FUZZY STABILITY OF QUARTIC MAPPINGS[1]

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Abstract. We establish some stability results concerning the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

in the setting of fuzzy normed spaces that in turn generalize a Hyers–Ulam stability result in the framework of classical normed spaces.

2000 Mathematics Subject Classification: Primary 46S40; Secondary 39B52, 39B82, 26E50, 46S50.

Keywords and phrases: Fuzzy normed space; quartic functional equation; normed space; Hyers–Ulam–Rassias stability; fuzzy stability.

1. Introduction and preliminaries

In 1984, Katrasas [11] defined a fuzzy norm on a linear space at the same year Wu and Fang [27] introduced fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [7], [14], [26]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [13]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [5] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the following notion of a fuzzy norm.

Let X be a real linear space. A function $N: X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,c)=0$ for $c \le 0$;
(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;

[1]This research was in part supported by a grant from center for research in Modeling and Computation of Linear and non-linear Systems (RMCS).

(N3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N6) For $x \neq 0$, $N(x,.)$ is (upper semi)continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x,t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'.

Example 1.1. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Example 1.2. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{t}{\|x\|} & 0 < t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

is a fuzzy norm on X.

A sequence $\{x_n\}$ in a fuzzy normed linear space (X, N) is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the fuzzy limit of the sequence $\{x_n\}$ and we denote it by $\mathrm{N} - \lim x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be fuzzy complete and the fuzzy normed space is called a fuzzy Banach space.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940 S.M. Ulam [25] posed the first stability problem. In the next year, D.H. Hyers [8] gave an affirmative answer to the question of Ulam. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. The concept of the Hyers–Ulam–Rassias stability was originated from Th.M. Rassias' paper [23] for the stability of the linear mappings and its importance in the proof of further results in functional equations. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [6], [9], [10], [24] and references therein.

The functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.1}$$

is called the *quartic functional equation*, since the function $f(x)=x^4$ is a solution of the functional equation. Note that f is called quartic because of the identity

$$(2x+y)^4+(2x-y)^4=4(x+y)^4+4(x-y)^4+24x^4-6y^4.$$

Every solution of the quartic functional equation is said to be a *quartic mapping*. In [15] it is proved that a function $f : X \to Y$ between real normed spaces is quartic if and only if there exists a symmetric biquadratic function $F : X \times X \to Y$ such that $f(x) = F(x,x)$ for all $x \in X$. The first result on the stability of the quartic functional equation was obtained by J.M. Rassias [22]. Also L. Cădariu [4], H.-M. Kim [12], S.H. Lee, S.M. Im and I.S. Hwang [15], Najati [20] and C. Park [21] investigated the stability of quartic functional equation.

The first result on fuzzy stability of functional equations was given by the present author and M.S. Moslehian [17]. Later, several various fuzzy version of stability concerning Jensen, cubic and quadratic functional equations were investigated [16], [18], [19]. In the next section we prove the Hyers–Ulam–Rassias stability of the quartic functional equation (1.1) in the setting of fuzzy normed spaces that in turn generalize Hyers–Ulam stability results ([15, Theorem 3.1] and [22]) in the framework of classical normed spaces.

2. Stability of quartic mappings in the fuzzy setting

Let

$$Df(x,y)=f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \tag{1.1}$$

The central theorem is a fuzzy generalized Hyers–Ulam–Rassias type theorem for the quartic functional equation (1.1).

Theorem 1.3. *Let X be a linear space and let (Z,N') be a fuzzy normed space and $\varphi : X\times X \to Z$ be a function. Let (Y,N) be a fuzzy Banach space and let $f : X \to Y$ be a φ-approximately quartic mapping in the sense that*

$$N(Df(x,y),t) \geq N'(\varphi(x,y),t). \tag{1.2}$$

If for some $\alpha < 16$,

$$N'(\varphi(2x,0),t) \geq N'(\alpha\varphi(x,0),t) \tag{1.3}$$

$f(0)=0$ and $\lim_{n\to\infty} N'(2^{-4n}\varphi(2^n x,2^n y),t)=1$ for all x,y in X and $t>0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N\left(Q(x)-f(x),t\right) \geq N'\left(\varphi(x,0),2(16-\alpha)t\right) \tag{1.4}$$

Proof. Let $y = 0$ in (1.2), then we have

$$N(2f(2x) - 2^5 f(x), t) \geq N'(\varphi(x, 0), t). \tag{1.5}$$

Replacing x by $2^{k-1}x$ in (1.5) and using (1.3), we obtain

$$\begin{aligned} N\left(\frac{f(2^k x)}{2^{4k}} - \frac{f(2^{k-1}x)}{2^{4(k-1)}}, t\right) &\geq N'\left(\frac{\varphi(2^{k-1}x, 0)}{2^{4k+1}}, t\right) \\ &\geq N'\left(\frac{\alpha^{k-1}}{2^{4k+1}}\varphi(x, 0), t\right) \\ &= N'(\varphi(x, 0), 2^{4k+1}t/\alpha^{k-1}). \end{aligned} \tag{1.6}$$

Substituting t by $\frac{\alpha^{k-1}t}{2^{4k+1}}$ in (1.6), we get

$$N\left(\frac{f(2^k x)}{2^{4k}} - \frac{f(2^{k-1}x)}{2^{4(k-1)}}, \frac{\alpha^{k-1}t}{2^{4k+1}}\right) \geq N'(\varphi(x, 0), t).$$

This together with

$$2^{-4n}f(2^n x) - 2^{-4m}f(2^m x) = \sum_{k=m+1}^{n} \left(2^{-4k}f(2^k x) - 2^{-4(k-1)}f(2^{k-1}x)\right) \qquad (n > m)$$

yields

$$N\left(2^{-4n}f(2^n x) - 2^{-4m}f(2^m x), \sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{2^{4k+1}}\right) \geq N'(\varphi(x, 0), t) \tag{1.7}$$

By replacing t by $t/\left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}}{2^{4k+1}}\right)$ in (1.7), we observe that

$$\begin{aligned} N(2^{-4n}f(2^n x) - 2^{-4m}f(2^m x), t) &\geq N'\left(\varphi(x, 0), t/\Big(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}}{2^{4k+1}}\Big)\right) \\ &= N'\left(\varphi(x, 0), 32t/\Big(\sum_{k=m+1}^{n} (\frac{\alpha}{16})^{k-1}\Big)\right) \end{aligned} \tag{1.8}$$

Now the Cauchy criterion for convergence and (N5) show that $\{f(2^n x)/2^{4n}\}$ is a Cauchy sequence in (Y, N), since $\sum_{n=1}^{\infty} \left(\frac{\alpha}{16}\right)^n < \infty$. Due to the assumption that (Y, N) is a fuzzy Banach space, the above sequence converges to some point of Y. Put

$$Q(x) := \mathrm{N} - \lim_{n\to\infty} f(2^n x)/2^{4n} \qquad (x \in X).$$

Set $m = 0$ in (1.8) and use the notion of fuzzy limit to obtain

$$N(2^{-4n}f(2^n x) - f(x), t) \geq N'(\varphi(x, 0), 2(16 - \alpha)t).$$

Therefore for each $\varepsilon > 0$ and large enough n

$$\begin{aligned} N\left(Q(x)-f(x), t+\varepsilon\right) &\geq \min\{N\left(Q(x)-f(2^n x)/2^{4n}, \varepsilon\right), N\left(f(2^n x)/2^{4n}-f(x), t\right)\} \\ &\geq N'\left(\varphi(x,0), 2(16-\alpha)t\right). \end{aligned}$$

By (N6),

$$N\left(Q(x)-f(x), t\right) \geq N'(\varphi(x,0), 2(16-\alpha)t).$$

Replace x, y by $2^n x, 2^n y$ respectively in (1.2) to get

$$N(Df(2^n x, 2^n y)/2^{4n}, t) \geq N'(2^{-4n}\varphi(2^x, 2^n y), t).$$

By our assumption $\lim_{n\to\infty} N'(2^{-4n}\varphi(2^n x, 2^n y), t) = 1$, it follows that Q satisfies formula (1.1).

To prove the uniqueness, let us assume that there exists a quartic function $S : X \to Y$ which satisfies (1.4). Fix $x \in X$. Clearly $Q(2^n x) = 2^{4n}Q(x)$ and $S(2^n x) = 2^{4n}S(x)$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} N(Q(x)-S(x), t) &= N(2^{-4n}Q(2^n x) - 2^{-4n}S(2^n x), t) \\ &\geq \min\Big\{N\Big(2^{-4n}(Q(2^n x) - f(2^n x)), t/2\Big), \\ &\qquad N\Big(2^{-4n}(S(2^n x) - f(2^n x)), t/2\Big)\Big\} \\ &\geq N'(\varphi(2^n x, 0), 2^{4n}t(16-\alpha)) \\ &\geq N'\left(\varphi(x,0), \frac{t(16-\alpha)16^n}{\alpha^n}\right) \end{aligned}$$

Since $\lim_{n\to\infty} \dfrac{t(16-\alpha)16^n}{\alpha^n} = \infty$, the last term in the above inequality tends to 1 as $n \to \infty$. Hence $S = Q$. ∎

3. Applications

This section includes three applications of our main result. These are indeed generalizations of known results in [15] to the framework of fuzzy normed spaces.

Corollary 1.4. *Let X be a normed space, (Y, N) be a fuzzy Banach space, (Z, N') be a fuzzy normed space, p, q be nonnegative real numbers and let $z_0 \in Z$. Suppose that $f : X \to Y$ is a (p, q, z_0)-approximately quartic mapping in the sense that*

$$N(Df(x,y), t) \geq N'((\|x\|^p + \|y\|^q)z_0, t) \qquad (x, y \in X).$$

If $f(0) = 0$ and $p, q < 4$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), t) \geq N'(\|x\|^p z_0, 2t(16 - 2^p)),$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\varphi : X \times X \to Z$ be defined by $\varphi(x,y) = (\|x\|^p + \|y\|^q)z_0$. Then the Corollary is followed from Theorem 1.3 by $\alpha = 2^p$. ∎

Remark 1.5. A similar result to Corollary 1.4, where $p, q > 4$ can be formulated. For this one needs to state a similar result to Theorem 1.3, in which one deals with the sequence $\{2^{4n} f(2^{-n}x)\}$ and appropriate conditions on the control function φ.

Corollary 1.6. *Let X be a linear space, (Y,N) be a fuzzy Banach space, (Z,N') be a fuzzy normed space, $z_0 \in Z$ and $\varepsilon > 0$. Suppose that $f : X \to Y$ is an (ε, z_0)-approximately quartic mapping in the sense that*

$$N(Df(x,y),t) \geq N'(\varepsilon z_0, t)$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), t) \geq N'(\varepsilon z_0, 30t),$$

for all $x \in X$ and all $t > 0$.

Proof. The result is deduced from Theorem 1.3, by considering $\varphi : X \times X \to Z$ to be $\varphi(x,y) = \varepsilon z_0$. ∎

Corollary 1.7. ([15]) *Let X be a linear space and Y be a Banach space. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies*

$$||Df(x,y)|| \leq \delta \qquad (x, y \in X),$$

then there exists a unique quartic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \delta/30.$$

Proof. Consider the induced fuzzy norms N and N' on Y and $\mathbb{R}$, respectively, defined as in Example 1.1. Now apply Theorem 1.3 for $\varphi(x,y) = \delta$ and $\alpha = 1$. ∎

Acknowledgement. The author wishes to thank the referee for careful reading the manuscript.

References

[1] Aoki, T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64–66.

[2] Bag, T. and Samanta, S.K., *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., 11 (3) (2003), 687-705.

[3] Bag, T. and Samanta, S.K., *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, 151 (2005), 513-547.

[4] Cădariu, L., *Fixed points in generalized metric space and the stability of a quartic functional equation*, Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz., 50 (64) (2005), no. 2, 25-34.

[5] Cheng, S.C. and Mordeson, J.N., *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., 86 (1994), 429-436.

[6] Czerwik, S., *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[7] Felbin, C., *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, 48 (1992), 239-248.

[8] Hyers, D.H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., U.S.A., 27 (1941), 222-224.

[9] Hyers, D.H., Isac, G. and Rassias, Th.M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[10] Jung, S.-M., *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[11] Katsaras, A.K., *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, 12 (1984), 143-154.

[12] Kim, H.-M., *On the stability for mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl., 324 (2006), 358-372.

[13] Kramosil, I. and Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (1975), 326-334.

[14] Krishna, S.V. and Sarma, K.K.M., *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems, 63 (1994), 207-217.

[15] Lee, S.H., Im, S.M. and Hwang, I.S., *Quartic functional equations*, J. Math. Anal. Appl., 307 (2005), no. 2, 387-394.

[16] Mirmostafaee, A.K., Mirzavaziri, M. and Moslehian, M.S., *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems, 159 (2008) 730-738.

[17] Mirmostafaee, A.K. and Moslehian, M.S., *Fuzzy versions of Hyers–Ulam–Rassias theorem*, Fuzzy Sets and Systems, 159 (2008) 720-729.

[18] Mirmostafaee, A.K. and Moslehian, M.S., *Fuzzy almost quadratic functions*, Results in Math., 52 (2008), 161-177.

[19] Mirmostafaee, A.K. and Moslehian, M.S., *Fuzzy approximately cubic mappings*, Information Sciences 178 (2008) 3791-3798.

[20] Najati, A., *On the stability of a quartic functional equation*, J. Math. Anal. Appl., 340 (2008), no. 1, 569-574.

[21] Park, C., *On the stability of the orthogonally quartic functional equation* Bull. Iranian Math. Soc., 31 (2005), no. 1, 63-70.

[22] Rassias, J.M., *Solution of the Ulam stability problem for quartic mappings*, Glas. Mat. Ser. III, 34(54) (1999), no. 2, 243-252.

[23] Rassias, Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978), 297-300.

[24] Rassias, Th.M. (ed.), *Functional equations, inequalities and applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

[25] Ulam, S.M., *Problems in Modern Mathematics (Chapter VI, Some Questions in Analysis: §1, Stability)*, Science Editions, John Wiley & Sons, New York, 1964.

[26] Xiao, J.-X. and Zhu, X.-H., *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems, 133 (2003), 389-399.

[28] Wu, C. and Fang, J., *Fuzzy generalization of Klomogoroff's theorem* (in Chinese, English abstract), J. Harbin Inst. Tech., no.1 (1984), 1-7.

Accepted: 28.11.2008