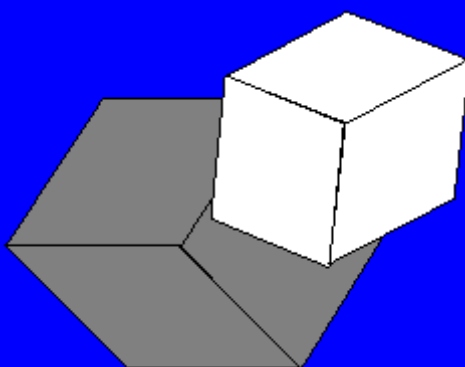


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## Papers Abstracts



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### Inequalities of DVT-type—the two-dimensional case

Barbora Batíková, Tomáš J. Kepka and Petr C. Němec

In this note, particular two-dimensional inequalities of Drápal-Valent type in integer numbers are investigated.

(pp. 1–8)

### Diameter estimate for generalized $m$ -quasi-Einstein manifolds

Deng Yi Hua

In this paper, we study the diameter estimate for generalized  $m$ -quasi-Einstein manifold. Using the Bochner formula and the Hopf maximum principle, we get a gradient estimate for the potential function of the generalized quasi-Einstein manifold. Based on the gradient estimate, we get a diameter estimates for generalized  $m$ -quasi-Einstein manifolds under suitable conditions.

(pp. 9–17)

### Relation between ultra matroid and linear decomposition

Takaaki Fujita

The notion of linear tangle was initially introduced as an obstacle to mixed searching number and Linear decomposition, both of which have significant connections to game theory and graph theory. In this concise paper, we introduce the concept of ultra matroid on a connectivity system, which combines the matroid concept defined on a set with the idea of an ultrafilter. Finally, we establish the equivalence between linear tangle and ultra matroid under certain conditions.

(pp. 18–24)

### On a Diophantine equation involving Mersenne number

William S. Gayo, Jr.

This research work focused on studying an exponential Diophantine equation involving Mersenne numbers. Specifically, it sought to find the nonnegative integer solutions  $(M_n, x, y, z)$  of the Diophantine equation  $M_n^x + (M_n + 1)^y = z^2$ . To obtain the solutions, a combination of modular arithmetic method and factoring method, together with some other results like Mihalescu's theorem, was utilized. Results of the Diophantine analysis revealed that aside from  $(3, 2, 2, 5)$  and  $(7, 0, 1, 3)$ , the equation has infinitely many solutions of the form  $(2^{2k} - 1, 1, 0, 2^k)$  where  $k$  is a positive integer.

(pp. 25–31)

### A compression of finite topological spaces based on homomorphism

Liqin He, Yidong Lin, Jinjin Li

The task of compressing large topologies into more manageable, smaller ones while leveraging the power of homomorphisms is a pivotal concern. Therefore, this paper focuses on the compression of topologies through the utilization of homomorphisms. In this paper, we devise a subbase-consistent function tailored specifically for topological spaces. To preserve certain properties of the original topological spaces, the concept of topological homomorphism is introduced. Additionally, we delve into the subbase reduction under homomorphism and subsequently construct the corresponding discernibility matrix. An experiment is conducted to validate the feasibility and effectiveness of our approach.

(pp. 32–46)

### On $IC\bar{s}$ -subgroups of finite groups

Huajie Zheng, Yong Xu

A subgroup  $H$  of a group  $G$  is said to be an  $IC\bar{s}$ -subgroup of  $G$  if the intersection of  $H$  and  $[H, G]$  is contained in  $H_{\bar{s}G}$ , where  $H_{\bar{s}G}$  is the maximal  $s$ -semipermutable subgroup of  $G$  contained in  $H$ . In this paper, we investigate the influence of  $IC\bar{s}$ -subgroups on the structure of finite groups. Some new results of  $p$ -nilpotency and supersolvability of finite groups are obtained.

(pp. 47–58)

### On the $SS$ -supplemented modules over Dedekind domains

Engin Kaynar, Hamza Çalışıcı, Ergül Türkmen

A module  $M$  is called *ss-supplemented* if every submodule  $U$  of  $M$  has a supplement  $V$  in  $M$  such that  $U \cap V$  is semisimple. In this paper, we completely determine the structure of (amply)  $ss$ -supplemented modules over Dedekind domains. In particular, we prove that an abelian group  $M$  is  $ss$ -supplemented (as a  $\mathbb{Z}$ -module) if and only if  $M \cong (\bigoplus_{p \in I} \mathbb{Z}_p^{(v)}) \oplus (\bigoplus_{q \in J} \mathbb{Z}_{q^2}^{(\nu)})$ , where  $\mathbb{P}$  is the set of all prime integers,  $I, J$  are some subsets of  $\mathbb{P}$  and  $v, \nu$  are any index sets.

(pp. 59–66)

### Intuitionistic fuzzy threshold hypergraphs

Myithili Kothandapani, Nandhini Chandrasekar

A hypergraph is a graph that allows any number of vertices to be connected by an edge. A threshold hypergraph is a hypergraph  $\mathbb{H}$  for which there exists a function  $w : U(\mathbb{H}) \rightarrow \mathbb{N} \cup \{0\}$  and a non-negative integer  $s$  such that for all

$V \subseteq U(H)$ ,  $V$  is independent if and only if  $\sum_{v \in V} w(v) \leq s$ . We introduce Intuitionistic Fuzzy Threshold Hypergraph (IFTHG) along with some of the multiple forms of them such as core, simple, elementary, sectionally elementary IFTHGs and  $(\mu, \nu)$ -tempered IFTHGs, with few of its properties. It is further proven that if an IFTHG  $\mathbb{H}_{\mathbb{G}}$  is elementary, support simple and simply ordered, then  $\mathbb{H}_{\mathbb{G}}$  is a  $(\mu, \nu)$ -tempered IFTHG. Further, we discuss how this method can be used to recognize water wastage using IFTHG and identify regions where it can be reduced, demonstrating that it is more effective for controlling water management systems compared to other methodologies.

(pp. 67–85)

### Some criteria for solvability in finite groups

M. Kuang, J. Lu, W. Meng, B. Zhang

Let  $G$  be a finite group and  $m(G) = \sum_{g \in G} \frac{1}{o(g)}$ , where  $o(g)$  is the order of the element  $g \in G$ . In this paper, we show that if  $G$  is a finite nonsolvable group with  $m(G) = m(A_5)$ , then  $G \cong A_5$ . Furthermore, we show that if  $m(G) < m(\text{SL}(2, 5))$  and  $m(G) \neq m(A_5)$ , then  $G$  is solvable.

(pp. 86–92)

### SEP elements and the solution to constructed equations

Siyuan Liu, Fanfan Ding, Shaokun Huang, Junchao Wei

For the last five years, *SEP* elements in a ring with involution have been discussed by many authors. In this paper, we obtain many new characterizations of *SEP* elements by using group inverse and Moore-Penrose inverse, also by constructing a lot of equations and discussing the expression forms of solution to these equations in certain given set.

(pp. 93–109)

### The $(p, q, r)$ -generations of the Mathieu sporadic simple group $M_{23}$

Malebogo J. Motalane, Ayoub B. M. Basheer, Thekiso T. Seretlo

A finite group  $G$  is called  $(l, m, n)$ -generated, if it is a quotient group of the triangle group  $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$ . In [25], Moori posed the question of finding all the  $(p, q, r)$  triples, where  $p$ ,  $q$  and  $r$  are prime numbers, such that a non-abelian finite simple group  $G$  is a  $(p, q, r)$ -generated. In this paper we establish all the  $(p, q, r)$ -generations of the Mathieu sporadic simple group  $M_{23}$ . GAP [16] and the Atlas of finite group representations [30] are used in our computations.

(pp. 110–137)

### Bi-endomorphism induces new types of derivations on BH-algebras

Patchara Muangkarn, Cholatis Suanoom, Thongsouk Saybounheang,  
Wilaiwan Sriyotha, Aiyared Iampan

In this paper, we study the concepts of an  $(l, r)$  and an  $(r, l)$ - $\tau$ -derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism and we provide important properties. In addition, the relationship among those derivations is also considered.

(pp. 138–152)

### Students' difficulties with eigenvalues and eigenvectors. An exploratory study

Margherita Piroi, Ferdinando Arzarello

In this paper we study the problem of students' difficulties in interpreting the definition of eigenvalues and eigenvectors, extending the research of other scholars. We analyse the activity of small groups of students, during an optional specifically designed extra class, while trying to make sense of the definitions seen in class. Using Sfard's notions of process/object, we scrutinise students' speech and written productions to detect those aspects in the formal definition of eigenvectors and eigenvalues that might hinder their understanding. We consider also how commognitive conflicts present in students' discourse with their classmates may encourage difficulties in grasping the meaning of these mathematical concepts.

(pp. 153–168)

### Bifurcation dynamics in a modified Leslie type predator-prey model with predator harvesting and delay

Wei Liu

In this article, the bifurcation behaviors of a modified Leslie type predator-prey model with harvesting and gestation delay of predator are discussed. The model takes the form of delayed differential-algebra equations. First, the existence of Hopf bifurcations in the model is studied by choosing the delay as a bifurcation parameter. It reveals that a sequence of stability switches and Hopf bifurcations can occur as the delay increases monotonously from zero. Next, the direction of the Hopf bifurcations and the stability of the bifurcating periodic orbits are also investigated. Moreover, we present several numerical simulations to support the theoretical results with the help of Matlab software. Lastly, the significances of our findings are discussed.

(pp. 169–193)

### Finite groups with $124p$ elements of the largest order

Yihu Xie, Li Gao, Guiyun Chen

We study a finite group with  $124p$  elements of the largest order, where  $p$  is a prime greater than 5 and not equal to 31, and prove that such group is either a solvable group, or has a normal series like  $1 \leq H \leq K \leq G$  such that  $H$  is a nilpotent  $\{2, 3\}$ -group,  $G/H \leq \text{Aut}(S)$ , where  $S \cong L_2(7), L_2(8), U_3(3)$ .

(pp. 194–204)

### The MPBT inverse of a complex matrix based on the Hartwig-Spindelböck decomposition

Sanzhang Xu, Qingyuan Xu, Nan Zhou

Let  $A$  be a square complex matrix. A new generalized inverse of  $A$  is introduced by using the Moore-Penrose inverse and B-T inverse of  $A$ , named the MPBT inverse of  $A$ . The formula of the MPBT inverse can be got by using the Hartwig-Spindelböck decomposition of  $A$ . A relationship between the MPBT inverse and the inverse along two given matrix is investigated.

(pp. 205–218)

### Application of new distance and entropy measures for probabilistic dual hesitant fuzzy sets in pattern recognition and VIKOR-based multi-criteria decision-making

Yueyue Zhang, Han-Liang Huang

Probabilistic dual hesitant fuzzy sets (PDHFSs) are constantly used to handle uncertainty problems. However, measures associated with PDHFSs have yet to be widely studied and may have limitations in application. In this paper, a weak equality is suggested to facilitate the conditional reflexivity of distance measures. Then, the consistency, probabilistic fuzziness and Hausdorff distances are defined to integrate different information in PDHFSs. Based on the above definitions, distance and similarity measures for PDHFSs that are not limited by the number of elements are presented. A simple example indicates that new distance measures can effectively differentiate between PDHFSs. Furthermore, a new entropy measure is built for PDHFSs by combining the sine function. Finally, the practicality and feasibility of development methods are demonstrated through two practical examples: pattern recognition and VIKOR-based multi-criteria decision-making (MCDM). Moreover, parameter analyses and comparative analyses indicate that these measures are superior to their existing counterparts.

(pp. 219–244)

### Codes from the incidence matrix of the essential graph

Ahmad Yousefian Darani, Fatemeh Soheilnia, Emel A. Ugurlu

Let  $R$  be a commutative ring and  $M$  be an  $R$ -module. The essential graph over  $M$ , denoted by  $EG(M)$ , is defined as a graph associated to  $M$  with vertex set  $Z(M) \setminus \text{Ann}_R(M)$ , and a pair of distinct vertices  $x$  and  $y$  are adjacent if and only if  $\text{Ann}_M(xy)$  is an essential submodule of  $M$ . In this paper, we investigate the linear codes with respect to the Hamming weight from incidence matrix of the essential graphs over  $M$ . If  $\mathbb{Z}_n$  be the ring of integer module  $n$ , then  $EG(\mathbb{Z}_n)$  is a linear code. Let  $p_1$  and  $p_2$  be distinct prime numbers. It is shown that if  $n = p_1 p_2$ , then  $C_2(EG(\mathbb{Z}_n)) = [(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}]_2$ . Moreover if  $n = p_1^{\alpha_1} p_2^{\alpha_2}$  with  $\alpha_i \geq 1$  for  $i = 1, 2$ , then  $C_2(EG(\mathbb{Z}_n)) = [|E|, |V| - 1, \min\{p_1 + 1, p_2 + 1\}]_2$ .

(pp. 245–254)

Editorial for the Special Issue on “The role of digital technologies in supporting the teaching of mathematics: experiences of good practices at the university level”

Giovannina Albano, Annalisa Cusi, Maria Antonietta Lepellere, Eugenia Taranto

(pp. 255–259)

Digital resources to promote individualization processes at university level: the role played by the integration between automatic scaffolding and expert scaffolding

Annalisa Cusi, Agnese I. Telloni

This paper reports the results of a study focused on the use of a digital resource designed to promote individualization processes at the university level. The aim of the study was to investigate the interaction between a group of university students enrolled in the first year of a Master’s Degree in Management Engineering and the digital resource, with the support of an expert tutor. Specifically, these interactions were analyzed to examine the integration and interplay between the automatic digital scaffolding provided by the digital resource and the scaffolding provided by the expert to help the students overcome the moments of impasse they encountered while solving a series of tasks focused on probability. The analysis we conducted allowed us to highlight the key role of the expert in reinforcing the individualization tools provided by the digital resource.

(pp. 260–280)

### [A blended teaching and learning environment for developing attitude towards mathematics](#)

Giovannina Albano, Maria Alessandra Mariotti, Anna Pierri

This paper reports on the experimentation of a blended learning approach to mathematics at the University level. A first result regards how the teacher's design exploited the didactical potentialities offered by the facilities of a platform, allowing the conceiving specific task based alternatively, on quiz and assignments. Teacher's interventions aim to make students reflect, but mainly to make students focus and interrelate the three fundamental aspects of mathematical thinking: the intuitive, the algorithmic and the formal.

(pp. 281–295)

### [GeoGebra applets and homework tasks to help engineering students with 3D visualization in a multivariable calculus course](#)

Maria Antonietta Lepellere

In this article, we will examine, for 82 first-year Management Engineering students at Udine University, the difficulties of graphically representing regions in space, surface boundaries of these regions, and curve boundaries of these surfaces, and suggest GeoGebra Applets to overcome them. This included the design, and the analysis of the strategy used, and difficulties encountered in three unfamiliar tasks given as homework, two at the beginning and one during the course, and a non-anonymous Moodle questionnaire. The homework tasks were followed by the design and the use in class of ad-hoc GeoGebra Applets, left available to the students for personal use. The activities will be presented by specifying the aspects indicated in the DiGIMATH good practice guidelines. The potential and limitations of GeoGebra for these subjects will also be illustrated. Since the proposed object of study has not been studied a lot, our research is qualitative and exploratory, based on Duval's theory of registers of semiotic representation and the Visual-analytical model. Thematic analysis will be used to analyse the responses to the questionnaire.

(pp. 296–326)

### [Design and use of learning graphs through ASYMPTOTE system](#)

Eugenia Taranto

This study investigates the integration of the ASYMPTOTE system into a Mathematics course for pre-service teachers. In particular, the study focuses on pre-service teachers' task design and its impact on their understanding of course topics and exam performance. Pre-service teachers embraced the task design process, producing more tasks than anticipated. They recognized the



value of task design for comprehensive topic review and deepening understanding. Collaborative efforts between instructors and students were crucial, resulting in a significant portion of participants passing exams. Initial data show that this direct involvement not only enriches the pre-service teachers' mathematical knowledge but also promises to strengthen their future teaching skills.

(pp. 327–341)

### [Prospective mathematics teachers' subjectification processes: joint labor in a Moodle environment through Quick Chat plugin](#)

Dello Iacono Umberto, Santi George

In this paper, we describe a study involving master's degree students in mathematics taking the "Mathematics Education" course. The students, potential prospective teachers, worked in presence on an activity designed to make them reflect on teaching practices and methodologies. They first responded individually to an online questionnaire on misconceptions in mathematics. Then, in small groups, they discussed in the Moodle environment through multiple communication channels using the Quick Chat plugin. The plugin allowed the teacher to monitor all conversations in real time. We analyzed a specific context of practice characterized by the introduction of the Quick Chat plugin within the framework of Objectification Theory, resorting to the notions of joint labor and subjectification.

(pp. 342–355)

### [A learning experience in and from practice using digital technologies: the mathematical discussion](#)

Antonella Montone, Michele Giuliano Fiorentino

In this contribution we report on an experimental research developed with Pre-Service primary Teachers (PSTs) in the Project Digimath, with the aim to give PSTs the opportunity to learn in and from the practice. For doing this we foster the Mathematical Discussion (MD) theoretical framework's learning (Bartolini Bussi, 1989). We explored the effectiveness of a new theoretical approach that combines Shulman's (1986) research about the Pedagogical Content Knowledge (PCK), Ball's Mathematical Knowledge for Teaching (MKT) (2008) and further research on how practice can be used for teachers' learning and what is relevant to learn in and from practice (Ball & Even, 2009), in a digital environment.

We noticed that to become aware of professional education and development of mathematics teachers and of specific characteristics of the teachers' actions, playing different roles (student - PST) is fundamental.

Our work was developed with 160 PSTs attending the Mathematics Education Course at the University of Bari. The MD has been at the same time the subject of the lecture and the methodology used during the lecture, both in a

digital environment and personally. This choice has been done with two different aims: the construction of the MD's characteristics and the way to manage MD in the future professional practice. The experimental activities have been held using hybrid teaching and were videotaped. Conversations have been transcribed and have been used to analyze the teaching experiments.

As we expected, from the transcriptions' analysis, it emerged that the PTSs became aware of the MD's value. In and from their own practice they learned how to manage MD in class, in a future perspective.

(pp. 356–369)

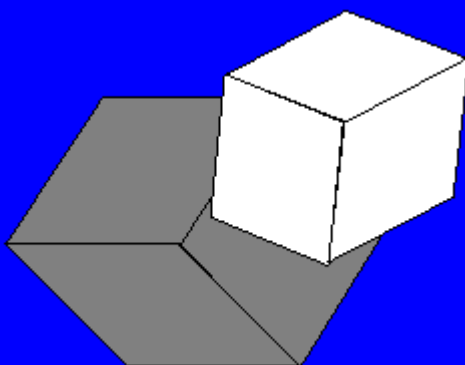
### Engineering students' use of scaffolding elements within digital tasks concerning planar integration domains

Francesca Alessio, Agnese Ilaria Telloni

In this paper, we focus on a sequence of digital tasks involving the representation of subsets of the plane as normal domains. We investigate the role of scaffolding elements provided to supports Engineering students' activities with the tasks. The study focuses on 63 volunteer Engineering freshman. Using qualitative methods, we carried on the combined analysis of video recordings of the students' interaction with the tasks in small groups and their individual answers to a survey after the activity. The main result of the data analysis is the identification of three categories of students' behaviours concerning their use of the provided scaffolding elements. These categories reflect different ways of grasping the scaffolding, purposes and levels of awareness by the students.

(pp. 370–387)

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