ON GENERALIZED PRE-CLOSURE SPACES AND SEPARATION FOR SOME SPECIAL TYPES OF FUNCTIONS

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Abstract. In this paper, we show that a pointwise symmetric pre-isotonic pre-closure functions is uniquely determined by the pairs of sets it separates. We then show that when the pre-closure function of the domain is pre-isotonic and the pre-closure function of the codomain is pre-isotonic and pointwise-pre-symmetric, functions which separate only those pairs of sets which are already separated are pre-continuous.

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1. Introduction

Throughout the paper, (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure, interior and complement of A in X are denoted by $Cl(A), Int(A)$ and $X - A$, respectively. By $PO(X, \tau)$ and $PC(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of (X, τ), respectively. Let A be a subset of a topological space (X, τ). A is preopen [4] or locally dense [1] if $A \subset Int(Cl(A))$. A is preclosed [4] if $X - A$ is preopen or equivalently if $Cl(Int(A)) \subset A$. The intersection of all preclosed sets containing A is called the preclosure of A [2] and is denoted by $pCl(A)$.

Definition 1.

(1) A generalized pre-closure space is a pair (X, pCl) consisting of a set X and a pre-closure function pCl, a function from the power set of X to itself.

(2) The pre-closure of a subset A of X, denoted pCl, is the image of A under pCl.

(3) The pre-exterior of A is $pExt(A) = X\backslash pCl(A)$, and the pre-Interior of A is $pInt(A) = X\backslash pCl(X\backslash A)$.

(4) We say that A is pre-closed if $A = pCl(A)$, A is pre-open if $A = pInt(A)$ and N is a pre-neighborhood of x if $x \in pInt(N)$.

Definition 2. We say that a pre-closure function pCl defined on X is:

(1) pre-grounded if $pCl(\emptyset) = \emptyset$.

(2) pre-isotonic if $pCl(A) \subseteq pCl(B)$ whenever $A \subseteq B$.

(3) pre-enlarging if $A \subseteq pCl(A)$ for each subset A of X.

(4) pre-idempotent if $pCl(A) = pCl(pCl(A))$ for each subset A of X.

(5) pre-sub-linear if $pCl(A \cup B) \subseteq pCl(A) \cup pCl(B)$ for all $A, B \subseteq X$.

(6) pre-additive if $\cup_{i\in I} pCl(A_i) = pCl(\cup_{i\in I} A_i)$ for $A_i \subseteq X$.

Throughout this paper, we will assume that pCl is pre-enlarging.

Definition 3.

(1) Subsets A and B of X are said to be pre-closure-separated in a generalized pre-closure space (X, pCl) (or simply, pCl-separated) if $A \cap pCl(B) = \emptyset$ and $pCl(A) \cap B = \emptyset$, or equivalently, if $A \subseteq pExt(B)$ and $B \subseteq pExt(A)$.

(2) *pExterior* points are said to be pre-closure-separated in a generalized pre-closure space (X, pCl) if for each $A \subseteq X$ and for each $x \in pExt(A)$, $\{x\}$ and A are pCl-separated.

Theorem 1.1. *Let (X, pCl) be a generalized pre-closure space in which pExterior points are pCl-separated and let S be the pairs of pCl-separated sets in X. Then, for each subset A of X, the pre-closure of A is $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$.*

Proof. In any generalized pre-closure space $pCl(A)) \subseteq \{x \in X : \{\{x\}, A\} \notin S\}$. Really suppose that $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$, that is, $\{\{y\}, A\} \in S$. Then $\{y\} \cap pCl(A) = \emptyset$, and so $y \notin pCl(A)$.

Suppose now that $y \notin pCl(A)$. By hypothesis, $\{\{y\}, A\} \in S$, and hence,

$$y \notin \{x \in X : \{\{x\}, A\} \notin S\}.$$

2. Some fundamental properties

Definition 4. A pre-closure function pCl defined on a set X is said to be pointwise pre-symmetric when, for all $x, y \in X$, if $x \in pCl(\{y\})$, then $y \in pCl(\{x\})$.

A generalized pre-closure space (X, pCl) is said to be pre-R_0 when, for all $x, y \in X$, if x is in each pre-neighborhood of y, then y is in each pre-neighborhood of x.

Corollary 2.1. *Let (X, pCl) a generalized pre-closure space in which pExterior points are pCl-separated. Then pCl is pointwise pre-symmetric and (X, pCl) is pre-R_0.*

Proof. Suppose that *pExterior* points are pCl-separated in (X, pCl).

If $x \in pCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not pCl-separated and hence, $y \in pCl(\{x\})$. Hence, pCl is pointwise pre-symmetric.

Suppose that x belongs to every pre-neighborhood of y, that is, $x \in M$ whenever $y \in pInt(M)$. Letting $A = X\backslash M$ and rewriting contrapositively, $y \in pCl(A)$ whenever $x \in A$.

Suppose $x \in pInt(N)$. $x \notin pCl(X\backslash N)$, so x is pCl-separated from $X\backslash N$. Hence $pCl(\{x\}) \subseteq N$. $x \in \{x\}$, so $y \in pCl(\{x\}) \subseteq N$. Hence (X, pCl) is pre-R_0.

While these three axioms are not equivalent in general, they are equivalent when the pre-closure function is pre-isotonic:

Theorem 2.2. *Let (X, pCl) be a generalized pre-closure space with pCl pre-isotonic. Then the following are equivalent:*

(1) *pExterior points are pCl-separated.*

(2) *pCl is pointwise pre-symmetric.*

(3) *(X, pCl) is pre-R_0.*

Proof. Suppose that (2) is true. Let $A \subseteq X$, and suppose $x \in pExt(A)$. Then, as pCl is pre-isotonic, for each $y \in A$, $x \notin pCl(\{y\})$, and hence, $y \notin pCl(\{x\})$. Hence $A \cap pCl(\{x\}) = \emptyset$. Hence (2) implies (1), and by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let $x, y \in X$ such that x is in every pre-neighborhood of y, that is, $x \in N$ whenever $y \in pInt(N)$. Then $y \in pCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in pCl(\{x\})$. Hence $x \in pCl(\{y\})$. Thus if $y \in B$, then $x \in pCl(\{y\}) \subseteq pCl(B)$, as pCl is pre-isotonic. Hence, if $x \in pInt(C)$, then $y \in C$, that is, y is in every pre-neighborhood of x. Hence, (2) implies (3).

Finally, suppose that (X, pCl) is pre-R_0 and suppose that $x \in pCl(\{y\})$. Since pCl is pre-isotonic, $x \in pCl(B)$ whenever $y \in B$, or, equivalently, y is in every pre-neighborhood of x. Since (X, pCl) is pre-R_0, $x \in N$ whenever $y \in pInt(N)$. Hence, $y \in pCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in pCl(\{x\})$. Hence (3) implies (2).

Theorem 2.3. *Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$,*

(1) *if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and*

(2) *if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.*

Then the pre-closure function pCl on X, defined by $pCl(A)=\{x \in X:\{\{x\}, A\}\notin S\}$ for every $A \subseteq X$, is pointwise pre-symmetric pre-isotonic and also, pre-closure-separates the elements of S.

Proof. Define pCl by $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in pCl(A)$, then $\{\{x\}, A\} \notin S$. Hence, $\{\{x\}, B\} \notin S$, that is, $x \in pCl(B)$. Hence pCl is pre-isotonic. Also, $x \in pCl(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in pCl(\{x\})$, and thus pCl is pointwise pre-symmetric.

Suppose that $\{A, B\} \in S$. Then $A \cap pCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \emptyset$. Similarly, $pCl(A) \cap B = \emptyset$. Hence, if $\{A, B\} \in S$, then A and B are pCl-separated.

Now suppose that A and B are pCl-separated.

Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap pCl(B) = \emptyset$ and $\{x \in B : \{\{x\}, A\} \notin S\} = pCl(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, and thus, $\{A, B\} \in S$.

Furthermore, many properties of pre-closure functions can be expressed in terms of the sets they separate:

Theorem 2.4. *Let S be the pairs of pCl-separated sets of a generalized pre-closure space (X, pCl) in which pExterior points are pre-closure-separates. Then pCl is*

(1) *pre-grounded if and only if for all $x \in X$ $\{\{x\}, \emptyset\} \in S$.*

(2) *pre-enlarging if and only if for all $\{A, B\} \in S$, A and B are disjoint.*

(3) *pre-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.*

Moreover, if pCl is pre-enlarging and for all $A, B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then pCl is pre-idempotent. Also, if pCl is pre-isotonic and pre-idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. Recall that, by Theorem 1.1, $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X$, $\{\{x\}, \emptyset\} \in S$. Then $pCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\} = \emptyset$. Hence pCl is pre-grounded.

Conversely, if $\emptyset = pCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\}$, then $\{\{x\}, \emptyset\} \in S$, for all $x \in X$.

Suppose that for all $\{A, B\} \in S$, A and B are disjoint. Since $\{\{a\}, A\} \notin S$ if $a \in A$, $A \subseteq pCl(A)$ for each $A \subseteq X$. Hence, pCl is pre-enlarging. Conversely, suppose that pCl is pre-enlarging and $\{A, B\} \in S$. Then $A \cap B \subseteq pCl(A) \cap B = \emptyset$.

Suppose that $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$. Let $x \in X$ and $B, C \subseteq X$ such that $\{\{x\}, B \cup C\} \notin S$. Then $\{\{x\}, B\} \notin S$ or $\{\{x\}, C\} \notin S$. Hence $pCl(B \cup C) \subseteq pCl(B) \cup pCl(C)$, and therefore, pCl is pre-sub-linear. Conversely, suppose that pCl is pre-sub-linear, and let $\{A, B\}, \{A, C\} \in S$. Then $pCl(B \cup C) \cap A \subseteq (pCl(B) \cup pCl(C)) \cap A = (pCl(B) \cap A) \cup (pCl(C)) \cap A) = \emptyset$ and $(B \cup C) \cap pCl(A) = (B \cap pCl(A)) \cup (C \cap pCl(A)) = \emptyset$. Suppose that pCl is pre-enlarging and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$. Then $pCl(pCl(A)) \subseteq pCl(A)$: If $x \in pCl(pCl(A))$, then $\{\{x\}, pCl(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in pCl(A)$; hence $\{\{x\}, A\} \notin S$. And since pCl is pre-enlarging, $pCl(A) \subseteq pCl(pCl(A))$. Thus $pCl(pCl(A)) = pCl(A)$, for each $A \subseteq X$.

Finally, suppose that pCl is pre-isotonic and pre-idempotent. Let $x \in X$ and $A, B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and, for each $y \in B$, $\{\{y\}, A\} \notin S$. Then $x \in pCl(B)$ and for each $y \in B$, $y \in pCl(A)$, that is, $B \subseteq pCl(A)$. Hence, $x \in pCl(B) \subseteq pCl(pCl(A)) = pCl(A)$.

Definition 5. If (X, pCl_X) and (Y, pCl_Y) are generalized pre-closure spaces, then a function $f : X \to Y$ is said to be

(1) pre-closure-preserving if $f(pCl_X(A)) \subseteq pCl_Y(f(A))$ for each $A \subseteq X$.

(2) pre-continuous if $pCl_X(f^{-1}(B)) \subseteq f^{-1}(pCl_Y(B))$ for each $B \subseteq Y$.

In general, neither condition implies the other. However, we easily obtain the following result:

Theorem 2.5. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces and let $f : X \to Y$.*

(1) *If f is pre-closure-preserving and pCl_Y is pre-isotonic, then f is pre-continuous.*

(2) *If f is pre-continuous and pCl_X is pre-isotonic, then f is pre-closure-preserving.*

Proof. Suppose that f is pre-closure-preserving and pCl_Y is pre-isotonic.

Let $B \subseteq Y$. $f(pCl_X(f^{-1}(B))) \subseteq pCl_Y(f(f^{-1}(B))) \subseteq pCl_Y(B)$ and hence,

$$pCl_X(f^{-1}(B)) \subseteq f^{-1}(f(pCl_X(f^{-1}(B)))) \subseteq f^{-1}(pCl_Y(B)).$$

Suppose that f is pre-continuous and pCl_X is pre-isotonic.

Let $A \subseteq X$. $pCl_X(A) \subseteq pCl_X(f^{-1}(f(A))) \subseteq f^{-1}(pCl_Y(f(A)))$, and hence

$$f(pCl_X(A)) \subseteq f(f^{-1}(pCl_Y(f(A)))) \subseteq pCl_Y(f(A)).$$

Definition 6. Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces and let $f : X \to Y$ be a function. If for all $A, B \subseteq X$, $f(A)$ and $f(B)$ are not pCl_Y-separated whenever A and B are not pCl_X-separated, then we say that f is non-preseparating.

Note that f is non-preseparating if and only if A and B are pCl_X-separated whenever $f(A)$ and $f(B)$ are pCl_Y-separated.

Theorem 2.6. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces, and let $f : X \to Y$.*

(1) *If pCl_Y is pre-isotonic, and f is non-preseparating, then $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated whenever C and D are pCl_Y-separated.*

(2) *If pCl_X is pre-isotonic, and $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated whenever C and D are pCl_Y-separated, then f is non-preseparating.*

Proof. Let C and D be pCl_Y-separated subsets, where pCl_Y is pre-isotonic. Let $A = f^{-1}(C)$ and let $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$, and since pCl_Y is pre-isotonic, $f(A)$ and $f(B)$ are also pCl_Y-separated. Hence, A and B are pCl_X-separated in X.

Suppose that pCl_X is pre-isotonic, and let $A, B \subseteq X$ such that $C = f(A)$ and $D = f(B)$ are pCl_X-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated and since pCl_X is pre-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are pCl_X-separated as well.

Theorem 2.7. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces and let $f : X \to Y$ be a function. If f is pre-closure-preserving, then f is non-preseparating.*

Proof. Suppose that f is pre-closure-preserving and $A, B \subseteq X$ are not pCl_X-separated. Suppose that $pCl_X(A) \cap B \neq \emptyset$. Then $\emptyset \neq f(pCl_X(A) \cap B) \subseteq f(pCl_X(A)) \cap f(B) \subseteq pCl_Y(f(A)) \cap f(B)$. Similarly, if $A \cap pCl_X(B) \neq \emptyset$, then $f(A) \cap pCl_Y(f(B)) \neq \emptyset$. Hence $f(A)$ and $f(B)$ are not pCl_Y-separated.

Corollary 2.8. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces with pCl_X pre-isotonic and let $f : X \to Y$. If f is pre-continuous, then f is non-preseparating.*

Proof. If f is pre-continuous and pCl_X pre-isotonic, then, by Theorem 2.5 (2), f is pre-closure-preserving. Hence, by Theorem 2.7, f is non-preseparating.

Theorem 2.9. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces which pExterior points pCl_Y-separated in Y and let $f : X \to Y$ be a function. Then f is pre-closure-preserving if and only if f non-preseparating.*

Proof. By Theorem 2.7, if f is pre-closure-preserving, then f is non-preseparating. Suppose that f is non-preseparating and let $A \subseteq X$. If $pCl_X = \emptyset$, then

$$f(pCl_X(A)) = \emptyset \subseteq pCl_Y(f(A).$$

Suppose $pCl_X(A) \neq \emptyset$. Let S_X and S_Y denote the pairs of pCl_X-separated subsets of X and the pairs of pCl_Y-separated subsets of Y, respectively. Let $y \in f(pCl_X(A))$, and let $x \in pCl_X(A) \cap f^{-1}(\{y\})$. Since $x \in pCl_X(A)$, $\{\{x\}, A\} \notin S_X$, and since f non-preseparating, $\{\{y\}, f(A)\} \notin S_Y$. Since *pExterior* points are pCl_Y-separated, $y \in pCl_Y(f(A))$. Thus $f(pCl_X(A)) \subseteq pCl_Y(f(A))$, for each $A \subseteq X$.

Corollary 2.10. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces with pre-isotonic closure functions and with pCl_Y-pointwise-presymmetric and let $f : X \to Y$. Then f is pre-continuous if and only if f non-preseparating.*

Proof. Since pCl_Y is pre-isotonic and pointwise-presymmetric, *pExterior* points are pre-closure separated in (Y, pCl_Y) (Theorem 2.2 (1)). Since both pre-closure functions are pre-isotonic, f is pre-closure-preserving (Theorem 2.5) if and only if f is pre-continuous. Hence, we can apply the Theorem 2.9.

3. Preconnected generalized pre-closure spaces

Definition 7. Let (X, pCl) be a generalized pre-closure space. X is said to be preconnected if X is not a union of disjoint nontrivial pre-closure-separated pair of sets.

Theorem 3.1. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl. Then, the following are equivalent:*

(1) *(X, pCl) is preconnected,*

(2) *X can not be a union of nonempty disjoint preopen sets.*

Proof. (1)⇒(2): Let X be a union of nonempty disjoint preopen sets A and B. Then, $X = A \cup B$ and this implies that $B = X \backslash A$ and A is a preopen set. Thus, B is preclosed and hence $A \cap pCl(B) = A \cap B = \varnothing$. By using similar way, we obtain $pCl(A) \cap B = \varnothing$. Hence, A and B are pre-closure-separated and hence X is not preconnected. This is a contradiction.

(2)⇒(1): Suppose that X is not preconnected. Then $X = A \cup B$, where A, B are disjoint pre-closure-separated sets, i.e $A \cap pCl(B) = pCl(A) \cap B = \varnothing$. We have $pCl(B) \subset X \backslash A \subset B$. Since pCl is pre-enlarging, we obtain $pCl(B) = B$ and hence, B is preclosed. By using $pCl(A) \cap B = \varnothing$ and similar way, it is obvious that A is preclosed. This is a contradiction.

Definition 8. Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl. Then, (X, pCl) is called a T_1-pre-grounded pre-isotonic space if $pCl(\{x\}) \subset \{x\}$ for all $x \in X$.

Theorem 3.2. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl. Then, the following are equivalent:*

(1) (X, pCl) *is preconnected,*

(2) *Any precontinuous function* $f : X \to Y$ *is constant for all* T_1*-pre-grounded pre-isotonic spaces* $Y = \{0, 1\}$.

Proof. (1)⇒(2): Let X be preconnected. Suppose that $f : X \to Y$ is precontinuous and it is not constant. Then there exists a set $U \subset X$ such that $U = f^{-1}(\{0\})$ and $X \backslash U = f^{-1}(\{1\})$. Since f is precontinuous and Y is T_1-pre-grounded pre-isotonic space, then we have $pCl(U) = pCl(f^{-1}(\{0\})) \subset f^{-1}(pCl\{0\}) \subset f^{-1}(\{0\}) = U$ and hence $pCl(U) \cap (X \backslash U) = \varnothing$. By using similar way we have $U \cap pCl(X \backslash U) = \varnothing$. This is a contradiction. Thus, f is constant.

(2)⇒(1): Suppose that X is not preconnected. Then there exist pre-closure-separated sets U and V such that $U \cup V = X$. We have $pCl(U) \subset U$ and $pCl(V) \subset V$ and $X \backslash U \subset V$. Since pCl is pre-isotonic and U and V are pre-closure-separated, then $pCl(X \backslash U) \subset pCl(V) \subset X \backslash U$. If we consider the space (Y, pCl) by $Y = \{0, 1\}$, $pCl(\varnothing) = \varnothing$, $pCl(\{0\}) = \{0\}$, $pCl(\{1\}) = \{1\}$ and $pCl(Y) = Y$, then the space (Y, pCl) is a T_1-pre-grounded pre-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X \backslash U) = \{1\}$. Let $A \neq \varnothing$ and $A \subset Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $pCl(X) = pCl(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $pCl(U) = pCl(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{1\}$, then $f^{-1}(A) = X \backslash U$ and hence $pCl(X \backslash U) = pCl(f^{-1}(A)) \subset X \backslash U = f^{-1}(A) = f^{-1}(pCl(A))$. Hence, f is precontinuous. Since f is not constant, this is a contradiction.

Theorem 3.3. *Let* $f : (X, pCl) \to (Y, pCl)$ *and* $g : (Y, pCl) \to (Z, pCl)$ *be precontinuous functions. Then,* $gof : X \to Z$ *is precontinuous.*

Proof. Suppose that f and g are precontinuous. For all $A \subset Z$ we have $pCl(gof)^{-1}(A) = pCl(f^{-1}(g^{-1}(A))) \subset f^{-1}(pCl(g^{-1}(A))) \subset f^{-1}(g^{-1}(pCl(A))) = (gof)^{-1}(pCl(A))$. Hence, $gof : X \to Z$ is precontinuous.

Theorem 3.4. *Let* (X, pCl) *and* (Y, pCl) *be generalized pre-closure spaces with pre-grounded pre-isotonic* pCl *and* $f : (X, pCl) \to (Y, pCl)$ *be a precontinuous function onto* Y*. If* X *is preconnected, then* Y *is preconnected.*

Proof. Suppose that $\{0, 1\}$ is a generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $g : Y \to \{0, 1\}$ is a precontinuous function. Since f is precontinuous, by Theorem 3.3, $gof : X \to \{0, 1\}$ is precontinuous. Since X is preconnected, gof is constant and hence g is constant. By Theorem 3.2, Y is preconnected.

Definition 9. Let (Y, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl and more than one element. A generalized pre-closure space (X, pCl) with pre-grounded pre-isotonic pCl is called Y-preconnected if any precontinuous function $f : X \to Y$ is constant.

Theorem 3.5. *Let (Y, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl and more than one element. Then every Y-pre-connected generalized pre-closure space with pre-grounded pre-isotonic is preconnected.*

Proof. Let (X, pCl) be a Y-preconnected generalized pre-closure space with pre-grounded pre-isotonic pCl. Suppose that $f : X \to \{0, 1\}$ is a precontinuous function, where $\{0, 1\}$ is a T_1-pre-grounded pre-isotonic space. Since Y is a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl and more than one element, then there exists a precontinuous injection $g : \{0, 1\} \to Y$. By Theorem 3.3, $gof : X \to Y$ is precontinuous. Since X is Y-preconnected, then gof is constant. Thus, f is constant and hence, by Theorem 3.2, X is preconnected.

Theorem 3.6. *Let (X, pCl) and (Y, pCl) be generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is Z-preconnected, then Y is Z-preconnected.*

Proof. Suppose that $g : Y \to Z$ is a precontinuous function. Then $gof : X \to Z$ is precontinuous. Since X is Z-preconnected, then gof is constant. This implies that g is constant. Thus, Y is Z-preconnected.

Definition 10. A generalized pre-closure space (X, pCl) is strongly preconnected if there is no countable collection of pairwise pre-closure-separated sets $\{A_n\}$ such that $X = \cup A_n$.

Theorem 3.7. *Every strongly preconnected generalized pre-closure space with pre-grounded pre-isotonic pCl is preconnected.*

Theorem 3.8. *Let (X, pCl) and (Y, pCl) be generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is strongly preconnected, then Y is strongly preconnected.*

Proof. Suppose that Y is not strongly preconnected. Then, there exists a countable collection of pairwise pre-closure-separated sets $\{A_n\}$ such that $Y = \cup A_n$. Since $f^{-1}(A_n) \cap pCl(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(pCl(A_m)) = \varnothing$ for all $n \neq m$, then the collection $\{f^{-1}(A_n)\}$ is pairwise pre-closure-separated. This is a contradiction. Hence, Y is strongly preconnected.

Theorem 3.9. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces. Then, the following are equivalent for a function $f : X \to Y$*

(1) *f is pre-continuous,*

(2) *$f^{-1}(pInt(B)) \subseteq pInt(f^{-1}(B))$ for each $B \subseteq Y$.*

Theorem 3.10. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-additive pCl. Then (X, pCl) is strongly preconnected if and only if (X, pCl) Y-preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl).*

Proof. ($\Rightarrow$): Let (X, pCl) be strongly connected. Suppose that (X, pCl) is not Y-preconnected for some countable T_1-pre-grounded pre-isotonic space (Y, pCl). There exists a precontinuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$.

Since f is precontinuous and Y is pre-grounded, then for each $n \neq m$, $U_n \cap pCl(U_m) = f^{-1}(\{y_n\}) \cap pCl(f^{-1}(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(pCl(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \varnothing$. This contradict with the strong preconnectedness of X. Thus, X is Y-preconnected.

($\Leftarrow$): Let X be Y-preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl). Suppose that X is not strongly preconnected. There exists a countable collection of pairwise pre-closure-separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space (Z, pCl), where Z is the set of integers and $pCl : P(Z) \to P(Z)$ is defined by $pCl(K) = K$ for each $K \subset Z$. Clearly (Z, pCl) is a countable T_1-pre-grounded pre-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \backslash U_k) = \{y\}$ where $x, y \in Z$ and $x \neq y$. Since $pCl(U_k) \cap U_n = \varnothing$ for all $n \neq k$, then $pCl(U_k) \cap \cup_{n \neq k} U_n = \varnothing$ and hence $pCl(U_k) \subset U_k$. Let $\varnothing \neq K \subset Z$. If $x, y \in K$ then $f^{-1}(K) = X$ and $pCl(f^{-1}(K)) = pCl(X) \subset X = f^{-1}(K) = f^{-1}(pCl(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $pCl(f^{-1}(K)) = pCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(pCl(K))$. If $y \in K$ and $x \notin K$ then $f^{-1}(K) = X \backslash U_k$. On the other hand, for all $n \neq k$, $U_k \cap pCl(U_n) = \varnothing$ and hence $U_k \cap \cup_{n \neq k} pCl(U_n) = \varnothing$. This implies that $U_k \cap pCl(\cup_{n \neq k} U_n) = \varnothing$. Thus, $pCl(X \backslash U_k) \subset X \backslash U_k$. Since $pCl(K) = K$ for each $K \subset Z$, we have $pCl(f^{-1}(K)) = pCl(X \backslash U_k) \subset X \backslash U_k = f^{-1}(K) = f^{-1}(pCl(K))$. Hence we obtain that f is precontinuous. Since f is not constant, this is a contradiction with the Z-preconnectedness of X. Hence, X is strongly preconnected.

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