SOME RESULTS ON NON-COMMUTING GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite non-abelian group. We define a graph Γ_G, called the non-commuting graph of G, with vertex set $G - Z(G)$ such that two vertices x and y adjacent if and only if $xy \neq yx$. In this paper some results on the number of edges of Γ_G and also its chromatic number are obtained in general. For some special group G we will prove that if H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$ and in some cases $G \cong H$.

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1. Introduction

Let G be a group. There are several ways to associate a graph to G. The one we will consider in this paper is denoted by Γ_G and is called the non-commuting graph of G. The vertex set of Γ_G is $V(\Gamma_G) = G - Z(G)$, where $Z(G)$ is the center of G and the edge set $E(\Gamma_G)$ contains (x, y) as an edge if and only if $xy \neq yx$. Since we consider simple graphs, hence (x, y) and (y, x) are the same edge and there is no edge of the shape (x, x) in $E(\Gamma_G)$. It is clear that if G is abelian, then Γ_G is the null graph, hence in what follows we will assume that G is a non-abelian group.

According to [8] the non-commuting graph of a finite group G was first introduced by Paul Erdös in connection with the following problem: let G be a group whose non-commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By [8] the answer to this question is positive and this was the origin of many similar questions and research.

In [1], relation between some graph theoretical properties of Γ_G and the group theory properties of the group G are studied. In particular the following two conjectures are raised:

Conjecture 1 *Let G be a finite non-abelian group. If there is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Conjecture 2 *Let S be a finite non-abelian simple group. Let G be a group such that $\Gamma_G \cong \Gamma_S$, then $G \cong S$.*

Our aim in this paper is to verify the above conjectures for some classes of finite groups. We also obtain some results about the number of edges of the graph Γ_G. Our notation for graphs is standard and [2] is a general reference.

2. Some results on the number of edges

Let G be a finite non-abelian group. The number of conjugacy classes of G is denoted by $k(G)$. By Lemma 3.27 in [1], the number of edges in Γ_G is

$$|E(\Gamma_G)| = \frac{1}{2}\,|G|\,(|G| - k(G)).$$

Proposition 1 *Let G be a non-abelian finite group and let Γ_G be its non-commuting graph. Then $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. Moreover equality holds if and only if G is the direct product of an abelian group and a 2-group H such that $|H| \geqslant 8$ and H is indecomposable with $\frac{k(H)}{|H|} = \frac{5}{8}$.*

Proof. On the contrary assume that $|E(\Gamma_G)| < \frac{3}{10}k(G)\,|G|$. Then by substituting $|E(\Gamma_G)| = \frac{1}{2}\,|G|\,(|G| - k(G))$ we will obtain $|G| < \frac{8}{5}k(G)$. Therefore $\frac{k(G)}{|G|} > \frac{5}{8}$. Now, by [4], the probability of commuting two randomly chosen elements of a finite group G is equal to $\frac{k(G)}{|G|}$, and $\frac{k(G)}{|G|} > \frac{5}{8}$ implies that G is abelian, a contradiction. Therefore, $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. Equality holds if and only if $\frac{k(G)}{|G|} = \frac{5}{8}$, which by [4] the conclusion follows. ∎

Proposition 2 *For any finite non-abelian group G we have $|E(\Gamma_G)| \neq 2\,|G|$.*

Proof. Assume $|E(\Gamma_G)| = 2\,|G|$. Then, by substituting the value of $E(\Gamma_G)$ we will obtain $|G| = k(G) + 4$. But by Proposition 1 we have $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$, hence $k(G) \leqslant \frac{20}{3} < 7$. Therefore, $k(G) = 1, 2, \ldots, 6$. Now, from $|G| = k(G) + 4$, we obtain $(k(G), |G|) = (1, 5), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10)$. Next a routine examination of the group orders eliminates all the above possibilities. ∎

Lemma 1 *Let G be a finite non-abelian group and n be a natural number. If $|G| > \frac{16n}{3}$, then $|E(\Gamma_G)| > n\,|G|$.*

Proof. By Proposition 1 we have $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. If $|E(\Gamma_G)| \leqslant n\,|G|$, then $k(G) \leqslant \frac{10n}{3}$. But $|E(\Gamma_G)| = \frac{|G|\,(|G| - k(G))}{2} \leqslant n\,|G|$ implies $|G| \leqslant 2n + k(G)$. Hence $|G| \leqslant 2n + k(G) \leqslant 2n + \frac{10n}{3} = \frac{16n}{3}$ a contradiction. ∎

Corollary 1 *Let G be a non-abelian finite group. If $|E(\Gamma_G)| \leqslant 31\,|G|$ and G is simple, then $G \cong \mathbb{A}_5$.*

Proof. If $|E(\Gamma_G)| \leqslant 31\,|G|$, then, by Lemma 1, we obtain $|G| \leqslant 166$. Since G is assumed to be simple, hence $G \cong \mathbb{A}_5$. ∎

Theorem 1 *Let G be a finite non-abelian group. Then $|E(\Gamma_G)| \geqslant \frac{3}{2}\,|G|$. Moreover equality holds if and only if $G \cong S_3, D_8$ or Q_8.*

Proof. Assume $|E(\Gamma_G)| < \frac{3}{2}\,|G|$. Substituting the value of $E(\Gamma_G)$ we will obtain $|G| - k(G) < 3$. But $|G| - k(G) \geqslant 1$ since G is non-abelian, hence$|G| - k(G) = 1$ or 2. By [4] for any non-abelian finite group we have $\frac{k(G)}{|G|} \leqslant \frac{5}{8}$. Now combining $|G| - k(G) = 1$ or 2 with the last inequality we will obtain $|G| \leqslant 2$ or 5 respectively; and in both cases G will be non-abelian, a contradiction.

If $|E(\Gamma_G)| = \frac{3}{2}G$, then $|G| - k(G) = 3$. Using $\frac{k(G)}{|G|} \leqslant \frac{5}{8}$ we obtain $|G| \leqslant 8$. Now, examination of non-abelian groups of order less than or equal to 8 yields $G \cong S_3, D_8$ or Q_8. ∎

3. Partition of non-commuting graph

Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is called r-partite if V admits a partition into r classes that every edge has its ends in different classes and vertices in the same partition class must not be adjacent. A 2-partite graph is said be bipartite. Any graph $G(V, E)$ is a $|V(G)|$-partite. In this position every vertex is a class of partition. this partition is said to be the trivial partition.

In the following by a finite group G we mean a non-abelian finite group.

Proposition 3 *Let G be a group, then Γ_G is not bipartite.*

Proof. Assume that Γ_G is bipartite and V_1 and V_2 are classes of this partition. There are $x_1 \in V_1$ and $x_2 \in V_2$ such that $x_1x_2 \neq x_2x_1$, so we have $G \neq C_G(x_1) \cup C_G(x_2)$. Then there is a $y \in G - Z(G)$ such that $y \notin C_G(x_1) \cup C_G(x_2)$. So y is adjacent to both x_1 and x_2. So $y \notin V_1 \cup V_2 = V(\Gamma_G)$. Therefore $y \in Z(G)$ and it is a contradiction.Thus G is not bipartite. ∎

Proposition 4 *Let G be a group. Then Γ_G is not a complete graph.*

Proof. Assume that Γ_G is a complete graph. In this case first we will prove that any non-central element of G has order 2. If $x \in G - Z(G)$, then $d(x) = |G| - |C_G(x)|$ and by the above discussion we have $d(x) = |G| - |Z(G)| - 1$. So $|G| - |C_G(x)| = |G| - |Z(G)| - 1$ and then $|C_G(x)| = |Z(G)| + 1$. But we know $|Z(G)| \Big| |C_G(x)|$, here $|Z(G)| \Big| |Z(G)| + 1$ and then $|Z(G)| = 1$. Therefore $|C_G(x)| = 2$ and the order of x is 2. If $y \in G - Z(G)$ and $y \neq x$, then $yx \in G - Z(G) = G - \{1_G\}$ and by above discussion y is adjacent to x. So $o(yx) = 2$ and we have $yxyx = 1$. This implies that $yx = xy$ and this means that $y \in C_G(x)$ and it contradicts the fact that x is adjacent to y. So Γ_G is not a complete graph and the proof is complete. ∎

4. Verification of the conjectures for certain groups

For certain non-abelian finite group G we want to show that if H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$, and in the case that G is a non-abelian simple group, then $G \cong H$. Note that $\Gamma_G \cong \Gamma_H$ is a graph isomorphism, that is a one-to-one correspondence $\varphi : G - Z(G) \longrightarrow H - Z(H)$ such that φ preserves edges, i.e. if $x, y \in G - Z(G)$, and $xy \neq yx$, then $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$. Equivalently if we consider the complimentary graph of Γ_G we have the following condition: $x, y \in G - Z(G), xy = yx \Longrightarrow \varphi(x)\varphi(y) = \varphi(y)\varphi(x)$

The isomorphism $\Gamma_G \cong \Gamma_H$ implies that $|G - Z(G)| = |H - Z(H)|$. Since G is assumed to be non-abelian, hence $0 \neq |G - Z(G)| = |H - Z(H)|$, implying that H is non-abelian. We also have $|Z(H)| \leqslant |H - Z(H)|$, so $Z(H)$ is a finite group. Therefore H is a finite non-abelian group.

The degree of a vertex v in graph Γ is defined to be the number of edges adjacent to v, and is denoted by $d(v)$. Now it is easy to see that the degree of a vertex g in the graph Γ_G is equal to $d(g) = |G| - |C_G(g)|$. Our first result concerns the degree properties of the commuting graph of G.

Proposition 5 *Let G be a finite non-abelian group such that there is an element $g \in G - Z(G)$ with $d(g) = p^n$, $n \in \mathbb{N}$, p a prime number. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We know $d(g) = |G| - |C_G(g)| = |C_G(g)|\,([G : C_G(g)] - 1) = p^n$. Therefore $|C_G(g)| = p^m$ for some m, $1 \leqslant m \leqslant n$. Hence we obtain $|G| = p^n + p^m$. Let

$g' \in H - Z(H)$ be the corresponding vertex in Γ_H under the isomorphism $\Gamma_G \cong \Gamma_H$. Therefore $d(g') = p^n$ and similarly we will obtain $|C_H(g)| = p^{m'}$, $1 \leqslant m' \leqslant n$, and finally $|H| = p^n + p^{m'}$.

From $Z(G) \lneqq C_G(g)$ we can put $|Z(G)| = p^l$ where $1 \leqslant l < m$. Similarly since $Z(H) \lneqq C_G(g')$ we put $|Z(H)| = p^{l'}$, $1 \leqslant l' < m'$. Now using the equality $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^m - p^l = p^{m'} - p^{l'}$, therefore $p^m(p^{m-l} - 1) = p^{m'}(p^{m'-l'} - 1)$. Since $p^{m-l} - 1$ and $p^{m'-l'} - 1$ are non-zero and relatively prime to p we deduce $m = m'$, whence $|G| = |H|$ and the Proposition is proved. ∎

Proposition 6 *Let G be a finite non-abelian group such that there is an element $g \in G - Z(G)$ with $d(g) = pq$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. From $pq = |C_G(g)|\,([G : C_G(g)] - 1)$ we deduced that $|C_G(g)| = p, q$ or pq, hence $|G| = pq + p$, $pq + q$ or $2pq$. Since the corresponding element $g' \in H - Z(H)$ has also degree pq we will obtain $|H| = pq + p$, $pq + q$ or $2pq$. Therefore to prove $|G| = |H|$ it is enough to prove, for example, $|G| = pq + p$ and $|H| = 2pq$ are impossible. From $|G| = pq + p$ we obtain $|C_G(g)| = p$, hence $|Z(G)| = 1$. Therefore using the equality $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $|Z(H)| = pq - p + 1$. we must have $|Z(H)| = pq - p + 1 \mid |H| = 2pq$ and an easy calculation shows that this is impossible. ∎

Proposition 7 *Let G be a non-abelian group and Γ_G be the non-commuting graph of G and let $g \in G - Z(G)$ be an element such that $d(g) = pqr$ where p, q and r are distinct primes where $p < q < r$ and $q \nmid r - 1$ and $p \nmid r - 1$. If H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. From $d(g) = |G| - |C_G(g)|$ it follows that

$$pqr = d(g) = |C_G(g)| \left(\left| \frac{G}{C_G(g)} \right| - 1 \right)$$

and so $|C_G(g)| = p, q, r, pq, pr, qr$ or pqr. Therefore by $|G| = d(g) + |C_G(g)|$ we will obtain $|G| = pqr + p, pqr + q, pqr + r, pqr + pq, pqr + pr, pqr + qr$ or $pqr + pqr$. Because of $\Gamma_G \cong \Gamma_H$, if $h \in H - Z(H)$ is the corresponding element with g, then all of cases will happen for $|C_H(h)|$ and $|H|$. Now we consider different cases and show that $|G| = |H|$.

Case 1. $|G| = pqr + p$ and $|H| = pqr + r$. In this case $|C_G(g)| = p$ and $|C_H(h)| = r$ and then $|Z(G)| = 1 = |Z(H)|$. Now by $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $|G| = |H|$. Therefore in all cases that $|C_G(g)|$ and $|C_H(h)|$ are prime we can show $|G| = |H|$.

Case 2. $|G| = pqr + p$ and $|H| = pqr + pq$. In this case $|C_G(g)| = p$ and $|C_H(h)| = pq$ and so $|Z(G)| = 1$ and $|Z(H)| = p, q$ or 1. By $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $pqr + p - 1 = pqr + pq - |Z(H)|$, hence $|Z(H)| = pq - p + 1 = p(q - 1) + 1$. If $|Z(H)| = p$, then $p = p(q - 1) + 1$ which is a contradiction. If $|Z(H)| = q$, then $q = p(q-1)+1$. Thus $p(q-1) = q-1$ that implies $p = 1$ and this is a contradiction. So we will obtain that $|Z(H)| = 1$ and then $|Z(G)| = 1 = |Z(H|$ implying that $|G| = |H|$.

Case 3. $|G| = pqr + +p$ and $|H| = pqr + qr$. In this case $|C_G(g)| = p$ and so $|Z(G)| = 1$ and from $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $p - 1 = qr - |Z(H)|$. Now, by $|C_H(h)| = qr$, we have $|Z(H)| = q, r$ or 1. If $|Z(H)| = q$, then $p-1 = qr-q = q(r-1)$ which contradicts $p < q$. If $|Z(H)| = r$, then by the same argument as in the case of $|Z(H)| = q$ we obtain a contradiction. Therefore $|Z(H)| = 1$ and $|G| = |H|$.

Case 4. $|G| = pqr + q$ and $|H| = pqr + pr$. Then with the same argument as in the case 3 we have $|C_G(g)| = q$ and so $|Z(G)| = 1$. Also from $|C_H(h)| = pr$ we have $|Z(H)| = p, r$ or 1. If $|Z(H)| = p$, then $q-1 = pr-p = p(r-1)$ and this contradicts $q < r$. If $|Z(H)| = r$, then $q-1 = pr-r = p(r-1)$ which contradicts $q < r$. So $|Z(H)| = 1$ hence $|G| = |H|$.

Case 5. $|G| = pqr + r$ and $|H| = pqr + pq$. In this case $|C_G(g)| = r$ and so $|Z(G)| = 1$. From $|C_H(h)| = pq$ we will obtain $|Z(H)| = p, q$ or 1. If $|Z(H)| = p$, then $r-1 = pq-p = p(q-1)$ that implies $p|r-1$ and this is a contradiction. If $|Z(H)| = q$, then $r-1 = pq-q = q(p-1)$ that implies $q|r-1$ and this is a contradiction. Thus $|Z(H)| = 1$ and from $|Z(H)| = 1 = |Z(G)|$ we deduce that $|G| = |H|$.

Case 6. $|G| = pqr+p$ and $|H| = pqr+pqr$. In this case $|C_G(g)| = p$ and $|C_H(h)| = pqr$. So $|Z(G)| = 1$ and by $|G| - |Z(G)| = |H| - |Z(H)|$ we have $|Z(H)| = pqr - (p-1)$. Now, by Lemma 3.1 in [1], we know $|Z(H)| \Big| |C_G(g)| - |Z(G)|$, hence $pqr - (p-1)|p-1$ and this is a contradiction because of $pqr - (p-1) > p-1$.

The cases $|G| = pqr + q, |H| = pqr + pqr$ or $|G| = pqr + r, |H| = pqr + pqr$ are same as the case 6 and are omitted.

Case 7. $|G| = pqr + pq$ and $|H| = pqr + pqr$. In this case $|C_G(g)| = pq$ and $|C_H(h)| = pqr$ and so $|Z(G)| = p, q$ or 1. If $|Z(G)| = p$, then from $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $|Z(H)| = pqr - pq + p$. On the other hand we know $|Z(H)| \Big| |C_G(g)| - |Z(G)|$ which implies that $|Z(H)| \Big| p-1$. So $p(qr-q+1)|p-1$ and this is a contradiction. If $|Z(G)| = q$, by the same argument we obtain a contradiction and therefore $|Z(G)| = 1$. Hence $|Z(H)| = pqr - pq + 1 = pq(r-1)+1$. But from $|Z(H)| \Big| |C_G(g)| - |Z(G)|$ we get $pq(r-1)+1|pq-1$ which contradicts $pq(r-1)+1 > pq-1$. So this case is impossible. Thus in all of the cases we have $|G| = |H|$ and the proof is completed. ∎

Proposition 8 *Let G be a finite non-abelian group such that there is a $g \in G - Z(G)$ in the non-commuting graph of G with degree $d(g) = p^n q$ where p, q are primes and $p > q$. If H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We have $d(G) = p^n q = |G| - |C_G(g)| = |C_G(g)|(|\frac{G}{C_G(g)}| - 1)$. So $|C_G(g)| = p^{n''}, q$ or $p^{n'}q$ such that n', n'' are non-negative integers and $n' \leq n$ and $n'' \leq n$.

Hence $|G| = p^nq + p^{n''}, p^nq + q$ or $p^nq + p^{n'}q$. If the corresponding element with g in $\Gamma_G \cong \Gamma_H$ is $h \in H - Z(H)$, then h has also degree p^nq and we deduce that $|H| = p^nq + p^{m''}, p^nq + q$ or $p^nq + p^{m'}q$ such that m', m'' are non-negative integers such that $m'' \leq m$ and $m' \leq m$. Now we consider different cases.

Case 1. $|G| = p^nq + p^{n''}, n'' \leq n$ and $|H| = p^nq + q$. In this case we have $|C_G(g)| = p^{n''}$ and $|C_H(h)| = q$, so $|Z(G)| = p^{n_1}$ such that $n_1 < n''$ and $|Z(H)| = 1$. From $|G| - |Z(G)| = |H| - |Z(H)|$ we get $p^nq + p^{n''} - p^{n_1} = p^nq + q - 1$. So $p^{n''} - p^{n_1} = q - 1$ and then $p^{n_1}(p^{n''-n_1} - 1) = q - 1$. Because of assumption $p > q$, we must have $n_1 = 0$ and then $p^{n''} - 1 = q - 1$. Thus $p^{n''} = q$ which is a contradiction. So this case is impossible.

Case 2. $|G| = p^nq+p^{n''}, n'' \leq n$ and $|H| = p^nq+p^{m''}, m'' \leq n$. Then $|C_G(g)| = p^{n''}$ and $|C_H(h)| = p^{m''}$. Now from $|Z(G)| \Big| |C_G(g)|$ we obtain $|Z(G)| = p^{n_1}$ where $n_1 < n''$. Thus from $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^nq + p^{n''} - p^{n_1} = p^nq + p^{m''} - |Z(H)|$ and so $|Z(H)| = p^{m''} + p^{n_1} - p^{n''}$ such that $p^{m''} > p^{n_1}$(if $p^{m''} < p^{n_1}$, then $p^{m''}+p^{n_1} < 2p^{n_1}$ and so $|Z(H)| = p^{m''}+p^{n_1}-p^{n''} \leq 2p^{n_1}-p^{n''} \leq 0$ which is a contradiction). Therefore $|Z(H)| = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$ and $m'' - n_1 \neq 0$ and $n'' - n_1 \neq 0$. On the other hand we know $Z(H) \leq C_H(h)$, so $|Z(H)| = p^{m_2}$ such that $m_2 < m''$ and then $p^{m_2} = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$. Hence $p^{m''-n_1} - p^{n''-n_1} + 1 = pk+1$ where k is a non-negative integer. But $gcd(pk+1, p)$=1 and by $p^{m_2} = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$ we have $pk = p^{m'-n_1} - p^{n'-n_1} = 0$. So we have $m'' = n''$ and in this case $|G| = |H|$.

Case 3. $|G| = p^nq + q$ and $|H| = p^nq + p^{m'}q$ where $m' \leq n$. Then $|C_G(g)| = q$ and $|C_G(h)| = p^{m'}q$ and so $|Z(G)| = 1$. Now, by Lemma 3.1 in [1], $|Z(H)| \big| q - 1$ and thus $gcd(|Z(H)|, q) = 1$ and from $|C_H(h)| = p^{m'}q$ we obtain $|Z(H)| = p^{m_1}$ where $m_1 \leq m'$. Therefore from $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $q - 1 = p^{m'}q - p^{m_1}$ and so $p^{m_1}(p^{m'-m_1}q - 1) = q - 1$. From this equality we must have $m_1 = 0$ and $m' - m_1 = 0$, so $|Z(H)| = 1$ and then $|Z(G)| = |Z(H)| = 1$ completes this case.

Case 4. $|G| = p^nq + p^{n''}$ and $|H| = p^nq + p^{m'}q$ where $n'' \leq n$ and $m' \leq n$. Then $|C_G(g)| = p^{n''}$ and $|C_H(h)| = p^{m'}q$ and so $|Z(G)| = p^{n_1}$ where $n_1 < n''$. By $|G| - |Z(G)| = |H| - |Z(H)|$ we have $p^{n''} - p^{n_1} = p^{m'}q - |Z(H)|$, hence $|Z(H)| = p^{m'}q + p^{n_1} - p^{n''}$. From lemma 3.1 in [1] we obtain $|Z(H)| \Big| p^{n''} - p^{n_1}$ and so $|Z(H)| \Big| p^{n_1}(p^{n''-n_1} - 1)$. But we know that $|Z(H)| \Big| |C_H(h)|$ and so $|Z(H)| = p^{s'}q$ or p^s where $s' < m'$ and $s \leq m'$. Now, by $|Z(G)| \Big| |C_H(h)| - |Z(H)|$ we will obtain $p^{n_1} \Big| p^{s'}q(p^{m'-s'} - 1)$ or $p^s(p^{m'-s}q - 1)$. Therefore from $gcd(p^{m'-s'}q - 1, p) = 1$ and $p > q$ we obtain $p^{n_1} \Big| p^{s'}$ or $p^{n_1} \Big| p^s$, so $n_1 \leq s'$ or $n_1 \leq s$. By considering $|Z(H)| \Big| |C_G(g)| - |Z(G)|$ we obtain $p^{s'} \Big| p^{n_1}(p^{n'-n_1} - 1)$ or $p^s \big| p^{n_1}(p^{n"-n_1} - 1)$ and so $s' \leq n_1$ or $s \leq n_1$. Hence $n_1 = s'$ or $n_1 = s$ and so $|Z(H)| = p^{n_1}q$ or p^{n_1}.

If $|Z(H)| = p^{n_1}$, then $p^{m'}q + p^{n_1} - p^{n''} = p^{n_1}$ and so $p^{m'}q = p^{n''}$ which is a contradiction.

If $|Z(H)| = p^{n_1}q$, then $p^{m'}q + p^{n_1} - p^{n''} = p^{n_1}q$ and so $p^{n_1}(p^{m'-n_1}q - q) = p^{n_1}(p^{n''-n_1} - 1)$ which implies that $p^{m'-n_1}q - q = p^{n'-n_1} - 1$ and hence $q = \frac{p^{n''-n_1}-1}{p^{m'-n_1}-1}$. Now, by $1 < q = \frac{p^{n''-n_1}-1}{p^{m'-n_1}-1}$ and $p \leq \frac{p^{n''-n_1}-1}{p^{m'-n1}-1}$ we will obtain $p \leq q$ that is impossible. Thus this case is impossible.

Case 5. $|G| = p^n q + p^{n'}q$ and $|H| = p^n q + p^{m'}q$ where $n' \leq n$ and $m' \leq n$. Then $|C_G(g)| = p^{n'}q$ and $|C_H(h)| = p^{m'}q$, so $|Z(G)| = p^{n_1}q$ or p^{n_2} where $n_1 < n'$ and $n_2 \leq n'$. By $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^{n'}q - |Z(G)| = p^{m'}q - |Z(H)|$. If $|Z(G)| = p^{n_1}q$, then $|Z(H)| = p^{m'}q + p^{n_1}q - p^{n'}q = qp^{n_1}(p^{m'-n_1} - p^{n'-n_1} + 1)$. (we have $m' > n_1$ because of if $m' \leq n_1$, then $|Z(H)| = p^{m_1} + p^{n_1} - p^{n'} < 2p^{n_1} - p^{n'} \leq 0$ which is contradiction). Thus $|Z(H)| = qp^{n_1}(p^{m'-n_1} - p^{n'-n_1} + 1)$ where $m' - n_1 > 0, n' - n_1 > 0$. So we have $p^{m'-n_1} - p^{n'-n_1} + 1 = pk + 1$ where k is a non-negative integer. If $k = 0$ then, $p^{m'-n_1} - p^{n'-n_1} = 0$ and so $m' = n'$ which implies $|G| = |H|$. If $k > 0$, then from $|Z(H)| \Big| |C_H(h)|$ we will obtain $qp^{n_1}(pk + 1)|qp^{m'}$ and so $pk + 1|p^{m'-n_1}$ which contradicts $gcd(p, pk + 1) = 1$.

If $|Z(G)| = p^{n_2}$ where $n_2 \leq n$, then $|Z(H)| = p^{m'}q + p^{n_2} - p^{n'}q$ and by the same argument as above we have $m' > n_2$. Also by $|C_H(h)| = p^{m'}q$ we get $|Z(H)| = p^{m_1}$ or p^{m_2} where $m_1 < m'$ and $m_2 \leq m'$. But from $|Z(H)|$ in this case we obtain q does not divide $|Z(H)|$ and so $|Z(H)| = p^{m_2}$ where $m_2 \leq m'$. Thus from different possibilities for the order of $Z(H)$ we get

$$p^{m_2} = p^{m'}q + p^{n_2} - p^{n'}q = p^{n_2}(p^{m'-n_2}q - p^{n'-n_2}q + 1) = p^{n_2}(qpk + 1),$$

where k is a non-negative integer. If $k = 0$, then $m' - n_2 = n' - n_2$ and so $m' = n'$, which implies $|G| = |H|$. If $k > 0$, then $p^{m_2} = p^{n_2}(qpk + 1)$ which contradicts $gcd(qpk+1, p) = 1$ and this case can not happen. This last contradiction completes the proof. ∎

Next, we turn to the groups $PSL_3(q)$ and $PSU_3(q^2)$. The conjugacy classes and the character tables of these groups are calculated in [10] and since we need the size of the centralizer orders for elements of these groups hence we state the following Lemma using [10].

Lemma 2 *The size of centralizer orders for elements of $PSL_3(q)$ and $PSU_3(q^2)$ are one of the following numbers: $q^3r'rst$, q^2r', q^2, $qr'rs$, qr', r^2, $r'r$, $r's$, t'. We have $r = q - \delta$, $s = q + \delta$, $t = q^2 + \delta q + 1$, $r' = \frac{r}{d}$, $t' = \frac{t}{d}$ and $d = (3, r)$, where $\delta = 1$ for $PSL_3(q)$ and $\delta = -1$ for $PSU_3(q^2)$.*

Proposition 9 *Let $G{=}PSL_3(q)$ or $PSU_3(q^2)$. If H is a group such that $\Gamma_G{\cong}\Gamma_H$, then $|G| = |H|$.*

Proof. We consider 2 cases.

Case a. $G = PSL_3(q)$. First suppose $3 \mid q - 1$. Then, by Lemma 2, there are elements $x, y, z \in G$ such that $|C_G(x)| = (q - 1)^2$, $|C_G(y)| = \frac{1}{3}(q^2 + q + 1)$ and $|C_G(g)| = q^2$. Since G is a simple group, we have $Z(G) = 1$.

Let $\varphi : \Gamma_H \to \Gamma_G$ be the given isomorphism of graphs. For each $h \in H - H(Z)$ clearly $|Z(H)|$ divides $|C_H(h)|$ and we have

$$|C_H(h)| - |Z(H)| = |C_G(\varphi(h))| - |Z(G)| .$$

Therefore, for all $\alpha \in G - Z(G)$, we have $|Z(H)| \mid |C_G(\alpha)| - |Z(G)|$.

Using the above divisibility condition, $|Z(H)|$ divides the numbers $|C_G(x)| - 1 = q^2 - 2q$, $|C_G(y)| - 1 = \frac{1}{3}(q^2 + q - 2)$, $|C_G(z)| - 1 = q^2 - 1$. From these we will obtain $|Z(H)| = 1$, and consequently $|G| = |H|$.

Secondly we assume $3 \nmid q - 1$. In this case by Lemma 2 there are elements x, y in G such that $|C_G(x)| = q^2$ and $|C_G(y)| = q^2 - 1$. Since $|Z(H)|$ must divide $|C_G(x)| - 1$ and $|C_G(y)| - 1$ we deduce $|Z(H)| = 1$ and hence $|G| = |H|$.

Case b. $G = PSU_3(q^2)$. First suppose $3 \mid q + 1$. By Lemma 2 there are elements x, y, z and t in G such that $|C_G(x)| = \frac{1}{3}(q^2 + q + 1)$, $|C_G(y)| = \frac{1}{3}(q^2 + q)$, $|C_G(z)| = q^2$ and $|C_G(t)| = (q + 1)^2$. Since $|Z(H)|$ divides $|C_G(\alpha)| - 1$ for all $\alpha \in G - 1$, we will obtain $|Z(H)| \mid \gcd(q^2 - q - 2,\ q^2 + q - 3,\ q^2 - 1,\ q^2 - 2q)$. But it is easy to verify that the greatest common divisor (gcd) written above is 1, hence $|Z(H)| = 1$, implying $|G| = |H|$.

Next assume $3 \nmid q+1$. In this case we consider elements x and y in G such that $|C_G(x)| = q^2 - 1$ and $|C_G(y)| = q^2$. In this case $|Z(H)| \mid \gcd(q^2 - 2,\ q^2 - 1) = 1$. Therefore $|Z(H)| = 1$, consequently $|G| = |H|$ and the proposition is proved. ■

Lemma 3 *Let G and H be finite centerless groups. If $\Gamma_G \cong \Gamma_H$, then $k(G)=k(H)$.*

Proof. Since $|Z(G)| = |Z(H)| = 1$, we will obtain $|G| = |H|$. From $|E(G)| = \frac{1}{2}|G|\,(|G| - k(G)) = |E(H)| = \frac{1}{2}|H|\,(|H| - k(H))$ we deduce that $k(G) = k(H)$. ■

Proposition 10 *If $\Gamma_{\mathbb{A}_5} \cong \Gamma_G$, then $G \cong \mathbb{A}_5$.*

Proof. Let G be a finite group such that $\Gamma_{\mathbb{A}_5} \cong \Gamma_G$. By [1] we obtain $|G| = |\mathbb{A}_5|$ and therefore $Z(G) = 1$. By Lemma 3 we have $k(G) = k(\mathbb{A}_5)$. The group $\mathbb{A}_5$ has five conjugacy classes whose representatives may be taken as $x_1 = 1$, x_2, x_3, x_4 and x_5 with centralizer orders 60, 4, 3, 5, 5 respectively. Since $\Gamma_G \cong \mathbb{A}_5$, hence for each $x_i \neq 1$ there is $g_i \in G$ such that $d(x_i) = d(g_i)$, where d denotes the degree of an element as a graph vertex. But $d(x_i) = |\mathbb{A}_5| - |C_{\mathbb{A}_5}(x_i)| = d(g_i) = |G| - |C_G(g_i)|$ which implies $|C_G(g_i)| = |C_{\mathbb{A}_5}(x_i)|$. Therefore we obtain elements g_2, g_3, g_4 in G such that $|C_G(g_2)| = 4$, $|C_G(g_3)| = 3$, $|C_G(g_4)| = 5$. Let g_1 be the identity element of G. Comparing the centralizer orders we deduce that no pair of the elements g_1, g_2, g_3, g_4 are conjugate in G. Since $k(G) = k(\mathbb{A}_5) = 5$, hence there is exactly one other class representative of G, which we denote by g_5 and $|C_G(g_5)| = 5$. Therefore G has class representatives $g_1, \ldots, g_5$ with class sizes: 1, 15, 20, 12, 12. Now if N is a normal subgroup of G, then $|N|$ must be a sum of the above numbers including 1 as a summand. But an easy calculation shows that $|N| = 1$ or 60. Hence G is a simple group. But it is well known that a simple group of order 60 must be isomorphic to $\mathbb{A}_5$. Therefore $G \cong \mathbb{A}_5$ and the proposition is proved. ■

Proposition 11 *If* $\Gamma_{\mathbb{A}_6} \cong \Gamma_G$*, then* $G \cong \mathbb{A}_6$.

Proof. Similar to Proposition 6 we will obtain $|G| = |\mathbb{A}_6|$ and $k(G) = k(\mathbb{A}_6) = 7$. By [3], representatives of the conjugacy classes of $\mathbb{A}_6$ may be taken as $x_1 = 1A$, $x_2 = 2A$, $x_3 = 3A$, $x_4 = 3B$, $x_5 = 4A$, $x_6 = 5A$, $x_7 = 5B$. Considering the degree of vertices in G corresponding to x_i we will obtain elements $g_1 = 1$, g_2, g_3, g_4, g_5, g_6, g_7 of G with centralizer orders 360, 8, 9, 9, 4, 5, and 5 respectively. Certainly the elements g_1, g_2, g_3, g_5 and g_6 are not conjugate in G because they have different centralizer orders. Let g and h be the other two representatives of the conjugacy classes of G with centralizer orders α and β respectively. From the class equation we will obtain $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{14}{45}$. Considering the degrees of g and h as vertices of the graph Γ_G we will see that α as well as β must be one of the centralizer orders in $\mathbb{A}_6$. Therefore we will obtain $\alpha = 9$ and $\beta = 5$. Therefore G has conjugacy classes whose sizes are the same as the size of conjugacy classes in $\mathbb{A}_6$. Now with the same reasoning as at the end of proof of Proposition 6 we can proves that G is a simple group. But any simple group of order 360 is isomorphic to $\mathbb{A}_6$, hence $G \cong \mathbb{A}_6$ and the Proposition is proved. ∎

Proposition 12 *Let G be a finite p-group such that* $\frac{|G|}{|Z(G)|} = p^2$ *and let A be a finite abelian group. If* $\Gamma_{A\times G} \cong \Gamma_H$ *for some group H, then* $|A \times G| = |H|$ *and* $H = Q \times B$*, where B is an abelian group and Q is a non-abelian p-group.*

Proof. By Lemma 3.1 in [1], H is a finite group. if $|G| = p^n$, then $|Z(G)| = p^{n-2}$ and the center of $A \times G$ is of order $p^{n-2}|A|$ and the centralizer of every non-central element of $A \times G$ is of order $p^{n-1}|A|$. It follows that $\Gamma_{A\times G}$ is a regular graph. So Γ_H is a regular graph and since $d(x) = |H| - |C_H(x)|$ for any vertex x, then $|C_H(x)| = |C_H(y)|$ for any non-central element $x, y \in H$. It follows that the conjugacy classes of H have only two sizes. Now, Theorem 1 in [6] implies that H is a nilpotent group and is isomorphic to a direct product of a non-abelian q-subgroup Q (q is a prime) with an abelian group B. Let $|Q|{=}q^s$, $|Z(Q)|{=}q^t$, $|B| = b$ and $|A| = a$. Note that $|C_Q(x_1)| = |C_Q(x_2)| = q^r$, for all non-central element $x_1, x_2 \in Q$ and $s > r > t > 0$. Now, using the hypothesis $\Gamma_{A\times G} \cong \Gamma_H$, we obtain

(i) $$p^{n-1}a - p^{n-2}a = q^r b - q^t b$$

and $p^n a - p^{n-2}a = q^s b - q^t b$. Thus, $q^s b - q^t b = (p^{n-1}a - p^{n-2}a)(p + 1) = (q^r b - q^t b)(p + 1)$ and so, $(p + 1)(q^r b - q^t b) = q^s - q^t$. It follows that

(ii) $$p(q^r - q^t) = q^s - q^r$$

and so, $p(q^{r-t} - 1) = q^{s-t} - q^{r-t} = q^{r-t}(q^{s-r} - 1)$. Thus, $q^{r-t}|p(q^{r-t} - 1)$ and $gcd(q^{r-t}, q^{r-t} - 1) = 1$ and so $q^{r-t}|p$, hence, $q = p$ and $r - t = 1$. Now, (ii) implies that $p(p^{t+1} - p^t) = p^s - p^{t+1}$, so $p^{t+2} = p^s$. Therefore, $s = t + 2$ and now, (i) yields that $p^{n-1}a - p^{n-2}a = p^{t+1}b - p^t b$, so $p^{n-2}a = p^t b$. In this situation we consider three cases.

Case 1. If $t = n - 2$, then $a = b$ and $|H| = bp^s = ap^{t+2} = ap^n = |A \times G|$.

Case 2. if $n - 2 > t$, then $b = ap^{n-2-t}$ and $|H| = bp^s = bp^{t+2} = ap^{n-2-t}p^{t+2} = ap^n = |A \times G|$.

Case 3. If $n - 2 < t$, then $a = p^{t-n+2}$ and $|A \times G| = ap^n = p^{t-n+2}bp^n = bp^{t+2} = bp^s = |H|$.

Now, in every case we have $|A \times G| = |H|$ and $q = p$, so Q is a non-abelian p-group. This completes the proof. ■

5. Chromatic number of the non-commuting graph

First, we recall some terminology from graph theory. Let $\Gamma = (V, E)$ be a simple graph. A subset X of V is called an independent set if the induced subgraph on X is the null graph. Let k be a natural number. A k-vertex coloring of Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertex has the same color. The chromatic number of the graph Γ is the minimum natural number k for which Γ has a k-vertex coloring, this number is denoted by $\chi(\Gamma)$.

Lemma 4 *Let G be a finite non-abelian group and $N \trianglelefteq G$. Then $\chi(\Gamma_{\frac{G}{N}}) \leqslant \chi(\Gamma_G)$.*

Proof. By [1] $\chi(\Gamma)$ is equal to the minimum number of the abelian subgroups of G which cover G. Therefore if $\chi(\Gamma_G) = n$, then $G = \bigcup_{i=1}^{n} H_i$, where H_i is an abelian subgroup of G and G can not be covered by less than n abelian subgroup of G. Since N is a normal subgroup of G, hence $\frac{G}{N} = \bigcup_{i=1}^{n} \frac{NH_i}{N}$. But $\frac{NH_i}{N} \cong \frac{H_i}{N \cap H_i}$ is abelian from which we deduce $\chi(\Gamma_{\frac{G}{N}}) \leqslant n = \chi(\Gamma_G)$. ■

Proposition 13 *Let G be a finite non-abelian group. Then $\chi(\Gamma_G) \leqslant \frac{|V(\Gamma_G)|}{|Z(G)|}$. If equality holds then $G \cong P \times A$, where P is a p-group, A is an abelian group and G is a nilpotent group of class at most 3.*

Proof. Suppose $[G : Z(G)] = n$ and $\{x_1 = 1, x_2, \ldots, x_n\}$ be a set of left transversals of $Z = Z(G)$ in G. It is clear that $x_i Z$ is a commutative subset of G, hence $x_i Z$ is an independent subset in Γ_G for each i. Since $G - Z(G) = \bigcup_{i=2}^{n} x_i Z(G)$ we deduce that $\chi(\Gamma_G) \leqslant n - 1$. But $n - 1 = \frac{|V(\Gamma_G)|}{|Z(G)|}$ and the inequality stated in the proposition is proved.

Now, assume equality holds, i.e., $\chi(\Gamma_G) = \left|\frac{G}{Z(G)}\right| - 1$. If x_i and x_j commutes, then clearly elements of $x_i Z$ commute with elements of $x_j Z$. Therefore the above equality implies that $\{x_2, \ldots, x_n\}$ forms the vertices of a complete subgraph of Γ_G. Now, for any t in G, there is an i such that $t = x_i z$, where $z \in Z(G)$. Now, it is easy to verify that $C_G(t) = C_G(x_i) = Z \cup x_i Z$. Therefore, if $t \in Z(G)$, we have $|C_G(t)| = |Z|$ and if $t \notin Z\,|G|$, then $|C_G(t)| = 2\,|Z|$. Hence, Γ_G is a regular graph and the result follows by Proposition 2.6 in [1]. ∎

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