ANALYSIS OF A TWO-STEP METHOD FOR NUMERICAL SOLUTION OF FUZZY ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Recently, fuzzy initial value problems or fuzzy differential equations have received considerable amount of attentions ([3], [4] and [5]). In all of them, one-step numerical methods have been considered, but in this paper we have a *two-step method* for solving fuzzy ordinary differential equations. In the first section, we present the necessary and introductory materials to deal with the fuzzy initial value differential equations. In the second section, a modified two-step Simpson method and the corresponding convergence theorem of our method are presented. In the last section, we will present an example of fuzzy differential equations. Our numerical results can compare with the results of the existing methods.

Keywords: fuzzy differential equations; two-step methods; Simpson method; ordinary differential equations.
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1. Preliminaries

A general definition of fuzzy numbers may be found in [1]. However, our fuzzy numbers will be almost always triangular or triangular shaped fuzzy numbers. Let T be the set of all triangular or triangular shaped fuzzy numbers and $u \in T$. We define the r-level sets:

$$[u]_r = \{x : u(x) \geq r\} \quad , \quad 0 \leq r \leq 1 \tag{1.1}$$

which are closed bounded intervals and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. For more details see [1], [2].

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Let S be the set of all closed bounded intervals in R and $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be two members of S. The interval metric d_I on S is defined as:

$$d_I(I_1, I_2) = \frac{|a_1 - a_2| + |b_1 - b_2|}{2}. \tag{1.2}$$

Consider the first-order one dimensional fuzzy initial value differential equation given by

$$\begin{cases} y'(t) = f(t, y(t)) & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{1.3}$$

where y is a fuzzy function of t, $f(t, y(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. Therefore we have a fuzzy Cauchy problem [4]. We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level sets of $y(t)$ for $t \in [t_0, T]$ is $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$. Also

$$[y'(t)]_r = [\underline{y}'(t;r), \overline{y}'(t;r)] \quad , \quad [f(t, y(t))]_r = [\underline{f}(t, y(t);r), \overline{f}(t, y(t);r)]. \tag{1.4}$$

We write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ such that $\underline{f}(t, y) = F[t, \underline{y}, \overline{y}]$ and $\overline{f}(t, y) = G[t, \underline{y}, \overline{y}]$. Because of $y' = f(t, y)$ we have:

$$\begin{aligned} \underline{y}'(t;r) &= \underline{f}(t, y(t);r) = F[t, \underline{y}(t;r), \overline{y}(t;r)] \\ \overline{y}'(t;r) &= \overline{f}(t, y(t);r) = G[t, \underline{y}(t;r), \overline{y}(t;r)]. \end{aligned} \tag{1.5}$$

Also we write

$$[y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)] \quad , \quad [y_0]_r = [\underline{y}_0(r), \overline{y}_0(r)] \tag{1.6}$$

where $\underline{y}(t_0;r) = \underline{y}_0(r)$ and $\overline{y}(t_0;r) = \overline{y}_0(r)$. By integration of the system (1.3), from t_{n-1} to t_{n+1} and using the Simpson method for the right hand side of the equation

$$\int_{t_{n-1}}^{t_{n+1}} y'(s)ds = \int_{t_{n-1}}^{t_{n+1}} f(s, y(s))ds, \tag{1.7}$$

we will have

$$\begin{aligned} y(t_{n+1}) &= y(t_{n-1}) + \frac{h}{3} f(t_{n-1}, y(t_{n-1})) + \frac{4h}{3} f(t_n, y(t_n)) \\ &+ \frac{h}{3} f(t_{n+1}, y(t_n) + hf(t_n, y(t_n))) \\ &+ \frac{h^3}{6} f'(\xi_2, y(\xi_2)) f_y(t_{n+1}, \xi_3) - \frac{h^5}{90} f^{(4)}(\xi_1, y(\xi_1)) \end{aligned} \tag{1.8}$$

where $t_{n-1} \le \xi_1 \le t_{n+1}$, $t_n \le \xi_2 \le t_{n+1}$ and ξ_3 is between $y(t_n) + hf(t_n, y(t_n))$ and

$y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$. We have the modified two-step Simpson method

$$y_{n+1} = y_{n-1} + \frac{h}{3} f(t_{n-1}, y_{n-1}) + \frac{4h}{3} f(t_n, y_n) + \frac{h}{3} f(t_{n+1}, y_n + hf(t_n, y_n)) \tag{1.9}$$

for numerical solutions of the fuzzy differential equation (1.3) with initial value $y_0 = y(t_0)$ and $y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f(t_0, y_0)$.

2. A modified explicit two-step Simpson method of order two

We note that throughout each integration step, the value of r is unchanged. We calculate the exact and Simpson approximation solution at grid points $t_n = t_0 + nh$, $0 \leq n \leq N$ where $h = \frac{T - t_0}{N}$. Let $y(t_n)$ be the exact solution and y_n be the Simpson approximation solution of the fuzzy initial value problem (1.3) at t_n. We denote the exact and approximation solution at t_n by:

$$[y(t_n)]_r = [\underline{y}(t_n; r), \overline{y}(t_n; r)] \quad , \quad [y_n]_r = [\underline{y}_n(r), \overline{y}_n(r)] \qquad 0 \leq n \leq N \tag{2.1}$$

respectively. We know that the exact solution satisfies to:

$$\begin{aligned} \underline{y}(t_{n+1}; r) = \underline{y}(t_{n-1}; r) &+ \frac{h}{3} F[t_{n-1}, \underline{y}(t_{n-1}; r), \overline{y}(t_{n-1}; r)] \\ &+ \frac{4h}{3} F[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &+ \frac{h}{3} F[t_{n+1}, \underline{y}(t_n; r) + hF[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &\quad , \overline{y}(t_n; r) + hG[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)]] + h^3 \underline{A}(r) \end{aligned} \tag{2.2}$$

$$\begin{aligned} \overline{y}(t_{n+1}; r) = \overline{y}(t_{n-1}; r) &+ \frac{h}{3} G[t_{n-1}, \underline{y}(t_{n-1}; r), \overline{y}(t_{n-1}; r)] \\ &+ \frac{4h}{3} G[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &+ \frac{h}{3} G[t_{n+1}, \underline{y}(t_n; r) + hF[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &\quad , \overline{y}(t_n; r) + hG[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)]] + h^3 \overline{A}(r) \end{aligned} \tag{2.3}$$

where $A = [\underline{A}, \overline{A}]$, $[A]_r = [\underline{A}(r), \overline{A}(r)]$ and:

$$[A]_r = \left[\frac{1}{6} f'(\xi_2, y(\xi_2)) f_y(t_{n+1}, \xi_3) - \frac{h^2}{90} f^{(4)}(\xi_1, y(\xi_1)) \right]_r. \tag{2.4}$$

In order to approximate the solution of the fuzzy differential equation (1.3), we will use the two-step explicit Simpson method:

$$\begin{aligned} \underline{y}_{n+1}(r) = \underline{y}_{n-1}(r) &+ \frac{h}{3} F[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3} F[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ &+ \frac{h}{3} F[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]] \end{aligned} \tag{2.5}$$

$$\begin{aligned}\overline{y}_{n+1}(r) = \overline{y}_{n-1}(r) + \frac{h}{3}G[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3}G[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ + \frac{h}{3}G[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]].\end{aligned} \tag{2.6}$$

The following lemmas will be applied to show the convergence of our method. For more details see [4].

Lemma 2.1. *Suppose a sequence of non negative numbers* $\{W_n\}_{n=0}^N$ *satisfy:*

$$W_n \leq AW_{n-1} + B \quad , \quad 1 \leq n \leq N \tag{2.7}$$

where A *and* B *are two given positive constants. Then, for* $s = 0, 1, 2, ..., n,$

$$W_n \leq A^{n-s}W_s + B\frac{A^{n-s} - 1}{A - 1} \quad , \quad s \leq n \leq N. \tag{2.8}$$

Lemma 2.2. *Suppose that a sequence of non negative numbers* $\{P_n\}_{n=0}^N$ *satisfy*

$$P_{n+1} \leq AP_n + BP_{n-1} + C \quad , \quad 1 \leq n \leq N - 1 \tag{2.9}$$

for some given positive constants A , B *and* C*. Then, for* $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$, *we have*

$$P_{n+1} + (\alpha - A)P_n \leq \alpha^n[P_1 + (\alpha - A)P_0] + C\frac{\alpha^n - 1}{\alpha - 1}. \tag{2.10}$$

Proof. It is obvious that $A = \dfrac{\sqrt{A^2 + 4B} + A}{2} - \dfrac{\sqrt{A^2 + 4B} - A}{2}$. Therefore, we have:

$$\begin{aligned}P_{n+1} + \frac{\sqrt{A^2 + 4B} - A}{2}P_n \\ \leq \frac{\sqrt{A^2 + 4B} + A}{2}\left(P_n + \frac{\sqrt{A^2 + 4B} - A}{2}P_{n-1}\right) + C.\end{aligned} \tag{2.11}$$

If we set $T_{n+1} = P_{n+1} + \dfrac{\sqrt{A^2 + 4B} - A}{2}P_n$ and $\alpha = \dfrac{\sqrt{A^2 + 4B} + A}{2}$, then

$$T_{n+1} \leq \alpha T_n + C \quad , \quad 1 \leq n \leq N - 1. \tag{2.12}$$

By using Lemma 2.1 with $s = 1$, the proof is completed. ∎

Let $F[t, u, v]$ and $G[t, u, v]$ be the functions which are given by the equations (1.5) where u and v are constants and $u \leq v$. Thus, the domain of F and G are defined as $K = \{(t, u, v) : t_0 \leq t \leq T, -\infty < u \leq v, -\infty < v < +\infty\}$. Now, we will present the convergence theorem.

Theorem 2.1. *Let $F[t,u,v]$ and $G[t,u,v]$ belong to $C^1(K)$ and suppose that the partial derivatives of F and G be bounded on K. Then for arbitrary fixed $0 \le r \le 1$ the Simpson approximations y_N converge to the exact solution $y(T)$ uniformly in t. In other words,*

$$\lim_{h\to 0} d_I([\underline{y}_N(r), \overline{y}_N(r)], [\underline{y}(t_N;r), \overline{y}(t_N;r)]) = 0. \tag{2.13}$$

Proof. Let $W_n = |\underline{y}(t_n;r) - \underline{y}_n(r)|$ and $V_n = |\overline{y}(t_n;r) - \overline{y}_n(r)|$. By using the equations (2.2), (2.3), (2.5) and (2.6) we conclude that [4]:

$$\begin{aligned} W_{n+1} \le W_{n-1} &+ \frac{2Lh}{3}\max\{W_{n-1}, V_{n-1}\} + \frac{8Lh}{3}\max\{W_n, V_n\} \\ &+ \frac{2Lh}{3}[2Lh\max\{W_n, V_n\} + \max\{W_n, V_n\}] + h^3\underline{M} \end{aligned} \tag{2.14}$$

$$\begin{aligned} V_{n+1} \le V_{n-1} &+ \frac{2Lh}{3}\max\{W_{n-1}, V_{n-1}\} + \frac{8Lh}{3}\max\{W_n, V_n\} \\ &+ \frac{2Lh}{3}[2Lh\max\{W_n, V_n\} + \max\{W_n, V_n\}] + h^3\overline{M} \end{aligned} \tag{2.15}$$

where $\underline{M}$ and $\overline{M}$ are upper bound for $\underline{A}(r)$ and $\overline{A}(r)$ respectively which

$$[A]_r = \left[\underline{A}(r), \overline{A}(r)\right] = \left[\frac{1}{6}f'(\xi_2, y(\xi_2))f_y(t_{n+1}, \xi_3) - \frac{h^2}{90}f^{(4)}(\xi_1, y(\xi_1))\right]_r. \tag{2.16}$$

We see that $\max\{W_i, V_i\} \le W_i + V_i$. Therefore,

$$\begin{aligned} W_{n+1} \le W_{n-1} &+ \frac{2Lh}{3}(W_{n-1} + V_{n-1}) + \frac{8Lh}{3}(W_n + V_n) \\ &+ \frac{2Lh}{3}(1 + 2Lh)(W_n + V_n) + h^3\underline{M} \end{aligned} \tag{2.17}$$

$$\begin{aligned} V_{n+1} \le V_{n-1} &+ \frac{2Lh}{3}(W_{n-1} + V_{n-1}) + \frac{8Lh}{3}(W_n + V_n) \\ &+ \frac{2Lh}{3}(1 + 2Lh)(W_n + V_n) + h^3\overline{M}. \end{aligned} \tag{2.18}$$

By adding above two equations and setting $U_n = W_n + V_n$, we obtain

$$U_{n+1} \le \frac{4Lh}{3}(5 + 2Lh)U_n + \left(1 + \frac{4Lh}{3}\right)U_{n-1} + 2h^3M \tag{2.19}$$

where $M = \max\{\underline{M}, \overline{M}\}$. By using Lemma 2.2, we have:

$$U_{n+1} + (\alpha - A)U_n \le \alpha^n[U_1 + (\alpha - A)U_0] + C\frac{\alpha^n - 1}{\alpha - 1} \tag{2.20}$$

where $\alpha = \dfrac{\sqrt{A^2+4B}+A}{2}$. Because of $U_0 = 0$, for $n = N-1$ we have:

$$\lim_{h\to 0}\left[\alpha^{N-1}[U_1 + (\alpha - A)U_0] + C\frac{\alpha^{N-1}-1}{\alpha - 1}\right] = 0. \tag{2.21}$$

Therefore, we have $\lim_{h\to 0}[U_N + (\alpha - A)U_{N-1}] = 0$ and consequently $\lim_{h\to 0} U_N = 0$. In other words, $\lim_{h\to 0} W_N = \lim_{h\to 0} V_N = 0$ and the proof is completed. ■

3. Numerical result

In this section, we will present a numerical example. For this example, the theoretical exact solution and the numerical solutions via our method are shown in the figures and tables at the end of this section. As well as the convergence theorem shows, the numerical results also show that for smaller stepsize h we get smaller errors and hence better results. This example has chosen in comparison with the results of other methods [4].

Example 3.1. Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = ty(t), \quad t \in [-1,1] \\ [y(-1)]_r = [0.5\sqrt{r} - 0.3, 0.2\sqrt{1-r} + 0.2]. \end{cases} \tag{3.1}$$

The exact solution is separated between two steps. If $t < 0$ then with $t_0 = -1$ we have:

$$\begin{cases} \underline{y}(t;r) = \dfrac{A+B}{2}\underline{y}_0(r) + \dfrac{A-B}{2}\overline{y}_0(r) \\ \overline{y}(t;r) = \dfrac{A-B}{2}\underline{y}_0(r) + \dfrac{A+B}{2}\overline{y}_0(r), \end{cases} \tag{3.2}$$

where

$$A = \frac{1}{2}e^{\frac{t^2-t_0^2}{2}} \quad , \quad B = \frac{1}{A}, \tag{3.3}$$

and if $t \geq 0$ with $t_0 = 0$ we have:

$$\underline{y}(t;r) = \underline{y}_0(r)e^{\frac{t^2-t_0^2}{2}} \quad , \quad \overline{y}(t;r) = \overline{y}_0(r)e^{\frac{t^2-t_0^2}{2}}. \tag{3.4}$$

By using the fuzzy two-step modified Simpson method approximation and denoting

$$\begin{aligned} \underline{y}_1(r) &= \underline{y}_0(r) + ht_0\overline{y}_0 + \frac{h^2}{2}(1+t_0^2)\underline{y}_0(r) \\ \overline{y}_1(r) &= \overline{y}_0(r) + ht_0\underline{y}_0 + \frac{h^2}{2}(1+t_0^2)\overline{y}_0(r) \end{aligned} \tag{3.5}$$

as the initial values, we have

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{4h}{3}t_i\overline{y}_i(r) + \frac{h}{3}t_{i+1}(\overline{y}_i(r) + ht_i\underline{y}_i(r)) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{4h}{3}t_i\underline{y}_i(r) + \frac{h}{3}t_{i+1}(\underline{y}_i(r) + ht_i\overline{y}_i(r)) \end{aligned} \tag{3.6}$$

where $t_i < 0$ and

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{h}{3}t_{i+1}\underline{y}_i(r) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{h}{3}t_{i+1}\overline{y}_i(r) \end{aligned} \tag{3.7}$$

where $t_i = 0$ and

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{4h}{3}t_i\underline{y}_i(r) + \frac{h}{3}t_{i+1}(\underline{y}_i(r) + ht_i\underline{y}_i(r)) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{4h}{3}t_i\overline{y}_i(r) + \frac{h}{3}t_{i+1}(\overline{y}_i(r) + ht_i\overline{y}_i(r)) \end{aligned} \tag{3.8}$$

where $t_i > 0$. The theoretical exact solution and the numerical solutions via our method with different stepsize h are shown in Figures 3.1, 3.2 and 3.3. Also the r-level sets of the fuzzy modified two-step Simpson approximations for $r = 0.2$ and $r = 0.7$ are given in Tables 3.1 and 3.2, respectively.

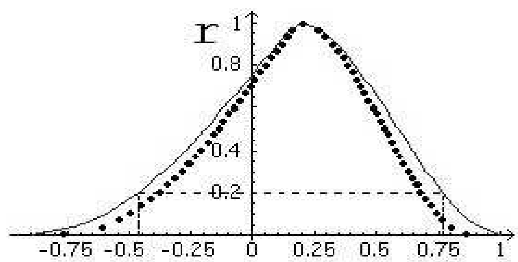

Figure 3.1: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^0$.

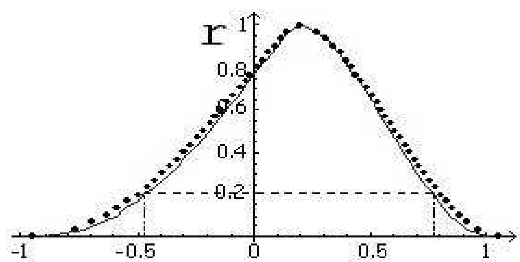

Figure 3.2: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^{-1}$.

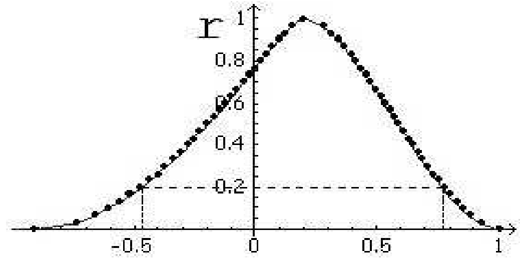

Figure 3.3: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^{-2}$.

h or N ($N = \frac{2}{h}$)	$\underline{y}_N(0.2)$	$\overline{y}_N(0.2)$
$h = 1$ or $N = 2$	-0.37991229588	0.68240453183
$h = 0.5$ or $N = 4$	-0.50158297633	0.79864855758
$h = 0.25$ or $N = 8$	-0.47712052458	0.77860199758
$h = 0.1$ or $N = 20$	-0.46916539153	0.77158097615
$h = 0.01$ or $N = 200$	-0.46755835823	0.77005051182
$h = 0.001$ or $N = 2000$	-0.46754187646	0.77003411232
Exact solution	-0.46754170963	0.77003394558

Table 3.1: 0.2-levelsets

h or N ($N = \frac{2}{h}$)	$\underline{y}_N(0.7)$	$\overline{y}_N(0.7)$
$h = 1$ or $N = 2$	-0.00914631888	0.43702084365
$h = 0.5$ or $N = 4$	-0.06294574081	0.48314427584
$h = 0.25$ or $N = 8$	-0.05047583804	0.47692064109
$h = 0.1$ or $N = 20$	-0.04667023792	0.47443633957
$h = 0.01$ or $N = 200$	-0.04595721883	0.47383162709
$h = 0.001$ or $N = 2000$	-0.04595025566	0.47382478031
Exact solution	-0.04595018555	0.47382471032

Table 3.2: 0.7-levelsets

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