

ON A FINER TOPOLOGICAL SPACE THAN τ_θ AND SOME MAPS

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Abstract. In 1943, Fomin [7] introduced the notion of θ -continuity. In 1966, the notions of θ -open subsets, θ -closed subsets and θ -closure were introduced by Veličko [18] for the purpose of studying the important class of H -closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ -open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets.

In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ -open and ω_θ -closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Key words and phrases: topological spaces, θ -open sets, θ -closed sets, ω_θ -open sets, ω_θ -closed sets, anti locally countable, ω_θ -continuity.

2000 Mathematics Subject Classification: 54B05, 54C08; Secondary: 54D05.

1. Introduction

In 1982, Hdeib [8] introduced the notion of ω -closedness by which he introduced and investigated the notion of ω -continuity. In 1943, Fomin [7] introduced the notion of θ -continuity. In 1966, the notions of θ -open subsets, θ -closed subsets and θ -closure were introduced by Veličko [18] for the purpose of studying the important class of H -closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ -open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets. In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ -open and ω_θ -closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Throughout this paper, by a space we will always mean a topological space. For a subset A of a space X , the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. A subset A of a space X is said to be α -open [14] (resp. preopen [13], regular open [17], regular closed [17]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$, $A = int(cl(A))$, $A = cl(int(A))$).

A point $x \in X$ is said to be in the θ -closure [18] of a subset A of X , denoted by $\theta-cl(A)$, if $cl(U) \cap A \neq \emptyset$ for each open set U of X containing x . A subset A of a space X is called θ -closed if $A = \theta-cl(A)$. The complement of a θ -closed set is called θ -open. The θ -interior of a subset A of X is the union of all open sets of X whose closures are contained in A and is denoted by $\theta-int(A)$. Recall that a point p is a condensation point of A if every open set containing p must contain uncountably many points of A . A subset A of a space X is ω -closed [8] if it contains all of its condensation points. The complement of an ω -closed subset is called ω -open. It was shown that the collection of all ω -open subsets forms a topology that is finer than the original topology on X . The union of all ω -open sets of X contained in a subset A is called the ω -interior of A and is denoted by $\omega-int(A)$.

The family of all ω -open (resp. θ -open, α -open) subsets of a space (X, τ) is denoted by $\omega O(X)$ (resp, $\tau_\theta = \theta O(X)$, $\alpha O(X)$).

A function $f : X \rightarrow Y$ is said to be ω -continuous [9] (resp. θ -continuous [7]) if $f^{-1}(V)$ is ω -open (resp. θ -open) in X for every open subset V of Y . A function $f : X \rightarrow Y$ is called weakly ω -continuous [6] if for each $x \in X$ and each open subset V in Y containing $f(x)$, there exists an ω -open subset U in X containing x such that $f(U) \subset cl(V)$.

2. A finer topology than τ_θ

Definition 1 A subset A of a space (X, τ) is called ω_θ -open if for every $x \in A$, there exists an open subset $B \subset X$ containing x such that $B \setminus \theta-int(A)$ is countable. The complement of an ω_θ -open subset is called ω_θ -closed.

The family of all ω_θ -open subsets of a space (X, τ) is denoted by $\omega_\theta O(X)$.

Theorem 2 $(X, \omega_\theta O(X))$ is a topological space for a topological space (X, τ) .

Proof. It is obvious that $\emptyset, X \in \omega_\theta O(X)$. Let $A, B \in \omega_\theta O(X)$ and $x \in A \cap B$. There exist the open sets $U, V \subset X$ containing x such that $U \setminus \theta - int(A)$ and $V \setminus \theta - int(B)$ are countable. Then

$$\begin{aligned} & (U \cap V) \setminus \theta - int(A \cap B) \\ &= (U \cap V) \setminus [\theta - int(A) \cap \theta - int(B)] \subset [(U \setminus \theta - int(A)) \cup (V \setminus \theta - int(B))]. \end{aligned}$$

Thus, $(U \cap V) \setminus \theta - int(A \cap B)$ is countable and hence $A \cap B \in \omega_\theta O(X)$. Let $\{A_i : i \in I\}$ be a family of ω_θ -open subsets of X and $x \in \cup_{i \in I} A_i$. Then $x \in A_j$ for some $j \in I$. This implies that there exists an open subset B of X containing x such that $B \setminus \theta - int(A_j)$ is countable.

Since $B \setminus \theta - int\left(\bigcup_{i \in I} A_i\right) \subset B \setminus \bigcup_{i \in I} \theta - int(A_i) \subset B \setminus \theta - int(A_j)$, then $B \setminus \theta - int\left(\bigcup_{i \in I} A_i\right)$ is countable. Hence, $\bigcup_{i \in I} A_i \in \omega_\theta O(X)$. ■

Theorem 3 Let A be a subset of a space (X, τ) . Then A is ω_θ -open if and only if for every $x \in A$, there exists an open subset U containing x and a countable subset V such that $U \setminus V \subset \theta - int(A)$.

Proof. Let $A \in \omega_\theta O(X)$ and $x \in A$. Then there exists an open subset U containing x such that $U \setminus \theta - int(A)$ is countable.

Take $V = U \setminus \theta - int(A) = U \cap (X \setminus \theta - int(A))$. Thus, $U \setminus V \subset \theta - int(A)$.

Conversely, let $x \in A$. There exists an open subset U containing x and a countable subset V such that $U \setminus V \subset \theta - int(A)$. Hence, $U \setminus \theta - int(A)$ is countable. ■

Remark 4 The following diagram holds for a subset A of a space X :

$$\begin{array}{ccc} \omega_\theta\text{-open} & \longrightarrow & \omega\text{-open} \\ \uparrow & & \uparrow \\ \theta\text{-open} & \longrightarrow & \text{open} \end{array}$$

The following examples show that these implications are not reversible.

Example 5

- (1) Let R be the real line with the topology $\tau = \{\emptyset, R, R \setminus (0, 1)\}$. Then the set $R \setminus (0, 1)$ is open but it is not ω_θ -open.
- (2) Let R be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where Q' is the set of irrational numbers. Then the set $A = Q' \cup \{1\}$ is ω -open but it is not ω_θ -open.

Example 6 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $A = \{a, b, d\}$ is ω_θ -open but it is not open.

Theorem 7 Let A be an ω_θ -closed subset of a space X . Then $\theta\text{-cl}(A) \subset K \cup V$ for a closed subset K and a countable subset V .

Proof. Since A is ω_θ -closed, then $X \setminus A$ is ω_θ -open. For every $x \in X \setminus A$ there exists an open set U containing x and a countable set V such that

$$U \setminus V \subset \theta\text{-int}(X \setminus A) = X \setminus \theta\text{-cl}(A).$$

Hence,

$$\theta\text{-cl}(A) \subset X \setminus (U \setminus V) = X \cap ((X \setminus U) \cup V) = (X \setminus U) \cup V.$$

Take $K = X \setminus U$. Thus, K is closed and $\theta\text{-cl}(A) \subset K \cup V$. ■

Definition 8 The intersection of all ω_θ -closed sets of X containing a subset A is called the ω_θ -closure of A and is denoted by $\omega_\theta\text{-cl}(A)$. The union of all ω_θ -open sets of X contained in a subset A is called the ω_θ -interior of A and is denoted by $\omega_\theta\text{-int}(A)$.

Lemma 9 Let A be a subset of a space X . Then

- (1) $\omega_\theta\text{-cl}(A)$ is ω_θ -closed in X .
- (2) $\omega_\theta\text{-cl}(X \setminus A) = X \setminus \omega_\theta\text{-int}(A)$.
- (3) $x \in \omega_\theta\text{-cl}(A)$ if and only if $A \cap G \neq \emptyset$ for each ω_θ -open set G containing x .
- (4) A is ω_θ -closed in X if and only if $A = \omega_\theta\text{-cl}(A)$.

Definition 10 A subset A of a topological space (X, τ) is said to be an (ω_θ, ω) -set if $\omega_\theta\text{-int}(A) = \omega\text{-int}(A)$.

Definition 11 A subset A of a topological space (X, τ) is said to be an (ω_θ, θ) -set if $\omega_\theta\text{-int}(A) = \theta\text{-int}(A)$.

Remark 12 Every ω_θ -open set is an (ω_θ, ω) -set and every θ -open set is an (ω_θ, θ) -set but not conversely.

Example 13

- (1) Let R be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where Q' is the set of irrational numbers. Then the natural number set N is an (ω_θ, ω) -set but it is not ω_θ -open.
- (2) Let R be the real line with the topology $\tau = \{\emptyset, R, (2, 3)\}$. Then the set $A = (1, \frac{3}{2})$ is an (ω_θ, θ) -set but it is not θ -open.

Theorem 14 *Let A be a subset of a space X . Then A is ω_θ -open if and only if A is ω -open and an (ω_θ, ω) -set.*

Proof. Since every ω_θ -open is ω -open and an (ω_θ, ω) -set, it is obvious.

Conversely, let A be an ω -open and (ω_θ, ω) -set. Then

$$A = \omega - \text{int}(A) = \omega_\theta - \text{int}(A).$$

Thus, A is ω_θ -open. ■

Theorem 15 *Let A be a subset of a space X . Then A is θ -open if and only if A is ω_θ -open and an (ω_θ, θ) -set.*

Proof. Necessity. It follows from the fact that every θ -open set is ω_θ -open and an (ω_θ, θ) -set.

Sufficiency. Let A be an ω_θ -open and (ω_θ, θ) -set. Then

$$A = \omega_\theta - \text{int}(A) = \theta - \text{int}(A).$$

Thus, A is θ -open. ■

Recall that a space X is called locally countable if each $x \in X$ has a countable neighborhood.

Theorem 16 *Let (X, τ) be a locally countable space and $A \subset X$.*

- (1) $\omega_\theta O(X)$ is the discrete topology.
- (2) A is ω_θ -open if and only if A is ω -open.

Proof. (1) : Let $A \subset X$ and $x \in A$. Then there exists a countable neighborhood B of x and there exists an open set U containing x such that $U \subset B$. We have $U \setminus \theta - \text{int}(A) \subset B \setminus \theta - \text{int}(A) \subset B$. Thus $U \setminus \theta - \text{int}(A)$ is countable and A is ω_θ -open. Hence, $\omega_\theta O(X)$ is the discrete topology.

(2) : Necessity. It follows from the fact that every ω_θ -open set is ω -open.

Sufficiency. Let A be an ω -open subset of X . Since X is a locally countable space, then A is ω_θ -open. ■

Corollary 17 *If (X, τ) is a countable space, then $\omega_\theta O(X)$ is the discrete topology.*

A space X is called anti locally countable if nonempty open subsets are uncountable. As an example, observe that in Example 5 (1), the topological space (R, τ) is anti locally countable.

Theorem 18 *Let (X, τ) be a topological space and $A \subset X$. The following hold:*

- (1) *If X is an anti locally countable space, then $(X, \omega_\theta O(X))$ is anti locally countable.*

(2) If X is anti locally countable regular space and A is θ -open, then

$$\theta - cl(A) = \omega_\theta - cl(A).$$

Proof. (1) : Let $A \in \omega_\theta O(X)$ and $x \in A$. There exists an open subset $U \subset X$ containing x and a countable set V such that $U \setminus V \subset \theta\text{-int}(A)$. Thus, $\theta\text{-int}(A)$ is uncountable and A is uncountable.

(2) : It is obvious that $\omega_\theta\text{-cl}(A) \subset \theta\text{-cl}(A)$.

Let $x \in \theta\text{-cl}(A)$ and B be an ω_θ -open subset containing x . There exists an open subset V containing x and a countable set U such that $V \setminus U \subset \theta\text{-int}(B)$. Then $(V \setminus U) \cap A \subset \theta\text{-int}(B) \cap A$ and $(V \cap A) \setminus U \subset \theta\text{-int}(B) \cap A$. Since X is regular, $x \in V$ and $x \in \theta\text{-cl}(A)$, then $V \cap A \neq \emptyset$. Since X is regular and V and A are ω_θ -open, then $V \cap A$ is ω_θ -open. This implies that $V \cap A$ is uncountable and hence $(V \cap A) \setminus U$ is uncountable. Since $B \cap A$ contains the uncountable set $\theta\text{-int}(B) \cap A$, then $B \cap A$ is uncountable. Thus, $B \cap A \neq \emptyset$ and $x \in \omega_\theta\text{-cl}(A)$. ■

Corollary 19 Let (X, τ) be an anti locally countable regular space and $A \subset X$. The following hold:

(1) If A is θ -closed, then $\theta - \text{int}(A) = \omega_\theta - \text{int}(A)$.

(2) The family of (ω_θ, θ) -sets contains all θ -closed subsets of X .

Theorem 20 If X is a Lindelof space, then $A \setminus \theta - \text{int}(A)$ is countable for every closed subset $A \in \omega_\theta O(X)$.

Proof. Let $A \in \omega_\theta O(X)$ be a closed set. For every $x \in A$, there exists an open set U_x containing x such that $U_x \setminus \theta - \text{int}(A)$ is countable. Thus, $\{U_x : x \in A\}$ is an open cover for A . Since A is Lindelof, it has a countable subcover $\{U_n : n \in \mathbb{N}\}$. Since $A \setminus \theta - \text{int}(A) = \bigcup_{n \in \mathbb{N}} (U_n \setminus \theta - \text{int}(A))$, then $A \setminus \theta - \text{int}(A)$ is countable. ■

Theorem 21 If A is ω_θ -open subset of (X, τ) , then $\omega_\theta O(X)|_A \subset \omega_\theta O(A)$.

Proof. Let $G \in \omega_\theta O(X)|_A$. We have $G = V \cap A$ for some ω_θ -open subset V . Then for every $x \in G$, there exist $U, W \in \tau$ containing x and countable sets K and L such that

$$U \setminus K \subset \theta - \text{int}(V) \quad \text{and} \quad W \setminus L \subset \theta - \text{int}(A).$$

We have $x \in A \cap (U \cap W) \in \tau|_A$. Thus, $K \cup L$ is countable and

$$\begin{aligned} A \cap (U \cap W) \setminus (K \cup L) &\subset (U \cap W) \cap (X \setminus K) \cap (X \setminus L) \\ &= (U \setminus K) \cap (W \setminus L) \subset \theta - \text{int}(V) \cap \theta - \text{int}(A) \cap A \\ &= \theta - \text{int}(V \cap A) \cap A \\ &= \theta - \text{int}(G) \cap A \subset \theta - \text{int}_A(G). \end{aligned}$$

Hence, $G \in \omega_\theta O(A)$. ■

3. Continuities via ω_θ -open sets

Definition 22 A function $f : X \rightarrow Y$ is said to be ω_θ -continuous if for every $x \in X$ and every open subset V in Y containing $f(x)$, there exists an ω_θ -open subset U in X containing x such that $f(U) \subset V$.

Theorem 23 For a function $f : X \rightarrow Y$, the following are equivalent:

- (1) f is ω_θ -continuous.
- (2) $f^{-1}(A)$ is ω_θ -open in X for every open subset A of Y ,
- (3) $f^{-1}(K)$ is ω_θ -closed in X for every closed subset K of Y .

Proof. (1) \Rightarrow (2) : Let A be an open subset of Y and $x \in f^{-1}(A)$. By (1), there exists an ω_θ -open set B in X containing x such that $B \subset f^{-1}(A)$. Hence, $f^{-1}(A)$ is ω_θ -open.

(2) \Rightarrow (1) : Let A be an open subset in Y containing $f(x)$. By (2), $f^{-1}(A)$ is ω_θ -open. Take $B = f^{-1}(A)$. Hence, f is ω_θ -continuous.

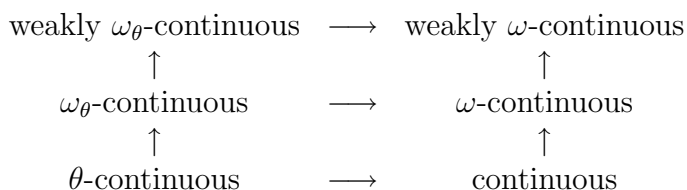
(2) \Leftrightarrow (3) : Let K be a closed subset of Y . By (2), $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$ is ω_θ -open. Hence, $f^{-1}(K)$ is ω_θ -closed. ■

Theorem 24 The following are equivalent for a function $f : X \rightarrow Y$:

- (1) f is ω_θ -continuous.
- (2) $f : (X, \omega_\theta O(X)) \rightarrow (Y, \sigma)$ is continuous.

Definition 25 A function $f : X \rightarrow Y$ is called weakly ω_θ -continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω_θ -open subset U in X containing x such that $f(U) \subset cl(V)$. If f is weakly ω_θ -continuous at every $x \in X$, it is said to be weakly ω_θ -continuous.

Remark 26 The following diagram holds for a function $f : X \rightarrow Y$:



The following examples show that these implications are not reversible.

Example 27 Let R be the real line with the topology $\tau = \{\emptyset, R, (2, 3)\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$.

Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & , \text{ if } x \in (0, 1) \\ b & , \text{ if } x \notin (0, 1) \end{cases} .$$

Then f is weakly ω_θ -continuous but it is not ω_θ -continuous.

Example 28 Let R be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where Q' is the set of irrational numbers. Let $Y = \{a, b, c, d\}$ and $\sigma = \{Y, \emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (R, \tau) \rightarrow (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & , \text{ if } x \in Q' \cup \{1\} \\ b & , \text{ if } x \notin Q' \cup \{1\} \end{cases} .$$

Then f is ω -continuous but it is not weakly ω_θ -continuous.

Example 29 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$, $f(d) = a$. Then f is ω_θ -continuous but it is not θ -continuous.

For the other implications, the contra examples are as shown in [6, 9].

Definition 30 A function $f : X \rightarrow Y$ is said to be (ω_θ, ω) -continuous if $f^{-1}(A)$ is an (ω_θ, ω) -set for every open subset A of Y .

Definition 31 A function $f : X \rightarrow Y$ is said to be (ω_θ, θ) -continuous if $f^{-1}(A)$ is an (ω_θ, θ) -set for every open subset A of Y .

Remark 32 Every θ -continuous function is (ω_θ, θ) -continuous and every ω_θ -continuous function is (ω_θ, ω) -continuous but not conversely.

Example 33 Let R be the real line with the topology $\tau = \{\emptyset, R, Q'\}$ where Q' is the set of irrational numbers. Define a function $f : (R, \tau) \rightarrow (R, \tau)$ as follows:

$$f(x) = \begin{cases} \pi & , \text{ if } x \in N \\ 1 & , \text{ if } x \notin N \end{cases} .$$

Then f is (ω_θ, ω) -continuous but it is not ω_θ -continuous.

Example 34 Let R be the real line with the topology $\tau = \{\emptyset, R, (2, 3)\}$. Let $A = (1, \frac{3}{2})$ and $\sigma = \{R, \emptyset, A, R \setminus A\}$. Define a function $f : (R, \tau) \rightarrow (R, \sigma)$ as follows:

$$f(x) = \begin{cases} \frac{5}{4} & , \text{ if } x \in (1, 2) \\ 4 & , \text{ if } x \notin (1, 2) \end{cases} .$$

Then f is (ω_θ, θ) -continuous but it is not θ -continuous.

Definition 35 A function $f : X \rightarrow Y$ is coweakly ω_θ -continuous if for every open subset A in Y , $f^{-1}(fr(A))$ is ω_θ -closed in X , where $fr(A) = cl(A) \setminus int(A)$.

Theorem 36 Let $f : X \rightarrow Y$ be a function. The following are equivalent:

- (1) f is ω_θ -continuous,
- (2) f is ω -continuous and (ω_θ, ω) -continuous,
- (3) f is weakly ω_θ -continuous and coweakly ω_θ -continuous.

Proof. (1) \Leftrightarrow (2) : It is an immediate consequence of Theorem 14.

(1) \Rightarrow (3) : Obvious.

(3) \Rightarrow (1) : Let f be weakly ω_θ -continuous and coveakly ω_θ -continuous. Let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Since f is weakly ω_θ -continuous, then there exists an ω_θ -open subset U of X containing x such that $f(U) \subset cl(V)$. We have $fr(V) = cl(V) \setminus V$ and $f(x) \notin fr(V)$. Since f is coveakly ω_θ -continuous, then $x \in U \setminus f^{-1}(fr(V))$ is ω_θ -open in X . For every $y \in f(U \setminus f^{-1}(fr(V)))$, $y = f(x_1)$ for a point $x_1 \in U \setminus f^{-1}(fr(V))$. We have $f(x_1) = y \in f(U) \subset cl(V)$ and $y \notin fr(V)$. Hence, $f(x_1) = y \notin fr(V)$ and $f(x_1) \in V$. Thus, $f(U \setminus f^{-1}(fr(V))) \subset V$ and f is ω_θ -continuous. ■

Theorem 37 *The following are equivalent for a function $f : X \rightarrow Y$:*

- (1) f is θ -continuous,
- (2) f is ω_θ -continuous and (ω_θ, θ) -continuous.

Proof. It is an immediate consequence of Theorem 15. ■

Theorem 38 *Let $f : X \rightarrow Y$ be a function. The following are equivalent:*

- (1) f is weakly ω_θ -continuous,
- (2) $\omega_\theta-cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$ for every subset K of Y ,
- (3) $\omega_\theta-cl(f^{-1}(int(A))) \subset f^{-1}(A)$ for every regular closed set A of Y ,
- (4) $\omega_\theta-cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for every open set A of Y ,
- (5) $f^{-1}(A) \subset \omega_\theta-int(f^{-1}(cl(A)))$ for every open set A of Y ,
- (6) $\omega_\theta-cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for each preopen set A of Y ,
- (7) $f^{-1}(A) \subset \omega_\theta-int(f^{-1}(cl(A)))$ for each preopen set A of Y .

Proof. (1) \Rightarrow (2) : Let $K \subset Y$ and $x \in X \setminus f^{-1}(cl(K))$. Then $f(x) \in Y \setminus cl(K)$. This implies that there exists an open set A containing $f(x)$ such that $A \cap K = \emptyset$. We have, $cl(A) \cap int(cl(K)) = \emptyset$. Since f is weakly ω_θ -continuous, then there exists an ω_θ -open set B containing x such that $f(B) \subset cl(A)$. We have $B \cap f^{-1}(int(cl(K))) = \emptyset$. Thus, $x \in X \setminus \omega_\theta-cl(f^{-1}(int(cl(K))))$ and $\omega_\theta-cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$.

(2) \Rightarrow (3) : Let A be any regular closed set in Y . Thus, $\omega_\theta-cl(f^{-1}(int(A))) = \omega_\theta-cl(f^{-1}(int(cl(int(A)))) \subset f^{-1}(cl(int(A))) = f^{-1}(A)$.

(3) \Rightarrow (4) : Let A be an open subset of Y . Since $cl(A)$ is regular closed in Y , $\omega_\theta-cl(f^{-1}(A)) \subset \omega_\theta-cl(f^{-1}(int(cl(A)))) \subset f^{-1}(cl(A))$.

(4) \Rightarrow (5) : Let A be any open set of Y . Since $Y \setminus cl(A)$ is open in Y , then $X \setminus \omega_\theta-int(f^{-1}(cl(A))) = \omega_\theta-cl(f^{-1}(Y \setminus cl(A))) \subset f^{-1}(cl(Y \setminus cl(A))) \subset X \setminus f^{-1}(A)$. Thus, $f^{-1}(A) \subset \omega_\theta-int(f^{-1}(cl(A)))$.

(5) \Rightarrow (1) : Let $x \in X$ and A be any open subset of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta\text{-int}(f^{-1}(cl(A)))$. Take $B = \omega_\theta\text{-int}(f^{-1}(cl(A)))$. Thus $f(B) \subset cl(A)$ and f is weakly ω_θ -continuous at x in X .

(1) \Rightarrow (6) : Let A be any preopen set of Y and $x \in X \setminus f^{-1}(cl(A))$. Then there exists an open set W containing $f(x)$ such that $W \cap A = \emptyset$. We have $cl(W \cap A) = \emptyset$. Since A is preopen, then $A \cap cl(W) \subset int(cl(A)) \cap cl(W) \subset cl(int(cl(A)) \cap W) \subset cl(int(cl(A) \cap W)) \subset cl(int(cl(A \cap W))) \subset cl(A \cap W) = \emptyset$. Since f is weakly ω_θ -continuous and W is an open set containing $f(x)$, there exists an ω_θ -open set B in X containing x such that $f(B) \subset cl(W)$. We have $f(B) \cap A = \emptyset$ and hence $B \cap f^{-1}(A) = \emptyset$. Thus, $x \in X \setminus \omega_\theta\text{-cl}(f^{-1}(A))$ and $\omega_\theta\text{-cl}(f^{-1}(A)) \subset f^{-1}(cl(A))$.

(6) \Rightarrow (7) : Let A be any preopen set of Y . Since $Y \setminus cl(A)$ is open in Y , then $X \setminus \omega_\theta\text{-int}(f^{-1}(cl(A))) = \omega_\theta\text{-cl}(f^{-1}(Y \setminus cl(A))) \subset f^{-1}(cl(Y \setminus cl(A))) \subset X \setminus f^{-1}(A)$. Hence, $f^{-1}(A) \subset \omega_\theta\text{-int}(f^{-1}(cl(A)))$.

(7) \Rightarrow (1) : Let $x \in X$ and A any open set of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta\text{-int}(f^{-1}(cl(A)))$. Take $B = \omega_\theta\text{-int}(f^{-1}(cl(A)))$. Then $f(B) \subset cl(A)$. Thus, f is weakly ω_θ -continuous at x in X . ■

Theorem 39 *The following properties are equivalent for a function $f : X \rightarrow Y$:*

- (1) $f : X \rightarrow Y$ is weakly ω_θ -continuous at $x \in X$.
- (2) $x \in \omega_\theta\text{-int}(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$.

Proof. (1) \Rightarrow (2) : Let A be any neighborhood of $f(x)$. There exists an ω_θ -open set B containing x such that $f(B) \subset cl(A)$. Since $B \subset f^{-1}(cl(A))$ and B is ω_θ -open, then $x \in B \subset \omega_\theta\text{-int}(B) \subset \omega_\theta\text{-int}(f^{-1}(cl(A)))$.

(2) \Rightarrow (1) : Let $x \in \omega_\theta\text{-int}(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$. Take $U = \omega_\theta\text{-int}(f^{-1}(cl(A)))$. Thus, $f(U) \subset cl(A)$ and U is ω_θ -open. Hence, f is weakly ω_θ -continuous at $x \in X$. ■

Theorem 40 *Let $f : X \rightarrow Y$ be a function. The following are equivalent:*

- (1) f is weakly ω_θ -continuous,
- (2) $f(\omega_\theta\text{-cl}(K)) \subset \theta\text{-cl}(f(K))$ for each subset K of X ,
- (3) $\omega_\theta\text{-cl}(f^{-1}(A)) \subset f^{-1}(\theta\text{-cl}(A))$ for each subset A of Y ,
- (4) $\omega_\theta\text{-cl}(f^{-1}(int(\theta\text{-cl}(A)))) \subset f^{-1}(\theta\text{-cl}(A))$ for every subset A of Y .

Proof. (1) \Rightarrow (2) : Let $K \subset X$ and $x \in \omega_\theta\text{-cl}(K)$. Let U be any open set of Y containing $f(x)$. Then there exists an ω_θ -open set B containing x such that $f(B) \subset cl(U)$. Since $x \in \omega_\theta\text{-cl}(K)$, then $B \cap K \neq \emptyset$. Thus, $\emptyset \neq f(B) \cap f(K) \subset cl(U) \cap f(K)$ and $f(x) \in \theta\text{-cl}(f(K))$. Hence, $f(\omega_\theta\text{-cl}(K)) \subset \theta\text{-cl}(f(K))$.

(2) \Rightarrow (3) : Let $A \subset Y$. Then $f(\omega_\theta\text{-cl}(f^{-1}(A))) \subset \theta\text{-cl}(A)$. Thus, $\omega_\theta\text{-cl}(f^{-1}(A)) \subset f^{-1}(\theta\text{-cl}(A))$.

(3) \Rightarrow (4) : Let $A \subset Y$. Since $\theta\text{-cl}(A)$ is closed in Y , then $\omega_\theta\text{-cl}(f^{-1}(int(\theta\text{-cl}(A)))) \subset f^{-1}(\theta\text{-cl}(int(\theta\text{-cl}(A)))) = f^{-1}(cl(int(\theta\text{-cl}(A)))) \subset f^{-1}(\theta\text{-cl}(A))$.

(4) \Rightarrow (1) : Let U be any open set of Y . Then $U \subset \text{int}(cl(U)) = \text{int}(\theta-cl(U))$. Thus, $\omega_\theta-cl(f^{-1}(U)) \subset \omega_\theta-cl(f^{-1}(\text{int}(\theta-cl(U)))) \subset f^{-1}(\theta-cl(U)) = f^{-1}(cl(U))$. By Theorem 38, f is weakly ω_θ -continuous. ■

Recall that a space is rim-compact [10] if it has a basis of open sets with compact boundaries.

Theorem 41 *Let $f : X \rightarrow Y$ be a function with the closed graph. Suppose that X is regular and Y is a rim-compact space. Then f is weakly ω_θ -continuous if and only if f is ω_θ -continuous.*

Proof. Let $x \in X$ and A be any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set B of Y such that $f(x) \in B \subset A$ and ∂B is compact. For each $y \in \partial B$, $(x, y) \in X \times Y \setminus G(f)$. Since $G(f)$ is closed, there exist open sets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$, $y \in V_y$ and $f(U_y) \cap V_y = \emptyset$. The family $\{V_y\}_{y \in \partial B}$ is an open cover of ∂B . Then there exist a finite number of points of ∂B , say, y_1, y_2, \dots, y_n such that $\partial B \subset \cup\{V_{y_i}\}_{i=1}^n$. Take $K = \cap\{U_{y_i}\}_{i=1}^n$ and $L = \cup\{V_{y_i}\}_{i=1}^n$. Then K and L are open sets such that $x \in K$, $\partial B \subset L$ and $f(K) \cap \partial B \subset f(K) \cap L = \emptyset$. Since f is weakly ω_θ -continuous, there exists an ω_θ -open set G containing x such that $f(G) \subset cl(B)$. Take $U = K \cap G$. Then, U is an ω_θ -open set containing x , $f(U) \subset cl(B)$ and $f(U) \cap \partial B = \emptyset$. Hence, $f(U) \subset B \subset A$ and f is ω_θ -continuous.

The converse is obvious. ■

Definition 42 If a space X can not be written as the union of two nonempty disjoint ω_θ -open sets, then X is said to be ω_θ -connected.

Theorem 43 *If $f : X \rightarrow Y$ is a weakly ω_θ -continuous surjection and X is ω_θ -connected, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets U and V of Y such that $Y = U \cup V$ and $U \cap V = \emptyset$. This implies that U and V are clopen in Y . By Theorem 38, $f^{-1}(U) \subset \omega_\theta\text{-int}(f^{-1}(cl(U))) = \omega_\theta\text{-int}(f^{-1}(U))$. Hence $f^{-1}(U)$ is ω_θ -open in X . Similarly, $f^{-1}(V)$ is ω_θ -open in X . Hence, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $X = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Thus, X is not ω_θ -connected. ■

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