# A FUNCTIONAL ASSOCIATED WITH TWO BOUNDED LINEAR OPERATORS IN HILBERT SPACES AND RELATED INEQUALITIES

**S.S. Dragomir**

*Mathematics*
*School of Engineering & Science*
*Victoria University*
*PO Box 14428, Melbourne City, MC 8001*
*Australia*
*e-mail: sever.dragomir@vu.edu.au*
*URL: http://www.staff.vu.edu.au/rgmia/dragomir*

**Abstract.** In this paper, several inequalities for the functional

$$\mu(A,B) := \sup_{\|x\|=1} \{\|Ax\| \, \|Bx\|\}$$

under various assumptions for the operators involved, including operators satisfying the uniform $(\alpha,\beta)$-property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive, are given.

**Keywords and phrases:** numerical radius, operator norm, Banach algebra.

**1991 Mathematics Subject Classification:** 47A12, 47A30, 47A63.

## 1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator $T$ is the subset of the complex numbers $\mathbb{C}$ given by [9, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator $T$ on $H$ is given by [9, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [9, p. 9]:

$$w(T) \le \|T\| \le 2w(T) \tag{1.2}$$

for any $T \in B(H)$

For other results on numerical radii, see [10], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [11], [12].

If $A, B$ are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then

$$w(AB) \leq 4w(A)\,w(B). \tag{1.3}$$

In the case that $AB = BA$, then

$$w(AB) \leq 2w(A)\,w(B). \tag{1.4}$$

The following results are also well known [9, p. 38]:

If $A$ is a unitary operator that commutes with another operator $B$, then

$$w(AB) \leq w(B). \tag{1.5}$$

If $A$ is an isometry and $AB = BA$, then (1.5) also holds true.

We say that $A$ and $B$ *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators $A$ and $B$ double commute, then [9, p. 38]

$$w(AB) \leq w(B)\,\|A\|. \tag{1.6}$$

As a consequence of the above, we have [9, p. 39]:

If $A$ is a normal operator commuting with $B$, then

$$w(AB) \leq w(A)\,w(B). \tag{1.7}$$

For other results and historical comments on the above see [9, pp. 39–41].

For two bounded linear operators $A, B$ in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we define the functional

$$\mu(A,B) := \sup_{\|x\|=1}\{\|Ax\|\,\|Bx\|\}\,(\geq 0). \tag{1.8}$$

It is obvious that $\mu$ is symmetric and sub-additive in each variable, $\mu(A,A)=\|A\|^2$, $\mu(A,I) = \|A\|$, where $I$ is the identity operator, $\mu(\alpha A, \beta B) = |\alpha\beta|\,\mu(A,B)$ and $\mu(A,B) \leq \|A\|\,\|B\|$. We also have the following inequalities

$$\mu(A,B) \geq w(B^*A) \tag{1.9}$$

and

$$\mu(A,B)\,\|A\|\,\|B\| \geq \mu(AB, BA). \tag{1.10}$$

Inequality (1.9) follows by the Schwarz inequality $\|Ax\|\|Bx\|\geq|\langle Ax, Bx\rangle|$, $x\in H$, while (1.10) can be obtained by multiplying the inequalities $\|ABx\| \leq \|A\|\,\|Bx\|$ and $\|BAx\| \leq \|B\|\,\|Ax\|$.

From (1.9) we also get

$$\|A\|^2 \geq \mu(A, A^*) \geq w\left(A^2\right) \quad \text{for any } A. \tag{1.11}$$

Motivated by the above results we establish in this paper several inequalities for the functional $\mu(\cdot,\cdot)$ under various assumptions for the operators involved, including operators satisfying the uniform $(\alpha,\beta)$ −property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive.

## 2. General inequalities

The following result concerning some general power operator inequalities may be stated:

**Theorem 2.1** *For any $A, B \in B(H)$ and $r \geq 1$ we have the inequality*

$$\mu^r(A,B) \leq \frac{1}{2}\|(A^*A)^r + (B^*B)^r\|. \tag{2.1}$$

*The constant $\frac{1}{2}$ is best possible.*

**Proof.** Using the arithmetic mean - geometric mean inequality and the convexity of the function $f(t) = t^r$ for $r \geq 1$ and $t \geq 0$ we have successively

$$\begin{aligned} \|Ax\|\,\|Bx\| &\leq \frac{1}{2}\left[\langle A^*Ax, x\rangle + \langle B^*Bx, x\rangle\right] \\ &\leq \left[\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right]^{\frac{1}{r}} \end{aligned} \tag{2.2}$$

for any $x \in H$.

It is well known that if $P$ is a positive operator, then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [13])

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle. \tag{2.3}$$

Applying this inequality to the positive operators $A^*A$ and $B^*B$ we deduce that

$$\left[\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right]^{\frac{1}{r}} \leq \left\langle \frac{\left[(A^*A)^r + (B^*B)^r\right]x}{2}, x\right\rangle^{\frac{1}{r}} \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

Now, on making use of the inequalities (2.2) and (2.4) we get

$$\|Ax\|\,\|Bx\| \leq \left\langle \frac{\left[(A^*A)^r + (B^*B)^r\right]x}{2}, x\right\rangle^{\frac{1}{r}} \tag{2.5}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we obtain the desired result (2.1).

For $r = 1$ and $B = A$ we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). ∎

**Corollary 2.1** *For any $A \in B(H)$ and $r \geq 1$ we have the inequality*

$$\mu^r(A, A^*) \leq \frac{1}{2} \|(A^*A)^r + (AA^*)^r\| \tag{2.6}$$

*and the inequality*

$$\|A\|^r \leq \frac{1}{2} \|(A^*A)^r + I\|, \tag{2.7}$$

*respectively.*

The following similar result for powers of operators can be stated as well:

**Theorem 2.2** *For any $A, B \in B(H)$, any $\alpha \in (0, 1t)$ and $r \geq 1$ we have the inequality*

$$\mu^{2r}(A, B) \leq \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (B^*B)^{r/(1-\alpha)} \right\|. \tag{2.8}$$

*The inequality is sharp.*

**Proof.** Observe that, for any $\alpha \in (0, 1)$ we have

$$\begin{aligned} \|Ax\|^2 \|Bx\|^2 &= \langle (A^*A)x, x\rangle \langle (B^*B)x, x\rangle \\ &= \left\langle \left[(A^*A)^{1/\alpha}\right]^{\alpha} x, x \right\rangle \left\langle \left[(B^*B)^{1/(1-\alpha)}\right]^{1-\alpha} x, x \right\rangle, \end{aligned} \tag{2.9}$$

where $x \in H$.

It is well known that (see for instance [13]), if $P$ is a positive operator and $q \in (0, 1)$, then

$$\langle P^q x, x\rangle \leq \langle Px, x\rangle^q. \tag{2.10}$$

Applying this property to the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$, where $\alpha \in (0, 1)$, we have

$$\begin{aligned} &\left\langle \left[(A^*A)^{1/\alpha}\right]^{\alpha} x, x \right\rangle \left\langle \left[(B^*B)^{1/(1-\alpha)}\right]^{1-\alpha} x, x \right\rangle \\ &\qquad \leq \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^{\alpha} \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^{1-\alpha} \end{aligned} \tag{2.11}$$

for any $x \in H$ with $\|x\| = 1$.

Now, by using the weighted arithmetic mean-geometric mean inequality, i.e.,

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b, \quad \text{where } \alpha \in (0, 1) \text{ and } a, b \geq 0,$$

we get

$$\left\langle (A^*A)^{1/\alpha} x, x \right\rangle^{\alpha} \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^{1-\alpha} \le \alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, by the elementary inequality

$$\alpha a + (1-\alpha) b \le (\alpha a^r + (1-\alpha) b^r)^{1/r}, \quad \text{where } \alpha \in (0,1) \text{ and } a, b \ge 0;$$

we have successively

$$\begin{aligned} &\alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle \\ &\le \left[ \alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^r + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^r \right]^{\frac{1}{r}} \\ &\le \left[ \alpha \cdot \left\langle (A^*A)^{r/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{r/(1-\alpha)} x, x \right\rangle \right]^{\frac{1}{r}}, \end{aligned} \tag{2.13}$$

for any $x \in H$ with $\|x\| = 1$, where for the last inequality we have used the property (2.3) for the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$.

Now, by making use of the identity (2.9) and the inequalities (2.11)-(2.13), we get

$$\|Ax\|^2 \|Bx\|^2 \le \left[ \left\langle \left[ \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (B^*B)^{r/(1-\alpha)} \right] x, x \right\rangle \right]^{\frac{1}{r}}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we deduce the desired result (2.8).

Notice that the inequality is sharp since for $r = 1$ and $B = A$ we get in both sides of (2.8) the same quantity $\|A\|^4$. ■

**Corollary 2.2** *For any $A \in B(H)$, any $\alpha \in (0,1)$ and $r \ge 1$, we have the inequalities*

$$\mu^{2r}(A, A^*) \le \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (AA^*)^{r/(1-\alpha)} \right\|,$$

$$\|A\|^{2r} \le \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot I \right\|$$

*and*

$$\|A\|^{4r} \le \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (A^*A)^{r/(1-\alpha)} \right\|,$$

*respectively.*

The following reverse of inequality (1.9) may be stated as well:

**Theorem 2.3** *For any $A, B \in B(H)$ we have the inequality*

$$(0 \leq)\, \mu(A, B) - w(B^*A) \leq \frac{1}{2}\|A - B\|^2 \tag{2.14}$$

*and the inequality*

$$\mu\left(\frac{A+B}{2}, \frac{A-B}{2}\right) \leq \frac{1}{2}w(B^*A) + \frac{1}{4}\|A - B\|^2, \tag{2.15}$$

*respectively.*

**Proof.** We have

$$\begin{aligned} \|Ax - Bx\|^2 &= \|Ax\|^2 + \|Bx\|^2 - 2\mathrm{Re}\langle B^*Ax, x\rangle \\ &\geq 2\|Ax\|\|Bx\| - 2|\langle B^*Ax, x\rangle|, \end{aligned} \tag{2.16}$$

for any $x \in H$, $\|x\| = 1$, which gives the inequality

$$\|Ax\|\|Bx\| \leq |\langle B^*Ax, x\rangle| + \frac{1}{2}\|Ax - Bx\|^2,$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we deduce the desired result (2.14).

By the parallelogram identity in the Hilbert space $H$, we also have

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \frac{1}{2}\left(\|Ax + Bx\|^2 + \|Ax - Bx\|^2\right) \\ &\geq \|Ax + Bx\|\|Ax - Bx\|, \end{aligned}$$

for any $x \in H$.

Combining this inequality with the first part of (2.16), we get

$$\|Ax + Bx\|\|Ax - Bx\| \leq \|Ax - Bx\|^2 + 2|\langle B^*Ax, x\rangle|,$$

for any $x \in H$. Taking the supremum in this inequality over $\|x\| = 1$ we deduce the desired result (2.15). ∎

**Corollary 2.3** *Let $A \in B(H)$. If*

$$\mathrm{Re}(A) := \frac{A + A^*}{2} \text{ and } \mathrm{Im}(A) := \frac{A - A^*}{2i}$$

*are the real and imaginary parts of $A$, then we have the inequality*

$$(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) \leq 2 \cdot \|\mathrm{Im}(A)\|^2$$

*and*

$$\mu(\mathrm{Re}(A), \mathrm{Im}(A)) \leq \frac{1}{2}w\left(A^2\right) + \|\mathrm{Im}(A)\|^2,$$

*respectively.*

*Moreover, we have*

$$(0 \leq)\, \mu(\mathrm{Re}(A), \mathrm{Im}(A)) - w(\mathrm{Re}(A)\mathrm{Im}(A)) \leq \frac{1}{2}\|A\|^2.$$

**Corollary 2.4** *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we have the inequality* (see also [6])

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{2|\lambda|} \|A - \lambda I\|^2 . \tag{2.17}$$

For a bounded linear operator $T$ consider the quantity

$$\ell(T) := \inf_{\|x\|=1} \|Tx\| .$$

We can state the following result as well.

**Theorem 2.4** *For any $A, B \in B(H)$ with $A \neq B$ and such that $\ell(B) \geq \|A - B\|$ we have*

$$(0 \leq) \mu^2(A, B) - w^2(B^* A) \leq \|A\|^2 \|A - B\|^2 . \tag{2.18}$$

**Proof.** Denote $r := \|A - B\| > 0$. Then for any $x \in H$ with $\|x\| = 1$ we have $\|Bx\| \geq r$ and by the first part of (2.16) we can write that

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \leq 2|\langle B^* Ax, x\rangle| \tag{2.19}$$

for any $x \in H$ with $\|x\| = 1$.

On the other hand, we have

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \geq 2 \cdot \|Ax\| \sqrt{\|Bx\|^2 - r^2} \tag{2.20}$$

for any $x \in H$ with $\|x\| = 1$.

Combining (2.19) with (2.20), we deduce

$$\|Ax\| \sqrt{\|Bx\|^2 - r^2} \leq |\langle B^* Ax, x\rangle|$$

which is clearly equivalent to

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^* Ax, x\rangle|^2 + \|Ax\|^2 \|A - B\|^2 \tag{2.21}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum in (2.21) over $x \in H$ with $\|x\| = 1$, we deduce the desired inequality (2.18). ■

**Corollary 2.5** *For any $A \in B(H)$ a non-self-adjoint operator in $B(H)$ and such that $\ell(A^*) \geq \|\mathrm{Im}(A)\|$ we have*

$$(0 \leq) \mu^2(A, A^*) - w^2\left(A^2\right) \leq 4 \cdot \|A\|^2 \|\mathrm{Im}(A)\|^2 . \tag{2.22}$$

**Corollary 2.6** *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $|\lambda| \geq \|A - \lambda I\|$ we have the inequality* (see also [6])

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{|\lambda|^2} \cdot \|A\|^2 \|A - \lambda I\|^2$$

*or, equivalently,*

$$(0 \leq) \sqrt{1 - \frac{\|A - \lambda I\|^2}{|\lambda|^2}} \leq \frac{w(A)}{\|A\|} (\leq 1).$$

## 3. Inequalities for operators satisfying the uniform $(\alpha, \beta)$-property

The following result that may be of interest in itself holds:

**Lemma 3.1** *Let $T \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. The following statements are equivalent:*

(i) *We have*

$$\operatorname{Re} \langle \beta y - Tx, Tx - \alpha y \rangle \geq 0 \tag{3.1}$$

*for any $x, y \in H$ with $\|x\| = \|y\| = 1$;*

(ii) *We have*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\alpha - \beta| \tag{3.2}$$

*for any $x, y \in H$ with $\|x\| = \|y\| = 1$.*

**Proof.** This follows by the following identity

$$\operatorname{Re} \langle \beta y - Tx, Tx - \alpha y \rangle = \frac{1}{4} |\alpha - \beta|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\|^2,$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$. ∎

**Remark 3.1** *For any operator $T \in B(H)$ if we choose $\alpha = a \|T\| (1 + 2i)$ and $\beta = a \|T\| (1 - 2i)$ with $a \geq 1$, then*

$$\frac{\alpha + \beta}{2} = a \|T\| \quad \text{and} \quad \frac{|\alpha - \beta|}{2} = 2a \|T\|$$

*showing that*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \|Tx\| + \left| \frac{\alpha + \beta}{2} \right| \leq \|T\| + a \|T\|$$

$$\leq 2a \|T\| = \frac{1}{2} \cdot |\alpha - \beta|,$$

*that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$, i.e., $T$ satisfies condition* (3.1) *with the scalars $\alpha$ and $\beta$ given above.*

**Definition 3.1** For given $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and $y \in H$ with $\|y\| = 1$, we say that the operator $T \in B(H)$ has the $(\alpha, \beta, y)$-property if either (3.1) or, equivalently, (3.2) holds true for any $x \in H$ with $\|x\| = 1$. Moreover, if $T$ has the $(\alpha, \beta, y)$-property for any $y \in H$ with $\|y\| = 1$, then we say that this operator has the uniform $(\alpha, \beta)$-property.

**Remark 3.2** The above Remark 3.1 shows that any bounded linear operator has the uniform $(\alpha, \beta)$-property for infinitely many $(\alpha, \beta)$ appropriately chosen. For a given operator satisfying an $(\alpha, \beta)$-property, it is an open problem to find the possibly nonzero lower bound for the quantity $|\alpha - \beta|$.

The following results may be stated:

**Theorem 3.1** *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$ . For $y \in H$ with $\|y\| = 1$ assume that $A^*$ has the $(\alpha, \beta, y)$-property while $B^*$ has the $(\gamma, \delta, y)$-property. Then*

$$|\|Ay\| \|By\| - \|BA^*\|| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{3.3}$$

*Moreover, if $A^*$ has the uniform $(\alpha, \beta)$-property and $B^*$ has the uniform $(\gamma, \delta)$-property, then*

$$|\mu(A, B) - \|BA^*\|| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{3.4}$$

**Proof.** $A^*$ has the $(\alpha, \beta, y)$-property while $B^*$ has the $(\gamma, \delta, y)$-property, then on making use of Lemma 3.1 we have that

$$\left\| A^*x - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\beta - \alpha| \quad \text{and} \quad \left\| B^*z - \frac{\gamma + \delta}{2} \cdot y \right\| \leq \frac{1}{2} |\gamma - \delta|$$

for any $x, z \in H$ with $\|x\| = \|z\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [7, p. 43]:

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0 \tag{3.5}$$

or, equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha| , \qquad \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma| . \tag{3.6}$$

Then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma| . \tag{3.7}$$

Applying (3.7) for $u = A^*x$, $v = B^*z$ and $e = y$ we deduce

$$|\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle| \le \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.8}$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$, which is an inequality of interest in itself.

Observing that

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \le |\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle|,$$

then by (3.8) we deduce the inequality

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \le \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$. This is equivalent with the following two inequalities

$$|\langle BA^*x, z\rangle| \le |\langle x, Ay\rangle \langle z, By\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.9}$$

and

$$|\langle x, Ay\rangle \langle z, By\rangle| \le |\langle BA^*x, z\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.10}$$

for any $x, z \in H, \|x\| = \|z\| = 1$.

Taking the supremum over $x, z \in H$, $\|x\| = \|z\| = 1$, in (3.9) and (3.10) we get the inequalities

$$\|BA^*\| \le \|Ay\| \|By\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.11}$$

and

$$\|Ay\| \|By\| \le \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.12}$$

which are clearly equivalent to (3.3).

Now, if $A^*$ has the uniform $(\alpha, \beta)$-property and $B^*$ has the uniform $(\gamma, \delta)$-property, then the inequalities (3.11) and (3.12) hold for any $y \in H$ with $\|y\| = 1$. Taking the supremum over $y \in H$ with $\|y\| = 1$ in these inequalities we deduce

$$\|BA^*\| \le \mu (A, B) + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$\mu (A, B) \le \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

which are equivalent to (3.4). ∎

**Corollary 3.7** *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that $A$ has the $(\alpha, \beta, y)$-property while $A^*$ has the $(\gamma, \delta, y)$-property. Then*

$$\left| \|A^* y\| \, \|Ay\| - \left\| A^2 \right\| \right| \leq \frac{1}{4} |\beta - \alpha| \, |\gamma - \delta| \, .$$

*Moreover, if $A$ has the uniform $(\alpha, \beta)$-property and $A^*$ has the uniform $(\gamma, \delta)$-property, then*

$$\left| \mu (A, A^*) - \left\| A^2 \right\| \right| \leq \frac{1}{4} |\beta - \alpha| \, |\gamma - \delta| \, .$$

The following results may be stated as well:

**Theorem 3.2** *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$. For $y \in H$ with $\|y\| = 1$ assume that $A^*$ has the $(\alpha, \beta, y)$-property while $B^*$ has the $(\gamma, \delta, y)$-property. Then*

$$|\|Ay\|\|By\| - \|BA^*\|| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha||\delta - \gamma|}{\sqrt{|\beta + \alpha||\delta + \gamma|}} \sqrt{(\|A\| + \|Ay\|)(\|B\| + \|By\|)}. \tag{3.13}$$

*Moreover, if $A^*$ has the uniform $(\alpha, \beta)$-property and $B^*$ has the uniform $(\gamma, \delta)$-property, then*

$$|\mu (A, B) - \|BA^*\|| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{\|A\| \, \|B\|}. \tag{3.14}$$

**Proof.** We make use of the following inequality obtained by the author in [5] (see also [7, p. 65]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$ and such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0$$

or, equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha| \, , \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma| \, .$$

Then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{(\|u\| + |\langle u, e \rangle|) (\|v\| + |\langle v, e \rangle|)}. \tag{3.15}$$

Applying (3.15) for $u = A^* x$, $v = B^* z$ and $e = y$ we deduce

$$|\langle BA^* x, z \rangle - \langle x, Ay \rangle \langle By, z \rangle| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{(\|A^* x\| + |\langle x, Ay \rangle|) (\|B^* z\| + |\langle z, By \rangle|)}$$

for any $x, y, z \in H$, $\|x\| = \|y\| = \|z\| = 1$.

Now, on making use of a similar argument to the one from the proof of Theorem 3.1, we deduce the desired results (3.13) and (3.14). The details are omitted. ■

**Corollary 3.8** *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$. For $y \in H$ with $\|y\| = 1$ assume that $A$ has $(\alpha, \beta, y)$-property while $A^*$ has the $(\gamma, \delta, y)$-property. Then*

$$\left|\|A^*y\|\,\|Ay\| - \left\|A^2\right\|\right| \leq \frac{1}{4} \cdot \frac{|\beta-\alpha|\,|\delta-\gamma|}{\sqrt{|\beta+\alpha|\,|\delta+\gamma|}} \sqrt{(\|A\| + \|A^*y\|)\,(\|A\| + \|Ay\|)}.$$

*Moreover, if $A$ has the uniform $(\alpha, \beta)$-property and $A^*$ has the uniform $(\gamma, \delta)$-property, then*

$$\left|\mu\left(A, A^*\right) - \left\|A^2\right\|\right| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha|\,|\delta - \gamma|}{\sqrt{|\beta + \alpha|\,|\delta + \gamma|}} \|A\|.$$

## 4. The transform $C_{\alpha,\beta}(\cdot, \cdot)$ and other inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha,\beta}(T, U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$C_{\alpha,\beta}(T) := (T^* - \bar{\alpha}I)(\beta I - T) = C_{\alpha,\beta}(T, I),$$

where $I$ is the identity operator, which has been introduced in [8] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator $T$ on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Using the following identity

$$\begin{aligned} \operatorname{Re}\langle C_{\alpha,\beta}(T, U)x, x\rangle &= \operatorname{Re}\langle C_{\beta,\alpha}(T, U)x, x\rangle \\ &= \frac{1}{4}|\beta - \alpha|^2 \|Ux\|^2 - \left\|Tx - \frac{\alpha + \beta}{2} \cdot Ux\right\|^2, \end{aligned} \tag{4.1}$$

that holds for any scalars $\alpha, \beta$ and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

**Lemma 4.2** *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}(T, U)$ (or, equivalently, $C_{\beta,\alpha}(T, U)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\| Tx - \frac{\alpha+\beta}{2} \cdot Ux \right\| \leq \frac{1}{2} |\beta - \alpha| \|Ux\| \tag{4.2}$$

*for any* $x \in H$.

As a consequence of the above lemma, we can state

**Corollary 4.9** *Let* $\alpha, \beta \in \mathbb{C}$ *and* $T, U \in B(H)$. *If* $C_{\alpha,\beta}(T, U)$ *is accretive, then*

$$\left\| T - \frac{\alpha+\beta}{2} \cdot U \right\| \leq \frac{1}{2} |\beta - \alpha| \|U\|. \tag{4.3}$$

**Remark 4.3** *In order to give examples of linear operators* $T, U \in B(H)$ *and numbers* $\alpha, \beta \in \mathbb{C}$ *such that the transform* $C_{\alpha,\beta}(T, U)$ *is accretive, it suffices to select two bounded linear operator* $S$ *and* $V$ *and the complex numbers* $z, w$ $(w \neq 0)$ *with the property that* $\|Sx - zVx\| \leq |w| \|Vx\|$ *for any* $x \in H$, *and, by choosing* $T = S$, $U = V$, $\alpha = \frac{1}{2}(z + w)$ *and* $\beta = \frac{1}{2}(z - w)$, *we observe that* $T$ *and* $U$ *satisfy* (4.2), *i.e.,* $C_{\alpha,\beta}(T, U)$ *is accretive.*

We are able now to give the following result concerning other reverse inequalities for the case when the involved operators satisfy the accretivity property described above.

**Theorem 4.1** *Let* $\alpha, \beta \in \mathbb{C}$ *and* $A, B \in B(H)$. *If* $C_{\alpha,\beta}(A, B)$ *is accretive, then*

$$(0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|B\|^4. \tag{4.4}$$

*Moreover, if* $\alpha + \beta \neq 0$, *then*

$$(0 \leq) \mu(A, B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|B\|^2. \tag{4.5}$$

*In addition, if* $\mathrm{Re}\left(\alpha\bar{\beta}\right) > 0$ *and* $B^*A \neq 0$, *then also*

$$(1 \leq) \frac{\mu(A, B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\mathrm{Re}\left(\alpha\bar{\beta}\right)}} \tag{4.6}$$

*and*

$$(0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\mathrm{Re}\left(\alpha\bar{\beta}\right)}\right) w(B^*A) \|B\|^2, \tag{4.7}$$

*respectively.*

**Proof.** By Lemma 4.2, since $C_{\alpha,\beta}(A,B)$ is accretive, then

$$\left\| Ax - \frac{\alpha+\beta}{2} \cdot Bx \right\| \leq \frac{1}{2} |\beta - \alpha| \|Bx\| \tag{4.8}$$

for any $x \in H$.

We use the following reverse of the Schwarz inequality in inner product spaces obtained by the author in [3] (see also [7, p. 4]):

If $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $u, v \in H$ are such that

$$\operatorname{Re} \langle \Gamma v - u, u - \gamma v \rangle \geq 0 \tag{4.9}$$

or, equivalently,

$$\left\| u - \frac{\gamma+\Gamma}{2} \cdot v \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|v\|, \tag{4.10}$$

then

$$0 \leq \|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|v\|^4. \tag{4.11}$$

Now, by making use of (4.11) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$, we can write the inequality

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^* Ax, x\rangle|^2 + \frac{1}{4} |\beta - \alpha|^2 \|Bx\|^4,$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ in this inequality produces the desired result (4.4).

Now, by using the result from [5] (see also [7, p. 29]) namely:

If $\gamma, \Gamma \in \mathbb{K}$ with $\gamma + \Gamma \neq 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, then

$$0 \leq \|u\| \|v\| - |\langle u, v\rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|v\|^2. \tag{4.12}$$

Now, by making use of (4.12) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and using the same procedure outlined above, we deduce the second inequality (4.5).

The inequality (4.6) follows from the result presented below obtained in [4] (see also [7, p. 21]):

If $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, then

$$\|u\| \|v\| \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})}} |\langle u, v\rangle|, \tag{4.13}$$

by choosing $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and taking the supremum over $\|x\| = 1$.

Finally, by making use of the inequality (see [6])

$$\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2 \le \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right) |\langle u, v\rangle| \|v\|^2 \tag{4.14}$$

that is valid provided $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, we obtain the last inequality (4.7). The details are omitted. ■

**Remark 4.4** *Let $M, m > 0$ and $A, B \in B(H)$. If $C_{m,M}(A, B)$ is accretive, then*

$$\begin{aligned}
(0 \le)\, \mu^2(A, B) - w^2(B^*A) &\le \frac{1}{4} \cdot (M - m)^2 \|B\|^4, \\
(0 \le)\, \mu(A, B) - w(B^*A) &\le \frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \|B\|^2, \\
(1 \le)\, \frac{\mu(A, B)}{w(B^*A)} &\le \frac{1}{2} \cdot \frac{m + M}{\sqrt{mM}} \\
(0 \le)\, \mu^2(A, B) - w^2(B^*A) &\le \left(\sqrt{M} - \sqrt{m}\right)^2 w(B^*A) \|B\|^2,
\end{aligned}$$

*respectively.*

**Corollary 4.10** *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$. If $C_{\alpha,\beta}(A, A^*)$ is accretive, then*

$$(0 \le)\, \mu^2(A, A^*) - w^2\left(A^2\right) \le \frac{1}{4} \cdot |\beta - \alpha|^2 \|A\|^4.$$

*Moreover, if $\alpha + \beta \ne 0$, then*

$$(0 \le)\, \mu(A, A^*) - w\left(A^2\right) \le \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|A\|^2.$$

*In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right) > 0$ and $A^2 \ne 0$, then also*

$$(1 \le)\, \frac{\mu(A, A^*)}{w(A^2)} \le \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}}$$

*and*

$$(0 \le)\, \mu^2(A, A^*) - w^2\left(A^2\right) \le \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right) w\left(A^2\right) \|A\|^2,$$

*respectively.*

**Remark 4.5** *In a similar manner, if $N, n > 0$, $A \in B(H)$ and $C_{n,N}(A, A^*)$ is accretive, then*

$$\begin{aligned}
(0 \le)\, \mu^2(A, A^*) - w^2\left(A^2\right) &\le \frac{1}{4} \cdot (N - n)^2 \|A\|^4, \\
(0 \le)\, \mu(A, A^*) - w\left(A^2\right) &\le \frac{1}{4} \cdot \frac{(N - n)^2}{n + N} \|A\|^2, \\
(1 \le)\, \frac{\mu(A, A^*)}{w(A^2)} &\le \frac{1}{2} \cdot \frac{n + N}{\sqrt{nN}} \quad \text{(for } A^2 \ne 0\text{)}
\end{aligned}$$

*and*

$$(0 \leq) \mu^2 (A, A^*) - w^2 (A^2) \leq \left(\sqrt{N} - \sqrt{n}\right)^2 w (A^2) \|A\|^2 ,$$

*respectively.*

**References**

[1] Dragomir, S.S., *A generalisation of Grüss' inequality in inner product spaces and applications*, J. Math. Anal. Applic., 237 (1999), 74-82.

[2] Dragomir, S.S., *Some Grüss type inequalities in inner product spaces,* J. Inequal. Pure & Appl. Math., 4 (2) (2003), Article 42.

[3] Dragomir, S.S., *A counterpart of Schwarz's inequality in inner product spaces,* East Asian Math. J., 20 (1) (2004), 1-10.

[4] Dragomir, S.S., *Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces,* J. Inequal. Pure & Appl. Math., 5 (3) (2004), Article 76.

[5] Dragomir, S.S., *New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces,* Australian J. Math. Anal. & Appl., 1 (2004), Issue 1, Article 1, 1-18.

[6] Dragomir, S.S., *Reverses of the Schwarz inequality generalising a Klamkin-McLenaghan result,* Bull. Austral. Math. Soc., 73 (1) (2006), 69-78.

[7] Dragomir, S.S., *Advances in Inequalities of the Schwarz, Gruss and Bessel Type in Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2005.

[8] Dragomir, S.S., *New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces,* Linear Algebra Appl., 428 (2008), 2750-2760.

[9] Gustafson, K.E. and Rao, D.K.M., *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[10] Halmos, P.R., *A Hilbert Space Problem Book,* Springer-Verlag, New York, Heidelberg, Berlin, Second edition, 1982.

[11] Kittaneh, F., *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., 158 (1) (2003), 11-17.

[12] Kittaneh, F., *Numerical radius inequalities for Hilbert space operators,* Studia Math., 168 (1) (2005), 73-80.

[13] Kittaneh, F., *Notes on some inequalities for Hilbert space operators,* Publ. Res. Inst. Math. Sci., 24 (1988), 283–293.

Accepted: 5.03.2009