CERTAIN TRANSFORMATION AND SUMMATION FORMULAE FOR q-SERIES

Remy Y. Denis

Department of Mathematics
University of Gorakhpur
Gorakhpur-273009
India
e-mail: ryddry@gmail.com, ddry@rediffmail.com

S.N. Singh

Department of Mathematics
T.D.P.G. College
Jaunpur-222002
India

S.P. Singh

Department of Mathematics
T.D.P.G. College
Jaunpur-222002
India

Abstract. In this paper, making use of certain summation formulae, an attempt has been made to establish certain new and interesting transformation and summation formulae for q-series.

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1. Introduction

Bailey [1] established a simple but very useful identity:

If

$$\beta_n = \sum_{r=0}^{n} u_{n-r}\, v_{n+r}\, \alpha_r \tag{1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} u_{r-n}\, v_{r+n}\, \delta_r \tag{1.2}$$

where α_r, δ_r, u_r and v_r are any functions of r only such that series γ_n exists, then subject to the convergence of the series.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \ \delta_n. \tag{1.3}$$

Making use of (1.3), Slater [3] gave a long list of Rogers-Ramanujan type identities. Later on, a number of mathematicians, notably, Verma [7], Verma and Jain [9], Singh [5], Denis [2], Singh [6] and others made use of Bailey's identity (1.3) and established a number of transformation formulae and also Rogers-Ramanujan type identities of different moduli. In this paper, making use of certain known summation formulae due to Verma and Jain [9] and identity (1.3), an attempt has been made to establish certain very interesting transformation formulae for q-hypergeometric series.

In the last section of this paper, making use of the following identity due to Verma [8], viz.,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{[q;q]_n\,[\gamma q^n;q]_n} \sum_{k=0}^{\infty} \frac{[\alpha,\beta;q]_{n+k} B_{n+k}\, z^k}{[q;q]_k\,[\gamma q^{2n+1};q]_k} \sum_{j=0}^{n} \frac{[q^{-n}.\gamma q^n;q]_j\, A_j (wq)^j}{[q;q]_j\,[\alpha,\beta;q]_j} \\ = \sum_{n=0}^{\infty} A_n B_n \frac{(zw)^n}{[q;q]_n} \end{aligned} \tag{1.4}$$

and summation formulae due to Verma and Jain [9], an attempt has been made to establish certain new transformation and summation formulae for basic hypergeometric series.

2. Definitions and notations

A basic (q-) hypergeometric series is generally defined to be a series of the type $\sum_{n=0}^{\infty} a_n z^n$, where a_{n+1}/a_n is a rational function of q^n, q being a fixed complex-parameter, called the base of the series, usually with modules less than one. An explicit representation of such series is given by

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r & ; q; z \\ b_1, b_2, \cdots, b_s & ; q^i \end{matrix} \right] = \sum_{n=0}^{\infty} q^{in(n-1)/2} \frac{[a_1, a_2, \cdots a_r; q]_n\, z^n}{[q, b_1, b_2, \cdots, b_s; q]_n} \tag{2.1}$$

and

$$[a_1, a_2, \cdots, a_r; q]_n = [a_1;q]_n\,[a_2;q]_n \cdots [a_r;q]_n$$

with the q-shifted factorial defined by

$$[a;q]_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n = 1, 2, \cdots \end{cases} \tag{2.2}$$

For convergence of the series (2.1) we need $|q| < 1$ and $|z| < \infty$, when $i > 0$, or $\max\{|q|, |z|\} < 1$, when $i = 0$, provided no zeros appear in the denominator. Following summations are needed in our analysis,

$$(2.3)\quad {}_2\Phi_1\left[\begin{matrix} a, b & ; q & ; c/ab \\ c & & \end{matrix}\right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}. \qquad \text{(Slater [2; App.IV(IV.2)])}$$

$$(2.4)\quad {}_2\Phi_1\left[\begin{matrix} a, b & ; q & ; c/ab \\ cq & & \end{matrix}\right] = \frac{[cq/a,\, cq/b; q]_\infty}{[cq,\, cq/ab; q]_\infty}\left\{\frac{ab(1+c) - c(a+b)}{ab - c}\right\}.$$

(Verma [7; (1.4)])

$$(2.5)\quad {}_4\Phi_3\left[\begin{matrix} q^{-n},\, x^2y^2q^{n+1},\, x,\, -xq & ; q & ; q \\ xyq,\, -xyq,\, x^2q & & \end{matrix}\right] = \frac{x^n[q;q]_n\,[x^2q^2;q^2]_m\,[y^2q^2;q^2]_m}{[x^2q;q]_n\,[x^2y^2q^2;q^2]_m\,[q^2;q^2]_m},$$

where m is greatest integer $\leq n/2$.

$$(2.6)\quad {}_4\Phi_3\left[\begin{matrix} q^{-2n},\, b^2x^4y^2q^{2n+2},\, x^2,\, x^2q & ; q^2 & ; q^2 \\ bx^2q,\, bx^2q^2,\, x^4q^2 & & \end{matrix}\right] = \frac{x^{2n}[-q;q]_n\,[bq;q]_n}{[-x^2q;q]_n\,[bx^2q;q]_n},$$

[Verma and Jain [9; (2.32)])

$$(2.7)\quad \begin{aligned} &{}_4\Phi_3\left[\begin{matrix} q^{-n},\, bx^2q^{n+2},\, x,\, -xq & ; q & ; q \\ xq\sqrt{b},\, -xq\sqrt{b},\, x^2q^2 & & \end{matrix}\right] \\ &\quad = \frac{x^n[q;q]_n\,[bxq^2;q]_n\,[bx^2q^3;q^2]_m\,[bq^2;q^2]_m\,[xq^2;q]_{2m}}{[xq;q]_n\,[bx^2q^2;q]_n\,[q^2;q^2]_m\,[x^2q^3;q^2]_m\,[bxq^2;q]_{2m}}, \end{aligned}$$

[Verma and Jain [9; (3.2)])

where m is greatest integer $< n/2$.

$$(2.8)\quad {}_5\Phi_4\left[\begin{matrix} a,\, aq,\, aq^2,\, a^3q^{3n+3},\, q^{-3n} & ; q^3 & ; q^3 \\ (aq)^{3/2},\, -(aq)^{3/2},\, a^{3/2}q^3,\, -a^{3/2}q^3 & & \end{matrix}\right] = \frac{a^n[q^3;q^3]_n\,[aq;q]_n}{[a^3q^3;q^3]_n\,[q;q]_n},$$

[Verma and Jain [9; (4.2)])

$$(2.9)\quad \begin{aligned} &{}_5\Phi_4\left[\begin{matrix} x,\, \omega xq,\, \omega^2xq,\, x^3q^{n+4},\, q^{-n} & ; q & ; q \\ (xq)^{3/2},\, -(xq)^{3/2},\, x^{3/2}q^2,\, -x^{3/2}q^2 & & \end{matrix}\right] \\ &\quad = \frac{x^n[x^2q^4;q]_n\,[q;q]_n\,[x^3q^6;q^3]_n\,[xq^3;q]_{3m}}{[x^3q^4;q]_n\,[xq;q]_n\,[q^3;q^3]_m\,[x^2q^4;q]_{3m}}, \end{aligned}$$

[Verma and Jain [9; (4.4)])

where m is greatest integer $\le n/3$ and $\omega = e^{2\pi i/3}$.

$$(2.10)\quad {}_5\Phi_4\left[\begin{matrix} a^{1/3},\ \omega a^{1/3},\ \omega^2 a^{1/3},\ aq^{n+1},\ q^{-n} & ;q & ;q \\ q\sqrt{a},\ -q\sqrt{a},\ \sqrt{aq},\ -\sqrt{aq} & & \end{matrix}\right] = \frac{(\sqrt{a})^{n-m}\,[q;q]_n\,[aq^3;q^3]_m}{[aq;q]_n\,[q^3;q^3]_m},$$

[Verma and Jain [9; (4.5)])

where m is greatest integer $\le n/3$ and $\omega = e^{2\pi i/3}$.

$$(2.11)\quad {}_6\Phi_5\left[\begin{matrix} a^{1/3},\ \omega a^{1/3},\ \omega^2 a^{1/3},\ q\sqrt{a},\ aq^{n+1},\ q^{-n} & ;q & ;q \\ \sqrt{a},\ -\sqrt{a},\ \sqrt{aq},\ -\sqrt{aq},\ q^2\sqrt{a} & & \end{matrix}\right]$$
$$= \frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(\sqrt{a})^{n-m}}{[aq;q]_n\,[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m},$$

[Verma and Jain [9; (4.8)])

where m is greatest integer $\le n/3$.

3. Main Results

In this section, we shall establish the transformation formulae by making use of (1.3).

Taking $u_r = \dfrac{1}{[q;q]_r}$, $v_r = \dfrac{[aq;q]_r}{q^{r^2/2}}$ in (1.1) and (1.2), we get:

If

$$(3.1)\quad \beta_n = \frac{[aq;q]_n}{[q;q]_n\,q^{n^2/2}}\sum_{r=0}^{n}\frac{(-1)^r q^{r/2}[q^{-n};q]_r\,[aq^{n+1};q]_r\,\alpha_r}{q^{r^2}} \qquad \text{infty}$$

and

$$(3.2)\quad \gamma_n = \frac{[aq;q]_{2n}}{q^{2n^2}}\sum_{r=0}^{\infty}\frac{[aq^{2n+1};q]_r\,\delta_{r+n}}{[q;q]_r\,q^{r^2/2+2nr}},$$

then

$$(3.3)\quad \sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\,\delta_n,$$

provided the series involving are convergent. We shall now use (3.1), (3.2) and (3.3) in order to establish the required transformations.

(i) Replacing a by x^2y^2 in (3.1) and (3.2), and then taking

$$\alpha_r = \frac{[x,-xq;q]_r\,q^{r^2+r/2}(-)^r}{[q,xyq,-xyq,x^2q;q]_r}$$

in (3.1) and making use of (2.5), we have:

(3.4) $$\beta_n = \frac{[x^2y^2q;q]_n\, x^n [x^2q^2;q^2]_m\, [y^2q^2;q^2]_m}{q^{n^2/2}\,[x^2q;q]_n\,[x^2y^2q^2;q^2]_m\,[q^2;q^2]_m}$$

where m is the greatest integer $< n/2$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get after some simplification:

(3.5) $$\gamma_n = \frac{[x^2y^2zq;q]_\infty\,[x^2y^2q;q^2]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[x^2q^2zq;q]_n\,[q/z;q]_n}.$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get:

(3.6) $$\begin{aligned} &\frac{[x^2y^2zq;q]_\infty}{[z;q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} x,\, -xq,\, xyq^{1/2},\, -xyq^{1/2} & ;q\;;\,q \\ x^2q,\, x^2y^2zq,\, q/z & \end{matrix}\right] \\ &= {}_2\Phi_1\left[\begin{matrix} x^2y^2q,\, y^2q^2 & ;q^2\;;\,x^2z^2 \\ x^2q & \end{matrix}\right] \\ &+ \frac{xz(1-x^2y^2q)}{(1-x^2q)}\,{}_2\Phi_1\left[\begin{matrix} x^2y^2q^3,\, y^2q^2 & ;q^2\;;\,x^2z^2 \\ x^2q^3 & \end{matrix}\right], \qquad |xz|<1. \end{aligned}$$

(ii) Next, replacing a by b^2x^4 and q by q^2 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x^2,x^2q;q^2]_r\,q^{2r^2+r}(-)^r}{[bx^2q,bx^2q^2,x^4q^2;q^2]_r\,[q^2;q^2]_r}$ in (3.1) and making use of (2.6) we have:

(3.7) $$\beta_n = \frac{[b^2x^4q^2;q^2]_n\,x^{2n}\,[-q;q]_n\,[bq;q]_n}{[q^2;q^2]_n\,[-x^2q;q]_n\,[bx^2q;q]_n\,q^{n^2}}.$$

Again,setting $\delta_r = z^r q^{r^2}$ in (3.2) we have:

(3.8) $$\gamma_n = \frac{[b^2x^4zq^2;q^2]_\infty\,[b^2x^4q^2;q^2]_{2n}\,(-)^n\,q^n}{[z;q^2]_\infty\,q^{2n^2}\,[b^2x^4zq^2;q^2]_n\,[q^2/z;q^2]_n}.$$

Putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get the following transformation:

(3.9) $$\begin{aligned} &\frac{[b^2x^4zq^2;q^2]_\infty}{[z;q^2]_\infty}\,{}_4\Phi_3\left[\begin{matrix} x^2,\, x^2q,\, -bx^2q,\, -bx^2q^2 & ;q^2\;;\,q^2 \\ x^4q^2,\, b^2x^4zq^2,\, q^2/z & \end{matrix}\right] \\ &= {}_2\Phi_1\left[\begin{matrix} -bx^2q,\, bq & ;q\;;\,x^2z \\ -x^2q & \end{matrix}\right], \qquad |x^2z|<1. \end{aligned}$$

(iii) Again, putting $a = bx^2q$ in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x,-xq;q]_r\,(-)^r\,q^{r^2+r/2}}{[x^2q^2,xq\sqrt{b},-xq\sqrt{b};q]_r\,[q;q]_r}$ in (3.1) and making use of (2.7) we get:

$$\beta_n = \frac{x^n [bxq^2;q]_n\,[bx^2q^3;q^2]_m\,[bq^2;q^2]_m\,[xq^2;q]_{2m}}{q^{n^2/2}[xq;q]_n\,[q^2;q^2]_m\,[x^2q^3;q^2]_m\,[bxq^2;q]_{2m}}, \tag{3.10}$$

where m is the greatest integer $\leq n/2$.

Again, taking $\delta_r = z^r q^{r^2/2}$ in (3.2) we get:

$$\gamma_n = \frac{[bx^2q^2z;q]_\infty\,[bx^2q^2;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[bx^2q^2z;q]_n\,[q/z;q]_n}. \tag{3.11}$$

Now, putting these values in (3.3) we get the following transformation:

$$\begin{aligned}
&\frac{[bzx^2q^2;q]_\infty}{[z;q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} x,\,-xq,\,xq\sqrt{bq},\,-xq\sqrt{bq} & ;q\,;\,q \\ bzx^2q^2,\,x^2q^2,\,q/z & \end{matrix}\right] \\
&= {}_3\Phi_2\left[\begin{matrix} bx^2q^3,\,bq^2,\,xq^3 & ;q^2\,;\,x^2z^2 \\ xq,\,x^2q^3 & \end{matrix}\right] \\
&+\frac{xz(1-bxq^2)}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} bxq^4,\,bx^2q^3,\,bq^2 & ;q^2\,;\,x^2z^2 \\ x^2q^3,\,bxq^2 & \end{matrix}\right],\quad |x^2z^2|<1.
\end{aligned} \tag{3.12}$$

(iv) Next, replacing a by a^3 and q by q^3 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[a,aq,aq^2;q^3]_r\,q^{3r+3r/2}\,(-)^r}{[q^3,(aq)^{3/2},-(aq)^{3/2},\,a^{3/2}q^3,\,-a^{3/2}q^3;q^3]_r}$ in (3.1) and making use of (2.8) we have:

$$\beta_n = \frac{[aq;q]_n\,a^n}{q^{3n^2/2}\,[q;q]_n}. \tag{3.13}$$

Again, taking $\delta_r = z^r q^{3r^2/2}$ in (3.2) we get after some simplifications,

$$\gamma_n = \frac{[a^3q^3;q^3]_{2n}\,[a^3q^3z;q^3]_\infty\,(-)^n\,q^{3n/2}}{[z;q^3]_\infty\,[a^3q^3z;q^3]_n\,[q^3/z;q^3]_n}. \tag{3.14}$$

Substituting these values of α_n, β_n, γ_n and δ_n in (3.3) we get the following summation,

$${}_3\Phi_2\left[\begin{matrix} a,\,aq,\,aq^2 & ;q^3\,;\,q^3 \\ a^3q^3z,\,q^3/z & \end{matrix}\right] = \frac{[z;q^3]_\infty\,[a^2zq;q]_\infty}{[a^3q^3z;q^3]_\infty\,[az;q]_\infty}. \tag{3.15}$$

(v) Next, replaying a by x^3q^3 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x,\omega xq,\omega^2xq;q]_r q^{r^2+r/2}(-)^r}{[q,(xq)^{3/2},-(xq)^{3/2},x^{3/2}q^2,-x^{3/2}q^2;q]_r}$ in (3.1) and making use of (2.9) we get:

$$\beta_n = \frac{x^n [x^2q^4;q]_n\,[x^3q^6;q^3]_m\,[xq^3;q]_{3m}}{q^{n^2/2}\,[xq;q]_n\,[q^3;q^3]_m\,[x^2q^4;q]_{3m}}, \tag{3.16}$$

where m is the greatest integer $\le n/3$ and $\omega = e^{2\pi i/3}$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2) we get:

$$\gamma_n = \frac{[x^3zq^4;q]_\infty\,[x^3q^4;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[x^3zq^4;q]_n\,[q/z;q]_n}. \tag{3.17}$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we have:

$$\begin{aligned}
&\frac{[x^3zq^4;q]_\infty}{[z;q]_\infty}\,{}_5\Phi_4\left[\begin{matrix} x,\,\omega xq,\,\omega^2xq,\,x^{3/2}q^{5/2},\,-x^{3/2}q^{5/2} & ;q\,;\,q \\ (xq)^{3/2},\,-(xq)^{3/2},\,x^3zq^4,\,q/z & \end{matrix}\right] \\
&\quad= {}_3\Phi_2\left[\begin{matrix} x^2q^5, x^2q^6, x^3q^6\,;\,q^3\,;\,x^3z^3 \\ xq,\,xq^2 \end{matrix}\right] \\
&\quad+\frac{xz(1-x^2q^4)}{(1-xq)}\,{}_4\Phi_3\left[\begin{matrix} x^2q^5, x^2q^6, x^2q^7, x^3q^6\,;q^3\,;\,x^3z^3 \\ xq^2,\,xq^4,\,x^2q^4 \end{matrix}\right] \\
&\quad+\frac{x^2z^2(1-x^2q^4)(1-x^2q^5)}{(1-xq)(1-xq^2)}\,{}_4\Phi_3\left[\begin{matrix} x^2q^6, x^2q^7, x^2q^8, x^3q^6\,;q^3\,;\,x^3z^3 \\ xq^4,\,xq^5,\,x^2q^4 \end{matrix}\right], \\
&\qquad\qquad |x^3z^3| < 1.
\end{aligned} \tag{3.18}$$

(vi) Taking $\alpha_r = \dfrac{[a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q]_r q^{r^2+r/2}(-)^r}{[q, q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q]_r}$ in (3.1) and making use of (2.10) we get:

$$\beta_n = \frac{(a)^{n-m/2}[aq^3;q^3]_m}{q^{n^2/2}\,[q^3;q^3]_m}, \tag{3.19}$$

where m is the greatest integer $\le n/3$ and $\omega = e^{2\pi i/3}$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get:

$$\gamma_n = \frac{[azq;q]_\infty\,[aq;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[azq;q]_n\,[q/z;q]_n}. \tag{3.20}$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get:

$$\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} a^{1/3},\,\omega a^{1/3},\,\omega^2a^{1/3},\,-q\sqrt{a}\,;q\,;\,q \\ azq,\,q/z,\,-\sqrt{a} \end{matrix}\right] \\
&\qquad= \frac{[z;q]_\infty[a^2z^3q^3;q^3]_\infty}{[azq;q]_\infty[az^3;q^3]_\infty}\,\{1+a^{1/2}z+az^2\}.
\end{aligned} \tag{3.21}$$

(vii) Lastly, taking $\alpha_r = \dfrac{[a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}, q\sqrt{a}; q]_r (-)^r q^{r^2+r/2}}{[q, q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a}; q]_r}$ in (3.1) and making use of (2.11), we get:

$$\beta_n = \frac{[\sqrt{a}; q]_n\, [aq^3; q^3]_m\, [q^6\sqrt{a}; q^3]_m\, (\sqrt{a})^{n-m}}{q^{n^2/2}\, [q^2\sqrt{a}; q]_n\, [q^3; q^3]_m\, [\sqrt{a}; q^3]_m}, \tag{3.22}$$

where m is the greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

Again, taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get:

$$\gamma_n = \frac{[azq; q]_\infty\, [aq; q]_{2n}\, (-)^n\, q^{n/2}}{[z; q]_\infty\, q^{n^2}\, [azq, q/z; q]_n}. \tag{3.23}$$

Now, putting these values in (3.3), we get:

$$\begin{aligned} &\frac{[azq; q]_\infty}{[z; q]_\infty}\, {}_6\Phi_5\left[\begin{matrix} a^{1/3},\ \omega a^{1/3},\ \omega^2 a^{1/3},\ q\sqrt{a},\ -q\sqrt{a},\ q\sqrt{a} & ;q\ ;\ q \\ azq,\ q/z,\ \sqrt{a},\ -\sqrt{a},\ q^2\sqrt{a} & \end{matrix}\right] \\ &= {}_3\Phi_2\left[\begin{matrix} q\sqrt{a},\ aq^3,\ q^6\sqrt{a}\ ;\ q^3\ ;\ az^3 \\ q^3\sqrt{a},\ q^4\sqrt{a} \end{matrix}\right] \\ &+ \frac{za^{1/2}(1-\sqrt{a})}{(1-q^2\sqrt{a})}\, {}_3\Phi_2\left[\begin{matrix} q\sqrt{a},\ aq^3,\ q^6\sqrt{a}\ ;\ q^3\ ;\ az^3 \\ q^4\sqrt{a},\ \sqrt{a} \end{matrix}\right] \\ &+ \frac{az^2(1-\sqrt{a})(1-q\sqrt{a})}{(1-q^2\sqrt{a})(1-q^3\sqrt{a})}\, {}_3\Phi_2\left[\begin{matrix} q^2\sqrt{a},\ q^3\sqrt{a},\ aq^3\ ;\ q^3\ ;\ az^3 \\ q^5\sqrt{a},\ \sqrt{a} \end{matrix}\right], \\ &\qquad |az^3| < 1. \end{aligned} \tag{3.24}$$

4. Certain transformations and summations

In this section, we shall make use of (1.4) and summation formulae (2.3)–(2.11) to establish certain transformation and summation formulae for q-series.

If we take $B_n = 1$, $z = \gamma q/\alpha\beta$ in (1.4) and make use of (2.3) to sum of inner ${}_2\Phi_1$ series, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[\gamma, \alpha, \beta; q]_n\, (1-\gamma q^{2n})\, (-\gamma q/\alpha\beta)^n\, q^{n(n-1)/2}}{[q, \gamma q\alpha, \gamma q/\beta]_n (1-\gamma)} \sum_{j=0}^{n} \frac{[q^{-n},\ \gamma q^n; q]_j\, A_j (wq)^j}{[q,\ \alpha,\ \beta; q]_j} \\ &= \frac{[\gamma q,\ \gamma q/\alpha\beta; q]_\infty}{[\gamma q/\alpha,\ \gamma q/\beta; q]_\infty} \sum_{n=0}^{\infty} A_n \frac{(w\gamma q/\alpha\beta)^n}{[q; q]_n}. \end{aligned} \tag{4.1}$$

Again taking $B_n = 1$, $z = \gamma/\alpha\beta$ in (1.4) and making use of (2.4) in order to sum the inner ${}_2\Phi_1$ series, we get:

$$\sum_{n=0}^{\infty} \frac{[\gamma, \alpha, \beta; q]_n\,(1-\gamma q^{2n})\,(-\gamma q/\alpha\beta)^n\,q^{n(n-1)/2}}{[q, \gamma q/\alpha, \gamma q/\beta]_n(1-\gamma)} \times \left\{\frac{\alpha\beta(1+\gamma q^{2n}) - \gamma q^n(\alpha+\beta)}{\alpha\beta-\gamma}\right\} \sum_{j=0}^{n} \frac{[q^{-n},\, \gamma q^n; q]_j\, A_j (wq)^j)}{[q,\, \alpha,\, \beta; q]_j} = \sum_{n=0}^{\infty} A_n \frac{(w\gamma/\alpha\beta)^n}{[q;q]_n}. \tag{4.2}$$

We shall make use of (4.1) and (4.2) in order to establish our main results.

(i) Replacing q by q^2 and then taking $\gamma = b^2x^4q^2$, $\alpha = bx^2q$, $\beta = bx^2q^2$, $w = 1$ and $A_j = \dfrac{[x^2, x^2q; q^2]_j}{[x^4q^2; q^2]_j}$ in (4.1), we get:

$$\sum_{n=0}^{\infty} \frac{[b^2x^4q^2,\, bx^2q,\, x^2q^2; q^2]_n(1-b^2x^4\,q^{4n+2}\,(-)^n\,q^{n(n-1)/2}\,q^n}{[q^2,\, bx^2q^3,\, b^2x^2q^2; q^2]_n\,(1-b^2x^4q^2)} \times {}_4\Phi_3\left[\begin{matrix} q^{-2n}, b^2x^4q^{2n+2}, x^2, x^2q\ ; q^2\ ; q^2 \\ bx^2q,\, bx^2q^2,\, x^4q^2 \end{matrix}\right] = \frac{[b^2x^4q^4,\, q; q^2]_\infty}{[bx^2q^3, bx^2q^2; q^2]_\infty}\, {}_2\Phi_1\left[\begin{matrix} x^2, x^2q\ ; q^2\ ;\ q \\ x^4q^2 \end{matrix}\right]. \tag{4.3}$$

Now, summing the inner ${}_4\Phi_3$-series on the left hand side and ${}_2\Phi_1$ on the right hand side of (4.3) with the help of (2.6) and (2.3), respectively, we get:

$$ {}_2\Phi_1\left[\begin{matrix} bq,\, -bx^2q^3 & ; q\ ;\ -x^2q \\ -\,x^2q & ; q^2 \end{matrix}\right] = \frac{[bx^2q; q]_\infty}{[-x^2q; q]_\infty}. \tag{4.4}$$

(ii) Taking $\gamma = bx^2q^2$, $\alpha = xq\sqrt{b}$, $\beta = -xq\sqrt{b}$, $A_j = \dfrac{[x, -xq; q]_j}{[x^2q^2; q]_j}$ and $w = 1$ in (4.1), we get:

$$\sum_{n=0}^{\infty} \frac{[bx^2q^2,\, xq\sqrt{b},\, -xq\sqrt{b}; q]_n(1-bx^2\,q^{2n+2})\,q^{n(n-1)/2}\,q^n}{[q,\, xq^2\sqrt{b},\, -xq^2\sqrt{b}^2; q]_n\,(1-bx^2q^2)} \times {}_4\Phi_3\left[\begin{matrix} q^{-n}, bx^2q^{n+2}, x, -xq\ ; q\ ; q \\ xq\sqrt{b},\, -xq\sqrt{b},\, x^2q^2 \end{matrix}\right] = \frac{[bx^2q^3,\, -q; q]_\infty}{[xq^2\sqrt{b},\, -xq^2\sqrt{b}; q^2]_\infty}\, {}_2\Phi_1\left[\begin{matrix} x, -xq\ ; q\ ;\ -q \\ x^2q^2 \end{matrix}\right]. \tag{4.5}$$

Now, summing the inner ${}_4\Phi_3$-series on the left hand side and ${}_2\Phi_1$ series on the right hand side of (4.5), with the help of (2.7) and (2.3) respectively, we get the following summation formula:

$$(4.6)\quad {}_3\Phi_2\left[\begin{matrix} bx^2q^3,\, bq^2,\, xq^3 & ;q^2\,;\,x^2q^3 \\ xq,\, x^2q^3 & ;q^4 \end{matrix}\right] + \frac{xq}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} bx^2q^3,\, bq^2,\, bxq^4 & ;q^2\,;\,x^2q^5 \\ x^2q^3,\, bxq^2 & ;q^4 \end{matrix}\right] = \frac{[bx^2q^3,\, -xq,\, xq^2;q]_\infty}{[x^2q^2,\, xq^2\sqrt{b},\, -xq^2\sqrt{b};q]_\infty}.$$

(iii) Replacing q by q^3 and then taking $\gamma = a^3q^3$, $\alpha = (aq)^{3/2}$, $\beta = -(aq)^{3/2}$, $A_j = \dfrac{[a, aq, aq^2; q^3]_j}{[a^{3/2}q^3, -a^{3/2}q^3; q^3]_j}$ and $w = 1$ in (4.1), we get:

$$(4.7)\quad \sum_{n=0}^{\infty} \frac{[a^3q^3,\, (aq)^{3/2},\, -(aq)^{3/2}; q^3]_n (1-a^3q^{6n+3})\, q^{3n}\, q^{3n(n-1)/2}}{[q^3,\, (aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_n\, (1-a^3q^3)} \times {}_5\Phi_4\left[\begin{matrix} q^{-3n}, a^3q^{3n+3}, a, aq, aq^2 & ;q^3\;;q^3 \\ (aq)^{3/2},\, -(aq)^{3/2},\, a^{3/2}q^3,\, -a^{3/2}q^3 & \end{matrix}\right] = \frac{[a^3q^6,\, -q^3;\, q^3]_\infty}{[(aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_\infty}\,{}_3\Phi_2\left[\begin{matrix} a,\, aq,\, aq^2 & ;q^3\,;\,-q^3 \\ a^{3/2}q^3,\, -a^{3/2}q^3 & \end{matrix}\right].$$

Now, summing the inner ${}_5\Phi_4$-series on the right hand side of (4.7) with the help of (2.8), we get:

$$(4.8)\quad {}_1\Phi_0\left[\begin{matrix} aq & ;q\,;\,aq^3 \\ - & ;q^3 \end{matrix}\right] = \frac{[-q^3,\, a^3q^6;\, q^3]_\infty}{[(aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_\infty}\,{}_3\Phi_2\left[\begin{matrix} a,\; aq,\; aq^2; q^3 & ;\,-q^3 \\ a^{3/2}q^3, -a^{3/2}q^3 & \end{matrix}\right].$$

(iv) Taking

$$\gamma = x^3q^4,\ \alpha = x^{3/2}q^2,\ \beta = -x^{3/2}q^2,\ w = 1 \text{ and } A_j = \frac{[x,\, \omega xq,\, \omega^2xq; q]_j}{[(xq)^{3/2},\, -(xq)^{3/2}; q]_j}$$

in (4.1), we get:

$$(4.9)\quad \sum_{n=0}^{\infty} \frac{[x^3q^4;q]_n\, q^{n+n(n-1)/2}}{[q;q]_n}\,{}_5\Phi_4\left[\begin{matrix} q^{-n}, x^3q^{n+4}, x, \omega xq,\, \omega^2xq & ;q\;;q \\ x^{3/2}q^2,\, -x^{3/2}q^2,\, (xq)^{3/2},\, -(xq)^{3/2} & \end{matrix}\right] = \frac{[x^3q^5,\, -q; q]_\infty}{[x^{3/2}q^3,\, -x^{3/2}q^3; q]_\infty}\,{}_3\Phi_2\left[\begin{matrix} x,\, \omega xq,\, \omega^2xq & ;q\,;\,-q \\ (xq)^{3/2},\, -(xq)^{3/2} & \end{matrix}\right],$$

where $\omega = e^{2\pi i/3}$.

Now, summing the inner ${}_5\Phi_4$-series on the right hand side of (4.9) with the help of (2.9), we get:

$$
\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, xq^4,\, xq^5 & ;q^3\,;\, x^3q^6 \\ xq,\, xq^2 & ;q^9 \end{matrix}\right] \\
&+\frac{xq(1-x^2q^4)}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, x^2q^7,\, xq^5 & ;q^3\,;\, x^3q^9 \\ xq^2,\, x^2q^4 & ;q^9 \end{matrix}\right] \\
&+\frac{x^2q^3(1-x^2q^4)(1-x^2q^5)}{(1-xq)(1-xq^2)}\,{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, x^2q^7,\, x^2q^8 & ;q^3\,;\, x^3q^{12} \\ x^2q^4,\, x^2q^5 & ;q^9 \end{matrix}\right] \\
&=\frac{[x^3q^5,\,-q;q]_\infty}{[x^{3/2}q^3,\,-x^{3/2}q^3;q]_\infty}\,{}_3\Phi_2\left[\begin{matrix} x,\,\omega xq,\,\omega^2xq & ;q\,;\,-q \\ (xq)^{3/2},\,-(xq)^{3/2} & \end{matrix}\right],
\end{aligned}
\tag{4.10}
$$

where $\omega = e^{2\pi i/3}$ and $|x| < 1$, $|q| < 1$.

(v) Lastly, taking

$$\gamma = aq,\quad \alpha = \sqrt{aq},\quad \beta = -\sqrt{aq},\quad w = 1 \quad \text{and}$$

$$A_j = \frac{[a^{1/3},\, \omega a^{1/3},\, \omega^2 a^{1/3}, q\sqrt{a}; q]_j}{[\sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a}; q]_j}$$

in (4.1), we get:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{[aq;q]_n\, q^{n(n-1)/2}}{[q;q]_n}\,{}_6\Phi_5\left[\begin{matrix} q^{-n}, aq^{n+1}, a^{1/3}, \omega a^{1/3},\, \omega^2a^{1/3},\, q\sqrt{a}\ ;q\ ;q \\ \sqrt{aq},\, -\sqrt{aq},\, \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} \end{matrix}\right] \\
&\quad= \frac{[aq^2,\,-q;q]_\infty}{[q\sqrt{aq},\,-q\sqrt{aq};q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} a^{1/3}, \omega a^{1/3},\, \omega^2a^{1/3},\, q\sqrt{a}\ ;q\ ;-q \\ \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} \end{matrix}\right],
\end{aligned}
\tag{4.11}
$$

Now, summing the inner ${}_6\Phi_5$-series on the left hand side of (4.11) with the help of (2.11), we get:

$$
\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} aq^3,\, q\sqrt{a},\, q^6\sqrt{a} & ;q^3\,;\, aq^6 \\ q^3\sqrt{a},\, q^4\sqrt{a} & ;q^9 \end{matrix}\right] \\
&+\frac{q\sqrt{a}(1-\sqrt{a})}{(1-q^2\sqrt{a})}\,{}_4\Phi_3\left[\begin{matrix} q\sqrt{a},\, q^2\sqrt{a},\, aq^3,\, q^6\sqrt{a} & ;q^3\,;\, aq^9 \\ q^4\sqrt{a},\, q^5\sqrt{a},\, \sqrt{a} & ;q^9 \end{matrix}\right] \\
&+\frac{aq^3(1-\sqrt{a})(1-q\sqrt{a})}{(1-q^2\sqrt{a})(1-q^3\sqrt{a})}\,{}_3\Phi_2\left[\begin{matrix} q^2\sqrt{a},\, q^3\sqrt{a},\, aq^3 & ;q^3\,;\, aq^{12} \\ q^5\sqrt{a},\, \sqrt{a} & ;q^9 \end{matrix}\right] \\
&=\frac{[-q,\, aq^2;q]_\infty}{[q\sqrt{aq},\,-q\sqrt{aq};q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} a^{1/3},\, \omega a^{1/3},\, \omega^2a^{1/3}, q\sqrt{a} & ;q\,;\,-q \\ \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} & \end{matrix}\right],
\end{aligned}
\tag{4.12}
$$

where $\omega = e^{2\pi i/3}$, $|a| < 1$ and $|q| < 1$.

Proceeding in the same way one can also establish certain summation and transformation formulae for q-series by making use of the summation (2.5)–(2.11) and identity (4.2).

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