CONVEXITY IN NORMED HYPERVECTOR SPACES

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Abstract. In this paper we obtain some results on convexity in a normed hypervector space. We also investigate the concept of absorbing and balanced set and generalize the corresponding results of vector space.

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1. Introduction

In 1934 Marty [5] introduced a new mathematical structure as a generalization of groups and called it hypergroup. Subsequently, many authors worked on this new field and constructed some other generalizations such as hyperrings, hypermodules, and hyperfields. In 1988 the notion of hypervector space was given by Tallini [12]. She studied some algebraic properties of this new structure in [8], [9], [10], and [11]. A wealth of applications of these new constructions in: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. can be found in [2]. Recently, we studied hypervector spaces in the viewpoint of analysis and generalized some definitions and proved many interesting theorems about them in [6] and [7]. In this paper we focus on some other properties of these spaces. We define convex, strictly convex, balanced, and absorbing subsets of a normed hypervector space and prove some theorems about them.

Let $P(X)$ be the power set of a set X, $P^\star(X) = P(X)\backslash\{\varnothing\}$, and K a field. A *hypervector space* over K that is defined in [8], is a quadruplet $(X, +, \circ, K)$ such that $(X, +)$ is an abelian group and

$$\circ : K \times X \longrightarrow P^\star(X)$$

is a mapping that for all a, $b \in K$ and x, $y \in X$ the following properties holds:

(i) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$,

(ii) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$,

(iii) $a \circ (b \circ x) = (ab) \circ x$, where $a \circ (b \circ x) = \{\, a \circ y \ : \ y \in b \circ x \,\}$,

(iv) $(-a) \circ x = a \circ (-x)$

(v) $x \in 1 \circ x$.

Note that every vector space is a hypervector space and specially, every field is a hypervector space over itself.

A non-empty subset of a hypervector space X over a field K is called a *subspace* of X if the following holds:

(i) $H - H \subseteq H$,

(ii) $a \circ H \subseteq H$, for every $a \in K$.

Let $(X, +, \circ, K)$ be a hypervector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A *pseudonorm* on X that is defined in [9], is a mapping

$$\|\cdot\| : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and x, $y \in X$ has the following properties:

(i) $||0|| = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| = |a|\, ||x||$.

A pseudonorm on X is called a *norm*, if:

$$||x|| = 0 \Longleftrightarrow x = 0.$$

2. Main results

The norm that is defined on hypervector space $(X, +, \circ, \|\cdot\|, K)$ induced a topology on X as following:

Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space. For $x \in X$ and $\epsilon > 0$ the open ball $B_\epsilon(x)$ is defined as

$$B_\epsilon(x) = \{\, y \in X \,:\, ||x - y|| < \epsilon \,\}.$$

The $\{\, B_\epsilon(x) \,:\, x \in X, \epsilon > 0 \,\}$ is a basis for a topology on X which is the topology induced by this norm.

Clearly, the intersection of any collection of convex sets is convex.

Definition 2.1. A non-empty subset E of a hypervector space $X = (X, +, \circ, K)$ is called convex if for any $x,\ y \in E$ and $0 \leqslant \lambda \leqslant 1$,

$$\lambda \circ x + (1 - \lambda) \circ y \subseteq E.$$

Also X is called strictly convex if $\sup \left\| \frac{1}{2} \circ x + \frac{1}{2} \circ y \right\| < 1$, whenever $x,\ y \in X$, $x \neq y$, and $||x|| = ||y|| = 1$.

Lemma 2.2 *Let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space, $x \in X$, and $\epsilon > 0$. If*

$$B_\epsilon(x) = \{\, y \in X : ||x - y|| < \epsilon\},$$

then $B_\epsilon(x)$ is a convex set.

Proof. First, suppose that $x = 0$. Let $z,\ w \in B_\epsilon(x)$ and $0 \leqslant \lambda \leqslant 1$. Since for all $a,\ b \in X$, $||a + b|| \leqslant ||a|| + ||b||$ and by the definition of a norm, we have

$$\begin{aligned} \sup ||\lambda \circ z + (1 - \lambda) \circ w|| &\leqslant \sup ||\lambda \circ z|| + \sup ||(1 - \lambda) \circ w|| \\ &= \lambda\, ||z|| + (1 - \lambda)\, ||w|| \leqslant \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon. \end{aligned}$$

On the other hand, it is easy to check that $B_\epsilon(x) = B_\epsilon(0) + x$. So the proof is complete. ■

Theorem 2.3. *Let $X = (X, +, \circ, \| \cdot \|, K)$ be a normed hypervector space, $x, y \in X$, $x \neq y$, and $||x|| = ||y|| = 1$. If X is strictly convex, then*

$$\sup ||t \circ x + (1 - t) \circ y|| < 1,$$

for $0 < t < 1$. Also X is strictly convex if there is some t, $0 < t < 1$, such that

$$\sup ||t \circ x + (1 - t) \circ y|| < 1,$$

for $x,\ y \in X$ with $x \neq y$, and $||x|| = ||y|| = 1$.

Proof. Suppose $0 < t_0 < 1$ and

$$\sup ||t_0 \circ x + (1 - t_0) \circ y|| < 1.$$

If $0 < t < t_0$, then let $s = \frac{t}{t_0}$. So $0 < s < 1$ and

$$\begin{aligned} t \circ x + (1-t) \circ y &= (st_0) \circ x + [s(1-t_0) + 1 - s] \circ y \\ &\subseteq (st_0) \circ x + (s(1-t_0)) \circ y + (1-s) \circ y. \end{aligned}$$

Therefore

$$\begin{aligned} \sup ||t \circ x + (1-t) \circ y|| &\leqslant \sup ||(st_0) \circ x + (s(1-t_0)) \circ y + (1-s) \circ y|| \\ &\leqslant \sup ||(st_0) \circ x + (s(1-t_0)) \circ y|| + \sup ||(1-s) \circ y|| \\ &\leqslant s \sup ||\frac{1}{s}((st_0) \circ x + (s(1-t_0)) \circ y|| + (1-s)||y|| \\ &\leqslant s \sup ||t_0 \circ x + (1-t_0) \circ y|| + (1-s)||y|| \\ &< 1 - s + s = 1, \end{aligned}$$

because $a \sup ||b \circ x|| = ab||x|| = \sup ||(ab) \circ x||$, for every a, $b > 0$.

If $t_0 < t < 1$, let $s = \frac{1-t}{1-t_0}$. Then $0 < s < 1$ and

$$\begin{aligned} t \circ x + (1-t) \circ y &= (st_0 + 1 - s) \circ x + s(1-t_0) \circ y \\ &\subseteq (1-s) \circ x + (st_0) \circ x + s(1-t_0) \circ y. \end{aligned}$$

Hence

$$\begin{aligned} \sup ||t \circ x + (1-t) \circ y|| &\leqslant \sup ||(1-s) \circ x + (st_0) \circ x + s(1-t_0) \circ y|| \\ &\leqslant \sup ||(1-s) \circ x|| + \sup ||(st_0) \circ x + s(1-t_0) \circ y|| \\ &\leqslant (1-s)||x|| + s \sup ||t_0 \circ x + (1-t_0) \circ y|| \\ &< 1 - s + s = 1, \end{aligned}$$

as before. Thus

$$\sup ||t \circ x + (1-t) \circ y|| < 1,$$

for $0 < t < 1$, if $\sup ||t_0 \circ x + (1-t_0) \circ y|| < 1$, for some $0 < t_0 < 1$. On the other hand if X is strictly convex, then

$$\sup ||\frac{1}{2} \circ x + \frac{1}{2} \circ y|| < 1.$$

This proves both parts of the theorem. ∎

Before proving the next theorem, it is necessary to note the following useful remark.

Remark 2.4. If $X = (X, +, \circ, ||\cdot||, K)$ is a normed hypervector space such that for every x, $y \in X$, $||(x+y)|| = \sup ||1 \circ x + 1 \circ y||$, then for every $0 \neq a \in K$, we have $\sup ||a \circ (x+y)|| = \sup ||a \circ x + a \circ y||$. Because

$$\begin{aligned} \sup ||a \circ (x+y)|| &= |a| \sup ||1 \circ (x+y)|| \\ &= |a| \sup ||1 \circ x + 1 \circ y|| \\ &= \sup ||a \circ (1 \circ x + 1 \circ y)||, \end{aligned}$$

and therefore

$$\begin{aligned}\sup\|a\circ x+a\circ y\| &= |a|\sup\|\tfrac{1}{a}\circ(a\circ x+a\circ y)\|\\ &\leqslant |a|\sup\|1\circ x+1\circ y\|=\sup\|a\circ(x+y)\|\\ &\leqslant \sup\|a\circ x+a\circ y\|.\end{aligned}$$

It shows that

$$\sup\|a\circ(x+y)\|=\sup\|a\circ x+a\circ y\|.$$

Theorem 2.5. *Let $X=(X,+,\circ,\|\cdot\|,K)$ be a normed hypervector space such that for every x, $y\in X$ and $c\in K$,*

(i) *$x\in c\circ y$ implies $y\in c^{-1}\circ x$,*

(ii) *$x\in 1\circ y$ and $y\in 1\circ x$ implies $x=y$,*

(iii) *$\|(x+y)\|=\sup\|1\circ x+1\circ y\|$,*

(iv) *$\|c\circ x\|$ is a closed set.*

Then X is strictly convex if and only if for all non-zero elements x, $y\in X$, $\|x+y\|=\|x\|+\|y\|$ implies $x\in c\circ y$, for some $c>0$.

Proof. First, suppose the condition holds. Let x, $y\in X$ be such that $\|x\|=\|y\|=1$ and $\sup\|\frac{1}{2}\circ x+\frac{1}{2}\circ y\|=1$. So

$$1=\|x\|=\|y\|=\sup\|\frac{1}{2}\circ x+\frac{1}{2}\circ y\|=\frac{1}{2}\|x+y\|$$

and, therefore, $\|x+y\|=2=\|x\|+\|y\|$. Hence, there is $c>0$ such that $x\in c\circ y$. Since $y\in c^{-1}\circ x$, we have

$$1=\|x\|\leqslant\sup\|c\circ y\|=c\|y\|\leqslant c\sup\|c^{-1}\circ x\|=cc^{-1}\|x\|=1,$$

and, therefore, $c=1$. It means that $x\in 1\circ y$ and $y\in 1\circ x$. So, $x=y$ and X is strictly convex.

To complete the proof, suppose X is strictly convex and let non-zero elements $x,y\in X$ be such that $\|x+y\|=\|x\|+\|y\|$. Also let $x_1\in\frac{1}{\|x\|}\circ x$ and $y_1\in\frac{1}{\|y\|}\circ y$ be such that $\|x_1\|=\|y_1\|=1$. Since $x\in\|x\|\circ x_1$ and $y\in\|y\|\circ y_1$, then

$$\|x+y\|\leqslant\sup\|\frac{1}{\|x\|}\circ x_1+\frac{1}{\|y\|}\circ y_1\|\leqslant\sup\|1\circ x+1\circ y\|=\|x+y\|.$$

So $\sup\|\|x\|\circ x_1+\|y\|\circ y_1\|=\|x\|+\|y\|$. It means that

$$\sup\|\frac{\|x\|}{\|x\|+\|y\|}\circ x_1+\frac{\|y\|}{\|x\|+\|y\|}\circ y_1\|=1,$$

where $0 < \dfrac{||x||}{||x|| + ||y||} < 1$. By Theorem 2.3, we have $x_1 = y_1$. Hence

$$x \in ||x|| \circ x_1 = ||x|| \circ y_1 \subseteq \frac{||x||}{||y||} \circ y.$$

This completes the proof. ■

Example 2.6. Let $(\mathbb{R}^n, +)$ be the classical additive group over $\mathbb{R}^n$ and for every $a \in \mathbb{R}$,

$$a \circ x = \{\, tax \,:\, 0 \leqslant t \leqslant 1 \,\},$$

where tax is the classical multiplication of $\mathbb{R}$ over $\mathbb{R}^n$. Now, let $||x||$ be the distance of x from the origin in $\mathbb{R}^n$. Then it is easily seen that $(\mathbb{R}^n, +, \circ, ||\cdot||, \mathbb{R})$ is a normed hypervector space which is satisfied in the hypothesis of Theorem 2.5.

Definition 2.7. A non-empty subset E of a hypervector space $X = (X, +, \circ, K)$ is called balanced if $k \circ x \subseteq E$, whenever $x \in E$ and $k \in K$ with $|k| \leqslant 1$, and it is called absorbing if for every $x \in X$ there exists $r > 0$ such that

$$\frac{1}{r} \circ x \subseteq E.$$

Theorem 2.8. *Let $X = (X, +, \circ, ||\cdot||, K)$ be a normed hypervector space such that*

(i) $0 \circ x = k \circ 0 = \{0\}$,

(ii) $x \in k \circ y$ *implies that* $y \in k^{-1} \circ x$,

(iii) $\{\, k \circ x \,:\, k \in K \,\}$ *is a subspace of* X,

for every $x, y \in X$, $k \in K$. Also suppose that E is a convex, balanced and absorbing subset of X such that no non-zero subspace of X is contained in E. If

$$||x|| = \inf\{\, r > 0 \,:\, \frac{1}{r} \circ x \subseteq E \,\}, \tag{1}$$

then $||\cdot||$ is a norm on X, and that

$$\{\, x \in X : ||x|| < 1 \,\} \subseteq E \subseteq \{\, x \in X : ||x|| \leqslant 1\}.$$

Furthermore, for every norm in a hypervector space X, there is a convex, balanced and absorbing subset E of X such that it stabilizes (1).

Proof. Let $x \in X$ and suppose that $||x|| = \inf S_x$, where

$$S_x = \{\, r > 0 : \frac{1}{r} \circ x \subseteq E \,\}.$$

Then $S_x \subseteq (0, \infty)$, and hence $||x|| \geqslant 0$, for every $x \in X$. If $x \in E$ then $-x \in (-1) \circ x \subseteq E$, because E is balanced. So

$$\{0\} = \frac{1}{2} \circ 0 = \frac{1}{2} \circ (x - x) \subseteq \frac{1}{2} \circ x + \frac{1}{2} \circ (-x) \subseteq E,$$

because E is convex. Thus $\{0\} = \frac{1}{r} \circ 0 \subseteq E$, for every $r > 0$, and therefore $S_0 = (0, \infty)$ and $||0|| = \inf S_0 = 0$.

Now, suppose that $x \neq 0$. Since $0 \neq x \in 1 \circ x \subseteq Y$, then $Y = \{ k \circ x : k \in K \}$ is a non-zero subspace of X. So, Y is not contained in E. Hence there is $k_1 \in K$ such that $k_1 \circ x \nsubseteq E$. Clearly, $k_1 \neq 0$. Suppose that $0 < r < \dfrac{1}{|k_1|}$ and $r \in S_x$. Then, $\dfrac{1}{r} \circ x \subseteq E$, and since $|K_1 r| < 1$ and E is balanced, we have

$$k_1 \circ x = (k_1 r) \circ \left(\frac{1}{r} \circ x \right) \subseteq E.$$

This contradiction shows that if $r \in S_x$, then $\dfrac{1}{|k_1|} \leqslant r$. Thus, we have proved that $||x|| = 0$ if and only if $x = 0$.

Next, assume that $k \neq 0$ and $r \in S_{k \circ x} = \bigcup_{y \in k \circ x} S_y$. Then there is $y \in k \circ x$ such that $r \in S_y$. Therefore, $\dfrac{1}{r} \circ y \subseteq E$, $x \in k^{-1} \circ y$ and since E is balanced, we have

$$\frac{|k|}{r} \circ x = \frac{|k|}{k} \circ \left(\frac{k}{r} \circ x \right) \subseteq \frac{|k|}{k} \circ \left(\frac{k}{r} \circ (k^{-1} \circ y) \right) = \frac{|k|}{k} \circ \left(\frac{1}{r} \circ y \right) \subseteq E.$$

It means that $\dfrac{r}{|k|} \in S_x$ and then $||x|| \leqslant \dfrac{r}{|k|}$. So, for every $r \in S_y$ that $y \in k \circ x$, we have $|k| ||x|| \leqslant r$ and hence

$$|k| ||x|| \leqslant ||y||.$$

So, for every $k \neq 0$,

$$|k| ||x|| \leqslant \sup ||k \circ x||.$$

By changing k to $\frac{1}{k}$, we have

$$\left| \frac{1}{k} \right| \|y\| \leqslant \sup \left\| \frac{1}{k} \circ y \right\|,$$

for every $y \in k \circ x$. Therefore, there is $z \in \dfrac{1}{k} \circ y$ such that $\left| \dfrac{1}{k} \right| \|y\| \leqslant ||z||$. So,

$$|k| ||z|| \leqslant ||y|| \leqslant |k| ||z||.$$

It means that $\frac{1}{|k|}||y|| = ||z||$. Since $z \in \frac{1}{k} \circ y \subseteq \frac{1}{k} \circ (k \circ x) = 1 \circ x$, we conclude that

$$\sup ||k \circ x|| \leqslant |k| \sup ||1 \circ x||.$$

Now, we show that $\sup ||1 \circ x|| = ||x||$. Let $z \in 1 \circ x$. Then, $x \in 1 \circ z$ and $\frac{1}{r} \circ z \subseteq E$, for every $r \in S_z$, and also

$$\frac{1}{r} \circ x \subseteq \frac{1}{r} \circ (1 \circ z) = \frac{1}{r} \circ z \subseteq E.$$

It means that $r \in S_x$. Similarly, one can show that if $r \in S_x$, then $r \in S_z$. Hence $||z|| = ||x||$. So by the last inequality, we have

$$\sup ||k \circ x|| \leqslant |k|||x||,$$

and, therefore, we have shown that

$$\sup ||k \circ x|| = |k|||x||,$$

for every $k \neq 0$. Clearly, it is true for $k = 0$.

To prove the triangular inequality, let x, $y \in X$. Given $\epsilon > 0$, we can find $r \in S_x$ and $s \in S_y$ such that $r < ||x|| + \epsilon$ and $s < ||y|| + \epsilon$. Then $\frac{1}{r} \circ x \subseteq E$, $\frac{1}{r} \circ y \subseteq E$, and since E is convex, we have

$$\frac{1}{r+s} \circ (x+y) \subseteq \frac{1}{r+s} \circ x + \frac{1}{r+s} \circ y = \frac{r}{r+s} \circ \left(\frac{1}{r} \circ x\right) + \frac{s}{r+s} \circ \left(\frac{1}{s} \circ y\right) \subseteq E.$$

Hence $r + s \in S_{x+y}$, and

$$||x + y|| \leqslant r + s < ||x|| + ||y|| + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$||x + y|| \leqslant ||x|| + ||y||.$$

Thus $|| \cdot ||$ is a norm on X.

Next, let $||x|| < 1$, then there is $r \in S_x$ with $||x|| < r < 1$. Now, since E is balanced, $\frac{1}{r} \circ x \subseteq E$ and $x \in 1 \circ x = r \circ \left(\frac{1}{r} \circ x\right) \subseteq E$, and therefore

$$\{x \in X : ||x|| < 1\} \subseteq E.$$

If $x \in E$, then $1 \circ x \subseteq E$, because E is balanced and hence $||x|| \leqslant 1$. Thus

$$E \subseteq \{x \in X : ||x|| \leqslant 1\}.$$

For the last part of the theorem, let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space. Put $E = \{\, x \in X : ||x|| < 1\}$. Then by Lemma 2.2, E is convex. If $x \in E$ and $r \in K$ such that $|r| \leqslant 1$, then $\sup ||r \circ x|| = |r|||x|| < 1$ and therefore $r \circ x \subseteq E$ and E is balanced. For $0 \neq x \in X$, let $r = 2||x||$, then $\sup \left\| \frac{1}{2\|x\|} \circ x \right\| = \frac{1}{2}$ and hence E is absorbing. Let $x \in X$ and

$$S_x = \left\{ r > 0 : \frac{1}{r} \circ x \subseteq E \right\}.$$

If $r \in S_x$, then $\frac{1}{r} \circ x \subseteq E$ and so $\frac{1}{r}||x|| = \sup \left\| \frac{1}{r} \circ x \right\| < 1$. Thus $||x|| < r$, for every $r \in S_x$. This implies that $||x|| \leqslant \inf S_x$. If $||x|| < \inf S_x$, then choose r with $||x|| < r < \inf S_x$. Thus, $\sup \left\| \frac{1}{r} \circ x \right\| = \frac{1}{r}||x|| < 1$, and so $\frac{1}{r} \circ x \subseteq E$, that is $r \in S_x$. This is not possible since $r < \inf S_x$. So, $||x|| = \inf S_x$ and the proof is complete. ∎

Example 2.9. One can easily check that Example 2.6 satisfies in the hypothesis of Theorem 2.8, too.

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