ON HOMOMORPHISMS OF SOME MULTIPLICATIVE HYPERRINGS

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Abstract. A *homomorphism* of a multiplicative hyperring $(A, +, \circ)$ is a function $f : A \to A$ satisfying the conditions $f(x + y) = f(x) + f(y)$ and $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in A$. Denote by $\mathrm{Hom}(A, +, \circ)$ and $\mathrm{Hom}(A, +)$ the set of all homomorphisms of the multiplicative hyperring $(A, +, \circ)$ and the set of all homomorphisms of the group $(A, +)$. Then $\mathrm{Hom}(A, +, \circ) \subseteq \mathrm{Hom}(A, +)$. It is known that if $(R, +, \cdot)$ is a ring and I is an ideal of R, then $(R, +, \circ)$ is a strongly distributive hyperring where $x \circ y = xy + I$ for all $x, y \in R$, and we shall write $(R, +, I)$ for $(R, +, \circ)$. The purpose of this paper is to prove the following results for positive integers m, n : $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite. $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$ if and only if $m \leq 2$. If $m > 2$, then $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite. If $(m, n) > 1$, then $|\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{2n}{(m,n)}$. $\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n, +)$ if and only if $(m, n) \leq 2$. If $(m, n) > 2$, then $|\mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.

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1. Introduction

The cardinality of a set X is denoted by $|X|$. Let $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_n, +, \cdot)$ be the ring of integers and the ring of integers modulo n, respectively, under usual addition and multiplication. The residue class of $x \in \mathbb{Z}$ modulo n will be denoted by $\overline{x}$. Then

$$\mathbb{Z}_n = \{\overline{x} \mid x \in \mathbb{Z}\} = \{\overline{0}, \overline{1}, ..., \overline{n-1}\} \text{ and } |\mathbb{Z}_n| = n.$$

For $a \in \mathbb{Z}$, define $g_a : \mathbb{Z} \to \mathbb{Z}$ and $h_{\overline{a}} : \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$g_a(x) = ax \text{ and } h_{\overline{a}}(\overline{x}) = \overline{ax} \text{ for all } x \in \mathbb{Z}.$$

For groups G, G', let $\mathrm{Hom}(G, G')$ be the set of all homomorphisms from G into G' and let $\mathrm{Hom}(G)$ stand for Hom (G, G). Then

$$\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\} \text{ and } \mathrm{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}.$$

Since $g_a \neq g_b$ if $a \neq b$ and $h_{\overline{a}} \neq h_{\overline{b}}$ if $\overline{a} \neq \overline{b}$, it follows that $|\mathrm{Hom}(\mathbb{Z}, +)| = \aleph_0$ and $|\mathrm{Hom}(\mathbb{Z}_n, +)| = n$.

We know that for $I \subseteq \mathbb{Z}, I$ is an ideal of the ring $(\mathbb{Z}, +, \cdot)$ if and only if $I = m\mathbb{Z}$ for some $m \in \mathbb{Z}$. Since $x \mapsto \overline{x}$ is an epimorphism from the ring $(\mathbb{Z}, +, \cdot)$ onto the ring $(\mathbb{Z}_n, +, \cdot)$, it follows that for $I \subseteq \mathbb{Z}_n, I$ is an ideal of the ring $(\mathbb{Z}_n, +, \cdot)$ if and only if $I = m\mathbb{Z}_n$ for some $m \in \mathbb{Z}$ where $m\mathbb{Z}_n = \{m\overline{x} \mid x \in \mathbb{Z}\}(= \{\overline{mx} \mid x \in \mathbb{Z}\})$. It is easy to see that

$$m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \left\{\, \overline{0}, \overline{(m,n)}, ..., \left(\tfrac{n}{(m,n)} - 1\right)\overline{(m,n)} \,\right\} \text{ and } |m\mathbb{Z}_n| = \tfrac{n}{(m,n)}$$

where (m,n) denotes the g.c.d. of m and n.

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to \mathcal{P}(H) \smallsetminus \{\emptyset\}$ where $\mathcal{P}(H)$ is the power set of H. The value of (x, y) under the hyperoperation $\circ$ is denoted by $x \circ y$. The system $(H, \circ)$ is called a *hypergroupoid*. For $A, B \subseteq H$ and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all} \quad x, y, z \in H.$$

For a hypergroupoid $(H, \circ)$, a function $f : H \to H$ is called a *homomorphism* of $(H, \circ)$ if

$$f(x \circ y) \subseteq f(x) \circ f(y) \quad \text{for all } x, y \in H \quad ([1], \text{p.12}).$$

A *multiplicative hyperring* is a system $(A, +, \circ)$ such that

1. $(A, +)$ is an abelian group,
2. $(A, \circ)$ is a semihypergroup,
3. for all $x, y, z \in A, x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$,
4. for all $x, y \in A,\ x \circ (-y) = (-x) \circ y = -(x \circ y)$.

If in the condition 3, the equalities are valid, then the multiplicative hyperring $(A, +, \circ)$ is called *strongly distributive*. Several results on multiplicative hyperrings were provided by Rota [5] and [6] and Oslon and Ward [3]. In [7], quasi-hyperideals in multiplicative hyperrings were defined and studied. If $(A, +)$ is an abelian group and $\circ$ is the hyperoperation on A defined by

$$x \circ y = \mathbb{Z}x + \mathbb{Z}y \text{ (the subgroup of } (A, +) \text{ generated by } x \text{ and } y) \\ \text{for all } x, y \in A,$$

then $(A, +, \circ)$ is a multiplicative hyperring ([1], p.177) which is not generally strongly distributive. Note that if $f \in \mathrm{Hom}(A, +)$, then for all $x, y \in A$,

$$f(x \circ y) = f(\mathbb{Z}x + \mathbb{Z}y) = \bigcup_{k,\, l \in \mathbb{Z}} f(kx + ly) = \bigcup_{k,\, l \in \mathbb{Z}} (kf(x) + lf(y))$$
$$= \mathbb{Z}f(x) + \mathbb{Z}f(y) = f(x) \circ f(y).$$

We are interested in multiplicative hyperrings defined from rings as follows: Let $(R, +, \cdot)$ be a ring, I an ideal of R and $\circ$ the hyperoperation defined on R by

$$x \circ y = xy + I \quad \text{for all } x, y \in R.$$

Then $(R, +, \circ)$ is a strongly distributive multiplicative hyperring ([1], p.177). For convenience, the multiplicative hyperring $(R, +, \circ)$ will be denoted by $(R, +, I)$.

By a *homomorphism* of a multiplicative hyperring $(A, +, \circ)$ we mean a function $f : A \to A$ such that f is a homomorphism from the group $(A, +)$ into itself and a homomorphism from the semihypergroup $(A, \circ)$ into itself, that is,

$$f(x + y) = f(x) + f(y) \text{ and } f(x \circ y) \subseteq f(x) \circ f(y) \quad \text{for all } x, y \in A.$$

Denote by $\mathrm{Hom}(A, +, \circ)$ the set of all homomorphisms of $(A, +, \circ)$. Notice that $\mathrm{Hom}(A, +, \circ) \subseteq \mathrm{Hom}(A, +)$.

It can be seen from our previous observation that if $(A, +, \circ)$ is the multiplicative hyperring defined from an abelian group $(A, +)$ by $x \circ y = \mathbb{Z}x + \mathbb{Z}y$ for all $x, y \in A$, then $\mathrm{Hom}(A, +, \circ) = \mathrm{Hom}(A, +)$.

In the remainder of this paper, let m and n be positive integers. Notice that $(-m)\mathbb{Z} = m\mathbb{Z}$ and $(-m)\mathbb{Z}_n = m\mathbb{Z}_n$. In [2], the authors characterized the elements of $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ and $\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$ when m is a prime. In this case, $|\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})|$ and $|\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)|$ were also determined. Some results on homomorphisms of certain multiplicative hyperrings were given in [4]. The purpose of this paper is to provide the following facts.

1. $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite. $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$ if and only if $m \leq 2$. If $m > 2$, then $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite.
2. If $(m, n) > 1$, then $|\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{2n}{(m,n)}$. $\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n, +)$ if and only if $(m, n) \leq 2$. If $(m, n) > 2$, then $|\mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.

2. Main Results

To obtain the main results, the following series of lemmas is needed.

Lemma 2.1. *For $a \in \mathbb{Z}$, $g_a \in \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ if and only if $m \,|(a^2 - a)$.*

Proof. Assume that $g_a \in \mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$. Then $g_a(1\circ 1) \subseteq g_a(1)\circ g_a(1)$, so

$$a+am\mathbb{Z} = a(1+m\mathbb{Z}) = a(1\cdot 1+m\mathbb{Z}) = g_a(1\circ 1) \subseteq g_a(1)\circ g_a(1) = a\circ a = a^2+m\mathbb{Z}.$$

This implies that $a = a^2 + mt$ for some $t \in \mathbb{Z}$. Thus $m\,|(a^2-a)$.

Conversely, assume that $m\,|(a^2-a)$. Then $a^2 - a = mt$ for some $t \in \mathbb{Z}$, so $a = a^2 - mt$. Thus for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} g_a(x\circ y) &= g_a(xy+m\mathbb{Z}) = a(xy+m\mathbb{Z}) = axy + am\mathbb{Z} \\ &= (a^2-mt)xy + am\mathbb{Z} \subseteq a^2xy + m\mathbb{Z} + am\mathbb{Z} \\ &= a^2xy + m\mathbb{Z} = (ax)(ay) + m\mathbb{Z} = g_a(x)g_a(y) + m\mathbb{Z} = g_a(x)\circ g_a(y). \end{aligned}$$

Hence $g_a \in \mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$, as desired. ∎

Lemma 2.2. $\{\, g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} \subseteq \mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$.

Proof. If $a \in m\mathbb{Z}\cup(m\mathbb{Z}+1)$, then $m\,|\,a$ or $m\,|(a-1)$, so $m\,|(a^2-a)$. By Lemma 2.1, the lemma is proved. ∎

Lemma 2.3. *If* $m > 2$, *then* $\{\, g_a \mid a \in m\mathbb{Z}+2\} \subseteq \mathrm{Hom}(\mathbb{Z},+) \smallsetminus \mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$.

Proof. Assume $m > 2$ and let $a \in m\mathbb{Z}+2$. Then $a = mk+2$ for some $k \in \mathbb{Z}$. But

$$a^2 - a = m^2k^2 + 3mk + 2,$$

so $m \nmid (a^2-a)$. By Lemma 2.1, $g_a \notin \mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$. Hence the desired result follows. ∎

Lemma 2.4. *For* $a \in \mathbb{Z}$, $h_{\bar{a}} \in \mathrm{Hom}\,(\mathbb{Z}_n,+,m\mathbb{Z}_n)$ *if and only if* $(m,n)|(a^2-a)$.

Proof. Assume that $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)$. Then

$$\begin{aligned} \bar{a} + am\mathbb{Z}_n &= \bar{a}(\bar{1}\cdot\bar{1} + m\mathbb{Z}_n) = \bar{a}(\bar{1}\circ\bar{1}) = h_{\bar{a}}(\bar{1}\circ\bar{1}) \subseteq h_{\bar{a}}(\bar{1})\circ h_{\bar{a}}(\bar{1}) \\ &= \bar{a}\circ\bar{a} = \bar{a}^2 + m\mathbb{Z}_n = \bar{a}^2 + (m,n)\mathbb{Z}_n, \end{aligned}$$

so $\bar{a} - \bar{a}^2 = (m,n)\bar{s}$ for some $s \in \mathbb{Z}$. Hence $a - a^2 - (m,n)s = nt$ for some $t \in \mathbb{Z}$. Thus $a - a^2 = (m,n)s + nt$. But $(m,n)\,|\,((m,n)s+nt)$, so $(m,n)\,|\,(a^2-a)$.

For the converse, assume that $(m,n)\,|\,(a^2-a)$. Then $a^2 - a = (m,n)s$ for some $s \in \mathbb{Z}$, so $a = a^2 - (m,n)s$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} h_{\bar{a}}(\bar{x}\circ\bar{y}) &= h_{\bar{a}}(\overline{xy} + m\mathbb{Z}_n) \\ &= \bar{a}(\overline{xy} + m\mathbb{Z}_n) \\ &= \overline{axy} + am\mathbb{Z}_n \\ &= \overline{(a^2-(m,n)s)xy} + am\mathbb{Z}_n \\ &= \overline{a^2xy} - \overline{(m,n)sxy} + am\mathbb{Z}_n \\ &\subseteq \overline{a^2xy} + (m,n)\mathbb{Z}_n + am\mathbb{Z}_n \\ &= \overline{a^2xy} + m\mathbb{Z}_n + am\mathbb{Z}_n \\ &= \overline{a^2xy} + m\mathbb{Z}_n \\ &= \overline{ax}\cdot\overline{ay} + m\mathbb{Z}_n \\ &= h_{\bar{a}}(\bar{x})\cdot h_{\bar{a}}(\bar{y}) + m\mathbb{Z}_n \\ &= h_{\bar{a}}(\bar{x})\circ h_{\bar{a}}(\bar{y}). \end{aligned}$$

Hence $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$. ∎

Lemma 2.5. $\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$.

Proof. If $a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)$, then $(m,n) \mid a$ or $(m,n) \mid (a-1)$, thus $(m,n) \mid (a^2-a)$. Hence by Lemma 2.3, the result follows. ∎

Lemma 2.6. *If* $(m,n) > 2$, *then*

$$\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z}+2\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n).$$

Proof. If $(m,n) > 2$ and $a \in (m,n)\mathbb{Z}+2$, then $a = (m,n)k+2$, so

$$a^2 - a = (m,n)^2k^2 + 3(m,n)k + 2$$

which is not divided by (m,n), so by Lemma 2.4, $h_{\bar{a}} \notin \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$, that is, $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$. ∎

Theorem 2.7. *The following statements hold.*

(i) $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ *is infinite.*

(ii) $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$ *if and only if* $m \leq 2$.

(iii) *If* $m > 2$, *then* $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ *is infinite.*

(iv) *If* m *is a prime power, then* $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$.

Proof. (i) Since $g_a \neq g_b$ if $a \neq b$ in $\mathbb{Z}$, (i) follows from Lemma 2.2.

(ii) If $m > 2$, then by Lemma 2.3, $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \neq \emptyset$, so $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \neq \mathrm{Hom}(\mathbb{Z}, +)$. This shows that if $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$, then $m \leq 2$.

Assume that $m \leq 2$. Since $1\mathbb{Z} \cup (1\mathbb{Z}+1) = \mathbb{Z}$ and $2\mathbb{Z} \cup (2\mathbb{Z}+1) = \mathbb{Z}$, we have $m\mathbb{Z} \cup (m\mathbb{Z}+1) = \mathbb{Z}$. It follows that $\{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} = \mathrm{Hom}(\mathbb{Z}, +)$. Hence by Lemma 2.2, $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$.

(iii) follows directly from Lemma 2.3.

(iv) Assume that m is a prime power. Let $a \in \mathbb{Z}$ be such that $g_a \in \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$. By Lemma 2.1, $m \mid a^2 - a$. Since $a^2 - a = a(a-1)$ and a and $a-1$ are relatively prime, we have that $m \mid a$ or $m \mid a-1$. Therefore $a \in m\mathbb{Z}$ or $a - 1 \in m\mathbb{Z}$. Hence $a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)$. This shows that $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \subseteq \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$. This implies that $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$ by Lemma 2.2. ∎

Theorem 2.8. *The following statements hold.*

(i) *If* $(m,n) > 1$, *then* $|\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{2n}{(m,n)}$.

(ii) $\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n, +)$ *if and only if* $(m,n) \leq 2$.

(iii) *If* $(m,n) > 2$, *then* $|\mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.

(iv) *If (m,n) is a prime power, then*

$$\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}$$

and thus $|\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| = \frac{2n}{(m,n)}$.

Proof. (i) Assume that $(m,n) > 1$. Then $|(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)} < n$. This implies that $(m,n)\mathbb{Z}_n \cap ((m,n)\mathbb{Z}_n + 1) = \emptyset$. Since $h_{\bar{a}} \neq h_{\bar{b}}$ for all distinct $\bar{a}, \bar{b} \in \mathbb{Z}_n$, it follows that

$$\begin{aligned}|\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| &\geq |\{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}| \\ &= |\{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in (m,n)\mathbb{Z}_n \cup ((m,n)\mathbb{Z}_n + \bar{1})\}| \\ &= |(m,n)\mathbb{Z}_n| + |(m,n)\mathbb{Z}_n + \bar{1}| \\ &= \frac{n}{(m,n)} + \frac{n}{(m,n)} = \frac{2n}{(m,n)}.\end{aligned}$$

(ii) If $(m,n) > 2$, then by Lemma 2.6, $\mathrm{Hom}(\mathbb{Z}_n,+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \neq \emptyset$, so $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \neq \mathrm{Hom}(\mathbb{Z}_n,+)$. Hence if $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n,+)$, then $(m,n) \leq 2$.

Assume that $(m,n) \leq 2$. Then $(m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1) = \mathbb{Z}$. This implies that $\{h_{\bar{a}} | \, a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\} = \mathrm{Hom}(\mathbb{Z}_n,+)$. Therefore by Lemma 2.5, we have $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n,+)$.

(iii) Assume that $(m,n) > 2$. Then

$$\begin{aligned}&|\mathrm{Hom}(\mathbb{Z}_n,+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| \\ &\qquad \geq |\{h_{\bar{a}} | \, a \in (m,n)\mathbb{Z} + 2\}| \quad \text{from Lemma 2.6} \\ &\qquad = |\{h_{\bar{a}} | \, a \in \mathbb{Z} \text{ and } \bar{a} \in (m,n)\mathbb{Z}_n + \bar{2}\}| \\ &\qquad = |(m,n)\mathbb{Z}_n + \bar{2}| = |(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)}.\end{aligned}$$

(iv) Let (m,n) be a prime power and let $a \in \mathbb{Z}$ be such that $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)$. By Lemma 2.4, $(m,n) \mid a^2 - a$. But $a^2 - a = a(a-1)$ and $(a, a-1) = 1$, so $(m,n) \mid a$ or $(m,n) \mid a-1$. Thus $a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)$. This shows that $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \subseteq \{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\,\}$. Hence by Lemma 2.5, we have $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}$. ∎

Example 2.9. By Theorem 2.7(iv),

$$\mathrm{Hom}(\mathbb{Z},+,4\mathbb{Z}) = \{\, g_a \mid a \in 4\mathbb{Z} \cup (4\mathbb{Z}+1)\}$$

and hence

$$\mathrm{Hom}(\mathbb{Z},+) \smallsetminus \mathrm{Hom}(\mathbb{Z},+,4\mathbb{Z}) = \{\, g_a \mid a \in (4\mathbb{Z}+2) \cup (4\mathbb{Z}+3)\}.$$

By Theorem 2.8(iv), $|\mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20})| = \dfrac{2 \times 20}{(4,20)} = 10$ and

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20}) &= \{\, h_{\bar{a}} \mid a \in 4\mathbb{Z} \cup (4\mathbb{Z}+1)\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z},\ \bar{a} \in 4\mathbb{Z}_{20} \cup (4\mathbb{Z}_{20}+\bar{1})\} \\ &= \{\, h_{\bar{0}}, h_{\bar{4}}, h_{\bar{8}}, h_{\overline{12}}, h_{\overline{16}}, h_{\bar{1}}, h_{\bar{5}}, h_{\bar{9}}, h_{\overline{13}}, h_{\overline{17}}\}.\end{aligned}$$

Thus

$$\mathrm{Hom}(\mathbb{Z}_{20},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20}) = \{h_{\bar{2}}, h_{\bar{3}}, h_{\bar{6}}, h_{\bar{7}}, h_{\overline{10}}, h_{\overline{11}}, h_{\overline{14}}, h_{\overline{15}}, h_{\overline{18}}, h_{\overline{19}}\}.$$

It follows from Theorem 2.8(i) and (iii) that

$$|\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| \geq \frac{2\times 18}{(6,18)} = 6$$

and

$$|\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| \geq \frac{18}{(6,18)} = 3.$$

From Lemma 2.5 and Lemma 2.6, we have respectively that

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1)\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18} \cup (6\mathbb{Z}_{18}+\bar{1})\} \\ &= \{\, h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}\,\},\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+2\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18}+\bar{2}\} \\ &= \{\, h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}\,\}.\end{aligned}$$

Let us consider $h_{\bar{a}}$ where $a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4) \cup (6\mathbb{Z}+5)$. If $k \in \mathbb{Z}$, then

$$6 \mid (6k+3)^2 - (6k+3),\, 6 \mid (6k+4)^2 - (6k+4) \text{ and } 6 \nmid (6k+5)^2 - (6k+5),$$

so by Lemma 2.4,

$$\{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})$$

and

$$\{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+5\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}).$$

Consequently,

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &= \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1) \cup (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \\ &= \{h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}, h_{\bar{3}}, h_{\bar{9}}, h_{\overline{15}}, h_{\bar{4}}, h_{\overline{10}}, h_{\overline{16}}\},\end{aligned}$$

$$|\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| = 12,$$

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &= \{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+2) \cup (6\mathbb{Z}+5)\} \\ &= \{h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}, h_{\bar{5}}, h_{\overline{11}}, h_{\overline{17}}\},\end{aligned}$$

$$|\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| = 6.$$

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