_i \geq 2k(k-1), \quad \text{for } 1 \leq k \leq n,$$

with equality when $k = n$.

Some stronger inequalities for scores in tournaments are given by Brualdi and Shen [2], and for scores in oriented graphs are given by Pirzada and Samee [4].

2. Stronger Inequalities

A regular r-digraph on n vertices is one whose all vertices have marks $2r(n-1)$. The converse D' of an r-digraph D is obtained by reversing each arc of D. If u and v are vertices in an r-digraph, we denote by $u(x-y)v$ to mean that there are x arcs directed from u to v and y arcs directed from v to u. Clearly, $0 \leq x, y \leq 2$ and $0 \leq x+y \leq r$. A triple in an r-digraph is an induced r-digraph with three vertices and is of the form $u(x_1-x_2)v(y_1-y_2)w(z_1-z_2)u$, where for $1 \leq i \leq r$, $0 \leq x_i, y_i, z_i \leq r$, and $0 \leq \sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i, \sum_{i=1}^{2} z_i \leq r$. In an r-digraph, a 1-triple is an induced 1-subdigraph with three vertices. An 1-triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r-digraph is said to be transitive if every of its 1-triple is transitive. The inequalities given below in Theorems 2.1, 2.2, 2.3, 2.4 are the generalizations of the inequalities on scores in tournaments due to Brualdi and Shen [2].

The following result gives a lower bound for $\sum_{i \in I} p_i$.

Theorem 2.1. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, ..., n\}$,*

$$\sum_{i \in I} p_i \geq r \sum_{i \in I} (i-1) + r \binom{|I|}{2} \tag{2.1.1}$$

with equality when $|I| = n$.

Proof. *Sufficiency.* Let the sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order satisfy equation (2.1.1). Now, for any $I \subseteq [n]$, we have

$$\sum_{i \in I} (i-1) \geq \sum_{i=1}^{|I|} (i-1) = \binom{|I|}{2}.$$

Therefore, from equation (2.1.1), we have

$$\sum_{i \in I} p_i \geq r \sum_{i \in I} (i-1) + r \binom{|I|}{2} \geq r \binom{|I|}{2} + r \binom{|I|}{2} = 2r \binom{|I|}{2}.$$

Hence, by Theorem 1.4, P is a mark sequence.

Necessity. Assume that $P = [p_i]_1^n$ is a mark sequence of some r-digraph. For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i \in I} p_i - r \sum_{i \in I} (i-1) - r \binom{|I|}{2}.$$

Claim $I = \{i : 1 \leq i \leq |I|\}$. If not, then there exists $i \notin I$ and $j \in I$ such that $j = i + 1$. So, $p_i \leq p_j$.

For $j \in I$, we have

$$\begin{aligned} f(I) &= \sum_{t \in I} p_t - r \sum_{t \in I} (t-1) - r \tbinom{|I|}{2} \\ &= \sum_{\substack{t \in I \\ j \notin I}} p_t + p_j - r \left(\sum_{\substack{t \in I \\ j \notin I}} (t-1) + (j-1) \right) - r \tbinom{|I|}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} f(I) - f(I - \{j\}) &= p_j - r(j-1) - r \tbinom{|I|}{2} + r \tbinom{|I|-1}{2} \\ &= p_j - r(j-1) - r(|I|-1) = p_j - r(j + |I| - 2). \end{aligned}$$

Since $f(I) - f(I - \{j\}) < 0$, therefore $p_j - r(j + |I| - 2) < 0$.

Again $f(I \cup \{i\}) = \sum_{i \in I} p_t + p_i - r \left(\sum_{t \in I} (t-1) + (i-1) \right) - r \binom{|I|+1}{2}$.

So $f(I \cup \{i\}) - f(I) = p_i - r(i-1) - r\binom{|I|+1}{2} + r\binom{|I|}{2} = p_i - r(i+|I|-1)$.
As $f(I \cup \{i\}) - f(I) \geq 0$, therefore $p_i - r(i+|I|) - 1) \geq 0$.
Thus $p_j < r(j+|I|-2)$ and $p_i \geq r(i+|I|-1)$.
Therefore $r(i+|I|-1) \leq p_i \leq p_j < r(j+|I|-2)$.
Since $j = i+1$, therefore $r(i+|I|-1) < r(i+1+|I|-2)$.
That is, $2(i+|I|-1) < 2(i+|I|-1)$, which is a contradiction.
Hence

$$\begin{aligned} f(I) &= \sum_{i=1}^{|I|} p_i - r\sum_{i=1}^{|I|}(i-1) - r\binom{|I|}{2} = \sum_{i=1}^{|I|} p_i - r\binom{|I|}{2} - r\binom{|I|}{2} \\ &\geq r|I|(|I|-1) - 2r\binom{|I|}{2} = 0. \qquad \text{(by Theorem 1.4)} \end{aligned}$$

Thus

$$\sum_{i\in I} p_i - r\sum_{i\in I}(i-1) - r\binom{|I|}{2} \geq 0,$$

that is ,

$$\sum_{i\in I} p_i \geq r\sum_{i\in I}(i-1) + r\binom{|I|}{2}.$$

This proves the necessity. ∎

We note that equality can occur often in equation (2.1.1). For example, in the transitive r-digraph of order n with mark sequence $[0, 2r, 4r, ..., 2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1), r(n-1), ..., r(n-1)]$. We further observe that Theorem 2.1 is best possible, since for any real $\varepsilon > 0$, the inequality

$$\sum_{i\in I} p_i \geq (1+\varepsilon) r\sum_{i\in I}(i-1) + (1-\varepsilon) r\binom{|I|}{2}$$

fails for some I, and some r-digraphs. This can been seen, for example, in the transitive r-digraph of order n with mark sequence $[0, 2r, 4r, ..., 2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1), r(n-1), ..., r(n-1)]$.

The next result gives a set of upper bounds for $\sum_{i\in I} p_i$ and is equivalent to the set of lower bounds for $\sum_{i\in I} p_i$ in Theorem 2.1.

Theorem 2.2. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, ..., n\}$,*

$$\sum_{i\in I} p_i \leq r\sum_{i\in I}(i-1) + \frac{1}{2}\, r|I|(2n-|I|-1),$$

with equality when $|I| = n$.

Proof. We have $[n] = \{1, 2, ..., n\}$. Let $J = [n] - I$, so that $I + J = [n]$ and $|J| + |I| = n$. Therefore, by Theorem 2.1, P is a mark sequence if and only if

$$\sum_{i \in [n]} p_i = rn(n-1) \quad \text{and} \quad \sum_{i \in J} p_i \geq r \sum_{i \in J} (i-1) + r \binom{|J|}{2}$$

if and only if

$$\sum_{i \in I} p_i + \sum_{i \in J} p_i = rn(n-1) \;\; \text{and} \;\; \sum_{i \in J} p_i \geq r \sum_{i \in J} (i-1) + r \binom{|J|}{2}$$

if and only if

$$\begin{aligned}
\sum_{i \in I} p_i &= rn(n-1) - \sum_{i \in J} p_i \leq rn(n-1) - \left(r \sum_{i \in J} (i-1) \right) + r \binom{|J|}{2} \\
&= rn(n-1) - \left(r \, \frac{n(n-1)}{2} - r \sum_{i \in I} (i-1) \right) + r \tbinom{n-|I|}{2} \\
&\qquad \left(\text{because } r \sum_{i \in I} (i-1) + r \sum_{i \in J} (i-1) = r \tbinom{n}{2} \text{ and } |I| + |J| = n \right) \\
&= rn(n-1) - r \, \frac{n(n-1)}{2} + r \sum_{i \in I} (i-1) + \frac{r}{2} \, (n-|I|)(n-|I|-1) \\
&= r \sum_{i \in I} (i-1) + \frac{r}{2} \, |I|(2n-|I|-1),
\end{aligned}$$

which proves the result. ∎

We now have the following results.

Theorem 2.3. *If $P = [p_i]_1^n$ is a mark sequence of an r-digraph, then for each i,*

$$r(i-1) \leq p_i \leq r(n+i-2).$$

Proof. Let $I = \{i\}$ in Theorem 2.1 and Theorem 2.2. Then

$$\sum_{i \in I} p_i \geq r \sum_{i \in I} (i-1) + r \binom{|I|}{2}$$

implies that

$$p_i \geq r(i-1),$$

and

$$\sum_{i \in I} p_i \leq r \sum_{i \in I} (i-1) + \frac{r}{2} \, |I|(2n-|I|-1)$$

implies that

$$p_i \leq r(n+i-2).$$

Therefore

$$r(i-1) \leq p_i = r(n+i-2).$$

Second proof. We first show that $r(i-1) \leq p_i$. Suppose on the contrary that $p_i < r(i-1)$. Then, for every $k < i$, we have $p_k \leq p_i < r(i-1)$. That is, $p_1 < r(i-1)$, $p_2 < r(i-1)$,...,$p_i < r(i-1)$. Adding these inequalities, we have $\sum_{k=1}^{i} p_k < ri(i-1)$, which is a contradiction to Theorem 1.4. Therefore $r(i-1) \leq p_i$.

The second inequality is dual to the first. In the converse r-digraph with mark sequence $P' = [p'_i]_1^n$, we have

$$p'_{n-i+1} \geq r((n-i+1)-1) = r(n-i). \qquad \text{(by the first inequality)}.$$

But $p_i = 2r(n-1) - p'_{n-i+1}$. So $p_i \leq 2r(n-1) - r(n-i) = r(n+i-2)$. Therefore $p_i \leq r(n+i-2)$. Hence the result. ■

For any integers r and s with $r \leq s$, let $[r, s]$ denotes the set of all integers between r and s.

Theorem 2.4. *Let $P = [p_i]_1^n$ be a mark sequence of an r-digraph. If*

$$\sum_{i\in I} p_i = r\sum_{i\in I}(i-1) + r\binom{|I|}{2}, \tag{2.4.1}$$

for some $I \subseteq [n]$, then one of the following holds.

(a) $I = [1, |I|]$ *and* $\sum_{i=1}^{|I|} p_i = r|I|(|I|-1)$.

(b) $I = [t, t+|I|-1]$ *for some* $t,\ 2 \leq t \leq n-|I|+1$,

$$\sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$$

and $p_i = r(t+|I|-2)$ *for all* $i \leq t+|I|-1$.

(c) $I = [1, m] \cup [m+t, t+|I|-1]$ *for some r and t such that* $1 \leq m \leq |I|-1$ *and* $2 \leq t \leq n-|I|+1$,

$$\sum_{i=1}^{m} p_i = rm(m-1), \quad \sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$$

and $p_i = 2(m+t+|I|-2)$ *for all* $i,\ m+1 \leq i \leq t+|I|-1$.

Theorem 2.5. *If $P = [p_i]_1^n$ is a mark sequence of an r-digraph, then*

(a) $\sum_{i=1}^{t} p_i^2 \geq \sum_{i=1}^{t} (2rt - 2r - p_i)^2$, *for $1 \leq t \leq n$, with equality when $t = n$.*

(b) *For $1 < g < \infty$, $\frac{1}{g} + \frac{1}{h} = 1$, $\sum_{i=1}^{t} p_i^g \geq t(rt - r)^g$, where $1 \leq t \leq n$, with equality when $t = n$ and $p_1 = p_2 = \cdots = p_t$.*

Proof. (a). By Theorem 1.4, we have

$$rt(t-1) \leq \sum_{i=1}^{t} p_i, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n$$

or

$$\sum_{i=1}^{t} p_i^2 + 2(rt - 2r)rt(t-1) \leq \sum_{i=1}^{t} p_i^2 + 2(2rt - 2r) \sum_{i=1}^{t} p_i, \quad \text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$\sum_{i=1}^{t} p_i^2 + t(2rt - 2r)^2 - 2(2rt - 2r) \sum_{i=1}^{t} p_i \leq \sum_{i=1}^{t} p_i^2, \quad \text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$p_1^2 + \cdots + p_t^2 + \underbrace{(2rt - 2r)^2 + \cdots + (2rt - 2r)^2}_{k-\text{times}} - 2(2rt - 2r)p_1 - \cdots - 2(2rt - 2r)p_t \leq \sum_{i=1}^{t} p_i^2, \quad \text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$(2rt - 2r - p_1)^2 + \cdots + (2rt - 2r - p_t)^2 \leq \sum_{i=1}^{t} p_i^2, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$\sum_{i=1}^{t} (2rt - 2r - p_i)^2 \leq \sum_{i=1}^{t} p_i^2, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n.$$

(b) Again, by Theorem 1.4, we have

$$\begin{aligned} rt(t-1) &\le \sum_{i=1}^{t} p_i, \text{ for } 1 \le t \le n \text{ with equality when} t = n, \\ &= \sum_{i=1}^{t} p_t \cdot 1, \text{ for } 1 \le t \le n \text{ with equality when} t = n, \\ &\le \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{t} 1^h\right)^{\frac{1}{h}}, \\ &\qquad \text{for } 1 \le t \le n \text{ with equality when } t = n \text{ and } p_1 = p_2 = \cdots = p_k, \\ &\qquad\qquad \text{(by Holders inequality)} \\ &= \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}} k^{\frac{1}{h}}, \\ &\qquad \text{for } 1 \le t \le n \text{ with equality when } t = n \text{ and } p_1 = p_2 = \cdots = p_t. \end{aligned}$$

That is,

$$rt^{1-\frac{1}{h}}(t-1) \le \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}},$$

for $1 \le t \le n$ with equality when $t = n$ and $p_1 = p_2 = \cdots = p_t$.

Hence

$$\sum_{i=1}^{t} p_i^g \ge t(rt - r)^g,$$

for $1 \le t \le n$ with equality when $t{=}n$, and $p_1{=}p_2 = \cdots = p_t$, since $\dfrac{1}{g} + \dfrac{1}{h} = 1$. ■

Theorem 2.6. *Let D be an r-digraph on n vertices with mark sequence $[p_i]_1^n$. Then, for each $t \ge 1$, there exists an r-digraph on tn vertices with mark sequence $[p_i + r(t-1)n]_1^{tn}$.*

Proof. For each $i, 1 \le i \le t$, let D^i be a copy of D with n vertices. Define an r-digraph D_1 as

$$D_1 = D^1 \cup D^2 \cup \cdots \cup D^t,$$

such that vertices and arcs of D_1 are that of D^i, and let there be no arc between the vertices of D^i and D^j $(i \ne j)$. Then D_1 is an r-digraph on tn vertices with mark sequence $[p_i + r(t-1)n]_1^{tn}$. ■

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