HYPER K-ALGEBRAS INDUCED BY A DETERMINISTIC FINITE AUTOMATON

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Abstract. In this note first we define a hyper *K-algebra S* on the states of a deterministic finite automaton. Then we obtain some commutative hyper K-ideals of types 3, 4, 5, 6 and 9 and also positive implicative hyper K-ideals of types 1, 2, 3, 4, 5, 6, 7, 8 and 9 of S. Also we prove some theorems and obtain some results, to show that some properties of this hyper K-algebra. Then we define another hyper *K-algebra* on the states of a deterministic finite automaton which is simple and normal. Finally, we introduce a hyper K-algebra on the set of all equivalence classes of an equivalence relation on states.

Keywords: deterministic finite automaton, hyper K-algebra, (commutative, positive implicative) hyper K-ideal.

1. Introduction

The hyper algebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [6] in 1966 introduced the notion of BCK-algebra. Borzooei, Jun and Zahedi et.al. [1], [2], [13] applied the hyper structure to BCK-algebra and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Roodbari and Zahedi [12] introduced 27 different types of positive implicative hyper K-ideals, also they introduced 9 different types of commutative hyper K-ideals. Corsini and Leoreanu [4] found some connections between a deterministic finite automaton and the hyper algebraic structure theory. Now, in this note we define two hyper K-algebras on the states of a deterministic finite automaton. Then we obtain some properties of these hyper K-algebras. Finally, we define a hyper K-algebra on the set of all equivalence classes of an equivalence relation on states.

2. Preliminaries

Let H be a nonempty set and o be a hyper operation on H, that is o is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \backslash \phi$.

Definition 2.1 [2] We say that H is a hyper K-algebra if it contains a constant 0 and satisfies the following axioms:

(HK1) $(xoz)o(yoz) < xoy$,
(HK2) $(xoy)oz = (xoz)oy$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Rightarrow x = y$,
(HK5) $0 < x$.

For all $x, y, z \in H$, where $x < y$ is defined by $0 \in xoy$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A, \exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by AoB we mean the subset $\bigcup\limits_{a \in A, b \in B} aob$ of H.

Definition 2.2 [10] Let $(H, o, 0)$ be a hyper K-algebra. Then H is called:

(i) A weak implicative, if for all $x, y \in H$, $x < xo(yox)$,

(ii) An implicative, if for all $x, y \in H$, $x \in xo(yox)$,

(iii) A strong implicative, if for all $x, y \in H$, $xo0 \subseteq xo(yox)$.

Definition 2.3 [9] Let $(H, o, 0)$ be a hyper K-algebra and I be a subset of H and $\phi \neq S \subseteq H$. Then we say that I is an S-absorbing set, whenever $x \in I$ and $y \in S$ imply that $xoy \subseteq I$.

Definition 2.4 [10] Let I be a nonempty subset of a hyper K-algebra H. Then we say that I is closed, whenever $x < y$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.5 [2], [10] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then,

(i) I is called a weak hyper K-ideal of H if $xoy \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(ii) I is called a hyper K-ideal of H if $xoy < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(iii) I is called a strong hyper K-ideal of H if $(xoy) \cap I \neq \phi$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(iv) I is called an implicative hyper K-ideal, if for all $x, y, z \in H$, $(xoz)o(yox) < I$ and $z \in I$ imply that $x \in I$.

(v) I is called a weak implicative hyper K-ideal, if for all $x, y, z \in H$, $(xoz)o(yox) \subseteq I$ and $z \in I$ imply that $x \in I$.

Theorem 2.6 [2] *Any strong hyper K-ideal of a hyper K-algebra H is a hyper K-ideal and a weak hyper K-ideal. Also any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 2.7 [12] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then I is called a commutative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

Definition 2.8 [10] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then the following statements hold:

(i) If I is a commutative hyper K-ideal of type 4, then I is a commutative hyper K-ideal of type 6,

(ii) If I is a commutative hyper K-ideal of type 6, then I is a commutative hyper K-ideal of type 9,

(iii) If I is a commutative hyper K-ideal of type 5, then I is a commutative hyper K-ideal of type 6,

(iv) If I is a commutative hyper K-ideal of type 9, then I is a commutative hyper K-ideal of type 3.

Definition 2.9 [12] Let I be a nonempty subset of H such that $0 \in I$. Then I is called a positive implicative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) < I$,

(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

(x) type 10, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(xi) type 11, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(xii) type 12, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(xiii) type 13, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(xiv) type 14, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(xv) type 15, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(xvi) type 16, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) < I$,

(xvii) type 17, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(xviii) type 18, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

(xix) type 19, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(xx) type 20, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(xxi) type 21, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(xxii) type 22, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(xxiii) type 23, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(xxiv) type 24, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(xxv) type 25, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) < I$,

(xxvi) type 26, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(xxvii) type 27, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

Definition 2.10 [10] A hyper K-algebra $(H, o, 0)$ is called simple if for all distinct elements $a, b \in H - \{0\}$, $a \not< b$ and $b \not< a$.

Definition 2.11 [11] Let H be a hyper K-algebra and S be a nonempty subset of H. Then the sets

$${}_{l1}S = \{x \in H | a < (aox), \forall a \in S\}, \qquad {}_{l2}S = \{x \in H | a \in (aox), \forall a \in S\},$$
$$S_{r1} = \{x \in H | x < (xoa), \forall a \in S\} \quad \text{and} \quad S_{r2} = \{x \in H | x \in (xoa), \forall a \in S\}$$

are called left hyper K-stabilizer of type 1 of S, left hyper K-stabilizer of type 2 of S, right hyper K-stabilizer of type 1 of S and right hyper K-stabilizer of type 2 of S.

Definition 2.12 [11] A hyper K-algebra $(H, o, 0)$ is called left (right) hyper normal of type 1(2) if $_{li}a(a_{ri})$ of any element $a \in H$ is a hyper K-ideal of H for $i = 1$ or 2. Also if H is both left and right hyper normal of type 1 (2), then H is called hyper normal K-algebra of type 1 (2).

Definition 2.13 [5] A deterministic finite automaton consists of:

(i) A finite set of states, often denoted by S.

(ii) A finite set of input symbols, often denoted by M.

(iii) A transition function that takes as arguments a state and an input symbol and returns a state. The transition function will commonly be denoted by t, and in fact $t : S \times M \to S$ is a function.

(iv) A start state, one of the states in S such as s_0.

(v) A set of final or accepting states F. The set F is a subset of S.

For simplicity of notation, we write (S, M, s_0, F, t) for a deterministic finite automaton.

Remark 2.14 [5] Let (S, M, s_0, F, t) be a deterministic finite automaton. A word of M is the product of a finite sequence of elements in M, λ is empty word and M^* is the set of all words on M. We define recursively the extended transition function, $t^* : S \times M^* \to S$, as follows:

$\forall s \in S, \forall a \in M, t^*(s, a) = t(s, a),$
$\forall s \in S, t^*(s, \lambda) = s,$
$\forall s \in S, \forall x \in M^*, \forall a \in M, t^*(s, ax) = t^*(t(s, a), x).$

Note that the length $\ell(x)$ of a word $x \in M^*$ is the number of its letters; so $\ell(\lambda) = 0$ and $\ell(a_1a_2) = 2$, where $a_1, a_2 \in M$.

Definition 2.15 [4] The state s of $S - s_0$ will be called connected to the state s_0 of S if there exists $x \in M^*$, such that $s = t^*(s_0, x)$.

3. Hyper K-algebras induced by a deterministic finite automaton

In this paragraph, we present some relationships between hyper K-algebras and deterministic finite automata.

Definition 3.1 Let (S, M, s_0, F, t) be a deterministic finite automaton.

If $s \in S - \{s_0\}$ is connected to s_0, then the order of a state s is the natural number $l + 1$, where $l = \min\{\ell(x) | t^*(s_0, x) = s, x \in M^*\}$, and if $s \in S - \{s_0\}$ is not connected to s_0 we suppose that the order of s is 1. Also we suppose that the order of s_0 is 0.

We denote the order of a state s by ord s.

Now, we define the relation $\sim$ on the set of states S, as follows:

$$s_1 \sim s_2 \Leftrightarrow ords_1 = ords_2.$$

It is obvious that this relation is an equivalence relation on S.

Note that we denote the equivalence class of s by $\bar{s}$. Also we denote the set of all these classes by $\bar{S}$.

Theorem 3.2 *Let (S, M, s_0, F, t) be a deterministic finite automaton. We define the following hyper operation on S:*

$$\forall (s_1, s_2) \in S^2, s_1 o s_2 = \begin{cases} \bigcup\limits_{ords \leq ords_2} \bar{s} & , \text{if } ords_1 < ords_2,\ s_1, s_2 \neq s_0,\ s_1 \neq s_2 \\ \bigcup\limits_{s_0 \neq s, ords \leq ords_1} \bar{s} & , \text{if } ords_1 \geq ords_2,\ s_1, s_2 \neq s_0,\ s_1 \neq s_2 \\ \bigcup\limits_{ords \leq ords_1} \bar{s} & , \text{if } s_1 = s_2 \\ s_0 & , \text{if } s_1 = s_0,\ s_2 \neq s_0 \\ s_1 & , \text{if } s_2 = s_0,\ s_1 \neq s_0. \end{cases}$$

Then (S, o, s_0) is a hyper K-algebra and s_0 is the zero element of S.

Proof. It is easy to see that (S, o, s_0) satisfies (HK3). Since $tot = \bigcup\limits_{ords \leq ordt} \bar{s}$, we conclude that $s_0 \in tot$. So

$$t < t, \forall t \in S \tag{1}$$

By the definition of the hyper operation o, we know that $s_1 \in s_1 o s_2$, and so, $s_1 o s_2 \neq \phi$ for any $s_1, s_2 \in S$.

Since $s_1 \in s_1 o s_3$ and $s_2 o s_3 \neq \phi$, we obtain that $s_1 \in (s_1 o s_3) o (s_2 o s_3)$.

So, by (1) we get that

$$(s_1 o s_3) o (s_2 o s_3) < s_1 o s_2$$

That is (HK1) holds.

Now, we have to consider the following situations to prove (HK2).

(i) Let $s_1, s_2, s_3 \neq s_0$ and $ords_1 < ords_2 < ords_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

So, in this case (HK2) holds.

(ii) Let $s_1, s_2, s_3 \neq s_0$ and $ords_2 < ords_1 < ords_3$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}\right) os_3 = \bigcup_{ords\leq ords_3} \bar{s},$$

and

$$(s_1os_3)os_2 = \left(\bigcup_{ords\leq ords_3} \bar{s}\right) os_2 = \bigcup_{ords\leq ords_3} \bar{s},$$

since $s_2 \in \bigcup_{ords\leq ords_3} \bar{s}$.

Hence, in this case (HK2) holds.

(iii) Let $s_1, s_2, s_3 \neq s_0$ and $ords_2 < ords_3 < ords_1$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}\right) os_3 = \bigcup_{ords\leq ords_1} \bar{s},$$

since $s_3 \in \bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}$, and

$$(s_1os_3)os_2 = \left(\bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}\right) os_2 = \bigcup_{ords\leq ords_1} \bar{s},$$

since $s_2 \in \bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}$. Thus, in this case (HK2) holds.

The proofs of the following three situations are the same as (i), (ii) and (iii) respectively.

(iv) $s_1, s_2, s_3 \neq s_0$ and $ords_1 < ords_3 < ords_2$,

(v) $s_1, s_2, s_3 \neq s_0$ and $ords_3 < ords_1 < ords_2$,

(vi) $s_1, s_2, s_3 \neq s_0$ and $ords_3 < ords_2 < ords_1$.

(vii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2 < ords_3$ and $s_1 \neq s_2$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{s_0\neq s, ords\leq ords_2} \bar{s}\right) os_3 = \bigcup_{ords\leq ords_3} \bar{s},$$

and

$$(s_1os_3)os_2 = \left(\bigcup_{ords\leq ords_3} \bar{s}\right) os_2 = \bigcup_{ords\leq ords_3} \bar{s}.$$

Therefore, in this case (HK2) holds.

(viii) Let $s_1, s_2, s_3 \neq s_0$,$ords_1 = ords_2 > ords_3$ and $s_1 \neq s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_3 \in \bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s}$, and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_2 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$. So, in this case (HK2) holds.

The proofs of the following two situations are the same as (vii) and (viii), respectively.

(ix) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3 < ords_2$ and $s_1 \neq s_3$,

(x) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3 > ords_2$ and $s_1 \neq s_3$.

(xi) Let $s_1, s_2, s_3 \neq s_0$, $ords_2 = ords_3 > ords_1$ and $s_2 \neq s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s}.$$

Hence, in this case (HK2) holds.

(xii) Let $s_1, s_2, s_3 \neq s_0$, $ords_2 = ords_3 < ords_1$ and $s_2 \neq s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_1} \bar{s}.$$

Thus, in this case (HK2) holds.

(xiii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2 = ords_3$ and $s_1 \neq s_2 \neq s_3 \neq s_1$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_3 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$, and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_2 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$. Therefore, in this case (HK2) holds.

(xiv) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3$, $s_1 \neq s_3$ and $s_1 = s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s}.$$

So, in this case (HK2) holds.

The proof of the following situation is the same as (xiv).

(xv) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2$, $s_1 \neq s_2$ and $s_1 = s_3$.

(xvi) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2$, $s_1 \neq s_2$ and $s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_1} \bar{s}.$$

Hence, in this case (HK2) holds.

(xvii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_3$ and $s_1 = s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

Thus, in this case (HK2) holds.

(xviii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_3$ and $s_1 = s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_2 \in \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right)$. Therefore, in this case (HK2) holds.

The proofs of the following two situations are the same as (xvii) and (xviii) respectively.

(xix) $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_2$ and $s_1 = s_3$,

(xx) $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_2$ and $s_1 = s_3$.

(xxi) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_2$ and $s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

So, in this case (HK2) holds.

(xxii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_2$ and $s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \neq ords_1} \bar{s}.$$

Hence, in this case (HK2) holds.

(xxiii) Let $s_1 = s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = (s_1 o s_1) o s_1 = (s_1 o s_3) o s_2.$$

Thus, in this case (HK2) holds.

(xxiv) Let $s_1 = s_0$ and $s_2, s_3 \neq s_0$. Then $(s_1 o s_2) o s_3 = (s_0 o s_2) o s_3 = (s_0 o s_3) = s_0$ and $(s_1 o s_3) o s_2 = (s_0 o s_3) o s_2 = (s_0 o s_2) = s_0$. Therefore, in this case (HK2) holds.

(xxv) Let $s_2 = s_0$ and $s_1, s_3 \neq s_0$. Then $(s_1 o s_2) o s_3 = (s_1 o s_0) o s_3 = s_1 o s_3$ and $(s_1 o s_3) o s_2 = (s_1 o s_3) o s_0 = s_1 o s_3$. So, in this case (HK2) holds.

The proof of the following situation is the same as (xxv).

(xxvi) $s_3 = s_0$ and $s_1, s_2 \neq s_0$.

(xxvii) Let $s_1 \neq s_0$ and $s_2 = s_3 = s_0$. Then $(s_1 o s_2) o s_3 = (s_1 o s_0) o s_0 = (s_1 o s_0) = s_1$ and $(s_1 o s_3) o s_2 = (s_1 o s_0) o s_0 = (s_1 o s_0) = s_1$. Hence, in this case (HK2) holds.

(xxviii) Let $s_3 \neq s_0$ and $s_1 = s_2 = s_0$. Then $(s_1 o s_2) o s_3 = (s_0 o s_0) o s_3 = s_0 o s_3 = s_0$ and $(s_1 o s_3) o s_2 = (s_0 o s_3) o s_0 = s_0 o s_0 = s_0$. Thus, in this case (HK2) holds.

The proof of the following situation is the same as (xxviii).

(xxix) $s_2 \neq s_0$ and $s_1 = s_3 = s_0$. So, we obtain that (S, o, s_0) satisfies (HK2).

To prove (HK4), Let $s_1 < s_2$ and $s_2 < s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 < s_2$, there exist two cases:

(i) $ord s_1 < ord s_2$. Then $s_2 o s_1 = \bigcup_{s_0 \neq s, ord s \leq ord s_2} \bar{s}$. Therefore, $s_2 \not< s_1$, which is a contradiction.

(ii) $s_1 = s_0$, $s_2 \neq s_0$. Then, $s_2 o s_1 = s_2 o s_0 = s_2$. Thus, $s_2 \not< s_1$, which is a contradiction.

Now, to complete the proof, we should prove that (S, o, s_0) satisfies (HK5).

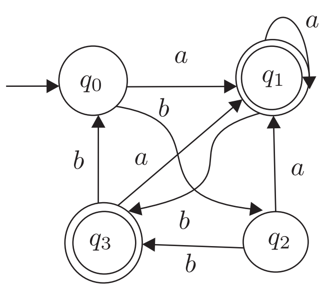
By the definition of the hyper operation o, we know that for any s_1 in S, $s_0 o s_1 = s_0$. Hence, $s_0 < s_1$.

Example 3.3 Let $A = (S, M, s_0, F, t)$ be a deterministic finite automaton such that $S = \{q_0, q_1, q_2, q_3\}$, $M = \{a, b\}$, $s_0 = q_0$, $F = \{q_1, q_3\}$ and t is defined by

$$t(q_0, a) = q_1, \quad t(q_0, b) = q_2, \quad t(q_1, a) = q_1, \quad t(q_1, b) = q_3$$

$$t(q_2, a) = q_1, \quad t(q_2, b) = q_3, \quad t(q_3, a) = q_1, \quad t(q_3, b) = q_0$$

It is clearly that $ord q_1 = ord q_2 = 2$, $ord q_3 = 3$ and $ord q_0 = 0$



According to the definition of the hyper operation "o", which is defined in Theorem 3.2, we have the following table.

o	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2, q_3\}$
q_2	q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2, q_3\}$
q_3	q_3	$\{q_1, q_2, q_3\}$	$\{q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$

Thus, (S, o, s_0) is a hyper K-algebra.

From now on, we let (S, o, s_0) be the hyper K-algebra, which is defined in Theorem 3.2.

Theorem 3.4 *(S, o, s_0) is a (weak, strong) implicative hyper K-algebra.*

Proof. By the definition of the hyper operation "o", we know that $s_1 \in s_1 o s_2$ and $s_1 o s_2 \neq \phi$ for all s_1, s_2 in S. So $s_1 \in s_1 o(s_2 o s_1)$, which implies that (S, o, s_0) is implicative.

Also, we show that $s_0 \in s_1 o s_1$ and $s_1 \in s_1 o(s_2 o s_1)$ for any s_1, s_2 in S. So, $s_1 < s_1 o(s_2 o s_1)$ and we obtain that (S, o, s_0) is weak implicative.

On the other hand, by the definition of the hyper operation "o", we have $s_1 o\ 0 = s_1$ and $s_1 \in s_1 o(s_2 o s_1)$. Thus, $s_1 o\ 0 \subseteq s_1 o(s_2 o s_1)$, which implies that (S, o, s_0) is strong implicative.

Remark 3.5 In (S, o, s_0), let F be a nonempty subset of $\bar{S}$, $s_0 = \overline{s_0} \in F$, $I = \bigcup_{\bar{t} \in F} \bar{t}$ and C be a nonempty subset of S. Then, I may not be a C-absorbing set. Because $s_1, s_2 \neq s_0$, $ords_1 < ords_2$, $F = s_0 \cup \overline{s_1}$ and $s_2 \in C$, then $s_1 \in I$ and $s_1 o s_2 = \bigcup_{ords \leq ords_2} \bar{s}$. So, $s_2 \in s_1 o s_2$ but $s_2 \notin I$.

Theorem 3.6 *In (S, o, s_0), any nonempty subset of S is an $\{s_0\}$-absorbing set.*

Proof. By definition of hyper operation "o" we know that for any s_1 in S, $s_1 o s_0 = s_1$. So it is clearly that for any nonempty subset I of S we have:
If $x \in I$ and $y = s_0 \Rightarrow xoy \subseteq I$.

Notation. We denote the class of all states which their order is n by $\overline{s_n}$.

Theorem 3.7 *For any $n \in N$, let $I_n = \{s \in S | s \in \bigcup_{i=0}^{n} \overline{s_i}\}$. Then I_n is:*

(i) *closed,*

(ii) *weak hyper K-ideal,*

(iii) *weak implicative hyper K-ideal for all $n \geq 1$.*

Proof.

(i) Suppose that $s_1 < s_2$ and $s_2 \in I_n$. Then $s_0 \in s_1 o s_2$. We have three cases:

1. $s_1, s_2 \neq s_0$ and $ords_1 < ords_2$.
   By definition of I_n, we can easily see that $s_1 \in I_n$.
2. $s_1 = s_2$.
   It is clear.
3. $s_1 = s_0$.
   By definition of I_n, it is obvious that $s_1 \in I_n$.

(ii) Assume that $s_1 o s_2 \subseteq I_n$ and $s_2 \in I_n$, then we have to consider the following situations:

1. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_2 < ords_1$.
   Since $s_1 o s_2 = \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \subseteq I_n$, we obtain that $ords_1 \leq n$. Hence $s_1 \in I_n$.
2. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_2 = ords_1$.
   By definition of I_n and the hyper operation "o", it is obvious that $s_1 \in I_n$.
3. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_1 < ords_2$.
   By definition of I_n and the hyper operation "o", it is easy to see that $s_1 \in I_n$.
4. $s_1 = s_2$.
   It is clear.
5. $s_2 = s_0$.
   Since $s_1 o s_2 = s_1 o s_0 = s_1$ and $s_1 o s_2 \subseteq I_n$, we obtain $s_1 \in I_n$.
6. $s_1 = s_0$.
   By definition of I_n, it is obvious that $s_1 \in I_n$.

(iii) Let $(s_1 o s_3) o (s_2 o s_1) \subseteq I_n$ and $s_3 \in I_n$. Since $s_1 \in s_1 o s_3$ and $s_2 o s_1 \neq \phi$ for any s_1, s_2, s_3 in S, we obtain that $s_1 \in (s_1 o s_3) o (s_2 o s_1)$. Therefore $s_1 \in I_n$.

Theorem 3.8 *Let I_n be a set, which is defined in Theorem 3.7. Then, I_n is a commutative hyper K-ideal of types 3,4, 5, 6 and 9.*

Proof. Let $(s_1 o s_2) o s_3 \subseteq I_n$ and $s_3 \in I_n$. Then, we should consider the following situations to prove that I_n is a commutative hyper K-ideal of type 4.

1. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 < ords_2$.
   Since I_n is a weak hyper K-ideal, $(s_1 o s_2) o s_3 \subseteq I_n$ and $s_3 \in I_n$, we obtain that $s_1 o s_2 = \bigcup_{ords \leq ords_2} \bar{s} \subseteq I_n$. Also we have: $s_2 o s_1 = \bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s}$, so

$$s_2 o (s_2 o s_1) = s_2 o \left(\bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s} \right) = \bigcup_{ords \leq ords_2} \bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = s_1o\left(\bigcup_{ords\leq ords_2} \bar{s}\right) = \bigcup_{ords\leq ords_2} \bar{s}.$$

It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

2. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 > ords_2$.
Since $s_0 \in I_n$ and $s_1os_2 = \bigcup_{s_0\neq s, ords\leq ords_1} \bar{s} \subseteq I_n$, we obtain that

$$\bigcup_{ords\leq ords_1} \bar{s} \subseteq I_n.$$

Also we have:

$$s_2os_1 = \bigcup_{ords\leq ords_1} \bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{ords\leq ords_1} \bar{s}\right) = \bigcup_{ords\leq ords_1} \bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = \bigcup_{ords\leq ords_1} \bar{s}.$$

Hence, $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

3. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 = ords_2$.
Since $s_0 \in I_n$ and $s_1os_2 = \bigcup_{s_0\neq s, ords\leq ords_1} \bar{s} \subseteq I_n$, we get that

$$\bigcup_{ords\leq ords_1} \bar{s} \subseteq I_n.$$

Also we have:

$$s_2os_1 = \bigcup_{s_0\neq s, ords\leq ords_1} \bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{s_0\neq s, ords\leq ords_1} \bar{s}\right) = \bigcup_{ords\leq ords_1} \bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = s_1o\left(\bigcup_{ords\leq ords_1} \bar{s}\right) = \bigcup_{ords\leq ords_1} \bar{s}.$$

It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

4. $s_1 = s_2$.
We know that $s_1os_2 = \bigcup_{ords\leq ords_1} \bar{s} \subseteq I_n$ and we have:

$$s_2os_1 = \bigcup_{ords\leq ords_1} \bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{ords\leq ords_1} \bar{s}\right) = \bigcup_{ords\leq ords_1} \bar{s}$$

and
$$s_1o(s_2o(s_2os_1)) = \bigcup_{ords \leq ords_1} \bar{s}.$$

Hence, $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

5. $s_1 = s_0$, $s_2 \neq s_0$.
Since $s_0 \in I_n$,
$$s_2os_1 = s_2os_0 = s_2,\ s_2o(s_2os_1) = s_2o(s_2os_0) = s_2os_2 = \bigcup_{ords \leq ords_2} \bar{s}$$
and
$$s_1o(s_2o(s_2os_1)) = s_0o\left(\bigcup_{ords \leq ords_2} \bar{s}\right) = s_0,$$
we obtain that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

6. $s_1 \neq s_0$, $s_2 = s_0$.
Since $s_1os_2 = s_1os_0 = s_1 \subseteq I_n$, we get that $s_1 \in I_n$. On the other hand we have: $(s_2os_1) = s_0os_1 = s_0$, $s_2o(s_2os_1) = s_0o(s_0os_1) = s_0os_0 = s_0$ and $s_1o(s_2o(s_2os_1)) = s_1os_0 = s_1$. It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$. So we obtain that I_n is a commutative hyper K-ideal of type 4.

Since $s_1o(s_2o(s_2os_1)) \neq \phi$ and I_n is a commutative hyper K-ideal of type 4, we get that I_n is a commutative hyper K-ideal of type 5.

On the other hand, by Theorem 2.8 we can easily see that I_n is a commutative hyper K-ideal of types 6, 9 and 3.

Theorem 3.9 *Let I_n be a set, which is defined in Theorem 3.7. Then I_n is a positive implicative hyper K-ideal of types 1, 2, 3, 4, 5, 6, 7, 8 and 9.*

Proof. Let for all s_1, s_2, s_3 in S, $(s_1os_2)os_3 \subseteq I_n$. By definition of hyper K-algebra we know that $(s_1os_2)os_3 = (s_1os_3)os_2$. So $(s_1os_3)os_2 \subseteq I_n$, and also we have for any s_1, s_2, s_3 in S, if $s_1os_2 \subseteq I_n$, then $s_1 \in I_n$, because $s_1 \in s_1os_2$. So if $(s_1os_3)os_2 \subseteq I_n$, then $s_1os_3 \subseteq I_n$. Therefore I_n is a positive implicative hyper K-ideal of types 1, 4 and 8. Also since for any s_1, s_3 in S, $s_1os_3 \neq \phi$ and $s_1os_3 \subseteq I_n$, we obtain that $s_1os_3 \cap I_n \neq \phi$ and $s_1os_3 < I_n$. So I_n is a positive implicative hyper K-ideal of types 2, 3, 5, 6, 7 and 9.

Remark 3.10 In (S, o, s_0), let $\exists n, m \in N$ such that $\overline{s_n} \neq \phi$, $\overline{s_m} \neq \phi$ and $m > n$. Here we give a subset I_n of S, which is not a positive implicative hyper K-ideal of types 10, 11, 12, , 26 and 27 but it is similar to the set I_n which is defined in Theorem 3.7.

Let $I_n = \bigcup_{i=0}^{n} \overline{s_i}$, $y \in \overline{s_n}$, $x \in \overline{s_m}$ and $z = s_0$, then we have:

1. Since $((xoy)oz) = (xoy)os_0 = xoy = \bigcup_{s_0 \neq s, ords \leq ordx} \bar{s}$, then $y \in \bigcup_{s_0 \neq s, ords \leq ordx} \bar{s}$. On the other hand we know that $y < y$. Thus $((xoy)oz) \cap I_n \neq \phi$ and $(xoy)oz < I_n$.

2. Since $yoz = yos_0 = y$, we obtain that $yoz \subseteq I_n$, $yoz < I_n$ and $yoz \cap I_n \neq \phi$.

3. Since $xoz = xos_0 = x$ and $x \notin I_n$, we get that $xoz \nsubseteq I_n$, $xoz \cap I_n = \phi$ and $x \nless I_n$.

By (1), (2), (3) and definition of the positive implicative hyper K-ideals of types 10, 11, 12,..., 26 and 27, we conclude that I_n is not a positive implicative hyper K-ideal of types 10,..., 27.

Theorem 3.11 (S, o, s_0) *is a hyper normal* K*-algebra of types 1 and 2 but it may not be simple.*

Proof. Since $a \in a$ and $a < a$, for any a, t in S, we have:

$$\begin{array}{ll} {}_{l1}a = \{t \in S | a < aot\} = S, & {}_{l2}a = \{t \in S | a \in aot\} = S, \quad \forall a \in S, \\ a_{r1} = \{t \in S | t < toa\} = S \quad \text{and} & a_{r2} = \{t \in S | t \in toa\} = S, \quad \forall a \in S. \end{array}$$

On the other hand, it is clear that S is a hyper K-ideal. So, (S, o, s_0) is a hyper normal K-algebra of types 1 and 2.

But, in Example 3.3, we saw that $q_0 \in q_1oq_3$ and $q_0 \notin q_3oq_1$. So $q_1 < q_3$ and $q_3 \nless q_1$. Hence, (S, o, s_0) may not be simple.

Theorem 3.12 *Let* (S', M, s'_0, F, t) *be a deterministic finite automaton. We define the following hyper operation on* S'*:*

$$\forall (s'_1, s'_2) \in S'^2, s'_1os'_2 = \begin{cases} s'_1 & , if \;\; s'_1 \neq s'_2, \;\; s'_1, s'_2 \neq s_0 \\ s'_1 & , if \;\; s'_2 = s'_0, \;\; s'_1 \neq s'_0 \\ s'_0 & , if \;\; s'_1 = s'_0, \;\; s'_2 \neq s'_0 \\ s'_0 \cup \overline{s'_1} & , if \;\; s'_1 = s'_2. \end{cases}$$

Then (S', o, s'_0) *is a hyper* K*-algebra and* s'_0 *is the zero element of* S'*.*

Proof. It is easy to see that (S', o, s'_0) satisfies (HK3), (HK4) and (HK5). Also the proof of (HK1) is similar to the proof of (HK1) in Theorem 3.2 by some suitable modifications. Now we consider the following situations to show that (S', o, s'_0) satisfies (HK2).

(i) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_2 \neq s'_3 \neq s'_1 \neq s'_2$. Then $(s'_1os'_2)os'_3 = s'_1os'_3 = s'_1$ and $(s'_1os'_3)os'_2 = s'_1os'_2 = s'_1$. So, in this case (HK2) holds.

(ii) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_3 \neq s'_1 = s'_2$. Then $(s'_1os'_2)os'_3 = (s'_0 \cup \overline{s'_1})os'_3 = s'_0 \cup \overline{s'_1}$ and $(s'_1os'_3)os'_2 = s'_1os'_2 = s'_0 \cup \overline{s'_1}$. Hence, in this case (HK2) holds.

The proof of the following case is the same as (ii).

(iii) $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_2 \neq s'_1 = s'_3$.

(iv) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_1 \neq s'_2 = s'_3$. Then, $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_2) o s'_2$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_2) o s'_2$. Thus, in this case (HK2) holds.

(v) Let $s'_1 = s'_2 = s'_3$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_1) o s'_1 = (s'_1 o s'_3) o s'_2$.
Therefore, in this case (HK2) holds.

(vi) Let $s'_1 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_0 o s'_2) o s'_3 = s'_0 o s'_3 = s'_0$ and $(s'_1 o s'_3) o s'_2 = (s'_0 o s'_3) o s'_2 = s'_0 o s'_2 = s'_0$. So in this case (HK2) holds.

(vii) Let $s'_2 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_0) o s'_3 = s'_1 o s'_3$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_3) o s'_0 = s'_1 o s'_3$. Hence, in this case (HK2) holds.

The proof of the following case is the same as (vii).

(viii) $s'_3 = s'_0$.

(ix) Let $s'_1 = s'_2 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_0 o s'_0) o s'_3 = s'_0 o s'_3 = s'_0$ and $(s'_1 o s'_3) o s'_2 = (s'_0 o s'_3) o s'_0 = s'_0 o s'_0 = s'_0$. Thus, in this case (HK2) holds.

The proof of the following case is the same as (ix).

(x) $s'_1 = s'_3 = s'_0$.

(xi) Let $s'_2 = s'_3 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_0) o s'_0 = s'_1 o s'_0 = s'_1$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_0) o s'_0 = s'_1 o s'_0 = s'_1$. So, in this case (HK2) holds.

Finally, we conclude that (S', o, s'_0) is a hyper K-algebra.

Example 3.13 Consider the deterministic finite automaton $A = (S, M, s_0, F, t)$ in Example 3.3. Then the structure of the hyper K-algebra (S, o, s_0) induced on the states of this automaton according to Theorem 3.12 is as follows:

o	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	$\{q_0, q_1, q_2\}$	q_1	q_1
q_2	q_2	q_2	$\{q_0, q_1, q_2\}$	q_2
q_3	q_3	q_3	q_3	$\{q_0, q_3\}$

Note that, if we compare the above table with the table of the Example 3.3, we see that the induced structures of the hyper K-algebras are different. So, the two methods give two different structures.

Theorem 3.14 *Let (S', o, s'_0) be the hyper K-algebra, which is defined in Theorem 3.12, F be a nonempty subset of $\overline{S'}$ and $s'_0 = \overline{s'_0} \in F$. Also, suppose that $I = \bigcup_{\bar{t} \in F} \bar{t}$ and C be a nonempty subset of S'. Then I is a C-absorbing set.*

Proof. Let $s' \in I$ and $t \in C$. then $s' o t = s'$ or $s' o t = s'_0 \cup \overline{s'}$. Since $s' \in I$, by definition of I we know that $\overline{s'} \subseteq I$ and $s'_0 \in I$. Hence $s' o t \subseteq I$.

Theorem 3.15 *Let (S', o, s'_0) be the hyper K-algebra, which is defined in Theorem 3.12. Then (S', o, s'_0) is*

1. *a hyper normal K-algebra of types 1 and 2,*
2. *a simple hyper K-algebra*

Proof.

1. Since $a \in aot$ and $a < a$, for any a, t in S', we have:

$$
\begin{array}{ll}
{}_{l1}a = \{t \in S' | a < (aot)\} = S', & {}_{l2}a = \{t \in S' | a \in (aot)\} = S', \quad \forall a \in S', \\
a_{r1} = \{t \in S' | t < (toa)\} = S' \quad \text{and} & a_{r2} = \{t \in S | t \in (toa)\} = S', \quad \forall a \in S'.
\end{array}
$$

   We know that S' is a hyper K-ideal. So, (S', o, s'_0) is a hyper normal K-algebra of types 1 and 2.

2. Let $s'_1 \neq s'_2$ and $s'_1, s'_2 \neq s'_0$, then $s'_1 o s'_2 = s'_1$ and $s'_2 o s'_1 = s'_2$. Hence, $s'_1 \not< s'_2$ and $s'_2 \not< s'_1$. So (S', o, s'_0) is a simple hyper K-algebra.

Theorem 3.16 *Let (S, M, s_0, F, t) be a deterministic finite automaton. We define the following hyper operation on $\bar{S}$:*

$$
\forall (\overline{s_1}, \overline{s_2}) \in \bar{S}^2, \overline{s_1} o \overline{s_2} = \begin{cases} \{\overline{s_1}, \overline{s_2}\}, & \text{if} \quad \overline{s_1} \neq \overline{s_2}, \quad \overline{s_1} \neq \overline{s_0} \neq \overline{s_2} \\ \{\overline{s_1}, \overline{s_0}\}, & \text{if} \quad \overline{s_1} = \overline{s_2} \\ \{\overline{s_2}, \overline{s_0}\}, & \text{if} \quad \overline{s_1} = \overline{s_0}, \quad \overline{s_2} \neq \overline{s_0} \\ \overline{s_1}, & \text{if} \quad \overline{s_1} \neq \overline{s_0}, \quad \overline{s_2} = \overline{s_0}. \end{cases}
$$

Then, $(\bar{S}, o, \overline{s_0})$ is a hyper K-algebra and $\overline{s_0}$ is the zero element of $\bar{S}$.

Proof. It is easy to see that $(\bar{S}, o, \overline{s_0})$ satisfies (HK3), (HK4) and (HK5). Also the proof of (HK1) is similar to the proof of (HK1) in Theorem 3.2 by some suitable modifications. Now we consider the following situations to show that $(\bar{S}, o, \overline{s_0})$ satisfies (HK2).

(i) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2} \neq \overline{s_3} \neq \overline{s_1}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_3} = \{\overline{s_1}, \overline{s_2}, \overline{s_3}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_3}\}$.
So, in this case (HK2) holds.

(ii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_2} \neq \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_0}\} o \overline{s_3} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_1} = \{\overline{s_1}, \overline{s_3}, \overline{s_0}\}$.
Hence, in this case (HK2) holds.

(iii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_3} \neq \overline{s_2}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_1} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_0}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$.
Thus, in this case (HK2) holds.

(iv) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_2} = \overline{s_3} \neq \overline{s_1}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$.
Therefore, in this case (HK2) holds.

(v) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_1} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_3} = \{\overline{s_2}, \overline{s_3}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_3}\} o \overline{s_2} = \{\overline{s_2}, \overline{s_3}, \overline{s_0}\}$.
So, in this case (HK2) holds.

(vi) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_3}$. Then,
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_3} = \{\overline{s_1}, \overline{s_3}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_0} = \{\overline{s_1}, \overline{s_3}\}$.
Hence, in this case (HK2) holds.

The proof of the following case is the same as (vi).

(vii) $\overline{s_1}, \overline{s_2} \neq \overline{s_0}, \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2}$.

(viii) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}, \overline{s_1} = \overline{s_0}$ and $\overline{s_2} = \overline{s_3}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$.
Thus in this case (HK2) holds.

(ix) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} = \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_1} = \{\overline{s_0}, \overline{s_1}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_1}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_1}\}$.
So in this case (HK2) holds.

The proof of the following case is the same as (ix).

(x) $\overline{s_1}, \overline{s_2} \neq \overline{s_0}, \overline{s_3} = \overline{s_0}$ and $\overline{s_1} = \overline{s_2}$.

(xi) Let $\overline{s_1} = \overline{s_2} = \overline{s_0}$ and $\overline{s_3} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_0} o \overline{s_3} = \{\overline{s_0}, \overline{s_3}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_3}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_3}\}$.
Hence, in this case (HK2) holds.

(xii) Let $\overline{s_1} = \overline{s_3} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_2}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \overline{s_0} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$.
Thus, in this case (HK2) holds.

(xiii) Let $\overline{s_2} = \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_0} = \overline{s_1}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \overline{s_1} o \overline{s_0} = \overline{s_1}$.
Hence, in this case (HK2) holds.

(xiv) Let $\overline{s_1} = \overline{s_2} = \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = (\overline{s_1} o \overline{s_1}) o \overline{s_1} = (\overline{s_1} o \overline{s_3}) o \overline{s_2}$.
Hence, in this case (HK2) holds.

Finally, we obtain that $(\bar{S}, o, \overline{s_0})$ is a hyper K-algebra.

Theorem 3.17 *Consider the deterministic finite automaton* $A = (S, M, s_0, F, t)$ *in Example* 3.3. *Then the structure of the hyper* K*-algebra* $(\bar{S}, o, \overline{s_0})$ *induced on* $\bar{S}$ *according to Theorem* 3.16 *is as follows:*

o	$\overline{q_0}$	$\overline{q_1}$	$\overline{q_3}$
$\overline{q_0}$	$\overline{q_0}$	$\{\overline{q_0}, \overline{q_1}\}$	$\{\overline{q_0}, \overline{q_3}\}$
$\overline{q_1}$	$\overline{q_1}$	$\{\overline{q_0}, \overline{q_1}\}$	$\{\overline{q_1}, \overline{q_3}\}$
$\overline{q_3}$	$\overline{q_3}$	$\{\overline{q_3}, \overline{q_1}\}$	$\{\overline{q_0}, \overline{q_3}\}$

Theorem 3.18 *Let* $(\bar{S}, o, \overline{s_0})$ *be the hyper* K*-algebra, which is defined in Theorem* 3.16. *Then* $(\bar{S}, o, \overline{s_0})$ *is*

(1) *a hyper normal* K*-algebra of types* 1 *and* 2*,*

(2) *a simple hyper* K*-algebra.*

Proof.

1. Since $\bar{a} \in \bar{a}o\bar{t}$ and $\bar{a} < \bar{a}$, for any $\bar{a}, \bar{t}$ in $\bar{S}$, we have:

$$_{l1}a = \{\bar{t} \in \bar{S} | \bar{a} < \bar{a}o\bar{t}\} = \bar{S}, \qquad _{l2}a = \{\bar{t} \in \bar{S} | \bar{a} \in \bar{a}o\bar{t}\} = \bar{S}, \quad \forall \bar{a} \in \bar{S},$$
$$\bar{a}_{r1} = \{\bar{t} \in \bar{S} | \bar{t} < \bar{t}o\bar{a}\} = \bar{S} \quad \text{and} \quad a_{r2} = \{\bar{t} \in \bar{S} | \bar{t} \in \bar{t}o\bar{a}\} = \bar{S}, \quad \forall \bar{a} \in \bar{S}.$$

It is easy to see that $\bar{S}$ is a hyper K-ideal. So $(\bar{S}, o, \overline{s_0})$ is a hyper normal K-algebra of types 1 and 2.

2. Let $\overline{s_1} \neq \overline{s_2}$ and $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, then $\overline{s_1} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}\}$ and $\overline{s_2} o \overline{s_1} = \{\overline{s_1}, \overline{s_2}\}$. Hence, $\overline{s_1} \not< \overline{s_2}$ and $\overline{s_2} \not< \overline{s_1}$. So $(\bar{S}, o, \overline{s_0})$ is a simple hyper K-algebra.

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