A NOTE ON CONTINUED FRACTIONS AND $_3\psi_3$ SERIES

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Abstract. The present paper concerns with the continued fraction representation for $_3\psi_3$ basic bilateral hypergeometric series. Several special cases are also discussed.

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1. Introduction

Continued fraction has been centre of attraction for applied mathematicians as well as pure mathematicians of previous centuries. In previous centuries there are so many results, which are established in terms of continued fraction. It is also a tool which act as a bridge between pure and applied mathematicians. So, the attraction of continued fraction for today's mathematicians has also amplified. R.P. Agrawal [1] has given the continued fraction representation for the basic hypergeometric series $_2\phi_1$. R.Y. Denis [4] has also developed a continued fraction representation for the ratio of two basic bilateral hypergeometric series $_2\psi_2$. There are also a number of researchers like R.P. Agarwal [2], G.E. Andrews and D. Bowman, [3], R.Y. Denis and S.N. Singh [5], P. Rai [7], S.N. Singh [8], etc., who have established a number of interesting results for hypergeometric function, basic hypergeometric function and basic bilateral hypergeometric function in terms of continued fraction.In this paper we are developing results for $_3\psi_3$ ratio's in terms of continued fraction and we also deduce some interesting special cases with the help of these results. These results may be helpful for further research in this area to developed more results in terms of continued fraction.

S.N. Singh [8] has derived an interesting transformation formula which transforms a basic bilateral hypergeometric function into basic hypergeometric function. In this paper, we shall use this transformation formula for the development of our main results. We shall use the following results in order to establish main results.

$$
\begin{aligned}
&{}_{r+3}\psi_{r+3}\left(\begin{matrix} \frac{a}{b}, & cq, & dq, & \frac{b_1}{b}q^{m_1}, & ...., & \frac{b_r}{b}q^{m_r} & \\ & & & & & & ;q, \quad z \\ \frac{q}{b}, & c, & d, & \frac{b_1}{b}, & ...., & \frac{b_r}{b} & \end{matrix}\right) \\
(1.1)\qquad &= \frac{\left(q, \frac{bq}{az}, \frac{az}{b}, \frac{q}{a}; q\right)_\infty}{\left(\frac{q}{b}, \frac{q}{az}, az, \frac{bq}{a}; q\right)_\infty} \frac{(1-bc)(1-bd)(b_1;q)_{m_1}...(b_r;q)_{m_r}}{(1-c)(1-d)\left(\frac{b_1}{b};q\right)_{m_1}...\left(\frac{b_r}{b};q\right)_{m_r}} \\
&\times {}_{r+3}\phi_{r+2}\left(\begin{matrix} a, & bcq, & bdq, & b_1q^{m_1}, & ...., & b_rq^{m_r} & \\ & & & & & & ;q, \quad z \\ & bc, & bd, & b_1, & ...., & b_r & \end{matrix}\right) [6;(11)]
\end{aligned}
$$

$$
(1.2)\qquad \frac{{}_3\phi_2\left(\begin{matrix} aq, & b, & c & \\ & & & ;q, \quad \frac{de}{abcq} \\ & d, & e & \end{matrix}\right)}{{}_3\phi_2\left(\begin{matrix} a, & b, & c & \\ & & & ;q, \quad \frac{de}{abc} \\ & d, & e & \end{matrix}\right)} = 1 + \frac{A_0}{(1{-}a)\left(1{-}\frac{de}{abcq}\right)+} \frac{B_0}{1+}\cdots \frac{A_n}{(1{-}a)\left(1{-}\frac{de}{abcq}\right)+} \frac{B_n}{1+}\cdots [1;(2)],
$$

where

$$
\begin{aligned}
A_n &= (de/abcq)(1-bq^n)(1-cq^n), \quad n = 0,1,2,... \\
B_n &= a[1-(dq^n/a)][1-(eq^n/a)], \quad n = 0,1,2,...
\end{aligned}
$$

$$
(1.3)\qquad \frac{{}_3\phi_2\left(\begin{matrix} \frac{xq}{a_1a_2}, & a_3, & a_4 & \\ & & & ;q, \quad \frac{xq}{a_3a_4} \\ & \frac{xq}{a_1}, & \frac{xq}{a_2} & \end{matrix}\right)}{{}_3\phi_2\left(\begin{matrix} \frac{xq^2}{a_1a_2}, & a_3, & a_4 & \\ & & & ;q, \quad \frac{xq}{a_3a_4} \\ & \frac{xq^2}{a_1}, & \frac{xq^2}{a_2} & \end{matrix}\right)} = \frac{T(x)P(x)}{S(x)Q(x)} + \cfrac{R(x)/S(x)Q(x)}{\frac{T(xq)P(xq)}{S(xq)Q(xq)} + \cfrac{R(xq)/S(xq)Q(xq)}{\frac{T(xq^2)P(xq^2)}{S(xq^2)Q(xq^2)} + ...}} \; [2;(3.15)],
$$

where

$$\begin{aligned} T(x) &= (1-xq^2/a_1)(1-xq^2/a_2)(1-xq^2/a_3/a_4), \\ S(x) &= (xq/a_1)_2(xq/a_2)_2(xq/a_3a_4)_2, \\ P(x) &= 1-xqS_1+\{(S_3-S_4)(1+q)+S_4q^2\}(x^2q^2+x^4q^5S_4) \\ &\quad +x^5q^7S_4^2S_1-x^6q^9S_4^3, \end{aligned}$$

$$\begin{aligned} S_M(x) &= (1-xq^{1+M}/a_1)(xq^{1+M}/a_2)_2(xq/a_3a_4)_2(xq/a_1a_2)_2, \\ Q(x) &= 1-x^2q^4S_4, \\ R(x) &= xq(1-x^2q^2S_4)(1-xq^2/a_1a_2)(1-xq^2/a_1a_3). \end{aligned}$$

2. Definition and notations

A continued fraction is a ratio of the type:

$$a_1+\frac{a_2}{a_3+}\,\frac{a_4}{a_5+}\,\frac{a_6}{a_7+}\,\frac{a_8}{a_9+}\,\frac{a_{10}}{a_{11}+}...,$$

where $a_1, a_2, a_3, a_4, ...$ are real or complex numbers.

A Basic hypergeometric series is denoted and defined as:

$${}_A\phi_B\left(\begin{matrix}(a)\\ \\ (b)\end{matrix};q,\quad z\right)=\sum_{n=0}^{\infty}\frac{[(a);q]_n}{[(b);q]_n}\frac{z^n}{(q;q)_n},\quad (|z|<1,|q|<1),$$

where (a) represents sequence of A parameters, (b) represents sequence of B parameters.

$$(a;q)_n=\begin{cases}(1-a)(1-aq)(1-aq^2)......(1-aq^{n-1}), & \text{when } n\neq 0;\\ 1, & \text{n=0.}\end{cases}$$

A Basic bilateral hypergeometric series is denoted and defined as:

$${}_r\psi_r\left(\begin{matrix}(a)\\ \\ (b)\end{matrix};q,\quad z\right)=\sum_{n=-\infty}^{\infty}\frac{[(a);q]_n}{[(b);q]_n}z^n,\quad (|z|<1,|q|<1),$$

where (a) and (b) represent sequences of r parameters.

All the parameters and variable may be real or complex numbers. The other notations appearing in this paper carry their usual meaning.

3. Main results

We shall establish the following results.

$$\frac{{}_3\psi_3\left(\begin{matrix}\frac{aq}{b}, & cq, & dq \\ & & & ;q, & \frac{1}{aq^3} \\ \frac{q}{b}, & c, & d\end{matrix}\right)}{{}_3\psi_3\left(\begin{matrix}\frac{a}{b}, & cq, & dq \\ & & & ;q, & \frac{1}{aq^2} \\ \frac{q}{b}, & c, & d\end{matrix}\right)} \tag{3.1}$$

$$= \frac{(a-1)}{(a-b)}\left[1+\frac{A_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\frac{B_0}{1+}\cdots\cdots\frac{A_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\frac{B_n}{1+}\cdots\right],$$

where

$$\begin{aligned} A_n &= \frac{1}{aq^3}(1-bcq^{n+1})(1-bdq^{n+1}), \quad n=0,1,2,\ldots \\ B_n &= a\left(1-\frac{bcq^n}{a}\right)\left(1-\frac{bdq^n}{a}\right), \quad n=0,1,2,\ldots \end{aligned}$$

and

$$\frac{{}_3\psi_3\left(\begin{matrix}\frac{azq}{a_1a_2b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} \\ & & & ;q, & \frac{a_1a_2}{azq^3} \\ \frac{q}{b}, & \frac{azq}{a_1b}, & \frac{azq}{a_2b}\end{matrix}\right)}{{}_3\psi_3\left(\begin{matrix}\frac{azq^2}{a_1a_2b}, & \frac{azq^3}{a_1b}, & \frac{azq^3}{a_2b} \\ & & & ;q, & \frac{a_1a_2}{azq^5} \\ \frac{q}{b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b}\end{matrix}\right)} \tag{3.2}$$

$$= \frac{b(azq-ba_1a_2)}{(azq-a_1a_2)}\times\frac{(1-azq/a_1)(1-azq/a_2)(1-azq^2/a_1b)}{(1-azq/a_1b)(1-azq/a_2b)(1-azq^2/a_1)}\times\frac{(1-azq^2/a_2b)}{(1-azq^2/a_2)}$$

$$\times\left[\frac{T(az)P(az)}{S(az)Q(az)}+\cfrac{R(az)/S(az)Q(az)}{\frac{T(azq)P(azq)}{S(azq)Q(azq)}+\cfrac{R(azq)/S(azq)Q(azq)}{\frac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)}+\ldots}}\right],$$

where

$$\begin{aligned}
T(az) &= (1-azq^2/a_1)(1-azq^2/a_2)(1-azq^2/a_1/a_2),\\
S(az) &= (azq/a_1)_2(azq/a_2)_2(azqa_1a_2/a^2z^2q^4)_2,\\
P(az) &= 1-azqS_1+\{(S_3-S_4)(1+q)+S_4q^2\}(a^2z^2q^2+a^4z^4q^5S_4)\\
&\quad +a^5z^5q^7S_4^2S_1-a^6z^6q^9S_4^3,\\
S_M(az) &= (1-azq^{1+M}/a_1)(azq^{1+M}/a_2)_2(azqa_1a_2/a^2z^2q^4)_2(azq/a_1a_2)_2,\\
Q(az) &= 1-a^2z^2q^4S_4,\\
R(az) &= azq(1-a^2z^2q^2S_4)(1-azq^2/a_1a_2)(1-azq^2a_2/a_1azq^2).
\end{aligned}$$

4. Proof of main results

Proof of (3.1). Taking $r=0$ in (1.1), we get

$$(4.1)\quad {}_3\psi_3\left(\begin{matrix} a/b, & cq, & dq \\ q/b, & c, & d\end{matrix}\,;q,\ z\right)=\frac{(q,bq/az,az/b,q/a;q)_\infty}{(q/b,q/az,az,bq/a;q)_\infty}\times\frac{(1-bc)(1-bd)}{(1-c)(1-d)}\times{}_3\phi_2\left(\begin{matrix} a, & bcq, & bdq \\ & bc, & bd\end{matrix}\,;q,\ z\right)$$

Replacing a by aq in (4.1), we get

$$(4.2)\quad {}_3\psi_3\left(\begin{matrix} aq/b, & cq, & dq \\ q/b, & c, & d\end{matrix}\,;q,\ z\right)=\frac{(q,b/az,aqz/b,1/a;q)_\infty}{(q/b,1/az,aqz,b/a;q)_\infty}\times\frac{(1-bc)(1-bd)}{(1-c)(1-d)}\times{}_3\phi_2\left(\begin{matrix} aq, & bcq, & bdq \\ & bc, & bd\end{matrix}\,;q,\ z\right)$$

Taking $z=1/aq^2$ in (4.1) and $z=1/aq^3$ in (4.2) and then taking ratio of (4.1) and (4.2) and using result (1.2), we get

$$\frac{{}_3\psi_3\left(\begin{matrix} \frac{aq}{b}, & cq, & dq \\ \frac{q}{b}, & c, & d\end{matrix}\,;q,\ \frac{1}{aq^3}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{a}{b}, & cq, & dq \\ \frac{q}{b}, & c, & d\end{matrix}\,;q,\ \frac{1}{aq^2}\right)} =\frac{(a-1)}{(a-b)}\left[1+\frac{A_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{B_0}{1+}\cdots\frac{A_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{B_n}{1+}\cdots\right].$$

Proof of (3.2). Now, replacing a by azq/a_1a_2, bc by azq/a_1, bd by azq/a_2, c by azq/a_1b, d by azq/a_2b, z by a_1a_2/azq^3 in (4.1), we get

$$\begin{aligned}
&{}_3\psi_3\left(\begin{matrix} azq/a_1a_2b, & azq^2/a_1b, & azq^2/a_2b \\ q/b, & azq/a_1b, & azq/a_2b \end{matrix}\, ; q,\quad a_1a_2/azq^3\right)\\
&= \frac{(q, bq^3, 1/bq^2, a_1a_2/az; q)_\infty}{(q/b, q^3, 1/q^2, ba_1a_2/az; q)_\infty} \times \frac{(1-azq/a_1)(1-azq/a_2)}{(1-azq/a_1b)(1-azq/a_2b)}\\
&\times {}_3\phi_2\left(\begin{matrix} azq/a_1a_2, & azq^2/a_1, & azq^2/a_2 \\ & azq/a_1, & azq/a_2 \end{matrix}\, ; q,\quad a_1a_2/azq^3\right).
\end{aligned} \tag{4.3}$$

Replacing a by azq^2/a_1a_2, bc by azq^2/a_1, bd by azq^2/a_2, c by azq^2/a_1b, d by azq^2/a_2b, z by a_1a_2/azq^5 in (4.1), we get

$$\begin{aligned}
&{}_3\psi_3\left(\begin{matrix} azq^2/a_1a_2b, & azq^3/a_1b, & azq^3/a_2b \\ q/b, & azq^2/a_1b, & azq^2/a_2b \end{matrix}\, ; q,\quad a_1a_2/azq^5\right)\\
&= \frac{(q, bq^4, 1/bq^3, a_1a_2/azq; q)_\infty}{(q/b, q^4, 1/q^3, ba_1a_2/azq; q)_\infty} \times \frac{(1-azq^2/a_1)(1-azq^2/a_2)}{(1-azq^2/a_1b)(1-azq^2/a_2b)}\\
&\times {}_3\phi_2\left(\begin{matrix} azq^2/a_1a_2, & azq^3/a_1, & azq^3/a_2 \\ & azq^2/a_1, & azq^2/a_2 \end{matrix}\, ; q,\quad a_1a_2/azq^5\right).
\end{aligned} \tag{4.4}$$

Now, taking ratio of (4.3) and (4.4), then simplifying and using the result (1.3), we get

$$\frac{{}_3\psi_3\left(\begin{matrix} \frac{azq}{a_1a_2b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} \\ \frac{q}{b}, & \frac{azq}{a_1b}, & \frac{azq}{a_2b} \end{matrix}\, ; q,\quad \frac{a_1a_2}{azq^3}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{azq^2}{a_1a_2b}, & \frac{azq^3}{a_1b}, & \frac{azq^3}{a_2b} \\ \frac{q}{b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} \end{matrix}\, ; q,\quad \frac{a_1a_2}{azq^5}\right)}$$

$$= \frac{b(azq - ba_1a_2)}{(azq - a_1a_2)} \times \frac{(1-azq/a_1)(1-azq/a_2)}{(1-azq/a_1b)(1-azq/a_2b)} \times \frac{(1-azq^2/a_1b)(1-azq^2/a_2b)}{(1-azq^2/a_1)(1-azq^2/a_2)}$$

$$\times \left[\frac{T(az)P(az)}{S(az)Q(az)} + \cfrac{R(az)/S(az)Q(az)}{\frac{T(azq)P(azq)}{S(azq)Q(azq)} + \cfrac{R(azq)/S(azq)Q(azq)}{\frac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)} + \dots}} \right]$$

5. Special cases

Here, we shall deduce certain interesting cases of our main results.

Putting $c = q$ in (3.1), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} aq/b, & q^2, & dq \\ & q/b, & d \end{matrix} ; q, \; 1/aq^3\right)}{{}_3\phi_2\left(\begin{matrix} a/b, & q^2, & dq \\ & q/b, & d \end{matrix} ; q, \; 1/aq^2\right)} \tag{5.1}$$

$$= \frac{(a-1)}{(a-b)} \left[1 + \frac{P_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{Q_0}{1+} \cdots \frac{P_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{Q_n}{1+} \cdots \right]$$

where

$$\begin{aligned} P_n &= \frac{1}{aq^3}(1-bq^{n+2})(1-bdq^{n+1}), \quad n = 0,1,2,... \\ Q_n &= a\left(1-\frac{bq^{n+1}}{a}\right)\left(1-\frac{bdq^n}{a}\right), \quad n = 0,1,2,... \end{aligned}$$

Putting $d = 1/b$ in (5.1), we get

$$\frac{{}_2\phi_1\left(\begin{matrix} aq/b, & q^2 \\ & 1/b \end{matrix} ; q, \; 1/aq^3\right)}{{}_2\phi_1\left(\begin{matrix} a/b, & q^2 \\ & 1/b \end{matrix} ; q, \; 1/aq^2\right)} \tag{5.2}$$

$$= \frac{(a-1)}{(a-b)} \left[1 + \frac{L_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{M_0}{1+} \cdots \frac{L_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{M_n}{1+} \cdots \right]$$

where

$$L_n = \frac{1}{aq^3}(1-bq^{n+2})(1-q^{n+1}), \quad n=0,1,2,...$$

$$M_n = a\left(1-\frac{bq^{n+1}}{a}\right)\left(1-\frac{q^n}{a}\right), \quad n=0,1,2,...$$

Replacing $1/b$ by q in (5.2), we get

$$\frac{{}_2\phi_1\left(\begin{matrix} aq^2, \quad q^2 \\ q \end{matrix} ;q, \quad 1/aq^3\right)}{{}_2\phi_1\left(\begin{matrix} aq, \quad q^2 \\ q \end{matrix} ;q, \quad 1/aq^2\right)} \tag{5.3}$$

$$= \frac{(a-1)q}{(aq-1)}\left[1+\frac{R_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\frac{S_0}{1+}\cdots\frac{R_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\frac{S_n}{1+}\cdots\right]$$

where

$$R_n = \frac{1}{aq^3}(1-q^{n+1})^2, \quad n=0,1,2,...$$

$$S_n = a\left(1-\frac{q^n}{a}\right)^2, \quad n=0,1,2,...$$

Putting $b=1$ in (3.2), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} azq/a_1a_2, \quad azq^2/a_1, \quad azq^2/a_2 \\ azq/a_1, \quad azq/a_2 \end{matrix} ;q, \quad a_1a_2/azq^3\right)}{{}_3\phi_2\left(\begin{matrix} azq^2/a_1a_2, \quad azq^3/a_1, \quad azq^3/a_2 \\ azq^2/a_1, \quad azq^2/a_2 \end{matrix} ;q, \quad a_1a_2/azq^5\right)} \tag{5.4}$$

$$= \frac{T(az)P(az)}{S(az)Q(az)}+\cfrac{R(az)/S(az)Q(az)}{\cfrac{T(azq)P(azq)}{S(azq)Q(azq)}+\cfrac{R(azq)/S(azq)Q(azq)}{\cfrac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)}+...}}$$

Replacing az by z in (5.4), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} zq/a_1a_2, & zq^2/a_1, & zq^2/a_2 \\ & zq/a_1, & zq/a_2 \end{matrix} ;q,\quad a_1a_2/zq^3\right)}{{}_3\phi_2\left(\begin{matrix} zq^2/a_1a_2, & zq^3/a_1, & zq^3/a_2 \\ & zq^2/a_1, & zq^2/a_2 \end{matrix} ;q,\quad a_1a_2/zq^5\right)} \tag{5.5}$$

$$= \frac{T(z)P(z)}{S(z)Q(z)} + \cfrac{R(z)/S(z)Q(z)}{\cfrac{T(zq)P(zq)}{S(zq)Q(zq)} + \cfrac{R(zq)/S(zq)Q(zq)}{\cfrac{T(zq^2)P(zq^2)}{S(zq^2)Q(zq^2)} + \dots}}$$

Similarly, some other interesting special cases could be deduced.

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