ON PERIODIC SOLUTIONS FOR NESTED POLYGON PLANAR $2N+1$-BODY PROBLEMS WITH ARBITRARY MASSES[1]

Liu Xuefei[2]

Department of Mathematics
Chongqing Three Gorges University
Chongqing, 404000
P.R. China

Zhang Shiqing

College of Mathematics
Sichuan University
Chengdu, 630000
P.R. China

Luo Jianmei

Department of Mathematics
Chongqing Three Gorges University
Chongqing, 404000
P.R. China

Abstract. In this paper we study some necessary conditions and sufficient conditions for the nested periodic polygon solutions of planar $2N+1$-body problem, in which N-body lie at the vertex of one regular polygon, other N-body lie at the vertex of another regular polygon with a running angle, and $2N+1$th body lies at their geometrical center (origin) of $2N$-body.

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1. Main results

This paper uses the same notations as the paper [6]. For $n \geq 2$, the equations of motion of the planar n-body problem ([1], [2], [3], [5], [6]) can be written in the form

$$\ddot{z}_k = -\sum_{\substack{j=1 \\ j\neq k}}^{n} m_j \frac{z_k - z_j}{|z_k - z_j|^3}, \tag{1.1}$$

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[2]Corresponding author. E-mail address: liu_xf666@163.com (X.F. Liu).

where z_k is the complex coordinate of the kth mass m_k in an inertial coordinate system.

Let ρ_k denote the N complex kth roots of unity; i.e.,

$$\rho_k = \exp(2\pi I k/N), \tag{1.2}$$

hereafter $I = \sqrt{-1}$. This equation will also serve to define ρ_k for any number k. We assume that the mass $m_k (k = 1, ..., N)$ locates at the vertex ρ_k of a regular polygon inscribed on the unit circle, and $\tilde{m}_k \ (k = 1, ..., N)$ locates at

$$\tilde{\rho}_k = a\rho_k \tag{1.3}$$

where $a > 0,\ 0 \leq \theta \leq 2\pi$, and $a \neq 1$ when $\theta = 0$ or 2π, and m_0 locates at the geometrical center(which is taken as the coordinate origin) of ρ_k and $\tilde{\rho}_k$ $(k = 1, ..., N)$. Then the center of masses $m_1, ..., m_N; \tilde{m}_1, ..., \tilde{m}_N, m_0$ is

$$z_0 = \frac{\sum_j (m_j\rho_j + \tilde{m}_j\tilde{\rho}_j)}{M} \tag{1.4}$$

where $M = \sum_j (m_j + \tilde{m}_j) + m_0$. In (1.4) and throughout this paper, unless specially restricted, all indices and summations will range from 1 to N. The functions describing their rotation about z_0 with angular velocity ω are then given by

$$\begin{aligned} z_k(t) &= (\rho_k - z_0)\exp(I\omega t), \qquad k = 1, ..., N & (1.5)\\ \tilde{z}_k(t) &= (a\rho_k e^{I\theta} - z_0)\exp(I\omega t), \quad k = 1, ..., N & (1.6)\\ \tilde{z}_0(t) &= (0 - z_0)\exp(I\omega t). & (1.7) \end{aligned}$$

Then the equations of motion of the planar $2N$-body problem can be written as the following form,

$$\ddot{z}_k = \sum_{j\neq k} m_j \frac{z_j - z_k}{|z_j - z_k|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - z_k}{|\tilde{z}_j - z_k|^3}, \tag{1.8}$$

$$\ddot{\tilde{z}}_k = \sum_j m_j \frac{z_j - \tilde{z}_k}{|z_j - \tilde{z}_k|^3} + \sum_{j\neq k} \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_k}{|\tilde{z}_j - \tilde{z}_k|^3}, \tag{1.9}$$

and

$$\ddot{\tilde{z}}_0 = \sum_j m_j \frac{z_j - \tilde{z}_0}{|z_j - \tilde{z}_0|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_0}{|\tilde{z}_j - \tilde{z}_0|^3}. \tag{1.10}$$

R. Moeckel and C. Simo ([5]) proved the following result:

Theorem (Moeckel-Simo). *If $\theta=0, m_0=0$ and $m_1=\cdots=m_N,\ \tilde{m}_1=\cdots=\tilde{m}_N$, then for every mass ratio $b = \dfrac{\tilde{m}_1}{m_1} \neq 1$, there are exactly two planar central configurations consisting of two nested regular N-gons. For one of these, the ratio of the sizes of the two polygons is less than 1, and for the other it is greater than 1.*

Zhang and Zhou also discussed periodic solutions for planar $2N$-body in [8], [9]. In this paper, we continue to study the inverse problem of the theorem (Moeckel-Simo) for $2N+1$-body problem and the following results are established.

Theorem 1. *For $N \geq 2$, $m_k, \tilde{m}_k, m_0 > 0$, the functions $z_k(t), \tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N+1$-body problem* (1.8)–(1.10), *then*

(i)
$$\begin{aligned}&\left(\sum_k m_k\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)\\&+\left(\sum_k \tilde{m}_k\right)\sum_{j}\left(\frac{1}{|1-a\rho_j e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_j e^{I\theta})\\&+m_0\cdot N\left(1-\frac{\omega^2}{M}\right)=0 \qquad (1.11)\end{aligned}$$

$$\begin{aligned}&\left(\sum_k m_k\right)\sum_{j}\left(\frac{1}{|ae^{I\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j)\\&+\left(\sum_k \tilde{m}_k\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta}\\&+m_0\cdot N\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)ae^{I\theta}=0, \qquad (1.12)\end{aligned}$$

(ii) $m_1 = m_2 = \cdots = m_N$ and $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N$. (1.13)

Theorem 2. *For $N \geq 2$, the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N+1$-body problem* (1.8)–(1.10), *if and only if the followings hold*

(i) $m_1 = m_2 = \cdots = m_N := m$ *and* $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N := \tilde{m}$, (1.14)

(ii)
$$\gamma := \frac{\omega^2}{M} = \frac{1}{N+bN+c}\left[\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{i\theta}|^3} + c\right] \qquad (1.15)$$

$$b = \frac{a^3 e^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^2\sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3-c)e^{I\theta}}{e^{I\theta}(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^3\sum_j\frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})}, \qquad (1.16)$$

where $b = \tilde{m}/m, c = m_0/m$.

Theorem 3. *When $\theta = 0$, for the given mass ratio: $b = \tilde{m}/m \neq 1$, and the arbitrary mass ratio: $c = m_0/m$, there exists two unique solutions in* (1.8)–(1.10) *satisfying* (1.14) *and one such that $0 < a < 1$, the other one such that $a > 1$.*

When $\theta = \pi/N$, for $b > 1$ there exists a unique solution in (1.8)–(1.10) *satisfying* (1.14) *and $0 < a < 1$, and for $1 > b > 0$ there exists a unique solution such that $a > 1$.*

Remark. It seems that only $\theta = 0$ or $\dfrac{\pi}{N}$, ω^2 and a are positive real numbers, but the proof seems very difficult.

Corollary 1. *For $N \geq 2$, $\theta = \pi/N, a = 1$, if the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N$+1-body problem* (1.8)–(1.10), *then*

$$\text{(i)} \quad b = 1, \ \textit{i.e.}, \ m_1 = m_2 = \cdots = m_N = \tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N, \tag{1.17}$$

$$\text{(ii)} \quad \gamma := \frac{\omega^2}{M} = \frac{1}{2N + c}\left[\frac{1}{4}\sum_{j \neq N} \csc\left(\frac{\pi j}{2N}\right) + c\right]. \tag{1.18}$$

It is the extension of Theorem 1 (Perko-Walter) [6].

Corollary 2. *Under the above assumptions,*

(i) *if $N = 2, \theta = 0, a > 1$, $z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2 + 2 + 1$-body problems* (1.8)–(1.10), *if and only If*

$$m_1 = m_2, \tilde{m}_1 = \tilde{m}_2,$$

$$\frac{\omega^2}{M} = \frac{1}{2(1 + b) + c}\left[\frac{1}{4} - \frac{4ab}{(a^2 - 1)^2} + c\right] \tag{1.19}$$

and

$$b = \frac{4c(a^3 - 1)(a^4 - 2a^2 + 1) + (a^7 - 2a^5 - 8a^4 + a^3 - 8a^2)}{17a^4 - 2a^2 + 1}, \tag{1.20}$$

hereafter $b = \tilde{m}_1/m_1, c = m_0/m_1$.

(ii) *If $N = 2, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2 + 2 + 1$-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2, \tilde{m}_1 = \tilde{m}_2,$$

$$\frac{\omega^2}{M} = \frac{1}{2(1 + b) + c}\left[\frac{1}{4} + \frac{2b(a^2 + 1)}{(a^2 - 1)^2} + c\right] \tag{1.21}$$

and

$$b = \frac{(a^7 - 2a^5 + 17a^3) + 4c(a^3 - 1)(a^4 - 2a^2 + 1)}{-8a^5 + a^4 - 8a^3 - 2a^2 + 1}. \tag{1.22}$$

(iii) *For $N = 2$ and $\theta = \dfrac{\pi}{2}$, b, c and a has the following relationship*

$$b = \frac{2^{-2} - 2(a^2 + 1)^{-3/2} + c - ca^{-3}}{2^{-2}a^{-3} - 2(a^2 + 1)^{-3/2}}. \tag{1.23}$$

Remark. When $m_0 = 0$, Corollary 2 is conclusions of MacMillan-Bartky [4] in some sense.

Corollary 3. *Under the above assumptions,*

(i) *If* $N = 3, \theta = 0, a > 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $3+3+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3,$$

$$\frac{\omega^2}{M} = \frac{1}{3(1+b)+c}\left[\frac{\sqrt{3}}{3} + \frac{b(2+a)}{(1+a+a^2)^{\frac{3}{2}}} - \frac{b}{(a-1)^2} + c\right] \tag{1.24}$$

and

$$b = \frac{\left[a^2\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} + \frac{1}{(a-1)^2}\right) - a^3\frac{\sqrt{3}}{3} - ca^3 + c\right]}{\left[a^3\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} - \frac{1}{(a-1)^2}\right) - \frac{\sqrt{3}}{3}\right]}. \tag{1.25}$$

(ii) *If* $N = 3, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $3+3+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3,$$

$$\frac{\omega^2}{M} = \frac{1}{3(1+b)+c}\left[\frac{\sqrt{3}}{3} + \frac{b(2+a)}{(1+a+a^2)^{\frac{3}{2}}} + \frac{b}{(a-1)^2} + c\right] \tag{1.26}$$

and

$$b = \frac{\left[a^2\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} - \frac{1}{(a-1)^2}\right) - a^3\frac{\sqrt{3}}{3} - ca^3 + c\right]}{\left[a^3\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} + \frac{1}{(a-1)^2}\right) - \frac{\sqrt{3}}{3}\right]}. \tag{1.27}$$

Corollary 4. *Under the above assumptions,*

(i) *If* $N = 4, \theta = 0, a > 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $4+4+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = \cdots = m_4, \tilde{m}_1 = \cdots = \tilde{m}_4,$$

$$\frac{\omega^2}{M} = \frac{1}{4(1+b)+c}\left[\frac{1}{4} + \frac{\sqrt{2}}{2} + b\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right) + c\right] \tag{1.28}$$

and

$$b = \frac{\left[a^3\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^2\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} + \frac{2(a^2+1)}{(a^2-1)^2}\right) + ca^3 - c\right]}{\left[\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^3\left(\frac{2}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right)\right]}. \tag{1.29}$$

(ii) *If* $N = 4, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $4+4+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = \cdots = m_4, \tilde{m}_1 = \cdots = \tilde{m}_4,$$

$$\frac{\omega^2}{M} = \frac{1}{4(1+b)+c}\left[\frac{1}{4} + \frac{\sqrt{2}}{2} + b\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{2(a^2+1)}{(a^2-1)^2}\right) + c\right] \tag{1.30}$$

and

$$b = \frac{\left[a^3\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^2\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right) + ca^3 - c\right]}{\left[\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^3\left(\frac{2}{(1+a^2)^{\frac{3}{2}}} + \frac{2(a^2+1))}{(a^2-1)^2}\right)\right]}. \tag{1.31}$$

2. Some lemmas

Definition 2.1. ([3]) If $N \times N$ matrix $A = (a_{i,j})$ satisfies

$$a_{i,j} = a_{i-1,j-1}, \quad 1 \le i, \quad j \le N, \tag{2.1}$$

where we assume $a_{i,0} = a_{i,N}$ and $a_{0,j} = a_{N,j}$, then we call A is a circular matrix.

Lemma 2.1. ([3])

(i) *If* A *and* B *are* $N \times N$ *circular matrices, for any numbers* α *and* β, *then* $A+B, A-B, AB, \alpha A + \beta B$ *are also circular matrices, and* $AB = BA$.

(ii) *Let* $A = (a_{i,j})$ *be a* $N \times N$ *circular matrix, then the eigenvalues* λ_k *and the eigenvectors* $\vec{v}_k$ *of* A *are*

$$\lambda_k(A) = \sum_j a_{1,j}\rho_{k-1}^{j-1}, \tag{2.2}$$

$$\vec{v}_k = (1, \rho_{k-1}, \rho_{k-1}^2, ..., \rho_{k-1}^{N-1})^T. \tag{2.3}$$

(iii) *Let* A, B *be circular matrices,* $\lambda_k(A)$, $\lambda_k(B)$ *are eigenvalues of* A, B. *Then the eigenvalues of* $A+B, A-B, A\cdot B$ *are*

$$\lambda_k(A) + \lambda_k(B), \lambda_k(A) - \lambda_k(B), \lambda_k(A)\cdot\lambda_k(B).$$

It is clear that

Lemma 2.2. *If* $A = (a_{i,j})$ *is a* $N \times N$ *circular matrices, and* $AX = 0$, *where* $X = (x_1, ..., x_n)^T, \quad x_i > 0 \quad (i = 1, ..., N)$, *then*

$$\begin{aligned} a_{1,j} + \cdots + a_{N,j} = 0, \quad 1 \le j \le N, \\ a_{i,1} + \cdots + a_{i,N} = 0, \quad 1 \le i \le N. \end{aligned} \tag{2.4}$$

Lemma 2.3. *Let* A, B *be* $N \times N$ *Hermite circular matrices, then* $A+B$, $A-B$, AB, $\alpha A + \beta B$ $(\alpha, \beta \in R)$ *are also Hermite circular matrices.*

Lemma 2.4. *Let* A *is a Hermite circular matrix, then the eigenvalues of* A *are real number.*

(i) *When* $n = 2m+1 (m \geq 1)$, *A can be denoted with*

$$A = A_{2m+1} = cir(a, b_1, b_2, ..., b_m, \bar{b}_m, ..., \bar{b}_2, \bar{b}_1),$$

where $a \in R$ *and* $\bar{b}_l$ *is a conjugate complex number of* b_l. *It has eigenvalues*

$$\lambda_0 = a + 2\sum_{l=1}^{m} Reb_l \tag{2.5}$$

$$\lambda_k = a + 2\sum_{l=1}^{m}\left[Reb_l cos\frac{2k\pi l}{2m+1} - Imb_l sin\frac{2k\pi l}{2m+1}\right] \quad 1 \leq k \leq 2m. \tag{2.6}$$

(ii) *When* $n = 2m (m \geq 1)$, *A can be denoted with*

$$A = A_{2m} = cir(a, b_1, b_2, ..., b_{m-1}, b_m, \bar{b}_{m-1}, ..., \bar{b}_2, \bar{b}_1,).$$

It has eigenvalues

$$\lambda_0 = a + 2\sum_{l=1}^{m-1} Reb_l + b_m, \tag{2.7}$$

$$\lambda_m = a + 2\sum_{l=1}^{m-1} (-1)^l Reb_l + (-1)^m b_m, \tag{2.8}$$

$$\lambda_k = a + 2\sum_{l=1}^{m-1}\left[Reb_l cos\frac{2k\pi l}{2m} - Imb_l sin\frac{2k\pi l}{2m}\right] + (-1)^k b_m \\ 1 \leq k \leq 2m-1,\ k \neq m. \tag{2.9}$$

Proof. This lemma can be simply proved by the properties of the circular matrix and the Hermite matrix.

Lemma 2.5. *The complex subspace* L *of* C^N *generated by* $X_1 = (1, 1, ..., 1)$, $X_2 = (1, \rho, ..., \rho^{N-1})$, *where* $N = 2k > 2$ $\left(\rho = \exp\dfrac{2\pi I}{N}\right)$, *and the complex subspace* $\tilde{L}$ *generated by* X_1, X_2 *and* $X_3 = (1, \rho^{(k+1)}, ..., \rho^{(N-1)(k+1)})$ *where* $N = 2k+1 > 3$, *are all contains no real vectors other than the multiples of* $(1, 1, ..., 1)$.

Proof. After some algebraic computation, it can be also simply proved.

Lemma 2.6. ([5]) *Let* $A = \dfrac{1}{4}\sum_{j\neq N} csc(\pi j/N)$, *then* $A(N)$ *has the following asymptotic expansion for* N *large:*

$$A(N) \sim \frac{N}{2\pi}\left(\gamma + log\frac{2N}{\pi}\right) + \sum_{k\geq 0}\frac{(-1)^k(2^{2k-1}-1)B_{2k}^2\pi^{2k-1}}{(2k)(2k)!}\frac{1}{N^{2k-1}}, \tag{2.10}$$

where γ *stands for the Euler-Mascheroni constant and* B_{2k} *stands for the Bernoulli numbers.*

Lemma 2.7. *Let* $\Phi_\lambda(x)=\sum_j \frac{1}{d_j^\lambda}$*, where* $\lambda>0, d_j=1+x^2-2xcos\left(\frac{2\pi j}{N}-\frac{\pi}{N}\right)$*, then, for* $0<x<1$*,* $\Phi_\lambda(x)$ *and all of its any order derivatives are positive. Moreover, the same is thus for* $\Psi_\lambda(x)=\sum_j \frac{cos\left(\frac{2\pi j}{N}-\frac{\pi}{N}\right)}{d_j^\lambda}$.

Proof. The conclusion and proof are similar to [5].

3. The proof of the main results

For two nested regular polygons, we define

$$\rho_k = \exp(2\pi Ik/N), \tag{3.1}$$

$$\tilde{\rho}_k = a\exp(2\pi Ik/N)e^{I\theta}, \tag{3.2}$$

$$z_0 = \sum_j (m_j\rho_j+\tilde{m}_j\tilde{\rho}_j)/M, \tag{3.3}$$

where

$$M=\sum_j(m_j+\tilde{m}_j)+m_0, \tag{3.4}$$

$$z_k(t)=(\rho_k-z_0)\exp(I\omega t), \qquad k=1,...,N, \tag{3.5}$$

$$\tilde{z}_k(t)=(a\rho_k e^{I\theta}-z_0)\exp(I\omega t), \qquad k=1,...,N, \tag{3.6}$$

and

$$\tilde{z}_0(t)=(0-z_0)\exp(I\omega t). \tag{3.7}$$

Proof of Theorem 1. (3.1)–(3.7) imply that the $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ are the solutions of (1.8) to (1.10) if and only if

$$\begin{aligned}&(\rho_k-z_0)\omega^2\exp(I\omega t)\\ &=\left(\sum_{j\neq k} m_j\frac{\rho_k-\rho_j}{|\rho_k-\rho_j|^3}+\sum_j \tilde{m}_j\frac{\rho_k-\tilde{\rho}_j}{|\rho_k-\tilde{\rho}_j|^3}+m_0\frac{\rho_k-0}{|\rho_k|^3}\right)\exp(I\omega t),\end{aligned} \tag{3.8}$$

$$\begin{aligned}&(\tilde{\rho}_k-z_0)\omega^2\exp(I\omega t)\\ &=\left(\sum_j m_j\frac{\tilde{\rho}_k-\rho_j}{|\tilde{\rho}_k-\rho_j|^3}+\sum_{j\neq k} \tilde{m}_j\frac{\tilde{\rho}_k-\tilde{\rho}_j}{|\tilde{\rho}_k-\tilde{\rho}_j|^3}+m_0\frac{\tilde{\rho}_k-0}{|\tilde{\rho}_k|^3}\right)\exp(I\omega t)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&(0-z_0)\omega^2\exp(I\omega t)\\ &=\left(\sum_{j\neq k} m_j\frac{0-\rho_j}{|0-\rho_j|^3}+\sum_j \tilde{m}_j\frac{0-\tilde{\rho}_j}{|0-\tilde{\rho}_j|^3}\right)\exp(I\omega t),\end{aligned} \tag{3.10}$$

or if and only if

$$\begin{aligned} &\sum_{j\neq k} m_j \left(\frac{1}{|\rho_k-\rho_j|^3}-\frac{\omega^2}{M}\right)(\rho_k-\rho_j) \\ &\qquad +\sum_j \tilde{m}_j \left(\frac{1}{|\rho_k-\tilde{\rho}_j|^3}-\frac{\omega^2}{M}\right)(\rho_k-\tilde{\rho}_j)+m_0\left(1-\frac{\omega^2}{M}\right)\rho_k=0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\sum_{j} m_j \left(\frac{1}{|\tilde{\rho}_k-\rho_j|^3}-\frac{\omega^2}{M}\right)(\tilde{\rho}_k-\rho_j) \\ &\qquad +\sum_{j\neq k} \tilde{m}_j \left(\frac{1}{|\tilde{\rho}_k-\tilde{\rho}_j|^3}-\frac{\omega^2}{M}\right)(\tilde{\rho}_k-\tilde{\rho}_j)+m_0\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)a\rho_k=0 \end{aligned} \tag{3.12}$$

and

$$z_0\omega^2=\left(\sum_{j\neq k} m_j\frac{\rho_j}{|\rho_j|^3}+\sum_j \tilde{m}_j\frac{\tilde{\rho}_j}{|\tilde{\rho}_j|^3}\right). \tag{3.13}$$

Multiplying both sides by ρ_{N-k} in (3.11), (3.12), noting that $|\rho_k-\rho_j| = |\rho_k||1-\rho_{j-k}|=|1-\rho_{j-k}|$ and using $\tilde{\rho}_k=a\rho_k e^{I\theta}$, we have

$$\begin{aligned} &\sum_{j\neq k} m_j \left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k}) \\ &\qquad +\sum_j \tilde{m}_j \left(\frac{1}{|1-a\rho_{j-k}e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{I\theta})+m_0(1-\frac{\omega^2}{M})=0, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\sum_{j} m_j \left(\frac{1}{|ae^{I\theta}-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_{j-k}) \\ &\qquad +\sum_{j\neq k} \tilde{m}_j \left(\frac{1}{|a-a\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-a\rho_{j-k})e^{I\theta}+m_0(\frac{1}{a^3}-\frac{\omega^2}{M})ae^{I\theta}=0 \end{aligned} \tag{3.15}$$

and

$$z_0\omega^2=\left(\sum_{j\neq k} m_j\frac{\rho_j}{|\rho_j|^3}+\sum_j \tilde{m}_j\frac{\tilde{\rho}_j}{|\tilde{\rho}_j|^3}\right). \tag{3.16}$$

Notice that every step from (3.8) to (3.16) can be conversed respectively, firstly we discuss (3.14)–(3.16). Now define the $N\times N$ circular matrices $A=[a_{k,j}]$, $B=[b_{k,j}]$, $C=[c_{k,j}]$, $D=[d_{k,j}]$ as follows:

$$a_{k,j}=0, \qquad \text{for} \quad k=j,$$

$$a_{k,j} = \left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k}), \qquad \text{for} \quad k\neq j, \tag{3.17}$$

$$b_{k,j} = \left(\frac{1}{|ae^{I\theta}-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_{j-k}), \tag{3.18}$$

$$c_{k,j} = \left(\frac{1}{|1-a\rho_{j-k}e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{I\theta}), \tag{3.19}$$

$$d_{k,j} = 0, \qquad \text{for} \quad k = j,$$

$$d_{k,j} = \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (a - a\rho_{j-k}) e^{I\theta} \quad \text{for} \quad k \neq j, \tag{3.20}$$

$$\vec{1} = (1, ..., 1)^T, \tag{3.21}$$

$$E = (1 - \frac{\omega^2}{M}) \cdot \vec{1}, \tag{3.22}$$

$$F = a e^{I\theta} (\frac{1}{a^3} - \frac{\omega^2}{M}) \cdot \vec{1}. \tag{3.23}$$

Then (3.14) and (3.15) hold if and only if the matrix equation

$$\begin{pmatrix} A & C & E \\ B & D & F \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_N \\ \tilde{m}_1 \\ \vdots \\ \tilde{m}_N \\ m_0 \end{pmatrix} = 0 \tag{3.24}$$

has a positive solution.

Let

$$\vec{m} = (m_1, ..., m_N)^T, \vec{\tilde{m}} = (\tilde{m}_1, ..., \tilde{m}_N)^T, \tag{3.25}$$

then (3.24) is equivalent to

$$A\vec{m} + C\vec{\tilde{m}} + E m_0 = \vec{0}, \tag{3.26}$$

$$B\vec{m} + D\vec{\tilde{m}} + F m_0 = 0. \tag{3.27}$$

Notice that A, B, C, D are $N \times N$ circular matrices, with the properties of circular matrix we know they must have positive real eigenvector $\vec{1}$. Each of (3.26), (3.27) left multiplies $\vec{1}^T = (1, 1, ..., 1)$, there are

$$\begin{aligned} &\left(\sum_k m_k \right) \sum_{j \neq N} \left(\frac{1}{|1 - \rho_j|^3} - \frac{\omega^2}{M} \right) (1 - \rho_j) \\ &+ \left(\sum_k \tilde{m}_k \right) \sum_j \left(\frac{1}{|1 - a\rho_j e^{I\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_j e^{I\theta}) \\ &+ m_0 \cdot N \left(1 - \frac{\omega^2}{M} \right) = 0, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\left(\sum_k m_k \right) \sum_j \left(\frac{1}{|a e^{I\theta} - \rho_j|^3} - \frac{\omega^2}{M} \right) (a e^{I\theta} - \rho_j) \\ &+ \left(\sum_k \tilde{m}_k \right) \sum_{j \neq N} \left(\frac{1}{|a - a\rho_j|^3} - \frac{\omega^2}{M} \right) (a - a\rho_j) e^{I\theta} \\ &+ m_0 \cdot N \left(\frac{1}{a^3} - \frac{\omega^2}{M} \right) a e^{I\theta} = 0. \end{aligned} \tag{3.29}$$

The conclusion (i) of Theorem 1 is proved.

(ii) By (3.26) and (3.27) we have

$$(AD-CB)\vec{m}+m_0\left[\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C\right]\vec{1}=\vec{0}, \tag{3.30}$$

$$(CB-AD)\vec{\tilde{m}}+m_0\left[\left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A\right]\vec{1}=\vec{0}. \tag{3.31}$$

From Lemma 2.1 we see that

$$\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C,\ \left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A$$

and $AD-CB$ are circular matrices, we know they must have positive real eigenvector $\vec{1}$. Using the properties of circular matrix, (3.30), (3.31) can be written as

$$(AD-CB)\cdot\vec{m}+\alpha_1\cdot\vec{1}=\vec{0}, \tag{3.32}$$

$$(CB-AD)\cdot\vec{\tilde{m}}+\alpha_2\cdot\vec{1}=\vec{0}, \tag{3.33}$$

where

$$\alpha_1\cdot\vec{1}=m_0\left[\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C\right]\cdot\vec{1}, \tag{3.34}$$

$$\alpha_2\cdot\vec{1}=m_0\left[\left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A\right]\cdot\vec{1}, \tag{3.35}$$

$$\begin{aligned}\alpha_1 &= m_0\Bigg[\left(1-\frac{\omega^2}{M}\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta} \\ &\quad -ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)\sum_j\left(\frac{1}{|1-a\rho_je^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_je^{I\theta})\Bigg],\end{aligned} \tag{3.36}$$

$$\begin{aligned}\alpha_2 &= m_0\Bigg[\left(1-\frac{\omega^2}{M}\right)\sum_{j}\left(\frac{1}{|ae^{i\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j) \\ &\quad -ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)\Bigg].\end{aligned} \tag{3.37}$$

From (3.28), (3.29), we have

$$\alpha_1\sum_k\tilde{m}_k+\alpha_2\sum_k m_k=0. \tag{3.38}$$

a. If $\alpha_1=0$, then $\alpha_2=0$ and

$$(AD-CB)\cdot\vec{m}=\vec{0}, \tag{3.39}$$

$$(AD - CB) \cdot \vec{\tilde{m}} = \vec{0}. \tag{3.40}$$

(3.39), (3.40) must have positive real solutions, i.e., the kernel K of circular matrix $AD - CB$ has positive vector(s).

By Lemmas 2.1, 2.2 we have the eigenvalue

$$\lambda_k(AD - CB) = \lambda_k(AD) - \lambda_k(CB) = \lambda_k(A)\lambda_k(D) - \lambda_k(C)\lambda_k(B). \tag{3.41}$$

Hence,

$$\lambda_k(AD - CB) = 0 \tag{3.42}$$

for some $1 \le k \le N$ if and only if

$$\lambda_k(A)\lambda_k(D) = \lambda_k(B)\lambda_k(C) \tag{3.43}$$

Since

$$\begin{aligned} &\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3} - \frac{\omega^2}{M}\right)(1-\rho_j)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3} - \frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta} \\ &= \sum_j\left(\frac{1}{|1-a\rho_j e^{i\theta}|^3} - \frac{\omega^2}{M}\right)(1-a\rho_j e^{I\theta})\sum_j\left(\frac{1}{|ae^{I\theta}-\rho_j|^3} - \frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j) \end{aligned} \tag{3.44}$$

by (3.36), (3.37), we see $\lambda_1(AD - CB) = 0$ and $\vec{v}_1 = (1, 1, ..., 1)^T \in K$. We know A, B, C, D and $AD - CB$ are all Hermite matrices when $\theta = 0 (a \neq 1), \frac{\pi}{N}$. In this case, by Lemma 2.4 , 2.5 and [7], after many complex calculations it implies that the kernel $K \subseteq L$ or $\subseteq \tilde{L}$ only contains such positive vectors as multiples of $v_1 = (1, 1, ..., 1)^T$. In a general way we shall obtain similar conclusion. Hence,

$$m_1 = m_2 = \cdots = m_N := m \qquad m > 0, \tag{3.45}$$

$$\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N := \tilde{m} \qquad \tilde{m} > 0. \tag{3.46}$$

b. If $\alpha_1 \neq 0$ then $\alpha_2 \neq 0$. From (3.32), (3.33), (3.38), we get

$$(CB - AD)\left[\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m}\right] = \vec{0}. \tag{3.47}$$

If

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m} = \vec{0}, \tag{3.48}$$

then $\tilde{m}_j = bm_j$, where $b = \sum_k \tilde{m}_k / \sum_k m_k$. Substitute it into (3.26) and (3.27). Similar to the proof in **a**, we also have (3.45) and (3.46).

If

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m} \neq \vec{0}, \tag{3.49}$$

let $G = CB - AD = (g_{ij})$, which is nonzero circular matrix, by Lemma 2.1 and 2.2, we have $\sum_j g_{ij} = \sum_i g_{ij} = 0$ and G has eigenvalue 0. Using the properties of circular matrix, we have $(AD-CB)\vec{1}=\vec{0}$ or $\vec{1}^T(AD-CB)=\vec{0}^T$. Let $\vec{1}^T$ left multiplies (3.32) and (3.33) respectively, we get $\alpha_1=\alpha_2=0$, which contradicts the supposition. So (3.48) holds. Hence, Theorem 1 is accomplished.

Proof of Theorem 2.

Proof of the Necessary. From Theorem 1, (1.14) holds. We only prove (1.15), (1.16).

Let $\tilde{m}=bm, m_0=cm$, from (3.28), (3.29), there are

$$\begin{aligned}&\left[\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)\right.\\&\left.+b\sum_j\left(\frac{1}{|1-a\rho_je^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_je^{I\theta})\right]+c\left(1-\frac{\omega^2}{M}\right)=0\end{aligned}\tag{3.50}$$

$$\begin{aligned}&\left[\sum_j\left(\frac{1}{|ae^{I\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j)\right.\\&\left.+b\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta}\right]+c\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)ae^{I\theta}=0.\end{aligned}\tag{3.51}$$

We know

$$\sum_j(1-\rho_j)=N,\tag{3.52}$$

$$\sum_j b(1-a\rho_je^{I\theta})=bN,\tag{3.53}$$

$$\sum_j(ae^{I\theta}-\rho_j)=aNe^{I\theta},\tag{3.54}$$

$$\sum_j b(ae^{I\theta}-a\rho_j)=abNe^{I\theta}.\tag{3.55}$$

By (3.50) and (3.51) there are

$$\begin{aligned}&\frac{\omega^2}{M}\left[\sum_j(1-\rho_j)+b\sum_j(1-a\rho_je^{I\theta})+c\right]\\&=\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-a\rho_je^{I\theta}}{|1-a\rho_je^{I\theta}|^3}+c,\end{aligned}\tag{3.56}$$

$$\begin{aligned}&\frac{\omega^2}{M}\left[\sum_j(ae^{I\theta}-\rho_j)+b\sum_j(a-a\rho_j)e^{I\theta}+cae^{I\theta}\right]\\&=\sum_j\frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3}+b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}e^{I\theta}+\frac{c}{a^2}e^{I\theta},\end{aligned}\tag{3.57}$$

and we have

$$\gamma := \frac{\omega^2}{M} = \frac{1}{N+bN+c}\left[\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3} + c\right], \tag{3.58}$$

$$\gamma := \frac{\omega^2}{M} = \frac{1}{ae^{I\theta}(N+bN+c)}\left[\sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}e^{I\theta} + \frac{c}{a^2}e^{I\theta}\right]. \tag{3.59}$$

Then

$$\begin{aligned} &ae^{I\theta}\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3} + c\right) \\ &\quad = \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}e^{I\theta} + \frac{c}{a^2}e^{I\theta}, \end{aligned} \tag{3.60}$$

and that

$$b = \frac{ae^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca - \frac{c}{a^2})e^{I\theta}}{e^{I\theta}(\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3} - a\sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})}, \tag{3.61}$$

i.e.

$$b = \frac{a^3 e^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^2\sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3 - c)e^{I\theta}}{e^{I\theta}(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^3\sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})}. \tag{3.62}$$

Namely (1.15) and (1.16) hold.

The proof of the Sufficiency. For $N \geq 2$, the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by (1.5) to (1.7) are solutions of the $2N$+1-body problem (1.8)–(1.10), if and only if (3.8) to (3.10), or (3.11) to (3.13) have positive solutions. Let

$$m_1 = m_2 = \cdots = m_N := m \qquad m > 0, \tag{3.63}$$

$$\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N := \tilde{m} \qquad \tilde{m} > 0, \tag{3.64}$$

where $\tilde{m} = bm, m_0 = cm$, then (3.13) holds, (3.11)–(3.12) or (3.14)–(3.15) are equivalence to (3.56) and (3.57). From the process of the necessary proof for theorem 2, we know that (3.56) and (3.57) are equivalence to (3.58) and (3.62), i.e. equivalence to (1.15)–(1.16). Hence the proof of the sufficiency is finished.

Proof of Theorem 3. Under the assumption of the Theorem 3, we know that the uniqueness of the periodic solution (1.5) to (1.7) or (1.8) to (1.10) is equivalence to that (1.15)–(1.16) or (3.58), (3.59) have a unique positive solution for $0 < a < 1$ or $a > 1$. We only prove the case of $\theta = \pi/N$. Obviously, when $\theta = \pi/N$ the right sides of (3.58)–(3.59) are positive. So the problem is that the following equation has a positive solution "a" ($0 < a < 1$ or $a \geq 1$) for given positive numbers b, c.

$$\begin{aligned} &a\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-a\rho_j e^{I\frac{\pi}{N}}}{|1-a\rho_j e^{I\frac{\pi}{N}}|^3} + c\right) \\ &\quad - = \sum_j \frac{a-\rho_j e^{-I\frac{\pi}{N}}}{|a-\rho_j e^{-I\frac{\pi}{N}}|^3} + b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3} + \frac{c}{a^2}. \end{aligned} \tag{3.65}$$

Let $a = x$,

$$\begin{aligned} f(x) = x\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-x\rho_j e^{I\frac{\pi}{N}}}{|1-x\rho_j e^{I\frac{\pi}{N}}|^3} + c\right) \\ -\sum_j \frac{x-\rho_j e^{-I\frac{\pi}{N}}}{|x-\rho_j e^{-I\frac{\pi}{N}}|^3} - b\sum_{j\neq N}\frac{x-x\rho_j}{|x-x\rho_j|^3} - \frac{c}{x^2}. \end{aligned} \tag{3.66}$$

We need to prove $f(x)$ has a unique zero for $0 < x < 1$ or for $x \geq 1$. Let

$$d_j^2 = 1 + x^2 - 2xcos\frac{\pi}{N}(2j-1), \tag{3.67}$$

$$\alpha(x) = \sum_j \frac{1}{d_j^3}, \tag{3.68}$$

$$\beta(x) = \sum_j \frac{cos\frac{\pi}{N}(2j-1)}{d_j^3}, \tag{3.69}$$

$$\xi = \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} = \begin{cases} \frac{1}{4}(2\sum_{j=1}^{\frac{N}{2}-1} csc\left(\frac{\pi j}{N}) + 1\right), & \text{when } N \text{ is even} \\ \frac{1}{2}\sum_{j=1}^{\frac{N-1}{2}} csc\left(\frac{\pi j}{N}\right), & \text{when } N \text{ is odd.} \end{cases} \tag{3.70}$$

Then

$$\lim_{x\to 0^+} f(x) = -\infty \tag{3.71}$$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-\rho_j e^{I\frac{\pi}{N}}}{|1-\rho_j e^{I\frac{\pi}{N}}|^3} \\ &\quad -\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - b\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} \\ &= (b-1)\left[\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-i\frac{\pi}{N}}|^3} - \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}\right]. \end{aligned} \tag{3.72}$$

But

$$\eta = \sum_{j\neq N}\frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} = \begin{cases} \frac{1}{4}\left(2\sum_{j=1}^{\frac{N}{2}-1} csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) + 1\right), & \text{when } N \text{ is even} \\ \frac{1}{2}\sum_{j=1}^{\frac{N-1}{2}} csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right), & \text{when } N \text{ is odd.} \end{cases} \tag{3.73}$$

Since, when $N \geq 2$ is even

$$\begin{aligned} &\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} \\ &= \frac{1}{2}\sum_{j=1}^{\frac{N}{2}-1}\left[csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) - csc\left(\frac{\pi j}{N}\right)\right] + \frac{1}{2}\left(csc\frac{\pi}{2N} + 1\right) > 0, \end{aligned} \tag{3.74}$$

and when $N \geq 2$ is odd, also

$$\begin{aligned}
&\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - \sum_{j\neq N} \frac{1-\rho_j}{|1-\rho_j|^3} \\
&= \frac{1}{2}\sum_{j=1}^{\frac{N-1}{2}} \left[csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) - csc\left(\frac{\pi j}{N}\right)\right] + \frac{1}{4} > 0.
\end{aligned} \tag{3.75}$$

Hence, when $b > 1$, we have

$$\lim_{x\to 1^-} f(x) > 0. \tag{3.76}$$

Obviously $f(x)$ is a continue function for $0 < x < 1$. Thus, to prove the existence of unique zero of $f(x)$, it suffices to show that $f(x)$ is increasing. Now $f(x)$ can be written as follow:

$$f(x) = \left(x - \frac{b}{x^2}\right)\xi + bx(\alpha(x) - x\beta(x)) + (\beta(x) - x\alpha(x)) + c\left(x - \frac{1}{x^2}\right). \tag{3.77}$$

Let

$$\Phi(x) = \sum_j \frac{1}{d_j}. \tag{3.78}$$

It follows from the definitions that

$$\Phi(x) = (1+x^2)\alpha(x) - 2x\beta(x), \tag{3.79}$$

and it implies

$$\alpha(x) - x\beta(x) = \alpha(x) + x(\beta(x) - x\alpha(x)). \tag{3.80}$$

Since

$$\frac{d\Phi}{dx} = \alpha(x) - x\beta(x), \tag{3.81}$$

then, also $f(x)$ can be written as

$$f(x) = \left(x - \frac{b}{x^2}\right)\xi + bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx} + c\left(x - \frac{1}{x^2}\right). \tag{3.82}$$

From Lemma 2.3, the first and the final terms are clearly increasing. Using Lemma 2.4 and its proof, we know $\Phi(x)$, $\frac{d\Phi}{dx}$ themselves and their derivatives are positive for $0 < x < 1$. But

$$\left[bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx}\right]' = b\Phi(x) + 3bx\frac{d\Phi}{dx} + (1+bx^2)\frac{d^2\Phi}{dx^2}, \tag{3.83}$$

so

$$\left[bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx}\right]' > 0, \qquad \text{for } 0 < x < 1. \tag{3.84}$$

Hence, the other two terms are increasing too for $0 < x < 1$, i.e. $f(x) = 0$ has a unique solution for $0 < x < 1$ when $b > 1$.

Let the solution of $f(x)$=0 for $0 < x < 1$ is $x_{problem}(b) := x_p(b), 0 < x_p(b) < 1$. For the case of $x > 1$: we don't directly discuss the equation $f(x) = 0$. Similarly we have a corresponding equation. By the symmetry of the problem, we have the solution $\tilde{x}_{problem}(b) := \tilde{x}_p(b)$ s.t. $\tilde{x}_p(b) = \dfrac{1}{x_p\left(\frac{1}{b}\right)} > 1$.

Remark. It implies for $b > 1$, then $0 < a < 1$; for $0 < b < 1$, then $a > 1$; and for $b = 1, \theta = \pi/N$, then $a = 1$.

Proof of Corollary 1. By Theorem 2, we easily prove Corollary 1.

Proof of Corollary 2. We only prove (iii), when $N = 2$, $\theta = \pi/2$,

$$\begin{aligned} & a^3 e^{I\theta} \sum_{j \neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^2 \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3 - c)e^{I\theta} \\ &= a^2 (\frac{2aI}{8} - \frac{aI-1}{(a^2+1)^{3/2}} - \frac{aI+1}{(a^2+1)^{3/2}}) + (ca^3-c)I \\ &= a^2 (\frac{2aI}{8} - \frac{2aI}{(a^2+1)^{3/2}}) + (ca^3-c)I \\ &= 2a^3 I \left(\frac{1}{8} - \frac{1}{(a^2+1)^{3/2}} \right) + (ca^3-c)I \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} & e^{I\theta} \left(\sum_{j \neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^3 \sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3} \right) \\ &= I \left(\frac{1+1}{2^3} - a^3 \left(\frac{1-aI}{|1-aI|^3} + \frac{1-a(-1)I}{|1-a(-1)I|^3} \right) \right) \\ &= 2a^3 I \left(\frac{1}{8a^3} - \frac{1}{(a^2+1)^{3/2}} \right). \end{aligned} \tag{3.86}$$

So

$$\begin{aligned} b &= \frac{a^3 e^{I\theta} \sum_{j\neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^2 \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3-c)e^{I\theta}}{e^{I\theta}(\sum_{j\neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^3 \sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})} \\ &= \frac{2^{-2} - 2(a^2+1)^{-3/2} + c - ca^{-3}}{2^{-2}a^{-3} - 2(a^2+1)^{-3/2}}. \end{aligned} \tag{3.87}$$

We omit other proofs.

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