FUZZY LIE IDEALS OVER A FUZZY FIELD

M. Akram

Punjab University College of Information Technology
University of the Punjab
Old Campus, Lahore-54000
Pakistan
e-mail: m.akram@pucit.edu.pk, makrammath@yahoo.com

K.P. Shum

Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
P.R. China (SAR)
e-mail: kpshum@math.cuhk.edu.hk

Abstract. The concept of fuzzy Lie ideals of a Lie algebra over a fuzzy field is introduced and some fundamental properties of such fuzzy Lie ideals are given. We then characterize the Artinian and Noetherian Lie algebras by considering their fuzzy Lie ideals over a fuzzy field.

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1. Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. Lie algebras were also discovered by him when he attempted to classify certain smooth subgroups of a general linear group. The importance of Lie algebras in mathematics and physics has become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. It is noted that Lie theory has applications not only in mathematics and physics but also in diverse fields such as continuum mechanics, cosmology and life sciences. Lie algebra has nowadays even been applied by electrical engineers in solving problems in mobile robot control.

On the other hand, Zadeh [13] introduced the notion of a fuzzy subset of a set in 1965. By using fuzzy sets, people have established the theory for study uncertainty. Fuzzy mathematics have become a vigorous area of research in different domains such as engineering, medical science, social science, artificial intelligence, signal processing, pattern recognition, computer networks,automata theory and so on. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were first introduced by Yehia in [12]. In this paper, the notion of fuzzy Lie ideals of a Lie algebra over a fuzzy field (in short, fuzzy Lie $\mathbb{F}$-ideals) is considered and some properties of fuzzy Lie $\mathbb{F}$-ideal of a Lie algebra are presented. By fuzzy Lie $\mathbb{F}$-ideals, we give characterizations for Artinian and Noetherian Lie algebras.

The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [1], [2], [3], [5], [8]-[12].

2. Preliminaries

We first review some elementary aspects which are useful in the sequel. Throughout this paper, L is a Lie algebra and X is a field. It is clear that the multiplication of a Lie algebra is not necessary associative, that is, $[[x, y], z] = [x, [y, z]]$ does not hold in general, however it is *anti- commutative*, that is, $[x, y] = -[y, x]$. Let μ be a *fuzzy set* on L, that is, a map $\mu : L \to [0, 1]$. For any fuzzy set μ in L and any $t \in [0, 1]$, we define set $U(\mu; t) = \{x \in L \mid \mu(x) \geq t\}$, which is called *upper t-level cut* of μ.

Definition 2.1. [14] A mapping $f : L \to L$ is called a *closure* if, for every $x, y \in L$,

(i) $x \geq y \Rightarrow f(x) \geq f(y)$ (monotony)

(ii) $x \leq f(x)$ (inclusion)

(iii) $f(f(x)) = f(x)$ (idempotence).

Definition 2.2. [1] A fuzzy set $\mu : L \to [0, 1]$ is called a *fuzzy Lie ideal* of L over a field X if the following conditions:

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(2) $\mu(\alpha x) \geq \mu(x)$,

(3) $\mu([x, y]) \geq \mu(x)$

hold for all $x, y \in L$ and $\alpha \in X$.

Definition 2.3. [10] A fuzzy set F of X is called a *fuzzy field* if the following conditions are satisfied:

- $(\forall\ m,\ n \in X)(F(m - n) \geq \min\{F(m), F(n)\})$,
- $(\forall m, n \in X, n \neq 0)(F(mn^{-1}) \geq \min\{F(m), F(n)\})$.

Lemma 2.4. [10] *If F is a fuzzy subfield of X, then*

$$F(0) \geq F(1) \geq F(m) = F(-m) \quad \text{for all } m \in X \qquad \text{and}$$
$$F(-m) = F(m^{-1}) \quad \text{for all } m \in X - \{0\}.$$

Lemma 2.5. [10] *Let F be a fuzzy subfield of X. Then for $t \in [0,1]$, the fuzzy-cut $U(F;t)$ is a crisp subfield of X.*

3. Fuzzy Lie ideals over a fuzzy field

Definition 3.1. Let μ be a fuzzy set of L and F a fuzzy field of X. Then μ is called a *fuzzy Lie ideal over a fuzzy field* F(briefly, fuzzy Lie $\mathbb{F}$-ideal) if the following conditions:

(a) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \max\{F(\alpha), \mu(x)\}$,

(c) $\mu([x,y]) \geq \mu(x)$

hold for all $x, y \in L$ and $\alpha \in X$.

From condition (b) above, it follows that $\mu(0) \geq F(0)$.

Example 3.2. Let $\Re^2 = \{(x,y) : x, y \in R\}$ be the set of all 2-dimensional real vectors. Then it is clear that $\Re^2$ endowed with the operation defined by $[x,y] = x \times y$ form a real Lie algebra. Define a fuzzy set $\mu : \Re^2 \to [0,1]$ by

$$\mu(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

and define $F : \mathbb{R} \to [0,1]$ for all $\alpha \in \mathbb{R}$ by

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \mathbb{Q}, \\ 1 & \text{if } \alpha \in \mathbb{R} - \mathbb{Q}(\sqrt{3}). \end{cases}$$

By routine computations, one can easily check that μ is a fuzzy Lie $\mathbb{F}$-ideal.

Definition 3.3. Let μ be a fuzzy set and $s \in [0,\ 1]$. Define:

(d) the *t-cut* of μ by the non-empty set $U(\mu;t) = \{x \in L \,|\, \mu(x) \geq t\}$.

(e) the strong *t-cut* of μ by the non-empty set ${}^{>}U(\mu;t) = \{x \in L \,|\, \mu(x) > t\}$.

(f) the image of μ by the set $t \in Im(\mu)$.

We now formulate the following theorem of fuzzy Lie $\mathbb{F}$-ideals of L.

Theorem 3.4. *Let μ be a fuzzy Lie $\mathbb{F}$-ideal of L and ν the closure of the image of μ. Then the following conditions are equivalent:*

(g) *μ is a fuzzy Lie $\mathbb{F}$-ideal of L,*

(h) *the non-empty strong level subset $^{>}U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in [0,1]$,*

(i) *the non-empty strong level subset $^{>}U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in Im(\mu) \setminus \nu$,*

(j) *the nonempty level subset $U(\mu;t)$ of μ is a Lie ideal of L,for all $t \in Im(\mu)$,*

(k) *the nonempty level subset $U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in [0,1]$.*

Proof. *(g)* $\Leftrightarrow$ *(h)*: Let $t \in [0,1]$ be such that the strong t-cut of μ is non-empty, that is, $^{>}U(\mu;t) \neq \emptyset$. Then for $x, y \in L$, $\alpha \in X$ satisfying the condition $x \in {}^{>}U(\mu;t)$, $y \in {}^{>}U(\mu;t)$, $\alpha \in {}^{>}U(F;t)$, we have $\mu(x) > t$ and $\mu(y) > t$, $F(\alpha) > t$. From Definition 3.1, it follows that

$$\begin{aligned} \mu(x+y) &\geq \min(\mu(x), \mu(y)) > t, \\ \mu(\alpha x) &\geq \max(F(\alpha), \mu(x)) > t, \\ \mu([x,y]) &\geq \mu(x) > t \end{aligned}$$

and hence $x + y \in {}^{>}U(\mu;t)$, $\alpha x \in {}^{>}U(\mu;t)$ and $[x,y] \in {}^{>}U(\mu;t)$.

(h) $\Leftrightarrow$ (i), (i) $\Leftrightarrow$ (j), (j) $\Leftrightarrow$ (k) are obvious.

(k) $\Leftrightarrow$ (g): suppose that $U(\mu;t) \neq \emptyset$ is a Lie ideal of L for every $t \in [0,1]$. If

$$\mu(x+y) < \min\{\mu(x), \mu(y)\}$$

for some x, $y \in L$, then by taking

$$s_0 := \frac{1}{2}\{\mu(x+y) + \min\{\mu(x) + \mu(y)\}\},$$

we have $\mu(x+y) < s_0 < \min\{\mu(x), \mu(y)\}$. This shows that $x + y \notin U(\mu;t)$, $x \in U(\mu;t)$ and $y \in U(\mu;t)$, however, this is a contradiction. Hence $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ for all x, $y \in L$. By using the same argumentations we can prove $\mu(\alpha x) \geq \max(F(\alpha), \mu(y))$, $\mu([x,y]) \geq \mu(x)$.

The proofs of the following Propositions are obvious.

Proposition 3.5.

(i) *Let A be a nonempty subset of L. Define a fuzzy set μ by*

$$\mu(x) = \begin{cases} \beta_2 & \text{if } x \in A \\ \beta_1 & \text{otherwise} \end{cases}$$

Clearly, μ with $0 \leq \beta_1 < \beta_2 \leq 1$ is a fuzzy Lie $\mathbb{F}$-ideal of L if and only if A is a Lie ideal of L.

(ii) *If μ and ν are fuzzy Lie $\mathbb{F}$-ideals of L, then $\mu + \nu$ and $\mu \cap \nu$ are clearly fuzzy Lie $\mathbb{F}$-ideals of L.*

(iii) *If* $\{\mu_i : i \in \Lambda\}$ *is a family of fuzzy Lie* $\mathbb{F}$*-ideals of* L*, then* $\bigcap_{i\in\Lambda} \mu_i$ *is also a fuzzy Lie* $\mathbb{F}$*-ideal of* L*.*

Proposition 3.6.

(iv) *Let* $f : L_1 \to L_2$ *be a homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_1*, then* $f(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*.*

(v) *A Lie algebra homomorphism image of a fuzzy Lie* $\mathbb{F}$*-ideal having the* sup *property is a fuzzy Lie* $\mathbb{F}$*-ideal.*

(vi) *Let* $f : L_1 \to L_2$ *be an onto homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*, then* $f^{-1}(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_1*.*

(vii) *Let* $f : L_1 \to L_2$ *be an onto homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*, then* $f^{-1}(\mu^c) = (f^{-1}(\mu))^c$*.*

Definition 3.7. Let L_1 and L_2 be Lie algebras. If f is a function of a fuzzy set μ in L_1, then the *image* of μ under f is a fuzzy set defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(t) \mid t \in L_1, f(t) = y\}, & \text{if } f^{-1}(y) \neq \emptyset \ , \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.8. Let L_1 and L_2 be any sets and $f : L_1 \to L_2$ any function. Then a fuzzy set μ is called f*-invariant* if and only if for $x, y \in L_1$, $f(x) = f(y)$ implies $\mu(x) = \mu(y)$.

Theorem 3.9. *Let* $f : L_1 \to L_2$ *be an epimorphism of Lie algebras. Then* μ *is an* f*-invariant fuzzy Lie* $\mathbb{F}$*-ideal of* L_1 *if and only if* $f(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*.*

Proof. Let $x, y \in L_2$ and $\alpha \in X$. Then there exist $a, b \in L_1$ such that $f(a) = x$, $f(b) = y$ and $x + y = f(a + b)$ with $\alpha x = \alpha f(a)$. Since μ is f-invariant, we conclude that

$$\begin{aligned} f(\mu)(x+y) &= \mu(a+b) \geq \min(\mu(a), \mu(b)) \\ &= \min(f(\mu)(x), f(\mu)(y)), \\ f(\mu)(\alpha x) &= \mu(\alpha a) \geq \max(F(\alpha), \mu(a)) \\ &= \max(f(F(\alpha)), f(\mu)(x)), \\ f(\mu)([x,y]) &= \mu([a,b]) = [\mu(a), \mu(b)] \geq \mu(a) = f(\mu)(x). \end{aligned}$$

Hence $f(\mu)$ is a fuzzy Lie $\mathbb{F}$-ideal of L_2.
Conversely, if $f(\mu)$ is a fuzzy Lie $\mathbb{F}$-ideal of L_2, then for any $x \in L_1$ we have

$$\begin{aligned} f^{-1}(f(\mu))(x) &= f(\mu)(f(x)) \\ &= \sup\{\mu(t) \mid t \in L_1, f(t) = f(x)\} \\ &= \sup\{\mu(t) \mid t \in L_1, \mu(t) = \mu(x)\} \\ &= \mu(x). \end{aligned}$$

This shows that $f^{-1}(f(\mu)) = \mu$ is a fuzzy Lie $\mathbb{F}$-ideal of L, by Proposition 3.6 (iv).

Definition 3.10. An ideal A of L is said to be a *characteristic* ideal of L if $f(A) = A$, for all $f \in \text{Aut}(L)$, where $\text{Aut}(L)$ is the set of all automorphisms of L. A fuzzy Lie $\mathbb{F}$-ideal μ of L is said to be *fuzzy characteristic* if $\mu^f(x) = \mu(x)$ for all $x \in L$ and $f \in \text{Aut}(L)$.

Definition 3.11. An ideal A of Lie algebra L is said to be *fully invariant* if $f(C) \subseteq C$ for all $f \in \text{End}(L)$, where $\text{End}(L)$ is the set of all endomorphisms of L. A fuzzy Lie $\mathbb{F}$-ideal μ is said to be *fuzzy fully invariant* if $\mu^f(x) \leq \mu(x)$, for all $x \in L$ and $f \in \text{End}(L)$.

Lemma 3.12. *Let μ be a fuzzy Lie $\mathbb{F}$-ideal of L. Then for any $x \in L$, $\mu(x) = s$ if and only if $x \in U(\mu; s)$ and $x \notin U(\mu; t)$ for all $s < t$.*

We now characterize the characteristic fuzzy Lie $\mathbb{F}$-ideals of L.

Theorem 3.13. *A fuzzy Lie $\mathbb{F}$-ideal μ of L is characteristic if and only if each of its level set is a characteristic Lie ideal of L.*

Proof. Suppose that μ is fuzzy characteristic over a fuzzy field F and $s \in Im(\mu)$, $f \in \text{Aut}(L)$ and $x \in U(\mu; s)$. Then $\mu^f(x) = \mu(x)$ implies $\mu(f(x)) \geq s$, and whence $f(x) \in U(\mu; s)$. Thus $f(U(\mu; s)) \subseteq U(\mu; s)$. On the other hand, if $x \in U(\mu; s)$ and $y \in L$ such that $f(y) = x$, then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq s \Rightarrow y \in U(\mu; s)$. Consequently, $y \in U(\mu; s)$ and so $x = f(y) \in U(\mu; s)$. This leads to $U(\mu; s) \subseteq f(U(\mu; s))$. Hence, $f(U(\mu; s)) = U(\mu; s)$, that is , $U(\mu; s)$ is a characteristic ideal .

Conversely, if each level Lie ideal of μ is a characteristic ideal of L with $x \in L$, $f \in Aut(L)$ and $\mu(x) = s$, then, by Lemma 3.12 , $x \in U(\mu; s)$ and $x \notin U(\mu; t)$ for all $s < t$. Hence, by our assumption, we have $f(x) \in f(U(\mu; s)) = U(\mu; s)$ and so $\mu^f(x) = \mu(f(x))) \leq s$. Let $t = \mu^f(x)$ and assume that $s < t$. Then $f(x) \in U(\mu; t) = f(U(\mu; t))$. This implies from the injectivity of f that $x \in U(\mu; t)$, a contradiction. This shows that $\mu^f(x) = \mu(f(x)) = s = \mu(x)$ and so μ is fuzzy characteristic over the fuzzy field F.

In view of the above Theorem, we deduce immediately the following theorem.

Theorem 3.14. *If μ is a fully invariant fuzzy Lie $\mathbb{F}$-ideal of L , then it is a characteristic ideal.*

Definition 3.15.

(i) A fuzzy relation on any set L is defined as a fuzzy set $\mu : L \times L \to [0, 1]$.

(ii) If μ is a fuzzy relation on a set L and ν is a fuzzy set in L, then μ is a *fuzzy relation* on ν if $\mu(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all x, $y \in L$.

(iii) Let μ and ν be the fuzzy sets in a set L. The *cartesian product* of μ and ν is defined by $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ for all x, $y \in L$.

Theorem 3.16. *If μ and ν are two fuzzy Lie $\mathbb{F}$-ideals of L, then $\mu \times \nu$ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.*

Proof. We restrict our proof on condition (b) of μ in Definition 3.1. Let $x = (x_1, x_2) \in L \times L$ and $\alpha \in X$. Then

$$\begin{aligned}(\mu \times \nu)(\alpha x) &= (\mu \times \nu)(\alpha(x_1, x_2)) = (\mu \times \nu)((\alpha x_1, \alpha x_2)) \\ &= \min(\mu(\alpha x_1), \nu(\alpha x_2)) \geq \min(\max(F(\alpha), \mu(x_1)), \max(F(\alpha), \nu(x_2))) \\ &= \min(\max(F(\alpha), F(\alpha)), \max(\mu(x_1), \nu(x_2))) \\ &= \max(F(\alpha), \min(\mu(x_1), \nu(x_2))) = \max(F(\alpha), (\mu \times \nu)(x_1, x_2) \\ &= \max(F(\alpha), (\mu \times \nu)(x)).\end{aligned}$$

The verifications for other conditions are analogous. Hence,$\mu \times \nu$ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.

Definition 3.17. Let ν be a fuzzy set in a set L. Then *the strongest fuzzy relation* on L is the fuzzy relation on ν is μ_ν which is defined by $\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}$, for all $x,\ y \in L$.

We now characterize the fuzzy Lie $\mathbb{F}$-ideal of L.

Theorem 3.18. *Let ν be a fuzzy set in L and μ_ν a strongest fuzzy relation on L. Then ν is a fuzzy Lie $\mathbb{F}$-ideal of L if and only if μ_ν is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.*

Proof. We restrict our proof on the verification of condition (b) of μ in Definition 3.1. Suppose that ν is a fuzzy Lie $\mathbb{F}$-ideal of L. Then For any $x = (x_1, x_2) \in L \times L$ and $\alpha \in X$,we have

$$\begin{aligned}\mu_\nu(\alpha x) &= \mu_\nu(\alpha(x_1, x_2)) = \mu_\nu(\alpha x_1, \alpha x_2) \\ &= \min\{\nu(\nu(\alpha x_1)), \nu(\alpha x_2)\} \\ &\geq \max\{\min(F(\alpha), \nu(x_1)), \min(F(\alpha), \nu(x_2))\} \\ &= \max\{F(\alpha), \min(\nu(x_1), \nu(x_2))\} \\ &= \max\{F(\alpha), \mu_\nu((x_1, x_2))\} \\ &= \max\{F(\alpha), \mu_\nu(x)\}.\end{aligned}$$

The verifications for other conditions are analogous and we omit the details. Hence μ_λ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$. The proof of the converse part is easy.

Definition 3.19. Let μ be a fuzzy Lie $\mathbb{F}$-ideal in L and $\mu_n = [\mu, \mu_{n-1}]$ for $n > 0$, where $\mu_0 = \mu$. If there exists a positive integer n such that $\mu_n = 0$, then a fuzzy Lie $\mathbb{F}$-ideal is called *nilpotent*.

Definition 3.20. Let μ be a fuzzy Lie $\mathbb{F}$-ideal in L. Define a sequence of fuzzy Lie $\mathbb{F}$-ideals in L by $\mu^0 = \mu$, $\mu^n = [\mu^{n-1}, \mu^{n-1}]$ for $n > 0$. If there exists a positive integer n such that $\mu^n = 0$, then a fuzzy Lie $\mathbb{F}$-ideal is called *solvable*.

By using similar method as in the proof of Theorem 3.21 in [1], we obtain the following Proposition.

Proposition 3.21.

(I) *The homomorphic image of a solvable fuzzy Lie* $\mathbb{F}$*-ideal is a solvable fuzzy Lie* $\mathbb{F}$*-ideal.*

(II) *The homomorphic image of a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal is a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal.*

(III) *If* μ *is a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal, then it is solvable.*

4. Artinian and Noetherian Lie algebras

Definition 4.1. An Lie algebra L is said to satisfy the *descending chain condition* for Lie ideals if for any sequence of Lie ideals $I_1, I_2, \cdots, I_i, \cdots$ of L such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_i \cdots ,$$

there exists an element $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \leq n$. $\mathbb{N} = \{1, 2, \cdots\}$ always denotes the set of natural numbers.

L is called *Artinian* if it satisfies the descending chain condition on its Lie ideals. Similarly, L is called *Noetherian* if it satisfies the ascending chain condition on its Lie ideals.

The following Lemma is immediate.

Lemma 4.2. *Let* μ *be a fuzzy Lie* $\mathbb{F}$*-ideal of a Lie algebra* L *with* $s, t \in Im(\mu)$. *Then* $U(\mu; s) = U(\mu; t) \iff s = t$.

Theorem 4.3. *Every fuzzy Lie* $\mathbb{F}$*-ideal of a Lie algebra* L *has finite number of values if and only if a Lie algebra* L *is Artinian.*

Proof. Suppose that every fuzzy Lie $\mathbb{F}$-ideal of a Lie algebra L has finite number of values but L is not Artinian. Then there exists a strictly descending chain

$$L = U_0 \supset U_1 \supset U_2 \supset \cdots$$

of ideals of L. Define a fuzzy set μ in L by μ be a fuzzy set in L defined by

$$\mu(x) := \begin{cases} \dfrac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1}, n = 0, 1, 2, ..., \\ 1 & \text{if } x \in \bigcap\limits_{n=0}^{\infty} U_n. \end{cases}$$

Let $x, y \in L$, $\alpha \in X$. Then $x+y$, αx, $[x, y] \in U_n \setminus U_{n+1}$ for some $n = 0, 1, 2, ...,$ and either $x \notin U_{n+1}$ or $y \notin U_{n+1}$. Now, let $y \in U_n \setminus U_{n+1}$ for $k \leq n$. Then by

Definition 3.1, we have

$$\begin{aligned}
\mu(x+y) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \min(\mu(x),\mu(y)),\\
\mu(\alpha x) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \max(F(\alpha),\mu(x)),\\
\mu([x,y]) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \mu(x).
\end{aligned}$$

Thus, μ is fuzzy Lie $\mathbb{F}$-ideal of L and μ has infinite number of different values. This contradiction proves that L is an Artinian Lie algebra.

Conversely, Suppose that L is an Artinian Lie algebra such that μ is a fuzzy Lie $\mathbb{F}$-ideal of L. If $Im(\mu)$ is infinite, then every subset of $[0,1]$ contains either a strictly increasing or a strictly decreasing sequence.

If $t_1 < t_2 < t_3 < \cdots$ is a strictly increasing sequence in $Im(\mu)$, then the following chain

$$U(\mu;t_1) \supset U(\mu;t_2) \supset U(\mu;t_3) \supset$$

is a strictly descending chain of ideals of L. Since L is Artinian, there exists a natural number i such that $U(\mu;t_i) = U(\mu;t_{i+n})$ for all $n \geq 1$. Since $t_i \in Im(\mu)$ for all i,it follows from Lemma 4.2 that $t_i = t_{i+n}$, for all $n \geq 1$. However, this is a contradiction because t_i are different.

On the other hand, if $t_1 > t_2 > t_3 > \cdots$ is a strictly decreasing sequence in $Im(\mu)$, then

$$U(\mu;t_1) \subset U(\mu;t_2) \subset U(\mu;t_3) \subset$$

is an ascending chain of ideals of L. Since L is Noetherian, there exists a natural number j such that $U(\mu;t_j) = U(\mu;t_{j+n})$ for all $n \geq 1$. Since $t_j \in Im(\mu)$ for all j, by Lemma 4.2, $t_j = t_{j+n}$, for all $n \geq 1$. This is again a contradiction because t_j are distinct. This shows that $Im(\mu)$ is finite.

Theorem 4.4. *Let L be an Artinian Lie algebra and μ a fuzzy Lie $\mathbb{F}$-ideal of L. Then $|U_\mu| = |Im(\mu)|$, where U_μ is a family of all level ideals of L with respect to μ.*

Proof. Since L is Artinian, by Theorem 4.3, $Im(\mu)$ is finite. Let $Im(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_1 < t_2 < \cdots < t_n$. Then,it suffices to show that U_μ consists of level ideals of L with respect to μ, for all $t_i \in Im(\mu)$, that is, $U_\mu = \{U(\mu;t_i) \mid 1 \leq i \leq n\}$. It is clear that $U(\mu;t_i) \in U_\mu$ for all $t_i \in Im(\mu)$. Let $0 \leq t \leq \mu(0)$ and $U(\mu;t)$ a level ideal of L with respect to μ. Assume that $t \notin Im(\mu)$. If $t < t_1$, then clearly $U(\mu;t) = U(\mu;t_1)$, and so $t_i < t < t_{i+1}$ for some i. Hence, $U(\mu;t_{i+1}) \subseteq U(\mu;t)$. Let $x \in U(\mu;t)$. Then $\mu(x) > t$ since $t \notin Im(\mu)$, and so $\mu(x) \geq U(\mu;t_{i+1})$. Thus $U(\mu;t) = U(\mu;t_{i+1})$. This shows that U_μ consists of the level ideals of L with respect to μ, for all $t_i \in Im(\mu)$. Hence $|U_\mu| = |Im(\mu)|$.

Theorem 4.5. *Let L be an Artinian Lie algebra. If μ and ν are fuzzy Lie $\mathbb{F}$-ideals of L, then $|U_\mu| = |U_\nu|$ and $Im(\mu)$=$Im(\nu)$ if and only if μ= ν.*

Proof. If $\mu = \nu$, then $U_\mu = U_\nu$ and $Im(\mu) = Im(\nu)$. Suppose that $U_\mu = U_\nu$ and $Im(\mu) = Im(\nu)$. Then, by Theorem 4.3 and 4.4, $Im(\mu) = Im(\nu)$ are finite and $|U_\mu| = |Im(\mu)|$ and $|U_\nu| = |Im(\nu)|$. Let

$$Im(\mu) = \{t_1, t_2, ..., t_n\} \quad \text{and} \quad Im(\nu) = \{s_1, s_2, ..., s_n\},$$

where $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$. Thus, $t_i = s_i$ for all i. We now prove that $U(\mu; t_i) = U(\nu; t_i)$, for all i. Observe that $U(\mu; t_1) = L = U(\nu; t_1)$. Consider $U(\mu; t_2)$, $U(\nu; t_2)$. If $U(\mu; t_2) \neq U(\nu; t_2)$, then $U(\mu; t_2) = U(\nu; t_k)$ for some $k > 2$ and $U(\nu; t_2) = U(\mu; t_j)$ for some $j > 2$. If there exists $x \in L$ such that $\mu(x) = t_2$, then

$$\mu(x) < t_j \ \text{ for all } j > 2. \tag{1}$$

Since $U(\mu; t_2) = U(\nu; t_k)$, $x \in U(\nu; t_k)$, $\nu(x) \geq t_k > t_2$, $k > 2$. Thus $x \in U(\nu; t_2)$. Now we have $x \in U(\mu; t_j)$ because $U(\nu; t_2) = U(\mu; t_j)$. Thus we deuce that

$$\mu(x) \geq t_j \ \text{ for some } j > 2. \tag{2}$$

Clearly, (1) and (2) contradict each other, and so $U(\mu; t_2) = U(\nu; t_2)$. Continuing in this manner, we deduce $U(\mu; t_i) = U(\nu; t_i)$ for all i, as required.

Now let $x \in L$. Suppose that $\mu(x) = t_i$ for some i. Then $x \notin U(\mu; t_j)$, for all $i + 1 \leq j \leq n$. This implies that $x \notin U(\nu; t_j)$ for all $i + 1 \leq j \leq n$. But then $\nu(x) < t_j$, for all $i + 1 \leq j \leq n$. Suppose that $\nu(x) = t_m$ for some $i \leq m \leq i$. If $i \neq m$, then $x \in U(\nu; t_i)$. On the other hand, since $\mu(x) = t_i$, $x \in U(\mu; t_i) = U(\nu; t_i)$, and hence we obtain a contradiction. Thus $i = m$ and $\mu(x) = t_i = \nu(x)$, and consequently $\mu = \nu$.

Theorem 4.6. *A Lie algebra L is Noetherian if and only if the set of values of any fuzzy Lie $\mathbb{F}$-ideal of L is a well ordered subset of* $[0, 1]$.

Proof. We first suppose that μ is a fuzzy Lie $\mathbb{F}$-ideal of L whose set of values is not a well ordered subset of $[0, 1]$. Then considering the strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Denote by U_n the set $\{x \in L \mid \mu(x) \geq \lambda_n\}$. Then $U_1 \subset U_2 \subset U_3 \cdots$ is a strictly ascending chain of ideals of L, which contradicts that L is Noetherian.

Conversely, assume that the set values of any fuzzy Lie $\mathbb{F}$-ideal of L is a well ordered subset of $[0, 1]$ and L is not Noetherian Lie algebra. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \cdots \tag{$*$}$$

of ideals of L. Define a fuzzy set μ in L by

$$\mu(x) := \begin{cases} \dfrac{1}{k} & \text{for } x \in U_k \backslash U_{k-1}, \\ 0 & \text{for } x \notin \displaystyle\bigcup_{k=1}^{\infty} U_k, \end{cases}$$

Now,by using similar argument as Theorem 4.4, one can easily show that μ is a fuzzy Lie $\mathbb{F}$-ideal of L. Since the chain $(*)$ is not terminating, μ has a strictly descending sequence of values which leads to a contradiction. Thus, L is Noetherian.

The following propositions follows easily and we omit their proofs.

Proposition 4.7. *Let $L = \{\lambda_n \in (0,1) \mid n \in \mathbb{N}\} \cup \{0\}$, where $\lambda_i > \lambda_j$ whenever $i < j$. If $\{U_n \mid n \in \mathbb{N}\}$ is a family of ideals of Lie algebra L such that $U_1 \subset U_2 \subset U_3 \subset \cdots$, then the fuzzy set μ in L defined by*

$$\mu(x) := \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1}, n = 2, 3, ... \\ 0 & \text{if } x \in L \setminus \bigcup_{n=1}^{\infty} U_n, \end{cases}$$

is a fuzzy Lie $\mathbb{F}$-ideal of L.

Proposition 4.8. *Let $L = \{\lambda_1, \lambda_2, ..., \lambda_n, ...\} \cup \{0\}$, where $\{\lambda_n\}$ is a fixed sequence, strictly decreasing to 0 and $0 < \lambda_n < 1$. Then a Lie algebra L is Noetherian if and only if for each fuzzy Lie $\mathbb{F}$-ideal μ of L, $Im(\mu) \subset L \Longrightarrow \exists\ n_0 \in \mathbb{N}$ such that $Im(\mu) \subset \{\lambda_1, \lambda_2, ..., \lambda_{n_0}\} \cup \{0\}$.*

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