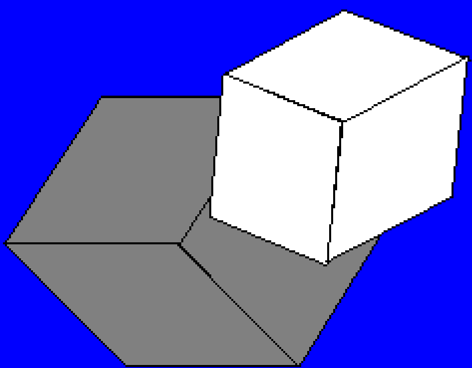
N° 43 – February 2020

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227



FORUM

In memoriam of Professor **Ivo Rosenberg**

The “Italian Journal of Pure and Applied Mathematics“ cannot more take advantage of the precious collaboration of prof. Ivo Rosenberg, who has passed away.

The members of Editorial Board express their deep sorrow for this loss.

The Chief Editor regrets the loss of Prof. Ivo Rosenberg. He has been a great man of science and a very dear friend.

All they who knew him will remember always his scientific value and his exquisite human qualities.

Piergiulio Corsini

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

Web Site

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Upper class functions on a controlled contraction principle in partial S-metric spaces

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Abstract. In this paper, we prove the existence and uniqueness of a fixed point of a self mapping on partial S-metric spaces under the partially α-contractive condition.
Keywords: common fixed point.

1. Introduction and mathematical preliminaries

The existence and uniqueness of a fixed point for a self mapping on different types of metric spaces were the main topic for many research papers [4-18]. The notion of S-metric space was introduced by Sedghi [3]. A generalization of S-metric space was given by Nabil in [1], where he introduced partial S-metric spaces. Moreover, he proved the existence of a fixed point for a self mapping in partial S-metric space. In this paper, we generalize the results in [1] by adding a control function to the contraction principle, which makes the results in [1] a direct consequences of our theorems.

Before proceeding to the main results, we set forth some definitions that will be used in the sequel.

Definition 1.1 ([4])**.** Let X be a nonempty set and $p : X \times X \longrightarrow [0, +\infty)$. We say that (X, p) is a *partial metric space* if for all $x, y, z \in X$ we have:

1. $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$;
2. $p(x, x) \leq p(x, y)$;

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3. $p(x,y) = p(y,x)$;

4. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Definition 1.2 ([3])**.** Let X be a nonempty set. An *S-metric space* on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$:

1. $S(x;y;z) \geq 0$,

2. $S(x;y;z) = 0$ if and only if $x = y = z$,

3. $S(x;y;z) \leq S(x;x;a) + S(y;y;a) + S(z;z;a)$.

The pair $(X;S)$ is called an *S-metric space.*

Next, we give the definition of partial S-metric space.

Definition 1.3 ([1])**.** Let X be a nonempty set. A *partial S-metric space* on X is a function $S_p : X^3 \to [0,\infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:
(i) $x = y$ if and only if $S_p(x,x,x) = S_p(y,y,y) = S_p(x,x,y)$;
(ii) $S_p(x,y,z) \leq S_p(x,x,t) + S_p(y,y,t) + S_p(z,z,t) - S_p(t,t,t)$;
(iii) $S_p(x,x,x) \leq S_p(x,y,z)$;
(iv) $S_p(x,x,y) = S_p(y,y,x)$.
The pair (X, S_p) is called a partial S-metric space.

Definition 1.4. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in (X, S_p) is called *p-Cauchy* if the limit $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m)$ exists and finite. The partial S-metric space (X, S_p) is called *complete* if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that $S_p(z,z,z) = \lim_n S_p(z,z,x_n) = \lim_{n,m} S_p(x_n,x_n,x_m)$.

Moreover, (X, S_p) is a complete partial S-metric space if and only if (X, S_p) is a complete S-metric space. A sequence $\{x_n\}_n$ in a partial S-metric space (X, S_p) is called 0-*Cauchy* if $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m) = 0$. We say that (X, S_p) is 0-*complete* if every 0-Cauchy in X converges to a point $x \in X$ such that $S_p(x,x,x) = 0$.

One can easily construct an example of a partial S-metric space by using the ordinary partial metric space.

Example 1.5 ([1])**.** Let $X = [0,\infty)$ and p be the ordinary partial metric space on X. Define the mapping on X^3 to be $S_p(x,y,z) = p(x,z) + p(y,z)$. Then S_p defines a partial S-metric space.

Now we introduce the notion of partially $\alpha-$contractive.

Definition 1.6. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is *partially* $\alpha-$*contractive* if there exists a constant $k \in [0, 1)$ and a function $\alpha : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y \in X$ we have

$$\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max\{kS_p(x, x, y), S_p(x, x, x), S_p(y, y, y)\}. \tag{1.1}$$

Definition 1.7. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is $R_\alpha-$*admissible* if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(x, Ty) \geq 1$. Also, we say that T is $\alpha-$*admissible* if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Example 1.8. Let $X = [0, +\infty)$. Define $T : X \longrightarrow X$ by $Tx = \sqrt{x}$ and $\alpha : X \times X \longrightarrow X$ by

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

It is a straightforward to verify that T is $\alpha-$admissible and $R_\alpha-$admissible.

Now, we set

$$\begin{aligned} \rho_{S_p}(\alpha) :&= \inf\{S_p(x, x, y) \mid x, y \in X : \alpha(x, y) \geq 1\} \\ &= \inf\{S_p(x, x, x) \mid x \in X : \alpha(x, x) \geq 1\}, \\ X_{S_p}(\alpha) &= \{x \in X \mid S_p(x, x, x) = \rho_{S_p}(\alpha)\}, \\ Z_{S_p}(\alpha) &= \{x \in X_{S_p} \mid \alpha(x, x) \geq 1\}. \end{aligned}$$

Definition 1.9. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is $R_\mu-$*subadmissible* if $x, y \in X$, $\mu(x, y) \leq 1$ implies that $\mu(x, Ty) \leq 1$.

2. Main result

In this section, we prove the existence of a fixed point in a partial S-metric space. We prove relevant corollary. This next theorem is considered to be our main result.

Definition 2.1 ([?])**.** Let $T : X \to X$ be a map and $\mu : X \times X \to [0, +\infty)$ be a function. We say that T is μ-subadmissible if $x, y \in X$, $\mu(x, y) \leq 1$ implies that $\mu(Tx, Ty) \leq 1$.

Definition 2.2. A map $T : X \to X$ is said to be triangular μ-subadmissible if the following holds:

$(T1)$ T is μ-subadmissible,

$(T2)$ $\mu(x, u) \leq 1$ and $\mu(u, y) \leq 1$ implies that $\mu(x, y) \leq 1$, $x, u, y \in X$.

Lemma 2.3. *Let $T : X \to X$ be a triangular $\mu -$ suborbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\mu(x_1, Tx_1) \leq 1$. Then there exists a sequence $\{x_n\}$ such that $\mu(x_n, x_m) \leq 1$ for all m, $n \in \mathbb{N}$ with $n < m$.*

The letter $\mathbb{N}$ represent the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, the nonnegative real numbers will be denoted by $\mathbb{R}_0^+ = [0, \infty)$.

In 2014 the concept of pair $(\mathcal{F}, h)$ is an upper class (see Definition 2.4 until 2.10) was introduced by A.H. Ansari in [19]

Definition 2.4 ([19, 20])**.** A function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a function of subclass of type I, if $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 2.5 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$;

(b) $h(x, y) = (x + l)^y, l > 1$;

(c) $h(x, y) = x^n y$, $n \in \mathbb{N}$;

(d) $h(x, y) = y$;

(e) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^n x^i\right) y$, $n \in \mathbb{N}$;

(f) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^n x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 2.6 ([19, 20])**.** Let $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and: (i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, (ii) $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ for all $t, y \in \mathbb{R}^+$.

Example 2.7 ([19, 20])**.** Define $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$ and $\mathcal{F}(s, t) = st + l$;

(b) $h(x, y) = (x + l)^y, l > 1$ and $\mathcal{F}(s, t) = (1 + l)^{st}$;

(c) $h(x, y) = x^m y$, $m \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;

(d) $h(x, y) = y$ and $\mathcal{F}(s, t) = t$;

(d) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^n x^i\right) y$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;

(e) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^n x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 2.8 ([19, 20])**.** A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a *function of subclass of type II*, if for all $x, y \geq 1$, we have $h(1, 1, z) \leq h(x, y, z)$, for all $z \in \mathbb{R}^+$.

Example 2.9 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y, z) = (z + l)^{xy}, l > 1;$

(b) $h(x, y, z) = (xy + l)^z, l > 1;$

(c) $h(x, y, z) = z;$

(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N};$

(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 2.10 ([19, 20])**.** Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an *upper class of type II*, if h is a subclass of type II and the following holds:

(i) if $0 \leq s \leq 1$ then we have $\mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,

(ii) if $h(1, 1, z) \leq \mathcal{F}(s, t)$ then we have $z \leq st$ for all $s, t, z \in \mathbb{R}^+$.

Example 2.11 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y, z) = (z + l)^{xy}, l > 1, \mathcal{F}(s, t) = st + l;$

(b) $h(x, y, z) = (xy + l)^z, l > 1, \mathcal{F}(s, t) = (1 + l)^{st};$

(c) $h(x, y, z) = z, F(s, t) = st;$

(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s, t) = s^p t^p$

(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, \mathcal{F}(s, t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

Notation.

$$\begin{aligned}
\rho_{S_p}(\alpha, \mu) &:= \inf\{S_p(x, x, y) \mid x, y \in X : \alpha(x, y) \geq 1, \mu(x, y) \leq 1\} \\
&= \inf\{S_p(x, x, x) \mid x \in X : \alpha(x, x) \geq 1, \alpha(x, x) \leq 1\}, \\
X_{S_p}(\alpha, \mu) &= \{x \in X \mid S_p(x, x, x) = \rho_{S_p}(\alpha, \mu)\}, \\
Z_{S_p}(\alpha, \mu) &= \{x \in X_{S_p} \mid \alpha(x, x) \geq 1, \mu(x, x) \leq 1\}.
\end{aligned}$$

Definition 2.12. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is *partially* $(\mathcal{F}, h, \alpha, \mu)$*-contractive* if there exists a constant $k \in [0,1)$ and a function $\alpha, \mu : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y \in X$ we have

$$h(\alpha(x,y), S_p(Tx,Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \max\{kS_p(x,x,y), S_p(x,x,x), S_p(y,y,y)\}). \tag{2.1}$$

where the pair $(\mathcal{F}, h)$ is an upper class of type I.

Theorem 2.13. *Let (X, S_p) be a complete partial S-metric space, T be a self mapping on X and assume that T is partially $(\mathcal{F}, h, \alpha, \mu)$-contractive. If T is $\alpha-$admissible ,$\mu-$subadmissible and $R_\alpha-$admissible, $R_\mu-$subadmissible and if $X_{S_p}(\alpha, \mu)$ is nonempty, then $Z_{S_p}(\alpha, \mu)$ is nonempty. Also, assume that there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1$, then there exists $a \in Z_{S_p}(\alpha)$ such that $Ta = a$.*

Moreover, if for all u, v in $Z_{S_p}(\alpha, \mu)$ with the property $Tu = u$ and $Tv = v$ we have $\alpha(u,v) \geq 1, \mu(u,v) \leq 1$, then T has a unique fixed point in $Z_{S_p}(\alpha, \mu)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$. Define a sequence $\{x_n\}$ for all $n \geq 0$ in X such that $x_1 = Tx_0, x_2 = Tx_1, \cdots, x_{n+1} = Tx_n, \cdots$. Since T is $\alpha-$admissible ,$\mu-$subadmissible and $R_\alpha-$admissible, $R_\mu-$*subadmissible*, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$,$\mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1$, and hence $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$, $\mu(x_1, x_2) = \mu(Tx_0, Tx_1) \leq 1$. So, by induction on n we get

$$\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1})) \leq 1,$$

for all $n \geq 0$. Also, since T is $R_\alpha-$admissible and $R_\mu-$*subadmissible*; $\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1$ implies $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1, \mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1$. By induction on n, we also conclude that

$$\alpha(x_0, x_n) \geq 1, \mu(x_0, x_n) \leq 1$$

for all $n \geq 0$. Also, given the fact that T is α-admissible and $\alpha(x_0, x_0) \geq 1$, it not difficult to prove that $\alpha(x_n, x_n) \geq 1$ for all $n \geq 0$. Hence,

$$\begin{aligned}
&h(1, S_p(x_1, x_1, x_1) = h(1, S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq h(\alpha(x_0, x_0), S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq \mathcal{F}(\mu(x_0, x_0), \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}) \\
&\leq \mathcal{F}(1, \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}).
\end{aligned}$$

This implies that

$$\begin{aligned}
S_p(x_1, x_1, x_1) &\leq \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\} \\
&= S_p(x_0, x_0, x_0).
\end{aligned}$$

By induction on n, we obtain:

$$S_p(x_{n+1},x_{n+1},x_{n+1}) \leq S_p(x_n,x_n,x_n).$$

Therefore, $\{S_p(x_n,x_n,x_n)\}_{\{n\geq 0\}}$ is a nonincreasing sequence. Define

$$r_0 := \lim_n S_p(x_n,x_n,x_n) = \inf_n S_p(x_n,x_n,x_n) \geq 0$$

and

$$M_0 := \frac{2}{1-k}S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_0).$$

Next, we show that $S_p(x_0,x_0,x_n) \leq M_0$, for any $n \geq 0$. If $n = 0$; the case is trivial. For $n = 1$ and using the fact that $k \in [0,1)$ we deduce that $S_p(x_0,x_0,x_1) \leq \dfrac{2}{1-k}S_p(x_0,x_0,x_1) \leq \dfrac{2}{1-k}S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_0) = M_0$. So, we may assume that it is true for all $n \leq n_0-1$ and prove it for $n = n_0 \geq 2$.

$$\begin{aligned}
&S_p(x_0,x_0,x_{n_0}) \leq S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_1) + S_p(x_{n_0},x_{n_0},x_1) - S_p(x_1,x_1,x_1)\\
&\leq 2S_p(x_0,x_0,x_1) + S_p(x_1,x_1,x_{n_0})\\
&\leq 2S_p(x_0,x_0,x_1) + \alpha(x_0,x_{n_0-1})S_p(Tx_0,Tx_0,Tx_{n_0-1})\\
&\leq 2S_p(x_0,x_0,x_1)+ \max\{kS_p(x_0,x_0,x_{n_0-1}), S_p(x_0,x_0,x_0), S_p(x_{n_0-1},x_{n_0-1},x_{n_0-1})\}\\
&\leq 2S_p(x_0,x_0,x_1) + \max\{kS_p(x_0,x_0,x_{n_0-1}), S_p(x_0,x_0,x_0)\}.
\end{aligned}$$

Also, by induction assumption, we have $S_p(x_0,x_0,x_{n_0-1}) \leq \frac{2}{1-k}S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_0)$. So, we have

$$\begin{aligned}
&S_p(x_0,x_0,x_{n_0}) \leq 2S_p(x_0,x_0,x_1)\\
&+\max\{\frac{2k}{1-k}S_p(x_0,x_0,x_1) + kS_p(x_0,x_0,x_0), S_p(x_0,x_0,x_0)\}\\
&\leq 2S_p(x_0,x_0,x_1) + \frac{2k}{1-k}S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_0)\\
&= \frac{2}{1-k}S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_0) = M_0.
\end{aligned}$$

Hence, we conclude that $S_p(x_0,x_0,x_n) \leq M_0$. Next, we need to show that

$$\lim_{n,m} S_p(x_n,x_n,x_m) = r_0.$$

For all n,m we have $S_p(x_n,x_n,x_m) \geq S_p(x_n,x_n,x_n) \geq r_0$. Let $\epsilon > 0$ find a natural number n_0 such that $S_p(x_{n_0},x_{n_0},x_{n_0}) < r_0+\epsilon$ and $2M_0k^{n_0} < r_0+\epsilon$. Now for any $n,m \geq 2n_0$, since T is $R_\alpha-$admissible and using the fact that $\alpha(x_n,x_{n+1}) \geq 1, \mu(x_n,x_{n+1}) \leq 1$ we deduce that $\alpha(x_n,x_m) \geq 1, \mu(x_n,x_m) \leq 1$. Hence,

$$h(1,S_p(x_n,x_n,x_m) \leq h(\alpha(x_n,x_m),S_p(x_n,x_n,x_m)) \leq \mathcal{F}(\mu(x_n,x_m),\theta) \leq \mathcal{F}(1,\theta),$$

where

$$\theta = \max\{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1})\}.$$

This implies that

$$\begin{aligned}
&S_p(x_n, x_n, x_m)\\
&\le \max\{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1})\}\\
&\le \max\{k^2S_p(x_{n-2}, x_{n-2}, x_{m-2}), S_p(x_{n-2}, x_{n-2}, x_{n-2}), S_p(x_{m-2}, x_{m-2}, x_{m-2})\}\\
&\le \cdots \le \max\{k^{n_0}S_p(x_{n-n_0}, x_{n-n_0}, x_{m-n_0}), S_p(x_{n-n_0}, x_{n-n_0}, x_{n-n_0}),\\
&S_p(x_{m-n_0}, x_{m-n_0}, x_{m-n_0})\}\\
&\le r_0 + \epsilon.
\end{aligned}$$

Hence,

$$\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.$$

Since (X, p) is a complete partial S-metric space; there exists $\widetilde{x} \in X$ such that

$$r_0 = S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = \lim_{n} S_p(\widetilde{x}, \widetilde{x}, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m).$$

Now, we show that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$. For each natural number n we have

$$S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) \le 2S_p(\widetilde{x}, \widetilde{x}, x_n) - S_p(x_n, x_n, x_n) + S_p(T\widetilde{x}, T\widetilde{x}, x_n).$$

Using the property that T is $\alpha-$contractive, we deduce that there exists a subsequence of natural numbers $\{n_l\}$ such that

$$\begin{aligned}
&h(1, S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le h(\alpha(\widetilde{x}, x_{n_l-1}), S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}))\\
&\le \mathcal{F}(\mu(\widetilde{x}, x_{n_l-1}), \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\})\\
&\le \mathcal{F}(1, \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\}),
\end{aligned}$$

and thus

$$S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\}.$$

So, for $l \ge 1$, we have either $S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1})$ or less than or equal $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x})$ or less than or equal $S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})$. In all of these three cases, if we take the limit as l goes toward ∞ we get $S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) \le S_p(\widetilde{x}, \widetilde{x}, \widetilde{x})$. But, we know by the property (ii) of the partial S-metric space definition that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) \le S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$. Therefore, $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$.

Now, we show that $X_{S_p}(\alpha, \mu)$ is nonempty. For each natural number l pick $x_l \in X$ with $\alpha(x_l, x_l) \ge 1$ and $S_p(x_l, x_l, x_l) < \rho_{S_p}(\alpha, \mu) + \frac{1}{l}$ and show that

$$\lim_{n,m} S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) = \rho_{S_p}(\alpha, \mu).$$

Let $\epsilon > 0$ put $n_0 := (\frac{3}{\epsilon(1-k)})+1$ if $l \geq n_0$ then we have: $\rho_{S_p}(\alpha,\mu) \leq S_p(\widetilde{x}_l, \widetilde{x}_l, T\widetilde{x}_l)$ $\leq S_p(\widetilde{x}_l, \widetilde{x}_l, T\widetilde{x}_l) \leq r_{x_l} \leq S_p(\widetilde{x}_l, \widetilde{x}_l, T\widetilde{x}_l) < \rho_{S_p}(\alpha,\mu) + \frac{1}{l} \leq \rho_{S_p}(\alpha,\mu) + \frac{1}{n_0} <$ $\rho_{S_p}(\alpha,\mu) + \frac{\epsilon(1-k)}{3}$. Hence, we deduce that:

$$U_l := S_p(\widetilde{x}_l, \widetilde{x}_l, \widetilde{x}_l) - S_p(T\widetilde{x}_l, T\widetilde{x}_l, T\widetilde{x}_l) < \frac{\epsilon(1-k)}{3},$$

for $i \geq n_0$.

Also, if $l \geq n_0$, then $S_p(\widetilde{x}_l, \widetilde{x}_l, \widetilde{x}_l) = r_{x_l} \leq S_p(x_l, x_l, x_l) < \rho_{S_p}(\alpha) + \frac{1}{n_0}$. Which implies that $S_p(\widetilde{x}_l, \widetilde{x}_l, \widetilde{x}_l) \leq \rho_{S_p}(\alpha,\mu) + \frac{\epsilon(1-k)}{3}$ for all $l \geq n_0$. Now, if $n, m \geq n_0$, then $S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) \leq 2S_p(\widetilde{x}_n, \widetilde{x}_n, T\widetilde{x}_n) + S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_m) +$ $2S_p(T\widetilde{x}_m, T\widetilde{x}_m, \widetilde{x}_m) - S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_n) - S_p(T\widetilde{x}_m, T\widetilde{x}_m, T\widetilde{x}_m)$.

We know that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$ which implies that

$$\begin{aligned}
&h(1, S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_m) \leq h(\alpha(\widetilde{x}_n, \widetilde{x}_m), S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_m)) \\
&\leq \mathcal{F}(\mu(\widetilde{x}_n, \widetilde{x}_m), \max\{kS_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m), S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n), S_p(\widetilde{x}_m, \widetilde{x}_m, \widetilde{x}_m)\}) \\
&\leq \mathcal{F}(1, \max\{kS_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m), S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n), S_p(\widetilde{x}_m, \widetilde{x}_m, \widetilde{x}_m)\})
\end{aligned}$$

Therefore,

$$\begin{aligned}
&S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_m) \leq \max\{kS_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m), S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n), S_p(\widetilde{x}_m, \widetilde{x}_m, \widetilde{x}_m), \\
&S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) \leq U_n + U_m + S_p(T\widetilde{x}_n, T\widetilde{x}_n, T\widetilde{x}_m) \\
&< U_n + U_m + \max\{kS_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m), S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n), S_p(\widetilde{x}_m, \widetilde{x}_m, \widetilde{x}_m)\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\rho_{S_p}(\alpha,\mu) \leq S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) \\
&\leq \max\{\frac{2}{3}\epsilon, \frac{2}{3}\epsilon(1-k) + S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n), \frac{2}{3}\epsilon(1-k) + S_p(\widetilde{x}_m, \widetilde{x}_m, \widetilde{x}_m)\} \\
&\leq \max\{\frac{2}{3}\epsilon, \rho_{S_p}(\alpha,\mu) + \epsilon(1-k)\} < \rho_{S_p}(\alpha,\mu) + \epsilon.
\end{aligned}$$

Thus,

$$\lim_{n,m} S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) = \rho_{S_p}(\alpha,\mu).$$

Since (X, S_p) is complete, there exists $a \in X$ such that,

$$S_p(a,a,a) = \lim_{n} S_p(a,a,\widetilde{x}_n) = \lim_{n,m} S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) = \rho_{S_p}(\alpha,\mu).$$

Therefore, we have $a \in X_{S_p}(\alpha,\mu)$ and thus $X_{S_p}(\alpha,\mu)$ is nonempty. This implies that, $Z_{S_p}(\alpha,\mu)$ is nonempty.

Now, let $x_0 \in Z_{S_p}(\alpha,\mu)$ be arbitrary. Then by the above argument we have

$$\rho_{S_p}(\alpha,\mu) \leq S_p(T\widetilde{x}, T\widetilde{x}, T\widetilde{x}) \leq S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = r_0 = \rho_{S_p}(\alpha,\mu).$$

Thus, $T\widetilde{x} = \widetilde{x}$, Now, assume that T has two fixed points $u, v \in Z_{S_p}(\alpha, \mu)$. By our hypothesis, we know that $\alpha(u,v) \geq 1, \mu(u,v) \leq 1$. Thus,

$$\begin{aligned} h(1, S_p(u,u,v) &\leq h(\alpha(u,v), S_p(Tu,Tu,Tv)) \\ &\leq \mathcal{F}(\mu(u,v), \max\{kS_p(u,u,v), S_p(u,u,u), S_p(v,v,v)\}) \\ &\leq \mathcal{F}(1, \max\{kS_p(u,u,v), S_p(u,u,u), S_p(v,v,v)\}). \end{aligned}$$

So we have,

$$S_p(u,u,v) \leq \max\{kS_p(u,u,v), S_p(u,u,u), S_p(v,v,v)\}.$$

Now, if $S_p(u,u,v) \leq kS_p(u,u,v)$ we deduce that $S_p(u,u,v) = 0$ and in this case $u = v$, or $S_p(u,u,v) \leq S_p(u,u,u) = S_p(v,v,v)$ and in this case by condition (ii) of the definition of the partial S-metric space we obtain $S_p(u,u,v) = S_p(u,u,u) = S_p(v,v,v)$ and hence by condition (i) of the same definition we conclude that $u = v$. Therefore, we obtain the uniqueness as desired. □

As a consequence of the above result, the following corollary follows easily.

Corollary 2.14. *Let (X, S_p) be a 0-complete partial S-metric space, $k \in [0,1)$ and consider the map $T : X \longrightarrow X$ to be α-admissible and R_α-admissible,and there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$, also for every $x, y \in X$ we have $\alpha(x,y)S_p(Tx,Tx,Ty) \leq kS_p(x,x,y)$. Then there exists $\widetilde{x} \in X$ such that $T\widetilde{x} = \widetilde{x}$.*

Proof. Using the same technique and notation in the proof of Theorem 2.13, we deduce that $S_p(x_n,x_n,x_n) \leq \alpha(x_n,x_n)S_p(x_n,x_n,x_n) \leq k^n S_p(x_0,x_0,x_0)$. Thus,

$$r_0 = S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = lim_n S_p(\widetilde{x},\widetilde{x},x_n) = lim_{n,m} S_p(x_n,x_n,x_m) = 0.$$

This implies that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = 0$. Since $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) = 0$, we have $\widetilde{x} = T\widetilde{x}$ as required. □

In closing, we change the contraction principle in Theorem 2.13, to show that there exist a unique fixed point in the whole space X.

Theorem 2.15. *Let (X, S_p) be a complete partial S-metric space, $k \in [0,1)$ and assume the there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq 1$. Consider the map $T : X \longrightarrow X$ to be $\alpha-$admissible and $R_\alpha-$admissible. Assume that for every $x, y \in X$ we have*

$$\alpha(x,y)S_p(Tx,Tx,Ty) \leq \max\{kS_p(x,x,y), \frac{S_p(x,x,x) + S_p(y,y,y)}{2}\}, \tag{2.2}$$

then there exists a unique $u \in X$ such that $Tu = u$.

Proof. Note that, for every $x, y \in X$ we have:

$$\begin{aligned} \alpha(x,y)S_p(Tx,Tx,Ty) &\leq \max\{kS_p(x,x,y), \frac{S_p(x,x,x) + S_p(y,y,y)}{2}\} \\ &\leq \max\{kS_p(x,x,y), S_p(x,x,x), S_p(y,y,y)\}. \end{aligned}$$

Thus, all conditions of Theorem 2.13 are satisfied. Hence, there exists $u \in X$ such that $Tu = u$. Assume that there exist two fixed points $u, v \in X$ for T such that $\alpha(u, v) \geq 1$. Hence,

$$\begin{aligned} S_p(u,u,v) = S_p(Tu,Tu,Tv) &\leq \alpha(u,v)S_p(Tu,Tu,Tv) \\ &\leq \max\{kS_p(u,u,v), \frac{S_p(u,u,u)+S_p(v,v,v)}{2}\}. \end{aligned}$$

Thus, we either have $S_p(u,u,v) \leq kS_p(u,u,v)$ which implies that $S_p(u,u,v) = 0$ and hence $u = v$, or $0 = 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v)$ which also implies that $u = v$ as desired. □

Example 3. Let (X, S_p) be a partial S-metric space, where $X = [0,1] \cup [2,3]$ and the partial S-metric space $S_p : X^3 \longrightarrow [0,+\infty)$ is defined by

$$S_p(x,y,z) = \begin{cases} \|\max\{x,y\} - z\|, & \text{if } \{x,y,z\} \cap [2,3] \neq \emptyset \\ |x-y-z|, & \text{if } \{x,y,z\} \subset [0,1]. \end{cases}$$

Define the functions $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ as follows $Tx = \dfrac{x+1}{2}$ if $0 \leq x \leq 1$, $T2 = 1$, and $Tx = \dfrac{x+2}{2}$ if $2 < x \leq 3$,

$$\alpha(x,y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

It is easy to see that T is $\alpha-$admissible and $R_\alpha-$admissible. Note that, we can always pick our x, y and z such that $\max\{x,y\} > z$. Also T is an increasing function. So, for every $x \geq y \in X$ we have:

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty) \leq \frac{1}{2}S_p(x,x,y),$$

if $x, y \in [0,1]$, and

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty) \leq \frac{S_p(x,x,x)+S_p(y,y,y)}{2},$$

$\{x,y\} \cap [2,3] \neq \emptyset$.

One can verify that the function T in this example satisfies the conditions of Theorem 2.15 and the unique fixed point will be 1.

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Accepted: 3.01.2017

Characterization of some linear groups by their conjugacy class sizes

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Abstract. Let G be a group and denote by $N(G)$ the set of conjugacy class sizes of G. In this paper, we proved that if $Z(G) = 1$ and $N(G) = N(PGL(3,\ q))$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$, then $G \cong PGL(3,\ q)$.

Keywords: finite group, conjugacy class sizes, Thompson's conjecture.

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1. Introduction

All groups considered in this paper are finite and simple groups are nonabelian. Let G be a group. For $x \in G$ we denote by x^G the conjugacy class of x, and by $|x^G|$ the size of x^G. Then set $N(G) = \{|x^G|| \ x \in G\}$. It is a well-established topic to investigate the relationship between the arithmetical properties of $N(G)$ and the structural properties of group G. More recently, there have appeared a number of papers addressing this research field. This paper is also a contribution along this line, which is related to an open conjecture of John G. Thompson (ref. to [15, Problem 12.38]):

Thompson's conjecture. *If S is a simple group and G is a group satisfying that $Z(G) = 1$ and $N(G) = N(S)$, then $G \cong S$.*

The *prime graph* of a group G is a simple graph whose vertices are the prime divisors of $|G|$ and where two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. Using the prime graph of simple group, the second author proved that Thompson's conjecture holds for all simple groups with disconnected prime graph in 1994 (see [1], also ref. to [2, 3, 4]). For the simple groups with connected prime graph, the conjecture has made considerable progress in recent years. Several mathematicians had proved the conjecture is true for the following simple groups: A_{10}, A_{16}, A_{22}, $U_4(4)$, $U_4(5)$, $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2D_n(q)$, and $E_7(q)$(see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

A group M is said to be an *almost simple* related to S if and only if $S \leq M \leq$Aut(S) for some simple group S. Naturally, one can put forward the following question: what are almost simple groups we can generalize Thompson's conjecture to? Some authors have generalized the conjecture to almost sporadic simple groups except Aut(J_2) and Aut(McL), symmetric groups S_n, where $n = p, p{+}1$, and p is an odd prime number, projective general linear groups $PGL(2, q)$, the automorphism groups of Suzuki-Ree groups (see[15, 16, 17, 18]). But they still used the second author's method, which is only valid for the groups with the disconnected prime graph.

In this paper, using Vasil'ev and Gorshkov's methods, we generalized Thompson's conjecture to projective general linear groups $PGL(3,\ q)$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$. Note that $PGL(3,\ 4)$ and $PGL(3,\ 7)$ have the connected prime graphs.

Our main result is the following theorem:

Main Theorem. *Let G be a group with $Z(G) = 1$ and M one of groups $PGL(3,\ q)$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$. If $N(G) = N(M)$, then $G \cong M$.*

By [19], we get $PGL(3,\ q) = L_3(q).d,\ d = gcd(3,\ q-1)$. Hence

$$M = PGL(3,\ q) = \begin{cases} L_3(q), & q = 2,\ 3,\ 5,\ 8,\ 9,\ 11, \\ L_3(q).3, & q = 4,\ 7. \end{cases}$$

Since simple groups $L_3(2)$, $L_3(3)$, $L_3(5)$, $L_3(8)$, $L_3(9)$, $L_3(11)$ have disconnected prime graphs, the second author in [1] has proved that Thompson's conjecture is right for these groups. Therefore, it is enough to prove Main Theorem for $PGL(3,\ 4)$ and $PGL(3,\ 7)$. We shall give the proofs on $PGL(3,\ 4)$ in Section 3 and $PGL(3,\ 7)$ in Section 4.

For convenience, we denote by $\pi(n)$ the set of all primes dividing n where n is a positive integer, and then n_π to denote π-part of n for $\pi \subseteq \pi(n)$. In addition, for a group G, we also denote by $\pi(G) = \pi(|G|)$, and $Soc(G)$ the socle of G which is a subgroup generated by all minimal normal subgroups of G. The other notation and terminologies in this paper are standard and the reader is referred to [19] and [21] if necessary.

2. Preliminaries

First, we cite here some known results which are useful in the sequel.

Lemma 2.1. *Let K be a normal subgroup of G and $\overline{G} = G/K$. Then*

(a) *If $\overline{x}$ is the image of an element x of G in the group $\overline{G}$, then $|x^K|\,\big|\,|x^G|$ and $|\overline{x}^{\overline{G}}|\,\big|\,|x^G|$.*

(b) *If $x \in G$ and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

(c) *If $x,\ y \in G$, $(|x|, |y|) = 1$, and $xy = yx$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

Lemma 2.2 ([5, Lemma 4]). *Let G be a group with trivial center, $p \in \pi(G)$ and p^2 not divide n for any $n \in N(G)$. Then a Sylow p-subgroup of G is elementary abelian.*

Lemma 2.3 ([9, Lemma 1.10]). *Let a Sylow p-subgroup of G be of order p, x be an element of order p, and $|x^G|$ be a number that is maximal with respect to divisibility in $N(G)$. Then $C_G(x)$ is abelian.*

Lemma 2.4 ([9, Lemma 1.9]). *Let G be a group, and p and q be two numbers in $\pi(G)$. If G satisfies the following conditions:*

(a)*$N(G)$ contains no number divisible by p^2 or q^2;*

(b)*$N(G)$ contains no number except 1 co-prime to pq;*

(c)*$N(G)$ contains a number h_q such that any n in $N(G)$ not divisible by q does not divide h_q and $N(G)$ contains no number divisible by h_q and n;*

(d)*$N(G)$ contains a number h_p such that any l in $N(G)$ not divisible by p does not divide h_p and $N(G)$ contains no number divisible by h_p and l.*

Then Sylow p-subgroups and q-subgroups of G are cyclic groups of prime order. In addition, G has no element of order pq.

Lemma 2.5 ([9, Lemma 1.12]). *Let G be a group, K the soluble radical of G, and $G/K = S$ a simple group. Suppose that there exists a prime p such that $p \in \pi(G) \setminus \pi(K)$. Assume that an element g of order p of G satisfies the following conditions:*

(a) *$|g^G| = |\overline{g}^S|$, where $\overline{g}$ is the image of an element g in the group S ;*

(b) *the number $|g^G|$ is maximal with respect to divisibility in $N(G)$;*

(c) *the subgroup* $C_G(g)$ *is abelian.*

Then $K \leq Z(G)$.

Let M be one of $PGL(3,\ 4)$ and $PGL(3,\ 7)$. Information on the set $N(M)$ and the order of M given in the next two lemmas is obtained via [19] or GAP [22].

Lemma 2.6. *Let* $M \cong PGL(3,\ 4)$*. Then:*

(1) $|M| = 2^6 \cdot 3^3 \cdot 5 \cdot 7$;

(2) $N(M) = \{n_1 = 1,\ n_2 = 3^2 \cdot 5 \cdot 7,\ n_3 = 2^4 \cdot 3 \cdot 7,\ n_4 = 2^6 \cdot 3 \cdot 5,\ n_5 = 2^6 \cdot 5 \cdot 7,\ n_6 = 2^6 \cdot 3^2 \cdot 5,$
$n_7 = 2^2 \cdot 3^3 \cdot 5 \cdot 7,\ n_8 = 2^6 \cdot 3^2 \cdot 7,\ n_9 = 2^4 \cdot 3^2 \cdot 5 \cdot 7\}$.

Especially,

(3) $N(M)$ *contains no number other than* n_1, n_4 *and* n_6 *not divisible by 7;*

(4) $N(M)$ *contains no number other than* n_1, n_3 *and* n_8 *not divisible by 5;*

(5) *For any* $n \in N(M)$ *and* $p \in \{5,\ 7\}$*, it follows that* $p^2 \nmid n$;

(6) $|x^M| = n_1,\ x \in M$ *if and only if* $x = 1$.

Lemma 2.7. *Let* $M \cong PGL(3,\ 7)$*. Then:*

(1) $|M| = 2^5 \cdot 3^3 \cdot 7^3 \cdot 19$;

(2) $N(M) = \{n_1 = 1,\ n_2 = 2^4 \cdot 3^2 \cdot 19,\ n_3 = 3 \cdot 7^2 \cdot 19,\ n_4 = 2^5 \cdot 3 \cdot 7^3,\ n_5 = 2^3 \cdot 7^3 \cdot 19,$
$n_6 = 2^5 \cdot 3^2 \cdot 7^3,\ n_7 = 2^5 \cdot 3^3 \cdot 7 \cdot 19,\ n_8 = 2 \cdot 3^2 \cdot 7^3 \cdot 19,\ n_9 = 2^4 \cdot 3^2 \cdot 7^2 \cdot 19,\ n_{10} = 2^3 \cdot 3 \cdot 7^3 \cdot 19\}$.

In particular,

(3) $N(M)$ *contains no number other than* n_1, n_4 *and* n_6 *not divisible by 19;*

(4) $N(M)$ *contains no number divided by* 19^2;

(5) $|x^M| = n_1,\ x \in M$ *if and only if* $x = 1$.

Lemma 2.8. *If* M *is one of* $PGL(3,\ 4)$ *and* $PGL(3,\ 7)$*, and* G *is a group with* $Z(G) = 1$ *and* $N(G) = N(M)$*, then* $|M|\big||G|$ *and* $\pi(G) = \pi(M)$.

Proof. Since the number in $N(G)$ divides $|G|$, under the hypothesis we see that $|M|\big||G|$ by Lemma 2.6 and Lemma 2.7. $\pi(M) = \pi(G)$ is the result of Lemma 1.2.1 in [1] or Lemma 3 in [5].

Lemma 2.9. *Let* S *be a simple group.*

(*i*) *If* $\pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$*, then* S *is isomorphic to one of simple groups of Table 1.*

(*ii*) *If* $\pi(S) \subseteq \{2,\ 3,\ 7,\ 19\}$*, then* S *is isomorphic to one of simple groups of Table 2.*

Proof. This is an immediate consequence of Theorem 2 in [23].

For convenience, we list all the cases of S in Lemma 2.9 as well as the orders of S, the orders of the outer automorphism of S in Table 1 and Table 2.

Table 1. Non-abeian simple groups S with $\pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$

S	Order of S	$\lvert\mathrm{Out}(S)\rvert$	S	Order of S	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\lvert S_3\rvert$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\lvert D_8\rvert$
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\lvert D_{12}\rvert$	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\lvert S_3\rvert$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2

Table 2. Non-abeian simple groups S with $\pi(S) \subseteq \{2,\ 3,\ 7,\ 19\}$

S	Order of S	$\lvert\mathrm{Out}(S)\rvert$	S	Order of S	$\lvert\mathrm{Out}(S)\rvert$
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	$\lvert S_3\rvert$
$U_3(8)$	$2^7 \cdot 3^4 \cdot 7 \cdot 19$	$\lvert 3 \times S_3\rvert$			

3. Proof of the main theorem for $PGL(3,\ 4)$

Theorem 3.1. *Let G be a group with trivial center. If $N(G) = N(PGL(3,\ 4))$, then $G \cong PGL(3,\ 4)$.*

Proof. We divide the proof of this theorem into six steps.

Step 1. Sylow 5-subgroups and Sylow 7-subgroups of G are cyclic groups of prime order and there are no elements of order 35 in G.

In view of $N(G) = N(PGL(3,\ 4))$ and Lemma 2.8, we can choose $p = 5$ and $q = 7$, and take $h_5 = n_6$ and $h_7 = n_8$ such that G satisfies the hypotheses of Lemma 2.4. Hence Sylow 5-subgroups and Sylow 7-subgroups of G are cyclic groups of prime order and there are no elements of order $5 \cdot 7$ in G.

Step 2. Let $g, h \in G$ be elements of orders 5 and 7, respectively. Then $|g^G| = n_8 = 2^6 \cdot 3^2 \cdot 7$ and $|h^G| = n_6 = 2^6 \cdot 3^2 \cdot 5$, and $C_G(g)$ and $C_G(h)$ are abelian.

Since the Sylow 5-subgroup of G is order of 5 by Step 1, one has that $5 \nmid |x^G|$ for any $1 \neq x \in C_G(g)$. Hence $|x^G| = n_3$ or n_8 by (4) and (6) of Lemma 2.6. Assume that $|g^G| = n_3 = 2^4 \cdot 3 \cdot 7$. Let H be a Sylow 3-subgroup of $C_G(g)$. Then H is a nontrivial group of order $|G|_3/3$ by Lemma 2.8. It follows that $Z(H) \neq 1$ and let $1 \neq y \in Z(H)$. Then $H \leq C_G(y)$, and so $|y^G|_3 \leq 3$. Thus $|y^G| = n_3$. Since $H \leq C_G(gy)$, we have that $|(gy)^G| = n_3$. In view of $C_G(gy) = C_G(g) \bigcap C_G(y)$, we see that $C_G(g) = C_G(y)$. The group $C_G(y)$

contains an element w from the center of a Sylow 3-subgroup of G, then $|w^G|$ is not divisible by 3, and so $|w^G| = n_5 = 2^6 \cdot 5 \cdot 7$ by (2) of Lemma 2.6. Thus $w \notin C_G(g)$, a contradiction. It follows that $|g^G| = n_8 = 2^6 \cdot 3^2 \cdot 7$. Since n_8 is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(g)$ is abelian.

In a similar way, we can show that $|h^G| = n_6 = 2^6 \cdot 3^2 \cdot 5$ and $C_G(h)$ is abelian.

In the following discussion, we assume that K is the soluble radical of a group G, and $\overline{G} = G/K$.

Step 3. G is non-soluble and has a unique composition factor S such that $5 \cdot 7\big||S|$ and $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$. Moreover, S may be isomorphic to one of the following groups:

$$A_7,\ A_8,\ A_9,\ L_4(3),\ S_6(2),\ U_4(3).$$

Assume that $5\big||K|$. Then $K/O_{\{5,7\}'}(K)$ has a normal subgroup T of order 5. Hence an element of $G/O_{\{5,7\}'}(K)$ of order 7 can act trivially on T, which implies that $G/O_{\{5,7\}'}(K)$ contains an element of order 35, so does G, contradicting with Step 1. Thus 5 does not divide $|K|$ and similarly we can prove that 7 does not divide $|K|$, and so G is not soluble.

Let $L = S_1 \times S_2 \times \cdots \times S_k$ be the socle of $\overline{G}$, where $S_1, S_2, \ldots,$ and S_k are simple groups. Let g be an element of order 5 of G and suppose that $5 \notin \pi(L)$. Then $\overline{g}$ is of order 5 in $\overline{G}$ and induces a nontrivial outer automorphism of the group L. Suppose that there exists i such that $S_i^{\overline{g}} \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s\ |s = s_1 s_1^{\overline{g}} s_1^{\overline{g}^2} s_1^{\overline{g}^3} s_1^{\overline{g}^4},\ s_1 \in S_1\rangle$. Then H lies in the centralizer of the element $\overline{g}$ and is isomorphic to S_1, but the centralizer of g is abelian by Step 2, a contradiction. Hence $\overline{g}$ induces a nontrivial outer automorphism of the group S_i such that $5\big||\mathrm{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 5, 7\}$ and by Table 1, the prime divisors of $|\mathrm{Out}(S_i)|$ are less than 5, a contradiction. Therefore $5\big||L|$ and similarly we can get $7\big||L|$.

If $k > 1$ and $\overline{g} \in S_i$, then $S_j < C_{\overline{G}}(\overline{g})$ for any $1 \leq j \leq k,\ j \neq i$, but $C_G(g)$ is abelian by Step 2, a contradiction. Thus $k = 1$. Let $S = S_1 = L$, and we get that G has a unique composition factor S such that $5 \cdot 7\big||S|$ and $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$. Since $\{5,\ 7\} \subseteq \pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$, $5 \parallel |S|$, and $7 \parallel |S|$, we can easily get that S can be isomorphic to one of the groups: $A_7,\ A_8,\ A_9,\ L_3(4),\ S_6(2),\ U_4(3)$ by Table 1.

Step 4. $S \cong L_3(4)$.

By Step 3, S may be isomorphic to one of groups $A_7, A_8, A_9, L_3(4), S_6(2), U_4(3)$. Recall that $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$.

If $S \cong A_7$, then $A_7 \trianglelefteq \overline{G} \leq \mathrm{Aut}(A_7) = S_7$ by Table 1. Since $2^6 \cdot 3^3\big||G|$, $2^4 \parallel |S_7|$, and $3^2 \parallel |S_7|$, we have $\pi(K) = \{2,\ 3\}$. Let $g, h \in G$ be elements of orders 5 and 7, and $\overline{g},\ \overline{h} \in \overline{G}$ be the image of the element g and h, respectively. If $\overline{G} \cong A_7$, then

$$|g^G| = 2^6 \cdot 3^2 \cdot 7,\ |h^G| = 2^6 \cdot 3^2 \cdot 5,$$

$$|\overline{g}^{\overline{G}}| = 2^3 \cdot 3^2 \cdot 7,\ |\overline{h}^{\overline{G}}| = 2^3 \cdot 3^2 \cdot 5.$$

Set $x = g,\ h$. Then $|x^G| = |\overline{x}^{\overline{G}}||x^K|$, and so $|K : C_K(x)| = 2^3$. It follows that g, h centralize every $3-$element of K, and thus there exists a $3-$element y in K such that $35\big||C_G(y)|$. By Lemma 2.6, one has that $|y^G| = n_1 = 1$, and so $y = 1$, a contradiction. If $\overline{G} \cong S_7$, then we also can get a contradiction in a similar way. Hence S is not isomorphic to A_7.

If $S \cong A_8$, then $A_8 \trianglelefteq \overline{G} \leq \mathrm{Aut}(A_8) = S_8$. By [19], S has an element x of order 6 satisfying with $|x^S| = 2^5 \cdot 3 \cdot 5 \cdot 7$ which does not divide any element of $N(G)$. Thus it is impossible that S is isomorphic to A_8.

Let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$. If S is one of $A_9, S_6(2)$ and $U_4(3)$, then by [19], $|\overline{x}^S|$ is a multiple of 3^4, so are $|\overline{x}^{\overline{G}}|$ and $|x^G|$. This contradicts with (2) of Lemma 2.6, and so $S \cong L_3(4)$.

Step 5. $\overline{G} = G/K \cong PGL(3,\ 4)$.

By virtue of $L_3(4) \leq G/K \leq \mathrm{Aut}(L_3(4))$, $\overline{G}$ may be isomorphic to one of the following groups: $L_3(4), L_3(4).2_1, L_3(4).2_2, L_3(4).2_3, L_3(4).3 = PGL(3,\ 4)$, $L_3(4).2_2.2_3$, $L_3(4).3.2_1, L_3(4).3.2_2, L_3(4).D_{12}$. Let $g, h \in G$ be elements of orders 5 and 7, and $\overline{g},\ \overline{h} \in \overline{G}$ be the image of the element g and h, respectively.

If $\overline{G} \cong L_3(4)$, then $3\big||K|$, and so $K \neq 1$. By Lemma 2.6 (2), Step 2 and [19], we have that n_8 is maximal in $N(G)$, $C_G(g)$ is abelian, and $|g^G| = |\overline{g}^S| = n_8$. Thus by Lemma 2.5, $K \leq Z(G) = 1$, a contradiction.

If $\overline{G}$ is one of $L_3(4).2_2,\ L_3(4).2_2.2_3,\ L_3(4).3.2_1,\ L_3(4).3.2_2$ and $L_3(4).D_{12}$, then by [22], there exists an element g of order 5 in G such that $|\overline{g}^S| = 8064 \nmid |g^G| = 4032$, a contradiction with Lemma 2.1.

If $\overline{G}$ is one of $L_3(4).2_1$ and $L_3(4).2_3$, also by [22], we can find an element h of order 7 in G such that $|\overline{h}^S| = 5760 \nmid |h^G| = 2880$, a contradiction again. Hence $\overline{G} = G/K \cong PGL(3,\ 4)$.

Step 6. K is a trivial group such that $G \cong PGL(3,\ 4)$.

Let $h \in G,\ |h| = 7$, and $\overline{h} \in G/K = \overline{G}$ be the image of the element h. In view of Lemma 2.6 (2), Step 2 and [19], we see that n_6 is maximal in $N(G)$, $C_G(h)$ is abelian and $|h^G| = |\overline{h}^S| = |\overline{h}^{\overline{G}}| = n_6$. Thus $K \leq C_G(h)$. If $K \neq 1$, then h centralizes an element from the center of a Sylow p-subgroup of G for some prime $p \in \pi(K)$, which is impossible by Lemma 2.6. Hence K is a trivial group, and so $G \cong PGL(3,\ 4)$.

4. Proof of the main theorem for $PGL(3,\ 7)$

Theorem 4.1. *Let G be a group with trivial center. If $N(G) = N(PGL(3,\ 7))$, then $G \cong PGL(3,\ 7)$.*

Proof. We divide the proof of this theorem into eight steps.

Step 1. The Sylow 19-subgroup P of G is order of 19.

Using Lemma 2.2 and (2) of Lemma 2.7, we derive that P is elementary abelian. Assume that 19^2 divides G. Since $N(G) = N(PGL(3,\ 7))$, the centralizer of every element of G contains an element of order 19 by (4) of Lemma 2.7. Considering an element y of G such that $|y^G| = n_2 = 2^4 \cdot 3^2 \cdot 19$.

Suppose that 19 does not divide $|y|$. Let x be an element of order 19 in $C_G(y)$. Then by (3) of Lemma 2.1, $C_G(xy) = C_G(x) \bigcap C_G(y)$, and so $lcm(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since P is abelian, $C_G(x)$ includes P up to conjugacy. Hence 19 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_4 = 2^5 \cdot 3 \cdot 7^3$ or $n_6 = 2^5 \cdot 3^2 \cdot 7^3$ by (3) and (5) of Lemma 2.7. In both cases, $2^5 \cdot 3^2 \cdot 7^3 \cdot 19$ divides $|(xy)^G|$, which is impossible by (2) of Lemma 2.7.

Suppose that 19 divides $|y|$. Let $|y| = 19t$. Since P is elementary abelian, one has that $gcd(19,\ t) = 1$. Put $u = y^{19}$ and $v = y^t$. Then $y = uv$, and so $C_G(uv) = C_G(u) \bigcap C_G(v)$ by Lemma 2.1. Therefore, $|v^G|$ divides $|y^G| = n_2 = 2^4 \cdot 3^2 \cdot 19$. On the other hand, the element v is order of 19, and thus $|v^G|$ is equal to $n_4 = 2^5 \cdot 3 \cdot 7^3$ or $n_6 = 2^5 \cdot 3^2 \cdot 7^3$ by Lemma 2.7, a contradiction. Hence P has order of 19.

Step 2. For an element $x \in G$ of order 19, it follows that $|x^G| = n_6 = 2^5 \cdot 3^2 \cdot 7^3$ and $C_G(x)$ is abelian.

By Step 1, for any $1 \neq y \in C_G(x)$ one has that $19 \nmid |y^G|$, and hence $|y^G| = n_4$ or n_6 by (3) and (5) of Lemma 2.7. Assume that $|x^G| = n_4 = 2^5 \cdot 3 \cdot 7^3$ and let H be a Sylow 3-subgroup of $C_G(x)$. Then H is a nontrivial group of order $|G|_3/3$ by Lemma 2.8. Hence there exists a 3-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 3$. Then $1 \neq H \bigcap Z(K) \leq C_G(x)$. Taking $1 \neq h \in H \bigcap Z(K)$, we have that $K \leq C_G(h)$, and so $|h^G|_3 = 1$. But $|h^G| = n_4$ or n_6, one has that $3\big||h^G|$, a contradiction. It follows that $|x^G| = n_6 = 2^5 \cdot 3^2 \cdot 7^3$.

Since n_6 is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(x)$ is abelian.

Step 3. Suppose that $q \in \{2,\ 3,\ 7\}$, Q is a Sylow q-subgroup of G. Then $19 \nmid |C_G(x)|$, $x \in Z(Q)$.

Let $1 \neq x \in Z(Q)$. Then q does not divide $|x^G|$, and by Lemma 2.7, $|x^G| = n_3 = 3 \cdot 7^2 \cdot 19$ while $q = 2$, $|x^G| = n_5 = 2^3 \cdot 7^2 \cdot 19$ while $q = 3$, and $|x^G|$ is equal to $n_2 = 2^4 \cdot 3^2 \cdot 19$ while $q = 7$. The Step 3 follows.

Step 4. G is non-soluble and $O_{2,\ 2'}(G) = O_2(G)$.

Let $K = O_2(G)$, $\overline{G} = G/K$, and denote by $\overline{x}$ the images of an element x of G in $\overline{G}$. If the statement is false, then there exists $r \in \{3,\ 7,\ 19\}$ such that $O_r(\overline{G}) \neq 1$.

If $O_{19}(\overline{G}) \neq 1$, then $|O_{19}(\overline{G})| = 19$ by Step 1. Let Q be a Sylow 7-subgroup of G and $y \in Z(Q)$ be an element of order 7. Obviously, the subgroup $O_{19}(\overline{G})\langle\overline{y}\rangle$ is cyclic. Hence 19 divides $|C_{\overline{G}}(\overline{y})|$. Since $(7,\ |K|) = 1$, Lemma 2.1(2) implies that 19 divides $|C_G(y)|$, which is impossible by Step 3. Thus, $O_{19}(\overline{G}) = 1$.

If $O_7(\overline{G}) \neq 1$, then $V = Z(O_7(\overline{G}))$ is a nontrivial normal subgroup of G. If x is an element of order 19 in G, then $V = C_V(\overline{x}) \times [V,\ \overline{x}]$ such that $\overline{x}$ acts fixed-point freely on $[\overline{x}, V]$, and then $|[\overline{x}, V]| - 1$ is divisible by 19. Lemma 2.1 (1) implies that $|\overline{x}^{\overline{G}}|$ is a divisor of $2^5 \cdot 3^2 \cdot 7^3$, and hence $|[V,\ \overline{x}]| = |V : C_V(\overline{x})|$ divides 7^3, which implies $[V,\ \overline{x}] = 1$ and $V = C_V(\overline{x})$. Let Q be a Sylow 7-subgroup of $\overline{G}$. Then $Z(Q)$ has a nontrivial intersection with V and let $\overline{z}$ be of order 7 from this intersection. Since $(|K|,\ 7) = 1$, there exists a pre-image z of $\overline{z}$ in G such that z lies in the center of a Sylow 7-subgroup of G by Lemma 2.1 (2). Further, the centralizer of z also contains an element of order 19, which contradicts Step 3. Therefore, $O_7(\overline{G}) = 1$.

Similarly, we can prove $O_3(\overline{G}) = 1$. The Step 4 holds.

Step 5. Let $K = O_2(G)$, $\overline{G} = G/K$. Then every minimal normal subgroup of $\overline{G}$ is non-soluble. Especially, $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(Soc(\overline{G}))$.

Let N be any minimal normal subgroup of $\overline{G}$ and assume that N is soluble. Then N is an elementary ablelian p-group for some $p \in \{3,\ 7,\ 19\}$, and so $N \leq O_p(\overline{G})$. It follows that $O_p(\overline{G})$ is nontrivial, contradicts Step 4. Hence every minimal normal subgroup of $\overline{G}$ is non-soluble. Let $N_1, N_2, \ldots, N_s$ be all minimal normal subgroups of $\overline{G}$, where s is a positive integer. Then $Soc(\overline{G}) = N_1 \times N_2 \times \cdots \times N_s$. We assert that $C_{\overline{G}}(Soc(\overline{G})) = 1$. Otherwise, $1 \neq C_{\overline{G}}(Soc(\overline{G})) \trianglelefteq \overline{G}$. But $C_{\overline{G}}(Soc(\overline{G})) \bigcap Soc(\overline{G})$=1 because $N_i (1 \leq i \leq s)$ are a direct product of some isomorphic simple groups. Hence $C_{\overline{G}}(Soc(\overline{G}))$ is soluble, a contradiction. By N/C theorem, we have $Soc(\overline{G}) \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(Soc(\overline{G})) \lesssim \mathrm{Aut}(Soc(\overline{G}))$.

Step 6. Let $L = Soc(\overline{G})$. Then L is a non-ableian simple group and $19\big||L|$.

By Step 5, we have that L is a direct product of non-ableian simple groups of $S_1, S_2, \ldots,$ and S_k. Let g be an element of order 19 of G and suppose that $19 \notin \pi(L)$. Then $\overline{g}$ is of order 19 in $\overline{G}$ and induces a nontrivial outer automorphism of the group L. Suppose that there exists a positive integer i satisfying $S_i^{\overline{g}} \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s \mid s = s_1 s_1^{\overline{g}} s_1^{\overline{g}^2} \cdots s_1^{\overline{g}^{18}},\ s_1 \in S_1 \rangle$. Then H lies in the centralizer of the element $\overline{g}$ and is isomorphic to S_1, but the centralizer of g is abelian by Step 2, a contradiction. Hence $\overline{g}$ induces a nontrivial outer automorphism of the group S_i such that $19\big||\mathrm{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 7, 19\}$ and Table 2, the prime divisors of $|\mathrm{Out}(S_i)|$ are not greater than 5, a contradiction. Therefore $19\big||L|$.

If $k > 1$ and $\overline{g} \in S_j$, then $S_i \leq C_{\overline{G}}(\overline{g})$ for any $1 \leq i \leq k,\ i \neq j$. On the other hand, $C_{\overline{G}}(\overline{g})$ is abelian by Step 2, a contradiction. Therefore $k = 1$, and so L is a non-ableian simple group and $19\big||L|$.

Step 7. $L \cong L_3(7)$.

By Step 6 and Step 1, we have that L is a non-ableian simple group satisfying $19 \parallel |M|$. Then by Table 2, L may be isomorphic to one of $U_3(8)$ and $L_3(7)$.

By Table 2 and Step 5, we see that $\pi(\mathrm{Out}(L)) \subseteq \{2,\ 3\}$ and $L \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(L)$. In view of $K = O_2(G)$ and $7^3\big||G|$, we have that 7^3 divides $|L|$. Hence L must be isomorphic to $L_3(7)$.

Step 8. $G \cong PGL(3,\ 7)$.

Let x be an element of order 19 in G and $\overline{x}$ be its image in $\overline{G}$. It is clear that $\overline{x} \in L$. By Lemma 2.1 and [19], we have that $|\overline{x}^L| = |\overline{x}^{\overline{G}}| = |x^G| = 2^5 \cdot 3^2 \cdot 7^3$ such that $K \leq C_G(x)$. If $K \neq 1$, then x centralizes an element from the center of a Sylow 2-subgroup of G, which is impossible by Step 3. Hence $G = \overline{G}$ and $L_3(7) \trianglelefteq G \lesssim \mathrm{Aut}(L_3(7))$. By [22], we see that $|N(G)| \neq |N(T)|$ for any group T except $PGL(3,\ 7)$, where T satisfies with $L_3(7) \leq T \leq \mathrm{Aut}(L_3(7))$. Therefore $G \cong PGL(3,\ 7)$.

Proof of Main Theorem. It follows directly from Theorem 3.1 and Theorem 4.1.

Acknowledgments

This Work is supported by Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education ([2017]3), Foundation of Chongqing Development and Reform Commission (2017[1007]), Scientifific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ1710254), and Foundation of Chongqing Three Gorges University (Grant Nos. 14ZD16, 18ZDPY07).

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Accepted: 4.01.2017

On new class of contra continuity in nano topology

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Abstract. The purpose of this paper is to introduce and study the stronger form of nano continuity called nano contra continuity. Further the concept of nano kernel and nano Bi-contra continuity is also discussed. These type of mappings can be used in Biotechnology, to study contra or two way contra effects. Here we applied both contra and Bi-contra mapping between a set of viruses to a set of antiviruses as a treatment for disease causing viruses.

Keywords: nano topology, nano α-open sets, nano rare set, nano contra continuity, nano Bi-contra continuity.

1. Introduction

Ganster and Reily [5] introduced and studied notion of LC-continuous functions. Dontchev [4] presented a new notion of continuous function called contra continuity, a stronger form of LC-continuity. Lellis Thivagar et al. [6] introduced a nano topological space with respect to a subset X of a universe which is defined in terms of lower and upper approximations of X. The elements of a nano topological space are called the nano open sets. But certain nano terms are satisfied simply to mean "very small". It comes from the Greek word 'Nanos' which means 'dwarf', in its modern scientific sense, an order to magnititude-one billionth of something. Nano car is an example. The topology introduced here is named so because of its size, since it has almost five elements in it. Nano

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continuous function [6] has been defined in terms of nano open sets and its characterisations were derived. Certain weak forms of nano α-open sets, nano semi-open sets and nano pre-open sets were also established[6]. In this paper nano contra continuity and nano Bi-contra continuity and their properties are discussed.

2. Preliminaries

The following recalls necessary concepts and preliminaries required in the sequel.

Definition 2.1 ([8]). Let $\mathcal{U}$ be a non-empty finite set of objects called the universe $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ named as the indiscerniblity relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.

(i) The Lower appproximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \left\{\bigcup_{x\in\mathcal{U}}\{R(x) : R(x) \subseteq X\}\right\}$, where R(x) denotes the equivalence class determined by x.

(ii) The Upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X) = \left\{\bigcup_{X\in\mathcal{U}}\{R(x) : R(x) \cap X \neq \emptyset\}\right\}$.

(iii) The Boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not -X with respect to R and it is denoted by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 ([9]). Let $\mathcal{U}$ be the universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{U}$ and $\tau_R(X)$satisfies the following axioms.

(i) $\mathcal{U}$ and $\emptyset \in \tau_R(X)$.

(ii) The union of the elements of any subcollection$\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology $\mathcal{U}$ called as the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called nano open sets. A set A is said to be nano closed if its complement is nano open.

Definition 2.3 ([6]). If $(\mathcal{U}, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq \mathcal{U}$ and if $A \subseteq \mathcal{U}$, then nano interior of A is defined as the union of all nano open subsets contained in A and its denoted by $\mathcal{N}Int(A)$. That is

$\mathcal{N}Int(A)$ is the largest nano open subset contained in A. The nano closure of A is defined as the intersection of all nano closed sets containing A and its denoted by $\mathcal{N}Cl(A)$. That is, $\mathcal{N}Cl(A)$ is the smallest nano closed set containing A.

Definition 2.4 ([6]). Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$. Then A is said to be,

(i) nano semi-open if $A \subseteq \mathcal{N}Cl(\mathcal{N}Int(A))$.

(ii) nano pre-open if $A \subseteq \mathcal{N}Int(\mathcal{N}Cl(A))$.

(iii) nano α-open if $A \subseteq \mathcal{N}Int(\mathcal{N}Cl(\mathcal{N}Int(A))$.

Definition 2.5 ([6]). Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$ is called nano α-closed (respectively,nano semi-closed,nano pre-closed) if its complement is nano α-open (nano semi-open, nano pre-open).

Definition 2.6 ([6]). Let $(\mathcal{U}, \tau_R(X))$ and $(\mathcal{V}, \tau_{R'}(Y))$ be nano topological spaces. Then a mapping f: $(\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano continuous on $\mathcal{U}$ if the inverse image of every nano open set in $\mathcal{V}$ is nano open in $\mathcal{U}$.

Throughout this paper, $\mathcal{U}$ and $\mathcal{V}$ are non empty finite universes, $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{V}$, where R and R' are equivalence relations on $\mathcal{U}$ and $\mathcal{V}$ respectively. $(\mathcal{U}, \tau_R(X))$ and $(\mathcal{V}, \tau_{R'}(Y))$ are the nano topological space with respect to X and Y, respectively.

3. Nano kernel

In this section, the notion of nano kernel is introduced and its properties are investigated.

Definition 3.1. Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$. The set $\mathcal{N}ker(A) = \bigcap\{$U: $A \subseteq$U, U $\in \tau_R(X)\}$ is called the nano kernel of A and is denoted by $\mathcal{N}ker(A)$.

Example 3.2. Let $\mathcal{U} = \{a, b, c, d, e\}$ with $\mathcal{U}/R = \{\{a, b\}, \{c\}, \{d, e\}\}$. Let $X = \{a\} \subseteq \mathcal{U}$ so that $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a, b\}\}$, then the $\mathcal{N}Ker\{a\} = \{a, b\}$, $\mathcal{N}Ker\{a, b\} = \{a, b\}$, $\mathcal{N}Ker\{a, b, c\} = \mathcal{U}$.

Theorem 3.3. *Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A, B \subseteq \mathcal{U}$. Then the following properties hold.*

(i) $x \in \mathcal{N}\ker A$ iff $A \cap F \neq \emptyset$ for any nano closed set containing x.

(ii) If $A \subseteq \mathcal{N}kerA$ and then $A = \mathcal{N}Ker(A)$ if A is nano open in $\mathcal{U}$.

(iii) If $A \subseteq B$, then $\mathcal{N}\ker A \subseteq \mathcal{N}\ker B$.

Proof. (i) $\Rightarrow$. If $x \in \mathcal{N}\ker A$, then $x \in A \subseteq F^c$, where F^c is a nano open set and $A \subseteq F^c, \Rightarrow A \cap F^c \neq \emptyset$. Hence $A \subseteq F^c \subseteq F \Rightarrow A \cap F \neq \emptyset$, F is nano closed. Hence $x \in A \cap F \neq \emptyset$, F is nano closed. Therefore $A \cap F \neq \emptyset$ for any nano closed set containing x.

$\Leftarrow$. Let $A \cap F \neq \emptyset$ for any nano closed set containing x.

Assume that $x \notin \mathcal{N}kerA$, hence their exist an nano open set F^c such that $A \subseteq F^c$ and $x \notin F^c$. Hence $A \subseteq F$ and $x \notin F$, where F is a nano closed which is a contradiction. Hence $x \in \mathcal{N}kerA$.

(ii) Let A be a nano open set. Since A is nano open $\mathcal{N}kerA \subseteq A$ holds.

Let B be any nano open set containing A, then we have $A \subseteq W$. Hence $A \subseteq W \cap A \subseteq A$ and $W \cap A$ is nano open. Now, $A \subseteq \cap\{W.A \subseteq W,\ W \in \tau_R(X)\}$. Therefore $A \subseteq \mathcal{N}kerA$ and hence $A = \mathcal{N}kerA$.

(iii) Let $A \subseteq B$, To prove that. $\mathcal{N}kerA \subseteq \mathcal{N}kerB$. Let $V \in \mathcal{N}kerA \Rightarrow A \subseteq V$ and V is nano open in $\{\mathcal{U}, \tau_R(X)\}$. Since $A \subseteq B$, so $A \subseteq B \subseteq V$, where V is a nano open set in $(\mathcal{U}, \tau_R(X))$. Therefore $V \in \mathcal{N}ker(B)$. Hence $V \in \mathcal{N}ker(A) \Rightarrow V \in \mathcal{N}ker(B)$ and thus $\mathcal{N}kerA \subseteq \mathcal{N}ker(B)$. □

4. Nano kernel

In this section we introduce and define the concept of nano no where dense sets and nano rare sets and some of their properties are investigated.

Definition 4.1. Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and let $A \subseteq \mathcal{U}$, then A is called nano nowhere dense if $\mathcal{N}Int[\mathcal{N}Cl(A)] = \emptyset$.

Example 4.2. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{a, c\}, \{b, d\}$ and let $X = \{a, c\} \subseteq \mathcal{U}$, $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a, c\}\}$, then the nano closed set are $\mathcal{U}, \emptyset, \{b, d\}$. Nano semi-closed sets are $\mathcal{U}, \emptyset, \{b, d\}, \{b\}, \{d\}$. Nano nowhere dense sets are $\mathcal{U}, \emptyset, \{b, d\}$, $\{b\}, \{d\}$.

Remark 4.3. The following theorem is the consequence of the above example.

Theorem 4.4. *Every nano nowhere dense set is nano semi-closed.*

Proof. Let A$\subseteq$ $\mathcal{U}$ be a nano nowhere dense set,then $\mathcal{N}Int[\mathcal{N}Cl(A)] = \emptyset$. Clearly, $\mathcal{N}Int[\mathcal{N}Cl(A)] = \emptyset \subseteq A \Rightarrow \mathcal{N}Int[\mathcal{N}Cl(A)] \subseteq A$. Hence A is nano semi-closed. □

Remark 4.5. The converse of Theorem 4.4 is not true which can be shown by the following example.

Example 4.6. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{c\}, \{b, d\}\}$, and let $X = \{a, b\} \subseteq \mathcal{U}$, $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}$. The set $\{b, c, d\}$ is a nano semi-closed set in $\mathcal{U}$ but not nano nowhere dense.

Definition 4.7. Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$,then A is said to be a nano rare set if $\mathcal{N}Int(A) = \emptyset$, otherwise it is known as nano non-rare set in $\mathcal{U}$.

Example 4.8. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\} \subseteq \mathcal{U}$. Then the nano topology $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}$. Then the nano rare sets are $\emptyset$, $\{b\}, \{c\}, \{d\}, \{c, d\}, \{b, c\}$. Nano non-rare sets are $\mathcal{U}$, $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

Theorem 4.9. *In a nano topological space $[\mathcal{U}, \tau_R(X)]$ if $L_R(X) = U_R(X)$, then $\mathcal{U}, L_R(X)$ and any set $A \supseteq L_R(X)$ are the only nano non-rare sets in $\mathcal{U}$.*

Proof. Since $L_R(X) = U_R(X) = X$, the nano topological space, $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X)\}$. If $A \neq \emptyset$ and $A \subset L_R(X)$, then $\mathcal{N}Int(A) = \emptyset$, since $\emptyset$ is the only nano open subset of A. Hence A is a nano non-rare subset of A. If $A \supset L_R(X)$, then $L_R(X)$ is the largest nano open subset of A and hence $\mathcal{N}Int(A) = L_R(X) \neq \emptyset$. Therefore, A is a nano non-rare set. Thus $\mathcal{U}, L_R(X)$ and any set $A \supset L_R(X)$ are the only nano non-rare sets in $\mathcal{U}$, if $L_R(X) = U_R(X)$. □

Theorem 4.10. *In a nano topological space if $L_R(X) = \emptyset$ and $U_R(X) \neq \mathcal{U}$, then $\mathcal{U}, U_R(X)$ and any set $A \supseteq U_R(X)$ are the only nano non-rare sets.*

Proof. Since $L_R(X) = \emptyset$ and $U_R(X) = X$, the nano topological space, $\tau_R(X) = \{\mathcal{U}, \emptyset, U_R(X)\}$ and the members of $\tau_R(X)$ are nano open in $\mathcal{U}$. Let $A \subset U_R(X)$, then $\mathcal{N}Int(A) = \emptyset$. Therefore A is a nano rare set in $\mathcal{U}$. If $A \supseteq U_R(X)$, then $U_R(X)$ is the largest nano open subset of A. Therefore $\mathcal{N}Int(A) = U_R(X)$ and hence A is a nano non-rare set in $\mathcal{U}$. Thus $\mathcal{U}$, and any set $A \supseteq U_R(X)$ are the only nano non-rare sets in $\mathcal{U}$. □

Theorem 4.11. *In $[\mathcal{U}, \tau_R(X)]$ if $U_R(X) = \mathcal{U}$ and $L_R(X) \neq \emptyset$ then $\mathcal{U}$ and any set $A \supseteq L_R(X), B_R(X)$ are the only nano non-rare sets in $\mathcal{U}$.*

Proof. Since $U_R(X) = \mathcal{U}$ and $L_R(X) \neq \emptyset$ the nano open sets in $\mathcal{U}$ are $\mathcal{U}, \emptyset$, $L_R(X)$ and $B_R(X)$. Let A $\subseteq \mathcal{U}$ such that $A \neq \emptyset$. If $A = \emptyset$, then A is nano rare open. Therefore, let $A \neq \emptyset$, when $A \subset L_R(X)$, then $\mathcal{N}Int(A) = \emptyset$, since the largest nano open subset of A is $\emptyset$. That is, A is nano rare open in $\mathcal{U}$. When $L_R(X) \subset A$, $\mathcal{N}Int(A) = L_R(X)$ and therefore A is nano non-rare open in $\mathcal{U}$. Similarly it can be shown that any set $A \subset B_R(X)$ is nano rare set in $\mathcal{U}$ and $A \supseteq B_R(X)$ is a nano non-rare set in $\mathcal{U}$. Thus $\mathcal{U}$ and any set $A \supseteq L_R(X), B_R(X)$ are the only nano non-rare sets in $\mathcal{U}$. □

Theorem 4.12. *If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \emptyset$ and $U_R(X) \neq \mathcal{U}$ in a nano topological space then any set $A \supseteq L_R(X), U_R(X)$ and $B_R(X)$ are the only nano non-rare sets in $\mathcal{U}$.*

Proof. The nano topology on $\mathcal{U}$ is given by $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$. Then $A \supseteq \mathcal{U}$ such that $A \supseteq U_R(X)$, then $\mathcal{N}Int(A) = U_R(X)$ and therefore any $A \supseteq U_R(X)$ is nano non-rare in $\mathcal{U}$. Similarly it can be shown that $A \supseteq B_R(X)$ and $A \subseteq L_R(X)$ are nano non-rare in $\mathcal{U}$. When $A \subset B_R(X)$, $\mathcal{N}Int(A) = \emptyset$ and hence A is nano rare in $\mathcal{U}$. When $A \subset U_R(X)$ such that A is neither a subset of $L_R(X)$ nor a subset of $B_R(X)$, $\mathcal{N}Int(A) = \emptyset$ and hence A is nano rare in $\mathcal{U}$. Thus any set $A \supseteq L_R(X), U_R(X)$ *and* $B_R(X)$ are the only nano non-rare sets in $\mathcal{U}$. □

Theorem 4.13. *Every nano nowhere dense set is nano rare.*

Proof. Let $A \subseteq \mathcal{U}$ be a nano nowhere dense set in $\mathcal{U}$, then by definition $\mathcal{N}Int[\mathcal{N}Cl(A)] = \emptyset$. We know that $A \subseteq \mathcal{N}Cl(A) \Longrightarrow \mathcal{N}Int(A) \subseteq \mathcal{N}Int[\mathcal{N}Cl(A)]$ $= \emptyset \Longrightarrow \mathcal{N}Int(A) = \emptyset$. That is, A is a nano rare set in $\mathcal{U}$. □

Remark 4.14. The converse of Theorem 4.13 is not true which can be seen from the following example.

Example 4.15. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b, c, d\}\}$ and $X = \{a, d\} \subseteq \mathcal{U}$. Then the nano topology $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{b, c, d\}\}$. Then the nano rare sets in $\mathcal{U}$ are $\{b\}, \{c\}, \{d\}, \{b, d\}, \{c, d\}$ but are not nano nowhere dense in $\mathcal{U}$.

Theorem 4.16. *If a set* $A \subseteq \mathcal{U}$ *in a nano topological space is both nano pre-open and nano semi-closed then it is a nano non-rare set.*

Proof. By definition of nano pre-open and nano semi-closed, $A \subseteq \mathcal{N}Int[\mathcal{N}Cl(A)]$ and $\mathcal{N}Int[\mathcal{N}Cl(A)] \subseteq A$ which implies $A = \mathcal{N}Int[\mathcal{N}Cl(A)]$. Hence A becomes nano regular open set in $\mathcal{U}$. Since every nano regular open set is nano open, then $\mathcal{N}Int(A) \neq \emptyset$. Hence A is not a nano rare set in $\mathcal{U}$ if it is both nano pre-open and nano semi-closed in $\mathcal{U}$. □

5. Nano contra continuous function

In this section, the notion of nano form of contra continuity is introduced and its properties are investigated. This nano contra continuity is also compared with other nano continuous functions.

Definition 5.1. Let $(\mathcal{U}, \tau_R(X))$ and $(\mathcal{V}, \tau_{R'}(Y))$ be nano topological spaces, then the mapping $f : (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano contra continuous, if the inverse image of every nano open set in $\mathcal{V}$ is nano closed in $\mathcal{U}$. That is, if $f^{-1}(B)$ is nano closed in $\mathcal{U}$ for every nano open set B of $\mathcal{V}$.

Example 5.2. Let $\mathcal{U} = \{a, b, c, d, e\}$ with $\mathcal{U}/R = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{d, e\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{d, e\}\}$. Then nano closed sets are $\mathcal{U}, \emptyset, \{a, b, c\}$. Let $\mathcal{V} = \{x, y, z, u, v\}$ with $\mathcal{V}/R' = \{\{x, y\}, \{z, v\}, \{u\}\}$. Let $Y = \{z, u, v\} \subseteq \mathcal{V}$. Then $\tau_{R'}(Y) = \{\mathcal{V}, \emptyset, \{z, u, v\}\}$. Define $f : \mathcal{U} \to \mathcal{V}$ by $f(a) = z, f(b) = u, f(c) = v, f(d) = v, f(d) = x, f(e) = y$. Then $f^{-1}(\{z, u, v\}) = \{a, b, c\}$ which is a nano closed set in $\mathcal{U}$. That is, the inverse image of every nano open set in $\mathcal{V}$ is nano closed in $\mathcal{U}$. Therefore f is nano contra continuous.

Definition 5.3. A function $f : \mathcal{U} \to \mathcal{V}$ is said to be

(i) Nano perfectly continuous. if $f^{-1}(V)$ is nano clopen in $\mathcal{U}$ for every nano open set V in $\mathcal{V}$.

(ii) Nano strongly continuous. if $f^{-1}(V)$ is nano clopen in $\mathcal{U}$ for every subset V in $\mathcal{V}$.

(iii) Nano α-continuous. if $f^{-1}(V)$ is nano α-open in $\mathcal{U}$ for every nano α- open set V in $\mathcal{V}$.

(iv) Nano pre-continuous. if $f^{-1}(V)$ is nano pre-open in $\mathcal{U}$ for every nano open set V in $\mathcal{V}$.

Theorem 5.4. *For a function $f : (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_R(Y))$ the following conditions are equivalent.*

(i) f is nano contra continuous.

(ii) The inverse image of each nano closed set in $\mathcal{V}$ is nano open in $\mathcal{U}$.

(iii) For each $x \in \mathcal{U}$ and each nano closed set B in $\mathcal{V}$ with $f(x) \in B$, there exists a nano open set A in $\mathcal{U}$ such that $f(A) \subseteq B$.

(iv) $f(\mathcal{N}(cl(A)) \subseteq \mathcal{N}Kerf(A)$ for every subset A of $\mathcal{U}$.

(v) $\mathcal{N}cl[f^{-1}(B)] \subseteq f^{-1}[\mathcal{N}Ker(B)]$.

Proof. $(i) \Rightarrow (ii)$. Let f be nano contra continuous. Let B be a nano closed set in $\mathcal{V}$ and therefore B^C is nano open in $\mathcal{V}$. By (i) $f^{-1}(B^C)$ is nano closed in $\mathcal{U}$. But, $f^{-1}(B^C) = \{f^{-1}(B)\}^C$. Hence $f^{-1}(B)$ is nano open in $\mathcal{U}$.

$(ii) \Rightarrow (i)$. Let B be a nano open set in $\mathcal{V}$. Then B^C is nano closed in $\mathcal{V}$. By (ii) $f^{-1}(B^C)$ is nano open in $\mathcal{U}$. Hence $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Hence f is nano contra continuous.

$(ii) \Rightarrow (iii)$. Let B be a nano closed set such that $f(x) \in B$. By (ii) $x \in f^{-1}(B)$ which is nano open. Let A $= f^{-1}(B)$. Then $x \in A$ and $f(A) \subseteq B$.

$(iii) \Rightarrow (ii)$. Let B be any nano closed set in $\mathcal{V}$ and $x \in f^{-1}(B)$. Then $f(x) \in B$ and there exists a nano open set $\mathrm{U}_x \in \mathcal{NO}(X, x)$ such that $f(\mathrm{U}_x) \subseteq B$. Therefore $f^{-1}(B) = \cup\{\mathrm{U}_x.x \in f^{-1}(B)\} \in \mathcal{N}O(X)$.

$(iii) \Rightarrow (iv)$. Let A be any subset of $\mathcal{U}$. If $y \notin kerf(A)$, then by Theorem 3.3 there exists $B \in \mathcal{N} \subseteq (\mathcal{V}, f(x))$ such that $f(A) \cap (B) = \emptyset$. Thus we have $A \cap f^{-1}(B)=\emptyset$ and since $f^{-1}(B)$ is nano open we have $\mathcal{N}cl(A) \cap f^{-1}(B) = \emptyset$. Therefore, we obtain $f(\mathcal{N}(cl(A)) \cap B = \emptyset$ and hence $y \notin f(\mathcal{N}cl(A))$. This implies that $f[\mathcal{N}(cl(A)] \subseteq \mathcal{N}Ker[f(A)]$.

$(iv) \Rightarrow (v)$. Let B be any subset of $\mathcal{V}$. By(iv) and Theorem 3.3 we have $f[\mathcal{N}cl(f^{-1}(B))] \subseteq kerf[f^{-1}(B)] \subseteq kerB$. Thus $\mathcal{N}cl[f^{-1}(B)] \subseteq f^{-1}[\mathcal{N}Ker(B)]$.

$(v) \Rightarrow (i)$. Let B be any nano open set of $\mathcal{V}$. Then by, Theorem 3.3 we have $\mathcal{N}cl[f^{-1}(B)] \subseteq f^{-1}[\mathcal{N}Ker(B)]$ and $\mathcal{N}cl[f^{-1}(B)] = f^{-1}(B)$. This proves that $f^{-1}(B)$ is nano closed in $\mathcal{U}$. $\square$

Remark 5.5. The concept of nano continuity and nano contra continuity are independent of each other as shown in the following example.

Example 5.6. Let $\mathcal{U} = \{a,b,c,d,e\}$ with $\mathcal{U}/R = \{\{a,c\},\{b\},\{d\},\{e\}\}$ and let $X = \{a,d,e\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U},\emptyset,\{d,e\},\{a,c,d,e\},\{a,c\}\}$. The nano closed sets are $\mathcal{U},\emptyset,\{a,b,c\},\{b\},\{a,c\}$. Let $\mathcal{V} = \{x,y,z,w\}$ with $\mathcal{V}/R' = \{\{x\},\{y,z\},\{w\}\}$. Let $Y = \{x,z\} \subseteq \mathcal{V}$.

Then $\tau_{R'}(Y) = \{\mathcal{V},\emptyset,\{x\},\{x,y,z\},\{y,z\}\}$. Define $f : \mathcal{U} \to \mathcal{V}$ as $f(a) = x, f(b) = w, f(c) = x, f(d) = z, f(e) = y$. Then $f^{-1}(\{x\}) = \{a,c\}$, $f^{-1}(\{x,y,z\}) = \{a,c,d,e\}$, $f^{-1}(\{y,z\}) = \{d,e\}$. Hence f is a nano continuous function, but not nano contra continuous function. Because, $f^{-1}(\{x\}) = \{a,c\}$ is not nano closed in $\mathcal{U}$,where $\{x\}$ is nano open in $\mathcal{U}$. Therefore f is nano continuous but not nano contra continuous.

Example 5.7. Let $\mathcal{U} = \{a,b,c,d,e\}$ with $\mathcal{U}/R = \{\{a\},\{b\},\{c\},\{d\},\{e\}\}$ and let $X = \{a,b\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U},\emptyset,\{a,b\}\}$. The nano closed sets are $\mathcal{U},\emptyset,\{c,d,e\}$. Let $\mathcal{V} = \{x,y,z,u,v\}$ with $\mathcal{V}/R' = \{\{u,v\},\{x,z\},\{y\}\}$. Let $Y = \{x,y,z\} \subseteq \mathcal{V}$. Then $\tau_{R'}(Y) = \{\mathcal{V},\emptyset,\{x,y,z\}\}$. Define f: $\mathcal{U} \to \mathcal{V}$ as $f(a) = u, f(b) = v, f(c) = x, f(d) = y, f(e) = z$. Thus $f^{-1}(\{x,y,z\}) = \{c,d,e\}$ is a nano closed in $\mathcal{U}$ and not a nano open set in $\mathcal{U}$. Hence f is nano contra continuous function, but not nano continuous function.

Theorem 5.8. *Every nano strongly continuous function is nano contra continuous.*

Proof. Let B be a subset of $\mathcal{V}$. Since f is nano strongly continuous, $f^{-1}(B)$ is a nano clopen in $\mathcal{U}$. That is $f^{-1}(B)$ is both nano closed and also nano open in $\mathcal{U}$. Since B is any subset of $\mathcal{V}$ and $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Then f is nano contra continuous. □

Remark 5.9. The converse of Theorem 5.8 need not be true which can be shown by the following example.

Example 5.10. Consider Example 5.7, the function f: $\mathcal{U} \to \mathcal{V}$ is nano contra continuous. Consider $\{y,z\}$ to be any subset of $\mathcal{U}$. Then $f^{-1}(\{y,z\}) = \{d,e\}$ where it is not nano clopen in $\mathcal{U}$. Therefore f is nano contra continuous but not strongly continuous.

Theorem 5.11. *Every nano perfectly continuous function is nano contra continuous function.*

Proof. Let B be a nano open subset of $\mathcal{V}$. Since f is nano perfectly continuous, $f^{-1}(B)$ is a nano clopen in $\mathcal{U}$. That is $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Since B is a nano open subset of $\mathcal{V}$ and $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Then f is nano contra continuous. □

Remark 5.12. The converse of Theorem 5.11 need not be true as shown in the following example.

Example 5.13. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b, c\}, \{d\}\}$ and let $X = \{a, d\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a, d\}\}$. Then nano closed sets are $\mathcal{U}, \emptyset, \{b, c\}$. Let $\mathcal{V} = \{x, y, z, w\}$ with $\mathcal{V}/R' = \{\{x, z\}, \{y, w\}\}$. Let $Y = \{x, z\} \subseteq \mathcal{V}$. Then $\tau_{R'}(Y) = \{\mathcal{V}, \emptyset, \{x, z\}\}$. Define $f\colon \mathcal{U} \to \mathcal{V}$ as $f(a) = y, f(b) = x, f(c) = z, f(d) = w$ then $f^{-1}(\{x, z\}) = \{b, c\}$ which is a nano closed set and not a nano open set in $\mathcal{U}$. Hence f is nano contra continuous and not nano perfectly continuous.

Theorem 5.14. *For a function $f\colon \mathcal{U} \to \mathcal{V}$ the following conditions are equivalent*

(*i*) *f is nano perfectly continuous.*

(*ii*) *f is nano continuous and nano contra continuous.*

(*iii*) *f is nano α-continuous and nano contra continuous.*

(*iv*) *f is nano pre-continuous and nano contra continuous.*

Proof. $(i) \Rightarrow (ii)$. Let B be a nano open set in $\mathcal{V}$. Since f is nano perfectly continuous. $f^{-1}(B)$ is a nano clopen set in $\mathcal{U}$. Hence $f^{-1}(B)$ is nano open and nano closed in $\mathcal{U}$. Therefore f is both nano continuous and nano contra continuous.

$(ii) \Rightarrow (iii)$. Given that f is nano continuous and nano contra continuous. Since, every nano continuous map is nano α-continuous, f is nano α-continuous.

$(iii) \Rightarrow (iv)$. Given that f is nano α-continuous and nano contra continuous. Since every nano α-continuous map is nano pre-continuous, f is nano pre-continuous.

$(iv) \Rightarrow (i)$. Given that f is nano pre-continuous and nano contra continuous. To prove that f is nano perfectly continuous. Let B be a nano open set in $\mathcal{V}$. by(iv) $f^{-1}(B)$ is both nano pre-open and nano closed in $\mathcal{V}$. Hence $f^{-1}(B) \subseteq \mathcal{N}Int\mathcal{N}cl[f^{-1}(B)] \subseteq \mathcal{N}Int[f^{-1}(B)]$. Since $f^{-1}(B)$ is nano closed in $\mathcal{V}$. But $\mathcal{N}Int[f^{-1}(B)] \subseteq f^{-1}(B)$. Therefore $\mathcal{N}Int[f^{-1}(B)] = f^{-1}(B)$ and hence $f^{-1}(B)$ is nano open in $\mathcal{V}$. So, $f^{-1}(B)$ is nano open and nano closed in $\mathcal{V}$. Thus f is nano perfectly continuous. □

Theorem 5.15. *Every nano strongly continuous function is both nano continuous and nano contra continuous.*

Proof. Let B be an arbitrary set in $\mathcal{V}$. Since f is nano strongly continuous. $f^{-1}(B)$ is nano clopen in $\mathcal{U}$. That is $f^{-1}(B)$ is both nano open as well as nano closed in $\mathcal{U}$. Since it holds for every subset of $\mathcal{V}$, it is also true for all the nano open sets in $\mathcal{V}$. Therefore clearly f is nano continuous and nano contra continuous. □

Remark 5.16. From the above discussion we have the following table which gives the relationship between different types of nano continuous functions. The symbol "1" in a cell means that a function corresponding row implies a function

Functions	A	B	C	D
A	1	0	0	0
B	1	1	0	1
C	1	1	1	1
D	0	0	0	1

(A) Nano continuous function (B) Nano perfectly continuous function (C) Nano strongly continuous function (D) Nano contra continuous function

on the corresponding column. The symbol "0" means that a function on the corresponding row does not imply a function on the corresponding column.

Remark 5.17. The composition of two nano contra continuous function need not be nano contra continuous as the following example shows.

Example 5.18. Let $\mathcal{U} = \{a, b, c, d, e\}$ with $\mathcal{U}/R' = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$, $\mathcal{U}/R" = \{\{a, b\}, \{c, e\}, \{d\}\}, \mathcal{U}/R''' = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$. Let $X = \{d, e\} \subseteq \mathcal{U}$, $Y = \{c, d, e\} \subseteq \mathcal{U}$, $Z = \{a, b\} \subseteq \mathcal{U}$. Then $\tau_{R'}(X) = \{\mathcal{U}, \emptyset, \{d, e\}\}$, $\tau_{R"}(X) = \{\mathcal{U}, \emptyset, \{c, d, e\}\}$, $\tau_{R'''}(X) = \{\mathcal{U}, \emptyset, \{a, b\}\}$. Then $f : (\mathcal{U}, \tau_{R'}(X)) \to (\mathcal{U}, \tau_{R''}(Y))$ defined by $f(a) = c, f(b) = d, f(c) = e, f(d) = a, f(e) = b$. $g : (\mathcal{U}, \tau_{R''}(Y)) \to (\mathcal{U}, \tau_{R'''}(Z))$ be defined as an identity function. Here f and g are nano contra continuous functions. But $(gof)^{-1}(\{a, b\}) = f^{-1}(\{g^{-1}(\{a, b\})\}) = f^{-1}(\{a, b\}) = \{d, e\}$, which is not nano closed in $[\mathcal{U}, \tau_{R'}(X)]$. Hence gof is not nano contra continuous.

Theorem 5.19. *Let f: $\mathcal{U} \to \mathcal{V}$ and g: $\mathcal{V} \to \mathcal{W}$ be the functions then gof is nano contra continuous if g is nano continuous and f is nano contra continuous.*

Proof. Let g be a nano continuous and f a nano contra continuous function. Suppose B is a nano open set in $\mathcal{W}$. Since g is nano continuous $g^{-1}(B)$ is nano open in $\mathcal{V}$. Since f is nano contra continuous, $f^{-1}(g^{-1}((B)))$ is nano closed in $\mathcal{U}$. That is $(gof)^{-1}(B)$ is nano closed in $\mathcal{U}$. Hence gof is nano contra continuous. □

6. Nano Bi-contra continuity

In this section, we define nano bi-contra continuous functions and derive some results involving its characterizations.

Definition 6.1. Let f: $\mathcal{U} \to \mathcal{V}$ be a surjective map. Then f is called a nano bi-contra continuous map if f is nano contra continuous and $f^{-1}(B)$ is nano open in $\mathcal{U}$ implies B is nano closed in $\mathcal{V}$.

Example 6.2. Let $\mathcal{U} = \{a, b, c, d, e\}$ with $\mathcal{U}/R = \{\{a, b\}, \{c, d\}, \{e\}\}$ and let $X = \{c, d\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{c, d\}\}$. The nano closed sets are $\mathcal{U}, \emptyset, \{a, b, e\}$. Let $\mathcal{V} = \{x, y, z, u, v\}$ with $\mathcal{V}/R' = \{\{x, y\}, \{z, u, v\}\}$. Let $Y =$

$\{z, u, v\} \subseteq \mathcal{V}$. Then $\tau_{R'}(Y) = \{\mathcal{V}, \emptyset, \{z, u, v\}\}$. Define $f \colon \mathcal{U} \to \mathcal{V}$ by $f(a) = u, f(b) = v, f(c) = x, f(d) = y, f(e) = z$. Now the inverse image of nano open set in $\mathcal{V}$ is $f^{-1}(\{z, u, v\}) = \{a, b, e\}$ which is a nano closed set in $\mathcal{U}$. Hence f is nano contra continuous. That is $f^{-1}(\{x, y\}) = \{c, d\}$ which is a nano open set in $\mathcal{U}$. Hence f is nano bi-contra continuous.

Theorem 6.3. *Let $f \colon \mathcal{U} \to \mathcal{V}$ be a nano bi-contra continuous and an onto mapping. If A is a subset of $\mathcal{U}$ which is both nano closed and nano open, then the restriction $f_A : \{A, \tau_R A(X)\} \to \{\mathcal{V}, \tau_{R'}(Y)\}$ is a nano bi-contra continuous map.*

Proof. Given that f is nano bi-contra continuous. Let B be a nano open set in $\mathcal{V}$, then $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Because f is nano contra continuous. Since A is nano closed and nano open, $f^{-1}(B) \cap A$ is nano closed in A. Also $f_A^{-1}(B) = f^{-1}(B) \cap A$ is nano closed in A. Hence $f_A^{-1}(B)$ be nano open in A and $f_A^{-1}(B) = f^{-1}(B) \cap A$. Since A is nano clopen, $f^{-1}(B)$ is nano open in $\mathcal{U}$ and since f is nano bi-contra continuous, B is nano closed in $\mathcal{V}$. □

Theorem 6.4. *Let $f \colon \mathcal{U} \to \mathcal{V}$ be nano open and nano bi-contra continuous, and $g \colon \mathcal{V} \to \mathcal{W}$ is nano continuous and nano bi-contra continuous, then the composition $gof \colon \mathcal{U} \to \mathcal{V}$ is nano bi-contra continuous map.*

Proof. Let B be nano open in $\mathcal{W}$. Since g is nano continuous $g^{-1}(B)$ is open in $\mathcal{V}$. Since f is nano bi-contra continuous $f^{-1}[g^{-1}(B)]$ is nano closed in $\mathcal{U}$. $(gof)^{-1}(B) = f^{-1}[g^{-1}(B)]$, so $(gof)^{-1}(B)$ is nano closed in $\mathcal{U}$. $f^{-1}[g^{-1}(B)]$ is nano open in $\mathcal{U}$, since f is nano open, $f(f^{-1}(g)^{-1}(B)) = g^{-1}(B)$ is nano open in $\mathcal{V}$. Again g is nano contra continuous. So B is nano closed in $\mathcal{W}$. Hence $(gof)^{-1}(B)$ is nano bi-contra continuous. □

7. Application

Nano contra continuous and nano bi-contra continuous function can be used to define contra and two way contra effects in bio-technology. To illustrate this we have shown a real life example.

Consider the anti-viruses as a treatment for the disease causing viruses. Let $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the universe of viruses of three diseases namely Flu-virus, Polio-virus and Hepatitis-virus. In the sequel, $v_1 - A/H5N1Flu$, $v_2 - A/HN1 - Flu$, $v_3 - Polio$, $v_4 - Hepatitis - C$, $v_5 - Hepatitis - B$, $v_6 - Hepatitis - D$ and let $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be the universe of anti-viruses for three diseases namely Flu-virus, Polio-virus and Hepatitis-virus are a_1 as Arbidol, a_2 as Amantadine, a_3 as Sabin, a_4 as Interferon, a_5 as Rebetol, a_6 as Alpha-Interferon. We know that anti-virus depends on the disease causing virus. Let $\mathcal{U}/R = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$ and $X = \{v_1, v_2, v_3\} \subseteq \mathcal{V}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{v_1, v_2, v_3\}\}$. The nano closed sets are $\mathcal{V}, \emptyset, \{v_4, v_5, v_6\}$. Let $\mathcal{A}/R' = \{\{a_1, a_2\}, \{a_3\}, \{a_4, a_5, a_6\}\}$ instead and $Y = \{a_4, a_5, a_6\} \subseteq \mathcal{A}$. Then $\tau_{R'}(Y) = \{\mathcal{A}, \emptyset, \{a_4, a_5, a_6\}\}$. Define $f \colon \mathcal{V} \to \mathcal{A}$ by $f(v_1) = a_1, f(v_2) = a_2$,

$f(v_3) = a_3, f(v_4) = a_4, f(v_5) = a_5, f(v_6) = a_6$. Then $f^{-1}(\mathcal{A}) = \mathcal{V}$, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a_4, a_5, a_6\}) = \{v_4, v_5, v_6\}$. That is, the inverse image of every nano open set in $\mathcal{A}$ is nano closed in $\mathcal{V}$. Therefore f is nano contra continuous. Also, $f^{-1}(\{a_1, a_2, a_3\}) = \{v_1, v_2, v_3\}$ which is a nano open set in $\mathcal{V}$. That is, the inverse image of every nano closed set in $\mathcal{A}$ is nano open in $\mathcal{V}$. Therefore, the anti-viruses as a function of treatment for disease causing viruses is both nano contra continuous and nano bi-contra continuous.

Conclusion

These types of mappings (nano contra continuous and nano bi-contra continuous mappings) will be of much use in biotechnology, where they need contra or two way contra effects. This mapping is obtained by fixing a contra mapping between a set of viruses to a set of anti-viruses and another contra mapping between the negative viruses to the positive viruses of the anti-virus set.

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Accepted: 23.01.2017

Heptavalent symmetric graphs of order $8p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $8p$ for each prime p. As a result, a connected heptavalent symmetric graph of order $8p$ with p a prime exists if and only if $p = 2$ or 3, and up to isomorphism, there are only two such graphs: one for each $p = 2$ and 3.
Keywords: symmetric graph, s-transitive graph, Cayley graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [20, 23] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an

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ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [7] and Potočnik [19]. Thus, classifying symmetric graphs with small valency has received considerable attention, see [8, 26, 27]. Following this structure given by Guo [9], a series of pentavalent symmetric graphs is classified in [14, 17, 18, 24, 25]. Recently, the structure of heptavalent case was determined by Guo [10]. Thus, as an application of this result, we classify heptavalent symmetric graphs of order $8p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [15, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [10, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*

(2) *For* $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ *or* $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;

(3) *For* $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ *or* $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.

In particular, a Sylow 3*-subgroup of* G_v *is elementary abelian.*

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of Z_p^* of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$.

Proposition 2.3. *Let* X *be a connected heptavalent symmetric graph of order* $2p$ *with* p *a prime. Then* X *is isomorphic to* $K_{7,7}$ *or* $G(2p, 7)$ *with* $7 \mid (p-1)$. *Furthermore,* $\mathrm{Aut}(G(2p, 7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

In view of [11, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let* X *be connected heptavalent symmetric graph of order* $4p$ *with* p *a prime. Then* X *is isomorphic to* K_8.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G, S)) = G$ and edge set $E(\mathrm{Cay}(G, S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G, S)$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [21]).

Example 2.5. Let $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ and $S = \{ab, a^2b, a^3b, a^5b, a^6b, a^7b, b\}$. We can define the Cayley graph:

$$\mathcal{G}_{16} = \mathrm{Cay}(G, S).$$

Then by Magma [3], $\mathcal{G}_{16} = K_{8,8} - 8K_2$ is 2-transitive and $\mathrm{Aut}(\mathcal{G}_{16}) = (\mathbb{Z}_8 \times \mathbb{Z}_2).\mathrm{S}_7 \cong \mathrm{S}_8 \times \mathbb{Z}_2$.

The next example is about a connected heptavalent symmetric graph of order 24.

Example 2.6. Define the Cayley graph on the symmetric group S_4:

$$\mathcal{G}_{24} = \mathrm{Cay}(\mathrm{S}_4, S).$$

where $S = \{(1,2,3,4),(1,4,3,2),(1,2,4),(1,4,2),(3,4),(2,4),(1,4)(2,3)\}$. By Magma [3], $\mathrm{Aut}(\mathcal{G}_{24}) = \mathrm{S}_4.D_{14} \cong \mathrm{PGL}(2,7)$ and $\mathcal{G}_{24}$ is a connected heptavalent 1-transitive graph.

3. Classification

This section is devoted to classifying heptavalent symmetric graphs of order $8p$ for p a prime.

Theorem 3.1. *Let X be a connected heptavalent symmetric graph of order $8p$ with p a prime. Then $X \cong \mathcal{G}_{16}$ or $\mathcal{G}_{24}$.*

Proof. By [16] and Magma [3], there is a unique connected heptavalent symmetric graphs of order 16 or 24. Thus, by Examples 2.5 and 2.6, we have that $X \cong \mathcal{G}_{16}$ and $\mathcal{G}_{24}$ for $p = 2$ and $p = 3$, respectively. Let $p \geq 5$ and $A = \mathrm{Aut}(X)$. Then we only need to prove that there are no new such graphs.

Case 1: A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then N is an elementary abelian q-group with $q = 2$ or p. Since X has order $8p$, by Proposition 2.1, N is semiregular on $V(X)$ and the quotient graph X_N of X relative to N is a heptavalent symmetric graph with A/N as an arc-transitive automorphism group. Clearly, the order of X_N is even and at least 8. This implies that $N = \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_p$.

Suppose that $N = \mathbb{Z}_2$. Then X_N is a heptavalent symmetric graph of order $4p$. Note that $p \geq 5$. Thus, by Proposition 2.4, there is no such graph, a contradiction.

Suppose that $N = \mathbb{Z}_2^2$. Then X_N is a heptavalent symmetric graph of order $2p$, and by Proposition 2.3, $X_N \cong G(2p, 7)$ or $K_{7,7}$.

Assume that $X_N \cong G(2p, 7)$. Then $A/N \leq (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Since $2 \cdot 7 \cdot p \,\big|\, |A/N|$, we have that $A = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Set $C = C_A(N)$. By "N/C-Theorem" (see [12, Chapter I, Theorem 4.5]), $A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(2,2)$. Since 7 and p does not divide the order of $\mathrm{GL}(2,2)$, we have that $\mathbb{Z}_p \rtimes \mathbb{Z}_7 \leq C$, that is, all p-elements and 7-elements commute with N. On the other hand, $\mathbb{Z}_2$ normalizes N and hence $\mathbb{Z}_2$ normalizes an element of order 2 in N. This implies that A has a normal subgroup of order 2, which contradicts that N is minimal normal.

Assume that $X_N \cong K_{7,7}$. Then $A/N \lesssim \mathrm{Aut}(K_{7,7}) \cong (\mathrm{S}_7 \times \mathrm{S}_7) \rtimes \mathbb{Z}_2$. By Magma [3], $K_{7,7}$ has two minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has a subgroup $M/N = \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. A similar argument as above, we can deduce that $\mathbb{Z}_2$ or $\mathbb{Z}_4$ normalizes an element of order 2 in N. It forces that M has a normal subgroup $T \cong \mathbb{Z}_2$, and X_T is a heptavalent M/T-symmetric graph of order $4p$. However, by Lemma 2.4, there is no heptavalent symmetric graph of order $4p$ with $p \geq 5$, a contradiction.

Suppose that $N = \mathbb{Z}_p$. Then $X_N \cong K_8$ and $A/N \leq \mathrm{S}_8$. Note that $8{\cdot}7 \,\big|\, |A/N|$. From the information in [4], we have that $A/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, $\mathbb{Z}_2^3 \rtimes F_{21}$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,7) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$, A_8 or S_8.

Assume that $A/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, $\mathbb{Z}_2^3 \rtimes F_{21}$, $\mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$, A_8 or S_8. Then A/N has an arc-transitive group $M/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. By "N/C-Theorem", $M/C_M(N) \lesssim \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. Thus, $2^2 \,\big|\, |C_M(N)|$. It forces that $C_M(N)$ has a characteristic subgroup $K \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. Since $C_M(N) \trianglelefteq M$, we have $K \trianglelefteq M$. Then X_K is a heptavalent M/K-symmetric graph of order $2p$ or p. Note that there is no heptavalent graph of order $p \geq 5$. Thus, $K = \mathbb{Z}_2^2$ and X_K is a heptavalent symmetric graph of order $2p$. By the above argument, this is also impossible.

Assume that $A/N = \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,7) \rtimes \mathbb{Z}_2$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$. Since $\mathrm{PSL}(2,7)$ is simple, we have that $\mathrm{PSL}(2,7)$ commutes with N. By Atlas [6], the Schur multiplier $\mathrm{Mult}(\mathrm{PSL}(2,7)) = \mathbb{Z}_2$. It implies that A has a normal subgroup $M = \mathrm{PSL}(2,7) \times \mathbb{Z}_p$ and M is arc-transitive. Clearly, $M_v \cong F_{21}$. Let $\mathrm{PSL}(2,7) \cong K \leq M$. Then $K \trianglelefteq M$. It follows that X_K is a heptavalent graph of order p. This is impossible because there is no heptavalent graph of order p.

Case 2: A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is non-solvable. Since every group of order $q^s{\cdot}r^t$ with q, r primes and s, t non-negative integers is solvable, the order $|N|$ has at least three different primes. Note that $|V(X)| = 8p$. Thus, $N_v \neq 1$. By Proposition 2.1, N acting on $V(X)$ has at most two orbits. Since A is arc-transitive, we have that $|A| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$ by Lemma 2.2. It follows that $|N| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$.

Since N is non-solvable, $N \cong T_1 \times T_2 \times \cdots \times T_n$ with $T \cong T_1 \cong T_2 \cong \cdots \cong T_n$ is a non-abelian simple group. Note that N has at most two orbits on $V(X)$. Thus, $4p \,\big|\, |N|$. Now we divide the prime p into the next three subcases: $p = 5$, $p = 7$ and $p > 7$.

Subcase 2.1: Let $p = 5$. Then $|T| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7$ and $4{\cdot}5 \,\big|\, |T|$.

By [6, pp.12-14] and [22, Theorem 2], a simple calculation implies that T is isomorphic to the following groups listed in Table 1:

Assume that $n \geq 2$. Then by Table 1, $n = 2$ and $T \cong \mathrm{A}_5$ or A_6. Note that $4{\cdot}5 \,\big|\, |N|$ or $8{\cdot}5 \,\big|\, |N|$. By Magma [3], N_v has a normal subgroup $M \cong \mathrm{A}_5$ or A_6 and N_v/M is solvable. Thus, M is also a normal subgroup of A_v. By Proposition 2.2, $A_v \cong \mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$ or $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$. It forces that $A_v/M \cong \mathrm{A}_7$ or S_7. However, N_v/M is solvable and normal in A_v/M, this is impossible. Thus, $n = 1$ and N is a non-abelian simple group listed in Table 1.

Suppose that $N \cong \mathrm{A}_5$. Then $N_v \cong \mathbb{Z}_3$ and N has two orbits on $V(X)$. Since $N_v \trianglelefteq A_v$, we have $A_v = F_{21} \times \mathbb{Z}_3$, $F_{42} \times \mathbb{Z}_3$ or $F_{42} \times \mathbb{Z}_6$. Let P be a Sylow 7-

Table 1: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7$

3-prime factor		4-prime factor	
T	Order	T	Order
A_5	$2^2 \cdot 3 \cdot 5$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
A_6	$2^3 \cdot 3^2 \cdot 5$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$PSU(4,2)$	$2^6 \cdot 3^4 \cdot 5$	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$
		A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$
		J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
		$PSL(3,4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
		$PSp(6,2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
		$PSU(3,5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$

subgroup of A_v. Then $P \trianglelefteq A_v$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong S_5$. Since $7 \nmid |S_5|$, we have that $P \leq C_A(N)$ and $P \trianglelefteq A_vN$. Since P is a Sylow 7-subgroup of A, we have that P is characteristic in A_vN. Note that N has two orbits on $V(X)$. By Proposition 2.1, X is bipartite and $|A : A_vN| = 2$. It implies that $A_vN \trianglelefteq A$. Thus, $P \trianglelefteq A$. However, P lies in the vertex stabilizer A_v and P cannot be normal, a contradiction.

Suppose that $N \cong A_6$. Then by Atlas [6], $N_v \cong \mathbb{Z}_3^2$ or $\mathbb{Z}_3^2 \times \mathbb{Z}_2$. Note that $N_v \trianglelefteq A_v$. By Proposition 2.2, A_v has no normal subgroup isomorphic to N_v, a contradiction.

Suppose that $N \cong PSU(4,2)$. Then $|N_v| = 2^3{\cdot}3^4$ or $2^2{\cdot}3^4$. By Atlas [6], a Sylow 3-subgroup of N_v is non-abelian. However, the Sylow 3-subgroups of A_v are elementary abelian by Proposition 2.2, a contradiction.

Suppose that $N \cong A_7$, A_8, A_9, A_{10}, J_2, $PSL(3,4)$, $PSp(6,2)$ or $PSU(3,5)$. Then $|N_v| = |N|/20$ or $|N|/40$. By Atlas [6], N has no subgroups of such orders, a contradiction.

Subcase 2.2: Let $p = 7$. Then $|N| \mid 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7^2$.

By [6, pp.12-14] and [22, Theorem 2], a simple calculation implies that T is isomorphic to the following groups listed in Table 2.

Assume that $n \geq 2$, then $n = 2$ and $N = PSL(2,7)^2$, $PSL(2,8)^2$, A_7^2, A_8^2 or $PSL(3,4)^2$. Note that $|N : N_v| = 28$ or 56. Thus, N_v is non-solvable and N_v has a normal subgroup isomorphic to $PSL(2,7)$, $PSL(2,8)$, A_7, A_8 or $PSL(3,4)$. By Proposition 2.2, A_v has no subgroups isomorphic to $PSL(2,8)$, A_8 or $PSL(3,4)$. Therefore, $N = PSL(2,7)^2$ or A_7^2. If $N = PSL(2,7)^2$, then $N_v = S_3 \times PSL(2,7)$ or $\mathbb{Z}_3 \times PSL(2,7)$. Since $N_v \trianglelefteq A_v$, we have A_v has a normal subgroup isomorphic to $S_3 \times PSL(2,7)$ or $\mathbb{Z}_3 \times PSL(2,7)$. This is impossible by Proposition 2.2. If $N = A_7^2$, then by Magma [3], A_7^2 has no subgroup of index 28 or 56, a contradiction.

Thus, $n = 1$ and N is a non-abelian simple group listed in Table 2.

Table 2: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7^2$

3-prime factor		4-prime factor	
T	Order	T	Order
PSL(2,7)	$2^3 \cdot 3 \cdot 7$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
PSL(2,8)	$2^3 \cdot 3^2 \cdot 7$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
PSU(3,3)	$2^5 \cdot 3^3 \cdot 7$	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$
		A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$
		J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
		PSL(2,49)	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$
		PSL(3,4)	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
		PSp(6,2)	$2^9 \cdot 3^4 \cdot 5 \cdot 7$

Suppose that $N = \mathrm{PSL}(2,7)$. Then $N_v = \mathbb{Z}_3$ or S_3. Since $N_v \trianglelefteq A_v$, we have $N_v = \mathbb{Z}_3$ and $A_v = F_{21} \times \mathbb{Z}_3$, $F_{42} \times \mathbb{Z}_3$ or $F_{42} \times \mathbb{Z}_6$. In this case, N is transitive on $V(X)$ and $A = NA_v$. Since $N \cap A_v = \mathbb{Z}_3$, we have that $N \rtimes F_{21} \leq A$ and $N \rtimes F_{21}$ is arc-transitive. Thus, A has a subgroup $B = N \rtimes \mathbb{Z}_7 = \mathrm{PSL}(2,7) \rtimes \mathbb{Z}_7$, and B is arc-transitive. However, $B_v = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, this is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2,8)$, $\mathrm{PSU}(3,3)$, A_9, A_{10}, J_2. Since $|N : N_v| = 28$ or 56, we have a Sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Proposition 2.2, a Sylow 3-subgroup of A_v is elementary abelian. However, by Atlas [6], a Sylow 3-subgroup of N is not elementary abelian, a contradiction.

Suppose that $N = \mathrm{A}_7$, $\mathrm{PSL}(2,49)$ or $\mathrm{PSp}(6,2)$. Then by Atlas [6], N has no subgroups of index 28 or 56, a contradiction.

Suppose that $N = \mathrm{A}_8$. Then by Atlas [6], $N_v = \mathrm{A}_6$, $(\mathrm{A}_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ or S_6. By Proposition 2.2, A_v has no normal subgroup isomorphic to $(\mathrm{A}_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Thus $N_v = \mathrm{A}_6$ or S_6. Since $N_v \trianglelefteq A_v$, we have that $A_v = \mathrm{A}_7 \times \mathrm{A}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$ or $\mathrm{S}_7 \times \mathrm{S}_6$ by Proposition 2.2. The normality of N in A implies that A has an arc-transitive subgroup $B = N \rtimes \mathbb{Z}_7$ and $B_v = \mathrm{A}_6 \times \mathbb{Z}_7$ or $\mathrm{S}_6 \times \mathbb{Z}_7$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(3,4)$. Then by Atlas [6], $N_v = \mathrm{A}_6$ and N is transitive on $V(X)$. The similar argument as above, A has an arc-transitive subgroup $B = N \rtimes \mathbb{Z}_7$ and $B_v = \mathrm{A}_6 \times \mathbb{Z}_7$. This is impossible by Proposition 2.2.

Subcase 2.3: Let $p > 7$. Then $|N| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$.

By [6, pp.12-14], [22, Theorem 2] and [13, Theorem A], a simple calculation implies that T is isomorphic to the following groups listed in Table 3. Since $4p \,\big|\, |N|$ and p^2 does not divide the orders of the groups listed in Table 3, we have $n = 1$ and N is a non-abelian simple group.

Suppose that $N = \mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, M_{12}, $\mathrm{PSU}(3,8)$, ${}^2F_4(2)'$, $\mathrm{PSL}(4,4)$, A_{11}, $\mathrm{P\Omega}^-(8,2)$, $G_2(4)$. Then $|N : N_v| = 4p$ or $8p$. It follows that a sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Atlas [6], a Sylow 3-

Table 3: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$

3-prime factor		4-prime factor		5-prime factor	
T	Order	T	Order	T	Order
PSL(2,17)	$2^4\cdot3^2\cdot17$	M_{11}	$2^4\cdot3^2\cdot5\cdot11$	$PSL(2,2^6)$	$2^6\cdot3^2\cdot5\cdot7\cdot13$
PSL(3,3)	$2^4\cdot3^3\cdot13$	M_{12}	$2^6\cdot3^2\cdot5\cdot11$	PSL(2,29)	$2^2\cdot3\cdot5\cdot7\cdot29$
		PSL(2,11)	$2^2\cdot3\cdot5\cdot11$	PSL(2,41)	$2^3\cdot3\cdot5\cdot7\cdot41$
		PSL(2,19)	$2^2\cdot3^2\cdot5\cdot19$	PSL(2,71)	$2^3\cdot3^2\cdot5\cdot7\cdot71$
		PSL(2,16)	$2^4\cdot3\cdot5\cdot17$	PSL(2,449)	$2^6\cdot3^2\cdot5^2\cdot7\cdot449$
		PSL(2,25)	$2^5\cdot3\cdot5^2\cdot13$	PSL(4,4)	$2^{12}\cdot3^4\cdot5^2\cdot7\cdot17$
		PSL(2,27)	$2^2\cdot3^3\cdot7\cdot13$	PSL(5,2)	$2^{10}\cdot3^2\cdot5\cdot7\cdot31$
		PSL(2,31)	$2^5\cdot3\cdot5\cdot31$	A_{11}	$2^7\cdot3^4\cdot5^2\cdot7\cdot11$
		PSL(2,81)	$2^4\cdot3^4\cdot5\cdot41$	M_{22}	$2^7\cdot3^2\cdot5\cdot7\cdot11$
		PSp(4,4)	$2^8\cdot3^2\cdot5^2\cdot17$	$P\Omega^-(8,2)$	$2^{12}\cdot3^4\cdot5\cdot7\cdot17$
		PSU(3,4)	$2^6\cdot3\cdot5^2\cdot13$	$G_2(4)$	$2^{12}\cdot3^3\cdot5^2\cdot7\cdot13$
		PSU(3,8)	$2^8\cdot3^4\cdot7\cdot19$		
		Sz(8)	$2^6\cdot5\cdot7\cdot13$		
		${}^2F_4(2)'$	$2^{11}\cdot3^3\cdot5^2\cdot13$		

subgroup of N is not elementary abelian. However, a Sylow 3-subgroup of A_v is elementary abelian by Proposition 2.2, a contradiction.

Suppose that $N = \mathrm{M}_{11}$, PSL(2,11), PSL(2,19), PSL(2,27), PSU(3,4), Sz(8), PSL$(2,2^6)$, PSL(2,29), PSL(2,41), PSL(2,71), PSL(2,449), M_{22}. Then by Atlas [6] and Magma [3], N has no subgroups of index $4p$ or $8p$, a contradiction.

Suppose that $N = \mathrm{PSL}(2,16)$. Then by Atlas [6], $N_v = D_{30}$ or A_5. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup isomorphic to D_{30} or A_5. However, by Proposition 2.2, A_v has no such normal subgroups, a contradiction.

Suppose that $N = \mathrm{PSL}(2,25)$. Then by Atlas [6], $N_v = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_{12}$. Clearly, N_v has a characteristic subgroup isomorphic to $\mathbb{Z}_5^2$. The normality of N_v in A_v implies that A_v has a normal subgroup isomorphic to $\mathbb{Z}_5^2$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2,81)$. Then $|N_v| = 2{\cdot}3^4{\cdot}5$ or $2^2{\cdot}3^4{\cdot}5$. By Atlas [6], N_v has a characteristic subgroup $\mathbb{Z}_3^4$. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup $\mathbb{Z}_3^4$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSp}(4,4)$. Then by Atlas [6], $N_v = (\mathrm{A}_5 \times \mathrm{A}_5) \rtimes \mathbb{Z}_2$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to N_v, a contradiction.

Suppose that $N = \mathrm{PSL}(5,2)$. Then by Atlas [6], $N_v = \mathbb{Z}_2^4 \rtimes \mathrm{A}_7$. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup isomorphic to $\mathbb{Z}_2^4 \rtimes \mathrm{A}_7$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2,31)$. Then by Atlas [6], $N_v = \mathrm{A}_5$. This implies that A_v has a normal subgroup isomorphic to A_5. However, by Proposition 2.2, this is impossible. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301154, 11601125).

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Accepted: 28.02.2017

On almost generalized pseudo-Ricci symmetric spacetime

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Abstract. The notion of an almost generalized pseudo-Ricci symmetric space time has been introduced and studied. The beauty of such spacetime is that it has the flavour of Ricci symmetric, Ricci recurrent , generalized Ricci recurrent, pseudo-Ricci symmetric, generalized pseudo-Ricci symmetric and almost pseudo-Ricci symmetric space. Having found, faulty example in [8] the present paper attempts to construct a nontrivial example of an almost pseudo Ricci symmetric spacetime.
Keywords: almost pseudo Ricci symmetric spacetime, quasi-Einstein.

1. Introduction

In the example given in ([8], page 2884-2885) authors have calculated or assumed the value of the covariant derivatives corresponding to the vanishing component of the Ricci tensor R_{13} & R_{14} (namely, $R_{13},_3$ & $R_{14},_4$) to be zero. But, those value are found to be $R_{13},_3 = \frac{2q^2(1-q)}{(1+2q)^3} = -R_{14},_4$ which are non-zero as $q \neq 0, 1$. Consequently for their [8] choice of the 1-forms

$$\begin{aligned} A_i(x) &= -\frac{q}{1+2q} \quad \text{for i=1,} \\ &= 0 \qquad \text{otherwise,} \\ B_i(x) &= \frac{1+q}{1+2q} \quad \text{for i=1,} \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

the relations

$$\begin{aligned} R_{13},_3 &= (A_3 + B_3)R_{13} + A_1R_{33} + A_3R_{13}, \\ R_{14},_4 &= (A_4 + B_4)R_{14} + A_1R_{44} + A_4R_{14}, \end{aligned}$$

do not stand. Hence, $(\mathbb{R}^4, g)$ under-considered metric ([8], equation 6.2, page 2884) can not be an almost pseudo-Ricci symmetric spacetime. Coming back to our present paper, we structured it as follows: Keeping in tune with Dubey[11], a new type of spacetime called an almost generalized pseudo-Ricci symmetric

∗. The author dedicates this work to the memory of Late Professor M. C. Chaki spacetime which is abbreviated by $A(GPRS)_n$-spacetime is introduced in section 2. Some interesting results of a conformally flat almost generalized pseudo-Ricci symmetric spacetime are obtained. A non-trivial example of an almost pseudo-Ricci symmetric spacetime is constructed in section 3. Finally, we ensured that there exists a spacetime $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for some choice of the 1-forms.

2. $A(GPRS)_n$-spacetime

In the sense of Chaki and Kawaguchi, a non-flat n-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$ is said to be an almost pseudo-Ricci symmetric manifold, [7] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the equation

$$(\nabla_X S)(Y, U) = [A(X) + B(X)]S(Y, U) + A(Y)S(X, U) + A(U)S(Y, X) \tag{2.1}$$

where $A(X)$ and $B(X)$ are two non-zero 1-forms defined by $A(X) = g(X, \theta)$ and $B(X) = g(X, \varrho)$, ∇ being the operator of the covariant differentiation. The local expression of the above equation is

$$R_{ik,l} = (A_l + B_l)R_{ik} + A_i R_{kl} + A_k R_{il}, \tag{2.2}$$

where A_l and B_l are two non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind is abbreviated by $A(PRS)_n$.

Generalizing the sense of Chaki and Kawaguchi, in the present paper, we attempt to introduce a new type of spacetime called almost generalized pseudo-Ricci symmetric spacetime which is abbreviated by $A(GPRS)_n$-spacetime and defined as follows:

A non-flat n-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$, is termed as almost generalized pseudo-Ricci symmetric manifold, if its Ricci tensor S of type $(0, 2)$ is not identically zero and admits the identity([2], [3])

$$\begin{aligned}(\nabla_X S)(Y, U) \;=\;& [A(X) + B(X)]S(Y, U) + A(Y)\, S(X, U) + A(U)\, S(X, Y)\\ & +[C(X) + D(X)]g(Y, U) + C(Y)\, g(X, U) + C(U)\, g(X, Y)\end{aligned} \tag{2.3}$$

where $A(X)$, $B(X)$, $C(X)$ and $D(X)$ are non-zero 1-forms defined by $A(X) = g(X, \theta)$, $B(X) = g(X, \varrho)$, $C(X) = g(X, \pi)$ and $D(X) = g(X, \delta)$. The beauty of such $A(GPRS)_n$-spacetime is that it has the flavour of

(a) Ricci symmetric space in the sense of Cartan (for $A = B = C = D = 0$),
(b) Ricci recurrent space by E. M. Patterson [14] (for $B \neq 0$ and $A = C = D = 0$),
(c) generalized Ricci recurrent space by De, Guha and Kamilya [9] (for $B \neq 0$, $D \neq 0$ and $A = C = 0$),
(d) pseudo-Ricci symmetric space by Chaki [6] (for $A = B \neq 0$ and $C = D = 0$),

(e) generalized pseudo-Ricci symmetric space, by Baishya [1] (for $A = B \neq 0$ and $C = D \neq 0$) and
(f) almost pseudo-Ricci symmetric space by Chaki and Kawaguchi [7] (for $A = B \neq 0$ and $C = D = 0$).

Next, if the vector fields associated to the 1-forms $A \& B$ are co-directional with that of C & D respectively, that is $C = \phi A$ & $D = \phi B$ where ϕ being constant, then the relation (2.3) turns into

$$(\nabla_X Z)(Y,U) = [A(X) + B(X)]Z(Y,U)] + A(Y)\ Z(X,U) + A(U)Z(X,U)$$

where $Z(X,Y) = S(X,Y) + \phi\ g(X,Y)$ is a well known Z-tensor introduced in ([12], [13]). This leads to the following:

Theorem 2.1 ([13])**.** *Every $A(GPRS)_n$-spacetime is an almost pseudo Z-symmetric spacetime provided that the vector fields associated to the 1-forms $A\&B$ are co-directional with that of C & D respectively.*

It is to be noted that the converse of the Theorem 2.1 is also true. Thus we can say that an almost pseudo Z-symmetric spacetime is a natural example of an almost generalized pseudo Z-symmetric spacetime.

Definition 2.1. A non-flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be a quasi-Einstein manifold [10] if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y) = \lambda g(X,Y) + \mu\psi(X)\psi(Y),$$

where λ, $\mu \in \mathbb{R}$ and ψ is a non-zero 1-form such that $g(X,U) = \psi(X)$, for all vector fields X, U being a unit vector field of the 1-form.

Now, contracting Y over U in (2.1) we obtain

$$\text{dr}(X) = r[A(X) + B(X)] + 2\bar{A}(X) + 6C(X) + 4D(X) \tag{2.4}$$

where $\bar{A}(X) = S(X,\theta)$. Again, from (2.1), one can easily bring out

$$\begin{aligned}(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) &= B(X)S(Y,U) - B(U)S(X,Y)\\ &\quad +D(X)g(Y,U) - D(U)g(X,Y)\end{aligned} \tag{2.5}$$

after further contraction which leaves

$$dr(X) = 2rB(X) - 2\bar{B}(X) + 6D(X), \tag{2.6}$$

where $\bar{B}(X) = S(X,\varrho)$.

It is known ([15], p, 41) that a conformally flat (M^4, g) spacetime possesses the relation

$$(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) = \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \tag{2.7}$$

By virtue of (2.5), (2.6) and (2.7) we find

$$\begin{aligned} &3[B(X)S(Y,U) - B(U)S(X,Y)] \\ &= [rB(X) - \bar{B}(X)]g(Y,U) - [rB(U) - \bar{B}(U)]g(X,Y) \end{aligned} \tag{2.8}$$

which yields

$$B(X)\bar{B}(U) = B(U)\bar{B}(X), \tag{2.9}$$

for $Y = \varrho$. Assuming the Ricci tensor of the spacetime as codazzi type (in the sense of [4]) and then making use of (2.6), we obtain from (2.9) that

$$B(X)D(U) = B(U)D(X) \ \forall \ X \text{ and } U. \tag{2.10}$$

This motivate us to state

Proposition 2.1. *In a conformally flat $A(GPRS)_4$-spacetime with codazzi type of Ricci tensor, the 1-forms B and D are co-directional.*

Again, for constant scalar curvature tensor (or codazzi type of Ricci tensor) by virtue of (2.6), (2.8) and (2.10), we can easily find out

$$S(Y,U) = -\frac{D(\varrho)}{B(\varrho)}g(Y,U) + \frac{1}{B(\varrho)}[rB(Y) + nD(Y)]B(U), \tag{2.11}$$

where $\frac{D(U)}{B(U)} = k \ \forall \ U$. If the 1-forms B and D are co-directional, then (2.11) takes the following form

$$S(Y,U) == \alpha g(Y,U) + \beta B(Y)B(U). \tag{2.12}$$

This leads to the followings:

Theorem 2.2. *A conformally flat $A(GPRS)_4$-spacetime with codazzi type of Ricci tensor, is a quasi-Einstein spacetime.*

But, it is proved in ([8], Theorem 3.1) that a conformally flat $A(GPRS)_4$-spacetime is always quasi-Einstein spacetime. In consequence of Corollary 3.1 in [8], we can state the following:

Corollary 2.1. *A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature can be considered as a model of the perfect fluid spacetime in general relativity.*

Corollary 2.2. *A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature is a space of quasi constant curvature.*

3. Existence of almost pseudo-Ricci symmetric spacetime

Example 3.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = e^{-x^1}[(dx^1)^2 - (dx^2)^2 + 2\ dx^3 dx^4\ ], \tag{3.1}$$

$(i,\ j = 1,\ 2,\ 3,\ 4)$.

The non-zero components of Riemannian curvature tensors, Ricci tensors (up to symmetry and skew-symmetry) and scalar curvature tensor are

$$\begin{aligned} R_{2324} &= \frac{1}{4}e^{-x^1} = R_{3434}, \\ R_{22} &= \frac{1}{2} = -R_{34}, \\ r &= -\frac{3}{2}e^{x^1}. \end{aligned}$$

Covariant derivatives of Ricci tensors (up to symmetry) is expressed as

$$\begin{aligned} R_{12,2} &= -R_{13,4} = -R_{14,3} = \frac{1}{4} \\ R_{22,1} &= -R_{34,1} = \frac{1}{2}. \end{aligned}$$

For the following choice of the1-forms

$$\begin{aligned} A_i &= \frac{1}{2}, \ \text{for } i = 1 \\ &= 0, \quad \text{otherwise} \\ B_i &= \frac{1}{2}, \ \text{for } i = 1 \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

one can easily verify the followings

$$\begin{aligned} R_{12,k} &= (A_k + B_k)\, R_{12} + A_1 R_{k2} + A_2 R_{1k}, \\ R_{13,k} &= (A_k + B_k)\, R_{13} + A_1 R_{k3} + A_3 R_{1k}, \\ R_{14,k} &= (A_k + B_k)\, R_{14} + A_1 R_{k4} + A_4 R_{1k}, \\ R_{23,k} &= (A_k + B_k)\, R_{23} + A_2 R_{k3} + A_3 R_{2k}, \\ R_{24,k} &= (A_k + B_k)\, R_{24} + A_2 R_{k4} + A_4 R_{2k}, \\ R_{34,k} &= (A_k + B_k)\, R_{34} + A_3 R_{k4} + A_4 R_{3k}, \\ R_{11,k} &= (A_k + B_k)\, R_{11} + A_1 R_{k1} + A_1 R_{1k}, \\ R_{22,k} &= (A_k + B_k)\, R_{22} + A_2 R_{k2} + A_2 R_{2k}, \\ R_{33,k} &= (A_k + B_k)\, R_{33} + A_3 R_{k3} + A_3 R_{3k}, \\ R_{44,k} &= (A_k + B_k)\, R_{44} + A_4 R_{k4} + A_4 R_{4k}, \end{aligned}$$

where $k = 1,\ 2,\ 3,\ 4$.

In consequence of the above, one can say that

Theorem 3.1. *There exists a spacetime* $(\mathbb{R}^4, g)$ *which is an almost pseudo-Ricci symmetric spacetime with the above mentioned choice of the 1-forms.*

It is obvious that the spacetime bearing the metric given by (3.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as almost generalized pseudo-Ricci symmetric spacetime.

4. Existence of $A(GPRS)_n$-spacetime

Example 4.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\ ] - (dx^4)^2, \tag{4.1}$$

$(i,\ j = 1,\ 2,\ 3,\ 4)$. The non-zero components of Ricci tensors (up to symmetry)

$$R_{11} = \frac{2}{3(x^4)^{2/3}} = R_{22} = R_{33},\ R_{44} = \frac{2}{3(x^4)^2}.$$

Covariant derivative (up to symmetry) $R_{ik,l}$ of Ricci tensors is expressed by

$$\begin{aligned} R_{11,4} &= -\frac{4}{3(x^4)^{5/3}} = R_{22,4} = R_{33,4}\,,\ R_{44,4} = -\frac{4}{3(x^4)^3} \\ R_{14,1} &= -\frac{8}{9(x^4)^{5/3}} = R_{24,2} = R_{34,3}\,. \end{aligned}$$

For following choice of the 1-forms

$$\begin{aligned} A_i &= \frac{1}{x^4},\ \text{ for } i = 4, \\ &= 0,\ \text{ otherwise} \\ B_i &= -\frac{19}{3x^4},\ \text{ for } i = 4, \\ &= 0,\ \text{ otherwise} \\ C_i &= -\frac{14}{9(x^4)^3},\ \text{ for } i = 4 \\ &= 0,\ \text{otherwise}\ ,, \\ D_i &= \frac{34}{9(x^4)^3},\ \text{ for } i = 4 \\ &= 0,\ \text{otherwise}\ , \end{aligned}$$

one can easily verify the followings

$$\begin{aligned}
R_{12},_k &= (A_k+B_k)\,R_{12}+A_1R_{k2}+A_2R_{1k}+(C_k+D_k)\,g_{12}+C_1g_{k2}+C_2g_{1k},\\
R_{13},_k &= (A_k+B_k)\,R_{13}+A_1R_{k3}+A_3R_{1k}+(C_k+D_k)\,g_{13}+C_1g_{k3}+C_3g_{1k},\\
R_{14},_k &= (A_k+B_k)\,R_{14}+A_1R_{k4}+A_4R_{1k}+(C_k+D_k)\,g_{14}+C_1g_{k4}+C_4g_{1k},\\
R_{23},_k &= (A_k+B_k)\,R_{23}+A_2R_{k3}+A_3R_{2k}+(C_k+D_k)\,g_{23}+C_2g_{k3}+C_3g_{2k},\\
R_{24},_k &= (A_k+B_k)\,R_{24}+A_2R_{k4}+A_4R_{2k}+(C_k+D_k)\,g_{24}+C_2g_{k4}+C_4g_{2k},\\
R_{34},_k &= (A_k+B_k)\,R_{34}+A_3R_{k4}+A_4R_{3k}+(C_k+D_k)\,g_{34}+C_3g_{k4}+C_4g_{3k},\\
R_{11},_k &= (A_k+B_k)\,R_{11}+A_1R_{k1}+A_1R_{1k}+(C_k+D_k)\,g_{11}+C_1g_{k1}+C_1g_{1k},\\
R_{22},_k &= (A_k+B_k)\,R_{22}+A_2R_{k2}+A_2R_{2k}+(C_k+D_k)\,g_{22}+C_2g_{k2}+C_2g_{2k},\\
R_{33},_k &= (A_k+B_k)\,R_{33}+A_3R_{k3}+A_3R_{3k}+(C_k+D_k)\,g_{33}+C_3g_{k3}+C_3g_{3k},\\
R_{44},_k &= (A_k+B_k)\,R_{44}+A_4R_{k4}+A_4R_{4k}+(C_k+D_k)\,g_{44}+C_4g_{k4}+C_4g_{4k},
\end{aligned}$$

where $k = 1, 2, 3, 4$.

In consequence of the above, one can say that

Theorem 4.1. *There exists a spacetime $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for the above mentioned choice of the1-forms.*

It is obvious that the spacetime bearing the metric given by (4.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as pseudo-Ricci symmetric.

Acknowledgements

We have made all the calculations in Wolfram mathematica. The author would also like to thank UGC, ERO Kolkata, for their financial support File No. PSW-194/15-16.

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Accepted: 4.05.2017

Some remarks on generalizations of prime submodules

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called $(n-1,n)$-ϕ-prime, if $r_1 \ldots r_{n-1}x \in N \setminus \phi(N)$ where $r_1, \ldots, r_{n-1} \in R$ and $x \in M$, then there exists $i \in \{1, ..., n-1\}$ such that $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}x \in N$ or $r_1 \ldots r_{n-1} \in (N : M)$ $(n \geq 2)$. In this work, $(n-1,n)$-ϕ-prime submodules are studied and some results are obtained. Also, the characterization of $(n-1,n)$-ϕ-prime submodules of a free multiplication module is given.

Keywords: ϕ-prime submodule, ϕ-prime ideal, $(n-1,n)$-ψ-prime ideal, multiplication module, $(n-1,n)$-almost prime, $(n-1,n)$- ϕ-$\mathbb{CP}$ submodule, $(n-1,n)$-ϕ-$\mathbb{FCP}$ module.

1. Introduction

Throughout the paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module and N be a submodule of M. The ideal $\{r \in R | rM \subseteq N\}$ will be denoted by $(N : M)$. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is said to be a $(n-1,n)$-ϕ-prime, if $r_1...r_{n-1}x \in N \setminus \phi(N)$, $r_1, ..., r_{n-1} \in R$ and $x \in M$ $(n \geqslant 2)$, then $r_1...r_{i-1}r_{i+1}...r_{n-1}x \in N$ for some $i \in \{1, ..., n-1\}$ or $r_1...r_{n-1} \in (N : M)$. Without loss of generality, throughout of the paper we will assume $\phi(N) \subseteq N$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \cap_{i=1}^{\infty}(N : M)^i N$), then the submodule N is called a $(n-1,n)$- prime (resp. $(n-1,n)$-weakly prime, $(n-1,n)$-almost prime, $(n-1,n)$-m-almost prime and $(n-1,n)$-ω -prime). Firstly, Anderson and Bataineh in [4] introduced various generalizations of prime ideals. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. We call a proper ideal I of R a ψ-prime ideal if $a, b \in R$ with $ab \in I \setminus \psi(I)$, then $a \in I$ or $b \in I$. If $\psi(I) = \emptyset$ (resp. $\psi(I) = 0$, $\psi(I) = I^2$, $\psi(I) = I^m$ and

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$\psi(I) = \cap_{m=1}^{\infty} I^m$), then ideal I is called a prime ideal (resp. weakly prime ideal, almost prime ideal, m-almost prime ideal and ω-prime ideal). Zamani in [11] used this concept for ϕ-prime submodule, in fact a proper submodule N of M is a prime submodule relative to ϕ or ϕ-prime submodule if $rx \in N \setminus \phi(N)$ where $r \in R$ and $x \in M$, then $x \in N$ or $r \in (N : M)$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \cap_{i=1}^{\infty}(N : M)^m N$), then a submodule N is a prime submodule (resp. weakly prime submodule, almost prime submodule, m-almost prime submodule and ω-prime submodule). Some properties of ϕ-prime submodules have been studied in [11]. Ebrahimpour and Nekooei defined $(n-1,n)$-ϕ-prime submodule and $(n-1,n)$-ψ-prime ideal (see [7]). A proper ideal I of R is $(n-1,n)$-ψ-prime if $r_1 \dots r_n \in I \setminus \psi(I)$, then $r_1 \dots r_{i-1} r_{i+1} \dots r_n \in I$ for some $i \in \{1, ..., n\}$. A number of results concerning $(n-1,n)$-ϕ- prime submodules have been established in [7]. Also some basic properties of prime submodules have been studied in [1,3, 5, 6, 9].

In this work, we continue the above studies in a special case, by alternation of n and ϕ. Again some other results lead us to conclude some corollaries and propositions and theorems to concerning the properties of $(n-1,n)$-ϕ-prime submodules. Also, for a free multiplication module M, the results are given in Section 4.

2. Some general results

The following propositions give some properties when we use the defintion $(n-1,n)$-ϕ-prime submodule.

Proposition 2.1. *Let R be a ring and M be an R-module. If N is a proper ϕ-prime submodule of M ((1, 2)-ϕ-prime), then N is $(n-1,n)$-ϕ-prime submodule ($n \geqslant 2$).*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1} m \in N \setminus \phi(N)$. Assume that $r_1 \dots r_{n-1} \notin (N : M)$. Since N is a ϕ-prime submodule of M, hence $m \in N$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$ for some $i \in \{1, ..., n-1\}$.

Proposition 2.2. *Let M be an R-module and $\phi_1, \phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M with $\phi_1 \leqslant \phi_2$ (i.e., for every submodule N of M, $\phi_1(N) \subseteq \phi_2(N)$). If N is $(n-1,n)$-ϕ_1-prime submodule, then N is $(n-1,n)$-ϕ_2-prime.*

Proof. It is clear.

Proposition 2.3. *Suppose that N is a $(n-1,n)$-ϕ-prime submodule of M, then N is a $(n,n+1)$-ϕ-prime submodule of M.*

Proof. Let $r_1 \dots r_n m \in N \setminus \phi(N)$ where $r_1, ..., r_n \in R$ and $m \in M$. Then $r_1 \dots r_{n-1}(r_n m) \in N \setminus \phi(N)$, so $r_1 \dots r_{n-1} \in (N : M)$ or $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1}(r_n m) \in N$ for some $i \in \{1, ..., n-1\}$.

Thus $r_1 \dots r_{n-1} r_n \in (N : M)$ or $r_1 \dots r_{i-1} r_{i+1} \dots r_n m \in N$ for some $i \in \{1, ..., n\}$.

Example 2.4. We know that if N is a prime submodule of M, then N is a $(n-1,n)$-prime submodule of M $(n \geqslant 2)$ by Proposition 2.1. But the converse of Proposition 2.1, is not true in general. For example, let $M = \mathbb{Z} \oplus \mathbb{Z}$ be a $\mathbb{Z}$-module and $N = \langle (3,0) \rangle$. Since $(N : M) = 0$ and $3(2,0) \in N$ but $3 \notin (N : M)$ and $(2,0) \notin N$, therefore N is not a prime submodule. We show that N is a $(2,3)$- prime submodule. Suppose that $r_1, r_2 \in \mathbb{Z}$, $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ with $r_1 r_2 (m,n) \in N$. We have $(r_1 r_2 m, r_1 r_2 n) \in \langle (3,0) \rangle$. If $r_1 = 0$ or $r_2 = 0$, then $0 = r_1 r_2 \in (N : \mathbb{Z} \oplus \mathbb{Z})$, so N is a $(2,3)$-prime submodule. Now, let $r_1 \neq 0$ and $r_2 \neq 0$, hence $0 \neq r_1 r_2 \notin (N : \mathbb{Z} \oplus \mathbb{Z})$. Since $(r_1 r_2 m, r_1 r_2 n) \in \langle (3,0) \rangle$, therefore $n = 0$ and $3 \mid r_1 r_2 m$. If $3 \mid m$, then $r_1(m,0) \in N$ and $r_2(m,0) \in N$. If $3 \nmid m$, then $3 \mid r_1 r_2$. Hence $3 \mid r_1$ or $3 \mid r_2$. Thus $r_1(m,0) \in N$ or $r_2(m,0) \in N$, as required.

Proposition 2.5 *Let M be an R-module and suppose that for every $x \in M$, $Ann(x) = 0$. If N is a $(n-1,n)$-weakly prime submodule of M, then N is a $(n-1,n)$-prime submodule .*

Proof. Let $r_1 \dots r_{n-1} x \in N$ where $r_1, ..., r_{n-1} \in R$, $x \in M$ $(n \geqslant 2)$, and suppose that $r_1 \dots r_{n-1} \notin (N : M)$. Since N is a $(n-1,n)$-weakly prime submodule, hence $0 \neq r_1 ... r_{n-1} x \in N$, implies that $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} x \in N$ for some $i \in \{1, ... n-1\}$. Hence N is a $(n-1,n)$-prime submodule. But if $x \neq 0$ and $r_1 \dots r_{n-1} x = 0$, then $r_1 \dots r_{n-1} = 0$, so $0 = r_1 \dots r_{n-1} \in (N : M)$, this is a contradiction.

Proposition 2.6 *Let $\varphi : R \to S$ be a ring homomorphism and M be an S-module. Suppose that N is a $(n-1,n)$-ϕ-prime submodule of S-module M, then N is a $(n-1,n)$-ϕ-prime submodule of R-module M.*

Proof. Let $r_1 \dots r_{n-1} x \in N \setminus \phi(N)$ where $r_1, \dots, r_{n-1} \in R$, $x \in M$ $(n \geqslant 2)$. We know that $r_1 \dots r_{n-1} x = \varphi(r_1 \dots r_{n-1}) x = \varphi(r_1) \dots \varphi(r_{n-1}) x \in N \setminus \phi(N)$ where $\varphi(r_i) \in S$, for all $i \in \{1, ..., n-1\}$ and $x \in M$ $(n \geqslant 2)$. It is clear that N is a $(n-1,n)$-ϕ-prime submodule of R-module M.

3. Main results

We state the following theorems and propositions which in the proofs of them, we use the definition $(n-1,n)$-ϕ-prime submodule.

The motivation of [7, Theorem 2.7], we introduce function $\phi_{R/I}$. Let M be an R-module and I be an ideal of R. Since $I \subseteq Ann_R(M/IM)$, so M/IM is an R/I-module. We know that $(r+I)(m+IM) = rm + IM$ where $r + I \in R/I$, $m + IM \in M/IM$. Now, for a submodule N of M with $IM \subseteq N$, let $\phi_{R/I} : S(M/IM) \to S(M/IM) \cup \{\emptyset\}$ be defined by $\phi_{R/I}(N/IM) = (\phi(N) + IM)/IM$ for $IM \subseteq N$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function.

Theorem 3.1. *Let M be an R-module , I be an ideal of R and N be a submodule of M with $IM \subseteq N$. If N is a $(n-1,n)$-ϕ-prime submodule M, then N/IM is a $(n-1,n)$-$\phi_{R/I}$-prime submodule of M/IM $(n \geq 2)$.*

Proof. See [7, Theorem 2.7].

Theorem 3.2. *Let M be an R-module, L and N be two submodules of M with $L \subseteq N \subset M$. Let $\phi_L : S(M/L) \to S(M/L) \cup \{\emptyset\}$ be defined by $\phi_L(N/L) = (\phi(N)+L)/L$ with $L \subseteq \phi(N)$. If N/L is a $(n-1,n)$-ϕ_L-prime submodule of M/L, then N is a $(n-1,n)$-ϕ-prime submodule of M $(n \geq 2)$.*

Proof. Let $r_1,\dots,r_{n-1} \in R$, $x \in M$ with $r_1 \dots r_{n-1}x \in N \setminus \phi(N)$ and $r_1 \dots r_{n-1} \notin (N : M)$. So $r_1 \dots r_{n-1}x \in N$ and $r_1 \dots r_{n-1}x \notin \phi(N)$. Hence $r_1 \dots r_{n-1}x + L \in N/L$ and $r_1 \dots r_{n-1}x + L \notin (\phi(N)+L)/L$. Since N/L is a $(n-1,n)$-ϕ_L-prime and $r_1 \dots r_{n-1} + L \notin (N/L :_R M/L)$, thus there exists $i \in \{1,\dots,n-1\}$ such that $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}x + L \in N/L$. Therefore $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}x \in N$ and so N is a $(n-1,n)$-ϕ-prime.

Corollary 3.4. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then N is a $(n-1,n)$-ϕ-prime submodule of M if and only if $\frac{N}{\phi(N)}$ is a $(n-1,n)$-weakly prime submodule of $\frac{M}{\phi(N)}$.*

Proof. It is straightforward.

Theorem 3.5. *Let $f : M \to M'$ be an R-module epimorphism, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*

(1) If N is a $(n-1,n)$-ϕ-prime submodule of M with $ker f \subseteq N$ and $f(\phi(N)) \subseteq \phi'(f(N))$, then $f(N)$ is a $(n-1,n)$-ϕ'-prime submodule of M'.

(2) If L is a $(n-1,n)$-ϕ'-prime submodule of M' and $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then $f^{-1}(L)$ is a $(n-1,n)$-ϕ-prime submodule of M.

Proof. (1) Let $r_1,\dots,r_{n-1} \in R$ and $m' \in M'$ with $r_1 \dots r_{n-1}m' \in f(N) \setminus \phi'(f(N))$. There exists $m \in M$ such that $f(m) = m'$. We have $r_1 \dots r_{n-1}f(m) \in f(N)$ and $r_1 \dots r_{n-1}f(m) \notin \phi'(f(N))$. It follows that $r_1 \dots r_{n-1}m \in N$ and $r_1 \dots r_{n-1}m \notin \phi(N)$, because $r_1 \dots r_{n-1}f(m) \notin f(\phi(N))$. Thus $r_1 \dots r_{n-1}m \in N \setminus \phi(N)$, so $r_1 \dots r_{n-1} \in (N : M)$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$ for some $i \in \{1,...,n-1\}$. Therefore $r_1 \dots r_{n-1} \in (f(N) : M')$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}f(m) \in f(N)$ for some $i \in \{1,...,n-1\}$.

(2) Let $r_1 \dots r_{n-1}m \in f^{-1}(L) \setminus \phi(f^{-1}(L))$ where $r_1,\dots,r_{n-1} \in R$ and $m \in M$. We have $r_1 \dots r_{n-1}m \in f^{-1}(L)$ and $r_1 \dots r_{n-1}m \notin \phi(f^{-1}(L))$. It follows that $r_1...r_{n-1}f(m) \in L \setminus \phi'(L)$. So $r_1 \dots r_{n-1} \in (L : M')$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}f(m) \in L$ for some $i \in \{1,...,n-1\}$, because L is a $(n-1,n)$-ϕ'-prime submodule of M'. Thus $r_1 \dots r_{n-1} \in (f^{-1}(L) : M)$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in f^{-1}(L)$ for some $i \in \{1,...,n-1\}$, as required.

Theorem 3.6. *Let M be a free R-module with a basis $\{m_\alpha\}_{\alpha\in\Lambda}$, $\phi : S(M) \to S(M)\cup\{\emptyset\}$ and $\psi : \mathcal{I}(R) \to \mathcal{I}(R)\cup\{\emptyset\}$ be two functions where $\mathcal{I}(R)$ is the set of all ideals of R. If P is a $(n-1,n)$-ψ-prime ideal of R with $\psi(P)M \subseteq \phi(PM)$, then PM is a $(n-1,n)$-ϕ-prime submodule of M and $(PM : M) = P$.*

Proof. Since P is a proper ideal of R, so PM is a proper submodule of M. Let $r_1,\dots,r_{n-1} \in R$, $x \in M$ with $r_1\dots r_{n-1}x \in PM \setminus \phi(PM)$. Since M is a free R-module with a basis $\{m_\alpha\}_{\alpha\in\Lambda}$, therefore $PM = \{\sum_{f.s} s_i m_i | s_i \in P, m_i \in \{m_\alpha\}_{\alpha\in\Lambda}\}$. We have $r_1\dots r_{n-1}x \in PM$ and $r_1\dots r_{n-1}x \notin \phi(PM)$ with $x = \sum_{f.s} r'{}_\alpha m_\alpha$ ($r'_\alpha \in R, m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$). Thus $\sum_{f.s}(r_1\dots r_{n-1}r'_\alpha)m_\alpha = \sum_{f.s} s_\alpha m_\alpha$, so $r_1\dots r_{n-1}r'_\alpha = s_\alpha \in P$ for every $\alpha \in \Lambda$. But $r_1\dots r_{n-1}r'_\alpha \notin \psi(P)$, otherwise $r_1\dots r_{n-1}r'_\alpha \in \psi(P)$. Thus for every $m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$, we have $r_1\dots r_{n-1}r'_\alpha m_\alpha \in \psi(P)m_\alpha$. So for every $\alpha \in \Lambda$, $r_1\dots r_{n-1}\sum_{f.s} r'_\alpha m_\alpha \in \psi(P)M$. Since $\psi(P)M \subseteq \phi(PM)$, therefore $r_1\dots r_{n-1}\sum_{f.s} r'_\alpha m_\alpha \in \phi(PM)$, so $r_1\dots r_{n-1}x \in \phi(PM)$. This is a contratiction. We showed that for every $\alpha \in \Lambda$, $r_1\dots r_{n-1}r'_\alpha \in P\setminus\psi(P)$. Since P is a $(n-1,n)$-ψ-prime ideal, so there exists $i \neq \alpha \in \{1,\dots n-1\}$, $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}r'_\alpha \in P$. For every $m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$, we have $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}r'_\alpha m_\alpha \in Pm_\alpha$. Therefore $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in PM$. But if $i = \alpha$, we have $r_1\dots r_{n-1} \in P$, so $r_1\dots r_{n-1} \in (PM : M)$. Thus PM is $(n-1,n)$-ϕ-prime submodule of M. It is clear that $(PM : M) = P$.

Now, the following corollary is given as a result of the above theorem. We recall that an R-module M is a multiplication module if for every submodule N of M, $N = IM$ for some ideal I of R (see [2], [8], [10]).

Corollary 3.7. *Let M be a free multiplication R-module and N be a proper submodule of M. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R)\cup\{\emptyset\}$ be a function where $\mathcal{I}$ is the set of all ideals of R and $\phi : S(M) \to S(M)\cup\{\emptyset\}$ be a function with $\psi(N : M)M \subseteq \phi((N : M)M)$. If $(N : M)$ is $(n-1,n)$-ψ-prime ideal of R, then N is a $(n-1,n)$-ϕ-prime submodule of M.*

Proof. Since $N = (N : M)M$, by Theorem 3.6 the proof is clear.

Definition 3.8. A proper submodule P of an R-module M is called *compactly packed* (or abbreviated by $\mathbb{CP}$ submodule) if for each family $\{P_\alpha\}_{\alpha\in\Lambda}$ of prime submodules of M with $P \subseteq \cup_{\alpha\in\Lambda}P_\alpha$, then $P \subseteq P_\beta$ for some $\beta \in \Lambda$. Whenever $P \subseteq \cup_{\alpha\in\Lambda}P_\alpha$ implies that there exist $\alpha_1\dots\alpha_n \in \Lambda$ such that $P \subseteq_{i=1}^{n} P_{\alpha_i}$, P is said *finitely compactly packed* (or abbreviated by $\mathbb{FCP}$) submodule. A module M is said to be $\mathbb{CP}$ ($\mathbb{FCP}$), if every proper submodule of M is a $\mathbb{CP}$ ($\mathbb{FCP}$) submodule. We will call a proper submodule N of M as ϕ-$\mathbb{CP}$ if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of ϕ-prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \Lambda$. Whenever $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$ implies that there exist $\alpha_1\dots\alpha_n \in \Lambda$ such that $N \subseteq_{i=1}^{n} N_{\alpha_i}$, N is said ϕ-$\mathbb{FCP}$ submodule. A module M is said to be ϕ-$\mathbb{CP}$ (ϕ-$\mathbb{FCP}$) if every proper submodule is a ϕ-$\mathbb{CP}$ (ϕ-$\mathbb{FCP}$).

Also, we call a proper submodule N of M as $(n-1,n)$-ϕ-$\mathbb{CP}$ if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of $(n-1,n)$-ϕ-prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \Lambda$. Whenever $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ implies that there exist $\alpha_1 \ldots \alpha_n \in \Lambda$ such that $N \subseteq_{i=1}^n N_{\alpha_i}$, then N is called $(n-1,n)$-ϕ-$\mathbb{FCP}$ submodule. A module M is said to be $(n-1,n)$-ϕ-$\mathbb{CP}$ ($(n-1,n)$-ϕ-$\mathbb{FCP}$) if every proper submodule is a $(n-1,n)$-ϕ-$\mathbb{CP}$ ($(n-1,n)$-ϕ-$\mathbb{FCP}$).

Proposition 3.9. *Let M be an R-module, ϕ_1 and $\phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M with $\phi_1 \leq \phi_2$ (i.e., for every submodule N, $\phi_1(N) \subseteq \phi_2(N)$). If M is a $(n-1,n)$-ϕ_2-$\mathbb{CP}$ ($(n-1,n)$-ϕ_2-$\mathbb{FCP}$) module, then M is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ ($(n-1,n)$-ϕ_1-$\mathbb{FCP}$) module.*

Proof. Let N be a proper submodule of M with $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ where N_α is a $(n-1,n)$-ϕ_1-prime submodule of M. By proposition 2.2., N_α is a $(n-1,n)$-ϕ_2-prime submodule of M, so $N \subseteq N_\beta$ for some $\beta \in \Lambda$, because M is a $(n-1,n)$-ϕ_2-$\mathbb{CP}$ module. Since N_β is a $(n-1,n)$-ϕ_1-prime submodule of M, so N is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ submodule of M. Thus M is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ module. Similarly, we can prove that M is a $(n-1,n)$-ϕ_1-$\mathbb{FCP}$ module.

Proposition 3.10. *Every $(n,n+1)$-ϕ-$\mathbb{CP}$ ($\mathbb{FCP}$) module is a $(n-1,n)$-ϕ-$\mathbb{CP}$ ($\mathbb{FCP}$) module.*

Proof. Apply Proposition 2.3.

Theorem 3.11. *Let $f : M \to M'$ be an R-module epimorphism, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*

(i) If M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ ($\mathbb{CP}$) module such that for every $(n-1,n)$-ϕ'-prime submodule L of M' with $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ ($\mathbb{CP}$) module.

(ii) If M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ ($\mathbb{CP}$) module such that every $(n-1,n)$-ϕ-prime submodule N of M with $kerf \subseteq N$ and $f(\phi(N)) \subseteq \phi'(f(N))$, then M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ ($\mathbb{CP}$) module.

Proof. (i) Let N' be a proper submodule of M' such that $N' \subseteq \cup_{\alpha\in\Lambda} L'_\alpha$, where L'_α is a $(n-1,n)$-ϕ'-prime submodule of M' for each $\alpha \in \Lambda$. We have $f^{-1}(N') \subseteq \cup_{\alpha\in\Lambda} f^{-1}(L'_\alpha)$. Since L'_α is a $(n-1,n)$-ϕ'-prime submodule of M' for each $\alpha \in \Lambda$ and $f^{-1}(\phi'(L'_\alpha) \subseteq \phi(f^{-1}(L'_\alpha))$, by Theorem 3.5. (2), $f^{-1}(L'_\alpha)$ is a $(n-1,n)$-ϕ-prime submodule of M for each $\alpha \in \Lambda$. But M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ module, thus there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $f^{-1}(N') \subseteq \cup_{i=1}^n f^{-1}(L'_{\alpha_i})$, hence $f^{-1}(N') \subseteq f^{-1}(\cup_{i=1}^n L'_{\alpha_i})$. Since f is an epimorphism R-module, so $N' \subseteq \cup_{i=1}^n L'_{\alpha_i}$. Therefore we showed that N' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ submodule of M'. Thus M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ module. Similarly, we can prove that N' is a $(n-1,n)$-ϕ'-$\mathbb{CP}$ submodule of M'. So M' is a $(n-1,n)$-ϕ'-$\mathbb{CP}$ module.

(ii) Assume that L is a proper submodule of M with $L \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ where N_α is a $(n-1,n)$-ϕ-prime submodule of M for each $\alpha \in \Lambda$. We have $f(L) \subseteq f(\cup_{\alpha\in\Lambda} N_\alpha) = \cup_{\alpha\in\Lambda} f(N_\alpha)$. Since N_α is a $(n-1,n)$-ϕ-prime submodule of M, $f(\phi(N_\alpha)) \subseteq \phi'(f(N_\alpha))$ and $kerf \subseteq N_\alpha$ for each $\alpha \in \Lambda$, by Theorem 3.5. (1),

$f(N_\alpha)$ is a $(n-1,n)$-ϕ'-prime submodule of M'. Since M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ module, so there exist $\alpha_1,\dots,\alpha_n \in \Lambda$ such that $f(L) \subseteq \cup_{i=1}^n f(N_{\alpha_i})$. Now, we prove that $L \subseteq \cup_{i=1}^n N_{\alpha_i}$.

Let $x \in L$, so $f(x) \in f(\cup_{i=1}^n N_{\alpha_i})$, hence $f(x) = f(t)$ for some $t \in \cup_{i=1}^n N_{\alpha_i}$. So $x - t \in kerf \subseteq N_{\alpha_j}$ and $t \in N_{\alpha_j}$ for some $\alpha_j \in \{\alpha_1,\dots,\alpha_n\}$. Therefore $x \in N_{\alpha_j}$, so $x \in \cup_{i=1}^n N_{\alpha_i}$. It follows that L is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ submodule of M. Thus M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ module. Similarly, we can prove that L is a $(n-1,n)$-ϕ-$\mathbb{CP}$ submodule of M. So M is a $(n-1,n)$-ϕ-$\mathbb{CP}$ module.

4. The generalization of prime submodules of free multiplication modules

Let M be a free multiplicaton R-module. We study several relations between various generalizations of $(n-1,n)$-ϕ-prime submodules among $(n-1,n)$-almost prime submodules, $(n-1,n)$-prime submodules and $(n-1,n)$-m-almost prime submodules.

Proposition 4.1. *Let M be a free multiplication R-module and N be a proper submodule of M. If N is a $(n-1,n)$-almost prime submodule of M such that $(N : M)^2$ is a prime ideal of R, then N is a $(n-1,n)$-prime submodule of M.*

Proof. Let M be a free multiplication R-module with a basis $\{x_\alpha\}_{\alpha\in\Lambda}$ and N be a proper submodule of M such that N is a $(n-1,n)$-almost prime submodule with $r_1\dots r_{n-1}x \in N$ where $r_1,\dots,r_{n-1} \in R$ and $x \in M$. If $r_1\dots r_{n-1}x \notin (N : M)N$, we have $r_1\dots r_{n-1}x \in N \setminus (N : M)N$, so $r_1\dots r_{n-1} \in (N : M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,...,n-1\}$. Thus N is a $(n-1,n)$-prime submodule of M. If $r_1\dots r_{n-1}x \in (N : M)N$, because $N = (N : M)M$, hence $r_1\dots r_{n-1}x \in (N : M)^2M$. Suppose that $r_1\dots r_{n-1} \notin (N : M)$, we prove that $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,...,n-1\}$. On the other hand, we have $x = \sum_{f.s} {r_\alpha}'x_\alpha$ that $\{x_\alpha\}_{\alpha\in\Lambda}$ is a basis for M. Also, we get $r_1\dots r_{n-1}x = \sum_{f.s} {r_\alpha}''x_\alpha$ with $r''_\alpha \in (N : M)^2$. Thus $\sum_{f.s}(r_1\dots r_{n-1}r'_\alpha)x_\alpha = \sum_{f.s} r''_\alpha x_\alpha$, so $r_1\dots r_{n-1}r'_\alpha = r''_\alpha$, for all $\alpha \in \Lambda$. Since M is a free module, so $r_1\dots r_{n-1}{r_\alpha}' = r''_\alpha$ for all $\alpha \in \Lambda$. Thus $r_1\dots r_{n-1}{r_\alpha}' \in (N : M)^2$, because $r_1\dots r_{n-1} \notin (N : M)$, hence $r_1\dots r_{n-1} \notin (N : M)^2$. But $(N : M)^2$ is a prime ideal of R, so ${r_\alpha}' \in (N : M)^2$ for every $\alpha \in \Lambda$. Therefore $x = \sum_{f.s} {r_\alpha}'x_\alpha \in (N : M)^2M$. Since $(N : M)^2M \subseteq (N : M)M = N$, so $x \in N$ and hence $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,\dots,n-1\}$.

Corollary 4.2. *Let M be a free multiplication R-module. If N is a $(n-1,n)$-m-almost prime submodule of M such that $(N : M)^m$ is a prime ideal of R, then N is a $(n-1,n)$-prime submodule of M.*

Proof. The proof is similar to the proof of Proposition 4.1.

Corollary 4.3. *Let M be a free multiplication R-module and I be a proper ideal of R such that $(IM : M)^2$ is a prime ideal of R. If IM is a $(n-1,n)$-*

almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule of M.

Proof. Apply Proposition 4.1.

Corollary 4.4. *Let M be a free multiplication R-module and I be a proper ideal of R such that $(IM:M)^m$ is a prime ideal of R. If IM is a $(n-1,n)$-m-almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule of M.*

Proof. Apply Corollary 4.2.

Proposition 4.5. *Let M be a faithful finitely generated R-module with a basis $\{x_\alpha\}_{\alpha\in\Lambda}$ and I be a proper radical ideal of R such that I^2 is a prime ideal of R. If IM is a $(n-1,n)$-almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule.*

Proof. Since M is a faithful finitely generated R-module and I is a radical ideal of R, so $(IM:M)=I$. Let $r_1\dots r_{n-1}x\in IM$ where $r_1,\dots,r_{n-1}\in R$ and $x\in M$. If $r_1\dots r_{n-1}x\notin(IM:M)IM$, then $r_1\dots r_{n-1}x\in IM\setminus(IM:M)IM$. Hence $r_1\dots r_{n-1}\in(IM:M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x\in IM$ for some $i\in\{1,\dots,n-1\}$. Now, assume that $r_1\dots r_{n-1}x\in(IM:M)IM$ and $r_1\dots r_{n-1}\notin(IM:M)$. Because $(IM:M)=I$, we have $r_1\dots r_{n-1}x\in I^2M$. On the other hand we get $x=\sum_{f.s}r_\alpha{}'x_\alpha$, since $\{x_\alpha\}_{\alpha\in\Lambda}$ is a basis is for M. Therefore $\sum_{f.s}(r_1\dots r_{n-1}r_\alpha{}')x_\alpha\in I^2M$, hence $\sum_{f.s}(r_1\dots r_{n-1}r_\alpha{}')x_\alpha=\sum_{f.s}r''_\alpha x_\alpha$ where $r''_\alpha\in I^2$ for all $\alpha\in\Lambda$. Thus $r_1\dots r_{n-1}r'_\alpha=r''_\alpha$ for all $\alpha\in\Lambda$, hence $r_1\dots r_{n-1}r'_\alpha\in I^2$. Because $r_1\dots r_{n-1}\notin(IM:M)=I$, so $r_1\dots r_{n-1}\notin I^2$. Since I^2 is a prime ideal of R, hence $r'_\alpha\in I^2$ for all $\alpha\in\Lambda$. Thus we proved that $x=\sum_{f.s}r'_\alpha x_\alpha\in I^2M$. Therefore $x\in IM$ and $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x\in IM$.

Proposition 4.6. *Let M be a free multiplication R-module and $f:M\to M'$ be an R-module epimorphism. Let N be a proper submodule of M with $ker f\subseteq N$ and $(N:M)^2$ be a prime ideal of R. If N is a $(n-1,n)$-almost prime submodule of M, then $f(N)$ is a $(n-1,n)$-prime submodule of M'.*

Proof. Let N be a proper submodule of M, then $f(N)$ is a proper submodule of M'. Suppose that $r_1\dots r_{n-1}m'\in f(N)$ where $r_1,\dots,r_{n-1}\in R$ and $m'\in M'$. Since $f(M)=M'$, so $f(m)=m'$, for some $m\in M$. So $f(r_1\dots r_{n-1}m)\in f(N)$. Since $ker(f)\subseteq N$, hence $r_1\dots r_{n-1}m\in N$. If $r_1\dots r_{n-1}m\notin(N:M)N$, then $r_1\dots r_{n-1}m\in N\setminus(N:M)N$. Since N is a $(n-1,n)$-almost prime submodule of M, so $r_1\dots r_{n-1}\in(N:M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}m\in N$ for some $i\in\{1,...,n-1\}$. Hence $r_1\dots r_{n-1}\in(f(N):f(M))$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}f(m)\in f(N)$ for some $i\in\{1,...,n-1\}$. Thus $f(N)$ is a $(n-1,n)$-prime submodule of M'. Now, if $r_1\dots r_{n-1}m\in(N:M)N$, so $r_1\dots r_{n-1}m\in(N:M)^2M$. Assume that $r_1\dots r_{n-1}\notin(N:M)$. Furthermore, since R-module M is free with a basis $\{x_i\}_{i\in\Lambda}$, so $m=\sum_{f.s}r'_ix_i$, hence $r_1\dots r_{n-1}\sum_{f.s}r'_ix_i\in(N:M)^2M$. Hence we have $\sum_{f.s}r_1\dots r_{n-1}r'_ix_i=\sum_{f.s}r''_ix_i$ where $r''_i\in(N:M)^2$. It follows that $r_1 \dots r_{n-1} r_i' = r_i''$. Since $(N : M)^2$ is a prime ideal of R, so $r_i' \in (N : M)^2$ for every $i \in \Lambda$. Thus $m = \sum_{f.s} r_i' x_i \in (N : M)^2 M$, therefore $m \in (N : M)M$, because of $(N : M)^2 M \subseteq (N : M)M$. Thus $m \in (N : M)M = N$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$ for some $i \in \{1, \dots, n-1\}$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} f(m) \in f(N)$ for some $i \in \{1, \dots, n-1\}$. Thus $f(N)$ is a $(n-1, n)$-prime submodule of M'.

Proposition 4.7. *Let M be a free multiplication R-module and $f : M \to M'$ be an R-module epimorphism. Let N be a proper submodule of M with $ker f \subseteq N$ and $(N : M)^m$ be a prime ideal of R. If N is a $(n-1, n)$-m-almost prime submodule of M, then $f(N)$ is a $(n-1, n)$- prime submodule of M'.*

Proof. The proof is similar to the proof of Proposition 4.6.

Theorem 4.8. *Let M be a free multiplication R-module with a basis $\{x_\alpha\}_{\alpha \in \Lambda}$ and N be a proper submodule of M. If $(N : M)$ is a $(n-1, n)$-prime ideal of R, then N is a $(n-1, n)$-prime submodule of M.*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $x \in M$ with $r_1 \dots r_{n-1} x \in N$. Since $N = (N : M)M$, so $r_1 \dots r_{n-1} x \in (N : M)M$. Because $\{x_\alpha\}_{\alpha \in \Lambda}$ is a basis for M, so $x = \sum_{f.s} r_\alpha' x_\alpha$. Therefore $\sum_{f.s} r_1 \dots r_{n-1} r_\alpha' x_\alpha \in (N : M)M$, hence $\sum_{f.s} r_1 \dots r_{n-1} r_\alpha' x_\alpha = \sum_{f.s} r_\alpha'' x_\alpha$ where $r_\alpha'' \in (N : M)$. It is clear that $r_1 \dots r_{n-1} r_\alpha' \in (N : M)$ for all $\alpha \in \Lambda$. Since $(N : M)$ is a $(n-1, n)$-prime ideal of R, we have two cases. The first case, $i \neq \alpha$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} r_\alpha' \in (N : M)$ for some $i \in \{1, ...n-1\}$, for all $\alpha \in \Lambda$. Therefore $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} \sum_{f.s} {r_\alpha}' x_\alpha \in (N : M)M$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} x \in (N : M)M = N$. The second case, $i = \alpha$ by elimination α, we have $r_1 \dots r_{n-1} \in (N : M)$. Finally, we showed that N is a $(n-1, n)$-prime submodule of M.

Theorem 4.9. *Let M be a free multiplication R-module and N be a proper submodule of M. If $(N : M)$ is a $(n-1, n)$-almost prime ideal of R, then N is a $(n-1, n)$-almost prime submodule of M.*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $x \in M$ with $r_1 \dots r_{n-1} x \in N \setminus (N : M)N$. Since $N = (N : M)M$, so $r_1 \dots r_{n-1} x \in (N : M)M \setminus (N : M)^2 M$, hence $r_1 \dots r_{n-1} x \in (N : M)M$ and $r_1 \dots r_{n-1} x \notin (N : M)^2 M$. Assume that $\{x_\alpha\}_{\alpha \in \Lambda}$ be a basis for M, so $x = \sum_{f.s} {r_\alpha}' x_\alpha$. It is clear that $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M)$ for all $\alpha \in \Lambda$. Also we get $r_1 \dots r_{n-1} {r_\alpha}' \notin (N : M)^2$ for all $\alpha \in \Lambda$, otherwise $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M)^2$, so $r_1 \dots r_{n-1} \sum_{f.s} r_\alpha' x_\alpha \in (N : M)^2 M$, hence $r_1 \dots r_{n-1} x \in (N : M)^2 M$, this is a contradiction. Thus we have $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M) \setminus (N : M)^2$. Since $(N : M)$ is a $(n-1, n)$-almost prime ideal of R, therefore $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} {r_\alpha}' \in (N : M)$ for some $i \in \{1, \dots, n-1, \alpha\}$. If $i \in \{1, \dots, n-1\}$ we get $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} r_\alpha' \in (N : M)$ for all $\alpha \in \Lambda$. Thus $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} \sum_{f.s} r_\alpha' x_\alpha \in (N : M)M$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} x \in (N : M)M = N$. But if $i = \alpha$, by elimination α, we have $r_1 \dots r_{n-1} \in (N : M)$ and hence we proved that N is a $(n-1, n)$-almost prime submodule of M.

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Accepted: 9.10.2017

A new view of closed-CS-module

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Abstract. This paper give a new fact about the extending module. A module M is called extending if every closed submodule N of M is a direct summand. Study of the concepts complement closed submodule ((Closed-N)^c) is achieved. Also we expose to a new way to obtain generalization of extending module by complement closed submodule.
Keywords: extending module, essential submodule, closed submodule, exact sequence.

1. Introduction

In (1976), Goodearl introduced the definition of complement closed submodule and Dungh, Huynh, Smith and Wisbauer [1], studied the extending modules. Wang [5] studied closed-CS-module. A submodule A of M is called essential submodule if A∩K≠0 for every non-zero submodule K of M, equivalently A is a essential in M if and only if every non-zero element of M has a non-zero multiple in A. Therefore if every submodule is essential in a direct summand of M, then M is called extending module. A module M is called extending if every closed submodule N of M is a direct summand of M. Extending modules has been studied in [1] and [2]. Let Z(M)= {I_x∈M:I_x=0, for some ideal I$\leq_{ess}$ R}. If Z(M)=M, then M is a singular. Thus we can define another set: Let $\frac{M}{N}$ be a quotient module and let Z($\frac{M}{N}$)={a+I_x∈($\frac{M}{N}$):I_x=0, for some ideal I$\leq_{ess}$ R}. If Z($\frac{M}{N}$)=$\frac{M}{N}$, then $\frac{M}{N}$ is singular. Therefore if Z($\frac{M}{N}$)≠$\frac{M}{N}$, this means the quotient module $\frac{M}{N}$ is non singular.

Remark 1.1. (a) We denote (Closed-N)^c to complement closed submodule N of M.

(b) Every semisimple R-module is an extending module. For example Z6 as Z-module.

(c) Not every module M has closed submodule is extending; for example; the module M=Z8⊕Z2 as a Z-module. Let A=(2,1) be the submodule generated by (2,1). Clear that A is closed in M but not a summand. Hence M is not extending.

(d) Let us take (Closed-B)^c belong to A; where A and B are submodules in an R-Module M. Then $\frac{A}{B}$ is essential in M.

(e) Every (Closed-N)^c is closed.

Theorem 1.2. *Any module K is singular if and only if there exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow L \longrightarrow K \longrightarrow 0$$

such that f is an essential monomorphism between N and L.

Definition 1.3. *(see [4]) Let M be a module. Then M is called closed-CS-module (generalization of extending module) if for every submodule N of M; the quotient module $\frac{M}{N}$ is non singular and is direct summand of M. (i.e. M has (Closed-N)^c and direct summand of M).*

This paper, contain two main sections. In the first section we give some properties of (Closed-N)^c and in the second section the closed-CS-module is investigated. We prove if K is maximal (Closed-K)^c of M, then $\frac{M}{K}$ is a projective and K is a direct summand of M. (see Proposition 2.13). On the other hand, we prove that an R-module M is closed-CS-module iff for every (Closed-N)^c of M, there is a decomposition M=M$_1$⊕M$_2$ such that A is a subset of M$_1$ and A^c =M$_2$∈M. (see Theorem 3.5).

2. Complement closed submodule

Let N be a submodule of an R-module M (N≤M). Then we can denote (Closed-N)^c of M to the complement closed submodule N and (closed-CS-module) means M has (Closed-N)^c. If every (Closed-N)^c of M is a direct summand, then we obtain a generalization of extending module M (closed-CS-module).

Remark 2.1. If the quotient module $\frac{M}{N}$ is non singular, then N is a (Closed-N)^c.

Definition 2.2. *For N subset of M and L subset of N such that L◁ N, then M≡($\frac{N}{L}$). So, if we have N as a module, then N is called generalization of extending module if the quotient module $\frac{N}{L}$ is non singular and is a direct summand in M.*

Note that, if (Closed-N)^c is a subset of M, then N subset of (Closed-K)^c and from the second isomorphism theorem, we have; N subset of (Closed-N)^c +K⟺(N∩K) is a subset of (Closed-K)^c. Also, by the third isomorphism theorem we can say: N is a subset of K and K is a subset of M⟹K is a subset of (Closed-N)^c of M ⟺ $\frac{K}{N}$ is a subset of (Closed-$\frac{K}{N}$)^c.

Lemma 2.3. *Let M be an R-module and let Bα in Λ, be an independent family of submodules of M and Aα is a subset of Bα, for all α in Λ. Then ⊕Aα is a subset of (Closed-N)^c of $B_\alpha\alpha$ if and only if Aα is a subset of (Closed-N)^c of Bα, for all α in Λ.*

Proof. Suppose that ⊕Aα is a subset of ⊕Bα. We have, $\frac{\oplus B\alpha}{\oplus A\alpha} \cong \frac{B\alpha}{A\alpha}$. Then A$\alpha$ subset of (Closed-N)^c of Bα, for all α in Λ. Conversely, Aα is a subset of (Closed-N)^c of Bα, for all α in Λ . Then $\frac{B\alpha}{A\alpha}$ is non-singular, for all αinΛ and hence ⊕ $\frac{B\alpha}{A\alpha}$ is non-singular. But ⊕$\frac{B\alpha}{A\alpha}$≃$\frac{\oplus B\alpha}{\oplus A\alpha}$. So A⊕$\alpha$ is a subset of (Closed-N)^c of ⊕ Bα. □

Theorem 2.4. *Let M be an R-module and let N and K are submodules of M. Then (N∩K) is a subset of (Closed-N)^c in M.*

Proof. Let N be a subset of (closed-CS-module) and let K be a subset of (closed-CS-M). We must prove that (N∩K) is a subset of (Closed-N)^c in M. Let us take an element m∈M such that m+(N∩K) belong to Z($\frac{M}{N}$∩K). Thus Annihilator of (m+N∩K) is a subset of (eR). Since Annihilator of (m+N∩K) is a subset of Annihilator of (m+N), then Annihilator of (m+N) is a subset of (eR). We have Z($\frac{M}{N}$)=0, therefore m+N=N. Similar, we get m+K=K. Thus m belong to N∩K and then Z($\frac{M}{N\cap K}$)=0. □

Lemma 2.5. *Let L and K be a submodules of an R-module M. If L is a subset of (Closed-K)^c and K is a subset of (closed-CS-module), then L is a subset of (closed-CS-module).*

Proof. Let L be a subset of (Closed-K)^c and let K be a subset of (closed-CS-module). Let us take short exact sequence:

$$0 \longrightarrow (\frac{K}{L}) \longrightarrow (\frac{M}{L}) \longrightarrow (\frac{M}{L})/(\frac{K}{L}) \longrightarrow 0.$$

Such that i is the inclusion map from ($\frac{K}{L}$) into ($\frac{M}{L}$) and π is the natural epimorphism from ($\frac{M}{L}$) into ($\frac{M}{L}$)/($\frac{K}{L}$). Since L is a subset of K and K is a subset of (closed-CS-module), then ($\frac{K}{L}$) is a subset of (Closed-N)^c of ($\frac{M}{L}$), (see Theorem 2.4). Since ($\frac{K}{L}$) and ($\frac{M}{L}$)/($\frac{K}{L}$) are non-singular, then $\frac{M}{L}$ is non-singular. □

Let M be an R-module such that L subset of K and K subset of M. If K subset of (Closed-N)^c of M, then L need not be (Closed-N)^c. See the following example:

Example 2.6. Consider Z as Z-module, it is clear that Z subset of (Closed-N)^c of Z. But Z(2Z ⊆Z)=Z(Z_2)= Z_2 is singular. On the other hand, if L subset of (Closed-N)^c of M, then K need not be (Closed-K)^c.

Example 2.7. Let 0 subset of 2Z and 2Z subset of Z. Clearly 0 subset of (closed-CS-Z). But Z($\frac{Z}{2Z}$)=Z(Z_2)=Z_2 is singular. Also, an epimorphic image of an (Closed-N)^c need not be (closed-CS-module). We have the natural epimorphism π:Z $\longrightarrow$ $\frac{Z}{4Z}$. That is means 0 subset of (Closed-N)^c of Z. On the other hand, since $\frac{Z}{4Z} \cong Z_4$ is a singular imply the image of zero always equal zero and moreover it is not (closed-CS-$\frac{Z}{4Z}$).

Proposition 2.8. *Let λ:M $\longrightarrow$ N be an epimorphism and L subset of (closed-CS-module). If ker(f) subset of L, then f(L) subset of (Closed-N)^c.*

Proof. Assume that L subset of (closed-CS-module). To show that f(L) subset of (Closed-N)^c. Let n belong to N such that Annihilator(n+f(L)) subset of eR. Since f is an epimorphism, then n=f(m), for some m ∈ M. Since ker(f) subset of L, then Annihilator(n+f(L)) subset of Annihilator(m+L) and hence Annihilator(n+f(L)) subset of eR. But L subset of (Closed-N)^c of M, so m ∈ L. Thus n=f(m)∈f(L). □

Theorem 2.9. *Let λ:M $\longrightarrow$ N be an R-homomorphism and K (Closed-N)^c, then for every singular submodule L of M, f(L) subset of K .*

Proof. Let μ:N $\longrightarrow \frac{N}{K}$ be the natural epimorphism. Let $\mu\circ\lambda$: M $\longrightarrow \frac{N}{K}$. Now $\mu\circ\lambda|_L$:L $\longrightarrow \frac{N}{K}$. But N is a singular and $\frac{N}{K}$ is non-singular. Thus $\mu\circ\lambda|_L$=0. So $\mu(\lambda$(L)) =0 and hence λ(L) subset of ker(μ)=K. □

As a result from Theorem 2.9, we introduce the following good corollary.

Corollary 2.10. *If N is a module and K subset of (Closed-N)^c. Then $\frac{Hom(M,N)}{M}$ subset of K, such that Z(M)=M.*

Example 2.11. Suppose that M is an R-module. Let L subset of (closed-CS-module). Then Z(M)=Z(L).

Proof. We must prove that Z(M) is a subset of Z(L). Let i:Z(M)$\longrightarrow$M be the inclusion map and μ:M$\longrightarrow\frac{M}{L}$ be the natural epimorphism from M into $\frac{M}{L}$. We take the map $\mu\circ$i:Z(M)$\longrightarrow\frac{M}{L}$. Since Z(M) is a singular and $\frac{M}{L}$ is non-singular, then $\mu\circ$i=0. So $\mu\circ$i:(Z(M))=μ(Z(M))=0. Thus Z(M) is a subset of ker(μ)=L. We know that Z(L)=Z(M)$\cap$A. So Z(L)=Z(M). □

Theorem 2.12. *Let M be an R-module and let L $\subseteq$ K $\subseteq$ M and N $\subseteq$ (closed-CS-module), then $\frac{M}{K}$ is a singular if and only K subset of (closed-CS-module).*

Proof. Let L subset of (Closed-N)^c of M and $\frac{M}{K}$ is singular. By the third isomorphism theorem $\frac{M}{K}\cong(\frac{M}{L})/(\frac{K}{L})$. Since $\frac{M}{L}$ is non-singular, then $(\frac{K}{L})$ subset of (closed-CS-$\frac{M}{N}$). Let μ:M$\longrightarrow\frac{M}{N}$ be the natural epimorphism. We have K=$\mu^{-1}(\frac{K}{L})$ is a subset of $\mu^{-1}(\frac{M}{L})$=M. The converse is clear by [3]. □

Proposition 2.13. *Let M be an R-module and K is maximal (Closed-K)^c of M. Then $\frac{M}{K}$ is projective and K is a direct summand of M.*

Proof. Since K is maximal submodule of M, then $\frac{M}{K}$ is simple and hence semisimple. But $\frac{M}{K}$ is non-singular, therefore $\frac{M}{K}$ is projective. Now consider the following short exact sequence 0$\longrightarrow$K$\longrightarrow$M$\longrightarrow\frac{M}{K}\longrightarrow$0; where i is the inclusion map and π is the natural epimorphism from M into $\frac{M}{K}$. Since $\frac{M}{K}$ is projective, then the sequence is splits, (see [6]). Thus K is a direct summand of M. Let M be an R-module and N subset of M. Recall that the resdual of M in N (denoted by [N:M]) is defined as follows: [N:M]=r$\in$R, rM$\subseteq$N, (see [7]). □

3. Closed-CS-module

In this section, we introduce main theorems which explain the new ways to obtain a generalization of extending module.

Proposition 3.1. *Let M be a (Closed-N)^c and N $\leq$ M, then the quotient module is a (Closed-N)^c of M*

Proof. Let $\frac{K}{N}$ subset of (Closed-N)^c of $\frac{M}{N}$. Then by Theorem 2.4 and Lemma 2.5, K is a subset of (Closed-N)^c in M. But M is a closed-CS-module. (i.e. has (Closed-N)^c) of M, therefore M=N⊕K, K is a subset of M. Since N is a subset of K, then one can easily show that $\frac{M}{N}$=($\frac{K}{N}$)⊕($\frac{K+N}{N}$). Thus $\frac{M}{N}$ is a closed-CS-module. □

Recall that a module M is called closed-CS-module if for any submodule N of M, there is a direct summand K of M such that N is a subset of K and $\frac{K}{N}$ is singular.

Let N subset of (Closed-N)^c. Since M is (Closed-N)^c, then there exists a direct summand K of M such that N is a subset of K and Z($\frac{K}{N}$)=($\frac{K}{N}$); ($\frac{K}{N}$ is a singular). But $\frac{K}{N}$ is a subset of $\frac{M}{N}$, so is non-singular. Thus K=N. So any (Closed-M)^c is closed-CS-module.

Theorem 3.2. *An R-module M is a closed-CS-module if and only if for every N submodule of M, (Closed-N)^c, there is a decomposition M=M_1⊕M_2 such that N is a subset of M_1 and M_2 is a complement of N in M.*

Proof. $\Longrightarrow$ Clear.

$\Longleftarrow$ Let N be a subset of (Closed-N)^c, then by our assumption, there exists decomposition M=M_1⊕M_2 such that N is a subset of M_1 and M_2 is a complement of N in M. So N⊕M_2 is a subset of (Closed-N)^c of M. Thus N is a subset of (Closed-N)^c of M_1 and hence Z($\frac{M_1}{N}$)=$\frac{M_1}{N}$; $\frac{M_1}{N}$ is singular). But N is a subset of M_1 and N is a subset of (Closed-N)^c of M, therefore N is a subset of (Closed-N)^c of M_1, (see Theorem 2.4). Thus N=M_1. □

Corollary 3.3. *Every (Closed-L)^c of closed-CS-module M is closed-CS-module.*

Proof. Let M be a closed-CS-module and let N be a subset of M . We must prove that N is a closed-CS-module. Let K subset of (Closed-N)^c, then by Theorem 2.4, L is a subset of (Closed-N)^c of M. But M is a closed-CS-module, therefore L is a direct summand of M and hence K is a direct summand of A. □

Lemma 3.4. *An R-module M is closed-CS-module if and only if every (Closed-N)^c of M is essential in a direct summand.*

Proof. $\Longrightarrow$ Clear.

$\Longleftarrow$ let N subset of (Closed-N)^c, we need to show that N is a direct summand of M. Since N subset of (Closed-N)^c of M, then by our assumption N is a subset of (Closed-N)^c of M, where D is a direct summand of M. Thus Z($\frac{D}{N}$)=$\frac{D}{N}$; ($\frac{D}{N}$ is singular). But $\frac{D}{N}$ subset of $\frac{M}{N}$, therefore $\frac{D}{N}$ is non-singular. Thus N=D and hence M is closed-CS-module. □

Theorem 3.5. *An R-module M is closed-CS-module if and only if for every (Closed-N)^c of M; there exists a decomposition M=M_1⊕M_2 such that N is a subset of M_1 and N⊕M_2 is a subset of (Closed-N)^c of M.*

Proof. $\Longrightarrow$ Clear .

$\Longleftarrow$ Let N be a subset of (Closed-N)^c of M, we need to show that N is a direct summand of M. Since N is a subset of (Closed-N)^c of M, then by assumption there exists a decomposition M=$M_1 \oplus M_2$ such that N⊆M$_1$ and (N⊆M_2) is a subset of (Closed-N)^c of M. So $\frac{M}{(N \oplus M_2)}$ is a singular. But N⊕M_1 and A are subset of (Closed-N)^c of M, therefore by Theorem 2.4, N is a subset of (Closed-N)^c of M_1. Since M_2 is a subset of (Closed-N)^c of M_2, then by Lemma 2.3, (N⊕M_2) is a subset of (Closed-N)^c of $M_1 \oplus M_2$=M. So $\frac{M}{(N \oplus M_2)}$ is non-singular. Thus M=N⊕M_2. □

Proposition 3.6. *An R-module M is a closed-CS-module if and only if for every direct summand A of the injective hull E(M) of M such that (A∩M)^c is a subset of (closed-CS-module), then (A∩M) is a direct summand of M.*

Proof. $\Longrightarrow$ Clear .

$\Longleftarrow$ Let N be a subset of (Closed-N)^c of M and let K be a relative complement of N, then (N⊕K) is a subset of (Closed-N)^c of M. Since M is a subset of (Closed-N)^c of E(M), then (N⊕K) is a subset of (Closed-N)^c of E(M). Thus E(N)⊕E(K)=E(N⊕K)=E(M). Since E(N) is a summand of E(M), then by our assumption E(N)∩M is a summand of M. Now N is a subset of (Closed-N)^c of E(N) and M is a subset of (Closed-N)^c of M, thus N=(N∩M) is a subset of (Closed-N)^c of E(M)∩M. Hence by Lemma 3.5, M is closed-CS-module. □

Theorem 3.7. *Let R be a ring, then R is a closed-CS-module if and only if every cyclic non-singular R-module is projective.*

Proof. Let R be a closed-CS-ring and M=Ra, a∈M be a nonsingular R-module. Let the following be a short exact sequence.

$$0 \longrightarrow \text{ Annihilator}\,(a) \longrightarrow R \longrightarrow Ra \longrightarrow 0,$$

where i is the inclusion homomorphisim and f is a map defined by f(r)= ra, r∈ R. So f is an epimorphisim and ker(f) equal Annihilator of (a). Hence from the first isomorphisim theorem, Annihilator of (a)R≅Ra. But Ra is non-singular, therefore Annihilator of (a) subset of (Closed-N)$_c$ of R. Since R is closed-CS-ring, then Annihilator of (a) is a direct summand of R, so the sequence is split. Thus R is equivalent to Annihilator of (a)⊕Ra. Since R is projective, then Ra is projective. Conversely, let A be a (Closed-N)^c of I, I an ideal in R, then $\frac{R}{A}$ is non-singular. Since R is cyclic, then $\frac{R}{A}$ is cyclic. By our assumption $\frac{R}{A}$ is a projective. Now consider the following short exact sequence:

$$0 \longrightarrow A \longrightarrow R \longrightarrow AR \longrightarrow 0,$$

where i is the inclusion homomorphisim and π is the natural epimorphisim from R into Ra. Since $\frac{R}{A}$ is projective, then the sequence is split. Thus A is a summand of R. Also a direct sum of closed-CS-module need not to be closed-CS-modules (see [4]). □

Proposition 3.8. *Let M and N be closed-CS-modules such that Annihilator of M+Annihilator of N equal R. Then M⊕N is closed-CS-module.*

Proof. Let A be a (Closed-N)$_c$ submodule of M⊕N.

Since Annihilator of M+Annihilator of N=R, then by the same way of the prove [9, Proposition 4.2, CH.1], A=C⊕D, where C is a submodule of M and D is a submodule of N. Since A=(C⊕D) is a subset of (closed-N)^c of M⊕N, then C and D are (Closed-N)^c of M and N respectively by Lemma 2.3. But M and N are closed-CS-modules, therefore C is a summand of M and D is a summand of N. So A=C⊕D is a summand of M⊕N. Thus M⊕N is a closed-CS-module. Recall that a submodule N of R-module M is called a fully invariant submodule of M, if for every endomorphism f:M⟶M, f(N) subset of N, (N is fully invariant) (see [8]). □

Corollary 3.9. *Let M=⊕M_i be an R-module, such that every (Closed-N)^c of M is fully invariant, then M is closed-CS-module if and only if M_i is closed-CS-module; i∈I.*

Proof. ⟹ Clear.

⟸ let S be a (Closed-N)^c of M. For each i∈I, let π_i:M ⟶ M$_i$ be the projection map. Let x∈S, then x=$\sum$m$_i$, m$_i$ ∈ M$_i$ and m$_i$=0 for all but finite many element of i∈I, π_i(x)=m_i. Since we have (Closed-S)^c, then by our assumption, S is fully invariant and hence π_i(x)=m$_i$ S∩M$_i$. So x∈$\bigotimes$(S∩M$_i$). Thus S subset of ⊕(S∩M$_i$). But ⊕(S∩M$_i$) subset of S, therefore S= ⊕(S∩M$_i$). Since S is a subset of (Closed-M)^c, then by Theorem 2.4, (S∩M$_i$) is a subset of (Closed-N)^c of M$_i$ ∀ i∈I. But M$_i$ closed-CS-modules for all i∈I, therefore (S∩M$_i$) is a direct summand of M$_i$. Thus S is a direct summand on M. □

An R-module M is called a distributive module if A∩(B+C)=(A∩B)+(A∩C), for all submodules A, B and C of M, (see [9]).

Corollary 3.10. *Let M=M_1⊕M_2 be distributive R-module. Then M is closed-CS-module if and only if M_1 and M_2 are closed-CS-module.*

Proof. ⟹ Clear.

⟸ Let K be a subset of (closed-N)^c in M. Since M=M$_1$⊕M$_2$, then K= K∩(M$_1$⊕M$_2$). But M is a distributive, therefore K=(K∩M$_1$)⊕(K∩M$_2$). By Lemma 2.3, (K∩M$_1$) is a subset of (Closed-N)^c of M$_1$ and (K∩M$_2$) is a subset of (Closed-N)^c. Since M$_1$ and M$_2$ are closed-CS-modules, then (K∩M$_1$) is a direct summand of M$_1$ and (K∩M$_2$) is a direct summand of M$_2$. Clearly that K=(K∩M$_1$)⊕(K∩M$_2$) is a direct summand of M. □

Corollary 3.11. *Let M be an R-module and let N be a subset of (closed-CS-M. Then [N:M] is a subset of (closed-CS-R).*

Proof. Let N be a subset of closed-CS-module. Assume that [N:M] is not closed-CS-module in R. So there exists r∈R such that [N:M]≠ r+[N:M]∈ Z($\frac{N}{[N:M]}$). Thus rM not subset of N and hence there exists m_0∈M such that rm_0 not in N. One can easily show that Annihilator of (r+[N:M]) is a subset of Annihilator of (rm_0+N).Since Annihilator of (r+[N:M]) is a subset of eR, then Annihilator of (rm_0+N) is a subset of eR. But $\frac{M}{N}$ is non-singular, therefore rm_o+N=N which is contradiction. □

Acknowledgments

The author would like to thank the referee, whose careful reading and thoughtful comments have helped improve the paper.

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Accepted: 21.10.2017

Hesitant fuzzy sets approach to ideal theory in ordered Γ-semigroups

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Abstract. The aim of this paper is to apply hesitant fuzzy sets theory in ordered Γ- semigroups. The hesitant fuzzy ideals in ordered Γ-semigroups are introduced and some related properties are explored. Using this concept, some characterizations on hesitant fuzzy left (right and bi-) ideals are given. The hesitant fuzzy interior ideal in oredered Γ-semigroup are defined and their related properties are studied. Finally, the characterization of a simple ordered Γ-semigroup in terms of a hesitant fuzzy simple ordered Γ-semigroup is presented.
Keywords: ordered Γ-semigroup, hesitant fuzzy ideal, hesitant fuzzy interior ideal.

1. Introduction

In 1981, the concept and notion of the Γ-semigroups was introduced by Sen [32]. Later on, this notion was futher studied by Sen and Saha in ([31], [30], [33]). In 1963, Fuchs had completed the first monograph on ordered algebras, in which ordered group, ordered semigroup, ordered ring and ordered field were involved

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[7]. Since ordered semigroup has a close relation with theoretical computer science, especially with formal language and theory of automata, it has been extensively investigated by many researcher (see e.g. [3], [8]). For the general development of gamma ordered semigroups, the ideal theory plays an important role.

In the year 1997, Kwon and Lee [19] introduced the concepts of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups and gave some characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups.

Since the concept of fuzzy sets was introduced by Zadeh in 1965[42], the theories of fuzzy sets and fuzzy systems developed rapidly. The study of the fuzzy algebraic structures has started in the pioneering paper of Rosenfeld [29] in 1971. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy group. Since then the literature of various fuzzy algebraic concepts has been growing very rapidly.

Kuroki ([16], [17]) applied the fuzzy sets theory to semigroups and Fuzzy sets in ordered semigroups were first considered by Kehayopulu and Tsingelis in ([13], [14], [15]).

Recently, to deal with hesitant and incongruous problems, Torra and Narukawa [38] and Torra [37] proposed the concept of hesitant fuzzy set(s) [briefly, HFS(s)], a new generalization of fuzzy sets, which allows the membership of an element of a set to be represented by several possible values. They also discussed relationships among hesitant fuzzy sets and other generalizations of fuzzy sets such as intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy multisets. Some set theoretic operations such as union, intersection and complement on hesitant fuzzy sets have also been proposed by Torra [37]. Hesitant fuzzy sets can be used as an efficient mathematical tool for modeling peoples hesitancy in daily life than the other classical extensions of fuzzy sets. The motivation for introducing these sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values. For example, two experts discuss the membership of x into A, and one wants to assign 0.3 and the other 0.4. So, the uncertainty on the possible values is somehow limited.

After the pioneering work of Torra, the HFS has received much attention from many authors and has been used in clustering analysis and decision-making (see [4], [28], [39], [40], [41]). For example, Chen et al. [4] systematically investigated the correlation coefficients of HFSs and applied them to clustering analysis, Xia and Xu [40] studied the aggregation operators of hesitant fuzzy sets and applied them to decision making.

Recently, hesitant fuzzy sets theory have been applied to different algebraic structures on various aspects viz., Jun et al. have applied the hesitant fuzzy sets theory to the theory of semigroups (see [9, 10, 11, 12]). Muhiuddin et al. have applied the hesitant fuzzy sets theory to different algebraic (see [20, 21, 22, 23, 24, 25, 26]). As follow-up, we have studied the ideals of ordered Γ-semigroups in terms of HFSs. In the present analysis, we first give the equivalent condition on hesitant fuzzy bi-ideal in a ordered Γ-semigroups. Moreover, we prove that a non-empty hesitant fuzzy subset H of an ordered Γ-semigroup S is hesitant fuzzy ideal of S if and only if the T-cut H_T of H is an ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided H_T is non-empty. Then we characterize simple ordered Γ-semigroup by means of a hesitant fuzzy simple ordered Γ-semigroup. We prove that in a simple ordered Γ-semigroup, every hesitant fuzzy interior ideal of S is a constant function and in a left zero (right zero) ordered Γ-semigroup, every hesitant fuzzy left (resp., hesitant fuzzy right) ideal is a constant function.

2. Preliminaries

Throughout the present paper, ordered Γ-semigroup will be denoted by po-Γ-semigroup. To develop our main results, we need the following notions.

Definition 2.1 ([5])**.** Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exist mappings $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$ and $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \to \alpha a\beta$ satisfying the following associative propeerties $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$, for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.2 ([6])**.** A Γ-semigroup S is a po-Γ-semigroup if S and Γ are posets such that:

(1) $a \leq b$ in S $\Rightarrow$ $\mathrm{a\alpha c \leq b\alpha c}$, $\mathrm{c\alpha a \leq c\alpha b}$ in S, for all $c \in S$ and $\alpha \in \Gamma$.

(2) $\alpha \leq \beta$ in $\Gamma \Rightarrow \mathrm{a\alpha b \leq a\beta b}$, for all $a, b \in S$.

In our further discussion, unless otherwise mentioned S will stand for a ordered Γ-semigroup. Torra [37] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain which can be defined as follows:

Definition 2.3 ([37])**.** Let S be a reference set, a hesitant fuzzy set on S is a function H that returns a subset of values in [0,1]:

$$H : S \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1].

Mathematically, we can represent HFS H on S as:

$$H = \{(a, h(a)) \ \forall a \in S : h(a) \text{ is the set of some different values in } [0,1]\}.$$

Let H and G be any two HFSs on S, we define $H \sqsubseteq G$ if $H(a) \subseteq G(a)$, for all $a \in S$. For any element a of S, we define

$$A_a = \{(b, c) \in S \times S \text{ and } \gamma \in \Gamma : \mathrm{a \leq b\gamma c}\}.$$

For any HFSs H and G in S. The hesitant fuzzy product of H and G is defined to be the HFS $H\tilde{o}G$ on S as follows:

$$(H\tilde{o}G)_a = \begin{cases} \bigcup\limits_{a \leq b\gamma c} (H_b \cap G_c) & \text{if } \ \mathrm{A_a} \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

For any two HFSs H and G on S, let $\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1], the hesitant union $H \sqcup G$ of H and G is defined to be HFS on S as follows:

$$H \sqcup G : S \to \mathcal{P}([0,1]), a \longmapsto H(a) \cup G(a)$$

and the hesitant intersection $H \sqcap G$ of H and G is defined to be HFS on S as follows:

$$H \sqcap G : S \to \mathcal{P}([0,1]), a \longmapsto H(a) \cap G(a).$$

Let A be any non empty subset of S. Recall that, we denote by H_A the characteristic HFS on S as follows:

$$H_A : S \ \to \ (\mathcal{P}[0,1]) \ , \ a \ \mapsto \ \begin{cases} [0,1], & \text{if } \ a \ \in \ A \\ \emptyset, & \text{if } \ a \ \notin \ A. \end{cases}$$

Let S be a po-Γ-semigroup, we define a hesitant fuzzy subset I of S as follows:

$$I : S \ \to \ (\mathcal{P}[0,1]) \mid x \to \ I_x := [0,1].$$

3. Main results

Definition 3.1. A hesitant fuzzy set H on po-Γ-semigroup S is called a hesitant fuzzy semigroup on S if it satisfies :

$$(\forall \ x, y \ \in S, \gamma \in \Gamma)(H(x\gamma y) \supseteq H(x) \cap H(y)).$$

Definition 3.2. A hesitant fuzzy set H on po-Γ-semigroup S is called a hesitant fuzzy left (resp., right) ideal on S if it satisfies:

(1) $(\forall \ x, y \in S) \ \ x \leq y \Rightarrow H(x) \supseteq H(y)$

(2) $(\forall \ x, y \ \in S, \gamma \in \Gamma)(H(x\gamma y) \supseteq H(y)(resp., H(x\gamma y) \supseteq H(x)).$

If a hesitant fuzzy set H on S is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on S, we say that H is a hesitant fuzzy (two-sided) ideal on S.

Example 3.3. Let $S = M_{1\times 2}(\mathbb{Z}_2)$ be a set of all 1×2 matrices over the field $\mathbb{Z}_2$ and $\Gamma = M_{2\times 1}(\mathbb{Z}_2)$ be a set of all 2×1 matrices over the field $\mathbb{Z}_2$. Then S is a Γ-semigroup with respect to usual matrix multiplication. Also S and Γ are posts with respect to $\leq$ defined by $(a_{ij}) \leq (b_{ij})$ if and only if $a_{ij} \leq b_{ij} \ \forall i, j$. Then, S is a po-Γ-semigroup.

Define a hesitant fuzzy subset H of S such that

$$H(x) = \begin{cases} [0,1], & \textit{if} \quad x = (0,0) \\ \{0.1\}, & \text{otherwise.} \end{cases}$$

Clearly, H is a hesitant fuzzy ideal of S.

Definition 3.4. A hesitant fuzzy subsemigroup H on po-Γ-semigroup S is called a hesitant fuzzy bi-ideal on S if it satisfies:

(1) $(\forall\, x, y \in S)\;\; x \leq y \Rightarrow H(x) \supseteq H(y)$;

(2) $(\forall\, x, y, z \in S, \alpha, \beta \in \Gamma)(H(x\alpha y\beta z) \supseteq H(x) \cap H(z))$.

Example 3.5. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be two non empty sets. Then S and Γ are posets with respect to the ordered relation $\leq$ given by $a \leq b \leq c \leq d$ and $\gamma \leq \alpha \leq \beta$ respectively. Clearly S is a po-Γ-semigroup with respect to the operations defined below:

α	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	b	a	c
d	c	c	c	d

β	a	b	c	d
a	a	a	a	a
b	a	a	a	c
c	a	a	a	c
d	c	c	c	c

γ	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	a	a

Let H be a hesitant fuzzy subset of S such that

$$H_a = [0,1];\;\; H_b = [0.1, 0.9];\;\; H_c = [0.2, 0.8];\;\; H_d = [0.3, 0.7]$$

Clearly H is a hesitant fuzzy bi-ideal of S. But H is neither a hesitant fuzzy right ideal nor a hesitant fuzzy left ideal of S, since $H(b\alpha d) = H(c) \not\supseteq H(b)$ and $H(d\alpha a) = H(c) \not\supseteq H(a)$ respectively.

Definition 3.6. A hesitant fuzzy subsemigroup H on po-Γ-semigroup S is called a hesitant fuzzy interior ideal on S if it satisfies:

(1) $(\forall\, x, y \in S)\;\; x \leq y \Rightarrow H(x) \supseteq H(y)$;

(2) $(\forall\, x, y, z \in S, \alpha, \beta \in \Gamma)(H(x\alpha y\beta z) \supseteq H(y))$.

Theorem 3.7. *Let H be a non empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the following conditions are equivalent:*

(1) H is a hesitant fuzzy bi-ideal of S;

(2) $H\tilde{o}H \sqsubseteq H$, $H\tilde{o}I\tilde{o}H \sqsubseteq H$ and $a \leq b \Rightarrow H(a) \supseteq H(b)\ \forall\, a, b \in S$.

Proof. (1) $\Rightarrow$ (2) Let H be a hesitant fuzzy bi-ideal of S. Then H is a hesitant fuzzy subsemigroup of S. Again, let $a \in S$. If for some $b, c \in S$ and $\gamma \in \Gamma$such that $a \leq b\gamma c$. Then, we have $(H\tilde{o}H)(a) = \bigcup_{a \leq b\gamma c} \{H(b) \cap H(c)\} \subseteq \bigcup_{a \leq b\gamma c} H(b\gamma c) \subseteq H(a)\ \forall\, a \in S$ implies $H\tilde{o}H \sqsubseteq H$. Other wise, $(H\tilde{o}H)(a) = \emptyset \subseteq H(a)$. Consequenly, $H\tilde{o}H \sqsubseteq H$.

Further to prove $H\tilde{o}I\tilde{o}H \sqsubseteq H$, let $a \in S$. Suppose $\exists\ u, v, p, q \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq u\alpha v$ and $u \leq p\beta q$. Again, since H is a hesitant fuzzy bi-ideal of S, we have

$$\begin{aligned}
(H\tilde{o}I\tilde{o}H)(a) &= \bigcup_{a\leq u\alpha v} \{(H\tilde{o}I)(u) \cap H(v)\} \\
&= \bigcup_{a\leq u\alpha v} \{ \bigcup_{u\leq p\beta q} \{H(p) \cap I(q)\} \cap H(v)\} \\
&= \bigcup_{a\leq u\alpha v} \{ \bigcup_{u\leq p\beta q} \{H(p) \cap [0,1]\} \cap H(v)\} \\
&= \bigcup_{a\leq p\beta q\alpha v} \{H(p) \cap H(v)\} \\
&\subseteq \bigcup_{a\leq p\beta q\alpha v} \{H(p\beta q\alpha v)\}(cf.Definition\ 3.4) \\
&\subseteq H(a)\ \forall\ a \in S.
\end{aligned}$$

implies $H\tilde{o}I\tilde{o}H \sqsubseteq H$. Other wise, $H\tilde{o}I\tilde{o}H(a) = \emptyset \subseteq H(a)$. Consequently, $H\tilde{o}I\tilde{o}H \sqsubseteq H$.

Moreover, for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$.

(2) $\Rightarrow$ (1). Assume that $H\tilde{o}H \sqsubseteq H$. Let $a, b \in S$ and $\gamma \in \Gamma$ we have $H(a\gamma b) \supseteq (H\tilde{o}H)(a\gamma b) \supseteq \{H(a) \cap H(b)\}$. Thus, H is a hesitant fuzzy subsemigroup of S.

Again, let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq a(\alpha b\beta)c := a\gamma c$, where $\gamma =: \alpha b\beta \in \Gamma$. Since $H\tilde{o}I\tilde{o}H \sqsubseteq H$, it follows that

$$\begin{aligned}
H(a(\alpha b\beta)c) &:= H(a\gamma c) \supseteq H(x) \\
&\supseteq \big((H\tilde{o}I)\tilde{o}H\big)(x) \\
&= \bigcup_{x\leq m\delta n} \{(H\tilde{o}I)(m) \cap H(n)\} \\
&= \bigcup_{x\leq m\delta n} \{ \bigcup_{m\leq p\eta q} \{H(p) \cap I(q)\} \cap H(n)\} \\
&= \bigcup_{x\leq m\delta n} \{ \bigcup_{m\leq p\eta q} \{H(p) \cap [0,1]\} \cap H(n)\} \\
&= \bigcup_{x\leq p(\eta q\delta)n} \{H(p) \cap H(n)\} \supseteq \{H(a) \cap H(c)\}.
\end{aligned}$$

Hence, H is a hesitant fuzzy bi-ideal of S. □

Theorem 3.8. *Let H be a non empty hesitant fuzzy subsemigroup of a po-Γ-semigroup S. Then the following conditions are equivalent:*

(1) *H is a hesitant fuzzy left (respecively right) ideal of S;*

(2) *$I\tilde{o}H \sqsubseteq H$. (resp; $H\tilde{o}I \sqsubseteq H$) and $a \leq b \Rightarrow H(a) \supseteq H(b)\ \forall\ a, b \in S$.*

Proof. (1) $\Rightarrow$ (2) Let H be a hesitant fuzzy left ideal of S. Again, let $a \in S$ and suppose there exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$, then

$$\begin{aligned} (I\tilde{o}H)(a) &= \bigcup_{a \leq b\gamma c} \{I(b) \cap H(c)\} \\ &\subseteq \bigcup_{a \leq b\gamma c} \{[0,1] \cap H(c)\} \\ &\subseteq \bigcup_{a \leq b\gamma c} \{H(c)\} \\ &\subseteq \bigcup_{a \leq b\gamma c} \{H(b\gamma c)\} \quad (cf. Definition\ 3.2) \\ &\subseteq H(a)\ \forall\ a \in S \quad (\text{since } a \leq b\gamma c). \end{aligned}$$

implies $I\tilde{o}H \sqsubseteq H$. Similar arguments for the right ideal. If there do not exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$, then $(I\tilde{o}H)(a) = \emptyset \subseteq H(a)$. Hence $I\tilde{o}H \sqsubseteq H$. Moreover, for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$, for all $a \in S$ implying that $I\tilde{o}H \sqsubseteq H$.

Furthermore, by definition of HF left ideal, we have for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$.

(2) $\Rightarrow$ (1). Assume that $I\tilde{o}H \subseteq H$ and $a \leq b \Rightarrow H(a) \supseteq H(b)$. Again, let $a, b \in S$ and $\gamma \in \Gamma$, then we have

$$\begin{aligned} H(a\gamma b) &= (I\tilde{o}H)(a\gamma b) \\ &\supseteq I(a) \cap H(b) \\ &= [0,1] \cap H(b) \\ &= H(b). \end{aligned}$$

Consequently, H is a hesitant fuzzy left ideal of S. In the similar way, we can prove the another case. □

Definition 3.9. Let H be a non-empty hesitant fuzzy subset of a S and $T \subseteq \mathcal{P}([0,1])$. Then the set $H_T := \{x \in S : H(x) \supseteq T\}$ is called the T-cut of H.

Theorem 3.10. *Let H be a non-empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the T-cut H_T of H is a left (respectively right) ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided it is non-empty if and only if H is a hesitant fuzzy left (resp., right) ideal of S.*

Proof. For every $T \subseteq \mathcal{P}([0,1])$, let H_T be a left ideal of S. We first show, H is a hesitant fuzzy subsemigroup of S. If possible $\exists\ x_0, y_0 \in S, \gamma_0 \in \Gamma$ such that $H(x_0\gamma_0y_0) \subset \{H(x_0) \cap H(y_0)\}$. Let $\{H(x_0) \cap H(y_0)\} = T_0$. Then $H(x_0) \supseteq T_0, H(y_0) \supseteq T_0$. Thus $x_0, y_0 \in H_{T_0}$ but $x_0\gamma_0y_0 \notin H_{T_0}$, a contradiction. Thus, $H(x_0\gamma_0y_0) \supseteq \{H(x_0) \cap H(y_0)\}$. Hence H is an hesitant fuzzy subsemigroup of

S. Again suppose that $\exists\ x_0, y_0 \in S, \gamma_0 \in \Gamma$ such that $H(x_0\gamma_0y_0) \subset H(y_0)$. Since $H(y_0) \subseteq \mathcal{P}([0,1])$. Let $H(y_0) := T_0 \subseteq \mathcal{P}([0,1])$. Thus, $y_0 \in H_{T_0}$ but $x_0\gamma_0y_0 \notin H_{T_0}$, a contradiction. Thus, $H(x_0\gamma_0y_0) \supseteq H(y_0)$. Moreover let $x_0, y_0 \in S$, be such that $x_0 \leq y_0$, then $H(x_0) \supseteq H(y_0)$. Indeed: since $H_T \neq \emptyset$, let $U = H(y_0)$ for any subset U of [0, 1]. Then $y_0 \in H(U)$. Since $H(U)$ is a left ideal of S, we have $x_0 \in H_U$. Therefore $H(x_0) \supseteq U = H(y_0)$. Hence H is a hesitant fuzzy left ideal of S.

Conversely, suppose that H is a hesitant fuzzy left ideal of S and $T \subseteq \mathcal{P}([0,1])$ such that H_T is non empty. Let $a, b \in H_T$ and $\gamma \in \Gamma$. Then $H(a) \supseteq T$ and $H(b) \supseteq T$ which implies $H(a) \cap H(b) \supseteq T$. Since H is a hesitant fuzzy left ideal, it is a hesitant fuzzy subsemigroup, and hence $H(a\gamma b) \supseteq H(a) \cap H(b) \supseteq T$. Consequently, $a\gamma b \in H_T$. Hence H_T is a subsemigroup of S. Now, let $a \in S; \gamma \in \Gamma$ and $b \in H_T$. Then $H(a\gamma b) \supseteq H(b) \supseteq T$ and so $a\gamma b \in H_T$. Furthermore, let $a, b \in S, a \in H_T$ such that $b \leq a$. Then $b \in H_T$. Indeed: since $a \in H_T, H(a) \supseteq T$ and H is a hesitant fuzzy left ideal of S, we have $H(b) \supseteq H(a) \supseteq T$, so $b \in H_T$. Hence H_T is a left ideal of S. Similarly we can prove for another case. □

Theorem 3.11. *Let H be a non-empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the T-cut H_T of H is an interior ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided it is non-empty if and only if H is a hesitant fuzzy interior ideal of S.*

Proof. For every $T \subseteq \mathcal{P}([0,1])$, let H_T be an interior ideal of S. Assume that $\exists\ a_0, x_0, y_0 \in S, \beta_0, \gamma_0 \in \Gamma$ such that $H(x_0\beta_0a_0\gamma_0y_0) \subset H(a_0)$. Since $H(a_0) \subseteq \mathcal{P}([0,1])$. Let $H(a_0) := T_0 \subseteq \mathcal{P}([0,1])$. Thus, $a_0 \in H_{T_0}$ but $x_0\beta_0a_0\gamma_0y_0 \notin H_{T_0}$, a contradiction.

Hence, $H(x_0\beta_0a_0\gamma_0y_0) \supseteq H(a_0)$.

Conversely, Assume that H is a hesitant fuzzy interior ideal of S and $T \subseteq \mathcal{P}([0,1])$ such that H_T is non empty. Let $a, b \in S; \beta, \gamma \in \Gamma$ and $x \in H_T$. Then $H(a\beta x\gamma b) \supseteq H(x) \supseteq T$ and so $a\beta x\gamma b \in H_T$.

The rest of the proof is a consequence of Theorem 3.10 □

Remark 3.12. Let H be a hesitant fuzzy ideal of a po-Γ-semigroup S and $T_1, T_2 \subseteq \mathcal{P}([0,1])$ such that $T_1 \subset T_2$, then $H_{T_1} \supseteq H_{T_2}$.

Definition 3.13. [27] A po-Γ-semigroup S is said to be regular if for each element $a \in S$, there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha s\beta a$.

Theorem 3.14. *Let H be a hesitant fuzzy set in a regular po-Γ-semigroup S then H is a hesitant fuzzy ideal of S if and only if H is a hesitant fuzzy interior ideal of S.*

Proof. Let H be a hesitant fuzzy ideal of S. For any x, y, $z \in S$ and $\alpha, \beta \in \Gamma$, we have $H(x\alpha y\beta z) = H\big((x\alpha(y\beta z)\big) \supseteq H(y\beta z) \supseteq Hy$. Hence H is a hesitant fuzzy interior ideal of S.

Conversely, let $a, b \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist elements $x \in S, \alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Since H is a hesitant fuzzy interior ideal of S, we have $H(a\gamma b) \supseteq H(a\alpha x\beta a\gamma b) \supseteq H_a$. So H is a hesitant fuzzy right ideal of S. Similarly, We can prove that H is a hesitant fuzzy left ideal of S. Hence H is a hesitant fuzzy ideal of S. $\square$

Theorem 3.15. *Let S be a regular po-Γ-semigroup and H be a hesitant fuzzy right ideal and G be a hesitant fuzzy left ideal S. Then $H\tilde{o}G = H \cap G$.*

Proof. Let H be a hesitant fuzzy right ideal and G be a hesitant fuzzy left ideal of S. Let $a \in S$. Suppose there exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$.

Then we have,

$$\begin{aligned}(H\tilde{o}G)(a) = \bigcup_{a \leq b\gamma c} \big\{H(b) \cap G(c)\big\} &\subseteq \bigcup_{a \leq b\gamma c} \big\{H(b\gamma c) \cap G(b\gamma c)\big\} \\ &\subseteq H(a) \cap G(a) = \big(H \cap G\big)(a) \ \forall\ a \in S.\end{aligned}$$

implies $H\tilde{o}G \subseteq H \cap G$. If there do not exist $b, c \in S$ such that $a \leq b\gamma c$. Then $(H\tilde{o}G)(a) = \emptyset \subseteq \big(H \cap G\big)(a)$. Thus $H\tilde{o}G \sqsubseteq H \cap G$.

Again, Let $a \in S$. Since S is regular, then there exists an element $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a := a\gamma a$ where $\gamma := \alpha x\beta \in \Gamma$. Therefore,

$$(H\tilde{o}G)(a) = \bigcup_{a \leq p\delta q} \big\{H(p) \cap G(q)\big\} \supseteq H(a) \cap G(a) = \big(H \cap G\big)(a)$$

Thus $H\tilde{o}G \sqsupseteq H \cap G$. Hence $H\tilde{o}G = H \cap G$. $\square$

In order to conclude the paper we obtain the following characterization of a simple po-Γ-semigroup by means of a hesitant fuzzy simple po-Γ-semigroup.

Definition 3.16 ([27])**.** A po-Γ-semigroup S is said to be simple if it does not contain any proper ideal. A po-Γ-semigroup S is said to be left (respectively right) simple if S has no proper left (respectively right) ideals.

Definition 3.17. A po-Γ-semigroup S is said to be hesitant fuzzy simple if every hesitant fuzzy ideal of S is a constant function. A po-Γ-semigroup S is called hesitant fuzzy left (respectively right) simple if every hesitant fuzzy left (respectively right) ideal of S is a constant function.

Theorem 3.18. *A po-Γ-semigroup S is simple if and only if it is hesitant fuzzy simple.*

Proof. Suppose that the po-Γ-semigroup S is simple. Let H be a hesitant fuzzy ideal of S and $x, y \in S$. Then, by Theorem 3.9, $H_{H(x)}$ and $H_{H(y)}$ are ideals of S. Since S is simple, $H_{H(x)} = S = H_{H(y)}$. Therefore, $x, y \in H_{H(x)}$ and $x, y \in H_{H(y)}$. In particular, $y \in H_{H(x)}$ and $x \in H_{H(y)}$, hence $H(y) \supseteq H(x)$ and $H(x) \supseteq H(y)$ implying $H(x) = H(y)$, for all $x, y \in S$. Hence H is a constant function. Consequently, S is a hesitant fuzzy simple.

Conversely, assume that S is a hesitant fuzzy simple. Let A be any ideal of S. Then its characteristic function H_A is a hesitant fuzzy ideal of S. Then H_A is a constant function. Let $a \in S$. Since A is non empty, therefore $H_A(a) = [0, 1]$ and so $a \in A$. Thus we obtain $S = A$. Hence S is simple. $\square$

Theorem 3.19. *For a left (resp., right) simple po-Γ-semigroup S, every hesitant fuzzy bi-ideal of S is a hesitant fuzzy right ideal (resp., hesitant fuzzy left ideal) of S.*

Proof. Let H be any hesitant fuzzy bi-ideal of S and $a, b \in S$. Then, $(S\Gamma a]$ is a left ideal of S and S is left simple po-Γ-semigroup, so $(S\Gamma a] = S$. So there exist $a_o \in S, \gamma \in \Gamma$ such that $b \leq a_o \gamma a$. Since H is hesitant fuzzy bi-ideal of S, we have for any $\alpha \in \Gamma$

$$H(a\alpha b) \supseteq H(a\alpha a_o \gamma a) \supseteq H(a) \cap H(a) = H(a).$$

Hence, H is a hesitant fuzzy right ideal of S. Similarly, we can prove the other case also. □

Theorem 3.20. *Let S be a left simple po-Γ-semigroup. Then every hesitant fuzzy left ideal of S is hesitant fuzzy left simple.*

Proof. Let H be any hesitant fuzzy left ideal of S and $a, b \in S$. Then, $(S\Gamma a]$, $(S\Gamma b]$ are left ideals of S and S is left simple po-Γ-semigroup, so $(S\Gamma a] = S = (S\Gamma b]$. So there exist $a_o, b_o \in S, \alpha, \beta \in \Gamma$ such that $b \leq a_o \alpha a, \quad a \leq b_o \beta b$. Since H is hesitant fuzzy left ideal of S, we have

$$H(b) \supseteq H(a_o \alpha a) \supseteq H(a) \text{ and } \mathrm{H(a) \supseteq H(b_o \beta b) \supseteq H(b)}.$$

Thus, $H(a) = H(b)$. Since a and b are any elements of S, which implies that H is a constant function. Hence H is a hesitant fuzzy left simple. □

Theorem 3.21. *Let S be a simple po-Γ-semigroup. Then every hesitant fuzzy interior ideal of S is a constant function.*

Proof. Let H be any hesitant fuzzy interior ideal of S and a and b any elements of S. Then, $(S\Gamma b\Gamma S]$ is an ideal of S and S is simple po-Γ-semigroup, so $(S\Gamma b\Gamma S] = S$. So, there exist elements a_o, b_o in S, $\alpha, \beta \in \Gamma$ such that $a \leq a_o \alpha b \beta b_o$. Since H is hesitant fuzzy interior ideal of S, we have

$$H(a) \supseteq H(a_o \alpha b \beta b_o) \supseteq H(b).$$

Similarly, we can see that $H(b) \supseteq H(a)$. Thus $H(a) = H(b)$. Since a and b are arbitrary elements of S, hence H is a constant function. □

Definition 3.22. A po-Γ-semigroup S is called left zero (resp., right zero), if $a \leq a\gamma b$ (resp., $\mathrm{b} \leq \mathrm{a}\gamma\mathrm{b}$), for all $a, b \in S, \gamma \in \Gamma$.

Theorem 3.23. *Let S be a left zero (resp., right zero) po-Γ-semigroup. Then every hesitant fuzzy left (resp., hesitant fuzzy right) ideal is a constant function.*

Proof. Let S be a left zero po-Γ-semigroup and H be a hesitant fuzzy left ideal of S. Let $a, b \in S$, then $a \leq a\gamma b$ and $b \leq b\gamma a$, for all $\gamma \in \Gamma$. Since H is a hesitant fuzzy left ideal of S, we have $H(a) \supseteq H(a\gamma b) \supseteq H(b)$ and $H(b) \supseteq H(b\gamma a) \supseteq H(a)$. Therefore $H(a) = H(b)$. Hence H is a constant function. □

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Accepted: 9.12.2017

Intuitionistic fuzzy rare α-continuity

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Abstract. In this paper, the concepts of intuitionistic fuzzy rare α-continuous function, intuitionistic fuzzy rarely continuous function, intuitionistic fuzzy rarely precontinuous function, intuitionistic fuzzy rarely semi-continuous function are introduced and studied in light of the concept of rare set in intuitionistic fuzzy setting. We present some basic properties.

Keywords: intuitionistic fuzzy rare set, intuitionistic fuzzy rarely α-continuous function, intuitionistic fuzzy rarely pre- continuous function, intuitionistic fuzzy almost α-continuous function, intuitionistic fuzzy weekly α-continuous function, intuitionistic fuzzy rarely semi continuous function.

1. Introduction

The study of fuzzy sets was initiated by Zadeh [20] in 1965. Thereafter the paper of Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In the course of time, Fuzzy Topology proved to be very beneficial in fixing many realistic problems.

Several mathematicians have tried almost all the pivotal concepts of General Topology for extension to the fuzzy settings. In 1981, Azad [1] gave fuzzy version of the concepts given by Levine ([12], [13]) and thus initiated the study of weak forms of several notions in fuzzy topological spaces.

Popa [17] introduced the notion of rare continuity as a generalization of weak continuity [12] which has been further investigated by Long and Herrington [14] and S. Jafari ([9], [10]). Noiri [15] introduced and investigated weakly α-continuity as a generalization of weak continuity. He also introduced and investigated almost α-continuity [16]. The concept of rare α-continuity was introduced by S. Jafari [11]. The concept of new weaker form of Popa's rare

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continuity via α-open sets [6] and rare s-precontinuity for multifunctions [7] are introduced by E. Ekici and S. Jafari. The concept of fuzzy rare α-continuity was introduced by R. Dhavaseelan and S. Jafari [5].

The purpose of the present paper is to introduce the concepts of intuitionistic fuzzy rare α-continuous function, intuitionistic fuzzy rarely continuous function, intuitionistic fuzzy rarely pre-continuous function, intuitionistic fuzzy rarely semi-continuous function in the light of the concept of rare set in a intuitionistic fuzzy setting.

Definition 1.1 ([4])**.** An intuitionistic fuzzy topology (IFT) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$;

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \tau$.

In this case the ordered pair (X, τ) or simply X is called an intuitionistic fuzzy topological space (IFTS) and each IFS in τ is called an intuitionistic fuzzy open set (IFOS). The complement $\overline{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 1.2 ([4])**.** Let A be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space X. Then:

$int(A) = \bigcup\{G \mid G$ is an IFOS in X and $G \subseteq A\}$ is called the intuitionistic fuzzy interior of A;

$cl(A) = \bigcap\{G \mid G$ is an IFCS in X and $G \supseteq A\}$ is called the intuitionistic fuzzy closure of A.

Definition 1.3 ([18])**.** Let X be a nonempty set. If $r \in I_0$,$s \in I_1$ are fixed real numbers such that $r + s \leq 1$, then the intuitionistic fuzzy set $x_{r,s}$ is called an intuitionistic fuzzy point(briefly IFP)in X given by

$$x_{r,s}(x_p) = \begin{cases} (r, s), & \text{if } x = x_p \\ (0, 1), & \text{if } x \neq x_p, \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,s}$, where r denotes the degree of membership value and s is the degree of non-membership value of $x_{r,s}$.

Definition 1.4 ([2, 8])**.** An intuitionistic fuzzy set A in an intuitionistic fuzzy topological space (X, T) is called

(1) an intuitionistic fuzzy semiopen set (briefly IFSOS) if $A \subseteq cl(int(A))$;

(2) an intuitionistic fuzzy α-open set(briefly IFαOS) if $A \subseteq int(cl(int(A)))$;

(3) an intuitionistic fuzzy preopen set (briefly IFPOS) if $A \subseteq int(cl(A))$;

(4) an intuitionistic fuzzy regular open set (briefly IFROS) if $A = int(cl(A))$.

An IFS A is called an intuitionistic fuzzy semiclosed set, intuitionistic fuzzy α-closed set, intuitionistic fuzzy preclosed set, and intuitionistic fuzzy regular closed set, respectively (briefly IFSCS, IFαCS, IFPCS, and IFRCS resp.), if the complement of A is an IFSOS, IFαOS, IFPOS, and IFROS, respectively.

Definition 1.5 ([19])**.** Let A be an IFS of an IFTS (X, T). The α-closure of A (briefly $\alpha cl(A)$) is defined as $\alpha cl(A) = \bigcap\{K|$ K is an $IF\alpha CS$ in X and $A \subseteq K\}$.

Definition 1.6 ([19])**.** Let A be an IFS of an IFTS (X, T). The α-interior of A (briefly $\alpha int(A)$) is defined as $\alpha int(A) = \bigcup\{K|$ K is an $IF\alpha OS$ in X and $K \subseteq A\}$.

2. Main results

Definition 2.1. An intuitionistic fuzzy set R is called intuitionistic fuzzy rare set if $IFint(R) = 0_\sim$.

Definition 2.2. An intuitionistic fuzzy set R is called intuitionistic fuzzy nowhere dense set if $IFint(IFcl(R)) = 0_\sim$.

Definition 2.3. Let (X, T) and (Y, S) be two intuitionistic fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called:

(i) intuitionistic fuzzy α-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G in Y containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U in X such that $f(U) \leq G$.

(ii) intuitionistic fuzzy almost α-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U such that $f(U) \leq IFint(IFcl(G))$.

(iii) intuitionistic fuzzy weakly α-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U such that $f(U) \leq IFcl(G)$.

Definition 2.4. Let (X, T) and (Y, S) be two intuitionistic fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called:

(i) intuitionistic fuzzy rarely α-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exist an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ and intuitionistic fuzzy α open set U in (X, T) such that $f(U) \leq G \cup R$.

(ii) intuitionistic fuzzy rarely continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exist an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ and intuitionistic fuzzy open set U in (X, T) such that $f(U) \leq G \cup R$.

(iii) intuitionistic fuzzy rarely pre-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exist an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ and intuitionistic fuzzy preopen set U in (X, T) such that $f(U) \leq G \cup R$.

(iv) intuitionistic fuzzy rarely semi-continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exist an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ and intuitionistic fuzzy semiopen set U in (X, T) such that $f(U) \leq G \cup R$.

Example 2.1. Let $X = \{a, b, c\}$. Define the intuitionistic fuzzy sets A, B and C as follows: $A = \langle x, (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}), (\frac{a}{1}, \frac{b}{1}, \frac{c}{0})\rangle$, $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{1})\rangle$ and $C = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{1})\rangle$. Then $T = \{0_\sim, 1_\sim, A\}$ and $S = \{0_\sim, 1_\sim, A, B, A \cup B\}$ are intuitionistic fuzzy topologies on X. So (X, T) and (X, S) are intuitionistic fuzzy topological spaces. Define $f : (X, T) \to (X, S)$ as an identity function. Clearly f is intuitionistic fuzzy rarely α-continuous function.

Proposition 2.1. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. For a function $f : (X, T) \to (Y, S)$, the following statements are equivalent:

(i) The function f is intuitionistic fuzzy rarely α-continuous function at $x_{r,s}$ in (X, T).

(ii) For each intuitionistic fuzzy open set G containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U in (X, T) such that $IFint(f(U) \cap \overline{G}) = 0_\sim$.

(iii) For each intuitionistic fuzzy open set G containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U in (X, T) such that $IFint(f(U)) \leq IFcl(G)$.

(iv) For each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exists an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ such that $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R))$.

(v) For each intuitionistic fuzzy open set G in (Y, S) containing $f(x_{r,s})$, there exists an intuitionistic fuzzy rare set R with $IFcl(G) \cap R = 0_\sim$ such that $x_{r,s} \in IFint_\alpha(f^{-1}(IFcl(G) \cup R))$.

(vi) For each intuitionistic fuzzy regular open set G in (Y, S) containing $f(x_{r,s})$, there exists an intuitionistic fuzzy rare set R with $IFcl(G) \cap R = 0_\sim$ such that $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R))$.

Proof. (i)$\Rightarrow$ (ii). Let G be an intuitionistic fuzzy open set in (Y, S) containing $f(x_{r,s})$. By $f(x_{r,s}) \in G \leq IFint(IFcl(G))$ and $IFint(IFcl(G))$ containing $f(x_{r,s})$, there exist an intuitionistic fuzzy rare set R with $IFint(IFcl(G)) \cap IFcl(R) = 0_\sim$ and an intuitionistic fuzzy α-open set U in (X, T) containing $x_{r,s}$ such that $f(U) \leq IFint(IFcl(G)) \cup R$. We have $IFint(f(U) \cap \overline{G}) = IFint(f(U)) \cap IFint(\overline{G}) \leq IFint(IFcl(G) \cup R) \cap \overline{(IFcl(G))} \leq IFcl(G) \cup IFint(R) \cap \overline{(IFcl(G))} = 0_\sim$.

(ii)$\Rightarrow$(iii). It is straightforward.

(iii)$\Rightarrow$ (i). Let G be an intuitionistic fuzzy open set in (Y, S) containing $f(x_{r,s})$. Then by (iii), there exists an intuitionistic fuzzy α-open set U containing $x_{r,s}$ such that $IFint(f(U) \leq IFcl(G)$. We have $f(U) = (f(U) \cap \overline{(IFint(f(U)))}) \cup IFint(f(U)) < (f(U) \cap \overline{(IFint(f(U)))}) \cup IFcl(G) = (f(U) \cap \overline{(IFint(f(U)))}) \cup G \cup (IFcl(G) \cap \overline{G}) = (f(U) \cap \overline{(IFint(f(U)))} \cap \overline{G}) \cup G \cup (IFcl(G) \cap \overline{G})$. Set $R_1 = f(U) \cap \overline{(IFint(f(U)))} \cap \overline{G}$ and $R_2 = IFcl(G) \cap \overline{G}$. Then R_1 and R_2 are intuitionistic fuzzy rare sets. More $R = R_1 \cup R_2$ is an intuitionistic fuzzy set such that $IFcl(R) \cap G = 0_\sim$ and $f(U) \leq G \cup R$. This show that f is intuitionistic fuzzy rarely α-continuous function.

(i)$\Rightarrow$ (iv). Suppose that G is an intuitionistic fuzzy open set in (Y, S) containing $f(x_{r,s})$. Then there exist an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ and U which is an intuitionistic fuzzy α open set in (X, T) containing $x_{r,s}$ such that $f(U) \leq G \cup R$. It follows that $x_{r,s} \in U \leq f^{-1}(G \cup R)$. This implies that $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R))$.

(iv)$\Rightarrow$ (v). Suppose that G is an intuitionistic fuzzy open set in (Y, S) containing $f(x_{r,s})$. Then there exists an intuitionistic fuzzy rare set R with $G \cap IFcl(R) = 0_\sim$ such that $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R))$. Since $G \cap IFcl(R) = 0_\sim$,$R \leq \overline{G}$, where $\overline{G} = \overline{(IFcl(G))} \cup (IFcl(G) \cap \overline{G})$. Put we have $R \leq R \cup \overline{(IFcl(G))} \cup (IFcl(G) \cap \overline{G})$. Now, $R_1 = R \cap \overline{(IFcl(G))}$. It follows that R_1 is an intuitionistic fuzzy rare set with $IFcl(G) \cap R_1 = 0_\sim$. Therefore $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R)) \leq IFint_\alpha(f^{-1}(G \cup R_1))$.

(v)$\Rightarrow$ (vi). Assume that G is an intuitionistic fuzzy regular open set in (Y, S) containing $f(x_{r,s})$. Then there exists an intuitionistic fuzzy rare set R with $IFcl(G) \cap R = 0_\sim$, such that $x_{r,s} \in IFint_\alpha(f^{-1}(IFcl(G) \cup R))$. Now $R_1 = R \cup (IFcl(G) \cup \overline{G})$. It follows that R_1 is an intuitionistic fuzzy rare set and $(G \cap IFcl(R_1)) = 0_\sim$. Hence $x_{r,s} \in IFint_\alpha(f^{-1}(IFcl(G) \cup R)) = IFint_\alpha(f^{-1}(G \cup (IFcl(G) \cap \overline{G})) \cup R) = IFint_\alpha(f^{-1}(G \cup R_1))$. Therefore $x_{r,s} \in IFint_\alpha(f^{-1}(G \cup R_1))$.

(vi)$\Rightarrow$ (ii). Let G be an intuitionistic fuzzy open set in (Y, S) containing $f(x_{r,s})$. By $f(x_{r,s}) \in G \leq IFint(IFcl(G))$ and the fact that $IFint(IFcl(G))$ is an intuitionistic fuzzy regular open in (Y, S), there exists an intuitionistic fuzzy rare set R and $IFint(IFcl(G)) \cap IFcl(R) = 0_\sim$ such that $x_{r,s} \in IFint_\alpha(f^{-1}(IFint(IFcl(G)) \cup R)$.

Let $U = IFint_\alpha(f^{-1}(IFint(IFcl(G)) \cup R)$. Hence U is an intuitionistic fuzzy α-open set in (X,T) containing $x_{r,s}$ and therefore $f(U) \leq IFint(IFcl(G)) \cup R$. Hence, we have $IFint(f(U) \cap \overline{G}) = 0_\sim$. □

Proposition 2.2. Let (X,T) and (Y,S) be any two intuitionistic fuzzy topological space. Then a function $f : (X,T) \to (Y,S)$ is an intuitionistic fuzzy rarely α-continuous function if and only if $f^{-1}(G) \leq IFint_\alpha(f^{-1}(G \cup R))$ for every intuitionistic fuzzy open set G in (Y,S), where R is an intuitionistic fuzzy rare set with $IFcl(R) \cap G = 0_\sim$.

Proof. Suppose that G is an intuitionistic fuzzy rarely α-open set in (Y,S) containing $f(x_{r,s})$. Then $G \cap IFcl(R) = 0_\sim$ and U be an intuitionistic fuzzy α-open set in (X,T) containing $x_{r,s}$ such that $f(U) \leq G \cup R$. It follows that $x_{r,s} \in U \leq f^{-1}(G \cup R)$. This implies that $f^{-1}(G) \leq IFint_\alpha(f^{-1}(G \cup R))$. □

Definition 2.5. A function $f : (X,T) \to (Y,S)$ is intuitionistic fuzzy $I\alpha$-continuous function at $x_{r,s}$ in (X,T) if for each intuitionistic fuzzy open set G in (Y,S) containing $f(x_{r,s})$, there exists an intuitionistic fuzzy α-open set U containing $x_{r,s}$ such that $IFint(f(U)) \leq G$.

If f has this property at each intuitionistic fuzzy point $x_{r,s}$ in (X,T), then we say that f is intuitionistic fuzzy $I\alpha$-continuous function on (X,T).

Example 2.2. Let $X = \{a,b,c\}$. Define the intuitionistic fuzzy sets A and B as follows: $A = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{1}, \frac{b}{0}, \frac{c}{1})\rangle$ and $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{1}, \frac{c}{1})\rangle$. Then $T = \{0_\sim, 1_\sim, A\}$ and $S = \{0_\sim, 1_\sim, B\}$ be intuitionistic fuzzy topologies on X. Suppose (X,T) and (X,S) are intuitionistic fuzzy topological spaces. Let $f : (X,T) \to (X,S)$ be defined by $f(a) = f(b) = b$ and $f(c) = c$. It is obvious that f is an intuitionistic fuzzy $I\alpha$-continuous function.

Proposition 2.3. Let (Y,S) be an intuitionistic fuzzy regular space [2]. Then the function $f : (X,T) \to (Y,S)$ is an intuitionistic fuzzy $I\alpha$-continuous function on X if and only if f is an intuitionistic fuzzy rarely α-continuous function on X.

Proof. $\Rightarrow$ It is obvious.
$\Leftarrow$ Let f be an intuitionistic fuzzy rarely α-continuous function on (X,T). Suppose that $f(x_{r,s}) \in G$, where G is an intuitionistic fuzzy open set in (Y,S) and $x_{r,s}$ an intuitionistic fuzzy point in X. By the intuitionistic fuzzy regularity of (Y,S), there exists an intuitionistic fuzzy open set G_1 in (Y,S) containing $f(x_{r,s})$ and $IFcl(G_1) \leq G$. Since f is intuitionistic fuzzy rarely α-continuous function, then there exists an intuitionistic fuzzy α-open set U such that $IFint(f(U)) \leq IFcl(G_1)$. This implies that $IFint(f(U)) \leq G$, which means that f is intuitionistic fuzzy $I\alpha$ continuous function on X. □

Definition 2.6. A function $f : (X, T) \to (Y, S)$ is called intuitionistic fuzzy pre-α-open if for every intuitionistic fuzzy α-open set U in X such that $f(U)$ is an intuitionistic fuzzy α-open in Y.

Proposition 2.4. If a function $f : (X, T) \to (Y, S)$ is an intuitionistic fuzzy pre-α-open and an intuitionistic fuzzy rarely α-continuous function then f is intuitionistic fuzzy almost α-continuous function.

Proof. Suppose that $x_{r,s}$ is an intuitionistic fuzzy point in X and G an intuitionistic fuzzy open set in Y containing $f(x_{r,s})$. Since f is intuitionistic fuzzy rarely α-continuous function at $x_{r,s}$, then there exists an intuitionistic fuzzy α-open set U in X, such that $IFint(f(U)) \subset IFcl(G)$. Since f is intuitionistic fuzzy pre-α-open, we have $f(U) \subset IFint(IFcl(IFint(f(U)))) \subset IFint(IFcl(G))$. Hence f is intuitionistic fuzzy almost α-continuous function. $\square$

For a mapping $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Proposition 2.5. Let $f : (X, T) \to (Y, S)$ be any function. If the $g : X \to X \times Y$ of f is intuitionistic fuzzy rarely α-continuous function then f is also intuitionistic fuzzy rarely α-continuous function.

Proof. Suppose that $x_{r,s}$ is an intuitionistic fuzzy point in X and an intuitionistic fuzzy open set W in Y containing $g(x_{r,s})$. It follows that there exists intuitionistic fuzzy open sets 1_X and V in X and Y respectively such that $(x_{r,s}, f(x_{r,s})) \in 1_X \times V \subset W$. Since f is intuitionistic fuzzy rarely α-continuous function, there exists an intuitionistic fuzzy α-open set G such that $IFint(f(G)) \subset IFcl(V)$. Let $E = 1_X \cap G$. It follows that E is an intuitionistic fuzzy α-open set in X, and we have $IFint(g(E)) \subset IFint(1_X \times f(G)) \subset 1_X \times IFcl(V) \subset IFcl(W)$. Therefore g is an intuitionistic fuzzy rarely α-continuous function. $\square$

3. Conclusion

In this paper, the notion of intuitionistic fuzzy rare α-continuity is introduced and studied.

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Accepted: 7.12.2017

En-semi prime subacts over monoids with zero

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Abstract. Let S be a monoids with zero and A_s be a right S-act . In this paper,we introduce the notion of En- semi prime subact of an S-act A_s which is define as : A proper subact B of an S-act A_s is said to be En- semi prime subact, if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s, and $a \in A_s$, then $f(a)S \subseteq B$. An S-act A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s. Also, we study and gives some related concepts of this notion like: semi prime subact, En-prime subact, En- pure subact and En-radical of subact.

Keywords: En-semi prime subact, En-Prime subact, En-pure subact and En-radadical of subact.

1. Introduction

Firstly we begin with some preliminary definitions and notions: "Recall that a nonempty set A is called a right S-act where S is monoid that is semigroup with identity element 1, if there exists a mapping $\phi : A \times S \to A$ define as $(a, s) \to as$ and satisfying $a.1 = a$ and $a(st) = (as)t$, for all $a \in A$and $s, t \in S$. We call A a right S-act or right act over S and write A_s" [1]. "Similarly, we define a left S-acts A and write ${}_sA$. If S is a commutative monoid, then every left S-act is right S-act. A non empty subset B of a right S-act A_s is called subact of A_s and written by $B \leq A_s$, if $bs \in B$ for all $b \in B$ and $s \in S$. An element $\theta \in A_s$ is called a zero of A_s or fixed element if $\theta_s = \theta$ for all $s \in S$, i.e. $\{\theta\}$ is a one-element subact [1]. In this paper θ is a unique fixed element of all S-act A_s. A nonempty subset I is called an ideal or two sided ideal of S (left and right) if $SI \subseteq I$ and $IS \subseteq I$. A mapping $f : A_s \to B_s$, where A_s and B_s are two right S-acts is called S- homomorphism if $f(as) = f(a)s$, for all $a \in A_s$ and $s \in S$. The set of all S-homomorphism from A in to B denoted by $Hom(A_s, B_s)$ or $Hom_s(A, B)$. An S-homomorphism $f : A_s \to A_s$ is called an endomorphism of A_s. The composition $g \circ f$ of homomorphism $f : A_s \to B_s$ and $g : B_s \to C_s$ of a right S-acts is a homomorphism of a right act,i.e. $g \circ f \in Hom(A_s, B_s)$" [1].

Now, A. A. Estaji and S. Tajnia in [3] introduce the concept of semi prime subact which is define as: "A proper subact B of an Sact A_s is called semi-prime, if whenever $as^k \in B$ for some $s \in S$, $a \in A_s$ and $k \in N$ implies that

$as \in B$" [2]. Shireen in [3] introduce the notion of En-prime subact, where: "A proper subact B of an Sact A_s is called En- prime subact of A_s if for any endomorphism f of A_s and $a \in A_s$ with $f(a)S \subseteq B$ implies that either $a \in B$ or $f(As) \subseteq B$. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact" [3].

In this paper, we introduce the concept of En- semi prime subact as a generalization of En-prime subact, where a proper subact B of an Sact A_s is said to be En- semi prime subact if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an Sact A_s and $a \in A_s$, then $f(a)S \subseteq B$. An Sact A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s. In section one, we study some basic properties of this notion and the relationships between this notion and some other related concepts like: semi prime, En-prime and maximal subacts.

2. En-semi prime subacts and some related concept

In this section, we introduce the concept of En- semi prime subacts and give some characterizations for this notion.

Definition 2.1. *A proper subact B of an S-act A_s is said to be En- semi prime subact, if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s and $a \in A_s$, then $f(a) \subseteq B$. An S-act A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s.*

Proposition 2.2. *Let B be a proper subact of an S-act A_s, then B is En- semi prime subact of A_s if and only if $f^k(a)S \subseteq B$ for some endomorphism f of an S-act A_s,$a \in A_s$ and for $k \geq 2$ then $f(a)S \subseteq B$.*

Proof. The proof by induction on the positive integer k. If $k = 2$, then the proposition is true by the definition of En- semi prime subact. Assume that our proposition is true for $k-1$ which means if $f^{k-1}(a)S \subseteq B$, then $f(a)S \subseteq B$. Now, suppose that $f^k(a)S \subseteq B$ and thus $f^2(f^{k-2}(a))S \subseteq B$ which implies that $f(f^{k-2}(a))S \subseteq B$. Therefore $f(a)S \subseteq B$ by our induction. □

Recall that a proper subact B of an Sact A_s is called semi-prime, if whenever $as^k \in B$ for some $s \in S$, $a \in A_s$ and $k \in N$ implies that $as \in B$ [2].

Proposition 2.3. *Let B be a proper subact of an S-act A_s. If B is En- semi prime subact of A_s, then B is semi prime subact.*

Proof. Suppose that $as^k \in B$, where $a \in A_s$, $s \in S$ and $k \in N$. Define $f : A_s \to A_s$ by $f(c) = cs$ for all $c \in A_s$ and $s \in S$. Now, $f(a) = as$ and so $f^k(a) = as^k \in B$. Hence $f^k(a)S \subseteq B$, but B is En-semi prime subact of A_s then $f(a)S \subseteq B$. Therefore $as \in B$. □

Remark 2.4. The converse of previous proposition is not true in general and we can show that by the following example: Let $Z \oplus Z$ be an $(Z, .)$act with multiplication by integers as operation and let $6Z \oplus Z$ be a subact of $Z \oplus Z_{(z,.)}$ which is semi prime subact of $Z \oplus Z_{(z,.)}$. We define a function as: $f : Z \oplus Z \to Z \oplus Z$ by $f(x, y) = (y, x)$, for all $x, y \in Z$. Now, $f^2(0, 3) = (0, 3) \in 6Z \oplus Z$ and hence $f^2(0, 3)Z \subseteq 6Z \oplus Z$. But $f(0, 3) = (3, 0) \notin 6Z \oplus Z$ which implies that $f(0, 3)Z \nsubseteq 6Z \oplus Z$. Thus $6Z \oplus Z$ is not En- semi prime subact of $Z \oplus Z_{(z,.)}$.

Recall that a proper subact B of an Sact A_s is called En- prime subact of A_s if for any endomorphism f of A_s and $a \in A_s$ with $f(a)S \subseteq B$ implies that either $a \in B$ or $f(A_s) \subseteq B$. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact [3].

Proposition 2.5. *Every En-prime subact of an Sact A_s is En-semi prime subact of A_s.*

Proof. Let B be a proper subact of an S-act A_s and suppose that $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s and $a \in A_s$. We have to prove that $f(a)S \subseteq B$. Now, $f^2(a)S = f(f(a))S \subseteq B$, but B is En- prime subact of A_s, then either $f(a) \in B$ or $f(A_s) \subseteq B$. That follows in any case $f(a) \in B$. Therefore $f(a)S \subseteq B$. □

Proposition 2.6. *Let A_s be an S-act and B a proper subact of A_s. If $B = \cap P_i$, where P_i is En- prime subact of A_s, then B is En- semi prime subact of A_s.*

Proof. Let $f^2(a)S \subseteq B$ for some endomorphism f of an Sact A_s and $a \in A_s$, then $f^2(a)S \subseteq P_i$ for each i. But P_i is En- prime subact of A_s, then by prop.(2.5) P_i is En- semi prime subact of A_s. Thus $f(a)S \subseteq P_i$ for each i which implies that $f(a)S \subseteq \cap P_i = B$. Therefore B is En- semi prime subact of A_s. □

Corollary 2.7. *The intersection of En-semi prime subacts of an S-act A_s is En-semi prime subact of A_s*

Proposition 2.8. *The union of any two En- semi prime subacts of A_s is En-semi prime subact of A_s.*

Proof. Let B_1 and B_2 be any two En- semi prime subacts of an Sact A_s. Suppose that for some endomorphism f of A_s and $a \in A_s$ we have $f^2(a)S \subseteq B_1 \cup B_2$. Now, since we have $f^2(a)S \subseteq B_1 \cup B_2$, then either $f^2(a)S \subseteq B_1$ or $f^2(a)S \subseteq B_2$. But B_1 and B_2 are En- semi prime subacts of A_s then either $f(a)S \subseteq B_1$ or $f(a)S \subseteq B_2$. Thus $f(a)S \subseteq B_1 \cup B_2$. □

Recall that a subact B of an S-act A_s is called fully invariant subact if $f(B) \subseteq B$ for every endomorphism f of A_s and A_s is called duo act if every subact of A_s is fully invariant .

Proposition 2.9. *Every maximal subact of duo act is En- semi prime subact.*

Proof. Let B be a maximal subact of duo act A_s . Then by [[3], corl.(2.7)] we have B is Enprime subact of A_s and by prop.(2.5) we get B is Ensemi prime subact of A_s. □

Definition 2.10. *A subact B of an S-act A_s is called En- pure subact if $f(A_s) \cap B = f(B)$ for any endomorphism f of A_s.*

Example 2.11. The one element subact zero θ and the act itself are En-pure subacts.

Proposition 2.12. *Let A_s be an S-act A_s such that every subact of A_s is En-pure subact of A_s, then each proper subact of A_s is En- semi prime subact of A_s.*

Proof. Let B be a proper subact of an Sact A_s and let $f^2(a)S \subseteq B$, where f an endomorphism of A_s and $a \in A_s$. Now, $f(a) \in f(A_s) \cap f(a)S = f(f(a)S) = f^2(a)S \subseteq B$. Hence $f(a) \in B$ and therefore $f(a)S \subseteq B$. □

Definition 2.13. *The intersection of all En-prime subacts of an S-act A_s containing a subact B of A_s is said to be En- radical of B and denoted by En-rad (B) . If B is not contained in any En- prime subact of A_s, then we put En-rad(B) = A_s.*

Recall that a proper subact B of an S-act A_s is said to be prime subact of A_s, if for every $s \in S$ and $a \in A_s$, $as \in B$ implies that $a \in B$ or $s \in (B : A_s)$ [2].
Recall that for a subact B of an Sact A_s, $rad_A(B)$ is the intersection of all prime subacts of A_s containing B and $rad(B) = B$, if B is not contained in any prime subact of A_s [2].

Proposition 2.14. *If B is a subact of an S-act A_s then:*

1. $B \subseteq En - rad(B)$.
2. $rad(B) \subseteq En - rad(B)$.

Proof.
1. It is clear.
2. Let C be En-prime subact of an S-act A_s containing B. Then By [**??**, prop.(2.3)] C is a prime subact of A_s, hence $rad(B) \subseteq C$. Also, $rad(B) \subseteq \cap C$ for all En- prime subact C containing B. Therefore $rad(B) \subseteq En - rad(B)$. □

Proposition 2.15. *Let B be a subact of an S-act A_s . If En-rad$(B) \neq A_s$, then En-rad(B) is En- semi prime subact of A_s.*

Proof. Let $f^2(a)S \subseteq En - rad(B)$, where f an endomorphism of A_s and $a \in A_s$. Now, we have $f^2(a)S \subseteq \cap P_i$, where P_i is En- prime subact of A_s containing B. Then $f^2(a)S \subseteq P_i$ for all i which implies that $f(a)S \subseteq P_i$ for all i. Therefore $f(a)S \subseteq \cap P_i. = En - rad(B)$. □

Definition 2.16. *Let B be a subact of an S-act A_s, then we define: En(B)={ $f(a)$, where f any endomorphism of A_s and $a \in A_s$ such that $f^k(a) \in B$ for some $k \in \mathbb{N}$ }.*

Recall that let B be a subact of an S-act A_s, then $E(B) = \{as : s \in S$ and $as^k \in B$ for some $k \in \mathbb{N}\}$ [2].

Proposition 2.17. *Let B be a subact of an S-act A_s, then:*

1. *$E(B) \subseteq En(B)$ and thus $B \subseteq En(B)$.*
2. *$En(B) \subseteq P$ for all En-prime subact P containing B and thus $En(B) \subseteq En - rad(B)$.*

Proof. 1. Let $x \in E(B)$, then $x = as$ where $a \in A_s$ and $s \in S$ and there exists a positive integer k such that $as^k \in B$. Define $f : A_s \to A_s$ by $f(b) = bs$ for all $b \in A_s$ and $s \in S$. Now, $x = f(a)$ and $f^k(a) = as^k \in B$ and hence $x \in En(B)$.

2. Suppose that $x \in En-(B)$, then there exists an endomorphism of A_s and $a \in A_s$ such that $x = f(a)$ and for some positive integer k we have $f^k(a) \in B$ and thus $f^k(a)S = f^{k-1}(f(a)) \subseteq B \subseteq P$ But P is En- prime subact of an S-act A_s, then either $f(a) \in P$ or $f^{k-1}(A_s) \subseteq P$, which follows that $x = f(a) \in P$. Consequently, $En(B) \subseteq En - rad(B)$. □

Proposition 2.18. *Let B be a subact of an S-act A_s, then B is En-semi prime subact of A_s if and only if $En(B) = B$.*

Proof. Let $x \in En(B)$, then there exists an endomorphism f of A_s and $a \in A_s$ such that $x = f(a)$ and for some positive integer k we have $f^k(a) \in B$ and thus $f^k(a)S \subseteq B$. But B is En-semi prime subact of A_s, then $f(a)S \subseteq B$ wich implies that $x = f(a) \in B$. Therefore $En(B) \subseteq B$ and the result follows from prop.(2.17)(1). Conversely, suppose that $En(B) = B$ and let $f^2(a)S \subseteq B$ for some endomorphism f of A_s and $a \in A_s$, then $f^2(a) \in B$ and thus $f(a) \in En - (B) = B$. Hence $f(a)S \subseteq B$. □

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Accepted: 9.12.2017

Join hesitant fuzzy filters of residuated lattices

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Abstract. The notions of join hesitant fuzzy filters and join hesitant fuzzy G-filters in residuated lattices are introduced, and related properties are investigated. Characterizations of join hesitant fuzzy filters and join hesitant fuzzy G-filters are discussed. Conditions for a join hesitant fuzzy filter to be a join hesitant fuzzy G-filter are provided, and a new join hesitant fuzzy filter is construct by the given join hesitant fuzzy filter.
Keywords: Residuated lattice, join hesitant fuzzy filter, join hesitant fuzzy G-filter.

1. Introduction

Hesiatnt fuzzy sets, as another extension of fuzzy sets, have been proposed in [16]. The motivation for introducing hesitant fuzzy sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values.

As a non-classical logic system, residuated lattices are a formal and useful tool for computer science to deal with uncertain and fuzzy information. In [20], Zhu and Xu discussed filter theory of residuated lattices. Moreover, Jun et al. applied the notion of hesitant fuzzy sets to MTL-algebras, BCK/BCI-algebras, EQ-algebras and semigroups (see [4], [5], [6] and [7]). Also, Muhiuddin et al. applied the notion of hesitant fuzzy sets to residuated lattices, lattice implication algebras and BCK/BCI-algebras (see [8], [9], [10], [11], [12], [13] and [14]).

In this paper, we introduce join hesitant fuzzy filters and join hesitant fuzzy G-filters in residuated lattices, and investigate their properties. We consider characterizations of join hesitant fuzzy filters and join hesitant fuzzy G-filters. We provide conditions for a join hesitant fuzzy filter to be a join hesitant fuzzy

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G-filter. Given a join hesitant fuzzy filter, we construct a new join hesitant fuzzy filter.

2. Preliminaries

We display well-known results on residuated lattices and hesitant fuzzy sets. We refer the reader to [1, 2, 3, 15, 16, 17, 18, 19] for further information regarding residuated lattices and hesitant fuzzy sets.

A *residuated lattice* is an algebra $\mathcal{L}{:=}(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.

(2) $(L, \otimes, 1)$ is a commutative monoid.

(3) $\otimes$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \otimes y \leq z)\,.$$

In a residuated lattice $\mathcal{L}$, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and x' will be reserved for $x \rightarrow 0$, and $x'' = (x')'$, etc. for all $x \in L$.

In a residuated lattice L, the following properties are valid.

$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1. \tag{2.1}$$

$$0 \rightarrow x = 1,\; x \rightarrow (y \rightarrow x) = 1. \tag{2.2}$$

$$x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.3}$$

$$x \leq y \;\Rightarrow\; z \rightarrow x \leq z \rightarrow y,\; y \rightarrow z \leq x \rightarrow z. \tag{2.4}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y). \tag{2.5}$$

$$z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \tag{2.6}$$

A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a *filter* of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \otimes y \in F)\,. \tag{2.7}$$

$$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,. \tag{2.8}$$

A nonempty subset F of $\mathcal{L}$ is called a *G-filter* of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x, y \in L)\,((x \otimes x) \rightarrow y \in F \;\Rightarrow\; x \rightarrow y \in F)\,. \tag{2.9}$$

Proposition 2.1 ([15])**.** *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.10}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \;\Rightarrow\; y \in F)\,. \tag{2.11}$$

Torra [16] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain.

Definition 2.2 ([16])**.** Let L be a reference set. Then we define *hesitant fuzzy set* on L in terms of a function $\mathcal{H}$ that when applied to L returns a subset of $[0,1]$.

For a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ and a subset τ of $[0,1]$, the *τ-exclusive set* of $\mathcal{H}$ is denoted by $e(\mathcal{H};\tau)$, and is defined to be the set

$$e(\mathcal{H};\tau) := \{x \in L \mid \mathcal{H}(x) \subseteq \tau\}\,.$$

3. Join hesitant fuzzy filters

In what follows, let $\mathcal{L}$ denote a residuated lattice unless otherwise specified, and we take L as a reference set.

Definition 3.1. A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is called a *join hesitant fuzzy filter* of $\mathcal{L}$ if it satisfies:

$$(\forall x, y \in L)\,(x \le y \;\Rightarrow\; x\mathcal{H} \supseteq y\mathcal{H})\,, \tag{3.1}$$

$$(\forall x, y \in L)\,(x\mathcal{H} \cup y\mathcal{H} \supseteq (x \otimes y)\mathcal{H})\,. \tag{3.2}$$

Proposition 3.2. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies:*

$$(\forall x \in L)\,(x\mathcal{H} \supseteq 1\mathcal{H})\,. \tag{3.3}$$

$$(\forall x, y \in L)\,(x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H})\,. \tag{3.4}$$

Proof. Let $x, y \in L$. Since $x \le 1$, we have $x\mathcal{H} \supseteq 1\mathcal{H}$ by (3.1). Since $x \otimes (x \to y) \le y$, it follows from (3.2) and (3.1) that

$$x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \otimes (x \to y))\mathcal{H} \supseteq y\mathcal{H}.$$

This completes the proof. □

Lemma 3.3. *If a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ satisfies two conditions* (3.3) *and* (3.4)*, then*

$$(\forall x, y, z \in L)\,(x \le y \to z \;\Rightarrow\; x\mathcal{H} \cup y\mathcal{H} \supseteq z\mathcal{H})\,, \tag{3.5}$$

$$(\forall x, y, z \in L)\,(x \otimes y \le z \;\Rightarrow\; x\mathcal{H} \cup y\mathcal{H} \supseteq z\mathcal{H})\,. \tag{3.6}$$

Proof. Assume that $x \leq y \to z$ for all $x, y, z \in L$. Then $x \to (y \to z) = 1$, and so

$$\begin{aligned} x\mathcal{H} \cup y\mathcal{H} &= (x\mathcal{H} \cup 1\mathcal{H}) \cup y\mathcal{H} \\ &= (x\mathcal{H} \cup (x \to (y \to z))\mathcal{H}) \cup y\mathcal{H} \\ &\supseteq y\mathcal{H} \cup (y \to z)\mathcal{H} \supseteq z\mathcal{H}. \end{aligned}$$

Since $x \leq y \to z \;\Leftrightarrow\; x \otimes y \leq z$, we know that (3.5) induces (3.6). □

We consider characterizations of join hesitant fuzzy filters.

Theorem 3.4. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies two conditions* (3.3) *and* (3.4)*.*

Proof. The necessity is from Proposition 3.2.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ that satisfies (3.3) and (3.4). Let x and y be elements of $\mathcal{L}$ such that $x \leq y$. Then $x \to y = 1$ and so

$$x\mathcal{H} = x\mathcal{H} \cup 1\mathcal{H} = x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}.$$

Since $x \otimes y \leq x \otimes y$ for all $x, y \in L$, it follows from (3.6) that $x\mathcal{H} \cup y\mathcal{H} \supseteq (x \otimes y)\mathcal{H}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 3.5. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.5)*.*

Proof. The necessity is from Lemma 3.3 and Theorem 3.4.

Conversely let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying (3.5). Since

$$x \leq x \to 1 \;\text{ and } x \to y \leq x \to y,$$

for all $x, y \in L$, it follows from (3.5) that

$$x\mathcal{H} = x\mathcal{H} \cup x\mathcal{H} \supseteq 1\mathcal{H} \text{ and } x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}$$

for all $x, y \in L$. Hence $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.6. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall x, y, z \in L)\left((x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H}\right). \tag{3.7}$$

Proof. Let $x, y, z \in L$. Using (2.3) and (2.5), we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

It follows from Theorem 3.5 that

$$(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H}.$$

This completes the proof. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if $\mathcal{H}$ satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,((x \to (y \to z))\mathcal{H} \cup y\mathcal{H} \supseteq (x \to z)\mathcal{H})\,. \tag{3.8}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Then the condition (3.3) is valid. Using (3.4) and (2.3), we have

$$\begin{aligned}(x \to z)\mathcal{H} &\subseteq y\mathcal{H} \cup (y \to (x \to z))\mathcal{H} \\ &= y\mathcal{H} \cup (x \to (y \to z))\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying (3.3) and (3.8). Taking $x := 1$ in (3.8) and using (2.1), we have

$$\begin{aligned}z\mathcal{H} = (1 \to z)\mathcal{H} &\subseteq (1 \to (y \to z))\mathcal{H} \cup y\mathcal{H} \\ &= (y \to z)\mathcal{H} \cup y\mathcal{H}\end{aligned}$$

for all $y, z \in L$. Thus $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.8. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall a, x \in L)\,(a\mathcal{H} \supseteq ((a \to x) \to x)\mathcal{H})\,. \tag{3.9}$$

Proof. If we take $y := (a \to x) \to x$ and $x := a$ in (3.4), then

$$\begin{aligned}((a \to x) \to x)\mathcal{H} &\subseteq a\mathcal{H} \cup (a \to ((a \to x) \to x))\mathcal{H} \\ &= a\mathcal{H} \cup ((a \to x) \to (a \to x))\mathcal{H} \\ &= a\mathcal{H} \cup 1\mathcal{H} = a\mathcal{H}.\end{aligned}$$

This completes the proof. □

Theorem 3.9. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the following conditions:*

$$(\forall x, y \in L)\,(x\mathcal{H} \supseteq (y \to x)\mathcal{H})\,, \tag{3.10}$$

$$(\forall x, a, b \in L)\,(a\mathcal{H} \cup b\mathcal{H} \supseteq ((a \to (b \to x)) \to x)\mathcal{H})\,. \tag{3.11}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Using (3.4), (2.3), (2.1) and (3.3), we have

$$(y \to x)\mathcal{H} \subseteq x\mathcal{H} \cup (x \to (y \to x))\mathcal{H} = x\mathcal{H} \cup 1\mathcal{H} = x\mathcal{H},$$

for all $x, y \in L$.

Using (3.8) and (3.9), we get

$$((a \to (b \to x)) \to x)\mathcal{H} \subseteq ((a \to (b \to x)) \to (b \to x))\mathcal{H} \cup b\mathcal{H} \subseteq a\mathcal{H} \cup b\mathcal{H},$$

for all $a, b, x \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying two conditions (3.10) and (3.11). If we take $y := x$ in (3.10), then $x\mathcal{H} \supseteq (x \to x)\mathcal{H} = 1\mathcal{H}$ for all $x \in L$. Using (3.11) induces

$$y\mathcal{H} = (1 \to y)\mathcal{H} = (((x \to y) \to (x \to y)) \to y)\mathcal{H} \subseteq (x \to y)\mathcal{H} \cup x\mathcal{H},$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.10. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if the nonempty τ-exclusive set of $\mathcal{H}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$.*

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ and let $\tau \in \mathcal{P}([0,1])$ be such that $e(\mathcal{H};\tau) \neq \emptyset$. Let $x, y \in L$ be such that $x \in e(\mathcal{H};\tau)$ and $x \to y \in e(\mathcal{H};\tau)$. Then $\tau \supseteq x\mathcal{H}$ and $\tau \supseteq (x \to y)\mathcal{H}$. It follows from (3.3) and (3.4) that $1\mathcal{H} \subseteq x\mathcal{H} \subseteq \tau$ and $y\mathcal{H} \subseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \subseteq \tau$. Hence $1 \in e(\mathcal{H};\tau)$ and $y \in e(\mathcal{H};\tau)$, and therefore $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ by Proposition 2.1.

Conversely, suppose that $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$ with $e(\mathcal{H};\tau) \neq \emptyset$. For any $x \in L$, let $x\mathcal{H} = \delta$. Then $x \in e(\mathcal{H};\delta)$ and $e(\mathcal{H};\delta)$ is a filter of $\mathcal{L}$. Hence $1 \in e(\mathcal{H};\delta)$ and so $x\mathcal{H} = \delta \supseteq 1\mathcal{H}$. For any $x, y \in L$, let $x\mathcal{H} = \delta_x$ and $(x \to y)\mathcal{H} = \delta_{x\to y}$. If we take $\delta = \delta_x \cup \delta_{x\to y}$, then $x \in e(\mathcal{H};\delta)$ and $x \to y \in e(\mathcal{H};\delta)$ which imply that $y \in e(\mathcal{H};\delta)$. Thus

$$x\mathcal{H} \cup (x \to y)\mathcal{H} = \delta_x \cup \delta_{x\to y} = \delta \supseteq y\mathcal{H}.$$

Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.11. *For a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$, let $\mathcal{H}^*$ be a hesitant fuzzy set of $\mathcal{L}$ which is given as follows:*

$$\mathcal{H}^* : L \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} x\mathcal{H}, & \text{if } x \in e(\mathcal{H};\tau), \\ [0,1], & \text{otherwise,} \end{cases}$$

where $\tau \in \mathcal{P}([0,1])$ with $\tau \neq [0,1]$. If $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$, then so is $\mathcal{H}^*$.

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Then $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$ with $e(\mathcal{H};\tau) \neq \emptyset$ by Theorem 3.10. Thus $1 \in e(\mathcal{H};\tau)$, and so $1\mathcal{H}^* = 1\mathcal{H} \subseteq x\mathcal{H} \subseteq x\mathcal{H}^*$ for all $x \in L$. Let $x, y \in L$. If $x \in e(\mathcal{H};\tau)$ and $x \to y \in e(\mathcal{H};\tau)$, then $y \in e(\mathcal{H};\tau)$. Hence

$$x\mathcal{H}^* \cup (x \to y)\mathcal{H}^* = x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H} = y\mathcal{H}^*.$$

If $x \notin e(\mathcal{H};\tau)$ or $x \to y \notin e(\mathcal{H};\tau)$, then $x\mathcal{H}^* = [0,1]$ or $(x \to y)\mathcal{H}^* = [0,1]$. Thus

$$x\mathcal{H}^* \cup (x \to y)\mathcal{H}^* = [0,1] \supseteq y\mathcal{H}^*.$$

Therefore $\mathcal{H}^*$ is a join hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 3.12. *If $\mathcal{H}$ is a join hesitant fuzzy filter of L, then the set*

$$\mathcal{L}_a := \{x \in L \mid a\mathcal{H} \supseteq x\mathcal{H}\}$$

is a filter of $\mathcal{L}$ for every $a \in L$.

Proof. Since $1\mathcal{H} \subseteq a\mathcal{H}$ for all $a \in L$, we have $1 \in \mathcal{L}_a$. Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $x\mathcal{H} \subseteq a\mathcal{H}$ and $(x \to y)\mathcal{H} \subseteq a\mathcal{H}$. Since $\mathcal{H}$ is a join hesitant fuzzy filter of L, it follows from (3.4) that

$$a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}$$

so that $y \in \mathcal{L}_a$. Hence $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.1. □

Theorem 3.13. *Let $a \in L$ and let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$. Then*

(1) *If $\mathcal{L}_a$ is a filter of L, then $\mathcal{H}$ satisfies the following condition:*

$$(\forall a, x, y \in L)\, (a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \;\Rightarrow\; a\mathcal{H} \supseteq y\mathcal{H}). \tag{3.12}$$

(2) *If $\mathcal{H}$ satisfies* (3.3) *and* (3.12)*, then $\mathcal{L}_a$ is a filter of L.*

Proof. (1) Assume that $\mathcal{L}_a$ is a filter of L. Let x and y be elements of $\mathcal{L}$ such that

$$a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H}.$$

Then $x \to y \in \mathcal{L}_a$ and $x \in \mathcal{L}_a$. Using (2.11), we have $y \in \mathcal{L}_a$ and so $a\mathcal{H} \supseteq y\mathcal{H}$.

(2) Suppose that $\mathcal{H}$ satisfies (3.3) and (3.12). Then $1 \in \mathcal{L}_a$ by (3.3). Let x and y be elements of $\mathcal{L}$ such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $a\mathcal{H} \supseteq x\mathcal{H}$ and $a\mathcal{H} \supseteq (x \to y)\mathcal{H}$, which imply that $a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H}$. Thus $a\mathcal{H} \supseteq y\mathcal{H}$ by (3.12), and so $y \in \mathcal{L}_a$. Therefore $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.1. □

4. Join hesitant fuzzy G-filters

Definition 4.1. A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is called a *join hesitant fuzzy G-filter* of $\mathcal{L}$ if it is a join hesitant fuzzy filter of $\mathcal{L}$ such that

$$(\forall x, y \in L)\, (((x \otimes x) \to y)\mathcal{H} \supseteq (x \to y)\mathcal{H})\,. \tag{4.1}$$

Note that the condition (4.1) is equivalent to the following condition:

$$(\forall x, y \in L)\, ((x \to (x \to y))\mathcal{H} \supseteq (x \to y)\mathcal{H})\,. \tag{4.2}$$

Example 4.2. Let $L := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\;\; a \wedge b = \min\{a, b\},$$
$$a \to b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise,} \end{cases} \quad \text{and} \quad a \otimes b = \min\{a, b\}.$$

Then $\mathcal{L} := (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a residuated lattice. Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ defined by

$$\mathcal{H} : L \rightarrow \mathcal{P}([0,1]),\ x \mapsto \begin{cases} (0, \frac{1}{2}], & \text{if } x \in [\frac{1}{2}, 1], \\ [0,1], & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$.

Theorem 4.3. *Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if it is a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the following condition:*

$$(\forall x, y, z \in L)\, ((x \rightarrow (y \rightarrow z))\mathcal{H} \cup (x \rightarrow y)\mathcal{H} \supseteq (x \rightarrow z)\mathcal{H})\,. \tag{4.3}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Note that $x \leq 1 = (x \rightarrow y) \rightarrow (x \rightarrow y)$, and thus $x \rightarrow y \leq x \rightarrow (x \rightarrow y)$ for all $x, y \in L$. It follows from (3.1) that $(x \rightarrow y)\mathcal{H} \supseteq (x \rightarrow (x \rightarrow y))\mathcal{H}$. Combining this and (4.2), we have

$$(x \rightarrow y)\mathcal{H} = (x \rightarrow (x \rightarrow y))\mathcal{H}, \tag{4.4}$$

for all $x, y \in L$. Using (3.7) and (4.4), we have

$$(x \rightarrow (y \rightarrow z))\mathcal{H} \cup (x \rightarrow y)\mathcal{H} \supseteq (x \rightarrow z)\mathcal{H},$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.3). If we put $y = x$ and $z = y$ in (4.3) and use (2.1) and (3.3), then

$$\begin{aligned} (x \rightarrow y)\mathcal{H} &\subseteq (x \rightarrow (x \rightarrow y))\mathcal{H} \cup (x \rightarrow x)\mathcal{H} \\ &= (x \rightarrow (x \rightarrow y))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \rightarrow (x \rightarrow y))\mathcal{H}, \end{aligned}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. $\square$

Theorem 4.4. *Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ that satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\, (x\mathcal{H} \cup ((y \rightarrow z) \rightarrow (x \rightarrow y))\mathcal{H} \supseteq y\mathcal{H})\,. \tag{4.5}$$

Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$.

Proof. If we take $z := 1$ in (4.5) and use (2.1), then

$$\begin{aligned} x\mathcal{H} \cup (x \rightarrow y)\mathcal{H} &= x\mathcal{H} \cup (1 \rightarrow (x \rightarrow y))\mathcal{H} \\ &= x\mathcal{H} \cup ((y \rightarrow 1) \rightarrow (x \rightarrow y))\mathcal{H} \\ &\supseteq y\mathcal{H}. \end{aligned}$$

Hence $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. Let $x, y, z \in L$. Since

$$x \to (y \to z) \leq (x \to y) \to (x \to (x \to z))$$

by (2.3), (2.4) and (2.5), we have

$$(x \to (y \to z))\mathcal{H} \supseteq ((x \to y) \to (x \to (x \to z)))\mathcal{H}$$

by (3.1). It follows that

$$\begin{aligned}(x \to y)\mathcal{H} \cup (x \to (y \to z))\mathcal{H} &\supseteq (x \to y)\mathcal{H} \cup ((x \to y) \to (x \to (x \to z)))\mathcal{H}\\ &\supseteq (x \to (x \to z))\mathcal{H}\\ &\supseteq (((x \to z) \to z) \to (x \to z))\mathcal{H}\\ &= (((x \to z) \to z) \to (1 \to (x \to z)))\mathcal{H}\\ &\supseteq (x \to z)\mathcal{H}.\end{aligned}$$

Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. □

The following example shows that any join hesitant fuzzy G-filter may not satisfy the condition (4.5).

Example 4.5. The join hesitant fuzzy G-filter $\mathcal{H}$ of $\mathcal{L}$ in Example 4.2 does not satisfy the condition (4.5) since

$$\tfrac{2}{3}\mathcal{H} \cup ((\tfrac{1}{3} \to \tfrac{1}{4}) \to (\tfrac{2}{3} \to \tfrac{1}{3}))\mathcal{H} = \tfrac{2}{3}\mathcal{H} \cup 1\mathcal{H} = \tau \not\supseteq U = \tfrac{1}{3}\mathcal{H}.$$

Proposition 4.6. *For a join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$, the condition* (4.5) *is equivalent to the following condition.*

$$(\forall x, y \in L)\,(((x \to y) \to x)\mathcal{H} \supseteq x\mathcal{H})\,. \tag{4.6}$$

Proof. Assume that the condition (4.5) is valid. It follows from (3.3) and (2.1) that

$$\begin{aligned}((x \to y) \to x)\mathcal{H} &= 1\mathcal{H} \cup ((x \to y) \to x)\mathcal{H}\\ &= 1\mathcal{H} \cup ((x \to y) \to (1 \to x))\mathcal{H}\\ &\supseteq x\mathcal{H}\end{aligned}$$

for all $x, y \in L$.

Conversely, suppose that the condition (4.6) is valid. It follows from (2.3) and (3.4) that

$$\begin{aligned}x\mathcal{H} \cup ((y \to z) \to (x \to y))\mathcal{H} &= x\mathcal{H} \cup (x \to ((y \to z) \to y))\mathcal{H}\\ &\supseteq ((y \to z) \to y)\mathcal{H} \supseteq y\mathcal{H}\end{aligned}$$

for all $x, y \in L$. □

Combining Theorem 4.4 and Proposition 4.6, we have the following result.

Theorem 4.7. *Every join hesitant fuzzy filter satisfying the condition* (4.6) *is a join hesitant fuzzy G-filter.*

Proposition 4.8. *Every join hesitant fuzzy filter* $\mathcal{H}$ *of* $\mathcal{L}$ *with the condition* (4.5) *satisfies the following condition.*

$$(\forall x, y \in L)\left(((x \to y) \to y)\mathcal{H} \supseteq ((y \to x) \to x)\mathcal{H}\right). \tag{4.7}$$

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.5) and let $x, y \in L$. Since $x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = (y \to x) \to 1 = 1$, that is, $x \le (y \to x) \to x$, we have $((y \to x) \to x) \to y \le x \to y$ by (2.4). It follows from (2.6), (2.3) and (2.4) that

$$\begin{aligned}(x \to y) \to y &\le (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x)\\ &\le (((y \to x) \to x) \to y) \to ((y \to x) \to x).\end{aligned}$$

Using (3.1), (3.3), (2.1), (2.3) and (4.5), we have

$$\begin{aligned}((x \to y) \to y)\mathcal{H} &\supseteq ((((y \to x) \to x) \to y) \to ((y \to x) \to x))\mathcal{H}\\ &= 1\mathcal{H} \cup (1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)))\mathcal{H}\\ &= 1\mathcal{H} \cup ((((y \to x) \to x) \to y) \to (1 \to ((y \to x) \to x)))\mathcal{H}\\ &\supseteq ((y \to x) \to x)\mathcal{H}.\end{aligned}$$

Hence the condition (4.7) is valid. □

Corollary 4.9. *Every join hesitant fuzzy filter* $\mathcal{H}$ *of* $\mathcal{L}$ *with the condition* (4.6) *satisfies the condition* (4.7).

Proposition 4.10. *Every join hesitant fuzzy G-filter* $\mathcal{H}$ *of* $\mathcal{L}$ *with the condition* (4.7) *satisfies the condition* (4.5).

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy G-filter of $\mathcal{L}$ that satisfies the condition (4.7). For any $x, y, z \in L$, we have

$$\begin{aligned}z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} &= z\mathcal{H} \cup (z \to ((x \to y) \to x))\mathcal{H}\\ &\supseteq ((x \to y) \to x)\mathcal{H}\\ &\supseteq ((x \to y) \to ((x \to y) \to y))\mathcal{H}\\ &\supseteq ((x \to y) \to y)\mathcal{H}\\ &\supseteq ((y \to x) \to x)\mathcal{H}\end{aligned}$$

by (2.3), (3.4), (2.6), (3.1), (4.2) and (4.7). Since

$$(x \to y) \to x \le y \to x \le z \to (y \to x),$$

it follows from (3.1) that $((x \to y) \to x)\mathcal{H} \supseteq (z \to (y \to x))\mathcal{H}$ and so from (3.4) that

$$\begin{aligned} z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} &\supseteq z\mathcal{H} \cup ((x \to y) \to x)\mathcal{H} \\ &\supseteq z\mathcal{H} \cup (z \to (y \to x))\mathcal{H} \\ &\supseteq (y \to x)\mathcal{H}. \end{aligned}$$

Therefore

$$z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} \supseteq (y \to x)\mathcal{H} \cup ((y \to x) \to x)\mathcal{H} \supseteq x\mathcal{H}.$$

Hence the condition (4.5) is valid. □

Theorem 4.11. *Let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall x \in L)\,((x \to (x \otimes x))\mathcal{H} = 1\mathcal{H})\,. \tag{4.8}$$

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy G-filter of L. Since $x \to (x \to (x \otimes x)) = 1$ for all $x \in L$, we have $(x \to (x \to (x \otimes x)))\mathcal{H} = 1\mathcal{H}$. It follows from (4.3) and (2.1) that

$$(x \to (x \otimes x))\mathcal{H} \subseteq (x \to (x \to (x \otimes x)))\mathcal{H} \cup (x \to x)\mathcal{H} = 1\mathcal{H}$$

and so from (3.3) that $(x \to (x \otimes x))\mathcal{H} = 1\mathcal{H}$ for all $x \in L$.

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ which satisfies the condition (4.8) and let $x, y \in L$. Since

$$x \to (x \to y) = (x \otimes x) \to y \leq (x \to (x \otimes x)) \to (x \to y)$$

by (2.3) and (2.5), it follows from (3.1) that

$$(x \to (x \to y))\mathcal{H} \supseteq ((x \to (x \otimes x)) \to (x \to y))\mathcal{H}.$$

Hence, we have

$$\begin{aligned} (x \to y)\mathcal{H} &\subseteq ((x \to (x \otimes x)) \to (x \to y))\mathcal{H} \cup (x \to (x \otimes x))\mathcal{H} \\ &\subseteq (x \to (x \to y))\mathcal{H} \cup (x \to (x \otimes x))\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H} \end{aligned}$$

by using (3.4), (4.8) and (3.3). Hence $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. □

Theorem 4.12. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if it is a join hesitant fuzzy filter of $\mathcal{L}$ with an additional condition:*

$$(\forall x, y \in L)\,((x \to y)\mathcal{H} = (x \to (x \to y))\mathcal{H})\,. \tag{4.9}$$

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Let $x, y \in L$. Since $x \to y \leq x \to (x \to y)$, we have

$$(x \to y)\mathcal{H} \supseteq (x \to (x \to y))\mathcal{H}$$

by (3.1). Hence $(x \to y)\mathcal{H} = (x \to (x \to y))\mathcal{H}$ by using (4.2).

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ with the condition (4.9). It follows from Proposition 3.6 that

$$(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H} = (x \to z)\mathcal{H}$$

for all $x, y, z \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. $\square$

Proposition 4.13. *Every join hesitant fuzzy G-filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following conditions:*

$$(\forall x, y, z \in L)\left((x \to (y \to z))\mathcal{H} \supseteq ((x \to y) \to (x \to z))\mathcal{H}\right). \tag{4.10}$$

$$(\forall x, y, z \in L)\left((x \to (y \to z))\mathcal{H} = ((x \to y) \to (x \to z))\mathcal{H}\right). \tag{4.11}$$

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy G-filter of $\mathcal{L}$. Using (2.3), (4.3), (2.5) and (3.3), we have

$$\begin{aligned}((x \to y) \to (x \to z))\mathcal{H} &= (x \to ((x \to y) \to z))\mathcal{H} \\ &\subseteq (x \to (y \to z))\mathcal{H} \cup (x \to ((y \to z) \to ((x \to y) \to z)))\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H} \cup ((y \to z) \to ((x \to y) \to (x \to z)))\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$. Thus (4.10) holds. Since $(x \to y) \to (x \to z) \leq x \to (y \to z)$ for all $x, y, z \in L$, it follows from (3.1) that $((x \to y) \to (x \to z))\mathcal{H} \supseteq (x \to (y \to z))\mathcal{H}$ and so that

$$(x \to (y \to z))\mathcal{H} = ((x \to y) \to (x \to z))\mathcal{H}$$

for all $x, y, z \in L$ by using (4.10). $\square$

Proposition 4.14. *Assume that $\mathcal{L}$ satisfies the divisibility, that is, $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. If $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ satisfying* (4.11)*, then the following equality is true.*

$$(\forall x, y, z \in L)\left(((x \otimes y) \to z)\mathcal{H} = ((x \wedge y) \to z)\mathcal{H}\right). \tag{4.12}$$

Proof. Using the divisibility and (2.3), we have

$$(x \wedge y) \to z = (x \otimes (x \to y)) \to z = (x \to y) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (2.3) and (4.11) that

$$\begin{aligned} ((x \otimes y) \to z)\mathcal{H} &= (x \to (y \to z))\mathcal{H} \\ &= ((x \to y) \to (x \to z))\mathcal{H} \\ &= ((x \wedge y) \to z)\mathcal{H} \end{aligned}$$

for all $x, y, z \in L$. □

Theorem 4.15. *Let $\mathcal{L}$ satisfy the divisibility, that is, $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. Then every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfying the condition* (4.12) *is a join hesitant fuzzy G-filter of $\mathcal{L}$.*

Proof. Using Proposition 3.6, (2.3) and (4.12), we have

$$\begin{aligned} &(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H} \\ &= ((x \otimes x) \to z)\mathcal{H} = ((x \wedge x) \to z)\mathcal{H} = (x \to z)\mathcal{H} \end{aligned}$$

for all $x, y, z \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. □

Theorem 4.16. *Let $\mathcal{H}$ and $\mathcal{G}$ be join hesitant fuzzy filters of $\mathcal{L}$ such that $1\mathcal{H} = 1\mathcal{G}$ and $\mathcal{H} \supseteq \mathcal{G}$, i.e., $x\mathcal{H} \supseteq x\mathcal{G}$ for all $x \in L$. If $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$, then so is $\mathcal{G}$.*

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Using (2.3) and (2.1), we have

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned} (x \to ((x \to (x \to y)) \to y))\mathcal{G} &\subseteq (x \to ((x \to (x \to y)) \to y))\mathcal{H} \\ &= (x \to (x \to ((x \to (x \to y)) \to y)))\mathcal{H} \\ &= 1\mathcal{H} = 1\mathcal{G} \end{aligned}$$

by hypotheses and (4.4), and so

$$(x \to ((x \to (x \to y)) \to y))\mathcal{G} = 1\mathcal{G}$$

for all $x, y \in L$ by (3.3). Since $\mathcal{G}$ is a join hesitant fuzzy filter of $\mathcal{L}$, it follows from (3.4), (2.3) and (3.3) that

$$\begin{aligned} (x \to y)\mathcal{G} &\subseteq (x \to (x \to y))\mathcal{G} \cup ((x \to (x \to y)) \to (x \to y))\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G} \cup (x \to ((x \to (x \to y)) \to y))\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G} \cup 1\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G} \end{aligned}$$

for all $x, y \in L$. Therefore $\mathcal{G}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. □

Acknowledgements

The authors are thankful to the anonymous referees for their valuable comments and suggestions.

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Accepted: 18.12.2017

Application of differential transformation method for solving prey predator model with holling type I

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Abstract. Nonlinear differential equations are used for describing many phenomena in the real world as prey predator interactions. Prey predator models are classified as one of the most important applications in applied mathematics. In this paper,

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modified structure of prey predator model is used, theoretical properties of the model are presented, the boundedness of the model is shown and the dynamical behavior of the model is proved as globally stable. Semi analytical solution by using differential transformation method (DTM) is obtained for non- dimensional prey predator model with Holling type I in the case of persistence dynamics of the model. The results seem to satisfy biological domain of the problem. We conclude that the results of differential transformation method is in good agreement with numerical results from interpolation method (IM) by using MATHEMATICA program.

Keywords: prey predator model, differential transform method, boundedness, stability.

1. Introduction

Nonlinear differential equations are used to describe many real world phenomena as prey predator interactions. Prey predator models are classified as one of the most important applications in applied mathematics, biology, and ecology sciences. Lotka-Volterra model is considered as the original model to formulate prey-predator interactions [1]. However, numerous extensions of the original model have been applied to describe particular scenarios which involve surroundings and nature of species by many researchers [2]-[11] problems of biological and ecological interest are described in the form of differential equations with appropriate initial or boundary conditions.

Usually, difficulty arises in the solution of nonlinear system, so many numerical methods and semi-analytical methods are developed for finding the solution of these problems by many researchers.

Differential transform method (DTM) is applied for solving specific kind of system of nonlinear differential equations. DTM is considered among the few semi analytical methods to overcome the difficulties that are caused by the nonlinear terms. The method gives an analytical solution in the form of a series for differential equations. It formulize the Taylor series in a totally different approach so it is a semi-numerical and semi-analytic method. Zhou [12] proposed firstly the idea of differential transform (see Ref. [1, 8, 9, 13, 14, 15, 16]) and it was used for solving linear and non-linear initial value problems in electric circuit analysis. Simultaneously, Pukhov [17] also studied differential transformation method. This method depends on transforming IVP or BVP into a recurrence relation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. The method is helpful to obtain both semi analytical and approximate solutions of linear and non-linear IVPs. This method avoids discretization, linearization or perturbation, big computational work and round-off errors.

Several authors have employed DTM for the solution of problems involved in non-linear phenomenon. Moon et al. [18] applied the DTM for solving some non-linear differential equations. Warade and Chopade [19] solved initial value problems involving fourth order ordinary differential equations by DTM. Chang and Chang [20] introduced developed algorithm for calculating one-dimensional differential transform of nonlinear functions. Iftikhar et al. [21] used the DTM for solving linear and nonlinear thirteenth order boundary value problems. Hatami and Ganji [22] used Lagrangian and high accuracy multi- step differential transformation method on the motion of a spherical particle. Hussain et al. [23] obtained analytical solution of magnetohydrodynamic flow of Newtonian fluids over a stretching sheet by using DTM. Hatami and Jing [24] used DTM for solving the coupled nonlinear differential equations in fluid mechanics problems. Sepasgozar et al. [25] applied DTM to obtain the solution of momentum and heat transfer equations of non-Newtonian fluid flow in an axisymmetric channel with porous wall. Through the literature, we observe that DTM is used for solving different real applications as semi analytical method.

Our objective for this paper is to introduce semi analytical solution of real world problem. Differential transform method (DTM) is applied for solving different style of prey predator model with Holling type I, which has been used in Alebraheem and Abu Hasan, [3]. The logistic law describes the growth rates of the model of the prey and predator. The persistence is considered as one of the main important dynamic behaviors. The new idea here we use differential transform method to solve the model with persistence dynamic behaviors. The persistence condition of this model is determined to give biological meaning and validation of the parametric values. We introduce comparison between differential transform method and approximation method in case of persistence dynamics of the model.

2. Mathematical model

The system of equations is written in non-dimensional form as

$$\frac{da}{dt} = a(1-a) - \alpha ab, \tag{2.1}$$

$$\frac{db}{dt} = -ub + e\alpha ab - e\alpha b^2 \tag{2.2}$$

The biological meanings of the parameters are explained as follows: The intrinsic growth rate of prey is 1. α measures efficiency of the search and the capture of predators is b, u is the death rates of predators. $f(a) = \alpha a$ denotes the functional response which is defined as consumption rate of prey by a predator, $g(a) = e\alpha a$, $g(a)$ represents numerical response of the predators y that characterize changing in the population of predators through prey consumption. e represents efficiency of converting consumed prey into predator births. Since the biological meaning is taken into consideration, all the parameters and initial conditions of the model are supposed positive values. The initial conditions of system (1) are:

$$a(0) = a_0, b(0) = b_0 \quad \text{where} \quad 0 < a_0, b_0 < 1. \tag{2.3}$$

However, the initial conditions are $0 < a_0, b_0 \leq 1$ because the carrying capacity of non dimensional system (2.3) is 1, so the maximum values of initial conditions are 1, while they are greater than zero because of biological meaning.

3. The boundedness of the model

3.1 Theorem

The solution of the system (2.1-2.2) for $t \geq 0$ in R^3_+ is bounded.

Proof. We show that the first equation of the system (2.1-2.2) is bounded through

$$\frac{da}{dt} \leq a(1+a). \tag{3.1}$$

The solution of the equation 2.2 is $a(t) = \frac{1}{(1+qe^(-t))}, q = \frac{(1-a_0)}{a_0}$ is the constant of integration. then $a(t) \leq 1 \;\; \forall \;\; t \geq 0$. Then, we prove that $a(t)+b(t) \leq L.\forall t \geq 0$. Let $D(t) = a(t) + b(t)$. The time derivative of the function D

$$\begin{aligned} \frac{dD}{dt} &= \frac{da}{dt} + \frac{db}{dt} \\ &= ((1-a) - \alpha b)a + (-u + e\alpha a - e\alpha b)b. \end{aligned} \tag{3.2}$$

However, the solutions initiating remain in nonnegative quadrant in R^3_+ and all the parameters are positive; it can be assumed the following

$$\frac{dD}{dt} \leq ((1-a))a + (-u + e\alpha a - e\alpha b)b. \tag{3.3}$$

It can be concluded that

$$\max_{(R_+)} a(1-a) = \frac{1}{4}. \tag{3.4}$$

By substituting in (3.3), it become as follows

$$\frac{dD}{dt} \leq \frac{1}{4} + (-u + e\alpha a - e\alpha b)b, \tag{3.5}$$

$$\frac{dD}{dt} \leq \frac{1}{4} + (-u + e\alpha a - e\alpha b)b + D(t) - D(t). \tag{3.6}$$

The equation (3.6) can be written as follows

$$\frac{dD}{dt} + D(t) \leq \frac{1}{4} + a + (-u + e\alpha a - e\alpha b + 1)b. \tag{3.7}$$

Since $a(t) \leq 1$, then

$$\frac{dD}{dt} + D(t) \leq \frac{5}{4} + (-u + e\alpha - e\alpha b + 1)b. \tag{3.8}$$

But

$$\max_{(R_+)} (-u + e\alpha - e\alpha b + 1)b = \frac{(1 + e\alpha - u)^2}{4e}. \tag{3.9}$$

So Eq. (3.8) becomes:

$$\frac{dD}{dt} + D(t) \leq L, \tag{3.10}$$

where

$$L = \frac{1}{4}(5 + \frac{(1 + e\alpha - u)^2}{4e\alpha}). \tag{3.11}$$

Consequently,

$$D(t) \leq L + \sigma e^{-t}, \tag{3.12}$$

where σ is a constant of integration. $t \to \infty$; Then $D(t) \leq L$.

4. The dynamic behavior

One of the main properties of dynamic systems is stability. The stability is studied to determine properties of solutions or equations in differential equations, consequently the dynamic behavior will be explained. The system has three non-negative equilibrium points that are: The first point is $E_0 = (0, 0)$ exists without conditions on parameters. The second point is $E_1 = (1, 0)$ exists without conditions on parameters. The third point is $E_2 = (\hat{a}, \hat{b}) = (\frac{(u+e)}{(e\alpha+e)}, \frac{(e\alpha-u)}{(e\alpha^2+e\alpha)})$, which is called persistence point. The equilibrium point E_2 is positive under the following condition: $e\alpha > u$ We are interested to study the dynamic behaviors in case the permanent coexistence of prey predator system, so we study the coexistence point $E_2 = (\hat{a}, \hat{b}) = (\frac{(u+e)}{(e\alpha+e)}, \frac{(e\alpha-u)}{(e\alpha^2+e\alpha)})$, it represents the permanent coexistence (i.e. persistence) of prey predator system.

4.1 Theorem

The persistence equilibrium point E_2 is globally asymptotically stable inside the positive quadrant of a-b plane.

Proof. Let $G(a, b) = \frac{1}{ab}$. G is a Dulac function, it is continuously differentiable in the positive quadrant of x-y plane $A = (a, b) \mid a > 0, b > 0$, Hsu [26].

$$\begin{aligned} N_1(a, b) &= a(1 - a) - \alpha ab, \\ N_2(a, b) &= -ub + e\alpha ab - \alpha eb^2. \end{aligned} \tag{4.1}$$

Thus, $\Delta(GN_1, GN_2) = \frac{(\partial(GN_1))}{\partial a} + \frac{(\partial(GN_2))}{\partial b} = \frac{-1}{b} - \frac{e\alpha}{a}$. It is observed that $\Delta(GN_1, GN_2)$ is not identically zero and does not change sign in the positive quadrant of a-b plane. So by Bendixson-Dulac criterion, there is no periodic solution inside the positive quadrant of $a - b$ plane. E_2 is globally asymptotically stable inside the positive quadrant of $a - b$ plane. In general, we conclude that the dynamic behavior of this system is stable. Kolmogorov analysis [27] is applied to find the persistence condition of the system (2.1-2.2), so the persistence condition is

$$0 < \frac{u}{e\alpha} < 1. \tag{4.2}$$

4.2 Corollary

The persistence dynamic behavior of the system (2.1-2.2) is globally asymptotically stable.

5. Approximate analytical solution

The basic definitions and rules [13] of the DTM are summarized below:

5.1 Basic definitions and concepts

If $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time t, as shown in equation

$$\frac{(d^n u(t))}{(dt^n)} = \phi(t,n), \quad \forall \ t \epsilon T. \tag{5.1}$$

If $\phi(t,n) = \phi(t_i,n)$, where n belongs to the set of non-negative integers, denoted as the n-domain. The Eq. (5.1) can be rewritten as:

$$U(n) = \phi(t_i,n) = [\frac{(d^n u(t))}{(dt^n)}] \mid_{(t=t_i)}, \tag{5.2}$$

where the spectrum of $u(t)$ at $t = ti.$ is denoted by $U(n)$. $u(t)$ can be represented by Taylor's series as follows:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n). \tag{5.3}$$

The Eq. 5.2 is inverse of $U(n)$. The combination of eq. 5.2 and eq. 5.3 yields:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n) = D^{-1} U(n). \tag{5.4}$$

The symbol "D" denotes the differential transformation process. By using the differential transformation, a differential equation $u(t)$ becomes:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n) + R_{(k+1)}(t), \tag{5.5}$$

where $R_{(k+1)}(t)$ is the remainder. If $F(n)$ and $G(n)$ are transformed functions corresponding to the given functions $f(t)$ and $g(t)$, the operation properties of DTM are as follows:

Base function	Transformed Functions
$q(t) = u(t) \pm v(t)$	$Q(n) = U(n) \pm V(n)$
$q(t) = \alpha u(t)$	$Q(n) = \alpha U(n)$
$q(t) = \frac{du(t)}{dt}$	$Q(n) = (n+1)U(n+1)$
$q(t) = \frac{d^2u(t)}{d(t)}$	$Q(n) = (n+1)(n+2)U(n+2)$
$q(t) = \frac{d^m u(t)}{d(t)_m}$	$Q(n) = (n+1)(n+2)...(n+m)U(n+m)$
$q(t) = u(t)v(t)$	$Q(n) = \sum_l^k = 0v(l)U(n-l)$
$q(t) = t^m$	$Q(n)\delta(n-m), \delta(n-m)$
$q(t) = exp(\gamma t)$	$Q(n) = \frac{\gamma^n}{n!}$
$Q(T) = (1+t)^m$	$Q(n) = \frac{m(m-1)...(m-n+1)}{n!}$
$q(t) = sin(\omega t + \alpha)$	$Q(n) = \frac{\omega^n}{n!} sin((\frac{\pi n}{2} + \omega))$
$q(t) = cos(\omega t + \alpha)$	$Q(n) = \frac{\omega^n}{n!} sin((\frac{\pi n}{2} + \omega))$

Table 1. Some basic transformations related to Differential Transform Method

By applying differential transformation method (DTM), the system (2.1) and (2.2) is transformed to yield recurrence relation

$$\begin{aligned}(n+1)A(n+1) &= A(n) - \sum_{(n_{1=0})}^{n} A(n_1)A(n-n_1) \\ &- \alpha \sum_{(n_{1=0})}^{n} A(n_1)B(n-n_1) \end{aligned} \tag{5.6}$$

$$\begin{aligned}(n+1)B(n+1) &= -uB(n) + e\alpha \sum_{(n_{1=0})}^{n} A(n_1)B(n-n_1) \\ &- e\alpha \sum_{(n_{1=0})}^{n} B(n_1)B(n-n_1). \end{aligned} \tag{5.7}$$

We choose the following appropriate values$A(0) = 0.5, B(0) = 0.2, e = 0.5, \alpha = 1.32, u = 0.52$, when $n = 0, 1, 2, 3, 4, 5, 6$.

Inverse relation

$$\begin{aligned}a(t) &= 0.5 + 0.118t + 0.005676t^2 - 0.0057915t^3 + 0.0019652t^4 \\ &- 0.00088551t^5 \end{aligned} \tag{5.8}$$

$$\begin{aligned}b(t) &= 0.2 - 0.0644t + 0.0224068t^2 - 0.0057254t^3 + 0.0013109t^4 \\ &- 0.00613653t^5 \end{aligned} \tag{5.9}$$

6. Results

In this section, the system of coupled non -linear differential equations (2.1) and (2.2) have been solved analytically by applying differential transform method.

The values of the parameters of interest have been chosen appropriately for satisfying the persistence of system (2.1)-(2.2). The accuracy of the results has been checked and found in good agreement by their comparison with the numerical results from interpolation method (IM) which is obtained by using MATHEMATICA program (see tables 2 and 3). In addition, the graphical patterns of the results do correspond to the biological configuration of the problem.

Time	DTM	IM	Δ=|DTM-IM|
0.0	0.5	0.5	0.00
0.1	0.511851	0.511850	0.000001
0.2	0.523784	0.523774	0.00001
0.3	0.535768	0.535732	0.000036
0.4	0.547779	0.547687	0.000092
0.5	0.559790	0.559602	0.000188
0.6	0.571778	0.571443	0.000335
0.7	0.583718	0.583175	0.000543
0.8	0.595582	0.594769	0.000813
0.9	0.607342	0.606194	0.001148
1.0	0.618964	0.617425	0.001539

Table 2. Comparative results of the prey equation (2.1) with absolute error obtained through DTM and IM.

Time	DTM	IM	Δ=|DTM-IM|
0.0	0.2	0.2	0.00
0.1	0.193778	0.193778	0.00
0.2	0.187971	0.187972	0.000001
0.3	0.182538	0.182552	0.000014
0.4	0.177429	0.177489	0.00006
0.5	0.172576	0.172758	0.000182
0.6	0.167882	0.168336	0.000454
0.7	0.163219	0.164201	0.000982
0.8	0.158415	0.160334	0.001919
0.9	0.153252	0.156715	0.003463
1.0	0.147456	0.153328	0.005872

Table 3. Comparative results of the predator equation (2.2) with absolute error obtained through DTM and IM

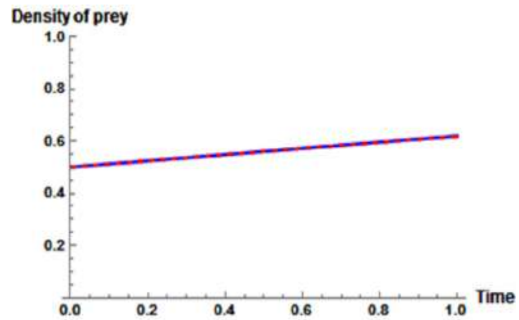

Figure 1: I M results of prey equation (1.a).

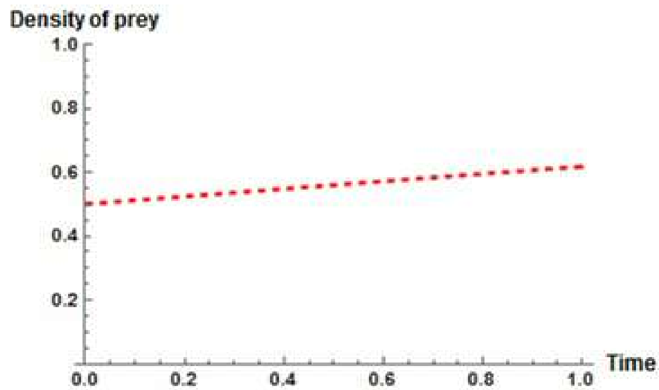

Figure 2: DTM results of prey equation (1.a).

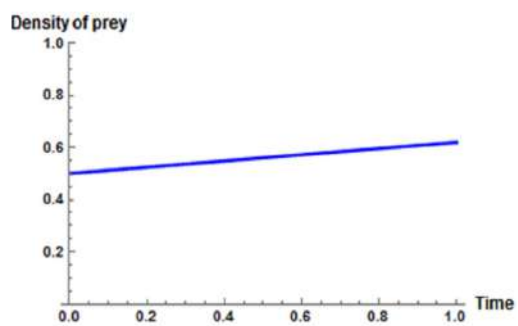

Figure 3: Comparative results of prey equation (1.a) when using IM and DTM.

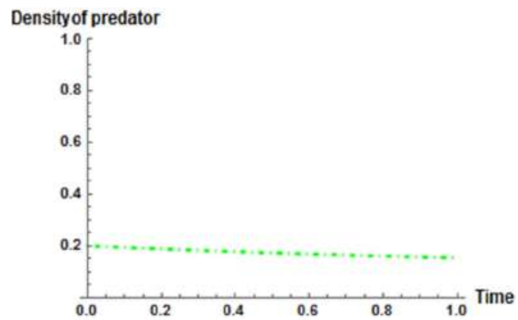

Figure 4: I M results of predator equation (1.b).

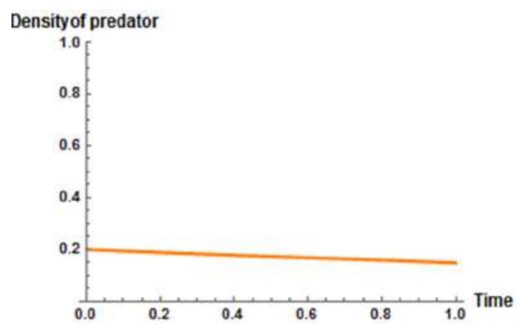

Figure 5: DTM results of predator equation (1.b).

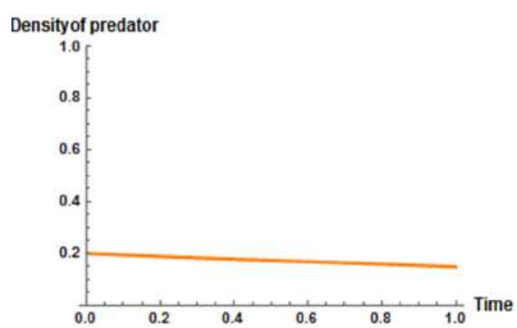

Figure 6: Comparative results of predator equation (1.b) when using IM and DTM.

7. Discussion

Fig 1 represents results of prey equation (2.1) through using IM by executing Mathematica, while the results of prey equation (2.1) using DTM are denoted through Fig 2. We notice through Fig 3 the comparison between both methods. In the same manner, Fig 4 represents results of predator equation (2.2) through using IM by executing Mathematica program, while the results of predator equation (2.2) using DTM are denoted through Fig 5. We notice that Fig 6 describes the comparison between both methods. The results show excellent approximations and the figures explain very close correspondence for both equations with only five terms of differential transformation method.

8. Conclusions

The prey predator model with Holing type I, has been considered for semi analytical solution, using differential transform method. The non-linear coupled equations have been solved smoothly without any rigorous computational work. DTM has been employed for solving the prey predator model in the persistence dynamics. The method worked well with efficacy and efficiency. We obtained close correspondence results of the model between the interpolation method (IM) and differential transform method (DTM) with only five terms. It is concluded that DTM can be used to solve applied problems of non-linear phenomena similar to this work.

9. Acknowledgements

This project was supported by the deanship of scientific research at Majmaah University.

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Accepted: 10.01.2018

Some classes of mappings on generalized quaternion metric spaces II

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Abstract. In connection with the theory of (m, q)-isometries mappings on metric spaces ([3]) and the theory of m-quaternion-valued G-isometric mappings ([1]), we introduce the concept of (m, ∞)-generalized isometric mappings on a generalized real-valued metric space. We present some essential properties of these classes of mappings.

Keywords: metric space, G-metric space, quaternion space, m-isometry.

1. Introduction and preliminaries results

In [9] and [10], the authors Mustafa and Sims introduced the concept of generalized metric spaces , which are called a real G-metric spaces as generalization of metric space $(E, d_{\mathbb{R}})$. A generalization of real metric and G-metric spaces to complex-valued metric space $(E, d_{\mathbb{C}})$ and to complex-valued G-metric space $(E, G_{\mathbb{C}})$ has been presented by many authors in the last years in papers [2], [4], [5], [8] and [12]. Very recently, in paper [1] the present author introduced and study the concept of quaternion-valued G-metric spaces $(E, G_{\mathbf{H}})$.

Definition 1.1 ([9], [10])**.** *Let E be an non-empty set and let $G_{\mathbb{R}} : E\times E\times E \longrightarrow \mathbb{R}_+ = [0,\ \infty)$ be a function satisfying the following conditions*

(1) $G_{\mathbb{R}}(u, v, w) = 0$ *if* $u = v = w$*;*

(2) $0 < G_{\mathbb{R}}(u, u, v)$ *for all* $u, v \in E$ *with* $u \neq v$*;*

(3) $G_{\mathbb{R}}(u, u, v) \leq G_{\mathbb{R}}(u, v, w)$ *for all* $u, v, w \in E$ *with* $v \neq w$*;*

(4) $G_{\mathbb{R}}(u, v, w) = G_{\mathbb{R}}(u, w, v) = G_{\mathbb{R}}(v, w, u) = \ldots$ *(symmetry in all three variables);*

(5) $G_{\mathbb{R}}(u, v, w) \leq G_{\mathbb{R}}(u, a, a) + G_{\mathbb{R}}(a, v, w)$*, for all* $u, v, w, a \in E$ *(rectangle inequality).*

Then the function $G_{\mathbb{R}}$ is called a real-valued generalized metric or, more specifically, a G-metric on E and the pair $(E, G_{\mathbb{R}})$ is called a real G-metric space.

Based on this notion, many fixed point results under different conditions have been obtained for a variety of mappings in this new setting.

A G-metric space $(E, G_{\mathbb{R}})$ is called symmetric G-metric space if $G_{\mathbb{R}}(u,v,v) = G_{\mathbb{R}}(u,u,v)$, for all $u, v \in E$. For $k = 1, \ldots, d$, let $(E_k, G^k_{\mathbb{R}})$ be G-metric spaces and let $E = E_1 \times E_2 \times \ldots \times E_d$, then natural definitions for G-metrics on the product space E would be

$$\begin{aligned} G^m_{\mathbb{R}}(u,v,w) &= G^m_{\mathbb{R}}((u_1,\ldots,u_d),(v_1,\ldots,v_d),(w_1,\ldots,w_d)) \\ &= \max_{k\in\{1,\ldots,d\}} \{G^k_{\mathbb{R}}(u_k,v_k,w_k)\} \end{aligned}$$

or

$$\begin{aligned} G^s_{\mathbb{R}}(u,v,w) &= G^s_{\mathbb{R}}((u_1,\ldots,u_d),(v_1,\ldots,v_d),(w_1,\ldots,w_d) \\ &= \sum_{k=1}^{d} G^k_{\mathbb{R}}(u_k,v_k,w_k)) \end{aligned}$$

for all $u = (u_1,\ldots,u_d), v = (v_1,\ldots,v_d), w = (w_1,\ldots,w_d) \in E$.

It was observed in [10] that in general $(E,\ G^m_{\mathbb{R}})$ and $(E,\ G^s_{\mathbb{R}})$ are not G-metrics spaces. However, it was proved that $(E,\ G^m_{\mathbb{R}})$ (resp.$(E,\ G^s_{\mathbb{R}})$) is a symmetric G-metric space if and only if each $(E_k,\ G^k_{\mathbb{R}})$ is a symmetric G-metric space for $k = 1, \ldots, d$.

The set of real quaternions, denoted by $\mathbf{H}$, is defined by

$$\mathbf{H} := \{a_0 + a_1 i + a_2 j + a_3 k,\ \ a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\ \},$$

where $i^2 = j^2 = k^2 = ijk = -1,\ ij = k, jk = i,\ ki = j$. Note that $ij = -ji,\ ik = -ki,\ jk = -kj$ and there is an operation on $\mathbf{H}$ called quaternionic conjugation which is defined by

$$\overline{(a_0 + a_1 i + a_2 j + a_3 k)} = a_0 - a_1 i - a_2 j - a_3 k.$$

Every element of $\mathbf{H}$ has an additive inverse − if $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathbf{H}$ then $-q = (-a_0) + (-a_1)i + (-a_2)j + (-a_3)k \in \mathbf{H}$. For all $q \in \mathbf{H}$, $Re(q) := \frac{1}{2}(q + \overline{q})$ is the real part of q and $Im(q) := \frac{1}{2}(q - \overline{q})$ is the imaginary part of q.

The sum of $q_1 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_2 = b_0 + b_1 i + b_2 j + b_3 k$ is defined as

$$q_1 + q_2 = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k \in \mathbf{H}.$$

We refer the reader to the paper [11] for more details.

In [6] the authors have considered a partial order on $\mathbf{H}$ as follows:

$$\begin{cases} q_1 \preccurlyeq q_2 \ \text{ if and only if } \ Re(q_1) \leq Re(q_2), \\ Im_s(q_1) \leq Im_s(q_2),\ q_1, q_2 \in \mathbf{H};\ s = i, j, k; \\ \text{where } Im_i(q_r) = a_{1r},\ Im_j(q_r) = a_{2r}, Im_k(q_r) = a_{3r}, \\ q_r = a_{0r} + a_{1r} i + q_{2r} j + a_{3r} k,\ \ r = 1, 2. \end{cases}$$

Following the partial order defined on $\mathbf{H}$, the present author has been apparently the first one introduce the notion of generalized quaternion metric space.

Definition 1.2 ([1]). *Let E be an non-empty set and let $G_{\mathbf{H}} : E \times E \times E \to \mathbf{H}$ be a function satisfying the following conditions:*

(1) $G_{\mathbf{H}}(u, v, w) = \mathbf{0_H}$ *if* $u = v = w$*;*

(2) $\mathbf{0_H} \prec G_{\mathbf{H}}(u, u, v)$ *for all* $u, v \in E$ *with* $u \neq v$;

(3) $G_{\mathbf{H}}(u, u, v) \preccurlyeq G_{\mathbf{H}}(u, v, w)$ *for all* $u, v, w \in E$ *with* $v \neq w$*;*

(4) $G_{\mathbf{H}}(u, v, w) = G_{\mathbf{H}}(u, w, v) = G_{\mathbf{H}}(v, w, u) = ...$ (*symmetry in all three variables*)*;*

(5) $G_{\mathbf{H}}(u, v, w) \preccurlyeq G_{\mathbf{H}}(u, a, a) + G_{\mathbf{H}}(a, v, w)$, (*for all* $u, v, w, a \in E$, (*rectangle inequality*).

Then the function $G_{\mathbf{H}}$ is called a quaternion-valued generalized metric or, more specifically, a quaternion-valued $G_{\mathbf{H}}$-metric on E and the pair $(E, G_{\mathbf{H}})$ is called a quaternion-valued G-metric space.

In the paper [3], Bermúdez, Martinón and Müller introduced the notion of (m, q)-isometry for maps on a real-valued metric space $(E, d_{\mathbb{R}})$. A map $S : E \to E$ is called an (m, q)-isometric mapping for positive integer $m \geq 1$ and for real $q > 0$, if it satisfies

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} d(S^k u, S^k v)^q = 0, \ \ \forall \ \ u, v \in E.$$

Definition 1.3 ([1]). (i) Let $(E, G_{\mathbb{R}})$ be a real-valued G-metric space. A map $S : E \to E$ is called an (m, q)-G-isometry for some positive integer m and $q > 0$ if, for all $u, v, w \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbb{R}}(S^r u, S^r v, S^r w)^q = 0.$$

(ii) Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space. A map $S : E \to E$ is called an m-quaternion-valued G-isometric map for some positive integer m if, for all $u, v, w \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}(S^r u, S^r v, S^r w) = \mathbf{0_H}.$$

In the following, we collect some properties of (m, q)-G-isometric mappings.

The proof of the following theorem is very similar to ([1], Proposition 3.1, Theorem 3.1, Theorem 3.3) we omit it.

Theorem 1.1. *Let $(E, G_{\mathbb{R}})$ be a real-valued G-metric space and let $S : E \longrightarrow E$ be an mapping. The following statements hold:*

(1) *if S is an (m, q)-G-isometry, then S is an (n, q)-G-isometry for all positive integer $n \geq m$.*

(2) *if S invertible (m, q)-G-isometry, then S^{-1} is an (m, q)-G-isometry.*

(3) *if S is an (m, q)-G-isometry, then T^n is an (m, q)-G-isometry for all $n = 1, 2, \ldots, .$*

Theorem 1.2. *Let $(X, G_{\mathbb{R}})$ be a real-valued* $\mathbf{G}$*-metric space and $S, R : E \to E$ be a maps such that $RS = SR$. If S is an (m, q)-G-isometry and R is an (n, q)-G-isometry, then SR is an $(m + n - 1, q)$-G-isometry.*

The proof of this theorem is very similar to ([1], Theorem 3.4).

For mare details about the concept of m-quaternion valued G-isometries, the reader can refer to [1].

2. (m, ∞)-G-isometric mappings in generalized real metric space

In this section, we present the definition of (m, ∞)-G-isometric mapping on a real-valued metric spaces and give the main results of this topic. Similar results for (m, ∞)-isometric operators on Banach space were proved by P. Hoffmann et al. in [7].

Let $S : E \to E$ be an (m, q)-G-isometric mapping. It obvious that for all $u, v, w \in E$

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q = 0$$
$$\Leftrightarrow \sum_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q = \sum_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q$$
$$\Leftrightarrow g\Bigg(\sum_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q\Bigg)^{\frac{1}{q}}$$
$$= g\Bigg(\sum_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q\Bigg)^{\frac{1}{q}}.$$

By taking the limit as $q \to \infty$, we make the following definition of an (m, ∞)-G-isometric mapping.

Definition 2.1. *Let m be a positive integer $m \geq 1$. An mapping S acting on a generalized real-valued metric space $(E, G_{\mathbb{R}})$ is called an (m, ∞)-G-isometry (or (m, ∞)-generalized isometry) if for all $u, v, w \in E$*

$$\max_{\substack{j\in\{0\dots,m\}\\ (j \text{ even})}}\{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} = \max_{\substack{j\in\{0,\dots,m\}\\ (j \text{ odd})}}\{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\}.$$

Remark 2.1. (i) Every $(1, \infty)$-G-isometric mapping S is an G-isometric mapping i.e.,S satisfies

$$G_{\mathbb{R}}(Su, Sv, Sw) = G_{\mathbb{R}}(u, v, w) \text{ for all } u, v, w \in E.$$

(ii) An mapping $S : E \to E$ is an $(2, \infty)$-G-isometric mapping if and only if

$$G_{\mathbb{R}}(Su, Sv, Sw) = \max\{G_{\mathbb{R}}(S^2 u, S^2 v, S^2 w), G_{\mathbb{R}}(u, v, w)\}, \ \forall\, u, v, w \in E.$$

(iii) An mapping $S : E \to E$ is an $(3, \infty)$-G-isometric mapping if and only if

$$\max\{G_{\mathbb{R}}(Su, Sv, Sw), G_{\mathbb{R}}(S^3u, S^3v, S^3w)\}$$
$$= \max\{G_{\mathbb{R}}(S^2u, S^2v, S^2w), G_{\mathbb{R}}(u, v, w)\}, \ \ \forall\, u, v, w \in E.$$

Example 2.1. Let $E = \mathbb{R}$ and let $G_{\mathbb{R}}$ be the G-metric on $E \times E \times E$ defined as follows

$$G_{\mathbb{R}}(u, v, w) = |u - v| + |v - w| + |u - w|.$$

Define a map $S : E \to E$ by $Su = u + 2$. Clearly we have $S^ku = u + 2k$ for all $k \in \mathbb{N}$. From which we get $G_{\mathbb{R}}(S^ku, S^kv, S^kw) = |u - v| + |v - w| + |u - w|$. Consequently,

$$\max_{\substack{j\in\{0\dots,m\}\\ (j\ \ \text{even})}}\{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\} = \max_{\substack{j\in\{0,\dots,m\}\\ (j\ \ \text{odd})}}\{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\}.$$

So, we have that S is an (m, ∞)-G-isometric mapping.

Proposition 2.1. *An mapping S acting on a real valued G-metric space E is an (m, ∞)-G-isometric if and only if $\forall u, v, w \in E, \forall l \in \mathbb{N}_0$*

$$\max_{\substack{j\in\{l,\dots,l+m\}\\ (j\ \ even)}}\{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\} = \max_{\substack{j\in\{l,\dots,l+m\}\\ (j\ \ odd)}}\{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\}.$$

Proof. For $l \in \mathbb{N}_0$, substituting S^lu, S^lv and S^lw for u, v and w in Definition 2.1 ,we obtained the desired characterizations. □

Lemma 2.1 ([7])**.** *For all $k \in \mathbb{N}_0$ let $\pi(k) = k \bmod 2$ denote the parity of k. Let further $m \in \mathbb{N}$ and $a = (a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$. The following are equivalent:*

(1) *a satisfies*

$$\max_{\substack{k\in\{l,\dots,m+l\}\\ k\ even}} a_k = \max_{\substack{k\in\{l,\dots,m+l\}\\ k\ odd}} a_k, \ \ \forall\, l \in \mathbb{N}_0.$$

(2) *a attains a maximum and*

$$\max_{k\in\mathbb{N}_0}(a_k) = \max_{\substack{k\in\{l,\dots,m+l\}\\ \pi(k)=\pi(m-1+l)}}(a_k), \ \ \forall\, l \in \mathbb{N}_0.$$

Corollary 2.1. *Let $S : E \to E$ be an mapping on a G-metric space E and $m \in \mathbb{N}$. Then S is an (m, ∞)-G-isometric mapping if and only if, for all $u, v, w \in E$*

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} = \max_{\substack{k\in\{j,\dots,m+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}, \ \ \forall\, j \in \mathbb{N}.$$

Proof. The proof in an immediate consequence of Lemma 2.1. □

Corollary 2.2. *Let $S : E \to E$ be an mapping on a G-metric space E such is an (m,∞)-G-isometric. Then for all $n \in \mathbb{N}_0$*

$$G_{\mathbb{R}}(S^n u, S^n v, S^n w) \leq \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k u, S^k w)\}, \forall\, u, v, w, \in E.$$

Proof. From Corollary 2.1, we have

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k u, S^k w)\} = \max_{\substack{k\in\{j,\ldots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^k u, S^k v, , S^k w)\},$$
$$\forall\, u, v, w \in E, \quad \forall\, j \in \mathbb{N}_0.$$

This gives that $\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} < \infty$. Further, we see that for all $n \in \mathbb{N}_0$

$$\begin{aligned} G_{\mathbb{R}}(S^n u, S^n v, S^n w)\} &\leq \max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &\leq \max_{0\leq k\leq m-1}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \forall\, u, v, w \in E. \end{aligned}$$

□

Theorem 2.1. *Let S be an (m,∞)-G-isometry on a real-valued G-metric space $(E, G_{\mathbb{R}})$ such that $(E, G_{\mathbb{R}})$ is symmetric. Then there exists a real-valued G-metric $G_{\mathbb{R}}^{\infty}$ on E such that S is an G-isometry on $(E, G_{\mathbb{R}}^{\infty})$. Moreover $G_{\mathbb{R}}^{\infty}$ is given by*

$$G_{\mathbb{R}}^{\infty}(u, v, w) = \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \quad \forall\; u, v, w \in E.$$

Proof. By the assumption that S is an (m,∞)-G-isometry, we have by Corollary 2.2 that

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\; \forall\, u, v, w \in E.$$

Define the map $G_{\mathbb{R}}^{\infty} : E \times E \times E \to \mathbb{R}_+$ by

$$G_{\mathbb{R}}^{\infty}(u, v, w) := \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\; \forall\, u, , v, w \in E.$$

A simple calculation shows that the map $G_{\mathbb{R}}^{\infty}$ satisfies the conditions (1)-(5) of Definition 1.1. Hence $G_{\mathbb{R}}^{\infty}$ is a real-valued G-metric on E. Furthermore we have

$$\begin{aligned} G_{\mathbb{R}}^{\infty}(u, v, w) &= \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &= \max_{k\in\mathbb{N}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &= \max_{k\in\{j,\ldots,m-1+j\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\forall\, u, v, w \in E, \;\forall\, j \in \mathbb{N}. \end{aligned}$$

Consequently, $G_{\mathbb{R}}^{\infty}(u, v, w) = G_{\mathbb{R}}^{\infty}(Su, Sv, Sw)$. So, S is an isometry on (E, G_{∞}) and the proof is complete. □

Proposition 2.2. *Let $S : E \to E$ be an mapping and $m \in \mathbb{N}, m \geq 2$. If S satisfies the following conditions.*

(i) $G_{\mathbb{R}}(S^m u, S^m v, S^m w) = G_{\mathbb{R}}(S^{m-1}u, S^{m-1}v, S^{m-1}w)$, for all $u, v, w \in E$.

(ii) $G_{\mathbb{R}}(S^m u, S^m v, S^m w) \geq G_{\mathbb{R}}(S^k u, S^k v, S^k w)$ for $k = 0, \ \ldots, m-2$, for all $u, v, w, \in E$, then S is an (m, ∞)-G-isometry.

Proof. By the assumptions (i) and (ii), we have for all $u, v, w \in E$,

$$G_{\mathbb{R}}(S^m, S^m v, S^m w) = G_{\mathbb{R}}(S^{m-1}u, S^{m-1}v, S^{m-1}w)$$

and

$$G_{\mathbb{R}}(S^m u, S^m v, S^m w) \geq G_{\mathbb{R}}(S^k u, S^k v, S^k w), k = 0. \ldots, m-2.$$

From which we conclude that

$$\max_{\substack{k \in \{0,\ldots,m\} \\ k \text{ even}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k \in \{0,\ldots,m\} \\ k \text{ odd}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

This implies that S is an (m, ∞)-G-isometry by Definition 2.1. □

The following lemma gives a characterization of $(2, \infty)$-G-isometric mapping on a real-valued G-metric space.

Lemma 2.2. *Let $S : E \longrightarrow E$ be an mapping on a G-metric space E. Then S is an $(2, \infty)$-G-isometric mapping if and only if S satisfies the following conditions*

$$\begin{cases} G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw), \ \ \forall \ \ u, v, w \in E, \\ G_{\mathbb{R}}(S^2u, S^2v, S^2w) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \ \ u, v, w \in E. \end{cases} \tag{2.1}$$

Proof. Firstly, assume that S is an $(2, \infty)$-G-isometry, then we have for all $u, v, w \in E$,

$$G_{\mathbb{R}}(Su, Sv, Sw) = \max\{G_{\mathbb{R}}(S^2u, S^2v, S^2w), G_{\mathbb{R}}(u, v, w)\}$$

and it follows that for all $u, v, w \in E$

$$G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w)$$

and

$$G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(S^2u, S^2v, S^2w).$$

Replacing $u \longrightarrow Su$, $v \longrightarrow Sv$ and $w \longrightarrow Sw$ we obtain that

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = \max\{G_{\mathbb{R}}(Su, Sv, Sw), G_{\mathbb{R}}(S^3u, S^3v, S^3w)\}, \ \ \forall \, u, v, w \in E.$$

Thus, for all $u, v, w \in E$

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) \geq G_{\mathbb{R}}Su, Sv, Sw).$$

So,

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \, u, v, w \in E.$$

This gives (2.1).

Conversely assume that S satisfies (2.1) in this case we have that S is an $(2, \infty)$-G-isometry by Proposition 2.2. □

Theorem 2.2. *Let $S : E \longrightarrow E$ be an mapping on a G-metric space E. If S is an $(2, \infty)$-G-isometric mapping, then S^n is an $(2, \infty)$-G-isometric mapping.*

Proof. Let S be an $(2, \infty)$-isometric mapping. We need to prove that S^n is an $(2, \infty)$-G-isometric mapping for all positive integer n. By Lemma 2.2 it suffices to show that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^n u, S^n v, S^n w) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \ \ u, v, w \in E.$$

Firstly, we prove by mathematical induction on n that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^n u, S^n v, S^n w), \ \ \forall \ \ u, v, w \in E.$$

For $n = 1$ it is true since S is an $(2, \infty)$-G-isometry. Assume that this equality is true for n and prove it for $n + 1$. In fact, we have

$$\begin{aligned} G_{\mathbb{R}}(S^{2n+2}u, S^{2n+1}v, S^{2n+2}w) &= G_{\mathbb{R}}(S^{2n}S^2u, S^{2n}S^2v, , S^{2n}S^2v) \\ &= G_{\mathbb{R}}(S^nS^2u, S^nS^2v, S^nS^2w) \\ &= G_{\mathbb{R}}(S^{n+1}u, S^{n+1}v, S^{n+1}w), \ \ \forall \ \ u, v, w \in E. \end{aligned}$$

Thus by induction, we proved that $G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^n u, S^n v, , S^n w)$, $\forall u, v, w \in E$ holds for all $n = 1, 2 \ldots.$

It remains to show that for all $u, v, w \in E : G_{\mathbb{R}}(S^n u, S^n v, S^n w) \geq G_{\mathbb{R}}(u, v, w,)$, for all $n = 1, 2, \ldots.$

Indeed, since $G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w,), \forall u, v, w \in E$, we have by using the same inequality that for all $u, v, w \in E$

$$\begin{aligned} G_{\mathbb{R}}(S^n u, S^n v, S^n w) &= G_{\mathbb{R}}(SS^{n-1}u, SS^{n-1}v, SS^{n-1}w) \\ &\geq G_{\mathbb{R}}(S^{n-1}u, S^{n-1}v, S^{n-1}w) \\ &= G_{\mathbb{R}}(SS^{n-2}u, SS^{n-2}v, SS^{n-2}w) \\ &\geq G_{\mathbb{R}}(S^{n-2}u, S^{n-2}v, S^{n-2}w) \\ &\geq \cdots \\ &\geq G_{\mathbb{R}}(Su, Sv, Sw) \\ &\geq G_{\mathbb{R}}(u, v, w). \end{aligned}$$

By induction on n it follows that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, , S^{2n}w) = G_{\mathbb{R}}(S^n u, S^n v, S^n w) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \ \ u, v, w \in E.$$

Thus S^n is an $(2, \infty)$-G-isometry. □

Theorem 2.3. *Let T and S are two mappings acting on a G-metric space $T, S : E \longrightarrow E$ such that $TS = ST$. If T is an (m,∞)-G-isometry and S is an $(2,\infty)$-G-isometry, then TS is an (m,∞)-G-isometry.*

Proof. Firstly, assume that T and S are both $(2,\infty)$-G-isometry.

Since S is an $(2,\infty)$-G-isometry, we have by Lemma 2.2

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u,v,w),\ \forall\, u,v,w \in E.$$

It follows that for all $u, v, w \in E$ we have

$$\begin{aligned}
&G_{\mathbb{R}}((TS)^2u, (TS)^2v, (TS)^2w)\\
&= G_{\mathbb{R}}(T^2S^2u, T^2S^2v, , T^2S^2w) = G_{\mathbb{R}}(TS^2u, TS^2v, TS^2w)\\
&= G_{\mathbb{R}}(S^2Tu, S^2Tv, S^2Tw) = G_{\mathbb{R}}(TSu, TSv, TSw)\\
&\geq G_{\mathbb{R}}(Su, Sv, Sw)\ \ (\text{since } T \text{ is an } \ (2,\infty)-\text{isometry})\\
&\geq G_{\mathbb{R}}(u,v,w)\ \ (\text{since } S \text{ is an } (2,\infty)-\text{isometry}).
\end{aligned}$$

This implies that,

$$G_{\mathbb{R}}((TS)^2u, (TS)^2v, (TS)^2w){=}G_{\mathbb{R}}(TSu, TSv, TSw) \geq G_{\mathbb{R}}(u,v,w), \forall u,v,w \in E$$

thus, we have TS is an $(2,\infty)$-G-isometry by the statement in Lemma 2.2.

If we assume that T is an (m,∞)-G-isometry for $m > 2$ and that S is an $(2,\infty)$-G-isometry, we have by this fact

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u,v,w)\ \ \forall\, u,v,w \in E$$

and also for all $k = 1, 2, \dots$

$$G_{\mathbb{R}}(S^ku, S^kv, S^kw) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u,v,w)\ \ \forall\ \ u,v,w \in E.$$

Thus we have for all $u, v, w \in E$

$$\begin{aligned}
G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw) &= G_{\mathbb{R}}(T^kS^ku, T^kS^kv, T^kS^kw)\\
&= G_{\mathbb{R}}(ST^ku, ST^kv, ST^kw)\\
&\geq G_{\mathbb{R}}(T^ku, T^kv, T^kw).
\end{aligned}$$

Using the above inequality, for all $u, v, w \in E$, we have

$$\begin{aligned}
&\max_{\substack{k\in\{1,\dots,m\}\\ k\ \text{even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}\\
&= \max_{\substack{k\in\{1,\dots,m\}\\ k\ \text{even}}}\{G_{\mathbb{R}}(T^kSu, T^kSv, T^kSw)\} \geq \max_{\substack{k\in\{1,\dots,m\}\\ k\ \text{even}}} G_{\mathbb{R}}(T^ku, T^kv, T^kw).
\end{aligned}$$

So that

$$\max_{\substack{k\in\{0,\dots,m\}\\ k\ \text{even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} \geq \max_{\substack{k\in\{0,\dots,m\\ k\ \text{even}}}\{G_{\mathbb{R}}(T^ku, T^kv, , T^kw)\}.$$

On the other hand, it is obvious that for all $u, v, w \in E$

$$\max_{\substack{k\in\{1,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^kSu, T^kSv, T^kSw)\} \leq \max_{\substack{k\in\{1,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

We get for all $u, v, w \in E$.

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} \leq \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

From the above inequality, we obtain for all $u, v, w \in E$

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kv)\} = \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}} G_{\mathbb{R}}(T^ku, T^kv, T^kw)$$

By a similar way we have also for all $u, v, w \in E$

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} = \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

Using the fact that T is an (m, ∞)-isometry, we deduce that for all $u, v, w \in E$.

$$\begin{aligned}&\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}\\ &= \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}.\end{aligned}$$

The conclusion that TS is an (m, ∞)-G-isometric mapping follows immediately from Definition 2.1. □

Proposition 2.3. *Let $S : E \to E$ be an (m, ∞)-G-isometry mapping on a real-valued generalized metric space E. Then S is an $(m + 1, \infty)$-G-isometric mapping.*

Proof. Assume that S is an (m, ∞)-G-isometry, then it follows that

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} = \max_{\substack{k\in\{j,\dots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}$$

$\forall, u, v, w \in E, \forall j \in \mathbb{N}$. This implies that for all $u, v, w \in E$ and $\forall j \in \mathbb{N}$ we have

$$\begin{aligned}\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} &= \max_{\substack{k\in\{j,\dots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}\\ &\leq \max_{\substack{k\in\{j,\dots,m+j\}\\ \pi(k)=\pi(m+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} \leq \max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}.\end{aligned}$$

Consequently

$$\max_{k\in\mathbb{N}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} = \max_{\substack{k\in\{j,\dots,m+j\}\\ \pi(k)=\pi(m+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}.$$

So, S is an $(m + 1, \infty)$-G-isometry. □

Proposition 2.4. *Let S be an mapping acting on a real-valued G-metric space $(E, G_{\mathbb{R}})$ such that S^n is an G-isometry for odd integer n, then S is an (m,∞)-G-isometry for $m \geq 2n-1$.*

Proof. As an consequence of Proposition 2.3, it suffices to show that S is an $(2n-1,\infty)$-G-isometric mapping.

Indeed, by the assumption that S^n is an G-isometry, it follows that

$$G_{\mathbb{R}}(S^{k+n}u, S^{n+k}v, S^{n+k}w) = G_{\mathbb{R}}(S^k u, S^k v, S^k w), \quad \forall\, u,v,w \in E,\ \forall\, k \in \mathbb{N}_0.$$

On the other hand , since n is odd integer we have for all $k \in \mathbb{N}$, k is even if and only if $n+k$ is odd. By assumption, S^n is an G-isometry it follows that

$$\begin{gathered}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w),\ k \in \{0,1,...,2n-1\},\ k \text{ even }\}\\ \shortparallel \\ \{G_{\mathbb{R}}(S^k u, S^k v, S^k w),\ k \in \{0,1,...,2n-1\},\ k \text{ odd }\}.\end{gathered}$$

and it follows that for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,...,2n-1\}\\ k \text{ even}}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k\in\{0,...,2n-1\}\\ k \text{ odd}}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

Consequently, S is an $(2n-1,\infty)$-G-isometry. □

Theorem 2.4. *If $S : E \to E$ is an invertible mapping on a real-valued G-metric space E such is an (m,∞) -G-isometry, then S^{-1} is an (m,∞)-G-isometry.*

Proof. Assume that S is an (m,∞)-G-isometry, then we have by Definition 2.1 for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,...,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k\in\{0,...,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

Replacing $u \to S^{-m}u$, $v \to S^{-m}v$ and $w \to S^{-m}w$ we obtain for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,...,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(S^{k-m}u, S^{k-m}v, S^{k-m}w)\} = \max_{\substack{k\in\{0,...,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}(S^{k-m}u, S^{k-m}v, S^{k-m}w)\}$$

or equivalently

$$\begin{aligned}&\max_{\substack{k\in\{0,...,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((S^{-1})^{m-k}u, (S^{-1})^{m-k}v, (T^{-1})^{m-k}w)\}\\ &= \max_{\substack{k\in\{0,...,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((S^{-1})^{m-k}u, (S^{-1})^{m-k}v, (S^{-1})^{m-k}w)\}\end{aligned}$$

this gives

$$\begin{aligned}&\max_{\substack{k\in\{0,...,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((S^{-1})^{k}u, (S^{-1})^{k}v, (S^{-1})^{k}w)\}\\ &= \max_{\substack{0\in\{0,...,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((S^{-1})^{k}u, (S^{-1})^{k}v, (S^{-1})^{k}w)\}, \quad \forall\, u,v,w \in E.\end{aligned}$$

Thus, S^{-1} is an (m,∞)-G-isometry. □

Theorem 2.5. *For $k = 1, 2, ..., d$, let (E_k, G_k) be a G-metric space which is symmetric, and let $S_k : E_k \to E_k$ be a map, $m_k \geq 1$. Denote by $E = E_1 \times E_2 \times ... \times E_d$ the product space endowed with the product G-metric*

$$G_{\mathbb{R}}((u_1, u_2, \dots, u_d), (v_1, v_2, \dots, v_d), (w_1, \dots, w_k)) := \max_{1 \leq k \leq d} g\{G^k_{\mathbb{R}}(u_k, v_k, w_k)g\}.$$

Let $S := S_1 \times S_2 \times ... \times S_d : E \to E$ be a mapping defined by

$$S(u_1, \dots, u_d) := (S_1 u_1, S_2 u_2, \dots, S_n u_d).$$

If each S_k is an (m_k, ∞)- G-isometry for $k = 1, 2, \dots, d$, then S is an (m, ∞)-G-isometry, where $m = \max\{m_1, \dots, m_d\}$.

Proof. Let $m = \max\{m_1, m_2, \dots, m_d\}$ and consider for all $u, v, w \in E$

$$\begin{aligned}
&\max_{\substack{j \in \{0,\dots,m\} \\ j \text{ even}}} \{G_{\mathbb{R}}(S^j u, S^j v, (S^j w)\} \\
&= \max_{\substack{j \in \{0,\dots,m\} \\ j \text{ even}}} g(\max_{1 \leq k \leq d} \{G^k_{\mathbb{R}}(S^j_k u_k, S^j_k v_k, S^j_k w_k)\} g) \\
&= \max_{1 \leq k \leq d} g(\max_{\substack{j \in \{0,\dots,m\} \\ j \text{ even}}} \{G^k_{\mathbb{R}}((S^j_k u_k, S^j_k v_k, S^j_k w_k)\} g)
\end{aligned}$$

By the assumption that each S_k is an (m_k, ∞)-G-isometry for each $k = 1, 2, ..., d$, it follows that S_k is an (m, ∞)-G-isometry for $k = 1, 2, \dots, d$ (by Proposition 2.3). Then we have

$$\begin{aligned}
&\max_{\substack{j \in \{0,\dots,.,m\} \\ j \text{ even}}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} \\
&= \max_{1 \leq k \leq d} g(\max_{\substack{j \in \{0,\dots,m\} \\ j \text{ odd}}} \{G^k_{\mathbb{R}}(S^j_k u_k, S^j_k v_k, S^j_k w_k)\} g) \\
&= \max_{\substack{j \in \{0,\dots,m\} \\ j \text{ odd}}} g(\max_{1 \leq k \leq d} \{G^k_{\mathbb{R}}(S^j_k u_k, S^j_k v_k, S^j_k w_k)\} g).
\end{aligned}$$

Thus, we have

$$\max_{\substack{j \in \{0,\dots,m\} \\ j \text{ even}}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} = \max_{\substack{j \in \{0,\dots,m\} \\ j \text{ odd}}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\}.$$

Consequently, S is an (m, ∞)-G-isometric mapping and the proof is completed. □

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Accepted: 15.01.2018

On two-sided group digraphs and graphs

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Abstract. In this paper, we consider a generalization of Cayley graphs and digraphs (directed graphs) introduced by Iradmusa and Praeger. For non-empty subsets L, R of group G, two-sided group digraph $\overrightarrow{2S}(G; L, R)$ has been defined as a digraph having the vertex set G, and an arc from x to y if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. This article has strived to answer some open problems posed by Iradmusa and Praeger related to these graphs. Further, we determine sufficient conditions by which two-sided group graphs to be non-planar, and then we consider some specific cases on subsets L, R. We prove that the number of connected components of $\overrightarrow{2S}(G; L, R)$ is equal to the number of double cosets of the pair L, R when they are two subgroups of G.

Keywords: Cayley digraph, Cayley graph, group.

1. Introduction

Let G be a finite group and $S \subseteq G$ such that $e \notin S$. The Cayley digraph is defined as a digraph with vertex set G and an arc (x, y) (from vertex x to vertex y) if and only if $x^{-1}y \in S$ denoted by $\overrightarrow{Cay}(G, S)$. The condition $e \notin S$ yields a digraph with no loops. Moreover, if $S = S^{-1}$ (where $S^{-1} = \{s^{-1} | s \in S\}$), then we have a simple undirected graph [4] called a Cayley graph and denoted by $Cay(G, S)$. In this definition, S can be considered an empty set, by which the related Cayley graph has no edges. It is proved that the Cayley graph is connected if and only if S generates G [4]. There are many applications of Cayley graphs in different fields such as biology, coding theory and computer [3, 7, 9, 11]. So far various generalizations of Cayley graphs have been introduced, for example: generalized Cayley graph [12], quasi-Cayley graphs [6], various kinds of groupoid graphs [13, 14], group action graphs [2], general semigroup graphs [10], and there are many graphs that have been defined on algebraic structures by which many authors have been motivated to reveal some properties of the algebraic structures [1]. In this paper, we study a generalization of Cayley digraphs introduced by Iradmusa and Praeger in 2016 [8]. They named it two-sided group digraph (graph) and denoted by $\overrightarrow{2S}(G;L,R)$ ($2S(G;L,R)$). Also, they found conditions for the adjacency relation defining the digraphs to be symmetric, transitive or connected, etc. and they posed eight problems in their article [8]. This paper has strived to answer a number of those problems which are as follows. We should emphasize that we have solved only Problem 2 completely.

Problem 1 ([8]). *Decide whether or not $\overrightarrow{2S}(G;L,R)$ can be a regular graph of valency strictly less than $|L|\,|R|$, and, if it is possible, find necessary and sufficient conditions for this to occur.*

Problem 2 ([8]). *Decide whether or not there exist G, L, R satisfying the hypothesis of Theorem 1.7 such that $G = \langle L\rangle\langle R\rangle$, and $\overrightarrow{2S}(G;L,R)$ has connected components of different sizes.*

Problem 3 ([8]). *Find necessary and sufficient conditions on L and R for a two-sided group digraph $\overrightarrow{2S}(G;L,R)$ to be connected, when at least one of L and R is not inverse-closed.*

Let G be a group and L, R be two non-empty subsets of G, then the two-sided group digraph $\overrightarrow{2S}(G;L,R)$ is defined with vertex set G and an arc (x,y) from x to y' if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. The connection set of $\overrightarrow{2S}(G;L,R)$ is defined as the set $\hat{S}(L,R) = \{\lambda_{l,r} : l \in L, r \in R\}$, where $\lambda_{l,r}$ is a permutation of the form $\lambda_{l,r} : g \mapsto l^{-1}gr$, for certain $l, r \in G$. Note that if there are no loops and the adjacency relation is symmetric, then $\overrightarrow{2S}(G;L,R)$ will be regarded as a simple graph, and will be named a two-sided group graph. Let $x \in G$ be an arbitrary element; we define an equivalence relation on $L \times R$ as follows: $(l_1,r_1) \sim_x (l_2,r_2)$ if and only if $(x)\lambda_{l_1,r_1} = (x)\lambda_{l_2,r_2}$; then equivalence class containing (l,r) is presented as $C_x(l,r) = \{(l',r')|(x)\lambda_{l',r'} = (x)\lambda_{l,r}, l' \in L, r' \in R\}$ and C_x is the set of all equivalence classes of $\sim_x$. It is obvious when $\Gamma = \overrightarrow{2S}(G;L,R)$ is an undirected graph, then valency(x) is equal to $|C_x|$. In other words, the *valency*(x) is corresponding to a partition of $|L||R|$.

Definition 1.1 ([8]). Let G be a group with identity element e and two subsets L, R. Then a pair (L,R) has *2S-graph-property* if both L and R are non-empty, and the following conditions hold:

(i) $L^{-1}xR = LxR^{-1}$ for each $x \in G$;

(ii) $L^x \cap R = \emptyset$ for each $x \in G$;

(iii) $(LL^{-1})^x \cap (RR^{-1}) = \{e\}$ for each $x \in G$.

(i) and (ii) in previous definition guarantee a two-sided digraph with these properties is a simple graph.

Theorem 1.2 ([8])**.** *Let G be a group, and L, R be non-empty, inverse-closed subsets of G. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is a two-sided graph, which is regular of valency $|L||R|$, if and only if (L, R) has the 2S-graph-property.*

The authors of [8] posed Problem 1 after Theorem 1.2. Although we didn't determine necessary and sufficient conditions by which a two-sided group graph would be a regular simple graph of valency strictly less than $|L||R|$, for which we present some recognized sufficient conditions and also necessary conditions independently. As some results of above theorem, we have the next corollaries.

Corollary 1.3. *Let G be a group and L, R be two non-empty subsets of G, that $L^{-1}xR = LxR^{-1}$ and $L^x \cap R = \emptyset$ for each $x \in G$. If $(LL^{-1}) \cap (RR^{-1}) = \{e\}$ and $LL^{-1} \trianglelefteq G$ or $RR^{-1} \trianglelefteq G$, then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency $|L||R|$.*

Corollary 1.4. *Let G be a group and L, R be two non-empty subsets of G, that $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ for each $x \in G$. If Γ is regular of valency strictly less than $|L||R|$, then the orders of elements of LL^{-1} and RR^{-1} are not relatively prime.*

Proof. It is obvious Γ is a simple graph. If the orders of elements of LL^{-1} and RR^{-1} are relatively prime, then it is true for $(LL^{-1})^x$ and RR^{-1} for all $x \in G$ and it implies that Γ is regular of valency $|L||R|$ by Theorem 1.2, which is a contradiction.

The next theorem answers Problem 1 by using above equivalence relation without presenting certain properties on L, R and G.

Theorem 1.5. *Let G be a group and L, R be two non-empty subsets of G, and $|L| > 1, |R| > 1$. Then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency strictly less than $|L|\,|R|$, if and only if $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$, $\{e\} \subsetneqq RR^{-1} \cap (LL^{-1})^x$, and $|C_x| = |C_e|$ for all $x \in G$.*

Corollary 1.6. *Let G be a group and L, R are two non-empty subsets of G. If at least one of L or R is normal in G, $L^{-1}xR = LxR^{-1}$ for each $x \in G$ and $|LL^{-1} \cap RR^{-1}| > 1$. Then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency strictly less than $|L|\,|R|$.*

Proof. By Proposition 4.1 from [8] $\Gamma = 2S(G; L, R)$ is a Cayley digraph and other assumptions guarantee Γ is a regular simple graph of valency strictly less than $|L||R|$.

Theorem 1.7 ([8])**.** *Let L, R be non-empty, inverse-closed subsets of a group G, and let $\Gamma = \overrightarrow{2S}(G; L, R)$. Then Γ is connected if and only if*

(1) *$G = \langle L\rangle\langle R\rangle$, and there exist words w in L and w' in R, with lengths of opposite parity, such that the evaluation $ww' = e$ in G.*

Further, if $G = \langle L\rangle\langle R\rangle$, but condition (1) does not hold, then Γ is disconnected with exactly two connected components.

Problem 2 is related to above theorem; to solve this problem, we obtain two theorems and one corollary (Theorem 1.8, Corollary 1.9 and Theorem 1.10).

Theorem 1.8. *Let G be a group, and L, R be non-empty, inverse-closed subsets of G. Let $G = \langle L\rangle\langle R\rangle$ and at least L or R including a non-identity element of odd order. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is connected.*

Corollary 1.9. *If G is a group of odd order, L, R are non-empty, inverse-closed subsets of G including at least a non-identity element and $G = \langle L\rangle\langle R\rangle$. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is connected.*

Theorem 1.10. *If G is a group of even order, L, R are non-empty, inverse-closed subsets of G and $G = \langle L\rangle\langle R\rangle$ but condition (1) of Theorem 1.7 does not hold. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ has two connected components of the same size.*

Theorem 1.11. *Let L, R be non-empty subsets of a group G such that at least one of them is inverse closed and let $\Gamma = \overrightarrow{2S}(G; L, R)$. Then Γ is connected if and only if*

(2) *$G = \langle L\rangle\langle R\rangle$, and there exist words w in $L \cup L^{-1}$ and w' in $R \cup R^{-1}$, with lengths of opposite parity, such that $ww' = e$ in G.*

Theorem 1.11 is not a complete answer to Problem 3. In fact, the only case which remains to be answered is when both L, R are not inverse-closed.

Moreover, in this paper, we present sufficient conditions by which $\Gamma = \overrightarrow{2S}(G; L, R)$ is a non-planar graph. Also, we consider the case in which both L, R are singleton and, in this case, necessary and sufficient conditions has been found by which $\Gamma = 2S(G; L, R)$ is a matching. Then we consider a particular case when L, R are both subgroups of G and we prove that the number of connected components is equal to the number of double cosets of the pair (L, R) when $\Gamma = 2S(G; L, R)$ is not connected. Further, if L and R are p-Sylow and q-Sylow subgroups of G, respectively, for prime numbers $p \neq q$ and if $L^{\#} = L - \{e\}, R^{\#} = R - \{e\}$, and the pair $(L^{\#}, R^{\#})$ has 2S-graph-property; therefore Γ, in this case, is a regular simple graph of valency $(|L| - 1)(|R| - 1)$.

For a vertex x of a two-sided group digraph $\overrightarrow{2S}(G; L, R)$, the arcs beginning with x, are the pair (x, y) with $y = (x)\lambda$, for some $\lambda \in \hat{S}(L, R)$, such elements y are called out-neighbors of x, and the number of distinct out-neighbors of x is called the out-valency of x. Similarly, the arcs ending in x are the pairs (y, x) with $(y)\lambda = x$, for some $\lambda \in \hat{S}(L, R)$, such elements y are called in-neighbors of x, and the number of distinct in-neighbors of x is called the in-valency of x. If there is a constant c such that each vertex x has out-valency c and in-valency c, then $\overrightarrow{2S}(G; L, R)$ is regular of valency c.

Remark 1.12. Let L be a non-empty subset of a group G. Then a word w in L is a string $w = l_1l_2...l_k$ with each $l_i \in L$; the integer k is called the length of w, denoted by $|w|$, and we often identify w with its evaluation in G (the element of G is obtained by multiplying together the l_i in the given order).

The following remark, from Iradmusa and Praeger [8], is used in our proof. They have shown that the connected components are the sets $\mathcal{C}_\delta = \{g|g \in G, \delta(g) = \delta\}$, for $\delta \in \{0, 1\}$.

Remark 1.13. Let G be a group and L, R be non-empty, inverse-closed subsets of G, and let $\Gamma = \overrightarrow{2S}(G; L, R)$. If $G = \langle L\rangle\langle R\rangle$ but condition (1) does not hold, then for each $g \in G$, we can present $g = ww'$, where w and w' are words in L and R, respectively, the parity of the sum $|w|+|w'|$ is independent of the words w, w', and depends only on g. Let $\delta\,(g) \in \{0, 1\}$, where $\delta\,(g) \equiv |w| + |w'|\,(\mathrm{mod}2)$.

2. Main results

Let G be a group with two non-empty subsets L, R. If l is an arbitrary element of L, so $l^{-1}lr = r$ and this relation means (l, r) is an arc in $\Gamma = \overrightarrow{2S}(G; L, R)$, for each $l \in L, r \in R$. Similarly, (r^{-1}, l^{-1}) is an arc as well. Thus, $\{l, r\}, \{r^{-1}, l^{-1}\}$ are edges in Γ, in the case that Γ is undirected, so $valency(l) \geq |R|$ and $valency(r^{-1}) \geq |L|$. Hence, if Γ is an undirected regular graph, we will have $valency(x) \geq \frac{|L|+|R|}{2}$ for each $x \in G$.

Proposition 2.1. Let G be a group with two non-empty subsets L, R and $\Gamma = 2S(G; L, R)$ is a regular, undirected graph, then $\frac{|L|+|R|}{2} \leq valency(x) \leq |L||R|$.

Remark 2.2. Let L, R be non-empty subsets of group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$ be a complete digraph (without regarding directions among all arcs). Then for each $e \neq g \in G$: (e, g) or (g, e) is an arc, so $l^{-1}er = g$ or $l^{-1}gr = e$, for some $l \in L$ and $r \in R$. Thus $l^{-1}r = g$ or $g = lr^{-1}$; therefore, $g \in L^{-1}R \cup LR^{-1}$, and it follows that $G = \langle L^{-1}R \cup LR^{-1}\rangle$. In particular, if L and R be inverse-closed, in this case, we have $G = \langle LR\rangle$; however the reverse is not true in general. The following example illustrates this point.

Example 2.3. Let $G = S_3$, $L = \{(12)\}$, $R = \{(123), (132)\}$ and $\Gamma = \overrightarrow{2S}(G; L, R)$. Figure 1 displays this graph. Clearly, $G = S_3 = \langle LR\rangle$, and Γ is not complete.

Theorem 2.4. *Let G be a group, and $|G| = p^\alpha q^\beta m$, where p and q are distinct prime numbers and $\gcd(m, p) = 1, \gcd(m, q) = 1$. Let L and R be p-Sylow subgroup and q-Sylow subgroup of G, respectively. Suppose that $L^\# = L - \{e\}, R^\# = R - \{e\}$ and $\Gamma = \overrightarrow{2S}(G; L^\#, R^\#)$, then pair $(L^\#, R^\#)$ has the 2S-graph-property; therefore, Γ is a simple graph and it is regular of valency $(p^\alpha - 1)(q^\beta - 1)$.*

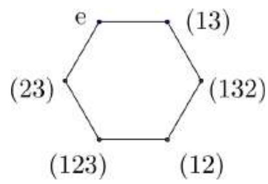

Figure 1: $2S(S_3, L, R)$.

Proof. It is clear that $L^{\#-1}gR^{\#} = L^{\#}gR^{\#-1}$. In addition, $L^{\#g} \cap R^{\#} = \emptyset$, for each $g \in G$, since L contains elements of a p-power order and R contains elements of a q-power order and $p \neq q$. Finally, we have $(L^{\#}L^{\#-1})^g \cap (R^{\#}R^{\#-1}) \subseteq L^g \cap R = \{e\}$. So pair $(L^{\#}, R^{\#})$ satisfies the third condition as well, therefore Γ is a graph. Thus by Theorem 1.2 Γ is regular of valency $(p^{\alpha} - 1)(q^{\beta} - 1)$.

A graph is planar if it can be drawn in such a way that no edges cross each other; by Kuratowski's theorem [6] we know that a graph is planar if and only if it contains no subgraph that is a subdivision of either K_5 or $K_{3,3}$; based on this theorem we have the next result.

Theorem 2.5. *Let L, R be non-empty subsets of a group G and let $\Gamma = 2S(G; L, R)$ be a two-sided group (undirected) graph. If we have $L \cap L^{-1}LR = \emptyset$, $R \cap L^{-1}RR = \emptyset$, $|L| \geq 3$ and $|R| \geq 3$, then Γ is non-planar.*

Proof. If $\{l, l'\}$ is an edge, for two arbitrary $l, l' \in L$, then $l' = l_1^{-1}lr_1$ for some $l_1 \in L$, $r_1 \in R$, thus $L \cap L^{-1}LR \neq \emptyset$ is a contradiction. So, for each $l, l' \in L$, $\{l, l'\}$ is not an edge. Similarly, assumption $R \cap L^{-1}RR \neq \emptyset$ implies that $\{r, r'\}$ is not an edge, for each $r, r' \in R$. And also for each $r \in R$ and $l \in L$ we have $r = l^{-1}lr$, so $\{r, l\}$ is an edge; therefore, Γ contains a complete bipartite graph $K_{|L|,|R|}$ as a subgraph. Since $|L| \geq 3$ and $|R| \geq 3$, so Γ contains $K_{3,3}$, hence Γ is non-planar. It is reminded that an independent set is a set of vertices in a graph, no two of which are adjacent.

Lemma 2.6. *Let L, R be two subsets of a group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$. If S is an independent subset of G, then $LS \cap SR = \emptyset$.*

Proof. Let $x \in LS \cap SR$ then $x = ls_1 = s_2r$ for some $s_1, s_2 \in S$ and $l \in L$, $r \in R$. Therefore we have $s_1 = l^{-1}s_2r$ so s_1 connected to s_2, and this is a contradiction.

Corollary 2.7. *Let L, R be two subsets of group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$. If S is an independent subset of G, then $|LS| + |SR| \leq |G|$.*

Proof. According to Lemma 2.6 we have : $LS \cap SR = \emptyset$; therefore we conclude: $|LS| + |SR| \leq |G|$.

In this part, we introduce some notations about two-sided group digraph. Let G be a group, and L, R be two non-empty subsets of G, and $\Gamma = \overrightarrow{2S}(G; L, R)$ be a two-sided group digraph of G with respect to L, R. Let $A = Aut(\Gamma)$, $Aut(G, L, R) = \{\alpha \in Aut(G) | L^\alpha = L, R^\alpha = R\}$, A_1 and 1^A be the stabilizer and the orbit of identity. It is reminded that $\mathcal{R}(G)$ and $\mathcal{L}(G)$ are considered as right and left representation respectively. Obviously, we have following results.

Proposition 2.8. (1) $\mathcal{R}(N_G(R))$, $\mathcal{L}(N_G(L))$ *are subgroups of* A.
(2) $N_G(L)N_G(R) \subseteq 1^A$.

Proof. It is clear.

Corollary 2.9. *Let G be a group and L, R be non-empty subsets of G, and let $\Gamma = \overrightarrow{2S}(G; L, R)$ be a two-sided group digraph. Then $Aut(G, L, R) \leq Aut(G) \cap A_1 \leq Aut(G, L^{-1}R)$.*

Proof. The first part of above inequality is clear. Now, assume that $\varphi \in Aut(G) \cap A_1$, therefore φ is a group homomorphism which keeps the adjacency relation and $\varphi(1) = 1$. Since 1 is connected to all elements of $L^{-1}R$; therefore, $\varphi(1) = 1$ connected to $\varphi(L^{-1}R)$. Thus $\varphi(L^{-1}R) \subseteq L^{-1}R$. It follows that $\varphi \in Aut(G, L^{-1}R)$.

2.1 Proof of Theorems

Proof of Theorem 1.5. First, if all given conditions are satisfied, it is clear, Γ is an undirected regular graph of valency strictly less than $|L|\,|R|$ by Lemma 3.1 [8] and argument which was presented before definition 1.1. Conversely, let Γ be a regular (undirected) graph of valency strictly less than $|L|\,|R|$. By Lemma 3.1 [8], it is clear that $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ and $|C_x| = |C_e|$ for each $x \in G$. On the other hand, for each $x \in G$, we have $\left|L^{-1}xR\right| < |L|\,|R|$, so there exist $(l_1, r_1), (l_2, r_2) \in L \times R$ such that $(l_1, r_1) \neq (l_2, r_2)$ $(|L| > 1, |R| > 1)$ and $l_1^{-1}xr_1 = l_2^{-1}xr_2$ $(l_1, l_2 \in L \text{ and } r_1, r_2 \in R)$, then $x^{-1}l_2l_1^{-1}x = r_2r_1^{-1}$. If $r_2r_1^{-1} = e$, then $r_1 = r_2$, and so $l_2 = l_1$ is a contradiction; therefore, $r_2r_1^{-1} \neq e$. Similarly if $l_1 = l_2$ then $r_1 = r_2$, i.e. $\{e\} \subsetneqq RR^{-1} \cap (LL^{-1})^x$.

Example 2.10. Let $G = S_3$, $L = \{(12), (23)\}$ and $R = \{(123), (132)\}$. It's easy to see that, this example has the mentioned above properties. This graph has been drawn by Figure 2, as you can see : $valency(x) = 3 < |L|\,|R| = 4$.

The next theorem gives a sufficient condition by which a simple two-sided group graph is regular of valency less than $|L||R|$.

Theorem 2.11. *Let G be a group, and L, R be two non-empty subsets of G. If G factorizes as $G = N_G(L)N_G(R)$ and $|LL^{-1} \cap RR^{-1}| > 1$, $\Gamma = 2S(G; L, R)$ be a simple graph, then Γ is regular of valency less than $|L|\,|R|$.*

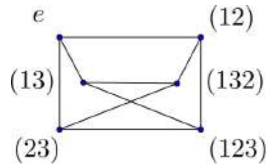

Figure 2: $2S(S_3, L, R)$.

Proof. By Theorem 1.13 of [8] Γ is vertex-transitive, thus Γ is regular. It is clear the valency of Γ is less than $|L|\,|R|$.

Theorem 2.12. *Let G be a group, and L, R are two non-empty subsets of G, $|G| = pm$ where p is a prime number, $gcd(p, m) = 1$, $|Syl_p(G)| > 1$, $LL^{-1} = RR^{-1}$ and $LL^{-1} \subseteq P$, for some $P \in Syl_p(G)$, $|L| > 1, |R| > 1$ and $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ for each $x \in G$, then $\Gamma = 2S(G; L, R)$ is a non-regular simple graph.*

Proof. It is clear Γ is a simple graph and $(LL^{-1})^x \subseteq P^x$, for all $x \in G$. Since $|Syl_p(G)| > 1$, so there is $x \in G$ such that $P^x \neq P$; therefore $|(LL^{-1})^x \cap RR^{-1}| \leq |P^x \cap P| = 1$. On the other hand, we have $|LL^{-1} \cap RR^{-1}| > 1$, thus Γ is not regular.

Proof of Theorem 1.8. Let $e \neq l \in L$ be an arbitrary element of odd order, and suppose m is its order. If $r \in R$ is an arbitrary element, so we have $l^m rr^{-1} = e$, and it means condition (1) of Theorem 1.7 holds, hence Γ is connected. Similarly, if R includes an element of odd order, then condition (1) holds.

Proof of Theorem 1.10. By Theorem 1.7 Γ is disconnected with exactly two connected components, and according to Remark 1.13 these two connected components are $\mathcal{C}_0$, and $\mathcal{C}_1$. We show that $|\mathcal{C}_0| = |\mathcal{C}_1|$. It is adequate to define function $\phi : \mathcal{C}_0 \to \mathcal{C}_1$ such that $\phi(g) = lg$, where $l \in L$ is an arbitrary element but fixed in L. It is clear that ϕ is well-defined, for $g \in \mathcal{C}_0$ and $g = x_g y_g$, we have $lg = lx_g y_g$, so $|lg| = |lx_g| + |y_g| = |x_g| + |y_g| \pm 1 \equiv 1(\text{mod}2)$, because $|x_g| + |y_g| \equiv 0(\text{mod}2)$, and it means $lg \in \mathcal{C}_1$. It is obvious ϕ is a one-to-one map, hence $|\mathcal{C}_0| = |\mathcal{C}_1|$.

Corollary 1.9 and Theorem 1.10 answer Problem 2, i.e. there are no G, L, R satisfying the hypothesis of Theorem 1.7 such that $G = \langle L\rangle\langle R\rangle$, and $\overrightarrow{2S}(G; L, R)$ has connected components of different sizes.

It should be reminded, Theorem 1.11 is a generalization of Theorem 1.7.

Proof of Theorem 1.11. If Γ is connected, then by Lemma 3.4 [8] condition (2) holds. Conversely, suppose that condition (2) holds. First, since $G = \langle L\rangle\langle R\rangle$, then we can write $g = x_g y_g$ for every $g \in G$ in which x_g and y_g are words in

$L \cup L^{-1}$ and $R \cup R^{-1}$, respectively. By condition (2) there are words x_e in $L \cup L^{-1}$ and y_e in $R \cup R^{-1}$ with lengths of opposite parity, such that $x_e y_e = e$. It implies, as it has been done in [8], for each $g \in G$ we can find x_g and y_g with the same length. Now, we suppose L is inverse-closed, then $x_g = l_k l_{k-1} ... l_1$ and $y_g = r_1 ... r_k$ such that $l_i \in L$ and $r_i \in R \cup R^{-1}$. If we put $g_i = l_i^{-1} g_{i-1} r_i$ and $g_0 = e$, then there is a path from e to g in Γ, because either (g_{i-1}, g_i) or (g_i, g_{i-1}) is an arc in Γ, and it depends on $r_i \in R$ or $r_i \in R^{-1}$. By a similar argument we can obtain a path from e to g when R is inverse-closed, thus Γ is connected.

2.2 Considering some specific case

Now, we consider the case in which L, R are singleton. Let L, R be non-empty subsets of group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$. It can be proved if $|L| = 1$ (or $|R| = 1$) then Γ is a regular digraph of valency $|R|$ ($|L|$). Furthermore, if $L = \{l\}, R = \{r\}, l \neq r$, then (L, R) has 2S-graph-property if and only if $l^2 = r^2, l^2 \in Z(G)$ and $r \neq x^{-1} l x$ for each $x \in G$; in particular $l \neq r$.

Example 2.13. Let $G = D_8 = \langle a, b | \ a^4 = b^2 = e, \ bab = a^{-1} \rangle$, be the dihedral group of order eight and $L = \{a^2\}$, $R = \{b\}$. Then $(a^2)^2 = b^2 = e$ and also $l^2 = (a^2)^2 = a^4 = e \in Z(D_8)$, and b, a^2 are not conjugate. Finally, Γ can be presented as follows.

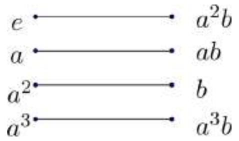

Figure 3: $\Gamma = 2S(D_8; L, R)$.

Proposition 2.14. *Let G be a group and L, R be non-empty subsets of G.*

(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ *is matching if and only if L and R are single-member having 2S-graph-property.*

(2) *If L, R are single-member subsets and pair (L, R) has the 2S-graph-property then the order of G is even.*

Proof. (1) Due to 2S-graph-property conditions and by Theorem 1.2, $\Gamma = \overrightarrow{2S}(G; L, R)$ is a regular graph of valency $|L|\,|R| = 1$, and it implies that graph Γ is a matching. Conversely, by Theorem 1.2 if $\Gamma = \overrightarrow{2S}(G; L, R)$ is a matching, thus Γ is a regular graph of valency 1 i.e. $|L|\,|R| = 1$, and so pair (L, R) has 2S-graph-property.

(2) By part (1), in this case, graph $\Gamma = \overrightarrow{2S}(G; L, R)$ is a matching and it concludes $|G|$ is even.

Proposition 2.15. *Let G be a group, $L = \{l\}, R = \{r\}$ be single-member subsets of G and $n \geq 3$ is an integer number. Then digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ has a cycle of length n (thus girth$\Gamma \leq n$) if and only if $l^n g = gr^n$ for some $g \in G$, and n is the least integer with this property.*

Proof. Suppose that digraph Γ has a cycle of length n, and this cycle is $x_1 x_2 ... x_n x_1$ in which $x_i \neq x_j$, when $i \neq j$. Because of $\Gamma = \overrightarrow{2S}(G; L, R)$ is a regular digraph of valency one, so we have: $x_2 = l^{-1} x_1 r$, $x_3 = l^{-1} x_2 r$, ..., $x_1 = l^{-1} x_n r$ and then $x_1 = \underbrace{l^{-1} l^{-1} ... l^{-1}}_{n-time} x_1 \underbrace{r ... rr}_{n-time} = l^{-n} x_1 r^n$, thus $l^n x_1 = x_1 r^n$.

Conversely, if $l^n g = gr^n$ for some $g \in G$, then $g = x_n \in G$ is satisfied with $x_n = l^{-n} x_n r^n$, and if we set $l^{-1} x_n r = x_1$, $l^{-1} x_1 r = x_2$, ..., $l^{-1} x_{n-2} r = x_{n-1}$ then $x_1 x_2 ... x_n x_1$ is a cycle of length n.

Let L, R be subgroups of group G and $\Gamma = \overrightarrow{2S}(G; L, R)$. Because of $L = L^{-1}$ and $R = R^{-1}$, adjacency relation in Γ is symmetric. However, the pair (L, R) doesn't have 2S-graph-property, because $\{e\} \subseteq L^x \cap R$ for each $x \in G$, so each vertex of Γ has a loop. In this case, because adjacency relation is symmetric, let us call Γ a graph for simply in spite of having loop on each vertex, and also we use words such as complete graph, regular graph, connected graph and domination number though we know, it is not a simple graph.

Let L, R be subgroups of group G and $H = \{\lambda_{l,r} | l \in L, r \in R\}$. It is clear H is a group (with the composition operation), and for each $x \in G$ and $\lambda_{l,r} \in H$ we have: $(x)\lambda_{l,r} = l^{-1} x r$, i.e. H acts on G, and stabilizer of x is $stab_H(x) = \{\lambda_{l,r} \in H | l^{-1} x r = x\} \leq H$, and the orbit of x; $orbit(x) = \{l^{-1} x r | \lambda_{l,r} \in H\} = LxR$ is a double coset of L and R, for each $x \in G$ and then $valency(x)$, in $\Gamma = \overrightarrow{2S}(G; L, R)$, is equal to $\frac{|H|}{|stab_H(x)|} = |LxR|$, and also an orbit is a connected component which is a complete subgraph with a loop on each vertex. In particular, the $orbit(e) = \{l^{-1} r | l \in L, r \in R\} = LR$, thus $|orbit(e)| = |LR| = \frac{|L||R|}{|L \cap R|}$, $stab_H(e) = \{\lambda_{r,r} \in H | r \in R \cap L\}$, so $|stab_H(e)| = |R \cap L|$, and hence $|H| = |L||R|$. In this case, if Γ is regular, then $|orbit(x)| = |orbit(e)| = |LR|$ for each $x \in G$. In other words, if Γ is a regular graph, then all double coset of L and R are the same size. In one specific case, if we consider $L = \{e\}$, then the connected component is the left coset R and the number of connected components is $\frac{|G|}{|R|}$ and Γ is a regular graph of valency $|R|$. By considering the action of H on G, we have: the kernel of this action contains all $\lambda_{l,l}$ such that $l \in L \cap R \cap Z(G)$, and it is faithful if and only if $L \cap R \cap Z(G) = \{e\}$, because $\lambda_{l,r}$ belongs to the kernel, if and only if $(x)\lambda_{l,r} = x$ for each $x \in G$, therefore $l^{-1} x r = x$ for each $x \in G$. Specially, if $x \in Z(G)$, then $l^{-1} x r = x$ and it concludes that $l = r$, it means $l \in L \cap R$. Now, $l^{-1} x l = x$ for each $x \in G$ implies that $l \in L \cap R \cap Z(G)$. If $G = LR$, then H acts on G transitively, and Γ is a complete graph. Therefore, we have the following theorem.

Theorem 2.16. *Let G be a group and L, R be subgroups of G, and $H = \{\lambda_{l,r} | l \in L, r \in R\}$. Then the group H acts on G, $\Gamma = \overrightarrow{2S}(G; L, R)$ is a graph with one* *loop on each vertex,* $valency(x) = \frac{|H|}{|stab_H(x)|}$ *for each* $x \in G$ *and* $|H| = |L||R|$*. In particular* Γ *is regular if and only if* $valency(x) = |LR|$ *for each* $x \in G$*. Graph* Γ *is connected if and only if* $G = LR$ *and otherwise, the number of connected components is equal to the number of double coset of the pair* (L, R)*.*

A dominating set for a graph is a subset D of its vertices such that every vertex which is not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating set for the graph [5].

Corollary 2.17. *Let* G *be a group, and* L, R *be subgroups of* G*, and* $\Gamma = \overrightarrow{2S}(G; L, R)$*. Then domination number of* Γ *is the number of double coset of the pair* (L, R) *.*

Proof. Since $L, R \leq G$, then connected components graph Γ are complete, therefore domination number of each connected component of graph Γ is one, so domination number of Γ is the number of connected components. It is trivial that the domination set contains one representative of each double coset of L and R.

Theorem 2.18. *Let* L, R *be two subgroups of a group* G*. Then:*

(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ *is a regular graph with one loop on each vertex, of valency strictly less than* $|L|\,|R|$*, if and only if* $|L \cap R| > 1$*.*

(2) *The valency of* e *in graph* $\Gamma = \overrightarrow{2S}(G; L, R)$ *is one, if and only if* $L = R = \{e\}$*.*

Proof. By Theorem 2.16, part (1) is clear.

If valency $(e) = 1$, then $\frac{|L||R|}{|L \cap R|} = 1$, therefore $L = R = \{e\}$, and the converse is clear. In this case $H = \{\lambda_{e,e}\} = \{id\}$ and $orbit(x) = \{x\}$ i.e. the graph Γ has only loops on each vertex.

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Accepted: 17.01.2018

A characterization of simple group $S_4(7)$

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Abstract. In this paper, we will use nse to give a new characterization of simple group $S_4(7)$.
Keywords: finite groups, characteristic, simple group, element orders.

1. Introduction

In this paper, all groups are finite. Let G be a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of G and $n_p(G)$ the number of Sylow p-subgroups of G, for $p \in \pi(G)$. Recall that G is a simple K_n-group if G is simple with $|\pi(G)| = n$. Write $M_t(G) := \{g \in G \mid g^t = 1\}$ with integer t. We call that groups G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$ for all t. All further unexplained notation is standard, readers may refer to [1].

This paper is related to Thompson's Problem.

Thompson's problem. Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Unfortunately, up to now, no one can prove the Thompson's Problem, even give a counterexample. Let $k \in \pi_e(G)$ and $m_k(G)$ be the number of elements

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of order k in G. Set $nse(G) := \{m_k(G) \big| k \in \pi_e(G)\}$. We call $nse(G)$ the set of numbers of elements with the same order. It is clear that if the assumption on Thompson's Problem holds, we always have $nse(G_1) = nse(G_2)$. So it is natural to investigate the Thompson's Problem by $nse(G)$, for instance, Asboei A.K, and Amiri S.S.S characterized PSL$(2, p)$ by the set nse in [6]. Recently, Liu characterized A_{26} by the set nse in [7].

In this present paper we provide a new method in characterizing the simple group $S_4(7)$. Our result is:

Main theorem. Let G be a group. Then $G \cong S_4(7)$ if and only if:
(1) $|G| = |S_4(7)|$;
(2) $nse(G) = nse(S_4(7))$.

2. Preliminaries

In this section, we give some lemmas which will be used in the sequel.

Lemma 2.1 ([2, Theorem 9.1.2]). *Let G be a group and t be a positive integer dividing $|G|$. Then $t \big| |M_t(G)|$.*

Lemma 2.2 ([2, Theorem 9.3.1]). *Let G be a solvable group of order mn, where $(m, n) = 1$. Then the number h_m of subgroups of order m may be expressed as a product of factors, each of which (a) is congruent to 1 modulo some prime factor of m, and (b) is a power of a prime and divides one of the chief factors of G.*

Lemma 2.3 ([3, Theorem 2]). *If G is a simple K_3-group, then G is isomorphic to one of the following groups: $A_5(2^2 \cdot 3 \cdot 5), A_6(2^2 \cdot 3^2 \cdot 5), L_2(7)(2^3 \cdot 3 \cdot 7), L_2(8)(2^3 \cdot 3^2 \cdot 7), L_2(17)(2^3 \cdot 3 \cdot 7 \cdot 17), L_3(3)(2^4 \cdot 3^3 \cdot 13)$, $U_3(3)(2^5 \cdot 3^3 \cdot 7)$ or $U_4(2)(2^6 \cdot 3^4 \cdot 5)$.*

Lemma 2.4 ([5, Lemma 2.3]). *Let G be a simple group of order $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, then $G \cong A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7), A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7), A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7), A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7), L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7), S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4), S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7), U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7), U_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 7)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$ or $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$.*

Lemma 2.5 ([5, Lemma 2.5]). *Let G be a group, $P \in Syl_p(G)$, where $p \in \pi(G)$. If G has a normal series: $K \trianglelefteq L \trianglelefteq G$ such that $P \leq L$ and $p \nmid |K|$, then the following hold:*
(1) $\boldsymbol{N}_{G/K}(PK/K) = \boldsymbol{N}_G(P)K/K$;
(2) $|G : \boldsymbol{N}_G(P)| = |L : \boldsymbol{N}_L(P)|$, that is, $n_p(G) = n_p(L)$;
(3) $|L/K : \boldsymbol{N}_{L/K}(PK/K)|t = |G : \boldsymbol{N}_G(P)| = |L : \boldsymbol{N}_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t. And $|\boldsymbol{N}_K(P)|t = |K|$.

3. Proof of the main theorem

Proof. The necessity of the proof is trivial, we only prove the other direction. By [1], we see that $|S_4(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ and $nse(S_4(7)) = \{1, 52675, 274400,$

2366700, 5531904, 6860000, 5764800, 10290000, 11524800, 5409600, 13171200, 23049600, 27659520, 9878400, 6585600}.

Let $t \in \pi_e(G)$ and k be the number of cyclic subgroups of G with order t. Then $m_t(G) = k\varphi(t)$. If $t > 2$, then $\varphi(t)$ is even, so is $m_t(G)$. This indicates that $m_2(G) = 52675$. Moreover, $m_3(G) = 274400$ or 6860000, $m_5(G) = 5531904$ and $m_7(G) = 5764800$ by Lemma 2.1.

We claim that G is non-solvable. Suppose false, then G is solvable. Let $\pi = \{3, 5, 7\}$ and H be a Hall π-subgroup of G. By Lemma 2.2, we see that $n_7(H) = 3^{a_0} \cdot 5^{b_0}$, where $3^{a_0} \equiv 1 (\text{mod } 7)$, $5^{b_0} \equiv 1 (\text{mod } 7)$ with $a_0 \leq 2$ and $b_0 \leq 2$, implying $n_7(H) = 1$. As a result, $m_7(G) \leq (7^4 - 1) \cdot 2^8 = 614400$ contradicting with $m_7(G) = 5764800$. Hence G is non-solvable, as claimed.

Let N be a maximal solvable normal subgroup of G and M/N be a minimal normal subgroup of G/N. Then M/N is non-solvable.

Assume that M/N is non-simple. Then $M/N = S_1 \times \cdots \times S_l$ is a direct product of isomorphic simple groups S_i, where $l \geq 2$. Moreover, by comparing the group orders, it follows by Lemma 2.3 and Lemma 2.4 that $l = 2$ and that $S_1 \cong A_5$ or $L_2(7)$. Assume that $S_1 \cong A_5$. Then $M/N = A_5 \times A_5$. Let $A/N := \mathbf{C}_{G/N}(M/N)$. Then $G/A \cong (G/N)/(A/N)$ is a subgroup of $\mathrm{Aut}(M/N)$, the automorphic group of M/N. Notice that $\mathrm{Aut}(A_5 \times A_5) = \mathrm{Aut}(A_5) \wr S_2$. Then $|A| \big| 2^4 \cdot 7^4$, which implies that A is solvable. Thus $A = N$ as N is the maximal solvable normal subgroup of G. Consequently, $G/N \leq \mathrm{Aut}(A_5) \wr S_2$, implying and $7^4 \big| |N|$ and $|N| \big| 2^4 \cdot 7^4$, this forces $m_7(G) = m_7(N) < |N|$, contrary to $m_7 = 5764800$. Hence $S_1 \ncong A_5$. Similarly, $S_1 \ncong L_2(7)$.

Consequently, M/N is a non-abelian simple group. By Lemma 2.3 and 2.4, it follows that $M/N \cong A_5, A_6, L_2(7), L_2(8), A_7, A_8, L_2(49), L_3(4)$ or $S_4(7)$. Next we prove that $\mathbf{C}_{G/N}(M/N) = 1$.

Suppose that $A/N := \mathbf{C}_{G/N}(M/N) > 1$. Since $A/N \cap M/N = Z(M/N) = 1$, we see that $A/N \times M/N \leq G/N$. This indicates that $|M/N| \big| |G/A|$. On the other hand, $G/A \cong (G/N)/(A/N) \leq \mathrm{Aut}(M/N)$ implies that $|G/A| \big| |\mathrm{Aut}(M/N)|$. That is, $|G|/|\mathrm{Aut}(M/N)|$ divides $|A|$ and $|A|$ divides $|G|/|M/N|$.

If $M/N \cong A_5$, then the argument above forces $2^5 \cdot 3 \cdot 5 \cdot 7^4 \big| |A|$ and $|A| \big| 2^6 \cdot 3 \cdot 5 \cdot 7^4$. Moreover, $A \trianglelefteq G$ implies that $m_7 = m_7(A) \leq |A| \leq 2^6 \cdot 3 \cdot 5 \cdot 7^4 = 2304960$, contrary to $m_7 = 5764800$. Similarly, $M/N \ncong A_6$.

Now we consider the case $M/N \cong L_2(7)$. The the same argument above gives $2^4 \cdot 3 \cdot 5^2 \cdot 7^3 \big| |A|$ and $|A| \big| 2^5 \cdot 3 \cdot 5^2 \cdot 7^3$. Notice that $A \trianglelefteq G$. It follows that $m_5(G) = m_5(A) < |A| \leq 2^5 \cdot 3 \cdot 5^2 \cdot (7^3 - 1) = 823200$, contradicting with $m_5 = 5531904$. Analogously, $M/N \ncong L_2(8)$.

As a result, M/N must be a simple K_4-group. Since A/N is non-solvable, we see that A/N has a section which is a simple group. Note that $A/N \times M/N \leq G/N$, in comparison the order of $A/N \times M/N$ with $|G|$, then only possibility is A/N has a section isomorphic to $L_2(7)$ and $M/N \cong L_2(49)$. In this case, $A/N \times M/N \leq G/N$ implies that $|N| \big| 14$. Note that $|M/N| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and $M \trianglelefteq G$, we obtain $m_5(G) = m_5(M)$. Recall that $m_5(M/N) = 4704$. Then

$m_5(M) \leq m_5(M/N) \cdot 14 = 65856 < m_5(G)$. This contradiction shows that $\mathbf{C}_{G/N}(M/N) = 1$, as required.

Furthermore, $G/N \leq \mathrm{Aut}(M/N)$. Recall that $M/N \cong A_5, A_6, L_2(7), L_2(8), A_7, A_8, L_2(49)$, $L_3(4)$ or $S_4(7)$, we will discuss it case by case.

Assume first that $M/N \cong A_5$. Then $G/N \leq S_5$, which implies that $G/N = A_5$ or S_5. Moreover, $|N| = 2^5 \cdot 3 \cdot 5 \cdot 7^4$ or $2^4 \cdot 3 \cdot 5 \cdot 7^4$. Since $N \trianglelefteq G$, we see that $m_7(G) = m_7(N)$. Moreover, by Lemma 2.2 we obtain that $n_7(N) = 1$ or 8. Therefore, $m_7 \leq 19200$, contrary to $m_7 = 5764800$. Similarly, $M/N \ncong A_6$.

If $M/N \cong L_2(7)$, then $G/N \leq \mathrm{Aut}(L_2(7))$ and thus $|N| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^3$ or $2^4 \cdot 3 \cdot 5^2 \cdot 7^3$. As $N \trianglelefteq G$, then $m_5(G) = m_5(N)$. By Lemma 2.2, we have $n_5(N) = 1$ or 2^4, which implies $m_5 \leq 400$, again contradiction. Similarly, $M/N \ncong L_2(8)$.

Now we consider the case that $M/N \cong A_7$. Then $G/N \leq S_7$, which implies that $G/N \cong A_7$ or S_7. Hence $|N| = 2^5 \cdot 5 \cdot 7^3$ or $|N| = 2^4 \cdot 5 \cdot 7^3$. By Lemma 2.5, we see that $n_3(G) = n_3(G/N)t$ for some integer t. As a result, $n_3(G) = 70t$ since $n_3(G/N) = 70$. Let P_3 be a Sylow 3-subgroup of G. As there is no element of order 9 in G/N, we see that P_3 is elementary abelian. This implies that $140t \leq m_3 \leq 560t$. If $m_3 = 274400$, then $490 \leq t \leq 1960$. On the other hand, $|\mathbf{N}_N(P_3)|t = |N|$ by Lemma 2.5, which indicates that $t \big| |N|$. Along with the fact that $3 \big| (70t - 1)$ by Sylow's Theorem, we obtain that $t = 490$ or 1960, yielding $|\mathbf{N}_N(P_3)| = 2^4 \cdot 7$. Notice that the action of P_3 on N by conjugation is coprime, there is a P_3-invariant group $N_5 \in \mathrm{Syl}_5(N)$. In N_5P_3, by applying Sylow's Theorem, we have $P_3 \trianglelefteq G$. Hence $N_5P_3 = N_5 \times P_3 \leq G$. This implies that $N_5 \leq N_N(P_3)$, a contradiction. Similarly, if $m_3 = 6860000$, we also get a contradiction.

Assume then that $M/N \cong L_2(49)$. In this case, $G/N \leq \mathrm{Aut}(L_2(49))$. Let P_5 be a Sylow 5-subgroup of G. Then P_5 is a cyclic subgroup of order 25 because M/N has an element of order 25. Moreover, $5 \nmid |G/M|$ and $M \trianglelefteq G$ imply $m_{25}(G) = m_{25}(M)$. By Lemma 2.5, we have $n_5(G) = n_5(M/N)t$ and $|\mathbf{N}_M(P_5)|t = |M|$ for some integer t. Furthermore, by [1], we see that $n_5(M/N) = 2^3 \cdot 3 \cdot 7^2$. As a result, $m_{25} = \varphi(25) \cdot n_5(G) = 20tn_5(M/N) = 2^5 \cdot 3 \cdot 5 \cdot 7^2 t = 27659520$, implying $t = 1176 = 2^3 \cdot 3 \cdot 7^2$. Thus $n_5(M) = 2^6 \cdot 3^2 \cdot 7^4$, yielding $|\mathbf{N}_M(P_5)| \big| 2^2 \cdot 5^2$. Since $|\mathbf{N}_N(P_5)|t = |N| \big| 2^4 \cdot 3 \cdot 7^2$, we have that $|\mathbf{N}_N(P_5)| \big| 2$. By a similar argument above, there is also contradiction.

Now we show that $M/N \ncong L_3(4)$. If not, then $M/N \cong L_3(4)$, implying $G/N \leq \mathrm{Aut}(L_3(4))$. Moreover, $|N| \big| 2^2 \cdot 5 \cdot 7^3$. By [1], we obtain $n_3(L_3(4)) = 2^3 \cdot 5 \cdot 7$. Moreover, $n_3(M) = n_3(M/N)t$ according to Lemma 2.5. Assume that $m_3 = 274400$. Then $2n_3(M) \leq m_3 \leq 8n_3(M)$, which implies that $122 < t \leq 490$. If $5 \big| t$, then $t = 140$, 245 or 490. Note that $3 \big| (n_3(G) - 1)$, we obtain $t = 490$ and thus $n_3(G) = 137200$. Hence $|\mathbf{N}_G(P_3)| = 2^4 \cdot 3^2 \cdot 7$. The similar argument in the case $G \cong A_7$ also gives a contradiction. Hence $5 \nmid t$. Moreover, $t = 196$ or 343. If the former holds, then $5 \big| |\mathbf{N}_N(P_3)|$. We assert that $N_5 \leq \mathbf{C}_G(P_3)$, where $N_5 \in \mathrm{Syl}_5(N)$. It follows that $m_{15} = 4 \cdot m_3 = 1646400$ since $N_5 \trianglelefteq G$, a contradiction. If $t = 343$, then $|\mathbf{N}_N(P_3)| \big| 2^4 \cdot 5$. Similarly, we get a contradiction. Therefore, $m_3 = 6860000$ and $3062 < t \leq 12250$, which yields that $t = 3430$ or 6860. As

$|G : \mathbf{N}_G(P_3)| = 2^3 \cdot 5 \cdot 7t$, $t = 3430$ and $|\mathbf{N}_N(P_3)| = 2^4$. Similar to the statement in the case $G \cong A_7$, we get a contradiction again.

Now we deal with the case $M/N \cong A_8$. In this situation, we have $G/N \leq S_8$. By [1], we have that $n_3(A_8) = 280$. Assume $m_3(G) = 274400$, then $490 > t \geq 122$. On the other hand, $|\mathbf{N}_N(P_3)|t = |N| \big| 2^2 \cdot 5 \cdot 7^3$ gives $t = 196$ or 343. If $t = 196$, then $|\mathbf{N}_G(P_3)| \big| 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Since $\mathbf{N}_G(P_3)/C_G(P_3)$ is isomorphic to a subgroup of $\mathrm{Aut}(P_3)$, we have $35 \big| |\mathbf{C}_G(P_3)|$, which forces $C_{35} \times P_3 \leq G$ and thus $m_{15}(G) = 4 \cdot m_3(G) = 1646400$, a contradiction. If $t = 343$, then $|\mathbf{N}_G(P_3)| \big| 2^5 \cdot 3^2 \cdot 5$. Similarly, we also get a contradiction. Assume $m_3(G) = 6860000$, then $12250 \geq t > 3062$. Note that $|\mathbf{N}_N(P_3)|t = |N| \big| 2^2 \cdot 5 \cdot 7^3$, we have that $t = 3430$ or 6860 and $|\mathbf{N}_N(P_3)| \big| 2$. This final contradiction shows that $M/N \cong S_4(7)$. Consequently, $G = M \cong S_4(7)$, the proof is completed. $\square$

Acknowledgements

The authors thank the referee for his/her valuable suggestions who pointed out several linguistic inaccuracies. It should be said that we could not have polished the final version of this paper well without their outstanding efforts.

This work was Supported by the Nature Science Fund of Shandong Province (No. ZR2019MA044) and the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2018QZJ04).

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Accepted: 19.01.2018

Redefined neutrosophic filters in BE-algebras

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Abstract. Neutrosophic set theory is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. In 2015, neutrosophic set theory is applied to BE-algebra, and the notion of neutrosophic filter is introduced. In this paper, some mistakes and deficiencies of original definition of neutrosophic filter are pointed out by some examples. Moreover, a new definition of neutrosophic filter is established, some basic properties are presented, and the relationships between fuzzy filters and neutrosophic filters are discussed. Finally, the concept of implicative neutrosophic filter in BE-algebra is introduced, and some necessary and sufficient conditions for a neutrosophic filter to be implicative are presented.

Keywords: neutrosophic set, neutrosophic filter, BE-algebra, implicative neutrosophic filter, fuzzy filter.

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1. Introduction

The neutrosophic set is a more general form of mathema- tical concepts that express uncertainty, such as fuzzy sets and intuitionistic fuzzy sets (see [17, and 18]). In the neutrosophic set, truth-membership, indeterminacy-member ship, and falsity-membership are represented independently. In this paper we work with special neutrosophic set (it is called single valued neutrosophic set, see [21]). The neutrosophic set theory is applied to many scientific fields (see [3033]), including algebraic systems (see [4, 10, 16, 28, and 29]), it is similar to the applications of fuzzy set and soft set theory in algebraic structures ([2, 14, 25, 27 and 34-37]).

As a generalization of dual BCK-algebra and related non-classical logic algebras ([23, 24, and 26]), Hee Sik Kim and Young Hee Kim introduced the notion of BE-algebra (see [7]). Since then, many scholars have conducted in-depth research on BE-algebras. For examples, the concept of ideal of BE-algebra is proposed and some characterizations are presented by the notion of upper set in [1]; a procedure which generated a filter by a subset in a transitive BE-algebra is established in [8]; the fuzzy filter (ideal) theory in BE-algebra is investigated in [5, 9, 19]; the theory of pseudo BE-algebra is constructed in [3], and so on (see [6, 11, 12, 13, 15, and 22]).

In this paper, we further study on the applications of neutrosophic sets to BE-algebras. We introduce the new definition of neutrosophic filters in BE-algebras, and investigate some basic properties and present relationships between neutrosophic filters and fuzzy filters. Moreover, we introduce the notion of implicative neutrosophic filters in BE-algebras. The relation between implicative neutrosophic filter and neutrosophic filter is investigated.

2. Basic concepts and properties

Definition 2.1 ([16, 17, 18])**.** Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x), I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$. That is, $T_A(x) : X \to]^{-}0, 1^{+}[, I_A(x) : X \to]^{-}0, 1^{+}[$, and $F_A(x) : X]^{-}0, 1^{+}[$. Thus, there is no restrictionon the sum of $T_A(x), I_A(x)$, and $F_A(x)$, so $^{-}0 \leq supT_A(x)+supI_A(x)+supF_A(x) \leq 3+$.

Definition 2.2 ([21])**.** Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. Then, a simple valued neutrosophic set A can be denoted by

$$A = \{x, TA(x), IA(x), FA(x)|x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0,1]$ for each point x in X. Therefore, the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.3 ([16])**.** A simple valued neutrosophic set A is contained in the other simple valued neutrosophic set B, denote $A \subseteq B$, if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$ for any x in X.

Remark 2.1. About the inclusion relation of neutrosophic sets, there are two different definitions in the literature. This article adopts the method in [16] (original definition byFlorentin Smarandache). Another way is given in [21], that is, $A \subseteq B$ if and only if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x)$ for any x in X.

Definition 2.4 ([16])**.** Two simple valued neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5 ([16])**.** The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity- membership functions are related to those of A and B by

$$T_C(x) = \max(T_A(x), T_B(x)), I_C(x) = \min(I_A(x), I_B(x)),$$
$$F_C(x) = \min(F_A(x), F_B(x)), \forall x \in X.$$

Definition 2.6 ([16])**.** The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity- membership functions are related to those of A and B by

$$T_C(x) = \min(T_A(x), T_B(x)), I_C(x) = \max(I_A(x), I_B(x)),$$
$$F_C(x) = \max(F_A(x), F_B(x)), \forall x \in X.$$

Definition 2.7 ([7])**.** By a BE-algebra we shall mean an algebraic structure $(X; \to, 1)$ of type (2, 0) satisfying the following axioms:

(BE1) $x \to x = 1$;

(BE2) $x \to 1 = 1$;

(BE3) $1 \to x = x$;

(BE4) $x \to (y \to z) = y \to (x \to z)$, for all$x, y, z \in X$.

For a BE-algebra $(X; \to, 1)$, we can define a relation $\leq$ on X by $x \leq y$ if and only if $x \to y = 1$.

Proposition 2.1[5,8] If $(X; \to, 1)$ is a BE-algebra, then for all $x, y \in X$,

(1) $1 \leq x \Rightarrow x = 1$.
(2) $x \to (y \to x) = 1$,or equivalently, $x \leq y \to x$.
(3)$x \to ((x \to y) \to y) = 1$,or equivalently, $x \leq (x \to y) \to y$.

A BE-algebra $(X; \to, 1)$ is said to be self distributive if $x \to (y \to z) = (x \to y) \to (x \to z)$, for all $x, y, z \in X$.

A BE-algebra $(X; \to, 1)$ is said to be commutative if $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in X$.

A BE-algebra $(X; \to, 1)$ is said to be transitive if for all $x, y, z \in X, y \to z \leq (x \to y) \to (x \to z)$.

Proposition 2.2 ([8])**.** If a BE-algebra $(X; \to, 1)$ is transitive, then for all $x, y, z \in X$,
(1) $y \leq z$ implies $x \to y \leq x \to z$.
(2) $y \leq z$ implies $z \to x \leq y \to x$.
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

Definition 2.8 ([7,8])**.** A subset F of BE-algebra $(X; \to, 1)$ is called a filter of X if it satisfies:
(F1) $1 \in F$;
(F2) $x \in F$ and $x \to y \in F \Rightarrow y \in F$.

Definition 2.9 ([7])**.** Let $(X; \to, 1)$ be a BE-algebra and let $x, y, z \in X$. Define $A(x, y) = \{z \in X | x \to (y \to z) = 1\}$. We call $A(x, y)$ an upper set of x and y. It is easy to see that $1, x, y \in A(x, y)$, for any $x, y \in X$.

Proposition 2.3 ([7])**.** *Let F be a non-empty subset of a BE- algebra* $(X; \to, 1)$*. Then F is a filter of X if and only if* $A(x, y) \subseteq F$*, for all* $x, y \in F$*.*

Definition 2.10 ([15])**.** A non-empty subset F of BE-algebra $(X; \to, 1)$ is called an implicative filter if satisfies the following conditions:
(IF1) $1 \in F$;
(IF2) $x \to (y \to z) \in F$ and$x \to y \in F$ imply that $x \to z \in F$, for all $x, y, z \in X$.

Definition 2.11 ([5,6,9])**.** A fuzzy set μ in BE-algebra $(X; \to, 1)$ is called a fuzzy filter of X if it satisfies:
(FF1) $\mu(1) \geq \mu(x)$;
(FF2)$\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$ for all $x, y \in X$.

Proposition 2.4 ([5,9])**.** *Let* μ *be a fuzzy filter of a BE-algebra* $(X; \to, 1)$*. Then, for any* $x, y \in X$*, if* $x \leq y$*, then* $\mu(x) \leq \mu(y)$*.*

Proposition 2.5 ([5,9])**.** *Let* μ *be a fuzzy set of a BE-algebra* $(X; \to, 1)$*.. Then the following conditions are equivalent.*
(1) μ *is a fuzzy filter in* X*;*

(2) *for all* $x \in X, \mu(1) \geq \mu(x)$*; and, for all* $x, y, z \in X$*,*$x \to (y \to z) = 1$ *implies* $\mu(z) \geq \min\{\mu(x), \mu(y)\}$*;*

(3) *for each* $\alpha \in [0, 1]$*, the level subset* $U(\mu; \alpha) = \{x \in X : \mu(x) \geq \alpha\}$ *is a filter of* X*, when* $U(\mu; \alpha) \neq \emptyset$*.*

3. Deficiencies of original definition of neutrosophic filter

In 2015, A. Rezaei, A. B. Saeid, and F. Smarandache [10] introduced the notion of neutrosophic filter in BE-algebras, and discussed some properties of neutrosophic filters.

Definition 3.1 (Definition 3.1 in [10])**.** A neutrosophic set A in a BE-algebra X is called a neutrosophic filter in X if satisfies the following conditions:

(NF1) $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$;

(NF2) $\min\{T_A(x), T_A(x \to y)\} \leq T_A(y), \min\{I_A(x), I_A(x \to y)\} \geq I_A(y)$ and $\min\{F_A(x), F_A(x \to y)\} \geq F_A(y)$,for all $x, y \in X$.

Proposition 3.1 (Theorem 3.4 in [10])**.** *Let* A *be a neutrosophic set of* X*. Then the following are equivalent:*

(i) *A is a neutrosophic filter in X;*

(ii) $(\forall t \in [0, 1])U(A; t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t, F_A(x) \leq t\} \neq \emptyset$ *imply* $U(A; t)$ *is a filter of* X*.*

Now, we give some counterexamples to show that Theorem 3.4 in [10] is not true and the original definition of neutrosophic filter in BE-algebras (Definition 3.1 in [10]) is not well-defined.

Example 3.1. Let $X = \{1, a, b, c, d\}$ be a set with the following operation table:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $(X; , 1)$ is a BE-algebra. Define a neutrosophic set A in X as follows:

$$T_A(x) = 0.79, for\ all\ x \in X;$$

$$I_A(x) = \begin{cases} 0.17, & if x = 1, a \\ 0.79, & \text{otherwise.} \end{cases}, F_A(x) = \begin{cases} 0.17, & if x = 1, a \\ 0.79, & \text{otherwise.} \end{cases}$$

Then we can verify that $(\forall t \in [0, 1])U(A; t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t, F_A(x) \leq t\} \neq \emptyset$ imply $U(A; t)$ is a filter of X. But A is not a neutrosophic filter in X, since

$$\min\{I_A(a), I_A(a \to d)\} = \min\{0.17, 0.79\} = 0.17 \ngeq I_A(d) = 0.79,$$
$$\min\{F_A(a), F_A(a \to d)\} = \min\{0.17, 0.79\} = 0.17 \ngeq F_A(d) = 0.79.$$

Example 3.2. Let $X = \{1, a, b\}$ be a set with the following operation table:

$\to$	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then $(X; , 1)$ is a BE-algebra. Define a neutrosophic set A in X as follows:

$T_A(1) = 0.9, T_A(a) = T_A(b) = 0.5; I_A(1) = 0.2, I_A(a) = I_A(b) = 0.35; F_A(1) = 0.1, F_A(a) = F_A(b) = 0$.

In [10], the authors said that A is a neutrosophic filter in X (see Example 3.1 in [10]). This is a mistake, since $\min\{I_A(a), I_A(a \to a)\} = \min\{0.35, 0.2\} = 0.2 \ngeq I_A(a) = 0.35, \min\{F_A(b), F_A(b \to b)\} = \min\{0, 0.1\} = 0 \ngeq F_A(1) = 0.1$.

Proposition 3.2. *Assume that the concept of neutrosophic filters in a BE-algebra X is defined by Definition 3.1. Let A be a neutrosophic filter in a BE-algebra X. Then:*

(i) $I_A(x) = I_A(1)$ *for all* $x \in X$*;*

(ii) $F_A(x) = F_A(1)$ *for all* $x \in X$*.*

Proof. For all $x \in X$, by Definition 3.1 (NF2) we have

$$I_A(1) \geq \min\{I_A(1), I_A(x)\} = \min\{I_A(1), I_A(1 \to x)\} \geq I_A(x).$$

That is, $I_A(x) \leq I_A(1)$. On the other hand, using (NF1) we have $I_A(x) \geq I_A(1)$. Hence $I_A(x) = I_A(1)$.

Similarly, we can get that $F_A(x) = F_A(1)$ for all $x \in X$.

By Proposition 3.2 we know that $I_A(x)$ and $F_A(x)$ are two constant valued functions for neutrosophic filter (it is defined by Definition 3.1 in [10]) in BE-algebras. Moreover, by Example 3.1 we know that a neutrosophic filter cannot be completely determined by its level subsets. All of these are inconsistent with the properties of the traditional fuzzy filter, so it is necessary to redefine the concept of neutrosophic filter in BE-algebras.

4. New definition of neutrosophic filters in BE-algebras

Definition 4.1. A neutrosophic set A in a BE-algebra X is called a neutrosophic filter in X if it satisfies:

(NSF1)$\forall x \in X, T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$;

(NSF2) $\forall x, y \in X, \min\{T_A(x), T_A(x \to y)\} \leq T_A(y), \max\{I_A(x), I_A(x \to y)\} \geq I_A(y)$ and $\max\{F_A(x), F_A(x \to y)\} \geq F_A(y)$;

Proposition 4.1 *Let A be a neutrosophic filter in BE- algebra X. Then:*
$(NSF3)\ \forall x, y \in X, x \leq y \Rightarrow T_A(x) \leq T_A(y), I_A(x) \geq I_A(y)$ *and* $\max F_A(x) \geq F_A(y)$.

Proof. If $x \leq y$, then $x \to y = 1$. It follows that $T_A(x \to y) = T_A(1)$. From this, using Definition 4.1 (NSF1) and (NSF2) we get

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A(x \to y)\} \leq T_A(y).$$

That is,$x \leq y \Rightarrow T_A(x) \leq T_A(y)$.

Similarly, we can get that $x \leq y \Rightarrow I_A(x) \geq I_A(y)$ and $x \leq y \Rightarrow F_A(x) \geq F_A(y)$.

It is easy to verify that the following proposition is true.

Proposition 4.2. *If A and B are two neutrosophic filters in a BE-algebra X, then $A \cap B$ is also a neutrosophic filter in X.*

Example 4.1. Let $(X; \to, 1)$ be the BE-algebra in Example 3.1. Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.86, & if x = 1, a \\ 0.13, & \text{otherwise} \end{cases}, I_A(x) = \begin{cases} 0.15, & if x = 1, a \\ 0.82, & \text{otherwise} \end{cases},$$

$$F_A(x) = \begin{cases} 0.15, & if x = 1, a \\ 0.82, & \text{otherwise.} \end{cases}$$

Then A is a neutrosophic filter in X.

Moreover, we can verify that $(\forall t \in [0,1]) U(A;t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t, F_A(x) \leq t\} \neq \emptyset$ imply $U(A;t)$ is a filter of X. This is a general result for every neutrosophic filter, it is proved as follows.

Proposition 4.3. *Let X be a BE-algebra, A be a neutrosophic filter in X. Then $(\forall t \in [0,1]) U(A;t) = \{x \in X : t \leq T_A(x), I_A(x) \leq t, F_A(x) \leq t\} \neq \emptyset$ imply $U(A;t)$ is a filter of X.*

Proof. Assume that A is neutrosophic filter in X and let $t \in [0,1]$ such that $U(A;t) \neq \emptyset$. Then there exists $x_0 \in X$ such that $t \leq T_A(x_0), I_A(x_0) \leq t$, and $F_A(x_0) \leq t$. By applying Definition 4.1 (NSF1) we have

$$t \leq T_A(x_0) \leq T_A(1), I_A(1) \leq I_A(x_0) \leq t \text{ and } F_A(1) \leq F_A(x_0) \leq t.$$

This means that $1 \in U(A;t)$. Let $x, y \in X, x \to y \in U(A;t)$ and $x \in U(A;t)$. Then $t \leq T_A(x \to y), I_A(x \to y) \leq t$ and $F_A(x \to y) \leq t$, $t \leq T_A(x), I_A(x) \leq t, and\ F_A(x) \leq t$. From these, using Definition 4.1 (NSF2) we have

$$t \leq \min\{T_A(x), T_A(x \to y)\} \leq T_A(y), I_A(y) \leq \max\{I_A(x), I_A(x \to y)\} \leq t,$$

$$F_A(y) \leq \max\{F_A(x), F_A(x \to y)\} \leq t.$$

This means that $y \in U(A;t)$. By Definition 2.8 we know that $U(A;t)$ is a filter of X.

The following example shows that the inverse of Proposition 4.3 is not true in general.

Example 4.2. Let $(X; \to, 1)$ be the BE-algebra in Example 3.1. Define a neutrosophic set B in X as follows:

$$T_B(x) = \begin{cases} 0.82, & if x = 1, a \\ 0.16, & \text{otherwise} \end{cases}, I_B(x) = \begin{cases} 0.16, & if x = 1, a \\ 0.82, & \text{otherwise} \end{cases},$$

$$F_B(x) = \begin{cases} 0.11, & if x = 1, a, b \\ 0.82, & \text{otherwise.} \end{cases}$$

Then we can get that:

If $t > 0.82, U(B;t) = \emptyset$;

If $0.82 \geq t > 0.16, U(B;t) = \{1, a\}$;

If $0.16 \geq t > 0.11, U(B;t) = \emptyset$;

If $0.11 \geq t, U(B;t) = \emptyset$.

This means that $(\forall t \in [0,1])U(B;t) = \{x \in X : t \leq T_B(x), I_B(x) \leq t, F_B(x) \leq t\} \neq \emptyset$ imply $U(B;t)$ is a filter of X. But B is not a neutrosophic filter in X, since $\max\{F_B(b), F_B(b \to d)\} = 0.11 \ngeq 0.82 = F_B(d)$.

Theorem 4.1. *Let A be a neutrosophic set in a BE-algebra X. Then A is a neutrosophic filter in X if and only if A satisfies:*

(i) T_A is a fuzzy filter of X;

(ii) $1 - I_A$ is a fuzzy filter of X, where $(1 - I_A)(x) = 1 - I_A(x), x \in X$;

(iii) $1 - F_A$ is a fuzzy filter of$X$, where $(1 - F_A)(x) = 1 - F_A(x), x \in X$.

Proof. Suppose that A is a neutrosophic filter in X. Then T_A is a fuzzy set on X; and using Definition 4.1 we have $\forall x, y \in X, T_A(x) \leq T_A(1), \min\{T_A(x), T_A(x \to y)\} \leq T_A(y)$. By Definition 2.11 we know that T_A is a fuzzy filter of X.

Moreover, it is easy to verify that $1 - I_A$ is a fuzzy set on X; and using Definition 4.1 we have: $\forall x, y \in X$,

$$(1 - I_A)(x) = 1 - I_A(x) \leq 1 - I_A(1) = (1 - I_A)(1);$$

$\min\{(1 - I_A)(x), (1 - I_A)(x \to y)\} = \min\{1 - I_A(x), 1 - I_A(x \to y)\} = 1 - \max\{I_A(x), I_A(x \to y)\} \leq 1 - I_A(y) = (1 - I_A)(y)$.

By Definition 2.11 we know that $1 - I_A$ is a fuzzy filter of X. Similarly, we can get that $1 - F_A$ is a fuzzy filter of X.

Conversely, suppose that neutrosophic set A satisfies the conditions (i), (ii) and (iii).Then by Definition 2.11 we have $(\forall x, y \in X)$,

$$T_A(x) \leq T_A(1), \min\{T_A(x), T_A(x \to y)\} \leq T_A(y);$$

$$(1-I_A)(x) \leq (1-I_A)(1), \min\{(1-I_A)(x), (1-I_A)(x \to y)\} \leq (1-I_A)(y);$$

$$(1-F_A)(x) \leq (1-F_A)(1), \min\{(1-F_A)(x), (1-F_A)(x \to y)\} \leq (1-F_A)(y).$$

It follows that, $\forall x, y \in X$,

$$\begin{aligned}
&T_A(x) \leq T_A(1),\\
&I_A(x) = 1-(1-I_A)(x) \geq 1-(1-I_A)(1) = I_A(1),\\
&F_A(x) = 1-(1-F_A)(x) \geq 1-(1-F_A)(1) = F_A(1),\\
&\min\{T_A(x), T_A(x \to y)\} \leq T_A(y),\\
&\max\{I_A(x), I_A(x \to y)\} = 1-\min\{(1-I_A)(x), (1-I_A)(x \to y)\}\\
&\geq 1-(1-I_A)(y) = I_A(y),\\
&\max\{F_A(x), F_A(x \to y)\} = 1-\min\{(1-F_A)(x), (1-F_A)(x \to y)\}\\
&\geq 1-(1-F_A)(y) = F_A(y).
\end{aligned}$$

From this, by Definition 4.1 we get that A is a neutrosophic filter in X.

Applying Theorem 4.1 and Proposition 2.5 (3) we can get:

Corollary 4.1. *Let A be a neutrosophic filter in a BE-algebra X. Then:*

(1) *for any $t \in [0,1], U(T_A;t) = \{x \in X | T_A(x) \geq t\}$ is a filter of X when $U(T_A;t) \neq \emptyset$;*

(2) *for any $t \in [0,1], U(1-I_A;t) = \{x \in X | 1-I_A(x) \geq t\}$ is a filter of X when $U(1-I_A;t) \neq \emptyset$;*

(3) *for any $t \in [0,1], U(1-F_A;t) = \{x \in X | 1-F_A(x) \geq t\}$ is a filter of X when $U(1-F_A;t) \neq \emptyset$.*

Definition 4.2 ([20]). Let A be a neutrosophic set in X and $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$ and (α, β, γ)-level set of A denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as:

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}.$$

Remark 4.1. In fact, the original definition of (α, β, γ)-level set in [20] is as follows:

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

In this paper, level set is defined as above in order to match the order relations of the neutrosophic set. Since this paper uses another ordering relationship (see Remark 2.1), the corresponding (α, β, γ)-level set uses the above Definition 4.2.

Theorem 4.2. *Let X be a BE-algebra, A be a neutrosophic set in X. Then A is a neutrosophic filter in X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are filters of X when $\alpha, \beta, \gamma \in [0,1]$ such that $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic filter in X and let $\alpha, \beta, \gamma \in [0,1]$ such that$A^{(\alpha,\beta,\gamma)} \neq \emptyset$. By Definition 4.2, we have $U(T_A;\alpha) \neq \emptyset, U(1-I_A;1-\beta) \neq \emptyset$, and $U(1-F_A;1-\gamma)\neq\emptyset$. Applying Theorem 4.1 and Proposition 2.5 (3), we get that $U(T_A;\alpha), U(1-I_A;1-\beta)$, and $U(1-F_A;1-\gamma)$ are filters of X. Thus $U(T_A;\alpha)\cap U(1-I_A;1-\beta)\cap U(1-F_A;1-\gamma)$ is also filter of X. Moreover, by Definition 4.2, it is easy to verify that

$$A^{(\alpha,\beta,\gamma)} = U(T_A;\alpha)\cap U(1-I_A;1-\beta)\cap U(1-F_A;1-\gamma).$$

Hence,(α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ is a filter of X.

Conversely, assume that all of (α,β,γ)-level sets $A^{(\alpha,\beta,\gamma)}$ are filters of X when $\alpha,\beta,\gamma\in[0,1]$such that $A^{(\alpha,\beta,\gamma)}\neq\emptyset$. For any $t\in[0,1]$, if $U(T_A;t)=\{x\in X|T_A(x)\geq t\}\neq\emptyset$, then there exists $x_0\in X$ such that $T_A(x_0)\geq t$. Obviously, $I_A(x_0)\leq 1, F_A(x_0)\leq 1$. It follows that $x_0\in A^{(t,1,1)}$, that is, $A^{(t,1,1)}=\{x\in X: t\leq T_A(x), I_A(x)\leq 1, F_A(x)\leq 1\}=\{x\in X|T_A(x)\geq t\}=U(T_A;t\neq\emptyset)$.

Hence, by the assumption $U(T_A;t) = A^{(t,1,1)}$ is a filter of X. Applying Proposition 2.5 we know thatT_A is a fuzzy filter of X. Moreover, for any $t\in[0,1]$, if $U(1-I_A;t)=\{x\in X|1-I_A(x)\geq t\}\neq\emptyset$, then there exists $x_0\in X$ such that $1-I_A(x_0)\geq t$, that is, $I_A(x_0)\leq 1-t$. Obviously, $T_A(x_0)\geq 0, F_A(x_0)\leq 1$. It follows that $x_0\in A^{(0,1-t,1)}$, that is, $A^{(0,1-t,1)}=\{x\in X: 0\leq T_A(x), I_A(x)\leq 1-t, F_A(x)\leq 1\}=\{x\in X|I_A(x)\leq 1-t\}=U(1-I_A;t)\neq\emptyset$.

Hence, by the assumption $U(1-I_A;t)=A^{(0,1-t,1)}$ is a filter of X. Applying Proposition 2.5 we know that$1-T_A$ is a fuzzy filter of X.

Similarly, we can get that $1-F_A$ is a fuzzy filter of X. Combining the above results, using Theorem 4.1, we know that A is a neutrosophic filter in X.

Applying Theorem 4.2 we can get

Corollary 4.2. *Let A be a neutrosophic filter in a BE-algebra X, we denote that:*

(1) $A_T=\{x\in X|T_A(x)=T_A(1)\}$;
(2) $A_I=\{x\in X|I_A(x)=I_A(1)\}$;
(3) $A_F=\{x\in X|F_A(x)=F_A(1)\}$.

Then A_T, I_T and F_T are are filters of X.

Corollary 4.3. *Let A be a neutrosophic filter in a BE-algebra X, we denote*

$$A_b=\{x\in X|T_A(x)\geq T_A(b), I_A(x)\leq I_A(b), F_A(x)\leq F_A(b)\}, b\in X.$$

Then A_b is a filter of X for every $b\in X$.

Theorem 4.3. *Let X be a BE-algebra, A be a neutrosophic set in X. Then A is a neutrosophic filter in X if and only if it satisfies (NSF1) and*

$(NSF4)$ $\forall x,y,z\in X$, *if* $x\to(y\to z)=1$, *then* $\min\{T_A(x),T_A(y)\}\leq T_A(z)$, $\max\{I_A(x),I_A(y)\}\geq I_A(z)$, *and* $\max\{F_A(x),F_A(y)\}\geq F_A(z)$.

Proof. Let A be a neutrosophic filter in X and let $x, y, z \in X$. Suppose that $x \to (y \to z) = 1$. Applying Definition 4.1 we have

$$\begin{aligned}
&T_A(y \to z) \geq \min\{T_A(x), T_A(x \to (y \to z))\} = \min\{T_A(x), T_A(1)\} = T_A(x),\\
&T_A(z) \geq \min\{T_A(y \to z), T_A(y)\} \geq \min\{T_A(x), T_A(y)\};\\
&I_A(y \to z) \leq \max\{I_A(x), I_A(x \to (y \to z))\} = \max\{I_A(x), I_A(1)\} = I_A(x),\\
&I_A(z) \leq \max\{I_A(y \to z), I_A(y)\} \leq \max\{I_A(x), I_A(y)\};\\
&F_A(y \to z) \leq \max\{F_A(x), F_A(x \to (y \to z))\} = \max\{F_A(x), F_A(1)\} = F_A(x),\\
&F_A(z) \leq \max\{F_A(y \to z), F_A(y)\} \leq \max\{F_A(x), F_A(y)\}.
\end{aligned}$$

That is, (NSF4) holds.

Conversely, let A satisfies (NSF1) and (NSF4). From Definition 2.7 (BE1) we have $(x \to y) \to (x \to y) = 1$. By (NSF4),

$\min\{T_A(x \to y), T_A(x)\} \leq T_A(y), \max\{I_A(x \to y), I_A(x)\} \geq I_A(y)$, and $\max\{F_A(x \to y), F_A(x)\} \geq F_A(y)$.

This means that A satisfy (NSF2). Using Definition 4.1 we get that Ais a neutrosophic filter in X.

5. Implicative neutrosophic filters

Definition 5.1. A neutrosophic set A in a BE-algebra X is called an implicative neutrosophic filter if it satisfies:$\forall x, y, z \in X$,

(1) $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$;

(2) $\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \leq T_A(x \to z), \max\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq I_A(x \to z)$, and $\max\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq F_A(x \to z)$.

Example 5.1. Let $X = \{a, b, c, 1\}$ be a BE-algebra with a binary operation given by the following table

$\to$	a	b	c	1
a	a	b	c	1
b	1	b	b	1
c	a	1	a	1
1	a	b	c	1

Define neutrosophic set A in X as following:

$$T_A(x) = \begin{cases} 0.87, & if x = 1 \\ 0.69, & if x = b \\ 0.11, & \text{otherwise} \end{cases}, I_A(x) = \begin{cases} 0.09, & if x = 1 \\ 0.15, & if x = b \\ 0.84, & \text{otherwise} \end{cases},$$

$$F_A(x) = \begin{cases} 0.05, & if x = 1 \\ 0.14, & if x = b \\ 0.79, & \text{otherwise.} \end{cases}$$

We can verify that A is an implicative neutrosophic filter.

When $x = 1$ in Definition 5.1 (2), we can get (NSF2) in Definition 4.1, this means that the following proposition is true.

Proposition 5.1. *Let A be an implicative neutrosophic filter in a BE-algebra X. Then A is a neutrosophic filter in X.*

The following example shows that the converse of Proposition 5.1 is not true in general.

Example 5.2. Let $(X; \to, 1)$ be the BE-algebra in Example 3.1, and A be the neutrosophic filter in Example 4.1. Then we can verify that A is not an implicative neutrosophic filter in$X$, since

$\min\{T_A(b \to (d \to c)), T_A(b \to d)\} = \min\{T_A(1), T_A(a)\} = \min\{0.86, 0.86\} = 0.86 \not\leq 0.13 = T_A(b) = T_A(b \to c)$.

Proposition 5.2. *Let A be an implicative neutrosophic filter in a BE-algebra X. Then A satisfies the following conditions:*

(i) $\forall x, y \in X, T_A(x \to y) = T_A(x \to (x \to y))$;

(ii) $\forall x, y \in X, I_A(x \to y) = I_A(x \to (x \to y))$;

(iii) $\forall x, y \in X, F_A(x \to y) = F_A(x \to (x \to y))$.

Proof. Putting $y = x$ and $z = y$ in Definition 5.1 (2), we can get that

$\min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y), \max\{I_A(x \to (x \to y)), I_A(x \to x)\} \geq I_A(x \to y)$, and $\max\{F_A(x \to (x \to y)), F_A(x \to x)\} \geq F_A(x \to y)$.

Applying Definition 2.7 (BE1) and Definition 5.1 (1) we have $T_A(x \to (x \to y)) = \min\{T_A(x \to (x \to y)), T_A(1)\} = \min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y)$, $I_A(x \to (x \to y)) = \max\{I_A(x \to (x \to y)), I_A(1)\} = \max\{I_A(x \to (x \to y)), I_A(x \to x)\} \geq I_A(x \to y)$, and $F_A(x \to (x \to y)) = \min\{F_A(x \to (x \to y)), F_A(1)\} = \min\{F_A(x \to (x \to y)), F_A(x \to x)\} \geq F_A(x \to y)$.

On the other hand, by Proposition 2.1 (2),we have $x \to y \leq x \to (x \to y)$. Using Proposition 4.1, $T_A(x \to y) \leq T_A(x \to (x \to y)), I_A(x \to y) \geq I_A(x \to (x \to y)), F_A(x \to y) \geq F_A(x \to (x \to y))$.

Combining the above two hands, we get that $T_A(x \to y) = T_A(x \to (x \to y)), I_A(x \to y) = I_A(x \to (x \to y)), F_A(x \to y) = F_A(x \to (x \to y))$.

Theorem 5.1. *Let A be a neutrosophic filter in a transitive BE-algebra X. Then A is an implicative neutrosophic filter in X if and only if it satisfies:*

(i) $\forall x, y \in X, T_A(x \to y) = T_A(x \to (x \to y))$;

(ii) $\forall x, y \in X, I_A(x \to y) = I_A(x \to (x \to y))$;

(*iii*) $\forall x, y \in X, F_A(x \to y) = F_A(x \to (x \to y))$.

Proof. If A is an implicative neutrosophic filter in X, then by Proposition 5.2 we know that the conditions (i), (ii) and (iii) hold.

Conversely, suppose that A satisfies the conditions (i), (ii) and (iii). For any $x, y, z \in X$, by the definition of a transitive BE-algebra and Definition 2.7 we have

$$x \to y \leq (y \to z) \to (x \to z) \leq (x \to (y \to z)) \to (x \to (x \to z)),$$

$$(x \to y) \to ((x \to (y \to z)) \to (x \to (x \to z))) = 1,$$

$$(x \to (y \to z)) \to ((x \to y) \to (x \to (x \to z))) = 1.$$

Applying Theorem 4.3 (NSF4) we have

$$\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \leq T_A(x \to (x \to z)),$$

$$\min\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq I_A(x \to (x \to z)),$$

$$\min\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq F_A(x \to (x \to z)).$$

From these, by (i), (ii) and (iii) we get

$$\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \leq T_A(x \to (x \to z)) = T_A(x \to z),$$

$$\max\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq I_A(x \to (x \to z)) = I_A(x \to z),$$

$$\max\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq F_A(x \to (x \to z)) = F_A(x \to z),$$

Hence, by Definition 5.1 we know that A is an implicative neutrosophic filter in X.

Theorem 5.2. *Let X be a self distributive BE-algebra. Then every neutrosophic filter in X is an implicative neutrosophic filter in X.*

Proof. Let A be a neutrosophic filter in X. Then

$$\forall x \in X, T_A(x) \leq T_A(1), I_A(x) \geq I_A(1) and F_A(x) \geq F_A(1).$$

For any $x, y, z \in X$, by the definition of a self distributive BE-algebra, $x \to (y \to z) = (x \to y) \to (x \to z)$. By applying (NSF2) in Definition 4.1 we have:

$$\begin{aligned} &\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \\ &= \min\{T_A((x \to y) \to (x \to z)), T_A(x \to y)\} \leq T_A(x \to z), \\ &\max\{I_A(x \to (y \to z)), I_A(x \to y)\} \\ &= \max\{I_A((x \to y) \to (x \to z)), I_A(x \to y)\} \geq I_A(x \to z), \end{aligned}$$

$$\max\{F_A(x \to (y \to z)), F_A(x \to y)\}$$
$$= \max\{F_A((x \to y) \to (x \to z)), F_A(x \to y)\} \geq F_A(x \to z).$$

By Definition 5.1 we know that A is an implicative neutrosophic filter in X.

Theorem 5.3. *Let X be a self distributive BE-algebra and A be a neutrosophic filter in X. Then the following conditions are equivalent:*

(1) *A is an implicative neutrosophic filter in X;*

(2) $\forall x, y \in X, T_A(x \to (x \to y)) \leq T_A(x \to y), I_A(x \to (x \to y)) \geq I_A(x \to y), F_A(x \to (x \to y)) \geq F_A(x \to y)$;

(3) $\forall x, y, z \in X, \min\{T_A(z \to (x \to (x \to y))), T_A(z)\} \leq T_A(x \to y)$, $\max\{I_A(z \to (x \to (x \to y))), I_A(z)\} \geq I_A(x \to y)$, and $\max\{F_A(z \to (x \to (x \to y))), F_A(z)\} \geq F_A(x \to y)$.

Proof. (1) $\Rightarrow$ (2).$\forall x, y \in X$, by Definition 5.1 (2), we have

$$\min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y),$$

$$\max\{I_A(x \to (x \to y)), I_A(x \to x)\} \geq T_A(x \to y),$$

$$\max\{F_A(x \to (x \to y)), F_A(x \to x)\} \geq F_A(x \to y).$$

Applying Definition 2.7 (BE1) and Definition 5.1 (1) we have $T_A(x \to (x \to y)) = \min\{T_A(x \to (x \to y)), T_A(1)\} = \min\{T_A(x \to (x \to y)), T_A(x \to x)\} \leq T_A(x \to y)$, $I_A(x \to (x \to y)) = \max\{I_A(x \to (x \to y)), I_A(1)\} = \max\{T_A(x \to (x \to y)), I_A(x \to x)\} \leq I_A(x \to y), and\ F_A(x \to (x \to y)) = \max\{F_A(x \to (x \to y)), F_A(1)\} = \max\{T_A(x \to (x \to y)), F_A(x \to x)\} \leq F_A(x \to y)$.

Hence, (2) holds.

(2) $\Rightarrow$ (3). $\forall x, y, z \in X$, by Definition 4.1 (NSF2), we have:

$$\min\{T_A(z \to (x \to (x \to y))), T_A(z)\} \leq T_A(x \to (x \to y)),$$

$$\max\{I_A(z \to (x \to (x \to y))), I_A(z)\} \geq I_A(x \to (x \to y)),$$

$$\max\{F_A(z \to (x \to (x \to y))), F_A(z)\} \geq F_A(x \to (x \to y)).$$

From these, using (2) we get

$$\min\{T_A(z \to (x \to (x \to y))), T_A(z)\} \leq T_A(x \to y),$$

$$\max\{I_A(z \to (x \to (x \to y))), I_A(z)\} \geq I_A(x \to y),$$

$$\max\{F_A(z \to (x \to (x \to y))), F_A(z)\} \geq F_A(x \to y).$$

Therefore, (3) holds.

(3) $\Rightarrow$ (1).$\forall x, y, z \in X$, by the definition of a self distributive BE-algebra and Definition 2.7, we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

By applying Proposition 4.1,

$$T_A(x \to (y \to z)) \leq T_A((x \to y) \to (x \to (x \to z))),$$

$$I_A(x \to (y \to z)) \geq I_A((x \to y) \to (x \to (x \to z))),$$

$$F_A(x \to (y \to z)) \geq F_A((x \to y) \to (x \to (x \to z))).$$

From these, using (3) we get $\min\{T_A(x \to (y \to z)), T_A(x \to y)\} \leq \min\{T_A((x \to y) \to (x \to (x \to z))), T_A(x \to y)\} \leq T_A(x \to z)$, $\max\{I_A(x \to (y \to z)), I_A(x \to y)\} \geq \max\{I_A((x \to y) \to (x \to (x \to z))), I_A(x \to y)\} \geq I_A(x \to z)$, $\max\{F_A(x \to (y \to z)), F_A(x \to y)\} \geq \max\{F_A((x \to y) \to (x \to (x \to z))), F_A(x \to y)\} \geq T_A(x \to z)$.

Hence, by Definition 5.1 we know that A is an implicative neutrosophic filter in X.

Definition 5.2. A fuzzy set μ in a BE-algebra X is called an implicative fuzzy filter if it satisfies:$x, y, z \in X$,

(1) $\mu(x) \leq \mu(1)$;

(2) $\min\{\mu(x \to (y \to z)), \mu(x \to y)\} \leq \mu(x \to z)$.

It is similar to Theorem 3.6 in [5] we can get the following theorem (the proof is omitted).

Theorem 5.4. *Let μ be a fuzzy set of a BE-algebra X. Then the following conditions are equivalent:*

(1) μ is an implicative fuzzy filter in X;

(2) for each $\alpha \in [0,1]$, the level subset $U(\mu;\alpha) = \{x \in X : \mu(x) \geq \alpha\}$ is an implicative filter of X, when$U(\mu;\alpha) \neq \emptyset$.

It is similar to Theorem 4.1 we can get the following theorem (the proof is omitted).

Theorem 5.5. *Let A be a neutrosophic set in a BE-algebra X. Then A is an implicative neutrosophic filter in X if and only if A satisfies:*

(i) T_A is a fuzzy implicative filter of X;

(ii) $1-I_A$ is a fuzzy implicative filter of X, where $(1-I_A)(x) = 1-I_A(x), x \in X$;

(iii) $1 - F_A$ is a fuzzy implicative filter of X, where $(1 - F_A)(x) = 1 - F_A(x), x \in X$.

It is similar to Theorem 4.2 we can get the following theorem (the proof is omitted).

Theorem 5.6. *Let X be a BE-algebra, A be a neutrosophic set in X. Then A is an implicative neutrosophic filter in X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are implicative filters of X when $\alpha, \beta, \gamma \in [0,1]$such that $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

6. Conclusion

This paper further studied the application of neutrosophic set theory to BE-algebras. First of all, we analyzed the defects of the original definition of neutrosophic filter in a BE-algebra, by using some examples we pointed out the following facts:

(1) An example of neutrosophic filter (Example 3.1 in [10]) is wrong;

(2) A theorem on neutrosophic filters (Theorem 3.4 in [10]) is wrong;

(3) The original definition of neutrosophic filter in BE- algebra is not normal, since the indeterminacy-membership function and falsity-membership function are constants for any neutrosophic filter (see Proposition 3.2).

In order to solve the above problems, we given a reasonable new definition of neutrosophic filter in BE- algebras, and through in-depth study its properties, we know that the new definition is good and overcomes the shortcomings of the original definition. Especially, some necessary and sufficient conditions are given, and an important fact is shown: a neutrosophic filter in BE-algebra can be completely determined by its (alpha, beta, gamma)- level sets. Moreover, the relationships between fuzzy filters and neutrosophic filters are investigated. Finally, the new concept of implicative neutrosophic filter in BE-algebra is introduced, and some necessary and sufficient conditions for a neutrosophic filter to be implicative neutrosophic filter are given. All these results are new and important, which can be used for reference to other research on non-classical logic algebra systems.

Acknowledgment

This work was supported by National Natural Science Foundation of China (Grant No. 61573240, 61976130).

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Accepted: 19.01.2018

More on almost countably compact spaces

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Abstract. We study a recent general space of countably compact space called almost countably compact. A topological space X is almost countably compact space if for every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^{m}$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{m} Cl(U_{n_i})$. In particular, we investigate this new class of spaces and some other properties in the view of regular cover notion and semiregularization topology.
Keywords: regularly open sets, regularly closed sets, nearly countably compact, semiregularization topology.

1 Introduction

Compactness has come to be one of the most important concepts in advanced mathematics. In the 19th century, many mathematical properties were used that would be later seen as consequences of compactness. This notion that arises from topology and metric spaces is very useful in analysis and so in applied mathematics. A generalization of compact spaces, the countable compact spaces arise in different study fields. For metrizable spaces, the countable compactness, sequential compactness, limit point compactness and compactness are all equivalent.

Not only compactness, but among various covering properties of topological spaces a lot of attention has been made to those covers involving regularly open sets and regularly closed sets. In 1959, weakly Lindelöf spaces were introduced by Frolik [5]. After that and as an analogous work on Lindelöfness; nearly compact spaces were defined in by Singal and Mathur in 1969 [8] as a generalization of compact spaces. By the definition, a topological space is nearly compact if for every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there is a finite subfamily $\{U_{\alpha_i}\}_{i=1}^{m}$ where $m \in \mathbb{N}$ and $X = \bigcup_{i=1}^{m} Int(Cl(U_{\alpha_i}))$. Further, nearly Lindelöf spaces are defined in by Balasubramanianin 1982 [2]. In the other way around, some generalizations as almost countbly compact and nearly countably compact of countably compact spaces are presented by different authors as Song and Zhao 2012 [11], and Altawallbeh and Al-Momani in their paper [1]. In addition to the mentioned references, the reader may take a look at [3] and [10]. By the definition, a topological space X is said to be nearly countably compact space if for every countable (regularly) open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^m Int(Cl(U_{n_i}))$ ($X = \bigcup_{i=1}^m U_{n_i}$). It is clear that every nearly countably compact space X is almost countably compact.

We know that a compact space is countably compact. In this paper, we investigate one more general space, called almost countably compact. In particular, we study some properties of almost countably compact spaces by using regularly open (closed) sets and semiregularization topology. Porter and Thomson [7] have shown the importance of semiregularization topologies in the study of H-closed and minimal Hausdorff spaces.

Throughout this paper, a space X stands for a topological space (X, τ). The interior and closure of a subset A in a space X are denoted by $Int(A)$ and $Cl(A)$, respectively. Regularly open sets are defined by Stone [12] in 1937 and investigated with more interesting results about the topic. A subset A is said to be regularly open if and only if $A = Int(Cl(A))$. It is obvious that every regularly open set is an open set. The complement of a regularly open set is called regularly closed set. In addition, we denote the set of all regularly open and closed sets in a space X by $RO(X)$ and $RC(X)$, respectively. In 1985 [6], Mršević, Rielly, and Vamanamurthy have been studied the topology of a space X whose base is the set of all regularly open sets in the space (X, τ) which is called semiregularization topology and denoted by τ^*. If $\tau = \tau^*$, then the space X is called semiregular. Furthermore, Cameron [4] has called a topological property ρ semiregular provided that the space (X, τ) has the property ρ. Moreover, a space X is called extremally disconnected if the closure of every open set in it is open.

The following lemma is well known and it is easily can be proved. In its results, it contains some preliminaries matching the regularly open and regularly closed sets of any topology and its semiregularization in such useful facts that can be appear in the text later.

Lemma 1.1. *For any topological space $(X, \tau)$and its' semiregularization (X, τ^*), we have:*

1. $RO(X, \tau) = RO(X, \tau^*)$.
2. $RC(X, \tau) = RC(X, \tau^*)$.
3. $Int_\tau(F) = Int_{\tau^*}(F)$ *for any* $F \in RC(X, \tau)$.
4. $Cl_\tau(O) = Cl_{\tau^*}(O)$ *for any* $O \in \tau$.

2 Almost countably comapct spaces

In this section, we study a recent class of generalized spaces of the countably compact spaces which is called the class of almost countably compact spaces. In particular, we prove that almost countably compact property is a semiregular property and a regularly closed subset of an almost countably compact space is an almost countably compact.

Definition 2.1. *A topological space X is said to be almost countably compact, if for every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^m Cl(U_{n_i})$.*

It is clear that every countably compact space X is almost countably compact but the converse needs not be true as it is shown in the follwoing example.

Example 1. Let $X = [0, \Omega] \times [0, \omega]/\{(\Omega, \omega)\}$ be the deleted Tychonoff plank space where Ω is the first infinite ordinal and ω is the first uncountable ordinal. Since the set $\{(n, \omega) : n \in \mathbb{N}\}$ is an infinite discrete closed subset of X, we deduce that X is not countably compact. On the other hand, it is easy to see that X is almost countably compact because there is the subset $\Omega \times [0, \omega]$ which is a dense countably compact subspace of X.

Proposition 0.1. *A topological space (X, τ) is almost countably compact space if and only if its' semiregularization (X, τ^*) is almost countably compact.*

Proof. Assume that (X, τ) is almost countably compact. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of (X, τ^*). Since $\tau^* \subseteq \tau$, the cover $\mathcal{U}$ is an open cover of (X, τ). From our assumption, the space (X, τ) is almost countably compact, and so there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, for some natural number m, of $\mathcal{U}$ such that

$$X = \bigcup_{i=1}^m Cl_\tau(U_{n_i}).$$

By using Lemma 1.1 (4), we get

$$X = \bigcup_{i=1}^m Cl_{\tau^*}(U_{n_i}).$$

This proves that (X, τ^*) is almost countably compact.

Conversely, we assume that (X, τ^*) is almost countably compact. Let $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be a countable open cover of (X, τ). From the definition of the topology τ^*, the family $\{Int_\tau(Cl_\tau(V_n)) : n \in \mathbb{N}\}$ is an open cover of the space (X, τ^*). Since (X, τ^*) is almost countably compact, there is a finite subfamily $\{Int_\tau(Cl_\tau(V_{n_i}))\}_{i=1}^m$ such that

$$X = \bigcup_{i=1}^m Cl_{\tau^*}(Int_\tau(Cl_\tau(V_{n_i}))).$$

By using Lemma 1.1 (4), we get

$$X = \bigcup_{i=1}^m Cl_\tau(Int_\tau(Cl_\tau(V_{n_i}))).$$

Since $Cl_\tau(Int_\tau(Cl_\tau(V_{n_i}))) \subseteq Cl_\tau(V_{n_i})$ for every set V_{n_i}, we have

$$X = \bigcup_{i=1}^{m} Cl_\tau(V_{n_i}).$$

This proves that (X, τ) is almost countably compact space and the proof is complete. $\square$

Corollary 2.1. *Almost countably compact property is a semiregular property.*

From Example 1, observe that the closed subset $\{(n, \omega) : n \in \mathbb{N}\}$ of a Tychonoff almost countably compact space X is not almost countably compact. Thus, the closed subset of an almost countably compact space need not be almost countably compact. Anyway, the following proposition shows that there is a positive result regarding that.

Proposition 0.2. *Every regularly closed subset of an almost countably compact space is almost countably compact.*

Proof. Let X be an almost countably compact space and let H be a regularly closed subset of X. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable open cover of H. So, for each n, there is an open subset V_n of X such that $V_n \bigcap H = U_n$. Now, we get

$$X = (\bigcup_{n\in\mathbb{N}} V_n) \bigcup (X/H).$$

It is clear that X/H is an open subset of X. Since X is almost countably compact, we deduce that there is a finite subfamily $\{U_{n_1}, U_{n2}, ..., U_{n_m}\}$ of the cover $\{U_n : n \in \mathbb{N}\}$ such that

$$X = (\bigcup_{i=1}^{m} Cl(V_{n_i})) \bigcup (Cl(X/H)).$$

Thus,

$$Int(H) \subseteq \bigcup_{i=1}^{m} Cl(V_{n_i}) = Cl(\bigcup_{i=1}^{m} V_{n_i}).$$

Since H is a regularly closed subset of X, we have $H = Cl(Int(H)) \subseteq Cl(\bigcup_{i=1}^{m} V_{n_i})$. That means

$$H{=}H\bigcap(Cl(\bigcup_{i=1}^{m} V_{n_i})){=}Cl_H(H\bigcap(\bigcup_{i=1}^{m} V_{n_i})){=}\bigcup_{i=1}^{m} Cl_H(V_{n_i}\bigcap H){=}\bigcup_{i=1}^{m} Cl_H(U_{n_i}).$$

That means H is almost countably compact space and the proof is completed. $\square$

3 Almost regular countably comapct spaces

We define one more general spaces called almost regular countably compact spaces by using the notion of regular covers. Besides that, we prove that almost regular countably compact property is a semiregular property with some other related results are presented.

Definition 3.1. *An open cover $\{U_\alpha : \alpha \in \Delta\}$ of a space X is called regular cover if, for every $\alpha \in \Delta$ there exists a nonempty regularly closed subset C_α in X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} Int(C_\alpha)$.*

In this paper and to make it clearer to the reader, we call the cover of regularly open sets a regularly open cover which is different from the regular cover.

Definition 3.2. *[9] A space X is called almost regular if any regularly closed set F and any singleton $\{x\}$ disjoint from F, then there exist two open disjoint sets U and V such that $F \subseteq U$ and $\{x\} \subseteq V$.*

Definition 3.3. *A space X is called almost regular countably compact if and only if for every countable regular cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_1}, U_{n}2, ..., U_{n_m}\}$ where $m \in \mathbb{N}$ of $\{U_n : n \in \mathbb{N}\}$ such that $X = \bigcup_{i=1}^{m} Cl(U_{n_i})$.*

It is obvious that every almost countably compact space is almost regular countably compact.

Proposition 0.3. *A topological space (X, τ) is almost regular countably compact space if and only if (X, τ^*) is almost regular countably compact space.*

Proof. Assume that (X, τ) is almost regular countably compact. Let $\{U_n : n \in \mathbb{N}\}$ be a countable regular cover of (X, τ^*). From definition of the regular cover, there is a nonempty regularly closed set C_n in (X, τ^*) such that $C_n \subseteq U_n$ and $X = \bigcup_{n\in\mathbb{N}} Int_{\tau^*}(C_n)$. From Lemma 1.1, and the fact that $\tau^* \subseteq \tau$, we get $\{U_n : n \in \mathbb{N}\}$ is a countable regular cover of (X, τ). From the assumption, (X, τ) is almost regular countably compact, there is a finite subfamily $\{U_{n_1}, U_{n}2, ..., U_{n_m}\}$ of the cover $\{U_n : n \in \mathbb{N}\}$ such that such that

$$X = \bigcup_{i=1}^{m} Cl_\tau(U_{n_i}).$$

Using 4 in Lemma 1.1, we get

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(U_{n_i}).$$

Thus (X, τ^*) is almost regular countably compact space. Conversely, Assume that is (X, τ^*) is almost regular countably compact and $\{U_n : n \in \mathbb{N}\}$ is a countable regular cover of (X,τ). We know that for every $U_n \in \tau$, we have $U_n \subseteq Int_\tau(Cl_\tau(U_n))$, and by using the same lemma, we get $\{Int_\tau(Cl_\tau(U_n)) : n \in \mathbb{N}\}$ is a countable regular cover of (X,τ^*). Since (X,τ^*) is almost regular countably compact, there is a finite subfamily $\{nt_\tau(Cl_\tau(U_{n_i}))\}_{i=1}^{m}$such that.

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(Int_\tau(Cl_\tau(U_{n_i}))).$$

Since $Int_\tau(Cl_\tau(U_{n_i}))$ is an open set in (X,τ) for each n_i and by using 4 in Lemma 1.1, we get

$$X = \bigcup_{i=1}^{m} Cl_\tau(Int_\tau(Cl_\tau(U_{n_i}))).$$

Thus, $X = \bigcup_{i=1}^{m} Cl_\tau(U_{n_i})$. This proves that (X,τ) is almost regular countably compact space. □

Corollary 3.1. *Almost regular countably compact property is a semiregular property.*

The next proposition shows that nearly regular countably compact property is a semiregular property where the proof is omitted because it is very similar to the proof of Proposition 0.3

Proposition 0.4. *A topological space (X,τ) is nearly regular countably compact space if and only if (X,τ^*) is nearly regular countably compact space.*

It is a direct result that is every nearly countably compact space is almost countably compact, and so it is almost regular countably compact space. Each of the following two propositions proves that the converse is true but with one more different strong condition in each of which.

Proposition 0.5. *Let X be an almost regular countably compact space and extremally disconnected then it is nearly countably compact space.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of X. Since X is extremally disconnected, we have $\{Cl(U_n) : n \in \mathbb{N}\}$ is a regular cover of X. Now from the assumption, X is almost regular countably compact and so it has a finite subfamily $\{Cl(U_{n_i})\}_{i=1}^{m}$ such that $X = \bigcup_{i=1}^{m} Cl(U_{n_i})$. Again since X is extremally disconnected, we get

$$X = \bigcup_{i=1}^{m} Int(Cl(U_{n_i})).$$

This proves that X is nearly countably compact space and the proof is done. □

Proposition 0.6. *Let X be an almost regular countably compact space and almost regular then it is nearly countably compact space.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable regularly open cover of X. That means for each $x \in X$ there is n_x such that $x \in U_{n_x}$. Since X is almost regular and by using theorem 2.2 in [9], there exists a regularly open set H_{n_x} such that $x \in H_{n_x} \subseteq Cl(H_{n_x}) \subseteq U_{n_x}$. Again, because X is almost regular and $x \in H_{n_x}$, then there is another regularly open set O_{n_x} such that $x \in O_{n_x} \subseteq Cl(O_{n_x}) \subseteq H_{n_x} \subseteq Cl(H_{n_x}) \subseteq U_{n_x}$. It is obvious that the subset $Cl(O_{n_x})$ is regularly closed subset where $X = \bigcup_{x\in X} Int(Cl(O_{n_x})) = \bigcup_{x\in X} O_{n_x}$. This makes the family $\{H_{n_x} : n_x \in \mathbb{N}\}_{x\in X}$ a countable regular cover of X. Since X is almost regular countably compact space, then there is a finite subfamily $\{H_{n_{x_i}} : n_{x_i} \in \mathbb{N}$ and $x_i \in X\}_{i=1}^{m}$ for some $m \in \mathbb{N}$ such that

$$X = \bigcup_{i=1}^{m} Cl(H_{n_{x_i}}).$$

Thus $X = \bigcup_{i=1}^{m} U_{n_{x_i}}$. This proves that X is nearly countably compact space and completes the proof. □

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Accepted: 20.01.2018

Fixed point results with Ω-distance by utilizing simulation functions

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Abstract. In this paper, we utilize the concept of simulation functions in sense of Khojasteh et al [10] and the notion of Ω-distance in the sense of Saadati et. al. [1] to introduce the notion of $(\Omega, \mathcal{Z})$-contraction and $(\Omega, \varphi, \mathcal{Z})$-contraction. We employ our contractions to formulate and prove many fixed point results for Ω-distance. Our results

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unify and improve many fixed point results in literature. Also, we give fixed point results of integral type as well as we support our result by introducing an example.
Keywords: fixed point theory, nonlinear contraction, simulation function, omega distance.

1. Introduction

The notion of Ω-distance in the sense of Saadati et al. [1] plays an important role in nonlinear analysis to extend and improve the Banach fixed point theorem to many directions. Saadati et al. [1] employed the notion of Ω-distance to prove many interesting results associated to the notion of G-metric spaces in the sense of Mustafa and Sims [2]. For some works in Ω-distance see [3]-[7] and all references cited their.

The definition of Ω-distance is given as follows:

Definition 1.1 ([1]). *Let (X, G) be a G-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an Ω-distance on X if the following conditions satisfied:*

(a) $\Omega(x, y, z) \le \Omega(x, a, a) + \Omega(a, y, z) \, \forall \, x, y, z, a \in X$;

(b) for any $x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \to X$ are lower semi continuous;

(c) for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \le \delta$ and $\Omega(a, y, z) \le \delta$ imply $G(x, y, z) \le \epsilon$.

Definition 1.2 ([1]). *Let (X, G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $M \ge 0$ such that $\Omega(x, y, z) \le M$ for all $x, y, x \in X$.*

The following lemma plays a crucial role in the development of our results.

Lemma 1.1 ([1]). *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let (x_n),(y_n) be sequences in X, (α_n),(β_n) be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:*

(1) If $\Omega(y, x_n, x_n) \le \alpha_n$ and $\Omega(x_n, y, z) \le \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \epsilon$ and hence $y = z$;

(2) If $\Omega(y_n, x_n, x_n) \le \alpha_n$ and $\Omega(x_n, y_m, z) \le \beta_n$ for any $m > n \in \mathbb{N}$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$;

(3) If $\Omega(x_n, x_m, x_l) \le \alpha_n$ for any $m, n, l \in \mathbb{N}$ with $n \le m \le l$, then (x_n) is a G-Cauchy sequence;

(4) If $\Omega(x_n, a, a) \le \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a G-Cauchy sequence.

Khojasteh et al. [10] in 2015 introduced the concept of simulation mappings in which they used it to unify several fixed point results in the literature.

Definition 1.3 ([10]). *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:*

(ζ1) $\zeta(0, 0) = 0$;

(ζ2) $\zeta(t, s) < s - t$ *for all $s, t > 0$;*

($\zeta 3$) *If* t_n *and* s_n *are sequences in* $[0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Hence forth, we denote by $\mathcal{Z}$ the set of all simulation functions.

Next, we list some examples of simulation functions, in the following ζ is defined from $[0, \infty) \times [0, \infty)$ to $\mathbb{R}$.

Example 1.1 ([10])**.** Let $h_1, h_2 : [0, \infty) \to [0, \infty)$ be two continuous functions such that $h_1(t) = h_2(t) = 0$ if and only if $t = 0$ and $h_2(t) < t \leq h_1(t)$ for all $t \in [0, \infty)$ and define $\zeta(t, s) = h_2(s) - h_1(t)$ for all $t, s \in [0, \infty)$. Then ζ is a simulation function.

Example 1.2 ([10])**.** Let $g : [0, \infty) \to [0, \infty)$ be a continuous function such that $g(t) = 0$ if and only if t=0 and define $\zeta(t, s) = s - g(s) - t$ for all $t, s \in [0, \infty)$. Then ζ is a simulation function.

2. Main result

We start our work by introducing the following definition:

Definition 2.1. *Let* (X, G) *be a* G*-metric space,* $\zeta \in \mathcal{Z}$ *and* Ω *be an* Ω*-distance on* X*. A self mapping* $f : X \to X$ *is said to be* $(\Omega, \mathcal{Z})$*-contraction with respect to* ζ *if* f *satisfies the following condition:*

$$\zeta(\Omega(fx, fy, fz), \Omega(x, y, z)) \geq 0 \quad for\ all \quad x, y, z \in X. \tag{2.1}$$

Lemma 2.1. *Let* (X, G) *be a* G*-metric space, and* Ω *be an* Ω*-distance on* X*. Let* $f : X \to X$ *be an* $(\Omega, \mathcal{Z})$*-contraction with respect to* $\zeta \in \mathcal{Z}$*. If* f *has a fixed point (say)* $u \in X$*, then it is unique.*

Proof. Assume that there is $v \in X$ such that $fv = v$. As f is $(\Omega, \mathcal{Z})$-contraction with respect to $\zeta \in \mathcal{Z}$, then by substituting $x = y = u$ and $z = v$ in 2.1 and taking into account ($\zeta 2$), we have

$$\begin{aligned} 0 &\leq \zeta(\Omega(fu, fu, fv), \Omega(u, u, v)) \\ &= \zeta(\Omega(u, u, v), \Omega(u, u, v)) \\ &< \Omega(u, u, v) - \Omega(u, u, v) = 0, \end{aligned}$$

a contradiction. Hence u is unique. □

Let (X, G) be a G-metric space, $x_0 \in X$ and $f : X \to X$ be a self mapping. Then the sequence (x_n) where $x_n = fx_{n-1}$ $n \in \mathbb{N}$ is called a picard sequence generated by f with initial point x_0.

Lemma 2.2. *Let* (X, G) *be a* G*-metric space,* $\zeta \in \mathcal{Z}$ *and* Ω *be an* Ω*-distance on* X*. If* $f : X \to X$ *is an* $(\Omega, \mathcal{Z})$*-contraction with respect to* ζ*, then*

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0. \tag{2.2}$$

for any initial point $x_0 \in X$ *where* (x_n) *is the picard sequence generated by* f *at* x_0*.*

Proof. Let $x_0 \in X$ be any point and (x_n) be the picard sequence generated by f at x_0. From 2.1 and $(\zeta 2)$, we have

$$\begin{aligned}0 &\leq \zeta(\Omega(fx_{n-1}, fx_n, fx_n), \Omega(x_{n-1}, x_n, x_n)) \\ &= \zeta(\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n-1}, x_n, x_n)) \\ &< \Omega(x_{n-1}, x_n, x_n) - \Omega(x_n, x_{n+1}, x_{n+1}).\end{aligned}$$

Thus, $(\Omega(x_n, x_{n+1}, x_{n+1}) : n \in \mathbb{N})$ is a non increasing sequence in $[0, \infty)$ and so there is $L \geq 0$ such that $\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = L$. Suppose to the contrary $L > 0$, then by 2.1 and $(\zeta 3)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n-1}, x_n, x_n)) < 0,$$

a contradiction and so $\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$. By the same way we can show that $\lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0$. □

Theorem 2.1. *Let (X, G) be a complete G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. Suppose that $f : X \to X$ is $(\Omega, \mathcal{Z})$-contraction with respect to ζ that satisfies the following condition*

$$\text{for all } u \in X \text{ if } fu \neq u, \text{ then } \inf\{\Omega(x, fx, u) : x \in X\} > 0. \tag{2.3}$$

Then f has a unique fixed point $x \in X$.

Proof. Let $x_0 \in X$ and consider the picard sequence (x_n) in X generated by f at x_0.

We claim that $\lim_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0$ for $m, n \in \mathbb{N}$ with $m > n$.

For this purpose assume to the contrary that $\lim_{n\to\infty} \Omega(x_n, x_m, x_m) \neq 0$. Hence, there is $\epsilon > 0$ and two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that (x_{m_k}) is chosen as the smallest index for which

$$\Omega(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon, \ k < n_k < m_k. \tag{2.4}$$

This implies that

$$\Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) < \epsilon. \tag{2.5}$$

Now, by using 2.4,2.5 and part (a) of the definition of Ω, we have

$$\begin{aligned}\epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq \Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}) \\ &< \epsilon + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}).\end{aligned}$$

Passing the limit as $n \to \infty$ and taking into account 2.2, we get

$$\lim_{n\to\infty} \Omega(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon.$$

Also,

$$\begin{aligned}\epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq \Omega(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + \Omega(x_{m_k+1}, x_{m_k}, x_{m_k})\end{aligned}$$

and

$$\begin{aligned}&\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) \\ &\leq \Omega(x_{n_k+1}, x_{n_k}, x_{n_k}) + \Omega(x_n, x_{m_k}, x_{m_k}) + \Omega(x_{m_k}, x_{m_k+1}, x_{m_k+1}).\end{aligned}$$

Passing the limit as $n \to \infty$ in the above two inequalities and taking into account 2.2, we get

$$\lim_{n\to\infty} \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) = \epsilon.$$

Now, by letting $s_n = \Omega(x_{n_k}, x_{m_k}, x_{m_k})$ and $t_n = \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1})$ then $(\zeta 3)$ and 2.1 yield that

$$0 \leq \limsup_{n\to\infty} \zeta(\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}), \Omega(x_{n_k}, x_{m_k}, x_{m_k})) < 0$$

which is a contradiction. Therefore $\lim_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0,\ m > n$. By the same argument we can show that $\lim_{n,m\to\infty} \Omega(x_n, x_n, x_m) = 0,\ m > n$.

For $l > m > n$ we have $\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_m, x_m) + \Omega(x_m, x_m, x_l)$.

By taking the limit as $n, m, l \to \infty$, we get $\lim_{n,m,l\to\infty} \Omega(x_n, x_m, x_l) = 0$. Thus by Lemma 1.1 (x_n) is a G-Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

By the lower semi-continuity of Ω, we get

$$\Omega(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leq \epsilon,\ \forall m \geq n.$$

Now, suppose that $fu \neq u$, then we get

$$\begin{aligned} 0 &< \inf\{\Omega(x, fx, u) : x \in X\} \\ &\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \in \mathbb{N}\} \\ &\leq \epsilon, \end{aligned}$$

for every $\epsilon > 0$ which is a contradiction. Therefore $fu = u$. The uniqueness of u follows from Lemma 2.1. □

We introduce the following example to support our main result.

Example 2.1. Let $X = [0, 1]$ and let $G : X \times X \times X \to [0, \infty)$, $\Omega : X \times X \times X \to [0, \infty)$, $f : X \to X$ and $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined as follow:
$G(x, y, z) = |x - y| + |y - z| + |x - z|$, $\Omega(x, y, z) = |x - y| + |x - z|$, $fx = ax$ and $\zeta(t, s) = bs - t$ where $0 \leq a \leq b < 1$. Then

(1) (X, G) is a complete G-metric space and Ω is an Ω-distance on X;

(2) $\zeta \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ

(3) for every $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Proof. We show (2) and (3)

(2) Clearly $\zeta \in \mathcal{Z}$.

To see that f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ let $x, y, z \in X$. Then

$$\begin{aligned}
\zeta(\Omega(fx, fy, fz), \Omega(x, y, z)) &= b\Omega(x, y, z) - \Omega(fx, fy, fz) \\
&= b(|x-y| + |x-z|) - (|ax-ay| + |ax-az|) \\
&= b(|x-y| + |x-z|) - a(|x-y| + |x-z|) \\
&= (b-a)(|x-y| + |x-z|) \\
&\geq 0
\end{aligned}$$

(3) If $fu \neq u$, then $u \neq 0$. Therefore

$$\begin{aligned}
\inf\{\Omega(x, fx, u) : x \in X\} &= \inf\{\Omega(x, \frac{1}{5}x, u) : x \in X\} \\
&= \inf\{|x-ax| + |x-u| : x \in X\} \\
&= \inf\{(1-a)|x| + |x-u| : x \in X\} \\
&= (1-a)u > 0.
\end{aligned}$$

Thus all hypotheses of Theorem 2.1 hold true. Hence f has a unique fixed point in X. Here the unique fixed point of f is 0. □

Now, we derive some interesting results based on our main result. To facilitate our work we define the following:

$$\Phi = \{\phi : [0, \infty) \to [0, \infty) : \phi \textit{ is continuous function}\}$$

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi \textit{ is lower semi continuous function}\},$$

where $\phi^{-1}(\{0\}) = \psi^{-1}(\{0\}) = \{0\}$ for all $\phi \in \Phi$ and $\psi \in \Psi$.

Corollary 2.1. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there are $\phi_1, \phi_2 \in \Phi$ where $\phi_1(t) < t \leq \phi_2(t)$ $\forall t > 0$ such that f satisfies the following condition:*

$$\phi_2\Omega(fx, fy, fz) \leq \phi_1\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.6}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_A : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_A(t, s) = \phi_1(s) - \phi_2(t)$. Clearly $\zeta_A \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ_A. Hence the result follows from Theorem 2.1 □

As a consequence result from Corollary 2.1, we have the following results:

Corollary 2.2. *Let (X,G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there is $\phi \in \Phi$ where $\phi(t) < t \ \forall t > 0$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \phi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.7}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Corollary 2.3. *Let (X,G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there is $\lambda \in [0,1)$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \lambda\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.8}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Corollary 2.4. *Let (X,G) be a complete G-metric space, Ω be an Ω-distance on X and Let $f : X \to X$ be a self mapping. Assume that there is $\psi \in \Psi$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \Omega(x, y, z) - \psi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.9}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_B(t,s) = s - \psi(s) - t$. Clearly $\zeta_B \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ_B. Hence the result follows from Theorem 2.1 □

As a consequence result from Corollary 2.4 we have the following result:

Corollary 2.5. *Let (X,G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there are $\phi \in \Phi$ and $\psi \in \Psi$ where $\phi(t) < t \ \forall t > 0$ such that f satisfies the following conditions:*

$$\Omega(fx, fy, fz) \leq \phi\Omega(x, y, z) - \psi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.10}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Definition 2.2. *The function $\varphi : [0,\infty) \to [0,\infty)$ is called a c-comparison function if the following properties are satisfied:*

(1) *φ is monotone increasing;*
(2) $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ *for all $t \geq 0$.*

It is clear that if φ is a c-comparison function then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Before, we present our second main results we introduce the following definition in order to facilitate our arguments.

Definition 2.3. *Let (X, G) be a G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. A self mapping $f : X \to X$ is said to be $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ if there is is a c-comparison function φ such that f satisfies the following condition:*

$$\zeta(2\Omega(fx, f^2x, fy), \varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y)) \geq 0 \ \forall x, y \in X. \tag{2.11}$$

Lemma 2.3. *Let (X, G) be a G-metric space,$\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. Let $f : X \to X$ be an $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ. If f has a fixed point (say) $u \in X$, then it is unique.*

Proof. First we show that for all $w \in X$ if $fw = w$, then $\Omega(w, w, w) = 0$. Assume that $\Omega(w, w, w) > 0$. From 2.11 and $\zeta 2$, we have

$$\begin{aligned} 0 &\leq \zeta(2\Omega(fw, f^2w, fw), \varphi\Omega(w, fw, w) + \varphi\Omega(w, fw, w)) \\ &= \zeta(2\Omega(w, w, w), 2\varphi\Omega(w, w, w)) \\ &< 2\varphi\Omega(w, w, w) - 2\Omega(w, w, w), \\ &< 2\Omega(w, w, w) - 2\Omega(w, w, w), \\ &= 0 \end{aligned}$$

a contradiction. Hence $\Omega(w, w, w) = 0$.

Now, assume that there is $v \in X$ such that $fv = v$ and $\Omega(u, v, v) > 0$. Since f is $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ, then by substituting $x = u$ and $y = v$ in 2.1 and taking into account $(\zeta 2)$, we have

$$\begin{aligned} 0 &\leq \zeta(2\Omega(fu, f^2u, fv), \varphi\Omega(u, fu, u) + \varphi\Omega(v, fv, v)) \\ &= \zeta(2\Omega(u, u, v), \varphi\Omega(u, u, u) + \varphi\Omega(v, v, v)) \\ &< \varphi\Omega(u, u, u) + \varphi\Omega(v, v, v) - 2\Omega(u, u, v) \\ &< \Omega(u, u, u) + \Omega(v, v, v) - 2\Omega(u, u, v). \end{aligned}$$

Hence $2\Omega(u, u, v) < \Omega(u, u, u) + \Omega(v, v, v) = 0 + 0 = 0$ a contradiction. Hence $\Omega(u, u, v) = 0$. Thus by the definition of Ω-distance we have $G(u, v, v) = 0$ and so $u = v$. □

Theorem 2.2. *Let (X, G) be a complete G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X such that X is Ω-bounded. Suppose that there is is a c-comparison function φ such that $f : X \to X$ is a $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ that satisfies the following condition*

$$\forall \ u \in X \ if \ fu \neq u, \ then \ \inf\{\Omega(x, fx, u) : x \in X\} > 0. \tag{2.12}$$

Then f has a unique fixed point in X.

Let $x_0 \in X$ and consider the picard sequence (x_n) in X generated by f at x_0.

Consider $s \geq 0$. Then by 2.11, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fx_{n-1}, f^2x_{n-1}, fx_{n+s-1}), \varphi\Omega(x_{n-1}, fx_{n-1}, x_{n-1}) \\
&\quad + \varphi\Omega(x_{n+s-1}, fx_{n+s-1}, x_{n+s-1})) \\
&= \zeta(2\Omega(x_n, x_{n+1}, x_{n+s}), \varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1})). \\
&< \varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1}) - 2\Omega(x_n, x_{n+1}, x_{n+s}).
\end{aligned}$$

Thus,

$$\Omega(x_n, x_{n+1}, x_{n+s}) < \frac{1}{2}[\varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1})]. \tag{2.13}$$

Now,

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fx_{n-2}, f^2x_{n-2}, fx_{n-2}), \varphi\Omega(x_{n-2}, fx_{n-2}, x_{n-2}) + \varphi\Omega(x_{n-2}, fx_{n-2}, x_{n-2})) \\
&= \zeta(2\Omega(x_{n-1}, x_n, x_{n-1}), 2\varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2})) \\
&< 2\varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2}) - 2\Omega(x_{n-1}, x_n, x_{n-1}).
\end{aligned}$$

So, $\Omega(x_{n-1}, x_n, x_{n-1}) < \varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2})$.

If we apply the previous steps repeatedly, we get

$$\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^{n-1}\Omega(x_0, x_1, x_0).$$

Therefore $\varphi\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^n\ \Omega(x_0, x_1, x_0)$. Since X is Ω-bounded, there is $M \geq 0$, such that $\Omega(x, y, z) \leq M,\ \forall x, y, z, \in X$. Thus,

$$\varphi\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^n\,(M).$$

In analogous manner, we can show that

$$\varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1}) \leq \varphi^n\,(M).$$

Thus, (2.13) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq \varphi^n(M). \tag{2.14}$$

Now, by using the definition of Ω-distance and (2.14), we have for all $l \geq m \geq n$

$$\begin{aligned}
\Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l) \\
&\leq \varphi^n(M) + \varphi^{n+1}(M) + \cdots + \varphi^{m-1}(M) \\
&= \sum_{k=n}^{m-1} \varphi^k(M) \\
&\leq \sum_{k=n}^{\infty} \varphi^k(M).
\end{aligned}$$

Since φ is c-comparison function, then the sequence $\left(\sum_{k=n}^{\infty} \varphi^k(M) : n \in \mathbb{N}\right)$ converges to 0. Thus for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\sum_{k=n}^{\infty} \varphi^k(M) < \epsilon \;\forall\; n \geq N$. Hence for $l \geq m \geq n \geq N$, we have

$$\Omega(x_n, x_m, x_l) \leq \sum_{k=n}^{m-1} \varphi^k(M) \leq \sum_{k=n}^{\infty} \varphi^k(M) < \epsilon \;\forall n \geq N.$$

By Lemma 1.1, (x_n) is a G-Cauchy sequence. Therefore there is $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Consider $\delta > 0$. Then there exists $r_0 \in \mathbb{N}$ such that $\Omega(x_n, x_m, x_l) \leq \delta \;\forall n, m, l \geq r_0$. Therefore, $\lim_{l\to\infty} \Omega(x_n, x_m, x_l) \leq \lim_{l\to\infty} \delta = \delta$.

By the lower semi continuity of Ω, we have

$$\Omega(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leq \delta \,\forall\, m, n \geq r_0.$$

Consider $m = n+1$. Then $\Omega(x_n, x_{n+1}, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_{n+1}, x_p) \leq \delta \;\forall n \geq r_0$.

If $fu \neq u$, then (2.12) implies that

$$\begin{aligned} 0 &< \inf\{\Omega(x, fx, u) : x \in X\} \\ &\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \geq r_0\} \\ &\leq \delta, \end{aligned}$$

for each $\delta > 0$ which is a contradiction. Therefore $fu = u$. The uniqueness follows from Lemma 2.3. □

Corollary 2.6. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a c-comparison function φ and an upper semi continuous function $\eta : [0, \infty) \to [0, \infty)$ where $\eta(t) < t \;\forall t > 0$ and $\eta(0) = 0$ such that f satisfies the following condition:*

$$2\Omega(fx, f^2x, fy) \leq \eta(\varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y)) \;\forall x, y \in X. \tag{2.15}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_{AA} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_{AA}(t, s) = \eta(s) - t$. Clearly $\zeta_{AA} \in \mathcal{Z}$ and f is $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ_{AA}. Hence the result follows from Theorem 2.2 □

Now, we introduce and prove the following fixed point theorems of integra type.

Theorem 2.3. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a function $\gamma : [0, \infty) \to [0, \infty)$ where $\int_0^\epsilon \gamma(u)du$ exists and $\int_0^\epsilon \gamma(u)du > \epsilon$ $\forall \epsilon > 0$ such that f satisfies the following condition:*

$$\int_0^{\Omega(fx,fy,fz)} \gamma(u)du \leq \Omega(x, y, z) \; \forall x, y, z \in X. \tag{2.16}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = s - \int_0^t \gamma(u)du$. Clearly $\zeta \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$. Hence the results follow from Theorem 2.1. □

Theorem 2.4. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a c-comparison function φ and a function $\gamma : [0, \infty) \to [0, \infty)$ where $\int_0^\epsilon \gamma(u)du$ exists and $\int_0^\epsilon \gamma(u)du > \epsilon$ $\forall \epsilon > 0$ such that f satisfies the following condition:*

$$\int_0^{2\Omega(fx,f^2x,fy)} \gamma(u)du \leq \varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y) \; \forall x, y \in X. \tag{2.17}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. The results follow from Theorem 2.2 by defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = s - \int_0^t \gamma(u)du$. and noting that $\zeta \in \mathcal{Z}$ and f is $(\Omega, \varphi, \mathcal{Z})$.

□

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Accepted: 20.01.2018

On coefficient inequalities for certain subclasses of meromorphic bi-univalent functions

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Abstract. In the present paper, we investigate and define two subclasses of meromorphic bi-univalent function class Σ' which are defined on the domain $\mathbb{U}^* = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Further, by using the well-known coefficients estimates of the Carathéodory functions (i.e functions with positive real part) we obtain the estimates on the coefficients $|b_0|$, $|b_1|$ and $|b_2 + b_0^3|$ for functions in these subclasses.

Keywords: analytic function, meromorphic function, univalent function, bi-univalent function, meromorphic bi-univalent function.

1. Introduction

Let the class $\mathcal{A} = \{f : \mathbb{U} \to \mathbb{C} : f$ is analytic in $\mathbb{U}$ and $f(0) = f'(0) - 1 = 0\}$ and its subclass $\mathcal{S} = \{f : \mathbb{U} \to \mathbb{C} : f \in \mathcal{A}$ and also univalent in $\mathbb{U}\}$ where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and such functions $f \in \mathcal{A}$ have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

In 1972, Ozaki and Nunokawa [14] proved the following Lemma (univalence criterion). In fact, this result is appeared in the paper by Aksentév [1] (also see the paper by Aksentév and Avhadiev [2]).

Lemma 1.1. *If for* $f(z) \in \mathcal{A}$

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

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then $f(z)$ *is univalent in* $\mathbb{U}$ *and hence* $f(z) \in \mathcal{S}$*.*

Also, a functions $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{T}(\mu)$, $(0 < \mu \leq 1)$ if

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in \mathbb{U})$$

and $\mathcal{T}(1) = \mathcal{T}$. Clearly, $\mathcal{T}(\mu) \subset \mathcal{T} \subset \mathcal{S}$. Further (see Kuroki et al. [10]), for $f(z) \in \mathcal{T}(\mu)$ see that:

$$\Re\left(\frac{z^2 f'(z)}{(f(z))^2} \right) > 1 - \mu \quad (z \in \mathbb{U}).$$

In particular, for initial coefficient estimates of bi-univalent function classes $\mathcal{T}_\Sigma(\mu)$ and $\mathcal{T}_\Sigma^\alpha$, see the paper by Naik and Patil [12].

In 1967, Lewin [11] introduced and studied the bi-univalent function class Σ. After which some researchers (viz. [3, 13]) found the initial coefficient estimates for the functions in Σ. Later, Srivastava et al. [17] revived it for the subclasses of Σ. Recently, the concept of bi-univalent functions is extend to meromorphic bi-univalent functions.

Let $\mathcal{S}'$ denote the class of meromorphic univalent functions g of the form:

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.2}$$

defined on the domain $\mathbb{U}^* = \{z : z \in \mathbb{C},\ 1 < |z| < \infty\}$. Clearly, $g \in \mathcal{S}'$ has an inverse say g^{-1}, defined by:

$$g^{-1}(g(z)) = z, \quad (z \in \mathbb{U}^*)$$

and

$$g(g^{-1}(w)) = w, \quad (0 < M < |w| < \infty),$$

which has a series expansion of the form:

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}, \quad (0 < M < |w| < \infty).$$

Some simple computations using equation (1.2) shows that:

$$g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots . \tag{1.3}$$

Let $\Sigma' = \{g \in \mathcal{S}' :$ both g and g^{-1} are meromorphic univalent in $\mathbb{U}^*\}$ denote the class of all meromorphic bi-univalent functions in $\mathbb{U}^*$. Recently the coefficient estimate on functions of various subclasses of Σ' were obtained by some researchers viz. Halim et al. [6], Hamidi et al. [7, 8], Panigrahi [15], Janani and Murugusundaramoorthy [9], Bulut [4], etc. In the present investigation, we define two new subclasses of the function class Σ' and obtain the estimate on $|b_0|$, $|b_1|$ and $\left|b_2 + b_0^3\right|$ for the functions in these new subclasses.

We need to recall the Carathéodory lemma in the following form to prove our main results (see [5], [16]).

Lemma 1.2. *If $p(z) \in \mathcal{P}$, the class of all functions analytic in $\mathbb{U}^*$, for which*

$$\Re(p(z)) > 0,$$

then $|p_n| \leq 2$ for each $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, where

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots, \quad (z \in \mathbb{U}^*).$$

2. Coefficient estimates

Definition 2.1. A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\mathcal{T}_{\Sigma'}(\mu)$ if the following conditions are satisfied:

$$\Re\left(\frac{z^2 g'(z)}{(g(z))^2}\right) > 1 - \mu, \quad (z \in \mathbb{U}^*;\ 0 < \mu \leq 1)$$

and

$$\Re\left(\frac{w^2 h'(w)}{(h(w))^2}\right) > 1 - \mu, \quad (w \in \mathbb{U}^*;\ 0 < \mu \leq 1),$$

where the function h is an inverse of g given by (1.3).

Theorem 2.2. *Let the function $g(z) \in \Sigma'$ given by (1.2) be in the class $\mathcal{T}_{\Sigma'}(\mu)$, where $0 < \mu \leq 1$. Then,*

$$|b_0| \leq \begin{cases} \mu; & \left(0 < \mu \leq \frac{2}{3}\right) \\ \sqrt{\frac{2\mu}{3}}; & \left(\frac{2}{3} \leq \mu \leq 1\right), \end{cases} \tag{2.1}$$

$$|b_1| \leq \frac{2\mu}{3}, \tag{2.2}$$

$$\left|b_2 + b_0^3\right| \leq \frac{\mu}{2}. \tag{2.3}$$

Proof. Let the function $g(z) \in \mathcal{T}_{\Sigma'}(\mu)$. See that clearly, the conditions given in the definition of meromorphic bi-univalent function class $\mathcal{T}_{\Sigma'}(\mu)$ can be written as:

$$\frac{z^2 g'(z)}{(g(z))^2} = (1 - \mu) + \mu\, s(z) \tag{2.4}$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = (1-\mu) + \mu\, t(w), \tag{2.5}$$

where $s(z), t(w) \in \mathcal{P}$ have the form:

$$s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \cdots, \ (z \in \mathbb{U}^*) \tag{2.6}$$

and

$$t(w) = 1 + \frac{t_1}{w} + \frac{t_2}{w^2} + \frac{t_3}{w^3} + \cdots, \ (w \in \mathbb{U}^*)\,. \tag{2.7}$$

Hence we have:

$$(1-\mu) + \mu\, s(z) = 1 + \frac{\mu s_1}{z} + \frac{\mu s_2}{z^2} + \frac{\mu s_3}{z^3} + \cdots$$

and

$$(1-\mu) + \mu\, t(w) = 1 + \frac{\mu t_1}{w} + \frac{\mu t_2}{w^2} + \frac{\mu t_3}{w^3} + \cdots.$$

Also, using (1.2) and (1.3) we obtain:

$$\frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3\left(b_0^2 - b_1\right)}{z^2} + \frac{8b_0 b_1 - 4b_2 - 4b_0^3}{z^3} + \cdots$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3\left(b_0^2 + b_1\right)}{w^2} + \frac{12b_0 b_1 + 4b_2 + 4b_0^3}{w^3} + \cdots.$$

Now, equating the coefficients in (2.4) and (2.5) we get:

$$-2b_0 = \mu s_1, \tag{2.8}$$

$$3\left(b_0^2 - b_1\right) = \mu s_2, \tag{2.9}$$

$$8b_0 b_1 - 4b_2 - 4b_0^3 = \mu s_3, \tag{2.10}$$

$$2b_0 = \mu t_1, \tag{2.11}$$

$$3\left(b_0^2 + b_1\right) = \mu t_2, \tag{2.12}$$

$$12b_0 b_1 + 4b_2 + 4b_0^3 = \mu t_3. \tag{2.13}$$

Clearly, equation (2.8) and (2.11) in light of Lemma 1.2 gives:

$$|b_0| \le \mu. \tag{2.14}$$

Also by adding (2.9) in (2.12), we obtain:

$$6b_0^2 = \mu\left(s_2 + t_2\right)$$

which, by using Lemma 1.2 gives:

$$\left|b_0^2\right| \leq \frac{2\mu}{3}. \tag{2.15}$$

Equation (2.14) and (2.15) together yields:

$$|b_0| \leq \min\left\{\mu, \sqrt{\frac{2\mu}{3}}\right\},$$

which, for $0 < \mu \leq 1$ gives the desired result (2.1).

Now, by subtracting (2.9) from (2.12), we get:

$$6b_1 = \mu\left(t_2 - s_2\right) \tag{2.16}$$

which, by using Lemma 1.2 gives:

$$|b_1| \leq \frac{2\mu}{3}.$$

This is the desired result (2.2).

Finally, for the last inequality subtracting (2.10) from (2.13), we get:

$$4b_0b_1 + 8b_2 + 8b_0^3 = \mu\left(t_3 - s_3\right). \tag{2.17}$$

Also, by adding (2.10) in (2.13), we get:

$$20b_0b_1 = \mu\left(s_3 + t_3\right). \tag{2.18}$$

Eliminating b_0b_1 from (2.17) and (2.18), we obtain:

$$40\left(b_2 + b_0^3\right) = \mu\left(4t_3 - 6s_3\right)$$

which, in light of Lemma 1.2, yields the desired inequality (2.3).

This completes the proof of Theorem 2.2. □

Definition 2.3. A function $g(z) \in \Sigma^{'}$ given by (1.2) is said to be in the class $\mathcal{T}_{\Sigma'}^{\alpha}$ if the following conditions are satisfied:

$$\left|arg\left(\frac{z^2g^{'}(z)}{\left(g(z)\right)^2}\right)\right| < \frac{\alpha\pi}{2} \quad \left(z \in \mathbb{U}^*; 0 < \alpha \leq 1\right)$$

and

$$\left|arg\left(\frac{w^2h^{'}(w)}{\left(h(w)\right)^2}\right)\right| < \frac{\alpha\pi}{2} \quad \left(w \in \mathbb{U}^*; 0 < \alpha \leq 1\right),$$

where the function h is an inverse of g given by (1.3).

Theorem 2.4. *Let the function $g(z) \in \Sigma'$ given by (1.2) be in the class $\mathcal{T}_{\Sigma'}^{\alpha}$, where $0 < \alpha \le 1$. Then,*

$$|b_0| \le \sqrt{\frac{2}{3}}\alpha, \tag{2.19}$$

$$|b_1| \le \frac{2}{3}\alpha^2, \tag{2.20}$$

$$\left|b_2 + b_0^3\right| \le \frac{\alpha\left(2\alpha^2+1\right)}{6}. \tag{2.21}$$

Proof. Since $g(z) \in \mathcal{T}_{\Sigma'}^{\alpha}$; for $s(z), t(w) \in \mathcal{P}$ the conditions given in the definition of the function class $\mathcal{T}_{\Sigma'}^{\alpha}$ can be written as:

$$\frac{z^2 g'(z)}{(g(z))^2} = [s(z)]^{\alpha} \tag{2.22}$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = [t(w)]^{\alpha}, \tag{2.23}$$

where $s(z)$ and $t(w)$ have the form as given in (2.6) and (2.7), respectively.

Clearly, we have:

$$[s(z)]^{\alpha} = 1 + \frac{\alpha s_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)s_1^2 + \alpha s_2}{z^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)s_1^3 + \alpha(\alpha-1)s_1 s_2 + \alpha s_3}{z^3} + \cdots$$

and

$$[t(w)]^{\alpha} = 1 + \frac{\alpha t_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)t_1^2 + \alpha t_2}{w^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)t_1^3 + \alpha(\alpha-1)t_1 t_2 + \alpha t_3}{w^3} + \cdots.$$

Also, just as in proof of Theorem 2.2 we have:

$$\frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3\left(b_0^2 - b_1\right)}{z^2} + \frac{8b_0 b_1 - 4b_2 - 4b_0^3}{z^3} + \cdots$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3\left(b_0^2 + b_1\right)}{w^2} + \frac{12b_0 b_1 + 4b_2 + 4b_0^3}{w^3} + \cdots.$$

Now, equating the coefficients in (2.22) and (2.23) we get:

$$-2b_0 = \alpha s_1, \tag{2.24}$$

$$3\left(b_0^2-b_1\right)=\frac{1}{2}\alpha\left(\alpha-1\right)s_1^2+\alpha s_2, \tag{2.25}$$

$$8b_0b_1-4b_2-4b_0^3=\frac{1}{6}\alpha\left(\alpha-1\right)\left(\alpha-2\right)s_1^3+\alpha\left(\alpha-1\right)s_1s_2+\alpha s_3, \tag{2.26}$$

$$2b_0=\alpha t_1, \tag{2.27}$$

$$3\left(b_0^2+b_1\right)=\frac{1}{2}\alpha\left(\alpha-1\right)t_1^2+\alpha t_2, \tag{2.28}$$

$$12b_0b_1+4b_2+4b_0^3=\frac{1}{6}\alpha\left(\alpha-1\right)\left(\alpha-2\right)t_1^3+\alpha\left(\alpha-1\right)t_1t_2+\alpha t_3. \tag{2.29}$$

Clearly, equation (2.24) and (2.27) in light of Lemma 1.2 gives:

$$|b_0|\le\alpha. \tag{2.30}$$

Also by adding (2.25) in (2.28), we obtain:

$$6b_0^2=\frac{1}{2}\alpha\left(\alpha-1\right)\left(s_1^2+t_1^2\right)+\alpha\left(s_2+t_2\right)$$

which, by using Lemma 1.2 gives:

$$\left|b_0^2\right|\le\frac{2}{3}\alpha^2. \tag{2.31}$$

Obviously, from (2.30) and (2.31) we can write:

$$|b_0|\le\sqrt{\frac{2}{3}}\alpha\le\alpha;\quad(0<\alpha\le1).$$

This gives the desired result (2.19).

Now, by subtracting (2.25) from (2.28), we get:

$$6b_1=\frac{1}{2}\alpha\left(\alpha-1\right)\left(t_1^2-s_1^2\right)+\alpha\left(t_2-s_2\right)$$

which, by using Lemma 1.2 gives:

$$|b_1|\le\frac{2}{3}\alpha^2.$$

This is the desired result (2.20).

Finally, subtracting (2.26) from (2.29), we get:

$$\begin{aligned}24\left(b_0b_1+2b_2+2b_0^3\right)=&\,\alpha\left(\alpha-1\right)\left(\alpha-2\right)\left(t_1^3-s_1^3\right)+\\&6\alpha\left(\alpha-1\right)\left(t_1t_2-s_1s_2\right)+6\alpha\left(t_3-s_3\right).\end{aligned} \tag{2.32}$$

Also, by adding (2.26) in (2.29), we get:

$$120b_0b_1 = \alpha(\alpha-1)(\alpha-2)(s_1^3+t_1^3)+6\alpha(\alpha-1)(s_1s_2+t_1t_2)+6\alpha(s_3+t_3). \tag{2.33}$$

Eliminating b_0b_1 from (2.32) and (2.33), we obtain:

$$\begin{aligned}240\left(b_2+b_0^3\right) = \alpha\left(\alpha-1\right)\left(\alpha-2\right)\left(4t_1^3-6s_1^3\right)+ \\ 6\alpha\left(\alpha-1\right)\left(4t_1t_2-6s_1s_2\right)+6\alpha\left(4t_3-6s_3\right)\end{aligned}$$

which, in light of Lemma 1.2, yields the desired inequality (2.21).

This completes the proof of Theorem 2.4. □

3. Conclusion

It is interesting that, for functions in both the subclasses $\mathcal{T}_{\Sigma'}(\mu)$ and $\mathcal{T}_{\Sigma'}^{\alpha}$, $(0<\mu,\alpha\leq 1)$; all the coefficient inequalities are similar in the following sense:

$$\max_{g\in\Sigma'}|b_0|\leq\sqrt{\frac{2}{3}},$$

$$\max_{g\in\Sigma'}|b_1|\leq\frac{2}{3},$$

$$\max_{g\in\Sigma'}\left|b_2+b_0^3\right|\leq\frac{1}{2}.$$

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Accepted: 20.01.2018

A note on unitarily invariant norm inequalities for accretive-dissipative operator matrices

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Abstract. In this paper, we present a unitarily invariant norm inequality for accretive-dissipative operator matrices, which is similar to an inequality obtained by Zhang in [J. Math. Anal. Appl. 412 (2014) 564-569]. Examples are provided to show that neither Zhang's inequality nor our inequality is uniformly better than the other.

Keywords: unitarily invariant norms, accretive-dissipative operators, inequalities.

1. Introduction

In this note, we use the same notation as in [11, 14]. For convenience, recall that, as usual, let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $\mathbf{H} := \mathcal{H} \oplus \mathcal{H}$ and $T \in \mathcal{B}(\mathbf{H})$, the operator T can be represented as a 2×2 operator matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with $T_{jk} \in \mathcal{B}(\mathcal{H})$, $j, k = 1, 2$.

For any $T \in \mathcal{B}(\mathbf{H})$, we can write

$$T = A + iB, \tag{1.1}$$

in which $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are Hermitian operators. This is the Cartesian decomposition of T. In this paper, we always represent the decomposition of (1.1) as follows,

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix}, \tag{1.2}$$

where $T_{jk}, A_{jk}, B_{jk} \in \mathcal{B}(\mathcal{H})$, $j, k = 1, 2$. Then $A_{12} = A_{21}^*$, $B_{12} = B_{21}^*$.

If T is a compact operator, we denote by $s_1(T) \geq s_2(T) \geq \cdots$ the eigenvalues of $(T^*T)^{\frac{1}{2}}$, which are called the singular values of T. Thus, whenever we talk about singular values, the operators are necessarily compact. We denote by

$W(A)$ the numerical range of A. A norm $\|\cdot\|_u$ on $\mathcal{B}(\mathcal{H})$ is unitarily invariant if $\|T\|_u = \|UTV\|_u$ for all unitaries $U, V \in \mathcal{B}(\mathcal{H})$. Every unitarily invariant norm is defined on an ideal in $\mathcal{B}(\mathcal{H})$. It will be implicity understood that the operator T is in this ideal when we talk of $\|T\|_u$. Recall that T with $T = A+iB$ is accretive-dissipative if both A and B are positive. For the study of accretive-dissipative matrices in matrix theory and numerical linear algebra, the readers can refer to [2, 3, 7, 8]. Recent works devoted to studying the accretive-dissipative operators or matrices are in [6, 9, 10].

Zhang [14, Theorem 2] obtained the following unitarily invariant norm inequality.

Theorem 1. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in (1.2). Then*

$$\|T\|_u \quad \leq \quad 2\|T_{11} + T_{22}\|_u \tag{1.3}$$

for any unitarily invariant norm $\|\cdot\|_u$.

However, there is a gap in the proof of Zhang [14, Theorem 2]. Since in the proof of Theorem 2 in [14] the author proves that the last equality

$$2\|A_{11} + B_{11} + i(A_{22} + B_{22})\|_u \quad = \quad 2\|T_{11} + T_{22}\|_u$$

holds, actually it is as follows:

$$\begin{aligned} 2\|A_{11} + B_{11} + i(A_{22} + B_{22})\|_u &\leq 2\|A_{11} + B_{11} + A_{22} + B_{22}\|_u \\ &\leq 2\sqrt{2}\|A_{11} + A_{22} + i(B_{11} + B_{22})\|_u \\ &= 2\sqrt{2}\|A_{11} + iB_{11} + A_{22} + iB_{22}\|_u \\ &= 2\sqrt{2}\|T_{11} + T_{22}\|_u. \end{aligned}$$

The purpose of this paper is to discuss unitarily invariant norm inequalities for the accretive-dissipative operator matrix (1.1), which are similar to the inequality (1.3) . Our main result is the following theorem.

Theorem 2. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in (1.2). Then*

$$\|T\|_u \leq \sqrt{2}[\|T_{11} + T_{22}\|_u + 2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] \tag{1.4}$$

for any unitarily invariant norm $\|\cdot\|_u$. Furthermore, if $0 \notin W(B_{12}+C_{12})$, then

$$\|T\|_u \leq \sqrt{2}[\|T_{11} + T_{22}\|_u + \|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}]. \tag{1.5}$$

2. Main results

Before proving the main theorem of this paper, we need a few auxiliary results.

Lemma 3 ([12]). *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then for any complex number z,*

$$\prod_{j=1}^{k} s_j(A + zB) \quad \leq \quad \prod_{j=1}^{k} s_j(A + |z|B)$$

for all $k = 1, 2, \ldots$. As a consequence,

$$\sum_{j=1}^{k} s_j(A + zB) \quad \leq \quad \sum_{j=1}^{k} s_j(A + |z|B)$$

for all $k = 1, 2, \ldots$.

Lemma 4 ([4, Corollary 2.1]). *If $A, B, X \in \mathcal{B}(\mathcal{H})$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive, then we have the following decomposition*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \quad = \quad U \begin{pmatrix} \dfrac{A+B}{2} + \mathrm{Re}X & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{A+B}{2} - \mathrm{Re}X \end{pmatrix} V^*$$

for some unitary operator matrices $U, V \in \mathcal{B}(\mathbf{H})$.

Lemma 5 ([13, p. 42]). *The operator matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is positive if and only if both A and C are positive and there exists a contraction W such that $B = A^{\frac{1}{2}} W C^{\frac{1}{2}}$.*

Lemma 6 ([14, Lemma 2]). *Let $P_i, Q_i \in \mathcal{B}(\mathcal{H})$ be positive and let $C_i \in \mathcal{B}(\mathcal{H})$ be contractive, $i = 1, 2, \ldots, m$. Then*

$$\sum_{j=1}^{k} s_j \left(\sum_{i=1}^{m} P_i C_i Q_i \right) \leq \sum_{j=1}^{k} s_j \left((\sum_{i=1}^{m} P_i^2)^{\frac{1}{2}} \right) s_j \left((\sum_{i=1}^{m} Q_i^2)^{\frac{1}{2}} \right),$$

for all $k = 1, 2, \ldots$.

Lemma 7 ([1, Theorem 1.1]). *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then*

$$s_j(A + B) \quad \leq \quad \sqrt{2} s_j(A + iB) for all j = 1, 2, \ldots.$$

Remark 8. Reverse inequality of Lemma 7 was given in [5].

Lemma 9. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in* (1.2). *Then*

$$\|B_{12} + C_{12}\|_u \leq \sqrt{2} \|T_{11}\|_u^{\frac{1}{2}} \|T_{22}\|_u^{\frac{1}{2}}.$$

Proof. Compute

$$\begin{aligned}
\|B_{12}+C_{12}\|_u &= \sum_{j=1}^{k}\alpha_j s_j(B_{12}+C_{12})\\
&= \sum_{j=1}^{\infty}\alpha_j s_j(B_{11}^{\frac{1}{2}}W_1B_{22}^{\frac{1}{2}}+C_{11}^{\frac{1}{2}}W_2C_{22}^{\frac{1}{2}}) && \text{(by Lemma 5)}\\
&\le \sum_{j=1}^{\infty}\alpha_j s_j((B_{11}+C_{11})^{\frac{1}{2}})s_j((A_{22}+B_{22})^{\frac{1}{2}}) && \text{(by Lemma 6)}\\
&= \sum_{j=1}^{\infty}\alpha_j (s_j(B_{11}+C_{11}))^{\frac{1}{2}}(s_j(A_{22}+B_{22}))^{\frac{1}{2}}\\
&\le \sum_{j=1}^{\infty}\alpha_j [\sqrt{2}s_j(B_{11}+iC_{11})]^{\frac{1}{2}}[\sqrt{2}s_j(B_{22}+iC_{22})]^{\frac{1}{2}} && \text{(by Lemma 7)}\\
&\le \sqrt{2}\sum_{j=1}^{\infty}\alpha_j [s_j(T_{11})]^{\frac{1}{2}}[s_j(T_{22})]^{\frac{1}{2}}\\
&\le \sqrt{2}(\sum_{j=1}^{\infty}\alpha_j s_j(T_{11}))^{\frac{1}{2}}(\sum_{j=1}^{\infty}\alpha_j s_j(T_{22}))^{\frac{1}{2}} && \text{(by Cauchy-Schwarz inequality)}\\
&= \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\|B_{12}+C_{12}\|_u \le \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}.$$

This completes the proof. □

Lemma 10 ([4, Corollary 2.6]). *If $A, B, X \in \mathcal{B}(\mathcal{H})$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive, then for $0 \notin W(X)$ we have*

$$\left\|\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right\|_u \le \|A+B\|_u + \|X\|_u$$

for any unitarily invariant norm.

Proof of Theorem 2. Compute

$$\begin{aligned}
&\|B+iC\|_u \le \|B+C\|_u && \text{(by Lemma 3)}\\
&\le \left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}+\mathrm{Re}(B_{12}+C_{12})\right\|_u\\
&+ \quad \left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}-\mathrm{Re}(B_{12}+C_{12})\right\|_u
\end{aligned}$$

$$
\begin{aligned}
& \text{(by Lemma 4 and triangle inequality)} \\
&\leq 2\left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}\right\|_u + 2\left\|\operatorname{Re}(B_{12}+C_{12})\right\|_u \text{ (by triangle inequality)} \\
&\leq \sqrt{2}\|B_{11}+B_{22}+i(C_{11}+C_{22})\|_u + 2\|\operatorname{Re}(B_{12}+C_{12})\|_u \qquad \text{(by Lemma 7)} \\
&\leq \sqrt{2}\|T_{11}+T_{22}\|_u + 2\sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}} \qquad \text{(by Lemma 9)} \\
&\leq \sqrt{2}[\|T_{11}+T_{22}\|_u + 2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}].
\end{aligned}
$$

Thus,

$$\|B+iC\| \leq \sqrt{2}\left[\|T_{11}+T_{22}\|_u + 2\,\|T_{11}\|_u^{\frac{1}{2}}\,\|T_{22}\|_u^{\frac{1}{2}}\right].$$

Furthermore, if $0 \notin W(B_{12}+C_{12})$, then we have

$$
\begin{aligned}
&\|B+iC\|_u \leq \|B+C\|_u && \text{(by Lemma 3)} \\
&\leq \|B_{11}+C_{11}+B_{22}+C_{22}\|_u + \|B_{12}+C_{12}\|_u && \text{(by Lemma 10)} \\
&\leq \sqrt{2}\|B_{11}+B_{22}+i(C_{11}+C_{22})\|_u + \|B_{12}+C_{12}\|_u && \text{(by Lemma 7)} \\
&= \sqrt{2}\|T_{11}+T_{22}\|_u + \|B_{12}+C_{12}\|_u && \\
&\leq \sqrt{2}\|T_{11}+T_{22}\|_u + \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}} && \text{(by Lemma 9)} \\
&= \sqrt{2}[\|T_{11}+T_{22}\|_u + \|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}].
\end{aligned}
$$

This completes the proof. □

The following examples show that neither (1.3) nor (1.4) is uniformly better than the other.

Example 1. Let

$$
\begin{aligned}
T &= B+iC \\
&= \begin{pmatrix} 0.001 & 0 \\ 0 & 2 \end{pmatrix} + i\begin{pmatrix} 0.001 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.001+0.001i & 0 \\ 0 & 2+1i \end{pmatrix},
\end{aligned}
$$

then $T_{11} = 0.001+0.001i$, $T_{22} = 2+i$.

For the right side of (1.3), $2\|T_{11}+T_{22}\|_u = 6.3283$. For the right side of (1.4), $\sqrt{2}[\|T_{11}+T_{22}\|_u+2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] = 3.3232$. This shows that (1.4) is better than (1.3) in some cases.

Example 2. If

$$\begin{aligned} T &= B + iC \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + i \begin{pmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0.001 \end{pmatrix} \\ &= \begin{pmatrix} 1 + i * 0.001 & -1 & 0 & 0 \\ -1 & 1 + i * 0.001 & 0 & 0 \\ 0 & 0 & 1 + i * 0.001 & 1 \\ 0 & 0 & 1 & 1 + i * 0.001 \end{pmatrix}, \end{aligned}$$

then

$$T_{11} = \begin{pmatrix} 1 + 0.001i & -1 \\ -1 & 1 + 0.001i \end{pmatrix}$$

and

$$T_{22} = \begin{pmatrix} 1 + 0.001i & 1 \\ 1 & 1 + 0.001i \end{pmatrix}.$$

For the right side of (1.3), $2\sqrt{2}\|T_{11} + T_{22}\|_2 = 5.6583$. For the right side of (1.4), $\sqrt{2}[\|T_{11} + T_{22}\|_u + 2\|T_{11}\|_2^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] = 8.4860$. This implies that (1.4) is weaker than (1.3) in some cases.

Acknowledgments

We are grateful to Dr. Limin Zou for fruitful discussions. The work is supported by Hainan Provincial Natural Science Foundation for High-level Talents grant no. 2019RC171 and the Ministry of Education of Hainan grant no. Hnky2019ZD-13.

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Accepted: 25.01.2018

Limiting direction of Julia sets and infinite radial order of solutions to complex linear differential equations

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Abstract. In this paper we find that for infinite order entire functions, the ray where it takes infinite radial order is a common limiting directions of Julia sets of their derivatives and their primitives. Applying this result to the solutions of some complex differential equations, we obtain the lower bound of the measure of sets of common limiting directions of Julia sets of the derivatives and integral primitives of any non-trivial solution of these equations, which give alternative proofs of previous results.
Keywords: radial order, limiting direction, Julia set, complex differential equation.

1. Introduction and main results

In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna theory in the complex plane $\mathbb{C}$ and in an angle domain; see [5, 9, 19]. We use $\sigma(g)$ and $\mu(g)$ to denote the order and lower order of meromorphic function g in the complex plane respectively; see [19, p.10] for the definitions. Following [5], we give the notations of Nevanlinna theory in the angle. Set

$$\Omega(\alpha,\beta)=\{z\in\mathbb{C}:\alpha<\arg z<\beta\},\ \Omega(\alpha,\beta,r)=\{z:z\in\Omega(\alpha,\beta),|z|<r\}$$

and denote by $\overline{\Omega}(\alpha,\beta)$ the closure of $\Omega(\alpha,\beta)$. Let $g(z)$ be meromorphic on the closed angle $\overline{\Omega}(\alpha,\beta)$, where $\beta-\alpha\in(0,2\pi]$. Define

$$\begin{aligned}
A_{\alpha,\beta}(r,g)&=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\}\frac{dt}{t};\\
B_{\alpha,\beta}(r,g)&=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta;\\
C_{\alpha,\beta}(r,g)&=2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_n-\alpha),
\end{aligned}$$

where $\omega=\pi/(\beta-\alpha)$, and $b_n=|b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. Thus, the Nevanlinna angular characteristic is

defined as
$$S_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g) + C_{\alpha,\beta}(r,g).$$
Moreover, the order of $S_{\alpha,\beta}(r,g)$ is defined by
$$\rho_{\alpha,\beta}(g) = \limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,g)}{\log r}.$$

In addition, if $g(z)$ is analytic on the angle $\overline{\Omega}(\alpha,\beta)$, we define the order of g on $\Omega(\alpha,\beta)$ by
$$\sigma_{\alpha,\beta}(g) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,\Omega(\alpha,\beta),g)}{\log r},$$
where $M(r,\Omega(\alpha,\beta),g) = \sup_{\alpha\leq\theta\leq\beta} |g(re^{i\theta})|$. If $g(z)$ is analytic on $\mathbb{C}$, the order $\sigma(g)$ of g satisfies $\sigma(g) \geq \sigma_{\alpha,\beta}(g)$. Moreover, the sectorial order $\sigma_{\theta,\varepsilon}(g)$ and the radial order $\sigma_\theta(g)$ are defined by
$$\sigma_{\theta,\varepsilon}(g) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g)}{\log r}, \quad \sigma_\theta(g) = \lim_{\varepsilon\to 0} \sigma_{\theta,\varepsilon}(g).$$
Similarly, the sectorial, respectively radial, exponent of convergence for zeros of $g(z)$ are defined by
$$\lambda_{\theta,\varepsilon}(g) = \limsup_{r\to\infty} \frac{\log^+ n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g=0)}{\log r}, \quad \lambda_\theta(g) = \lim_{\varepsilon\to 0} \lambda_{\theta,\varepsilon}(g),$$
where $n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g=0)$ stands for the number of zeros of $g(z)$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon,r)$ counting multiplicity.

Definition 1.1 *Let $f(z)$ be a transcendental meromorphic function of order σ. The ray* $\arg z = \theta$ *is called a Borel direction of f if for any $\varepsilon > 0$, $\lambda_{\theta,\varepsilon}(f-a) = \sigma$ with at most two exceptional value $a \in \mathbb{C}\cup\{\infty\}$.*

Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [3, 22]. We define $f^n, n \in \mathbb{N}$ denote the nth iterate of f. The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. It is well known that $F(f)$ is open and completely invariant under f, $J(f)$ is closed and non-empty. Given $\theta \in [0,2\pi)$, if $\Omega(\theta-\varepsilon,\theta+\varepsilon)\cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $meas\Delta(f)$ is the linear measure of $\theta \in [0,2\pi)$ such that $J(f)$ has the radial distribution with respect to $\arg z = \theta$.

Baker [2] first proved that the Julia set of a transcendental entire function can not lie in finite many rays emanating from the original point. Later, Qiao [11] proved the measure of limiting direction of Julia set of finite lower order transcendental entire function f satisfies $meas\Delta(f) \geq \min\{2\pi, \frac{\pi}{\mu(f)}\}$. Moreover, for a transcendental entire function of finite lower order, Qiao [12] also found out the lower bounded of the measure of the common limiting direction of Julia set of its derivatives and its primitive. There are also some other papers related to this aspect; see [10, 12, 13, 16].

For transcendental meromorphic function, Zheng [23] has given significant results. Recent years, many results about the measure of limiting directions of Julia sets of solutions of complex linear differential equations have been appeared; see [6, 7, 15, 20, 21].

For a transcendental entire function $f(z)$ of infinite order, it is easy to see that there exist some angular domains $\Omega(\alpha, \beta)$ such that $\sigma_{\alpha,\beta}(f) = \infty$. But, $\sigma(f) = \infty$ cannot guarantee $\sigma_{\alpha,\beta}(f) = \infty$ for any angular domain $\Omega(\alpha, \beta)$. For example, it's known that $\sigma(\exp\{\exp z\}) = \infty$, while $\sigma_{\frac{\pi}{2},\frac{3\pi}{2}}(\exp\{\exp z\}) = 0$. We know that a main study content about the complex differential equations is the estimation of the order of their solutions. There has many results about the infinite order of the solutions to complex differential equations under various conditions. Naturally, a question that how wide are the angular domains $\Omega(\alpha, \beta)$ such that $\sigma_\theta(f) = \infty$ for any ray $\theta = \arg z \in \Omega(\alpha, \beta)$ is raised. For convenience of the following, set $I(f) = \{\theta \in [0, 2\pi) : \sigma_\theta(f) = \infty\}$ for infinite order entire function f. By the idea of study the measure of Julia set of solution of complex differential equations, Huang and Wang [8] obtained the lower bound of $I(f)$ of the non-trivial solutions of second order linear differential equations. In order to give subsequent results, we firstly give the following theorem about the high order differential equation.

Theorem 1.1. *Let $A_i(z)(i = 0, 1, \ldots, n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $T(r, A_i) = o(T(r, A_0)), (i = 1, 2, \ldots, n-1)$ as $r \to \infty$. Then every non-trivial solution f of the equation*

$$f^{(n)} + A_{n-1}f^{(n-1)} + \ldots + A_0 f = 0 \tag{1.1}$$

satisfies $meas I(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$. Moreover, if $\sigma(A_i) < \sigma(A_0), (i = 1, 2, \cdots, n-1)$, then there exist a closed interval $I_0 \subseteq I(f)$ with $meas(I_0) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

Comparing the proof of [8, Theorem 1.3] and that of [7, Theorem 1.1], we can see that the method of finding the lower bound of $meas I(f)$ and $meas\Delta(f)$ of the solutions of complex differential equations is similar in some sense. Therefore, there may has special relations between the infinite radial order and limiting direction of Julia set for infinite order entire function f. Indeed, we find the following relationship.

Theorem 1.2 *Suppose that f is an entire function of infinite order, then for any $\theta \in I(f)$, the ray $\arg z = \theta$ is a common limiting direction of Julia sets of $f^{(n)}$, where $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f, for $n \geq 0$ or $n < 0$, respectively.*

Then by the above Theorems 1.1 and 1.2, we obtain the following result, which has been proved by Chen and Wang [15]. Our method is different from that in [15].

Corollary 1.3 *Suppose f is any non-trivial solution of* (1.1) *in Theorem 1.1, then the measure of set of common limiting direction of Julia sets of $f^{(n)}$ satisfies $meas(\bigcap_{n\in\mathbf{Z}}\Delta(f^{(n)}))\geq \min\{2\pi,\pi/\mu(A_0)\}$.*

In [15], the authors also proved that every solution of a second order linear differential equation, the coefficient of which has finite deficient value, is of infinite lower order, see the following.

Theorem 1.4 *Suppose that A_0 is a transcendental entire function and $T(r,A_0)\sim\log M(r,A_0)$ as $r\to\infty$ outside a set of finite logarithmic measure, $A_1(z)$ is a finite order entire function and has a finite deficient value a, i.e. $\delta(a,A_1)>0$, then every non-trivial solution f of*

$$f''+A_1(z)f'+A_0(z)f=0 \tag{1.2}$$

satisfies $\mu(f)=\infty$.

Together the proof of above theorem with the proof of [8, Theorem 1.3], we can obtain the lower bound of $measI(f)$ for the non-trivial solution of equation (1.2).

Theorem 1.5 *Suppose that f is a non-trivial solution of* (1.2)*, where $A_0(z)$ and $A_1(z)$ satisfy the conditions in Theorem 1.4, then*

$$measI(f)\geq\min\left\{2\pi,\frac{4}{\mu(A_1)}\arcsin\sqrt{\frac{\delta(a,A_1)}{2}}\right\}.$$

Combining the above theorem and Theorem 1.2, we can obtain the following result easily, which is proved in [15] by another method.

Theorem 1.6 *Under the hypotheses of Theorem 1.4, the measure of set of common limiting direction of Julia sets of $f^{(n)}$ satisfies*

$$meas(\bigcap_{n\in\mathbf{Z}}\Delta(f^{(n)}))\geq\min\left\{2\pi,\frac{4}{\mu(A_1)}\arcsin\sqrt{\frac{\delta(a,A_1)}{2}}\right\}.$$

2. Preliminary lemmas

We call W is a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ contains at least three points, where $\overline{\mathbb{C}}$ is the extended complex plane. For an $a\in\mathbb{C}\backslash W$, define

$$C_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It's well known that, if every component of W is simply connected, then $C_W(a)\geq 1/2$; see [23].

Lemma 2.1 ([23, Lemma 2.2]). *Let $f(z)$ be analytic in $\Omega(r_0, \theta_1, \theta_2)$, U be a hyperbolic domain, and $f : \Omega(r_0, \theta_1, \theta_2) \to U$. If there exists a point $a \in \partial U \backslash \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), \quad z \to \infty, \ z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon).$$

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of R-set; for reference, see [9]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$, then $\bigcup_{n=1}^{\infty} B(z_n, r_n)$ is called an R-set. Clearly, the set $\{|z| : z \in \bigcup_{n=1}^{\infty} B(z_n, r_n)\}$ is of finite linear measure.

Lemma 2.2 ([7, Lemma 2.2]). *Let $z = re^{i\psi}, r_0 + 1 < r$ and $\alpha \le \psi \le \beta$, where $0 < \beta - \alpha \le 2\pi$. Suppose that $n(\ge 2)$ is an integer, and that $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\rho_{\alpha,\beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, (\beta_j - \alpha_j)/2)(j = 1, 2, \ldots, n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \ldots, n-1,$$

there exists $K>0$ and $M>0$ only depending on g, $\varepsilon_1, \ldots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right| \le Kr^M (\sin k(\psi - \alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \le Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j)\right)^{-2},$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set D, where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)(j = 1, 2, \ldots, n-1)$.

Lemma 2.3 ([18, 22]). *Let $f(z)$ be a transcendental meromorphic function with lower order $\mu(f) < \infty$ and order $0 < \sigma(f) \le \infty$. Then, for any positive number λ with $\mu(f) \le \lambda \le \sigma(f)$ and any set H of finite measure, there exists a sequence $\{r_n\}$ satisfies:*

(1) $r_n \notin H, \lim_{n\to\infty} r_n/n = \infty$;
(2) $\liminf_{n\to\infty} \log T(r_n, f)/\log r_n \ge \lambda$;
(3) $T(r, f) < (1 + o(1))(2t/r_n)^{\lambda} T(r_n/2, f), t \in [r_n/n, nr_n]$;
(4) $t^{-\lambda-\varepsilon_n} T(t, f) \le 2^{\lambda+1} r_n^{-\lambda-\varepsilon_n} T(r_n, f), 1 \le t \le nr_n, \varepsilon_n = (\log n)^{-2}$.

Such $\{r_n\}$ is called a sequence of Pólya peaks of order λ outside H. The following lemma, which related to Pólya peaks, is called the spread relation; see [1].

Lemma 2.4 ([1]). *Let $f(z)$ be a transcendental meromorphic function with positive order and finite lower order, and has a deficient value $a \in \overline{\mathbb{C}}$. Then, for any sequence of Pólya peaks $\{r_n\}$ of order $\lambda > 0$, $\mu(f) \leq \lambda \leq \sigma(f)$, and any positive function$\Upsilon(r) \to 0$ as $r_n \to \infty$, we have*

$$\liminf_{r_n\to\infty} meas D_\Upsilon(r_n, a) \geq \min\left\{2\pi, \frac{4}{\lambda}\arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\},$$

where

$$D_\Upsilon(r,a) = \left\{\theta \in [0, 2\pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Upsilon(r)T(r,f)\right\},\ a \in \mathbb{C}$$

and

$$D_\Upsilon(r,\infty) = \left\{\theta \in [0, 2\pi) : \log^+ |f(re^{i\theta})| > \Upsilon(r)T(r,f)\right\}.$$

For the Borel directions of entire functions with infinite order, Sun [14] obtained the following lemma.

Lemma 2.5 ([14]). *Let g be an entire function of infinite order, then the ray* $\arg z = \theta$ *is a Borel direction of infinite order for g if and only if* $\arg z = \theta$ *is a Borel direction of infinite order for g'.*

The following lemma is a weaker version of Chuang's result.

Lemma 2.6 ([4]). *Let f be a meromorphic function of infinite order, then the ray* $\arg z = \theta$ *is one Borel direction of infinite order of f if and only if f satisfies the equality*

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\log r} = \infty,$$

for any $\varepsilon \in (0, \pi/2)$.

Lemma 2.7 ([21]). *Let $f(z)$ be a transcendental entire function. If $\sigma_\theta(f) = \sigma(f)$, then the ray* $\arg z = \theta$ *is a radial distribution of the Julia set of f.*

Lemma 2.8 ([17, Corollary 2.3.6]). *If $g(z)$ is an entire function with $0 < \sigma(g) < \infty$, then there exists an angular domain $\Omega(\theta_1, \theta_2)$ with $\theta_2 - \theta_1 \geq \pi/\sigma(g)$ such that*

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ |g(re^{i\theta})|}{\log r} = \sigma(g),$$

for any $\theta \in (\theta_1, \theta_2)$.

3. Proof of Theorems

Proof of Theorem 1.1

Suppose that f is a non-trivial solution of equation (1.1) under the hypotheses of this theorem. From [7, p.479] we know that $\sigma(f) \geq \mu(f) = \infty$. We assume that $measI(f) < \nu := \min\{2\pi, \pi/\mu(A_0)\}$, so $\zeta := \nu - measI(f) > 0$. Clearly $S = (0, 2\pi)\backslash\overline{I(f)}$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i := (\alpha_i, \beta_i), i = 1, 2, \cdots, m$ satisfying $[\alpha_i, \beta_i] \subset S$ and $meas(S\backslash\bigcup_{i=1}^{m} I_i) < \frac{\zeta}{4}$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see

$$\Omega(\alpha_i, \beta_i) \cap I(f) = \emptyset.$$

This implies that for each $i = 1, 2, \cdots, m$, we have $\sigma_{\alpha_i,\beta_i}(f) < \infty$, and from the definition of $\rho_{\alpha_i,\beta_i}(f)$ and [22, Corollary 2.2.2], we have $\rho_{\alpha_i,\beta_i}(f) < \infty$. Therefore, by Lemma 2.7, for sufficiently small $\varepsilon > 0$, there exist two constants $M > 0$ and $K > 0$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \leq Kr^M, s = 1, 2, \cdots, n, \tag{3.1}$$

for all $z \in \bigcup_{i=1}^{m} \Omega(\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside an R-set H.

Applying Lemma 2.3 to $A_0(z)$, there exist a sequence of Pólya peak $\{r_n\}$ of order $\mu(A_0)$ such that $r_n \notin \{|z|, z \in H\}$, and for sufficiently large n,

$$meas\{D_\Upsilon(r_n, \infty)\} \geq \nu - \frac{\zeta}{4}, \tag{3.2}$$

where we take the function $\Upsilon(r)$ as

$$\Upsilon(r) = \max\left\{\sqrt{\frac{\log r}{T(r, A_0)}}, \sqrt{\frac{T(r, A_1)}{T(r, A_0)}}, \cdots, \sqrt{\frac{T(r, A_{n-1})}{T(r, A_0)}}\right\}.$$

Without loss of generality, we assume that (3.2) holds for all n, and simplified denote $D(r_n) = D_\Upsilon(r_n, \infty)$. Obviously,

$$\begin{aligned} meas(D(r_n) \cap S) &= meas(D(r_n)\backslash(I(f) \cap D(r_n))) \\ &\geq measD(r_n) - measI(f) > \frac{3\zeta}{4}. \end{aligned} \tag{3.3}$$

Then, for each n we have

$$\begin{aligned} meas((\bigcup_{i=1}^{m} I_i) \cap D(r_n)) &= meas(S \cap D(r_n)) - meas((S\backslash\bigcup_{i=1}^{m} I_i) \cap D(r_n)) \\ &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2} > 0. \end{aligned} \tag{3.4}$$

This means there exist at least one open interval I_{i_0}=(α,β) of $I_i, (i=1,2,\cdots,m)$ such that for infinitely many j,

$$meas(D(r_j)\cap(\alpha,\beta)) > \frac{\zeta}{2m} > 0. \tag{3.5}$$

Set $G_j = D(r_j)\cap(\alpha+2\varepsilon,\beta-2\varepsilon)$, it follows from the definition of $D(r_j)$ in Lemma 2.4, $T(r,A_0)=m(r,A_0)$ and (3.5) that

$$\int_{G_j}\log^+|A_0(r_je^{i\theta})|d\theta \geq meas(G_j)\Upsilon(r_j)m(r_j,A_0) \geq \frac{\zeta}{4m}\Upsilon(r_j)m(r_j,A_0). \tag{3.6}$$

We rewrite (1.1) as

$$A_0 = -\left(\frac{f^{(n)}}{f} + A_{n-1}\frac{f^{(n-1)}}{f} + \cdots + A_1\frac{f'}{f}\right). \tag{3.7}$$

Substituting (3.1) into (3.7) yields

$$\begin{aligned}\int_{G_j}\log^+|A_0(r_je^{i\theta})|d\theta &\leq \int_{G_j}\left(\sum_{i=1}^{n-1}\log^+\left|A_i(r_je^{i\theta})\right|\right)d\theta + O(\log r_j)\\ &\leq \sum_{i=1}^{n-1}m(r_j,A_i) + O(\log r_j).\end{aligned} \tag{3.8}$$

This and (3.6) give out

$$\frac{\zeta}{4m}\Upsilon(r_j)m(r_j,A_0) \leq \sum_{i=1}^{n-1}m(r_j,A_i) + O(\log r_j) \tag{3.9}$$

which is impossible since A_0 is transcendental and $T(r,A_i) = o(T(r,A_0))(i = 1,2,\cdots,n-1)$ as $r\to\infty$. Thus, we deduce that $measI(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

In the following, we consider the case $\sigma(A_i) < \sigma(A_0)(i=1,2,\cdots,n-1)$. By Lemma 2.8, there exists an interval (a,b) with $b-a \geq \min\{2\pi, \frac{\pi}{\sigma(A_0)}\}$ such that, for $\theta\in(a,b)$,

$$\limsup_{r\to\infty}\frac{\log^+\log^+|A_0(re^{i\theta})|}{\log r} = \sigma(A_0). \tag{3.10}$$

We shall prove $[a,b]\subset I(f)$. Assume that $[a,b]\not\subset I(f)$, then $(a,b)\setminus I(f)$ is bounded and open, so there must have $(\alpha,\beta)\subset(a,b)$ such that $\sigma_{\alpha,\beta}(f) < +\infty$. Thus, (3.1) still holds for $z\in\Omega(\alpha+\varepsilon,\beta-\varepsilon)$ outside an R-set H for sufficiently small ε. Since $\{r=|z| : z=re^{i\theta}\in H\}$ is a set of finite linear measure, then the set of θ, where the ray $\theta=\arg z$ meets R-set H infinitely many times, is of measure zero. Thus, we can find $\theta_0\in(\alpha+\varepsilon,\beta-\varepsilon)\in(a,b)$ such that the ray $\arg z = \theta_0$ meets H finitely many times. Furthermore, combining (3.1) and (3.7) yields, for sufficiently large r,

$$\begin{aligned}\log^+|A_0(re^{i\theta_0})| &\leq \sum_{i=1}^{n}\log^+\left|\frac{f^{(i)}(re^{i\theta_0})}{f(re^{i\theta_0})}\right| + \sum_{i=1}^{n-1}\log^+\left|A_i(re^{i\theta_0})\right| + O(1)\\ &= \sum_{i=1}^{n-1}\log^+\left|A_i(re^{i\theta_0})\right| + O(\log r)\\ &\leq r^{\sigma(A_0)-\varepsilon} \qquad (3.11)\end{aligned}$$

which contradicts with (3.10). Hence, we prove that $[a,b] \in I(f)$. Thus, we complete the proof.

Proof of Theorem 1.2

Following Lemma 2.7, we can get that, for any $\theta \in I(f)$, the ray $\arg z = \theta$ must be a limiting direction of Julia set of f. Moreover, by Lemma 2.6, for any $\theta \in I(f)$, the ray $\arg z = \theta$ must be one infinite order Borel direction of f. Then, by Lemma 2.5, the ray $\arg z = \theta$ is also a Borel direction of infinite order for f' and F, setting which a primitive function of f. Applying Lemma 2.6 to f' and F, we obtain that the radial order of f' and F at $\arg z = \theta$ is infinity. Finally, by Lemma 2.7, we deduce that the ray $\arg z = \theta$ is not only a limiting direction of Julia sets of f', but also a limiting direction of Julia sets of F.

Repeating the above arguments infinitely many times, we can obtain that $\arg z = \theta$ is a common limiting direction of Julia set of $f^{(n)}$, where $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n \geq 0$ or $n < 0$, respectively.

Proof of theorem 1.5

We shall prove the conclusion by reduction to absurdity. We firstly assume that $measI(f) < \nu := \min\{2\pi, \frac{4}{\mu(A_1)}\arcsin\sqrt{\frac{\delta(a,A_1)}{2}}\}$, then $\zeta := \nu - measI(f) > 0$. For given $0 < c < 1$, set $I_c(r) = \{\theta \in [0,2\pi) : \log|A_0(re^{i\theta})| < c\log M(r,A_0)\}$. The definition of proximity function yields that

$$\begin{aligned}T(r,A_0) = m(r,A_0) &\leq \left(1 - \frac{measI_c(r)}{2\pi}\right)\log M(r,A_0)\\ &+ c\left(\frac{measI_c(r)}{2\pi}\right)\log M(r,A_0).\end{aligned}$$

Since $T(r,A_0) \sim \log M(r,A_0)$ outside a set F of finite linear measure, we have $measI_c(r) \to 0$ as $r \notin F \to \infty$. By Lemma 2.4, we can take an increasing and unbounded sequence $\{r_k\}$ such that $measD(r_k) \geq \nu - \frac{\zeta}{4}$, where $D(r) = \{\theta \in [0,2\pi) : \log|A_1(re^{i\theta}) - a| < 1\}$, all $r_j \notin \{|z| : z \in H\} \cup F$ with H being an $R-$ set. Clearly, $|A_1(r_ke^{i\theta})| \leq e + |a|$ for $\theta \in D(r_k)$. Similarly as in the proof of Theorem

1.1, there always exists an open interval $I_{i_0} = (\alpha, \beta)$ of $I_i, (i = 1, 2, \cdots, m)$ such that for infinitely many j,

$$meas(D(r_j) \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0, \tag{3.12}$$

and (3.1) still holds in $\Omega(r, \alpha + \varepsilon, \beta - \varepsilon)$. Hence, substituting (3.1) into

$$|A_0(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| \tag{3.13}$$

yields

$$M(r_k, A_0)^c \leq |A_0(r_k e^{i\theta})| \leq (|a| + e + 1)Kr_k^M,$$

for $\theta \in (D(r_k) \cap I_0) \setminus I_c(r_k)$. It is impossible since A_0 is transcendental. Then we complete the proof.

Acknowledgements

This work was supported by the key scientific research project for higher education institutions of Henan Province, China (no. 18A110002) and training program for young backbone teachers of colleges and universities in Henan Province, China (no. 2017GGJS126).

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Accepted: 24.02.2018

Intuitionistic fuzzy ideals on approximation systems

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Abstract. In this paper, we initiate the concept of intuitionistic fuzzy ideals on rough sets. Using a new relation we discuss some of the algebraic nature of intuitionistic fuzzy ideals of a ring.
Keywords: intuitionistic fuzzy ideals, lower and upper approximation, rough ideals.

1. Introduction

Rough set theory, proposed by Pawlak [25] is a new mathematical tool that supports uncertainty reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields. The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximation spaces are induced. Using this approximation knowledge hidden information may be revealed and expressed in the form of decision rules. A key notion in Pawlak rough set model is an equivalence relation. Atanassov [2] presented intuitionistic fuzzy sets in 1986 which is very effective to deal with vagueness. As a generalization of fuzzy set the concept of intuitionistic fuzzy set has played an important role in analysis of uncertainty of data. Various notions of intuitionistic fuzzy rough set were explored to extend rough set theory in the intuitionistic fuzzy environment. This paper concerns a relationship between rough sets, intuitionistic fuzzy sets and ring theory. We consider a ring as a universal set and assume the knowledge about objects is

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restricted by an intuitionistic fuzzy ideal. In fact, we apply the notion of intuitionistic fuzzy ideal of a ring for definitions of lower and upper approximations in a ring. Some of its characterizations are discussed.

2. Preliminaries

Definition 2.1 ([2]). *An intuitionistic fuzzy set (IFS in short) A in X is an object having the form* $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ *where the function* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non membership (namely* $\nu_A(x)$*) of each element* $x \in X$ *to the set A, respectively and* $0 \le \mu_A(x) + \nu_A(x) \le 1$ *for each* $x \in X$*. Denote by IFS(X) the set of all intuitionistic fuzzy set in X.*

Definition 2.2 ([2]). *Let A and B be IFS's of the form* $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)/x \in X\rangle\}$*. Then*

1. $A \subseteq B$ *if and only if* $\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$ *for all* $x \in X$.
2. $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.
3. $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)/x \in X\rangle\}$ *.(Complement of A)*
4. $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)/x \in X\rangle\}$.
5. $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)/x \in X\rangle\}$.

For the sake of simplicity we use the notion $A = \langle x, \mu_A, \nu_A\rangle$ *instead of* $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$.

The intuitionistic fuzzy set $0\sim = \{\langle x, 0\sim, 1\sim\rangle/x \in X\}$ *and* $1\sim = \{\langle x, 1\sim, 0\sim\rangle/x \in X\}$ *are respectively the empty set and the whole set of X.*

Definition 2.3 ([4]). *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be any two IFS of R. Then their sum* $A + B$ *is defined by*

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B)),$$

where $(\mu_A+\mu_B)(x) = \bigvee_{x=y+z}[\mu_A(y) \wedge \mu_B(z)]$ *and* $(\nu_A+\nu_B)(x) = \bigwedge_{x=y+z}[\nu_A(y) \vee \nu_B(z)]$ *for all* $x \in R$

Definition 2.4 ([4]). *An IFS* $A = (\mu_A, \nu_A)$ *is called an intuitionistic fuzzy ideal of R if for all* $x, y, i \in R$

$(IF1)$ $\mu_A(x - y) \ge \mu_A(x) \wedge \mu_A(y)$ *and* $\nu_A(x - y) \le \nu_A(x) \vee \nu_A(y)$;

$(IF2)$ $\mu_A(xy) \ge \mu_A(y)$ *and* $\nu_A(xy) \le \nu_A(y)$.

Definition 2.5 ([14]). *For an approximation space* (U, θ) *by a rough approximation in* (U, θ) *we mean a mapping* $(U, \theta, -) : P(U) \to P(U) \times P(U)$ *defined for every* $X \in P(U)$ *by* $(U, \theta, X) = ((\underline{U}, \theta, X), (\overline{U}, \theta, X))$ *where* $(\underline{U}, \theta, X) =$

$\{x \in U | [x]_\theta \subseteq X\}, (\overline{U}, \theta, X) = \{x \in U | [x]_\theta \cap X \neq \phi\}$ where $(\underline{U}, \theta, X)$ is called a lower rough approximation of X in (U, θ) , whereas $(\overline{U}, \theta, X)$ is called an upper approximation of X in (U, θ). Given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, θ) if and only if $(A, B) = (U, \theta, X)$ for some $X \in P(U)$. Let (U, θ) be an approximation space and X a non-empty subset of U:

(*i*) *If $(\underline{U}, \theta, X) = (\overline{U}, \theta, X)$, then X is called definable.*

(*ii*) *If $(\underline{U}, \theta, X) = \phi$ then X is called empty interior.*

(*iii*) *If $(\overline{U}, \theta, X) = U$, then X is called empty interior.*

The lower approximation of X in (U, θ) is the greatest definable set in U contained in X. The upper approximation of X in is the least definable set in U containing X. Therefore we have

$$(\underline{U}, \theta, X) = \bigcup \{S | S \subseteq X, S \text{ is definable}\}$$

$$(\overline{U}, \theta, X) = \bigcap \{S | X \subseteq S, S \text{ is definable}\}$$

A rough set of X is the family of all subsets of U having the same upper approximation of X.

3. Intuitionistic fuzzy ideals and congruence relations

Theorem 3.1. *For an intuitionistic fuzzy ideal A of a ring R we have the following*

(*i*) $\mu_A(0) \geq \mu_A(x)$ *and* $\nu_A(0) \leq \nu_A(x)$,

(*ii*) $\mu_A(-x) = \mu_A(x)$ *and* $\nu_A(-x) = \nu_A(x)$ *for all* $x \in R$.

Proof. (i) For any $x \in R$ we have $\mu_A(0) = \mu_A(x - x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x), \nu_A(0) = \nu_A(x - x) \leq \nu_A(x) \vee \nu_A(x) = \nu_A(x)$.

(ii) By using (i) we get $\mu_A(-x) = \mu_A(0 - x) \geq \mu_A(0) \wedge \mu_A(x) = \mu_A(x)$, $\nu_A(-x) = \nu_A(0 - x) \leq \nu_A(0) \vee \nu_A(x) = \nu_A(x)$.

Since x is arbitrary we conclude that $\mu_A(-x) = \mu_A(x)$ and $\nu_A(-x) = \nu_A(x)$. □

Theorem 3.2. *If an intuitioistic fuzzy set $A = (\mu_A, \nu_A)$ in R satisfies (IFI) then:*

(*i*) $\mu_A(x - y) = \mu_A(0) \Rightarrow \mu_A(x) = \mu_A(y)$,

(*ii*) $\nu_A(x - y) = \nu_A(0) \Rightarrow \nu_A(x) = \nu_A(y)$ *for all* $x, y \in R$.

Proof. Let $x, y \in R$ such that $\mu_A(x-y) = \mu_A(0)$. Then $\mu_A(x) = \mu_A(x-y+y) \geq \mu_A(x - y) \wedge \mu_A(y) = \mu_A(0) \wedge \mu_A(y) = \mu_A(y)$. Similarly $\nu_A(y) = \nu_A(x - x + y) = \nu_A(x - (x - y)) \leq \nu_A(x) \vee \nu_A(x - y) = \mu_A(x)$. So $\mu_A(x) = \mu_A(y)$.

(ii) $\nu_A(x-y) = \nu_A(0)$ for all $x, y \in R$ then $\nu_A(x) = \nu_A(x-y+y) \leq \nu_A(x) \vee \nu_A(y) \leq \nu_A(0) \vee \nu_A(y) = \nu_A(y)$.Similarly $\nu_A(x) \leq \nu_A(y)$ and so $\nu_A(x) = \nu_A(y)$. □

Definition 3.3. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on R and let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. Then the set $A_{\alpha,\beta} = \{x \in R | \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is called a (α, β)-level subset of A. The set of all $(\alpha, \beta) \in Im(\mu_A) \times Im(\nu_A)$ such that $\alpha + \beta \leq 1$ is called the image of $A = (\mu_A, \nu_A)$ denoted by $Im(A)$.*

Definition 3.4. *Let A be an intuitionistic fuzzy ideal of R for each $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ the set*

$$U(A_{(\alpha,\beta)}) = \{(a,b) \in R \times R | \mu_A(a-b) \geq \alpha, \nu_A(a-b) \leq \beta\}.$$

is called a (α, β)-level relation on A. An equivalence relation θ on a ring R is called a congruence relation if $(a,b) \in \theta \Rightarrow (a+x, b+x) \in \theta, (x+a, x+b) \in \theta$ for all $x \in R$.

Theorem 3.5. *Let A be an intuitionistic fuzzy ideal of R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ then $U(A_{(\alpha,\beta)})$ is a congruence relation on R.*

Proof. For any element $a \in R, \mu_A(a-a) = \mu_A(0) \geq \alpha$ and $\nu_A(a-a) = \nu_A(0) \leq \beta$ and so $(a,a) \in U(A_{(\alpha,\beta)})$ then $\mu_A(a-b) \geq \alpha$ and $\nu_A(a-b) \leq \beta$ implies $(a,b) \in U(A_{(\alpha,\beta)})$. Since A is an ideal of R $\mu_A(b-a) = \mu_A(-(a-b)) = \mu_A(a-b) \geq \alpha$. And $\nu_A(b-a) = \nu_A(-(a-b)) = \nu_A(a-b) \leq \beta$. Which yields $(b,a) \in U(A_{(\alpha,\beta)})$. If $(a,b) \in U(A_{(\alpha,\beta)})$ and $(b,c) \in U(A_{(\alpha,\beta)})$ then since A is an intuitionistic fuzzy ideal of R

$$\mu_A(a-c)=\mu_A((a-b)+(b-c))\geq \min\{\mu_A((a-b), \mu_A(b-c))\}\geq \min\{\alpha, \alpha\}=\alpha,$$
$$\nu_A(a-c) = \nu_A((a-b)+(b-c))\leq \max\{\nu_A((a-b), \nu_A(b-c))\}\leq \max\{\beta, \beta\}=\beta.$$

And hence $(a,c) \in U(A_{(\alpha,\beta)})$. Therefore $U(A_{(\alpha,\beta)})$ is an equivalence relation on R. Now let $(a,b) \in U(A_{(\alpha,\beta)})$ and x be an element of R. Then since $\mu_A(a-b) \geq \alpha, \nu_A(a-b) \leq \beta, \mu_A((a+x)-(b+x)) = \mu_A((a+x)+(-x-b)) = \mu_A(a+(x-x)-b) = \mu_A(a+0-b) = \mu_A(a-b) \geq \alpha, \nu_A((a+x)-(b+x)) = \nu_A((a+x)+(-x-b)) = \nu_A(a+(x-x)-b) = \nu_A(a+0-b) = \nu_A(a-b) \leq \beta$ and so $(a+x, b+x) \in U(A_(\alpha, \beta))$. Since $(R, +)$ is an abelian group, we have $(x+a, x+b) \in U(A_{(\alpha,\beta)})$. Therefore $U(A_{(\alpha,\beta)})$ is a congruence relation.

We denote $[x]_{A_{(\alpha,\beta)}}$ the equivalence class of $U(A_{(\alpha,\beta)})$ containing x of R. □

Lemma 3.6. *Let A be an intuitionistic fuzzy ideal of R. If $a, b \in R$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ then:*

(i) $[a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}} = [a+b]_{A_{(\alpha,\beta)}}$;

(ii) $[-a]_{A_{(\alpha,\beta)}} = -([a]_{A_{(\alpha,\beta)}})$.

Proof. (i) Suppose $x \in [a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}}$. Then there exists an $y \in [a]_{A_{(\alpha,\beta)}}$ and $[z] \in [b]_{A_{(\alpha,\beta)}}$ such that $x = y + z$. Since $(a, y) \in U(A_{(\alpha,\beta)})$ and $(b, z) \in U(A_{(\alpha,\beta)})$ we have $(a+b, y+z) \in U(A_{(\alpha,\beta)})$ or $(a+b, x) \in U(A_{(\alpha,\beta)})$ and so $x \in [a+b]_{A_{(\alpha,\beta)}}$.

Conversely let $x \in [a+b]_{A_{(\alpha,\beta)}}$ then $(x, a+b) \in U(A_{(\alpha,\beta)})$. Hence $(x-b, a) \in U(A_{(\alpha,\beta)})$and so $x - b \in U(A_{(\alpha,\beta)})$ or $x \in [a]_{A_{(\alpha,\beta)}} + [b] \Rightarrow x \in [a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}}$.

(ii) We have $x \in [-a]]_{A_{(\alpha,\beta)}} \Leftrightarrow (x, -a) \in U(A_{(\alpha,\beta)}) \Leftrightarrow (0, -a-x) \in U(A_{(\alpha,\beta)}) \Leftrightarrow (a, -x) \in U(A_{(\alpha,\beta)}) \Leftrightarrow -x \in [a]_{A_{(\alpha,\beta)}} \Leftrightarrow x \in -([a]_{A_{(\alpha,\beta)}}$. □

Lemma 3.7. *Let A and B be two intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ then $U((A \cap B)_{(\alpha,\beta)}) = U(A_{(\alpha,\beta)}) \cap U(B_{(\alpha,\beta)})$.*

Lemma 3.8. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. For any $a \in R$ we have $a + [0]_{A_{(\alpha,\beta)}} = [a]_{A_{(\alpha,\beta)}}$.*

Proof. Assume that $a \in R$ then we have $x \in a + [0]_{A_{(\alpha,\beta)}} \Leftrightarrow x - a \in [0]_{A_{(\alpha,\beta)}} \Leftrightarrow (x - a, 0) \in U(A_{(\alpha,\beta)}) \Leftrightarrow (x - a) \in U(A_{(\alpha,\beta)}) \Leftrightarrow x \in [a]_{A_{(\alpha,\beta)}}$ □

Lemma 3.9. *Let A and B be two intuitionistic fuzzy ideal of R such that $B \subseteq A$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$ for every $x \in R$.*

Proof. We have $y \in [x]_{A_{(\alpha,\beta)}} \Rightarrow (x, y) \in U(B_{(\alpha,\beta)}) \Rightarrow \mu_B(x - y) \geq \alpha$ and $\nu_B(x - y) \leq \beta \Rightarrow \mu_A(x - y) \geq \alpha$ and $\nu_A(x - y) \leq \beta \Rightarrow (x, y) \in U(A_{(\alpha,\beta)}) \Rightarrow y \in [x]_{A_{(\alpha,\beta)}}$. □

Definition 3.10. *Let A and B be two intuitionistic fuzzy ideals of a ring R. Then the composition of the congruence relation $U(A_{(\alpha,\beta)})$ and $U(B_{(\alpha,\beta)})$ is defined by*

$$U(A_{(\alpha,\beta)}) \circ U(B_{(\alpha,\beta)}) = \{(a, b) \in R \times R | \exists y \in R$$

such that $(a, c) \in U(A_{(\alpha,\beta)}), (c, b) \in U(B_{(\alpha,\beta)})\}$.

We have $U(A_{(\alpha,\beta)}) \circ U(B_{(\alpha,\beta)})$ is also a congruence relation. We denote the congruence relation by $U((A \circ B)_{(\alpha,\beta)})$

Lemma 3.11. *Let A and B be two intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then $U((A \circ B)_{(\alpha,\beta)}) \subseteq U((A + B)_{(\alpha,\beta)})$.*

Proof. Assume that (a, b) be an arbitrary element of $U((A \circ B)_{(\alpha,\beta)})$. Then there exist an element $c \in R$ such that $(a, c) \in U(A_{(\alpha,\beta)}))$ and $(c, b) \in U(B_{(\alpha,\beta)}))$. Therefore we have $\mu_A(a - c) \geq \alpha, \nu_A(a - c) \leq \beta, \mu_B(c - b) \geq \alpha, \nu_B(c - b) \leq \beta$. Then $\mu_A + \mu_B(a-b) = \bigvee_{u+v=a-b}(\mu_A(u) \wedge \mu_B(v)) = \mu_A(a-c) \wedge \mu_B(c-b) \geq \alpha \wedge \alpha = \alpha$, $(\nu_A + \nu_B)(a-b) = \bigwedge_{u+v=a-b}(\nu_A(u) \vee \nu_B(v)) = \nu_A(a-c) \vee \nu_B(c-b) \geq \beta \vee \beta = \beta$ and so $U((A + B)_{(\alpha,\beta)})$. □

Lemma 3.12. *Let A and B be two intuitionistic fuzzy ideals of a ring R with finite images and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then $U((A \circ B)_{(\alpha,\beta)}) = U((A + B)_{(\alpha,\beta)})$.*

Proof. Assume that $(a,b) \in U((A \circ B)_{(\alpha,\beta)})$. Then $(\mu_A + \mu_B)(a-b) \geq \alpha, (\nu_A + \nu_B)(a-b) \leq \beta$. Thus we have $\bigvee_{a-b=x+y}(\mu_A(x) \wedge \mu_B(y)) \geq \alpha, \bigwedge_{a-b=x+y}(\nu_A(u) \vee \nu_B(v)) \leq \beta$. Since $Im\mu_A$ and $Im\mu_B$ are finite $\mu_A(x_0) \wedge \mu_B(y_0) \geq \alpha$ for some $x_0, y_0 \in R$ such that $a - b = x_0 + y_0$. Thus

$$\mu_A(x_0) \geq \alpha \text{ and } \mu_A(y_0) \geq \alpha \Rightarrow \mu_A(x_0 - 0) \geq \alpha \text{ and } \mu_A(a - b - x_0) \geq \alpha \tag{1}$$

and $\nu_A(x_0) \vee \nu_B(y_0) \leq \beta$ for some $x_0, y_0 \in R$. Thus

$$\nu_A(x_0) \leq \beta \text{ and } \nu_B(y_0) \leq \beta \Rightarrow \nu_A(x_0 - 0) \leq \beta \text{ and } \nu_A(a - b - x_0) \leq \beta. \tag{2}$$

From (i) and (ii) $(x_0, 0) \in U(A_{(\alpha,\beta)}))$ and $(a - b, x_0) \in U(B_{(\alpha,\beta)})$. Therefore $(a - b, 0) \in U((A \circ B)_{(\alpha,\beta)}$. Since $U((A \circ B)_{(\alpha,\beta)}$ is a congruence relation we get $(a, b) \in U((A \circ B)_{(\alpha,\beta)}$. Thus $U((A \circ B)_{(\alpha,\beta)} = U((A + B)_{(\alpha,\beta)}$, if $Im\mu_A$ and $Im\mu_B$ are finite. □

4. Approximation based on intuitionistic fuzzy ideals

Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Since $U(A_{(\alpha,\beta)})$ is an equivalence(congruence) relation on R we use $(R, A_{(\alpha,\beta)})$ instead of the approximation space (U, θ) where $U = R$ and θ is the above equivalence relation.

Definition 4.1. *Let A be an intuitionistic fuzzy ideal of a ring R and $U(A_{(\alpha,\beta)})$ be an (α, β)-level congruence relation of A on R. Let X be a non-empty subset of R. Then the sets*

$$\underline{U}(A_{(\alpha,\beta)}, X) = \{x \in R | [x]_{A_{(\alpha,\beta)}} \subseteq X\},$$
$$\overline{U}(A_{(\alpha,\beta)}, X) = \{x \in R | [x]_{A_{(\alpha,\beta)}} \cap X \neq \phi\}.$$

are respectively the lower and upper approximation of the set X with respect to $U(A_{(\alpha,\beta)})$.

Proposition 4.2. *For every approximation space $(R, A_{(\alpha,\beta)})$ and every subset A, B of R, we have:*

(i) $\underline{U}(A_{(\alpha,\beta)}, B) \subseteq B \subseteq \overline{U}(A_{(\alpha,\beta)}, B)$;

(ii) $\underline{U}(A_{(\alpha,\beta)}, \phi) = \phi = \overline{U}(A_{(\alpha,\beta)}, \phi)$;

(iii) $\underline{U}(A_{(\alpha,\beta)}, R) = R = \overline{U}(A_{(\alpha,\beta)}, B)$;

(iv) *If $B \subset C$, then* $\underline{U}(A_{(\alpha,\beta)}, B) \subseteq \underline{U}(A_{(\alpha,\beta)}, C)$;$\overline{U}(A_{(\alpha,\beta)}, B) \subseteq \overline{U}(A_{(\alpha,\beta)}, C)$

(v) $\underline{U}(A_{(\alpha,\beta)}, \underline{U}(A_{(\alpha,\beta)}, B)) = \underline{U}(A_{(\alpha,\beta)}, B)$;

(vi) $\overline{U}(A_{(\alpha,\beta)}, \overline{U}(A_{(\alpha,\beta)}, B)) = \overline{U}(A_{(\alpha,\beta)}, B)$;

(*vii*) $\overline{U}(A_{(\alpha,\beta)}, \underline{U}(A_{(\alpha,\beta)}, B)) = \underline{U}(A_{(\alpha,\beta)}, B)$;

(*viii*) $\underline{U}(A_{(\alpha,\beta)}, \overline{U}(A_{(\alpha,\beta)}, B)) = \overline{U}(A_{(\alpha,\beta)}, B)$;

(*ix*) $\underline{U}(A_{(\alpha,\beta)}, B) = \overline{U}(A_{(\alpha,\beta)}, B^c)^c$;

(*x*) $\overline{U}(A_{(\alpha,\beta)}, B) = \underline{U}(A_{(\alpha,\beta)}, B^c)^c$;

(*xi*) $\underline{U}(A_{(\alpha,\beta)}, B \cap C) = \underline{U}(A_{(\alpha,\beta)}, B) \cap \underline{U}(A_{(\alpha,\beta)}, C)$;

(*xii*) $\overline{U}(A_{(\alpha,\beta)}, B \cap C) \subseteq \overline{U}(A_{(\alpha,\beta)}, B) \cap \overline{U}(A_{(\alpha,\beta)}, C)$;

(*xiii*) $\underline{U}(A_{(\alpha,\beta)}, B \cup C) \supseteq \underline{U}(A_{(\alpha,\beta)}, B) \cup \underline{U}(A_{(\alpha,\beta)}, C)$;

(*xiv*) $\overline{U}(A_{(\alpha,\beta)}, B \cup C) = \overline{U}(A_{(\alpha,\beta)}, B) \cup \overline{U}(A_{(\alpha,\beta)}, C)$;

(*xv*) $\underline{U}(A_{(\alpha,\beta)}, [x]_{A_{(\alpha,\beta)}}) = \overline{U}(A_{(\alpha,\beta)}, [x]_{A_{(\alpha,\beta)}})$ *for all* $x \in R$.

Proof. The proof is obvious. □

The converse of (xii) and (xiii) in proposition 4.2 need not be true seen from the following example.

Example 4.3. Let $R = \{0, x, y, z\}$ be a set with binary operations as follows:

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

.	0	x	y	z
0	0	0	0	0
x	0	x	y	z
y	0	x	y	z
z	0	0	0	0

Then clearly R is a ring with $x = -x, y = -y$ and $z = -z$. Now let $\mu_A(0) = \alpha_0, \nu_0 = \beta_0, \mu_A(z) = \alpha_1, \nu_A(z) = \beta_1, \mu_A(x) = \mu_A(y) = \alpha_2, \nu_A(x) = \nu_A(y) = \beta_2$, where $\alpha_1, \beta_1 \in [0,1], i = 0,1,2$ and $\alpha_2 < \alpha_1 < \alpha_0$ and $\beta_0 < \beta_1 < \beta_2$. We have $A_{(\alpha_0,\beta_0)} = \{(0,0),(x,x),(y,y),(z,z)\}$ $A_{(\alpha_1,\beta_1)} = \{(0,0),(x,x),(y,y),(z,z),(x,y),(y,x),(0,z),(z,0)\}$ $A_{(\alpha_2,\beta_2)} = R \times R$.

Now let B=$\{0,x\}$ and C=$\{0,y,z\}$. Then $\overline{U}(A_{(\alpha_1,\beta_1)}, B) = R; \overline{U}(A_{(\alpha_1,\beta_1)}, C) = R; \overline{U}(A_{(\alpha_1,\beta_1)}, (B \cap C) = \{0, z\}$; and $\underline{U}(A_{(\alpha_1,\beta_1)}, B) = \phi; \underline{U}(A_{(\alpha_1,\beta_1)}, C) = \{0, c\}$; $\underline{U}(A_{(\alpha_1,\beta_1)}, (B \cup C) = R$. Thus $\overline{U}(A_{(\alpha_1,\beta_1)}, B) \cap \overline{U}(A_{(\alpha_1,\beta_1)}, C) \nsubseteq \overline{U}(A_{(\alpha_1,\beta_1)}, (B \cap C)$; and $\underline{U}(A_{(\alpha_1,\beta_1)}, (B \cup C)) \nsubseteq \underline{U}(A_{(\alpha_1,\beta_1)}, B) \cup \underline{U}(A_{(\alpha_1,\beta_1)}, C)$;

Proposition 4.4. *Let A and B be intuitionistic fuzzy ideals of a ring R and let* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \leq 1$. *If X is a non-empty subset of R, then* $\overline{U}((A \cap B)_{A_{(\alpha,\beta)}}, X) \subseteq \underline{U}(A_{(\alpha,\beta)}, X) \cap \underline{U}(A_{(\alpha,\beta)}, X)$.

Proof. Let $x \in overline((A \cap B)_{A_{(\alpha,\beta)}}, X)$, $\Rightarrow [x]_{(A \cap B)_{(\alpha,\beta)}}) \cap X \neq \phi$, $\Rightarrow a \in [x]_{(A \cap B)_{(\alpha,\beta)}}) \cap X \Rightarrow (a, x) \in U(A \cap B)_{A_{(\alpha,\beta)}}$ and $a \in X$. $\Rightarrow (\mu_A \cap \mu_B)(a - x) \geq \alpha, (\nu_A \cup \nu_B) \leq \beta$ and $a \in X$. $\Rightarrow \min\{\mu_A(a-x), \mu_B(a-x)\} \geq \alpha, \max\{\nu_A(a -$

$x), \nu_B(a-x)\} \leq \beta$ and $a \in X$. $\Rightarrow \mu_A(a-x) \geq \alpha, \nu_A(a-x) \leq \beta$ and $\mu_B(a-x) \geq \alpha, \nu_B(a-x) \leq \beta$ and $a \in X$, $\Rightarrow (a,x) \in U(A_{(\alpha,\beta)})$ and $(a,x) \in U(B_{(\alpha,\beta)})$ and $a \in X$. $\Rightarrow (a,x) \in U(A_{(\alpha,\beta)}), a \in X$ and $(a,x) \in U(B_{(\alpha,\beta)}), a \in X$. $\Rightarrow a \in [x]_{A_{(\alpha,\beta)}} \cap X$ and $a \in [x]_{B_{(\alpha,\beta)}} \cap X$, $\Rightarrow x \in \overline{U}(A_{(\alpha,\beta)}, X)$ and $x \in \overline{U}(B_{(\alpha,\beta)}, X)$. The converse of the above proposition need not be true seen from the following example. $\square$

Example 4.5. Let $R = Z_6$(the ring of integers modulo 6). Let $B = Z_6 \to [0,1]$ and $C : Z_6 \to [0,1]$ with $\mu_B(0) = \nu_B(0) = \alpha_0, \mu_B(1) = \mu_B(2) = \mu_B(4) = \mu_B = \alpha_3, \nu_B(1) = \nu_B(2) = \nu_B(4) = \nu_B(5) = \beta_3, \mu_B(3) = \alpha_2, \nu_B(3) = \beta_2$;

$\mu_C = \alpha_1, \nu_c(0) = \beta_1; \mu_C(1) = \mu_C(3) = \mu_C(5) = \alpha_4, \nu_C(1) = \nu_C(2) = \nu_C(5) = \beta_4, \mu_C(2) = \mu_C(4) = \alpha_2, \nu_C(2) = \nu_C(4) = \beta_2$ where $\alpha_1, \beta_1 \in [0,1], i = 0,1,2,3,4$ and $\alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 < \alpha_0$ and $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$. We have $(\mu_B \cap \mu_C)(2) = (\mu_B \cap \mu_C)(4) = \alpha_3; (\mu_B \cap \mu_C)(1) = (\mu_B \cap \mu_C)(3) = (\mu_B \cap \mu_C)(5) = \alpha_4; (\mu_B \cap \mu_C)(0) = \alpha_1$; $(\nu_B \cup \nu_C)(2) = (\nu_B \cup \nu_C)(4) = \beta_3; (\nu_B \cup \nu_C)(1) = (\nu_B \cup \nu_C)(3) = (\nu_B \cup \nu_C)(5) = \beta_4; (\nu_B \cap \nu_C)(0) = \beta_1$;

Also $B_{(\alpha_0,\beta_0)}$=$\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$; $B_{(\alpha_2,\beta_2)}$=$\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(5,2),(2,5),(4,1)(1,4),(0,3),(3,0)\}$; $B_{(\alpha_3,\beta_3)} = Z_6 \times Z_6$; $C_{(\alpha_0,\beta_0)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$; $C_{(\alpha_2,\beta_2)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(5,3),(3,5),(4,2),(2,4),(1,3),(3,1),(0,2),(2,0),(5,1),(1,5),(0,4),(4,0)\}$; $C_{(\alpha_4,\beta_4)} = Z_6 \times Z_6$; $(B \cap C)_{(\alpha_0,\beta_0)}$=$\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$; $(B \cap C)_{(\alpha_3,\beta_3)}$=$\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(5,3),(3,5),(4,2),(2,4),(1,3),(3,1),(0,2),(2,0),(5,1),(1,5),(0,4),(4,0)\}$; $(B \cap C)_{(\alpha_4,\beta_4)} = Z_6 \times Z_6$.

Now let $X = \{1,2,3\}$, then $\overline{U}(B_{(\alpha_2,\beta_2)}, X) = Z_6; \overline{U}(C_{(\alpha_2,\beta_2)}, X) = Z_6; \overline{U}((B \cap C)_{(\alpha_2,\beta_2)}, X) = \{1,2,3\}$; and $\overline{U}((B \cap C)_{(\alpha,\beta)}, X) =\neq \overline{U}(B_{(\alpha,\beta)}, X) \cap \overline{U}(C_{(\alpha,\beta)}, X)$

Proposition 4.6. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If X is a non-empty subset of R, then $\underline{U}(A_{(\alpha,\beta)}, X) \cap \underline{U}(B_{(\alpha,\beta)}, X) = \underline{U}((A \cap B)_{(\alpha,\beta)}, X)$.*

Proof. Let $x \in \underline{U}(A_{(\alpha,\beta)}, X) \cap \underline{U}(B_{(\alpha,\beta)}, X)$, $\Rightarrow x \in \underline{U}(A_{(\alpha,\beta)}, X)$ and $x \in \underline{U}(B_{(\alpha,\beta)}, X)$, $\Rightarrow [x]_{A_{(\alpha,\beta)}} \subseteq X$ and $[x]_{B_{(\alpha,\beta)}} \subseteq X$, $\Rightarrow [x]_{(A \cap B)_{(\alpha,\beta)}} \subseteq X$, $\Rightarrow x \in \underline{U}((A \cap B)_{(\alpha,\beta)}, X)$. $\square$

Proposition 4.7. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If A is an ideal of R, then A is an upper rough ideal of R.*

Proof. Let $a, b \in \overline{U}(A_{(\alpha,\beta)})$ and $r \in R$, then $[a]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ and $[b]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ so there exists $x \in [a]_{A_{(\alpha,\beta)}} \cap B$ and $y \in [y]_{B_{(\alpha,\beta)}} \cap B$. Since B is an ideal of R we have $x - y \in B$ and $rx \in B$. Thus $x - y \in [a]_{A_{(\alpha,\beta)}} - [b]_{A_{(\alpha,\beta)}} = [a-b]_{A_{(\alpha,\beta)}}$. Hence $[a-b]_{(A \cap B)_{(\alpha,\beta)}} \cap B \neq \phi$ this implies $a - b \in \overline{U}(A_{(\alpha,\beta)}, B)$. Since $(x,a) \in$

$U(A)_{(\alpha,\beta)}$, then $\mu_A(x-a) \geq \alpha, \nu_A(x-a) \leq \beta$. Now we have

$$\mu_A(rx-ra) = \mu_A(r(x-a)) \geq \max\{\mu_A(r), \mu_A(x-a)\} \geq \mu_A(x-a) \geq \alpha,$$
$$\nu_A(rx-ra) = \nu_A(r(x-a)) \leq \min\{\nu_A(r), \nu_A(x-a)\} \leq \nu_A(x-a) \leq \beta.$$

Hence $(rx, ra) \in U(A_{(\alpha,\beta)})$ or $rx \in [ra]_{A_{(\alpha,\beta)}}$, thus $rx \in [ra]_{A_{(\alpha,\beta)}} \cap B \Rightarrow [ra]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ Therefore $ra \in \overline{U}(A_{(\alpha,\beta)}), B)$. Likewise $ar \in \overline{U}(A_{(\alpha,\beta)}), B)$. Therefore $\overline{U}(A_{(\alpha,\beta)}), B)$ is an ideal of R. □

Lemma 4.8. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$.If $\underline{U}(A_{(\alpha,\beta)}, B)$ is a non-empty set, then $[0]_{A_{(\alpha,\beta)}} \subseteq B$.*

Proof. Let $\underline{U}(A_{(\alpha,\beta)}, B) \neq 0$ then there exists $x \in \underline{U}(A_{(\alpha,\beta)}, B)$ or $[x]_{A_{(\alpha,\beta)}} \subseteq B$. So $[x]_{A_{(\alpha,\beta)}} \subseteq B$. So $-([x]_{A_{(\alpha,\beta)}}) \subseteq -B = \{-a | a \in B\} = B$.

$[0]_{A_{(\alpha,\beta)}} = [x+(-x)]_{A_{(\alpha,\beta)}}; = [x]_{A_{(\alpha,\beta)}} + [-x]_{A_{(\alpha,\beta)}}; = [x]_{A_{(\alpha,\beta)}} + (-[x]_{A_{(\alpha,\beta)}}) \subseteq B + B = B$. □

Proposition 4.9. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$.Let B be an intuitionistic fuzzy ideal of R. If $\underline{U}(A_{(\alpha,\beta)}, B)$ is a non-empty set then it is equal to B.*

Proof. We know $\underline{U}(A_{(\alpha,\beta)}, B) \subseteq B$. Assume tha t a is an arbitrary element of B. Since $[0]_{A_{(\alpha,\beta)}} \subseteq B$. Since A is an ideal of R, we have $a+[0]_{A_{(\alpha,\beta)}} \subseteq a+B \subseteq B$; $\Rightarrow [a]_{A_{(\alpha,\beta)}} \subseteq B$. $\Rightarrow a \in \underline{U}(A_{(\alpha,\beta)}, B)$. □

Corollary 4.10. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$.If B is an ideal of R, then $(\underline{U}(A_{(\alpha,\beta)}, B), \overline{U}(A_{(\alpha,\beta)}, B)$ is a rough ideal of R.*

Proposition 4.11. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$.If C is a non-empty subset of R, then*

(*i*) $\overline{U}(B_{(\alpha,\beta)}, C) \subseteq \overline{U}(A_{(\alpha,\beta)}, C)$;

(*ii*) $\underline{U}(A_{(\alpha,\beta)}, C) \subseteq \underline{U}(B_{(\alpha,\beta)}, C)$.

Proof. (*i*) Let x be an arbitrary element of $\overline{U}(B_{(\alpha,\beta)}, C)$ then $[x]_{B_{(\alpha,\beta)}} \cap C \neq \phi$, since $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$, we have $[x]_{A_{(\alpha,\beta)}} \cap C \neq \phi$, which implies $x \in \overline{U}(A_{(\alpha,\beta)}, C)$.

(*ii*) Let $x \in \underline{U}(A_{(\alpha,\beta)}, C)$, then $[x]_{A_{(\alpha,\beta)}} \subseteq C \Rightarrow [x]_{B_{(\alpha,\beta)}} \subseteq C$ thus $x \in \overline{U}(B_{(\alpha,\beta)}, C)$.

□

Proposition 4.12. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$.Let X be a non-empty subset of R.If $U(B_{(\alpha,\beta)}) \subseteq \overline{U}(A_{(\alpha,\beta)})$ then*

(i) $\overline{U}(B_{(\alpha,\beta)}, X) \subseteq \overline{U}(A_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}(A_{(\alpha,\beta)}, X) \subseteq \underline{U}(B_{(\alpha,\beta)}, X)$.

Proof. (i) Let x be an arbitrary element of $\overline{U}(B_{(\alpha,\beta)}, X)$ then there exists $a \in [x]_{B_{(\alpha,\beta)}} \cap C$. Then $a \in X$ and $(a, x) \in U(B_{(\alpha,\beta)}, X) \subseteq \overline{U}(A_{(\alpha,\beta)})$. Therefore $a \in [x]_{A_{(\alpha,\beta)}} \cap X$and so $x \in \overline{U}(A_{(\alpha,\beta)}, X)$.

(ii) Let x be an arbitrary element of $\underline{U}(A_{(\alpha,\beta)}, X)$, then $[x]_{A_{(\alpha,\beta)}} \subseteq X$. Since $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$ we get $[x]_{B_{(\alpha,\beta)}} \subseteq X$ implies $x \in \underline{U}(A_{(\alpha,\beta)}, X)$.

□

Proposition 4.13. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Let X be a non-empty subset of R then*

(i) $\overline{U}((A \circ B)_{(\alpha,\beta)}, X) \subseteq \overline{U}((A + B)_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}((A + B)_{(\alpha,\beta)}, X) \subseteq \underline{U}((A \circ B)_{(\alpha,\beta)}, X)$.

Proposition 4.14. *Let A and B be an intuitionistic fuzzy ideal of a ring R, with finite images and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. If X is a non-empty subset of R, then*

(i) $\overline{U}((A \circ B)_{(\alpha,\beta)}, X) \subseteq \overline{U}((A + B)_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}((A + B)_{(\alpha,\beta)}, X) \subseteq \underline{U}((A \circ B)_{(\alpha,\beta)}, X)$.

If A and B are non-empty subsets of R. Let A.B denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \ldots + a_nb_n, n \in N, a_i \in A, b_i \in B\}$.

Proposition 4.15. *Let A and B be intuitionistic fuzzy ideals of a ring R and $\alpha, \beta \in [0, 1]$ with $\alpha+\beta \leq 1$. If A is an ideal of R, then $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C) \subseteq \overline{U}((A \circ B)_{(\alpha,\beta)}, C)$*

Proof. Suppose that z be any element of $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C)$. Then $z = \sum_{i=1}^{n} a_ib_i$ for some $a_i \in \overline{U}(A_{(\alpha,\beta)}, C)$ and $b_i \in \overline{U}(B_{(\alpha,\beta)}, C)$. Thus $[a_i]_{A_{(\alpha,\beta)}} \cap C \neq \phi$ and $[b_i]_{B_{(\alpha,\beta)}} \cap C \neq \phi$ for $i = 1, 2, 3 \ldots n$. Since C is ideal of R then $\sum_{i=1}^{n} x_iy_i \in C$. Since $(x_i, a_i) \in U(A_{(\alpha,\beta)})$ and $(y_i, b_i) \in U(B_{(\alpha,\beta)})$ we have $\mu_A(x_i - a_i) \geq \alpha, \nu_A(x_i - a_i) \leq \beta$ and $\mu_B(y_i - b_i) \geq \alpha, nu_A(y_i - b_i \leq \beta$. Then $\mu_A(x_ib_i - a_ib_i) = \mu_A((x_i - a_i)b_i) \geq max\{\mu_A(x_i - a_i), \mu_A(b_i)\} \geq \mu_A(x_i - a_i) \geq \alpha$, $\nu_A(x_ib_i - a_ib_i) = \nu_A((x_i - a_i)b_i) \leq min\{\nu_A(x_i - a_i), \nu_A(b_i)\} \leq \nu_A(x_i - a_i) \leq \beta$, $\mu_B(x_iy_i - x_ib_i) = \mu_B((x_i(y_i - b_i)) \geq max\{\mu_B(x_i), \mu_B(y_i - b_i)\} \geq \mu_B(y_i - b_i) \geq \alpha$, $\nu_B(x_iy_i - x_ib_i) = \nu_B((x_i(y_i - b_i) \leq min\{\nu_B(x_i), \nu_B(y_i - b_i)\} \leq \nu_B(y_i - b_i) \leq \beta$. Hence $(x_ib_i, a_ib_i) \in U(A_{\alpha,\beta})$ and $(x_iy_i, x_ib_i) \in U(B)_{\alpha,\beta}$ and so $(x_iy_i, a_ib_i) \in U(A \circ B)_{\alpha,\beta}$ for all $i = 1, 2, ...n$. Since $U(A \circ B)_{\alpha,\beta}$ is a congruence relation we get $[\sum_{i=1}^{n} x_iy_i, \sum_{i=1}^{n} a_ib_i] \in U(A \circ B)_{(\alpha,\beta)}$ and so $\sum_{i=1}^{n} x_iy_i \in [\sum_{i=1}^{n} a_ib_i]_{(A \circ B)_{(\alpha,\beta)}}$. Therefore $[\sum_{i=1}^{n} a_ib_i]_{(A \circ B)_{(\alpha,\beta)}} \cap C \neq \phi \Rightarrow \sum_{i=1}^{n} a_ib_i \in \overline{U}(A \circ B)_{(\alpha,\beta)}$. □

Corollary 4.16. *Let A and B be intuitionistic fuzzy ideal of a ring R and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If A is an ideal of R, then $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C) \subseteq \overline{U}((A+B)_{(\alpha,\beta)}, C)$*

5. Conclusion

In this paper we considered the concept of intuitionistic fuzzy ideals on rough sets. The lower and upper approximation of rough sets were defined and a new definition is defined such that the approximation space satisfies the condition of the ideal. Using this new relation we have discussed some of the algebraic nature of intuitionistic fuzzy ideals of a ring.In future the authors may extend this paper to neutrosophic ideals in approximation systems.

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Accepted: 27.02.2018

On modular flats and pushouts of matroids

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Abstract. In this paper, a sufficient condition for a submatroid of a loopless matroid to be a modular flat is given. Moreover, it is shown that if the injective pushout of two loopless matroids relative to a common submatroid exists, then the join of the given matroids exists and is isomorphic to the indicated pushout.

Keywords: matroid, flat, modular flat, injective pushout.

1. Background

We follow the terminology of White [17] and Lawvere and Schanuel [18]. In particular, the *ground set* of a matroid M, the *rank* of M and the *closure* of a subset $A \subseteq E(M)$ are denoted by $E(M)$, $r(M)$, $\bar{A}$, respectively. A *loopless* matroid is a matroid which has no single element set with rank zero. Let A and B be flats of M. Then (A, B) is a *modular pair of flats* if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$. If F is a flat of M such that (F, A) is a modular pair for all flats A, then F is a *modular flat* of M.

By a *join* of two matroids M and N relative to a common submatroid S, we mean a matroid on the point set consisting of the disjoint union of $M - S$, $N - S$ and S, the flats of which are all subsets F such that $F \cap M$ is a flat of M and $F \cap N$ is a flat of N.

By an *injective pushout* of two matroids M and N relative to a common submatroid S, we mean a colimit for the diagram in Figure 1 where i_M and i_N are non-rank-decreasing injective strong maps. We will show the existence of the injective pushout guarantees the existence of the join. In fact, the join is isomorphic to the injective pushout.

$$\begin{tikzcd}
 & M \\
S \arrow[ur, "i_M"] \arrow[dr, "i_N"'] & \\
 & N
\end{tikzcd}$$

Fig. 1 Injective pushout of M and N relative to S.

For a complete background on the previous notions and the following ones, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

2. Joins of matroids

We begin this section by recalling the following result which is needed to prove Theorem 1:

Lemma 1 ([16]). *Let F be a flat of a matroid M.Then F is a modular flat if and only if $r(F)+r(A)=r(M)$ for all complements A of F.*

Theorem 1. *Let M be a loopless matroid with a submatroid S and suppose that $r(S)+r(X)\leq r(M)$ for all flats X of M disjoint from S. Then S is a modular flat.*

Proof. If $\bar{S}\neq S$, then there exists a point $c\in\bar{S}-S$. Let X be a subset of M satisfying $X\cap\bar{S}=\{c\}$ and $X\cup\bar{S}=M$. Then by the semimodularity of the rank and as $r(S)+r(X)=r(\bar{S})+r(X)$,

$$r(S)+r(X)\geq r(X\cup\bar{S})+r(X\cap\bar{S})=r(M)+r(c)>r(M), \tag{1}$$

and $S\cap X=\varnothing$, which is a contradiction to the assumption. Hence S is a flat. By the semimodularity of the rank for every complement X of S,

$$r(S)+r(X)\geq r(S\cup X)+r(S\cap X)=r(\overline{S\cup X})=r(M),$$

and then by assumption $r(S)+r(X)=r(M)$. Hence S is modular by Lemma 1. □

Next, we recall the following two results from [16]:

Lemma 2. *Suppose that T is a modular flat of $M$and every non-loop element of $\overline{T}-T$ is parallel to some element of T.Then T is fully embedded in M.*

Lemma 3. *Let M be a matroid on a set E and suppose that, for some subset T of E, the matroid $M/T=M1\oplus M2$. If T is a modular flat of the simple matroid associated with $M\setminus(E(M_2)$, then*

$$M=P_{M|T}(M\setminus(E(M_2),M\setminus(E(M_1)).$$

Next, we look at some sufficient conditions for a join to be exist. The proofs of the first two theorems follow from Lemma 2 and Lemma 3 combined with Theorem 1.

Theorem 2. *Let M_1 be a loopless matroid, M_2 be a matroid and T be the intersection of the ground sets of M_1 and M_2. If $\overline{T}^{M_1}$ satisfies the rank property in Theorem 1 and every loop element of $\overline{T}^{M_1}-T$ is parallel to some element of T, then the join of M_1 and M_2 relative to $M_1|T$ exists, termed the generalized parallel connection $P_{M_1|T}(M_1,M_2)$.*

Theorem 3. *Let M be a matroid on a set E and suppose that, for some subset T of E, the matroid $M/T = M_1 \oplus M_2$, $\widetilde{T}$ be the simple matroid associated with $M_1|T$ and $\widetilde{M\backslash E(M_2)}$ the simple matroid associated with $M\backslash E(M_2)$. If $r(\widetilde{T}) + r(X) \leq r(\widetilde{M\backslash E(M_2)})$ for all flats X of $\widetilde{M\backslash E(M_2)}$ disjoint from $\widetilde{T}$, then $P_{M_1|T}(M\backslash E(M_2), M\backslash E(M_1))$ exists, termed the matroid M.*

Injective pushouts of matroids M and N relative to a common submatroid S have been known to exist for S equal to the empty set in which case it is the direct sum; and for S equal to a point in which case it is the parallel connection. Let S be the rank zero matroid with the points consisting of the disjoint union of $M-S$, $N-S$ and S. Then the identity maps from M and N into S are strong, so that by the unique existence of the colimit map $P \longrightarrow S$, the points of an injective pushout when it exists can be identified with the point set consisting of the disjoint union of $M-S$, $N-S$ and S so that it is a combinatorial geometry. Now we are ready to prove our main theorem which is an extremal matroid result, that the existence of the injective pushout guarantees the existence of the join.

Theorem 4. *If P is an injective pushout of matroids M and N relative to a common submatroid S, then the join of M and N relative to S exists and is isomorphic to P.*

Proof. By assumption there are strong maps $j_M : M \longrightarrow P$ and $j_N : N \longrightarrow P$ such that $j_M i_M = j_N i_N$. Also if I is a matroid and $g : M \longrightarrow I$ and $h : N \longrightarrow I$ are strong maps for which $gi_M = hi_N$, then there exists a unique strong map $f : P \longrightarrow I$ which make the diagram in Figure 2 commutative. By the paragraph preceding this theorem, j_M and j_N are injective and the point set of P is consisting of the disjoint union of $M-S$, $N-S$ and S. Let $K \subseteq P$ and assume $j_M^{-1}(K \cap M)$ and $j_N^{-1}(K \cap N)$ are flats of M and N, respectively. We need only show K is a flat of P since then P is the join of submatroids isomorphic to M and N relative to a common submatroid isomorphic to S. Let I be the matroid with a single loop y and $(P-K)$ parallel elements. Define a strong map $g : M \longrightarrow I$ by $g(z) = z$ when $z \in M - j_M^{-1}(K)$, and $g(z) = y$ when $z \in j_M^{-1}(K)$. Define a strong map $h : N \longrightarrow I$ similarly. For the strong map $f : P \longrightarrow I$, which makes the diagram in Figure 2 commutative, we find that $f(z) = z$ when $z \in P - K$ and $f(z) = y$ when $z \in K$. It follows that $K = f^{-1}(y)$ is a flat, which was to be proved. $\square$

$$\begin{tikzcd}
S \arrow[r, "i_M"] \arrow[d, "i_N"'] & M \arrow[d, "j_M"'] \arrow[ddr, "g"] & \\
N \arrow[r, "j_N"] \arrow[drr, "h"'] & P \arrow[dr, "f"] & \\
& & I
\end{tikzcd}$$

3. Acknowledgment

The authors would like to thank the referees for useful comments and suggestions.

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Accepted: 1.03.2018

Chebyshev wavelet method (CWM) for the numerical solutions of fractional boundary value problems

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Abstract. This research article, is concerned with the numerical solutions of fractional boundary value problems by using Chebyshev Wavelet Method (CWM). Simulations based on (CWM), are better in terms of the numerical solutions of higher order boundary value problems. The results obtained by the proposed method are compared with the results of Optimal Homotopy Asymptotic Method (OHAM), Modified Optimal Homotopy Asymptotic Method (MOHAM), Variation Iteration Method (VIM) and exact solutions of the problems. By comparison, it is obvious that the current method improved the accuracy and is easy to implement. The numerical solutions of some examples are discussed to show the suitability of (CWM).

Keywords: fractional boundary value problem, Chebyshev wavelet method (CWM), iterative method, numerical solution.

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1. Introduction

Fractional calculus is the subject that dealing with the study of fractional order differential equations. Fractional order differential equations are the generalization of classical integer order differential equations. Gemant and Scot-Blair started the study of fractional calculus by proposing a fractional derivative models for viscoelasticity, anomalous stress and strain [1, 2]. Other applications of fractional calculus while modeling physical phenomena, such as nonlinear oscillation of earth quake [3], signal Processing [4], control theory [5] and fluid dynamics traffic [6] have made this area important for mathematicians and researchers.
The exact and numerical solutions are important to describe and analyze fractional order differential equations. Therefore a number of efficient techniques have been used to obtain the solution of fractional differential equation such as Adomian Decomposition Method (ADM) [7], the Variational Iteration Method (VIM) [8], Homotopy Analysis Method (HAM) [9], Homotopy Perturbation Method (HPM) [10, 11], Differential Transformation Method (DTM) [12], the Fractional Sub Equation Method (FSEM) [13], the First Integral Method [FIM] [14], Reproducing Kernel Hilbert Space Method (RKHSM) [15, 16], shifted Jacobi Polynomials Method [17], shifted Legendre polynomials [18, 19] and the Ex-Function Method (EFM) [20].

Recently, most of the researchers have shown great interest in wavelet theory [21, 22, 23, 24, 25, 26, 27, 28] The most relevant methods based on wavelets are Haar wavelet [25], Legendre wavelet [24] and Chebyshev wavelet [21, 23, 27]. Harmonic wavelet method [22]. CAS wavelet [26].

In the current work, we have used a numerical method based on Chebyshev Wavelets for the numerical solution of some fractional higher order differential equations. The numerical solution by (CWM) are compared with the results obtained by (OHAM), (MOHAM), (VIM) and exact solution of the problems. The numerical results have suggested that (CWM) has the higher degree of accuracy than other methods.

The organization of this paper is in the following manner. The definition and properties of fractional calculus will be given in Section 2 while in Section 3, we give some properties of the Chebyshev Wavelets. In section 4, we introduce the Chebyshev Wavelet Method (CWM). In Section 5, we give four numerical examples and finally in Section 6, we give the conclusion.

2. Preliminaries and definitions

In this section some of the important definitions and preliminary concepts are discussed for the continuation of the current work.

Definition 2.1. The Riemann fractional integral operator I of order μ on the usual Lebesgue space $L_1[a,b]$ is given by

$$(I^{\mu}g)(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\xi)^{\mu-1}g(\xi)d\xi, \mu > 0,\ (I^0 g)(t) = g(t),$$

This integral operator has the following properties

(a) $I^{\mu}I^{\eta} = I^{\mu+\eta}$,

(b) $I^{\mu}I^{\eta} = I^{\eta}I^{\mu}$,

(c) $I^{\mu}(t-a)^{\nu} = \dfrac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\mu+\nu}$

Where $\mu, \eta > 0, \nu > -1$.

Definition 2.2. The Riemann fractional derivative of order $\gamma > 0$ is defined as

$$(D^{\gamma}g)(t) = \Big(\frac{d}{dt}\Big)^n \Big(I^{(n-\gamma)}g)(t)\Big), n-1 < \gamma \leq n,$$

where n is an integer.

However the Riemann fractional derivative has certain disadvantages and therefore Caputo introduced a modified differential operator.

Definition 2.3. The Caputo definition of fractional differential operator is given by

$$(D^{\mu}g)(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-\xi)^{n-\mu-1}g^{(n)}(\xi)d\xi,\ n-1 < \mu < n.$$

Where $t > 0, n$ is an integer.

3. Properties of the Chebyshev wavelets

Wavelets consist of family of functions generated from the dilation a and translation b of a single function $\psi(x)$ called the mother wavelet. When the dilation a and translation parameter b change continuously, we get the following continuous family of Wavelet [27]

$$\psi_{a,b}(x) = |a|^{\frac{1}{2}}\psi(\frac{x-b}{a}), a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as

$$a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0,$$

where n, k are positive integers.

Then the following family of discrete wavelets is obtained

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\psi(a_0^k x - nb_0), k, n \in Z.$$

Especially when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthogonal basis.

The second kind of Chebyshev wavelets is constituted of four parameters, $\psi_{n,m}(x) = \psi(k, n, m, x)$, where $n = 1, 2, ..., 2^{k-1}, k$ is any nonnegative integer, m is the degree of the second Chebyshev polynomial. The Chebyshev wavelets are defined on the interval $0 \le x < 1$ as

$$\psi_{n,m} = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases}$$

Where $\tilde{T}_m(x) = \sqrt{\dfrac{2}{\pi}} T_m(x), m = 0, 1, 2..., M - 1.$

Here $T_m(x)$ are second Chebyshev polynomials of degree m with respect to the weight function $w(x) = \sqrt{1 - x^2}$ on the interval $[-1, 1]$, and satisfying the following recursive formula

$$T_0(x) = 1, T_1(x) = 2x, T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), m = 1, 2, 3, \ldots$$

Lemma 3.1. *If the Chebyshev Wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev Wavelet expansion converges to the function $f(x)$.*

Proof. See [29]. □

Theorem 3.2. *A function $f(x) \in L_2[0, 1]$, with bounded second derivative, say $|f''(x)| \le N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(x)$, that is,*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x).$$

Proof. See [29]. □

4. Chebyshev Wavelet Method (CWM)

In the present paper we consider the fractional boundary value of the form

$$D^\alpha y(x) = g(x) + f(y), a < x \le b, 9 < \alpha \le 10, \tag{4.1}$$

with the boundary conditions given by $y(a) = \alpha_0, y^{(2)}(a) = \alpha_1, y^{(4)}(a) = \alpha_2, y^{(6)}(a) = \alpha_3, y^{(8)}(a) = \alpha_4, y(b) = \beta_0, y^{(2)}(b) = \beta_1, y^{(4)}(b) = \beta_2, y^{(6)}(b) = \beta_3, y^{(8)}(b) = \beta_4$, where $g(x)$ is a source function, $f(y)$ is linear or nonlinear continuous function and α_i and β_i are real valued constants.

The solution to equation (4.1) can be extended by Chebyshev Wavelets series as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \tag{4.2}$$

The series in equation (4.2) is truncated to finite number of terms that is

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{4.3}$$

This shows that there are $2^{k-1}M$ conditions to determine $2^{k-1}M$ coefficients which are $c_{i,j}$.

Since, we have eighth boundary conditions; therefore eight conditions are obtained by these boundary conditions.

The conditions are

$$\begin{cases} y_{k,m} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_0, \\ \frac{d^2}{dx^2} y_{k,M}(a) = \frac{d^2}{dx^2} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_1, \\ \frac{d^4}{dx^4} y_{k,M}(a) = \frac{d^4}{dx^4} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_2, \\ \frac{d^6}{dx^6} y_{k,M}(a) = \frac{d^6}{dx^6} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_3, \\ \frac{d^8}{dx^8} y_{k,M}(a) = \frac{d^8}{dx^8} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_4, \\ y_{k,M}(b) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_0, \\ \frac{d^2}{dx^2} y_{k,M}(b) = \frac{d^2}{dx^2} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_1, \\ \frac{d^4}{dx^4} y_{k,M}(b) = \frac{d^4}{dx^4} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_2, \\ \frac{d^6}{dx^6} y_{k,M}(b) = \frac{d^6}{dx^6} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_3, \\ \frac{d^8}{dx^8} y_{k,M}(b) = \frac{d^8}{dx^8} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_4. \end{cases} \tag{4.4}$$

The remaining $2^{k-1}M - 10$ conditions can be obtained by substituting equation (4.4) in equation (4.3), we get

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x) = f(x) + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x) \tag{4.5}$$

Assume that equation (4.5) is exact at $2^{k-1}M - 10$ points, which we call it x_i, then

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x_i) = f(x_i) + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x_i) \tag{4.6}$$

The x_i are obtain by using the following formula $x_i = \frac{i-0.5}{2^{k-1}M}$, $i = 1, 2, 3, ..., 2^{k-1}M - 10$. The combination of equations (4.5) and (4.6) form the linear system of $2^{k-1}M$ linear equations. The solution of this linear system of equations determine the unknown coefficients $c_{i,j}$.

The same procedure can be repeated for other fractional boundary value problems of any order.

5. Method implementation

Example 5.1. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha}y(x) = \frac{d^3}{dx^3}y(x) + 2e^x y^2,\ 0 \le x \le 1, 9 < \alpha \le 10.$$

Subject to the boundary conditions $y(0) = 1, y(1) = \frac{1}{e}, y^{(2)}(1) = \frac{1}{e}, y^{(4)}(0) = 1, y^{(4)}(1) = \frac{1}{e}, y^{(6)}(0) = 1, y^{(6)}(1) = \frac{1}{e}, y^{(8)}(0) = 1, y^{(8)}(1) = \frac{1}{e}$. The exact solution of this problem is $y(x) = e^{-x}$.

Table 1 shows the comparison of the absolute error between exact solutions and approximate solutions for $\alpha = 10$, when $M = 20$ and $k = 1$. Here y_{exact} and y_{10} represent the exact solution and approximate solution of the problem at $\alpha = 10$. The numerical results given by the present method are also compared with (OHAM) solutions. From the table it is obvious that the results of the current method are far better than (OHAM) method.

Table 2 displays the approximate solutions $y_{9.25}, y_{9.5}, y_{9.75}$ and $y_{9.95}$ for different values of $\alpha = 9.25, 9.50, 9.75$ and 9.95 respectively. Error($y_{9.25}$), Error($y_{9.5}$), Error($y_{9.75}$) and Error(y_{10}) are the errors obtain for different values of $\alpha = 9.25,\ 9.5,\ 9.75$ and 9.95 respectively, comparing with the exact solutions of the given problem. The error associated with different fractional order differential equations shows that the error in each fractional order decreases as the order of the fractional order differential equation approaches to integer order. This phenomena shows the consistency and reliability of the fractional order solutions.

Table 1: Numerical results for Example 5.1 for $\alpha = 10$

x_i	y_{exact}	y_{10} (CWM)	Error (CWM)	Error (OHAM)
0.0	1.000000000000000000	0.999999999999999999	1.00E-21	1.10E-11
0.1	0.904837418035959573	0.904837418035960081	5.07E-16	8.39E-7
0.2	0.81873075307798185	0.818730753077982824	9.65E-16	1.59E-6
0.3	0.74081822068171786	0.740818220681719194	1.32E-15	2.19E-6
0.4	0.670320046035639300	0.670320046035640860	1.55E-15	2.58E-6
0.5	0.606530659712633423	0.606530659712635061	1.63E-15	2.71E-6
0.6	0.548811636094026432	0.54881163609402798	1.56E-15	2.58E-6
0.7	0.496585303791409514	0.496585303791410837	1.32E-15	2.20E-6
0.8	0.449328964117221591	0.449328964117222551	9.59E-16	1.60E-6
0.9	0.406569659740599111	0.406569659740599616	5.04E-16	8.41E-7
1.0	0.367879441171442321	0.367879441171442321	4.00E-16	4.85E-11

Table 2: Numerical results for Example 5.1 for different fractional order

x_i	$y_{9.25}$	Error ($y_{9.25}$)	$y_{9.5}$	Error ($y_{9.5}$)	$y_{9.75}$	Error ($y_{9.75}$)	$y_{9.95}$	Error ($y_{9.95}$)
0.0	1.000000	0.0	0.9999999	2.0E-20	0.99999999	3.0E-20	1.00000000	0.0
0.1	0.904839	2.05E-6	0.9048383	8.97E-7	0.90483767	2.59E-7	0.90483744	2.80E-8
0.2	0.818734	3.91E-6	0.8187324	1.70E-6	0.81873124	4.94E-7	0.81873080	5.34E-8
0.3	0.740823	5.39E-6	0.7408205	2.35E-6	0.74081890	6.80E-7	0.74081829	7.36E-8
0.4	0.670326	6.34E-6	0.6703228	2.76E-6	0.67032084	8.01E-7	0.67032013	8.68E-8
0.5	0.606537	6.67E-6	0.6065337	2.91E-6	0.60653150	8.43E-7	0.60653075	9.15E-8
0.6	0.548817	6.35E-6	0.5488144	2.77E-6	0.54881243	8.03E-7	0.54881172	8.72E-8
0.7	0.496590	5.40E-6	0.4965876	2.35E-6	0.49658530	6.84E-7	0.49658537	7.43E-8
0.8	0.449333	3.92E-6	0.4493306	1.17E-6	0.44932946	4.97E-7	0.44932896	5.41E-8
0.9	0.406571	2.06E-6	0.4065705	9.01E-7	0.40656992	2.61E-7	0.40656965	2.84E-8
1.0	0.367879	1.0E-20	0.3678794	4.0E-20	0.36787944	1.0E-20	0.36787944	5.0E-20

Example 5.2. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha} y(x) = y(x) - 15e^x - 10xe^x, \ 0 \leq x \leq 1, 4 < \alpha \leq 5.$$

Subject to the boundary conditions

$$y(0) = 0, y(1) = 0, y'(0) = 1, y'(1) = -e, y''(0) = 0.$$

The analytic solution of this problem is $y(x) = x(1-x)e^x$.

Table 3 illustrates the comparison of the absolute error between the present method and other numerical methods such as (MOHPM), (OHAM) and (VIM).

Figure 1: The solution graph, of example 5.1 for different fractional order $\alpha = 10, 9.95, 9.75, 9.5, 9.25$

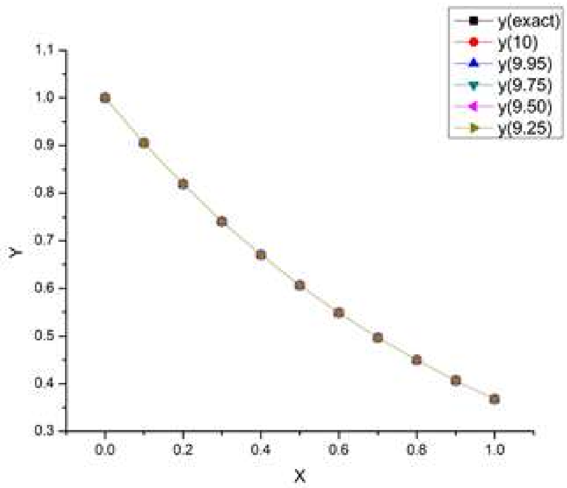


Table 3: Numerical results for Example 5.2 for $\alpha = 5$

x_i	y_{exact}	y_5 (CWM)	Error (CWM)	Error (MOHAM)	Error (OHAM)	Error (VIM)
0.0	0.000000000000000000	0.000000000000000002	2.53E-20	0.0000	0.00000	0.0000
0.1	0.099465382626808286	0.099465382626808625	3.39E-16	5.4E-14	9.00E-11	1.0E-9
0.2	0.195424441305627173	0.195424441305628649	1.47E-15	3.3E-13	4.00E-10	2.0E-9
0.3	0.283470349590960651	0.283470349590963518	2.86E-15	1.0E-13	5.00E-10	1.0E-9
0.4	0.358037927433904876	0.358037927433908857	3.98E-15	1.9E-12	2.00E-11	2.0E-9
0.5	0.412180317675032036	0.412180317675036519	4.48E-15	2.7E-12	1.00E-9	3.1E-8
0.6	0.437308512093722153	0.437308512093726389	4.23E-15	3.0E-12	2.00E-9	3.7E-8
0.7	0.422888068568800069	0.422888068568803370	3.30E-15	2.1E-12	2.00E-9	4.1E-8
0.8	0.356086548558794816	0.356086548558796758	1.94E-15	3.7E-12	1.00E-9	3.1E-8
0.9	0.221364280004125469	0.221364280004126090	6.20E-16	-3.2E-11	4.0E-10	1.4E-8
1.0	0.000000000000000000	0.000000000000000004	4.58E-20	-1.6E-10	0.00000	0.0000

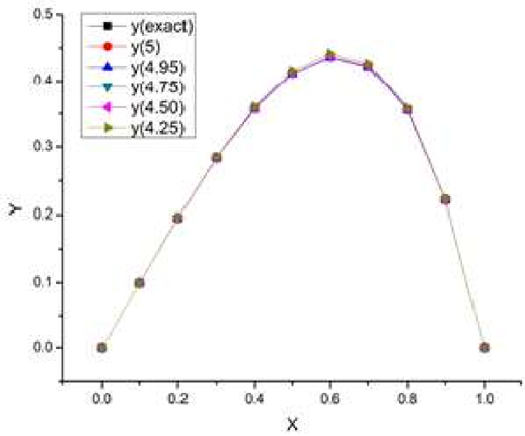
The results reveal that the present method has higher accuracy than any other method given the table. Here we have used $M = 17$ and $k = 1$, to obtain these results given in Table 3. y_{exact} and y_5 are the exact and approximate solution of example 5.2 at $\alpha = 5$. Error y_5, Error (MOHPM), Error (OHAM) and Error (VIM) are the corresponding errors in the present method, (MOHPM), (OHAM) and (VIM) respectively.

Table 4: Numerical results for Example 5.2 for different fractional order

x_i	$y_{4.25}$	Error ($y_{4.25}$)	$y_{4.5}$	Error ($y_{4.5}$)	$y_{4.75}$	Error ($y_{4.75}$)	$y_{4.95}$	Error ($y_{4.95}$)
0.0	0.999999	1.6E-19	0.0000000	7.2E-21	0.00000000	8.5E-21	0.00000000	1.87E-9
0.1	0.099502	4.54E-6	0.0994459	1.94E-5	0.09945966	4.08E-5	0.09943453	3.08E-5
0.2	0.195751	2.50E-5	0.1953345	7.91E-5	0. 19534526	2.50E-4	0.19522367	2.00E-4
0.3	0.284518	5.81E-5	0.2833612	1.09E-4	0.28336120	6.29E-4	0.28347034	5.34E-4
0.4	0.360170	9.26E-5	0.3579946	4.32E-5	0.35799467	1.07E-3	0.35707949	9.58E-4
0.5	0.415450	1.16E-4	0.4123023	1.22E-4	0.41230239	1.43E-3	0.41084175	1.33E-3
0.6	0.441314	1.20E-4	0.4376282	3.19E-4	0.43762826	1.56E-3	0.43578474	152E-3
0.7	0.426794	1.01E-4	0.4233281	4.40E-4	0.42332813	1.37E-3	0.42148765	1.40E-3
0.8	0.358885	6.39E-5	0.3564752	3.88E-4	0.35647526	9.04E-4	0.35513104	9.55E-3
0.9	0.222431	2.17E-5	0.2213553	1.71E-4	0.22153538	3.19E-4	0.22101464	3.49E-4
1.0	0.000000	0.0000	0.0000000	3.7E-20	0.00000000	2.2E-19	0.00000000	1.8E-9

Table 4 represents the solution of fractional order differential equations for different values of $\alpha = 4.25, 4.50, 4.75$ and 4.95. The solutions $y_{4.25}, y_{4.5}, y_{4.75}$ and $y_{4.95}$ and Error ($y_{4.25}$), Error ($y_{4.5}$), Error ($y_{4.75}$) and Error ($y_{4.95}$) are the corresponding errors of fractional order differential equations at $\alpha = 4.25, 4.50, 4.75, 4.95$ respectively.

Figure 2: The exact solution is represented by y(exact), while $y_5, y_{4.95}, y4.75$ and $y_{4.5}$ show (CWM) solutions at $\alpha = 5, 4.95, 4.75$ and 4.5 respectively.



Example 5.3. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha} y(x) - y^2 e^{-x} = 0, \ \ 0 \le x \le 1, 4 < \alpha \le 5.$$

Subject to the boundary conditions

$$y(0) = 1, y(1) = e, y'(0) = 1, y'(1) = e, y''(0) = 1,$$

The exact solution of this problem is $y(x) = e^x$.

Table 5: Numerical results for Example 5.3 for $\alpha = 5$

x_i	y_{exact}	y_5 (CWM)	Error (CWM)	Error (MOHPM)	Error (OHAM)	Error (VIM)
0.0	1.000000000000000000	1.0000000000000000001	1.00E-19	0.00000	0.000000	0.0000
0.1	1.1051709180756476248	1.1051709180756476237	1.10E-18	3.1E-15	1.9E-10	-8.0E-3
0.2	1.2214027581601698339	1.2214027581601698286	5.30E-18	1.9E-14	4.00E-9	-1.2E-3
0.3	1.3498588075760031040	1.3498588075760030936	1.04E-17	5.4E-14	5.00E-9	-5.0E-3
0.4	1.4918246976412703178	1.4918246976412703033	1.45E-17	1.0E-13	2.00E-9	3.0E-3
0.5	1.6487212707001281468	1.6487212707001281305	1.63E-17	1.4E-13	1.00E-9	8.0E-3
0.6	1.8221188003905089749	1.8221188003905089595	1.54E-17	1.6E-13	2.00E-8	6.0E-3
0.7	2.0137527074704765216	2.0137527074704765096	1.20E-17	1.5E-13	2.00E-8	1.0E-4
0.8	2.2255409284924676046	2.2255409284924675976	7.00E-18	9.9E-14	1.00E-9	9.0E-3
0.9	2.4596031111569496638	2.4596031111569496617	2.10E-18	1.1E-14	4.00E-9	-9.0E-3
1.0	2.7182818284590452354	2.7182818284590452354	0.000000	1.0E-13	0.00000	0.0000

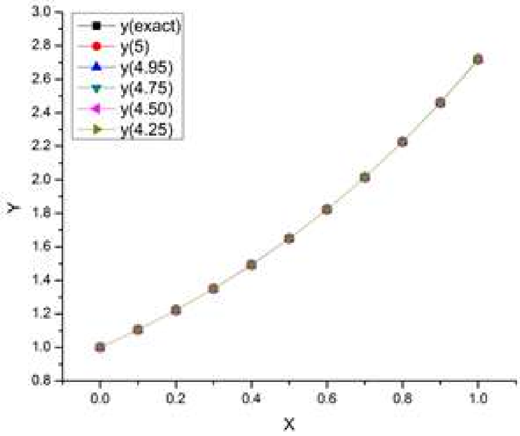
In Table 5, we have presented y_{exact} the exact solution and y_5 the approximate solution for example 5.3. Here we use $M = 17$ and $k = 1$ for the implementation of the current method. The accuracy is compared with other methods such as (MOHPM), (OHAM) and (VIM). This table reveals that the present method have the highest degree of accuracy than other methods. The exact and approximate solutions at $\alpha = 5$ are represented by y_{exact} and y_5 respectively. Error (y_5), Error (MOHPM), Error (OHAM) and Error (VIM) are the corresponding errors in the present method, (MOHPM), (OHAM) and (VIM) respectively.

Table 6: Numerical results for Example 5.3 for different fractional order

x_i	$y_{4.25}$	Error ($y_{4.25}$)	$y_{4.5}$	Error ($y_{4.5}$)	$y_{4.75}$	Error ($y_{4.75}$)	$y_{4.95}$	Error ($y_{4.95}$)
0.0	0.999999	1.6E-19	0.0000000	1.0E-19	0.00000000	1.0E-19	0.00000000	1.0E-19
0.1	1.105175	4.54E-6	1.1051747	3.84E-6	1.10517300	2.08E-6	1.10517133	4.20E-7
0.2	1.221427	2.50E-5	1.2214251	2.23E-5	1.22141525	1.24E-5	1.22140531	2.55E-6
0.3	1.349916	5.80E-5	1.3499128	1.09E-5	1.34988964	3.08E-5	1.34986520	6.39E-6
0.4	1.491917	9.26E-5	1.4919134	8.87E-5	1.49187625	5.15E-5	1.49183549	1.08E-5
0.5	1.648837	1.16E-4	1.6488356	1.14E-4	1.64872127	6.74E-5	1.64873551	1.42E-5
0.6	1.822239	1.20E-4	1.8222396	1.20E-4	1.82219088	7.20E-5	1.82213415	153E-5
0.7	2.013854	1.01E-4	2.0138561	1.03E-4	2.01381513	6.24E-5	2.01376610	1.33E-5
0.8	2.225604	6.39E-5	2.2256069	6.60E-5	2.22558116	4.02E-5	2.22554961	8.68E-6
0.9	2.459624	2.17E-5	2.4596257	2.26E-5	2.45961705	1.39E-5	2.45960661	3.02E-6
1.0	2.718281	0.0000	2.7182818	2.0E-19	2.71628182	1.0E-19	2.71828182	1.0E-19

Table 6 represents the solution of fractional order differential equations for different values of $\alpha = 4.25, 4.5, 4.75, 4.95$. The solutions $y_{4.25}, y_{4.5}, y_{4.75}$ and $y_{4.95}$ and Error $(y_{4.25})$, Error $(y_{4.5})$, Error $(y_{4.75})$ and Error $(y_{4.95})$ the corresponding errors of fractional order differential equations at $\alpha = 4.25, 4.5, 4.75, 4.95$ respectively.

Figure 3: The exact solution y_{exact}, while $y_5, y_{4.95}, y_{4.75}, y_{4.5}$ and $y_{4.25}$ show (CWM) solutions $\alpha = 5, 4.95, 4.75, 4.5$ and 4.25 respectively.



6. Conclusion

In this research paper, we have attempted to find the numerical solutions of fractional order boundary value problems by using Chebyshev Wavelet method. Three problems of different fractional order α such that $9 < \alpha \leq 10$ and $4 < \alpha \leq 5$ were considered for numerical treatment. The numerical simulation has shown that (CWM) has better accuracy than other methods which are under discussion.

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Accepted: 2.03.2018

Fuzzy zero suffix algorithm to solve fully fuzzy transportation problems by using element-wise operations

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Abstract. In this paper zero suffix method with element-wise operations of fuzzy numbers is proposed to solve fully fuzzy transportation problem. The proposed method assures the optimality, feasibility and positivity conditions of the fuzzy solution. The proposed method is easy to understand since it follows zero suffix algorithm and easy to compute since it considers the fuzzy numbers as ordered pairs as it uses the element-wise operations.

Keywords: fuzzy number, triangular fuzzy number, trapezoidal fuzzy number, fuzzy arithmetic operations, fuzzy transportation problems, fuzzy optimal solution.

1. Introduction

Transportation problem (TP) plays predominant role in supply chain management for reducing the transporting cost. The primary task of the algorithm is to optimize the transportation cost of commodity while transporting the commodity from sources to sinks. Hitchcock [20] constructed the transportation problem. Dantzig and Thapa [22] applied the simplex method to solve transportation problem. Charnes and Cooper [21] developed the stepping stone method as an alternative to the simplex method. The decision variables in the transporta-

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tion problem such as availability, requirement and the transportation cost per unit should be crisp to get a solution. Due to some unmanageable environment, these decision variables may not be precise. The unpredictability in determining the data may be designed using fuzzy variables which was introduced by Zadeh [1, 2] in the year 1965. If the variables are represented by fuzzy numbers in a TP, then the TP is called as a Fully Fuzzy TP or TP with fuzzy environment. Several authors proposed several approaches to solve a fuzzy TP. Chanas et al., [4, 5] applied parametric programming technique to solve fuzzy TP and also solved the given problem by converting the given problem into a bi-criterial TP with a crisp objective function. Liu Kao [3] used the extension principle for solving fuzzy TP. Verma et al., [10] solved the fuzzy TP with hyperbolic and exponential membership function by appling the fuzzy programming technique. Liang et al., [23], [11] proposed possibilistic linear programming technique for fuzzy TP and solved interactive multi objective transportation planning decision problems by using fuzzy linear programming. Nagoorgani et al., [6] approached a two stage cost minimizing fuzzy transportation problem by parametric technique. Pandian et al., [7] proposed fuzzy zero point method to solve fuzzy TP. Amit Kumar et al., [14] fuzzified least cost method, north west corner rule and VAM to solve fuzzy TP with generalized fuzzy numbers. Many authors [24], [25], [26], [27] used zero suffix method for solving transportation problem with crisp values and fuzzy values. All the existing methods transform the given problem in to crisp problem then implemented the zero suffix method. In this paper fuzzified version of zero suffix method and to order the fuzzy numbers Yager's ranking technique [8] is used. In the proposed method, the fuzzy zero suffix method is applied with element-wise addition, subtraction [9], [17], [18] and element-wise division to get the solution. In this paper, Section 2 deals with fuzzy preliminaries followed by Section 3 in which the proposed algorithm is given in detail. In Section 4, the implementation of the algorithm through example is explained. Finally,the conclusion is given in Section 5.

2. Preliminaries

Definition 1. *A fuzzy set can be obtained by mapping each possible individual in the universe of discourse to a value represented by its grade of membership.*

Definition 2. *A fuzzy number $\tilde{A}$ is a fuzzy set whose membership function is piecewise continuous, convex and normal.*

Definition 3. *A fuzzy number $\tilde{A} = (a, b, c)$ with membership function of the form*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x = b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & otherwise \end{cases}$$

is called a triangular fuzzy number and a fuzzy number $\tilde{A} = (a, b, c, d)$ *with membership function of the form*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0, & otherwise \end{cases}$$

is called a trapezoidal fuzzy number.

Definition 4. *Fuzzy Addition:*

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

Fuzzy Subtraction:

$$(a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2) = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$$

$$(a_1, b_1, c_1) - (a_2, b_2, c_2) = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$$

Definition 5. *Element-wise Addition:*
$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$
$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$
Element-wise Subtraction:
$(a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2) = (a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2)$
$(a_1, b_1, c_1) - (a_2, b_2, c_2) = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$
Element-wise Multiplication:
$(a_1, b_1, c_1, d_1) * (a_2, b_2, c_2, d_2) = (a_1 * a_2, b_1 * b_2, c_1 * c_2, d_1 * d_2)$
$(a_1, b_1, c_1) * (a_2, b_2, c_2) = (a_1 * a_2, b_1 * b_2, c_1 * c_2)$
Element-wise Division:
$(a_1, b_1, c_1, d_1)/(a_2, b_2, c_2, d_2) = (a_1/a_2, b_1/b_2, c_1/c_2, d_1/d_2)$
$(a_1, b_1, c_1)/(a_2, b_2, c_2) = (a_1/a_2, b_1/b_2, c_1/c_2)$

Definition 6. *The Yager's ranking of a fuzzy number* $\tilde{a}$ *is given by*

$$Y(\tilde{a}) = \int_0^1 (0.5)(a_U^\alpha + a_L^\alpha) d\alpha,$$

where a_L^α = *Lower* α*- level cut and* a_U^α = *Upper* α*- level cut.If* $Y(\tilde{s}) \leq Y(\tilde{i})$ *then* $\tilde{s} \leq \tilde{i}$.

Definition 7. *A Fully Fuzzy transportation problem is defined by*

$$\min \tilde{Z} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_{ij}$$

subject to

$$\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{s}_i \text{ for } i = 1, 2, 3 \ldots m,$$

$$\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{d}_j \text{ for } j = 1, 2, 3 \ldots n.$$

for all $\tilde{x}_{ij} \succ \tilde{0}$, *where* $i = 1, 2, 3 \ldots m$ *and* $j = 1, 2, 3 \ldots n$.

Here $\tilde{x}_{ij}$ *is the number of units to be transported from* i^{th} *source to* j^{th} *destination,* $\tilde{c}_{ij}$ *is the cost of transporting one unit from* i^{th} *source to* j^{th} *destination,* $\tilde{s}_i$ *is the number of units available in the* i^{th} *source and* $\tilde{d}_j$ *is the number of units required in the* j^{th} *destination.*

The matrix form of fuzzy transportation problem is given as follows

	A	B	C	$\cdots$	E	Supply
1	$\tilde{c}_{11}$	$\tilde{c}_{12}$	$\tilde{c}_{13}$	$\cdots$	$\tilde{c}_{1n}$	$\tilde{s}_1$
2	$\tilde{c}_{21}$	$\tilde{c}_{22}$	$\tilde{c}_{23}$	$\cdots$	$\tilde{c}_{2n}$	$\tilde{s}_2$
3	$\tilde{c}_{31}$	$\tilde{c}_{32}$	$\tilde{c}_{33}$	$\cdots$	$\tilde{c}_{3n}$	$\tilde{s}_3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
m	$\tilde{c}_{m1}$	$\tilde{c}_{m2}$	$\tilde{c}_{m3}$	$\cdots$	$\tilde{c}_{mn}$	$\tilde{s}_m$
Demand	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\cdots$	$\tilde{d}_n$	

3. Fuzzy zero suffix method

Step 1. Construct the fuzzy transportation table for the given fuzzy transportation problem and then,convert it into a balanced one, if it is not.Subtract each row entries of the fuzzy transportation table from the row minimum. Do the same for columns also.

Step 2. In the reduced cost matrix there will be at least one fuzzy zero in each row and column. Find fuzzy suffix value $\tilde{S}$ of all the fuzzy zeros in the reduced cost matrix by the ratio of addition fuzzy costs of nearest adjacent sides of fuzzy zeros which are greater than fuzzy zero to the number of fuzzy values added. Here we should take the denominator as fuzzy values. i.e., if the number of values is 3, we should take that as fuzzy number $(3, 3, 3)$. $\tilde{S}$ = Addition of the fuzzy costs of adjacent sides of fuzzy zero which are greater than fuzzy zero/number of fuzzy values added.

Step 3. Select the maximum of $\tilde{S}$, and supply to that fuzzy demand corresponding to that cell. If it has more equal fuzzy values then select any one and supply to that fuzzy demand maximum possible.

Step 4. After the above step, the exhausted fuzzy demands or fuzzy supplies to be trimmed. The resultant fuzzy matrix posses at least one fuzzy zero in each row and column else repeat Step 1.

Step 5. Repeat Step 3 to Step 4 until the optimal solution is obtained.

4. Numerical examples

4.1 Example

Consider the fully fuzzy transportation problem with triangular numbers given as follows

Solution:
Since the total supply $(4, 15, 27)$ is equal to the total demand $(4, 15, 27)$, this is a balanced fuzzy TP.

	A	B	C	D	Supply
1	(-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)	(0,3,6)
2	(4 9 16)	(4 8 12)	(2,5,8)	(1,4,7)	(2,7,13)
3	(2,7,13)	(0,5,10)	(0,5,10)	(4,8,12)	(2,5,8)
Demand	(1,4,7)	(0,3,5)	(1,4,7)	(2,4,8)	(4,15,27)

The given transportation problem rewritten as assignment problem.

	A	B	C	D
1	(-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)
2	(4,9,16)	(4,8,12)	(2,5,8)	(1,4,7)
3	(2,7,13)	(0,5,10)	(0,5,10)	(4,8,12)

The fuzzy costs and fuzzy units of fuzzy transportation table are given with their crisp values in the following table.

	A	B	C	D
1	(-2,3,8)(3)	(-2,3,8)(3)	(-2,3,8)(3)	(-1,1,4)(1.33)
2	(4,9,16)(9.667)	(4,8,12)(8)	(2,5,8)(5)	(1,4,7)(4)
3	(2,7,13)(7.33)	(0,5,10)(5)	(0,5,10)(5)	(4,8,12)(8)

Choose the smallest fuzzy number in each and every row and subtract it with the other elements in the corresponding row. Repeat the same for the columns also. It is noted that the reduced matrix has at least one fuzzy zero in each row and column. The reduced matrix is given in

	A	B	C	D
1	(0,0,0)	(-1,2,4)	(-1,2,4)	(0,0,0)
2	(4,3,5)	(3,4,5)	(1,1,1)	(0,0,0)
3	(3,0,-1)	(0,0,0)	(0,0,0	(4,3,2)

The fuzzy zeros are in the position $(1,1),(1,4),(2,4),(3,2),(3,3)$ of the reduced matrix. If we take the fuzzy zero in the $(1,1)$, the adjacent values $(-1,2,4)$ and $(4,3,5)$ which are greater than fuzzy zero. So the fuzzy suffix value for that position $(1,1)$ is given by $\frac{((-1,2,4)+(4,3,5))}{(2,2,2)} = (1.5, 2.5, 4.5)$, where the fuzzy number $(2,2,2)$ is the fuzzy value of the number of adjacent values which are greater than fuzzy zero added. Similarly find the fuzzy suffix value for all other fuzzy zeros. The values are given below : for the position $(1,4)$ is $(-1,2,4)$, for the position $(2,4)$ is $(2.5,2,1.5)$, for the position $(3,2)$ is $(3,2,2)$ and for the position $(3,3)$ is $(2.5,2,1.5)$. Out of all these fuzzy suffix value, the fuzzy suffix value of fuzzy zero in the position $(1,1)$ is maximum. Therefore allocate the corresponding fuzzy supply or fuzzy demand whichever is less to that $(1,1)$ position. From the problem it is noted that in that position the corresponding fuzzy supply $(0,3,6)$ is minimum. So allocate the corresponding fuzzy supply $(0,3,6)$ to that position and delete the corresponding row. This is given as follows.

	A	B	C	D	Supply
1	(0,3,6) (-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)	
2	(4 9 16)	(4 8 12)	(2,5,8)	(1,4,7)	(2,7,13)
3	(2,7,13)	(0,5,10)	(0,5,10)	(4,8,12)	(2,5,8)
Demand	(1,1,1)	(0,3,5)	(1,4,7)	(2,4,8)	(4,15,27)

After deleting the first row the reduced matrix is given as follows.

	A	B	C	D
2	(4,3,5)	(3,4,5)	(1,1,1)	(0,0,0)
3	(3,0,-1)	(0,0,0)	(0,0,0)	(4,3,2)

Again apply the first step the resultant reduced matrix is given as follows.

	A	B	C	D
2	(1,3,6)	(3,4,5)	(1,1,1)	(0,0,0)
3	(0,0,0)	(0,0,0)	(0,0,0)	(4,3,2)

The fuzzy zeros are in the position $(2,4),(3,1),(3,2),(3,3)$ of the reduced matrix. If we take the fuzzy zero in the $(2,4)$, the adjacent values $(1,1,1)$ and $(4,3,2)$ which are greater than fuzzy zero. So the fuzzy suffix value for that position $(2,4)$ is given by $\frac{((-1,1,1)+(4,3,2))}{(2,2,2)} = (2.5,2,1.5)$, where the fuzzy number $(2,2,2)$ is the fuzzy value of the number of adjacent values which are greater than fuzzy zero added. Similarly find the fuzzy suffix value for all other fuzzy zeros. The values are given below : for the position $(3,1)$ is $(1,3,6)$, for the position $(3,2)$ is $(3,4,5)$ and for the position $(3,3)$ is $(2.5,2,1.5)$. Out of all these fuzzy suffix value, the fuzzy suffix value of fuzzy zero in the position $(3,2)$ is maximum. Therefore allocate the corresponding fuzzy supply or demand whichever is less to that $(3,2)$ position. From the problem it is noted that in that position the corresponding fuzzy demand $(0,3,5)$ is minimum. So allocate the corresponding fuzzy demand $(0,3,5)$ to that position and delete the corresponding column. The matrix is given as follows.

	A	B	C	D	Supply
1	(0,3,6) (-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)	
2	(4 9 16)	(4 8 12)	(2,5,8)	(1,4,7)	(2,7,13)
3	(2,7,13)	(0,3,5) (0,5,10)	(0,5,10)	(4,8,12)	(2,2,3)
Demand	(1,1,1)		(1,4,7)	(2,4,8)	(4,15,27)

After deleting the second column, the reduced matrix is given as follows

	A	C	D
2	(1,3,6)	(1,1,1)	(0,0,0)
3	(0,0,0)	(0,0,0)	(4,3,2)

The fuzzy zeros are in the position $(2,4),(3,1),(3,3)$ of the reduced matrix. If we take the fuzzy zero in the $(2,4)$, the adjacent values $(1,1,1)$ and $(4,3,2)$ which are greater than fuzzy zero. So the fuzzy suffix value for that position $(2,4)$ is given by $\frac{((-1,1,1)+(4,3,2))}{(2,2,2)} = (2.5,2,1.5)$, where the fuzzy number $(2,2,2)$ is the fuzzy value of the number of adjacent values which are greater than fuzzy zero added. Similarly find the suffix value for all other fuzzy zeros. The values are given below : for the position $(3,1)$ is $(1,3,6)$, and for the position $(3,3)$ is $(2.5,2,1.5)$. Out of all these fuzzy suffix value, the fuzzy suffix value of fuzzy zero in the position $(3,1)$ is maximum. Therefore allocate the corresponding fuzzy supply or demand whichever is less to that $(3,1)$ position. From the problem it is noted that in that position the corresponding fuzzy demand $(1,1,1)$ is minimum. So allocate the corresponding fuzzy demand $(1,1,1)$ to that position and delete the corresponding column. This is given in the following table

	A	B	C	D	Supply
1	(0,3,6) (-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)	
2	(4,9,16)	(4 8 12)	(2,5,8)	(1,4,7)	(2,7,13)
3	(1,1,1) (2,7,13)	(0,3,5) (0,5,10)	(0,5,10)	(4,8,12)	(1,1,2)
Demand			(1,4,7)	(2,4,8)	(4,15,27)

After deleting the first column , the reduced matrix is given in the following table.

	C	D
2	(1,1,1)	(0,0,0)
3	(0,0,0)	(4,3,2)

The fuzzy zeros are in the position $(2,4),(3,3)$ of the reduced matrix. If we take the fuzzy zero in the $(2,4)$, the adjacent values $(1,1,1)$ and $(4,3,2)$ which are greater than fuzzy zero. So the fuzzy suffix value for that position $(2,4)$ is given by $\frac{((-1,1,1)+(4,3,2))}{(2,2,2)} = (2.5,2,1.5)$, where the fuzzy number $(2,2,2)$ is the fuzzy value of the number of adjacent values which are greater than fuzzy zero added. Similarly find the fuzzy suffix value for all other fuzzy zeros. The values are given below : for the position $(3,3)$ is $(2.5,2,1.5)$. Both the fuzzy suffix values are same, So we can take any position for allocation. Here we choose $(2,4)$ for allocation. Therefore allocate the corresponding fuzzy supply or demand whichever is less to that $(2,4)$ position. From the problem it is noted that in that position the corresponding fuzzy demand $(2,4,8)$ is minimum. So allocate the corresponding fuzzy demand $(2,4,8)$ to that position and delete the corresponding column. This is given as follows

	A	B	C	D	Supply
1	(0,3,6) (-2,3,8)	(-2,3,8)	(-2,3,8)	(-1,1,4)	
2	(4,9,16)	(4 8 12)	(2,5,8)	(2,4,8) (1,4,7)	(0,3,5)
3	(1,1,1) (2,7,13)	(0,3,5) (0,5,10)	(0,5,10)	(4,8,12)	(1,1,2)
Demand			(1,4,7)		(4,15,27)

After deleting the fourth column , the reduced matrix is given as follows

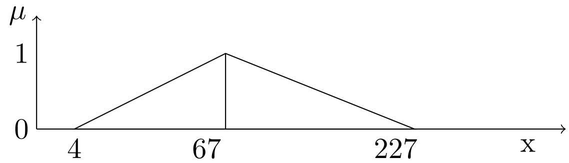
	C	
2	(1,1,1)	
3	(0,0,0)	

Applying the above mentioned procedure we get the optimal table which is given as follows.

	A	B	C	D	Supply
1	(-2,3,8)	(-2,3,8)	(-2,3,8)	(0,3,6) (-1,1,4)	(0,3,6)
2	(-1,2,4) (4,9,16)	(4,8,12)	(1,4,7) (2,5,8)	(2,1,2) (1,4,7)	(2,7,13)
3	(2,2,3) (2,7,13)	(0,3,5) (0,5,10)	(0,5,10)	(4,8,12)	(2,5 ,8)
Demand	(1,4,7)	(0,3,5)	(1,4,7)	(2,4,8)	(4,15,27)

Fuzzy T.C $\approx$ (-2,3,8) * (0,3,6) + (2,5,8) * (0,3,5) + (0,5,10) * (0,3,5) + (0,5,10) * (1,1,2) + (1,4,7) * (2,4,8) + (2,7,13) * (1,1,1)$\approx$ (4,67,227)

The membership function for the obtained result is



- According to the decision maker the minimum transportation cost will lie between 4 and 227

- The overall level of satisfaction of the decision maker about the statement that the minimum transportation cost will be 67 dollars is 100 percent.

- The overall level of satisfaction of the decision maker for the remaining values of minimum transportation cost can be obtained as follows:Let x_0 represents the minimum transportation cost then the overall level of satisfaction of the decision maker for x_0 is $\mu_{\tilde{A}}(x_0) \times 100$
  where

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-4}{63}, & 4 \leq x \leq 67 \\ 1, & x = 67 \\ \dfrac{227-x}{160}, & 67 \leq x \leq 227 \\ 0, & otherwise. \end{cases}$$

5. Conclusions

In this paper, an efficient method called Fuzzy zero suffix algorithm is proposed to solve fully fuzzy transportation problem with element-wise subtraction and element-wise division. The solution obtained by using the fuzzy zero suffix method satisfies the feasibility, optimality conditions and the positive values in all the allocated cells. An advantage of the proposed method is that it follows the zero suffix method which is easy to understand and to apply. It can also be used to solve special type of fuzzy transportation problem like unbalanced transportation problems and transportation problem with Degeneracy.

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Accepted: 14.03.2018

Pricing European call options with default risk under a jump-diffusion model via FFT transform

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Abstract. This paper considers the pricing of European call options with default risk within the framework of reduced-form model. We model the stock price and the default intensity by two dependent jump-diffusion models with common jumps. By using a Girsanov theorem, we give the explicit expression for the Fourier transform of the price of call options with default risk.
Keywords: vulnerable option, reduced-form model, jump-diffusion, FFT

1. Introduction

Vulnerable option is a kind of option with credit risk. Credit risk is the risk that the counterparty to a financial contract will default prior to the expiration of the contract and will not make all the payments required by the contract. There are two primary approaches for pricing credit derivatives, the structural approach and the reduced-form approach. Structural models, initially proposed by Black and Scholes (1973) and Merton (1974), could give an intuitive understanding for the credit risk by specifying a firm value process. Reduced-form models, introduced by Jarrow and Turnbull (1995), Duffie and Singleton (1999), and others, focus directly on the modeling of the default probability. This methodology does not intend to explain the default of a firm by means of an economic construction. Instead, the time of default is defined as the first jump time of a point process. Comparing with structural models, reduced-form models are more flexible and tractable in the real market. For more information on reduced-form models, we refer the interested reader to Bielecki and Rutkowski (2004) and Dong et al. (2014).

Extending the corporate bond default model of Merton (1974), Johnson and Stulz (1987) firstly proposed the conception of vulnerable option and investigated the option pricing with credit risk based on a structural model. Hull and White (1995) derived the price of vulnerable option by adopting a reduced-form

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approach under the assumption that the underlying asset and the counterparty asset were independent of each other. Extending Hull and White (1995), Klein (1996) relaxed the assumption of independence condition and deduced option pricing via a martingale method. By generalizing the results of Klein (1996), Klein and Inglis (2001) considered the stochastic default boundary which depends on options and counterparty debts for the discussion of option pricing. Wu and Dong (2019) investigated the pricing of European vulnerable option under a correlated diffusion process.

Most of the literature on vulnerable options assume that the dynamics of the assets follow the log-normal diffusion process. However, this assumption ignores sudden shocks in price due to the arrival of important new information. The purpose of this study is to provide a new pricing model for vulnerable options, where the dynamics of the underlying asset and the default intensity follow jump-diffusion processes with common jumps. The paper is organized as follows. In Section 2, we present the pricing model. In Section 3, we derive the price of the vulnerable options. Section 4 presents some numerical analysis by using FFT. Section 5 concludes.

2. The model

Consider a continuous-time model with a finite time horizon $\mathcal{T} = [0, T]$ with $T < \infty$. Let $\{\Omega, \Im, \{\Im_t\}_{0\leq t\leq T}, Q\}$ be a filtered complete probability space, where Q is the risk neutral measure such that the discounted asset price processes are martingales, and $\{\Im_t\}_{0\leq t\leq T}$ is a filtration satisfying the usual conditions. Throughout the paper, it is assumed that all random variables are well defined on this probability space and $\Im_T-$measurable.

Assume that the dynamics of the process B_t for the bank account are described by

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where the interest rate r_t is given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma_1 dW_1(t). \tag{2.1}$$

Here, $\kappa > 0, \theta > 0, \sigma_1 > 0$ are constants; $\{W_1(t), t \geq 0\}$ is a standard Brownian motion. From (2.1), we have

$$\int_t^T r_s ds = \theta(T-t) + (r_t - \theta)D(t,T) + \int_t^T \sigma_1 D(s,T)dW_1(s), \tag{2.2}$$

where $D(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}$.

Let S_t be the value of the asset at time t. Let τ denote the default time of the writer of the option with default intensity process λ_t. Suppose that the dynamics of the stock price S_t and default intensity λ_t follow

$$\frac{dS_t}{S_{t^-}} = (r_t - (\rho_1 + \rho_2)\xi)dt + \sigma_2 dW_2(t) + d\left(\sum_{i=1}^{N_1(t)+N_2(t)} (e^{Y_i} - 1)\right), \tag{2.3}$$

and

$$d\lambda_t = a(b-\lambda_t)dt + \sigma_3 dW_3(t) + d\sum_{i=1}^{N_1(t)+N_3(t)} Z_i, \tag{2.4}$$

where a, b, σ_2, σ_3 are all positive constants; $\{W_2(t), t \geq 0\}$ and $\{W_3(t), t \geq 0\}$ are two standard Brownian motions; $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N_3(t), t \geq 0\}$ are three mutually independent Poisson processes with arrival rates ρ_1, ρ_2 and ρ_3, respectively; $\{Y_i, i \geq 1\}$ is a sequence of independent identically distributed random variables with common density function given by f_y and $\xi = E(e^{Y_1})$; $\{Z_i, i \geq 1\}$ is also a sequence of independent identically distributed random variables with common density function given by f_z. Moreover, we suppose that $\{N_1(t), t \geq 0\}$,$\{N_2(t), t \geq 0\}$,$\{N_3(t), t \geq 0\}$,$\{Z_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ are mutually independent. Finally, we assume that the covariance matrix of the Brownian motion $(W_1(t), W_2(t), W_3(t))$ is

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} t$$

This model assumes that the firm value and the default intensity have common jumps, which describe the sudden changes in stock prices and default intensity due to the revealing of important new information which has a large effect on them. Note that, the default intensity λ_t can take negative values with positive probability. However, in practical applications, due to the low volatility, the probability λ_t takes negative values can be considered negligible.

We now specify the information structure of our model. Let $\Im_t = \mathcal{G}_t \vee \mathcal{H}_t$, where $\mathcal{G}_t = \Im_t^s \vee \Im_t^r \vee \Im_t^\lambda$ with $\Im_t^s = \sigma(S_s, s \leq t), \Im_t^r = \sigma(r_s, s \leq t), \Im_t^\lambda = \sigma(\lambda_s, s \leq t)$ and $\mathcal{H}_t = \sigma(1_{\{\tau \leq s\}}, s \leq t)$.

Let $J(t) = \sum_{i=1}^{N_1(t)+N_3(t)} Z_i$. Then from (2.3) and (2.4), we can obtain

$$S_T = S_t e^{\int_t^T (r_s - (\rho_1+\rho_2)\xi - \frac{1}{2}\sigma_2^2)ds + \int_t^T \sigma_2 dW_2(s) + \sum_{i=N_1(t)+N_2(t)}^{N_1(T)+N_2(T)} Y_i}, \tag{2.5}$$

and

$$\begin{aligned} \int_t^T \lambda_s ds &= b(T-t) + (\lambda_t - b)D_1(t,T) + \int_t^T \sigma_3 D_1(s,T) dW_3(s) \\ &\quad + \int_t^T D_1(s,T) dJ(s), \end{aligned} \tag{2.6}$$

where $D_1(t,T) = \frac{1-e^{-a(T-t)}}{a}$.

3. Pricing options with credit risk

In this section we consider the pricing of the European style option with credit risk. Assume that the recovery rate is a constant ω. When the writer of the

European option defaults, the payoff is given by ω times the payoff of the default-free option at maturity. By risk-neutral pricing theorem, the valuation of the vulnerable European call option at time t,with strike price K and maturity T is given by

$$C(t,T,K) = E\left[e^{-\int_t^T r_s ds}\left(\omega(S_T-K)^+1_{\{\tau\le T\}} + (S_T-K)^+1_{\{\tau>T\}}\right)\middle|\Im_t\right]$$

Then from Corollary 5.1.1 of Bielecki and Rutkowski (2004), we obtain the following expression:

$$\begin{aligned} C(t,T,K) &= \omega E\left[e^{-\int_t^T r_s ds}(S_T-K)^+\middle|\mathcal{G}_t\right] \\ &\quad +(1-\omega)1_{\{\tau>t\}}E\left[e^{-\int_t^T (r_s+\lambda_s)ds}(S_T-K)^+\middle|\mathcal{G}_t\right] \\ &\doteq \omega C_1(t,T,K) + 1_{\{\tau>t\}}(1-\omega)C_2(t,T,K) \end{aligned}$$

where

$$\begin{aligned} C_1(t,T,K) &= E\left[e^{-\int_t^T r_s ds}(S_T-K)^+|\mathcal{G}_t\right], \\ C_2(t,T,K) &= E\left[e^{-\int_t^T (r_s+\lambda_s)ds}(S_T-K)^+|\mathcal{G}_t\right]. \end{aligned}$$

Since it is difficult to compute $C_1(t,T,K)$ and $C_2(t,T,K)$, we will investigate the Fourier transform of the option price. We adopt the Fourier methods, introduced in Carr and Madan (1999), to investigate the option price. Following the notation in Carr and Madan (1999), we write $k = ln(K)$. For $a > 0$, define

$$\begin{aligned} c(t,T,k) &= e^{ak}C(t,T,K) \\ &\doteq \omega c_1(t,T,k) + 1_{\{\tau>t\}}(1-\omega)c_2(t,T,k), \end{aligned}$$

where

$$c_1(t,T,k) = e^{ak}C_1(t,T,K),\ c_2(t,T,k) = e^{ak}C_2(t,T,K).$$

Define

$$\begin{aligned} \zeta(u,t,T) &= \int_{-\infty}^{+\infty} e^{iuk}c(t,T,k)dk \\ &\doteq \omega\zeta_1(u,t,T) + 1_{\{\tau>t\}}(1-\omega)\zeta_2(u,t,T), \end{aligned}$$

where

$$\zeta_1(u,t,T) = \int_{-\infty}^{+\infty} e^{iuk}c_1(t,T,k)dk,\ \zeta_2(u,t,T) = \int_{-\infty}^{+\infty} e^{iuk}c_2(t,T,k)dk.$$

Proposition 3.1. *For $a > 0$, we have*

$$\zeta_1(u,t,T) = \frac{P(t,T)\eta^T(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u},$$

where

$$P(t,T) = E\left[e^{-\int_t^T r_s ds}\middle|\Im_t\right] = exp(-r_t D(t,T) + A(t,T)),$$

with $A(t,T) = (\theta - \frac{\sigma_1^2}{2\kappa^2})(D(t,T) - (T-t)) - \frac{\sigma_1^2}{4\kappa} D^2(t,T)$, *and*

$$\begin{aligned}\eta^T(v,t,T) &= e^{iv(s_t + \wedge(t,T) - \int_t^T \rho_{12}\sigma_1\sigma_2 D(u,T)du - \int_t^T \sigma_1^2 D^2(u,T)du)} \\ &\quad \times e^{-\int_t^T (\frac{\sigma_2^2 v^2}{2} + \frac{\sigma_1^2 D^2(u,T)}{2} + iv\rho_{12}\sigma_1\sigma_2 D(u,T))du} e^{(\rho_1+\rho_2)(T-t)(E[e^{ivY_1}]-1)},\end{aligned}$$

with $s_t = \ln S_t$ *and* $\wedge(t,T) = \theta(T-t) + (r_t - \theta)D(t,T) - (\frac{\sigma_2^2}{2} + (\rho_1+\rho_2)\xi)(T-t)$.

Proof. In the presence of a stochastic interest rate, we will define the forward-neutral measure Q^T equivalent to the risk-neutral measure Q by

$$\frac{dQ^T}{dQ} = \frac{P(T,T)}{P(0,T)B_T} = \frac{e^{-\int_0^T r_s ds}}{P(0,T)}.$$

where $P(t,T)$ denotes the value at time t of a T-maturity zero coupon bond whose face value is 1. It is well known that

$$P(t,T) = exp(-r_t D(t,T) + A(t,T))$$

and $P(t,T)$ satisfies

$$dP(t,T) = r_t P(t,T)dt - \sigma_1 D(t,T)P(t,T)dW_1(t).$$

So, the Radon-Nikodym derivative is given by

$$\frac{dQ^T}{dQ} = e^{-\int_0^T \sigma_1 D(t,T)dW_1(t) - \frac{1}{2}\int_0^T \sigma_1^2 D^2(t,T)dt}.$$

Girsanov's theorem implies that

$$W_1^T(t) = W_1(t) + \int_0^t \sigma_1 D(u,T)du; \quad W_2^T(t) = W_2(t) + \int_0^t \rho_{12}\sigma_1 D(u,T)du$$

are two standard Brownian motions under Q^T with the correlation coefficient ρ_{12}.

Therefore,

$$\begin{aligned}S_T &= S_t \exp(\wedge(t,T) + \int_t^T \sigma_2 dW_2^T(u) + \int_t^T \sigma_1 D(u,T)dW_1^T(u) \\ &\quad - \int_t^T \rho_{12}\sigma_1\sigma_2 D(u,T)du - \int_t^T \sigma_1^2 D^2(u,T)du + \sum_{i=N_1(t)+N_2(t)}^{N_1(T)+N_2(T)} Y_i).\end{aligned}$$

Moving to the forward measure yields

$$C_1(t,T,K) \quad = \quad P(t,T)E^T\Big[(S_T-K)^+|\mathcal{G}_t\Big].$$

In order to derive the formula for $\zeta_1(u,t,T)$, we first derive the expression for the characteristic function of s_T conditional on $\mathcal{G}_t$. Let $f^T(s)$ be the density function conditional on s_t under Q^T, then we have

$$\begin{aligned}
\eta^T(v,t,T) &= E^T\Big[e^{ivs_T}\Big|\mathcal{G}_t\Big] \\
&= e^{iv(s_t+\wedge(t,T)-\int_t^T\rho_{12}\sigma_1\sigma_2 D(u,T)du-\int_t^T\sigma_1^2D^2(u,T)du)} \\
&\quad \times E^T\Big[e^{iv(\int_t^T\sigma_2dW_2^T(u)+\int_t^T\sigma_1D(u,T)dW_1^T(u)+\sum_{j=N_1(t)+N_2(t)}^{N_1(T)+N_2(T)}Y_j)}\Big|\mathcal{G}_t\Big] \\
&= e^{iv(s_t+\wedge(t,T)-\int_t^T\rho_{12}\sigma_1\sigma_2 D(u,T)du-\int_t^T\sigma_1^2D^2(u,T)du)} \\
&\quad \times e^{-\int_t^T(\frac{\sigma_2^2v^2}{2}+\frac{\sigma_1^2D^2(u,T)}{2}+iv\rho_{12}\sigma_1\sigma_2D(u,T))du}\times e^{(\rho_1+\rho_2)(T-t)(E[e^{ivY_1}]-1)}
\end{aligned}$$

Hence,

$$\begin{aligned}
\zeta_1(u,t,T) &= \int_{-\infty}^{+\infty}e^{iuk}c_1(t,T,K)dk \\
&= \int_{-\infty}^{+\infty}\int_k^{+\infty}e^{(iu+a)k}P(t,T)(e^s-e^k)f^T(s)dsdk \\
&= \frac{P(t,T)\eta^T(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u}.
\end{aligned}$$

□

Proposition 3.2. *For $a>0$, we have*

$$\zeta_2(u,t,T)=\frac{P^\lambda(t,T)\eta^\lambda(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u},$$

where

$$\begin{aligned}
P^\lambda(t,T) &= e^{-(\theta+b)(T-t)-(r(t)-\theta)D(t,T)-(\lambda(t)-b)D_1(t,T)} \\
&\quad \times e^{(\rho_1+\rho_3)\int_t^T\int_{-\infty}^{+\infty}(e^{-D_1(u,T)}-1)f_z(z)dzdu} \\
&\quad \times e^{-\frac{1}{2}\int_t^T(\sigma_1^2D^2(u,T)+2\rho_{13}\sigma_1\sigma_3D(u,T)D_1(u,T)+\sigma_3^2D_1^2(u,T))du}
\end{aligned}$$

and

$$\begin{aligned}
\eta^\lambda(v,t,T) &= e^{iv(s_t+\wedge(t,T)-\int_t^T(\rho_{12}\sigma_2M_1(u)+\rho_{13}\sigma_2M_2(u)+\sigma_1D(u,T)M_1(u))du)} \\
&\quad \times e^{-\frac{v^2}{2}\sigma_2^2(T-t)-\int_t^T\frac{\sigma_1^2}{2}v^2D(u,T)du+\int_t^Tiv\sigma_1\sigma_2D(u,T)du} \\
&\quad \times e^{\int_t^T(\rho_1^\lambda(s)+\rho_2)(E[e^{ivY_1}]-1)ds}
\end{aligned}$$

with

$$M_1(t) = \sigma_1 D(t,T) + \rho_{13}\sigma_3 D_1(t,T), \qquad M_2(t) = \sigma_3 D_1(t,T) + \rho_{23}\sigma_1 D(t,T).$$

Proof. Define

$$\frac{dQ^\lambda}{dQ} = \frac{e^{-\int_0^T(\lambda_s+r_s)ds}}{E[e^{-\int_0^T(\lambda_s+r_s)ds}]}$$

From Eqs. (2.5)-(2.6) and some calculations, we can have

$$\begin{aligned} P^\lambda(t,T) &= E\left[e^{-\int_t^T(r_s+\lambda_s)ds}\middle|\mathcal{G}_t\right] \\ &= e^{-(\theta+b)(T-t)-(r(t)-\theta)D(t,T)-(\lambda(t)-b)D_1(t,T)} \\ &\quad \times e^{(\rho_1+\rho_3)\int_t^T\int_{-\infty}^{+\infty}(e^{-D_1(u,T)}-1)f_z(z)dzdu} \\ &\quad \times e^{-\frac{1}{2}\int_t^T(\sigma_1^2 D^2(u,T)+2\rho_{13}\sigma_1\sigma_3 D(u,T)D_1(u,T)+\sigma_3^2 D_1^2(u,T))du}. \end{aligned}$$

Then

$$\begin{aligned} \frac{dQ^\lambda}{dQ} &= e^{-\int_0^T\sigma_1 D(u,T)dW_1(u)-\int_0^T\sigma_3 D_1(u,T)dW_3(u)-\frac{1}{2}\int_0^T\sigma_1^2 D^2(u,T)du} \\ &\quad \times e^{-\frac{1}{2}\int_0^T\sigma_3^2 D_1^2(u,T)du-\rho_{13}\int_0^T\sigma_1\sigma_3 D(u,T)D_1(u,T)du} \\ &\quad \times e^{-\int_0^T D_1(u,T)dJ(u)-(\rho_1+\rho_3)\int_0^T\int_{-\infty}^{+\infty}(e^{-D_1(u,T)}-1)f_Z(z)dzdu} \end{aligned}$$

and Girsanov's theorem implies that

$$W_1^\lambda(t) = W_1(t) + \int_0^t \sigma_1 D(u,T)du + \int_0^t \rho_{13}\sigma_3 D_1(u,T)du$$

and

$$W_3^\lambda(t) = W_3(t) + \int_0^t \sigma_3 D_1(u,T)du + \int_0^t \rho_{13}\sigma_1 D(u,T)du$$

are standard Brownian motions under Q^λ, and the intensity of the jump process $N_1^\lambda(t)$ is given by $\rho_1^\lambda(t) = \rho_1\int_{-\infty}^{+\infty}e^{-zD_1(t,T)}f_Z(z)dz$. Therefore, under the measure Q^λ,

$$\begin{aligned} S_T &= S_t e^{\wedge(t,T)+\int_t^T\sigma_2 D(u,T)dW_2^\lambda(u)+\int_t^T\sigma_1 D(u,T)dW_1^\lambda(u)-\int_t^T\rho_{12}\sigma_2 M_1(u)du} \\ &\quad \times e^{-\int_t^T\rho_{23}\sigma_2 M_2(u)du-\int_t^T\sigma_1 D(u,T)M_1(u)du+\sum_{j=N_1^\lambda(t)+N_2(t)}^{N_1^\lambda(T)+N_2(T)}Y_j}. \end{aligned}$$

Similar to deriving $\eta^T(v,t,T)$, we have that the characteristic function of s_T under Q^λ is given by

$$\begin{aligned} \eta^\lambda(v,t,T) &= e^{iv(s_t+\wedge(t,T)-\int_t^T\rho_{12}\sigma_2 M_1(u)du-\int_t^T\rho_{23}\sigma_2 M_2(u)du-\int_t^T\sigma_1 D(u,T)M_1(u)du)} \\ &\quad \times e^{-\frac{v^2}{2}\sigma_2^2(T-t)-\int_t^T\frac{\sigma_1^2}{2}v^2 D(u,T)du+\int_t^T iv\sigma_1\sigma_2 D(u,T)du} \\ &\quad \times e^{\int_t^T(\rho_1^\lambda(s)+\rho_2)(E[e^{ivY_1}]-1)ds}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_2(u,t,T) &= \int_{-\infty}^{+\infty} e^{iuk} c_2(t,T,K)dk \\
&= P^{\lambda}(t,T)\int_{-\infty}^{+\infty}\int_{-\infty}^{s} e^{(iu+a)k}(e^s - e^k) f^{\lambda}(s)dkds \\
&= \frac{P^{\lambda}(t,T)\eta^{\lambda}(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u},
\end{aligned}$$

where $f^{\lambda}(s)$ is the density function conditional of s_t under Q^{λ}. □

From Propositions 3.1, 3.2, we can directly obtain the following result.

Corollary 3.1. *For* $a > 0$, *we have*

$$\begin{aligned}
\zeta(u,t,T) &= \omega\zeta_1(u,t,T) + 1_{\{\tau>t\}}(1-\omega)\zeta_2(u,t,T) \\
&= \omega\frac{P(t,T)\eta^{T}(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u} \\
&\quad +1_{\{\tau>t\}}(1-\omega)\frac{P^{\lambda}(t,T)\eta^{\lambda}(u-i(a+1),t,T)}{a^2+a-u^2+i(2a+1)u}.
\end{aligned}$$

4. FFT for vulnerable European option pricing

In this section, we shall carry out some numerical calculations for the pricing of options. An approach based on the fast Fourier transform (FFT) is widely used to numerically evaluate a price of a European-style call option. The main advantage of the FFT approach is that it computes the discrete Fourier transform (DFT) faster than other approaches. For the details of the fast Fourier transform, we refer to Carr and Madan (1999).

Let $u_j = \eta(j-1)$. Following Carr and Madan (1999), an approximation for $C(0,T,k)$ is

$$\begin{aligned}
C(0,T,k) &\approx \frac{e^{-ak}}{\pi}\Bigg(\omega\sum_{j=1}^{N} e^{-iu_jk}\zeta_1(u_j)\eta P(0,T) \\
&\quad +(1-\omega)\sum_{j=1}^{N} e^{-iu_jk}\zeta_2(u_j)\eta P^{\lambda}(0,T)\Bigg).
\end{aligned}$$

The FFT returns N values of modified logarithmic strike k given as follows: $k_v = -b + h(v-1), v = 1,\cdots,N$, where $b = \frac{1}{2}Nh$.

In order to apply FFT, we let $\eta h = \frac{2\pi}{N}$. To obtain an accurate integration with larger values of η, we incorporate Simpson's rule weightings into our summation. From Simpson's rule weightings , we obtain European call option prices as

$$\begin{aligned} C(0,T,k_v) &\approx \frac{e^{-ak_v}}{\pi}\Bigg(\omega\sum_{j=1}^{N} e^{-\frac{2\pi i(j-1)(i-1)}{N}} e^{ibu_j}\frac{\eta}{3}[3+(-1)^j-w_{j-1}]P(0,T) \\ &+ (1-\omega)\sum_{j=1}^{N} e^{-\frac{2\pi i(j-1)(i-1)}{N}} e^{ibu_j}\frac{\eta}{3}[3+(-1)^j-w_{j-1}]P^{\lambda}(0,T)\Bigg), \end{aligned}$$

where w_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The above summation is an exact application of the FFT.

In what follows, we give a numerical example. Assume the parameters are as follows: $\omega = 0.4, \kappa = 0.3, \theta = 0.05, r_0 = 0.02, a = 0.2, b = 0.02, \lambda_0 = 0.5, \sigma_1 = 0.2, \sigma_2 = 0.15, \sigma_3 = 0.25, \rho_1 = \rho_2 = \rho_3 = 0.25, \rho_{12} = 0.7, \rho_{13} = 0.5, \rho_{23} = 0.6, S_0 = 100, T = 1$, the densities f_y and f_z are given by $f_y(y) = 10e^{-20y}1_{\{y>0\}}+ 10e^{-20y}1_{\{y<0\}},\ f_z(z) = 5e^{-5z}, z > 0$. The numerical results for the option prices are presented in Tables 1, 2. From them we can see that the convergence rate of the FFT is comparatively fast.

Table 1 presents the relationship between the option price and K. From it we can see that the option price decreases with the strike K. This is because a high value of K leads to a decreasing probability that S_T is larger than K.

Table 1: Prices calculated by FFT

K	$N = 512$	$N = 1024$	$N = 2048$	$N = 4096$
90	8.2413	8.2415	8.2415	8.2415
95	6.5219	6.5221	6.5222	6.5223
100	5.2344	5.2346	5.2346	5.2346
105	4.4651	4.4652	4.4652	4.4652
110	3.5442	3.5343	3.5343	3.5343

Table 2 represents the impact of the jump intensity of the common jumps on the option price. From it we can observe that the price increases with λ_0. This is because a high value of λ_0 leads to an increasing volatility of S_t and λ_t, and S_t is more sensitive to λ_0.

Table 2: Impact of jump intensity of option prices for $K = 100$

λ_0	$N = 512$	$N = 1024$	$N = 2048$	$N = 4096$
0.4	4.4154	4.4155	4.4156	4.4156
0.5	4.8732	4.8733	4.8733	4.8733
0.6	5.2344	5.2346	5.2346	5.2346
0.7	5.7612	5.7613	5.7613	5.7613
0.8	6.1507	6.1508	6.1509	6.1509

5. Conclusions

In this paper, we consider a jump-diffusion model to analyze a vulnerable European call option within the reduced-form framework. We assume the default intensity and the stock price are modelled by two jump-diffusion processes with common jumps. The jump components describe the impact of macro-economy on the asset price and the default intensity. We adopt the measure of change and the fast Fourier transform (FFT) method to value options. Numerical examples illustrate the practicality of the method.

Funding The research of Yinghui Dong was supported by QingLan Project and the scholarship of Jiangsu Overseas Visiting Scholar Program. The research of Sang Wu was supported by the Graduate Innovation Program of USTS (SKCX18-Y06).

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Accepted: 5.04.2018

Estimation for the parameter of a class of diffusion processes

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Abstract. This paper is concerned with the parameter estimation problem for a stationary ergodic diffusion process with drift coefficient $a(X_t, \theta)$ and diffusion coefficient $b(X_t)$ under the case of continuous-time observations. Firstly, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. Secondly, we prove that the maximum likelihood estimator exists in the interval when the sample size is large enough. Finally, the strong consistency of the estimator and the asymptotic normality of the error of estimation are proved. All of the results are obtained by applying the maximal inequality for martingales, Borel-Cantelli lemma and uniform ergodic theorem.

Keywords: maximum likelihood estimation, ergodic diffusion processes, strong consistency, asymptotic normality.

1. Introduction

Diffusion processes have been widely used for model building in demographic theory, safety science, computer science and management science. Moreover, diffusion processes are the essential stochastic modeling tools in the modern financial theories and applications. Recently, diffusion models are applied to describe the dynamics of a financial asset, such as Black-Scholes-Merton option pricing formula (see e.g. [6, 21]), and Vasicek and Cox-Ingersoll-Ross pricing formulas for the zero coupon bond (see e.g. [25, 7, 8]). Statistical inference for diffusion processes is very important to the stochastic theories as well as to the applications in model building. In the stochastic models describing the actual systems, part or all of the parameters are always unknown, but the observed values are known. Therefore, parameter estimation becomes an important problem needed to be solved depending on observed values. Parameters in the stochastic models have to be estimated either from continuous-time observation if continuous observation is possible, or from discrete sampled data set for the process if the process is not observed continuously. As far as we know, it is of great importance for the special stochastic model to estimate the parameters for the purpose of obtaining a proper structure of the model no matter which method is used.

In earlier works, some methods such as maximum likelihood estimation, Bayes estimation and least-squares estimation have been employed to solve the parameter estimation problem for the diffusion processes and the asymptotic properties of the estimators have been discussed as well. In previous literatures, numerical approximation schemes have been employed together with the estimation methods to estimate the parameters based on discrete observations. For example, by applying Euler method to discrete the original continuous-time processes and employing least squares estimation (see e.g. [12, 11, 19]), generalized method of moments (see e.g. [10]), and maximum likelihood estimation (see e.g. [1, 14]), or by using Itô sum and Riemann sum to approximate the integrals in the continuous-time likelihood function and applying maximum likelihood estimation (see e.g. [28]), martingale estimation (see e.g. [4]), and estimation based on eigenfunctions (see e.g. [5]). When the process is observed partially, both the parameter estimation method and the state estimation method such as extended Kalman filter are employed to solve the estimation problem (see e.g. [13, 23, 18]). However, numerical approximation schemes have some disadvantages. For example, the discretized processes may not converge to the original continuous-time processes and the estimators obtained may not consistent when the time between observations is bounded away from zero. As a consequence, it is of great importance to estimate parameters for the original continuous-time diffusion processes based on continuous observations. In previous literatures, some methods have been employed to solve the estimation problem for continuous-time diffusion processes described by stochastic differential equations. For example, estimation of the drift parameter for a linear stochastic differential equation (see e.g. [22]), maximum likelihood estimation in the scalar parameter case and vector parameter case for a nonlinear stochastic differential equation (see e.g. [24, 2]), and other methods such as Moment estimation (see e.g. [17]), M-estimation method (see e.g. [27]), and minimum distance method (see e.g. [15, 9]). Moreover, Kutoyants([16]) and Wei ([26]) used likelihood ratio process and maximum likelihood estimation respectively to investigate the parameter estimation in probability for ergodic diffusion processes.

Although the parameter estimation has been studied by some authors, the almost sure convergence of the parameter estimator has not been discussed. In this paper, the parameter estimation problem for a class of stationary ergodic diffusion processes is investigated by applying maximum likelihood estimation under the case of continuous-time observations. The idea of solving the estimation problem for ergodic diffusion processes in this article is different from that in Kutoyants([16]) and Wei ([26]). In ([16]) and ([26]), only the weak convergence of the estimator has been considered, but in this paper, the strong convergence, namely the almost sure convergence of the estimator is considered. For the purpose of proving the existence of the maximum likelihood estimator, we find a compact set on which the likelihood function is continuous and does not attain the maximum at two endpoints of this compact set when the sample size is large enough. Hence, the likelihood function has a local maximum in this compact set and the existence of the maximum likelihood estimator is proved. The strong consistency of the parameter and the asymptotic normality of the error of estimation are proved by applying maximal inequality for martingale, Borel-Cantelli lemma, the dominated convergence theorem and the uniform ergodic theorem.

This paper is organized as follows. In Section 2, some assumptions are provided and the likelihood function is given based on the Girsanov theorem. The main results are given in Section 3 where the existence and strong consistency of the maximum likelihood estimator are proved and the limit distribution of the error of estimation is obtained. An example is given to verify the effectiveness of the estimator in Section 4. The conclusion is given in Section 5.

2. Problem formulation and preliminaries

In this paper, the one-dimensional stationary ergodic diffusion processes described by the following class of stochastic differential equation will be studied:

$$\begin{cases} dX_t = a(X_t, \theta)dt + b(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{1}$$

where $\theta \in \Theta$ a open subset of $\mathbb{R}$ is the unknown one-dimensional parameter.

Suppose (1) satisfies the conditions that ensure the existence and uniqueness of the solution, (see e.g. [20]). The process is observed over $[0, T]$. x_0 is distributed according to the stationary distribution of the process. The drift and diffusion coefficients are supposed to be known and do not depend on the time t. $(W_t, t \geq 0)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now the Girsanov theorem is introduced below.

Lemma 1 ([3])**.** *Let $Y(t)$ be an Itô process of the form*

$$dY(t) = a(t, \omega)dt + dB(t); t \leq T,$$

where $T \leq \infty$ is a given constant and $B(t)$ is Brownian motion. Put

$$M_t = \exp(-\int_0^t a(s, \omega)dB_s - \frac{1}{2}\int_0^t a^2(s, \omega)ds); t \leq T.$$

Assume that $a(s, \omega)$ satisfies Novikov's condition

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T a^2(s, \omega)ds)] < \infty,$$

where $\mathbb{E}$ is the expectation with respect to P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T dP(\omega).$$

Then $Y(t)$ is a Brownian motion with respect to the probability law Q, for $t \leq T$.

From now on we shall work under the assumptions below.

Assumption 1. *$\mathbb{P}_{\theta_1} \neq \mathbb{P}_\theta$ for $\theta_1 \neq \theta$ in Θ where θ denotes the true parameter.*

Assumption 2. *$|a(x,s)| \leq M(x)$ and $|a'(x,s)| \leq Q(x)$ for all $s \in I(\theta)$ where $I(\theta)$ is a closed interval containing θ and a' denotes the differential with respect to θ. Moreover, $\mathbb{E}_\theta[\frac{M(X_0)}{b(X_0)}]^2 < \infty$ and $\mathbb{E}_\theta[\frac{Q(X_0)}{b(X_0)}]^2 < \infty$.*

Assumption 3.

$$\mathbb{E}_\theta[\int_0^T (\frac{a(X_t,\theta)}{b(X_t)})^2 dt] < \infty,$$
$$\mathbb{E}_\theta[\int_0^T (\frac{a'(X_t,\theta)}{b(X_t)})^2 dt] < \infty$$

and

$$\mathbb{E}_\theta[\int_0^T (\frac{a''(X_t,\theta)}{b(X_t)})^2 dt] < \infty,$$

which ensure the existence of the stochastic integrals $\int_0^T \frac{a(X_t,\theta)}{b(X_t)} dW_t$, $\int_0^T \frac{a'(X_t,\theta)}{b(X_t)} dW_t$ and $\int_0^T \frac{a''(X_t,\theta)}{b(X_t)} dW_t$.

Remark 1. Assumption 1 means that the value the likelihood function takes at the true parameter is not equal to the value at other parameters. Assumptions 2 and 3 play a key role in applying the Borel-Cantelli lemma and uniform ergodic theorem.

Let P_θ^T be the probability measure generated by the process $\{X_t, 0 \leq t \leq T\}$ and P_W^T be the probability measure induced by the standard Wiener process. Then, by applying the Girsanov theorem, the log likelihood function is described as follows:

$$\ell_T(\theta) = \log \frac{dP_\theta^T}{dP_W^T} = \int_0^T \frac{a(X_t,\theta)}{b^2(X_t)} dX_t - \frac{1}{2}\int_0^T \frac{a^2(X_t,\theta)}{b^2(X_t)} dt. \tag{2}$$

Assume that $\ell_T(\theta)$ is continuous and differentiable with respect to θ.

In the next section two problems will be solved. One is that the existence and strong consistency of the maximum likelihood estimator at which the likelihood function attains a local maximum will be proved, the other one is that the limit distribution of the error of estimation will be discussed.

3. Main results and proofs

In the following theorem, the existence and strong consistency of the maximum likelihood estimator are proved by applying the maximal inequality for martingale, Borel-Cantelli lemma and the uniform ergodic theorem. In this case, the likelihood function attains a maximum on a compact set, which means that the likelihood function has a local maximum. The process of finding the compact set in which the likelihood function has a maximum plays a key role in the proof of this theorem.

Theorem 1. *Under Assumptions 1–3, there exists a solution of the equation* $\ell'_T(\theta) = 0$ *which is strongly consistent for* θ *as* $T \to \infty$.

Proof. Suppose θ denotes the true value of the parameter. According to the expression of the likelihood function, for any $\sigma > 0$ such that $\theta \pm \sigma \in \Theta$, it is easy to check that

$$\begin{aligned}
&\ell_T(\theta \pm \sigma) - \ell_T(\theta) \\
&= \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))}{b^2(X_t)} dX_t - \frac{1}{2} \int_0^T \frac{(a^2(X_t, \theta \pm \sigma) - a^2(X_t, \theta))}{b^2(X_t)} dt \\
&= \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))}{b(X_t)} dW_t - \frac{1}{2} \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))^2}{b^2(X_t)} dt.
\end{aligned}$$

First of all, we will prove that $\frac{1}{T} \int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t \overset{a.s.}{\to} 0$ as $T \to \infty$ where $s \in I(\theta)$.

By applying maximal inequality for martingale and the stationarity of the process, it follows that

$$\begin{aligned}
\mathbb{P}_\theta(\sup_{0<T\leq T_0} |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > \varepsilon) &\leq \frac{\mathbb{E}_\theta(\int_0^{T_0} \frac{a(X_t,s)}{b(X_t)} dW_t)^2}{\varepsilon^2} \\
&= \frac{T_0 \mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{\varepsilon^2}.
\end{aligned}$$

Let

$$\mathcal{B}_n = \{\sup_{2^{n-1}<T<2^n} \sup_s |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > 2^{\frac{n}{2}} n^\alpha\}, \tag{3}$$

where $n \geq 1$ and $\alpha > \frac{1}{2}$. Then

$$\begin{aligned}
\mathbb{P}_\theta(\mathcal{B}_n) &= \mathbb{P}_\theta(\sup_{0<T<2^{n-1}} \sup_s |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > 2^{\frac{n}{2}} n^\alpha) \\
&\leq \frac{2^{n-1}\mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{2^n n^{2\alpha}} = \frac{\mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{2} \frac{1}{n^{2\alpha}}.
\end{aligned}$$

It can be obtained that

$$\sum_{n=1}^\infty \mathbb{P}_\theta(\mathcal{B}_n) < \infty. \tag{4}$$

According to Borel-Cantelli lemma, it follows that

$$\mathbb{P}_\theta(\limsup_{n\to\infty} \mathcal{B}_n) = 0. \tag{5}$$

Therefore,

$$\limsup_{T\to\infty} \sup_s \frac{|\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t|}{T^{\frac{1}{2}}(\ln T)^\alpha} \le 2^{\frac{1}{2}}(\frac{1}{\ln 2})^\alpha \quad a.s. \tag{6}$$

Hence, for large T,

$$\sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t| \le \frac{(\ln T)^\alpha}{T^{\frac{1}{2}}} 2^{\frac{1}{2}}(\frac{1}{\ln 2})^\alpha, \tag{7}$$

with probability one.

It is easy to check that

$$\sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t| \overset{a.s.}{\to} 0, \tag{8}$$

as $T \to \infty$.

Since

$$|\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t| \le \sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t|, \tag{9}$$

it follows that

$$\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t \overset{a.s.}{\to} 0. \tag{10}$$

As a result, one has

$$\frac{1}{T}\int_0^T \frac{(a(X_t,\theta\pm\sigma) - a(X_t,\theta))}{b(X_t)} dW_t \overset{a.s.}{\to} 0, \tag{11}$$

as $T \to \infty$.

According to the uniform ergodic theorem (see e.g. [24]), it can be checked that

$$\frac{1}{T}\int_0^T \frac{(a(X_t,\theta\pm\sigma) - a(X_t,\theta))^2}{b^2(X_t)} dt \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma) - a(X_0,\theta))^2}{b^2(X_0)}], \tag{12}$$

as $T \to \infty$.

We assume that $\mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma) - a(X_0,\theta))^2}{b^2(X_0)}] > 0$.

Therefore,

$$\frac{1}{T}(\ell_T(\theta\pm\sigma) - \ell_T(\theta)) \overset{a.s.}{\to} -\frac{1}{2}\mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma) - a(X_0,\theta))^2}{b^2(X_0)}] < 0, \tag{13}$$

as $T \to \infty$.

Hence, for almost every $\omega \in \Omega$, σ and θ, there exists T_0 such that for $T \geq T_0$,

$$\ell_T(\theta \pm \sigma) < \ell_T(\theta). \tag{14}$$

Since $\ell_T(\theta)$ is continuous on the interval $[\theta-\sigma, \theta+\sigma]$, there exists an element $\overline{\theta}_T \in (\theta - \sigma, \theta + \sigma)$ such that $\ell_T(\theta)$ reaches the maximum at this element, that is to say, $\ell'_T(\overline{\theta}_T) = 0$. As $|\overline{\theta}_T - \theta| < \sigma$, it leads to the relation

$$\overline{\theta}_T \overset{a.s.}{\to} \theta, \tag{15}$$

as $T \to \infty$.

The proof is complete. □

In the following theorem, the limit distribution of the error of estimation is obtained by applying the maximal inequality for martingale, Borel-Cantelli lemma and uniform ergodic theorem.

Theorem 2. *Under Assumptions 1–3,* $\sqrt{T}(\overline{\theta}_T - \theta) \overset{d}{\to} N(0, \frac{1}{\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2}), (T \to \infty)$.

Proof. Expanding $\ell'_T(\theta)$ about $\overline{\theta}_T$, it follows that

$$\ell'_T(\theta) = \ell'_T(\overline{\theta}_T) + \ell''_T(\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))(\theta - \overline{\theta}_T), \tag{16}$$

where $0 < \lambda < 1$.

In view of Theorem 1, it is known that $\ell'_T(\overline{\theta}_T) = 0$, then

$$\ell'_T(\theta) = \ell''_T(\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))(\theta - \overline{\theta}_T). \tag{17}$$

Since θ is the true value of the parameter,

$$\begin{aligned}\ell'_T(\theta) &= \int_0^T \frac{a'(X_t,\theta)}{b^2(X_t)} dX_t - \int_0^T \frac{a(X_t,\theta)a'(X_t,\theta)}{b^2(X_t)} dt \\ &= \int_0^T \frac{a'(X_t,\theta)}{b(X_t)} dW_t,\end{aligned}$$

with the stationarity of the process, it can be checked that

$$\mathbb{E}_\theta[\ell'_T(\theta)] = 0, \tag{18}$$

and

$$\mathbb{E}_\theta[\ell'_T(\theta)]^2 = T\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2. \tag{19}$$

From the central limit theorem for stochastic integrals, one has

$$T^{-\frac{1}{2}}\ell'_T(\theta) \overset{d}{\to} N(0, \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2). \tag{20}$$

As

$$\frac{1}{T}\ell_T''(\theta) = \frac{1}{T}(\int_0^T \frac{a''(X_t,\theta)}{b^2(X_t)}dX_t - \int_0^T \frac{(a'^2(X_t,\theta) + a(X_t,\theta)a''(X_t,\theta))}{b^2(X_t)}dt) = \frac{1}{T}(\int_0^T \frac{a''(X_t,\theta)}{b(X_t)}dW_t - \int_0^T \frac{a'^2(X_t,\theta)}{b^2(X_t)}dt).$$

By applying the same method used in Theorem 1, it follows that

$$\frac{1}{T}\int_0^T \frac{a''(X_t,\theta)}{b(X_t)}dW_t \overset{a.s.}{\to} 0, \tag{21}$$

as $T \to \infty$.

By employing the uniform ergodic theorem, it can be obtained that

$$\frac{1}{T}\int_0^T \frac{a'^2(X_t,\theta)}{b^2(X_t)}dt \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2, \tag{22}$$

as $T \to \infty$.

From the above analysis, one has

$$\frac{1}{T}\ell_T''(\theta) \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2, \tag{23}$$

as $T \to \infty$.

Since

$$\begin{aligned}
&\frac{1}{T}(\ell_T''(\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T)) - \ell_T''(\theta)) \\
&= \frac{1}{T}(\int_0^T \frac{(a''(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T)) - a''(X_t,\theta))}{b^2(X_t)}dX_t \\
&+ \int_0^T \frac{a'^2(X_t,\theta) - a'^2(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T)) + a(X_t,\theta)a''(X_t,\theta)}{b^2(X_t)}dt \\
&- \int_0^T \frac{a(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))a''(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))}{b^2(X_t)}dt) \\
&= \frac{1}{T}(\int_0^T \frac{(a''(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T)) - a''(X_t,\theta))}{b(X_t)}dW_t \\
&+ \int_0^T \frac{a'^2(X_t,\theta) - a'^2(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))}{b^2(X_t)}dt \\
&+ \int_0^T \frac{a''(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))(a(X_t,\theta) - a(X_t,\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T)))}{b^2(X_t)}dt),
\end{aligned}$$

by applying the uniform ergodic theorem, the dominated convergence theorem together with $\overline{\theta}_T \xrightarrow{a.s.} \theta$ and (21), it follows that

$$\frac{1}{T}(\ell''_T(\overline{\theta} + \lambda(\theta - \overline{\theta}_T)) - \ell''_T(\theta)) \xrightarrow{a.s.} 0, \tag{24}$$

as $T \to \infty$.

Therefore,

$$\frac{1}{T}(\ell''_T(\overline{\theta} + \lambda(\theta - \overline{\theta}_T))) \xrightarrow{a.s.} \mathbb{E}_\theta[\frac{a'(X_0, \theta)}{b(X_0)}]^2, \tag{25}$$

as $T \to \infty$.

From the above analysis, we have

$$\sqrt{T}(\overline{\theta}_T - \theta) \xrightarrow{d} N(0, \frac{1}{\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2}), \tag{26}$$

as $T \to \infty$.

The proof is complete. □

4. Example

We consider the diffusion process described by the following stochastic differential equation:

$$\begin{cases} dX_t = & -\theta X_t[1 + \frac{1}{2}\sin(X_t)]dt + \sigma dW_t \\ X_0 \sim & u_\theta, \end{cases} \tag{27}$$

where $\theta > 0$,$\sigma > 0$, u_θ is the invariant measure.

It is easy to check that the likelihood function has the following expression

$$\ell_T(\theta) = \int_0^T \frac{-\theta X_t[1 + \frac{1}{2}\sin(X_t)]}{\sigma^2} dX_t - \frac{1}{2}\int_0^T \frac{\theta^2 X_t^2[1 + \frac{1}{2}\sin(X_t)]^2}{\sigma^2} dt. \tag{28}$$

Then, we obtain the estimator

$$\widehat{\theta_T} = \frac{-\int_0^T X_t[1 + \frac{1}{2}\sin(X_t)]dX_t}{\int_0^T X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 dt}. \tag{29}$$

Thus, the estimation error is

$$\widehat{\theta_T} - \theta = \frac{-\sigma \int_0^T X_t[1 + \frac{1}{2}\sin(X_t)]dW_t}{\int_0^T X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 dt}. \tag{30}$$

Since $X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 \le \frac{9}{4}X_t^2$ and $\mathbb{E}[X_0]^2 < \infty$, it is obviously that this process satisfy the Assumptions 1–3. Then, we have

$$\widehat{\theta_T} \xrightarrow{a.s.} \theta, \tag{31}$$

and

$$\sqrt{T}(\widehat{\theta_T} - \theta) \xrightarrow{d} N(0, \frac{\sigma^2}{\mathbb{E}[X_0^2[1 + \frac{1}{2}\sin(X_0)]^2]}). \tag{32}$$

5. Conclusion

In this paper, the existence and strong consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation have been proved with the help of the maximal inequality for martingale, Borel-Cantelli lemma and uniform ergodic theorem. In this paper, the likelihood function has attained a local maximum at the maximum likelihood estimator in a compact set. This paper has considered a class of processes driven by Brown Motion, one of the further research topics will study the parameter estimation for the processes driven by fractional Brown Motion or small Lévy noises.

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Accepted: 27.05.2018

Relationships between tropical eigenvectors and tropical fixed points of the group $GL(2, \mathbb{R})$

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Abstract. The eigenvalues, eigenvectors and fixed points of matrices have many applications in various branches of science and many mathematical disciplines. In this paper first we introduce the concept of tropical fixed points, then we calculate the tropical eigenvalues and tropical eigenvectors of $GL(2, \mathbb{R})$. Furthermore we give relationships between tropical eigenvectors and tropical fixed points of $GL(2, \mathbb{R})$.

Keywords: tropical eigenvalues, tropical eigenvectors, fixed points.

1. Introduction

The tropical semiring is the set $\mathbb{R} \cup \{\infty\}$ denoted by $\mathbb{T}$, with the two new operations $\oplus$ and $\odot$. The operation $\odot$ is defined as the classical + and $\oplus$ is defined to be the minimum of two elements of $\mathbb{T}$. That is for all $a, b \in \mathbb{T}$,

$$a \oplus b = \min\{a, b\}, \qquad a \odot b = a + b.$$

Following examples explain the tropical operations.

Example 1.1.1. Let $3, 7 \in \mathbb{T}$, $3 \oplus 7 = \min\{3, 7\} = 3$, and $3 \odot 7 = 3 + 7 = 10$.

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Example 1.1.2. The elements $\infty, 0 \in \mathbb{T}$, are identities under the tropical operations $\oplus$ and $\odot$ respectively, as for any $a \in \mathbb{T}$,

$$a \oplus \infty = \min\{a, \infty\} = a = \min\{\infty, a\} = \infty \oplus a,$$
$$a \odot 0 = a + 0 = a = 0 + a = 0 \odot a.$$

It is easy to verify that many of classical axioms remains true in tropical linear algebra, see page 10 of [2].

Definition 1.1. [2] Let $X = [x_{ij}] \in \mathbb{R}^{n\times r}$, $Y = [y_{ij}] \in \mathbb{R}^{r\times m}$ be tropical matrices, the tropical product of X and Y is defined as $X \odot Y = [z_{ij}]$ where $z_{ij} = \bigoplus(x_{ik} \odot y_{kj})$, where $k = 1, 2, ..., r$.

Example 1.1. Let $X = \begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 5 \\ 4 & 1 \end{bmatrix}$ be the tropical matrices in $\mathbb{R}^{2\times 2}$ then

$$\begin{aligned} X \odot Y &= \begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix} \odot \begin{bmatrix} -1 & 5 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \odot -1 \oplus 5 \odot 4 & 3 \odot 5 \oplus 5 \odot 1 \\ 8 \odot -1 \oplus 6 \odot 4 & 8 \odot 5 \oplus 6 \odot 1 \end{bmatrix} = \begin{bmatrix} 3 + (-1) \oplus 5 + 4 & 3 + 5 \oplus 5 + 1 \\ 8 + (-1) \oplus 6 + 4 & 8 + 5 \oplus 6 + 1 \end{bmatrix} \\ &= \begin{bmatrix} \min(2, 9) & \min(8, 6) \\ \min(7, 10) & \min(13, 7) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 7 & 7 \end{bmatrix} \end{aligned}$$

Definition 1.2. [2] Let $X = [x_{ij}] \in \mathbb{R}^{n\times r}$, be a tropical matrix, and $r \in \mathbb{R}$, the tropical scalar product is component wise just like the classical scalar product, that is $r \odot X = r \odot [x_{ij}] = [r \odot x_{ij}]$.

Example 1.2. Let $X = \begin{bmatrix} 7 & 5 & 1 \\ 8 & 6 & 5 \end{bmatrix}$ be a tropical matrix in $\mathbb{R}^{2\times 3}$ then

$$3 \odot X = 3 \odot \begin{bmatrix} 7 & 5 & 1 \\ 8 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3 \odot 7 & 3 \odot 5 & 3 \odot 1 \\ 3 \odot 8 & 3 \odot 6 & 3 \odot 5 \end{bmatrix} = \begin{bmatrix} 10 & 8 & 4 \\ 11 & 9 & 8 \end{bmatrix}$$

Definition 1.3. [2] Let A be an $n\times n$-matrix with entries in the tropical semiring $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$. An eigenvalue of A is a real number λ such that $A \odot v = \lambda \odot v$, for some $v \in \mathbb{R}^n$. We say that v is an eigenvector of the tropical matrix A .

Example 1.3. Consider $\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix} \in \mathbb{R}^{2\times 2}$,

$$\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix} \odot \begin{pmatrix} 2 \\ 13 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \end{pmatrix} = 2 \odot \begin{pmatrix} 2 \\ 13 \end{pmatrix}$$

The scalar 2 is tropical eigenvalue of $\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix}$, while $\begin{pmatrix} 2 \\ 13 \end{pmatrix}$ is corresponding tropical eigenvector.

Definition 1.4 ([3])**.** A graph G consists of two finite sets, $V(G)$ the set of vertices and $E(G)$ the set of edges. The edges connect the different vertices in a graph. A graph is said to be strongly connected if every vertex is reachable from every other vertex.

Theorem 1.1 ([2])**.** Let A be a tropical $n \times n$-matrix whose graph $G(A)$ is strongly connected. Then A has precisely one eigenvalue λ. That eigenvalue equals the minimal normalized length of any directed cycle in $G(A)$.

The power algorithm ([3])**.** Let A be a tropical matrix of order $n \times n$, with tropical eigenvalue λ and v be the corresponding tropical eigenvector then we calculate λ and v as:

(1) Choose $x(0) \in \mathbb{T}^n$ such that $x(0)$ contains at least one finite entry.

(2) Compute $x(k+1) = A \odot x(k)$, until a positive integer k is reached such that $x(k+p) = q \odot x(k)$ for some $p \in \mathbb{N}$ and $q \in \mathbb{R}$.

(3) Calculate $\lambda = \frac{q}{p}$.

(4) Calculate $v = \min(\lambda^{\odot(p-j)} \odot x(k+j-1))$ for $j = 1, 2, ..., p$.

Remarks. If $p = 1$ then $v = x(k)$, if $p = 2$ then $v = x(k+1) \oplus \lambda \odot x(k)$. Many authors have studied the eigenvalues, eigenvectors and fixed points of matrix groups, see for example [1,4].

2. Main results

Definition 2.1. Let $A \in GL(2, \mathbb{R})$, a vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2$ is called tropical fixed point of A if $A \odot X = X$.

Theorem 2.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, if $2a \leq b + c$ and $a \leq d$ then a is the tropical eigenvalue and $\begin{pmatrix} a \\ c \end{pmatrix}$ is the corresponding tropical eigenvector of A.

Proof. Let $x(0) = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number.

Now,

$$x(1) = A \odot x(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 0 \\ \infty \end{pmatrix} = \begin{pmatrix} a \odot 0 \oplus b \odot \infty \\ c \odot 0 \oplus d \odot \infty \end{pmatrix} = \begin{pmatrix} \min(a, \infty) \\ \min(c, \infty) \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

$$x(2) = A \odot x(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \min(2a, b+c) \\ \min(c+a, c+d) \end{pmatrix}$$

since $2a \leq b+c$ and $a \leq d$ therefore

$$x(2) = \begin{pmatrix} 2a \\ c+a \end{pmatrix} = a \odot \begin{pmatrix} a \\ c \end{pmatrix},$$

this implies $x(2) = x(1+1) = a \odot x(1)$, here $k = 1, q = a, p = 1$, so tropical eigenvalue $= \frac{q}{p} = \frac{a}{1} = a$ and tropical eigenvector$=\begin{pmatrix} a \\ c \end{pmatrix}$.

Example 2.1. Let $A = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = 3$, and $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is the tropical eigenvector.

Verification. $A \odot v = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \odot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} = 3 \odot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. This implies $A \odot v = \lambda \odot v$.

Corollary 2.1. If in Theorem 2.1 $a = 0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix} 0 \\ c \end{pmatrix}$ is the tropical eigenvector moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $a = 0$, $0 \leq b+c$ and $0 \leq d$, so matrix A becomes $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$, then by Theorem 2.1 tropical eigenvalue is zero and tropical eigenvector is $\begin{pmatrix} 0 \\ c \end{pmatrix}$. Now we show that $X = r \odot v = \begin{pmatrix} r \\ r+c \end{pmatrix}$ is the fixed point of A,

$$A \odot X = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} r \\ r+c \end{pmatrix} = \begin{pmatrix} \min(0+r, b+r+c) \\ \min(r+c, r+c+d) \end{pmatrix} = \begin{pmatrix} r \\ r+c \end{pmatrix} = X.$$

Hence the required result.

Theorem 2.2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, if $\frac{b+c}{2} \leq a$ and $\frac{b+c}{2} \leq d$ then $\frac{b+c}{2}$ is the tropical eigenvalue and $\begin{pmatrix} b+c \\ \frac{b+c}{2}+c \end{pmatrix}$ is the corresponding tropical eigenvector of A.

Proof. Let $x(0) = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number. Now

$$x(1) = A \odot x(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 0 \\ \infty \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$x(2) = A \odot x(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \min(2a, b+c) \\ \min(c+a, c+d) \end{pmatrix},$$

here $b + c \leq 2a$, for $\min(c + a, c + d)$ two cases arise:

Case I. If $c + a \leq c + d$ this implies $a \leq d$ then we have $x(2) = \begin{pmatrix} b+c \\ a+c \end{pmatrix}$,

$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c \\ a+c \end{pmatrix} = \begin{pmatrix} \min(a+b+c, a+b+c) \\ \min(b+2c, a+c+d) \end{pmatrix},$$

if $a + c + d < b + 2c$ this implies $a + d < b + c$ this means $2a < b + c$ (since $a \leq d$), which is not true therefore $b + 2c \leq a + c + d$, and we have

$$x(3) = \begin{pmatrix} a+b+c \\ b+2c \end{pmatrix}$$

$$x(3) = (b+c) \odot \begin{pmatrix} a \\ c \end{pmatrix},$$

this implies $x(3) = x(1 + 2) = (b + c) \odot x(1)$, here $k = 1, q = b + c, p = 2$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{b+c}{2}$$

and

$$\begin{aligned} \text{tropical eigenvector} = x(2) \oplus \lambda \odot x(1) &= \begin{pmatrix} b+c \\ a+c \end{pmatrix} \oplus \begin{pmatrix} a + \frac{b+c}{2} \\ c + \frac{b+c}{2} \end{pmatrix} \\ &= \begin{pmatrix} \min(b+c, a+\frac{b+c}{2}) \\ \min(a+c, c+\frac{b+c}{2}) \end{pmatrix}, \end{aligned}$$

if $a + \frac{b+c}{2} < b+c$ this means $a < \frac{b+c}{2}$, which is not true. Therefore $b+c \leq a+\frac{b+c}{2}$. If $a + c < c + \frac{b+c}{2}$ then again we get $a < \frac{b+c}{2}$ so $c + \frac{b+c}{2} \leq a + c$. Hence $v = \begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix}$.

Case II. If $c + d \leq c + a$ this implies $d \leq a$ then we have $x(2) = \begin{pmatrix} b+c \\ c+d \end{pmatrix}$,

$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c \\ c+d \end{pmatrix} = \begin{pmatrix} \min(a+b+c, b+c+d) \\ \min(b+c+c, c+d+d) \end{pmatrix},$$

if $a+b+c < b+c+d$ this implies $a < d$, which is not true therefore $b+c+d \leq a + b + c$, if $c + d + d < b + c + c$ then we have $2d < b + c$, which is not true so $b + c + c \leq c + d + d$ and we get $x(3) = \begin{pmatrix} b+c+d \\ b+2c \end{pmatrix}$,

$$x(4) = A \odot x(3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c+d \\ c+2c \end{pmatrix} = \begin{pmatrix} \min(a+b+c+d, 2b+2c) \\ \min(b+2c+d, b+2c+d) \end{pmatrix},$$

if $a+b+c+d < 2b+2c$ this implies $a+d < b+c$, this means $2d < b+c$ (since $d \leq a$) which is not true so $2b+2c \leq a+b+c+d$, we have $x(4) = \begin{pmatrix} 2b+2c \\ b+2c+d \end{pmatrix}$,

$$x(5) = A \odot x(4) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 2b+2c \\ b+2c+d \end{pmatrix} = \begin{pmatrix} \min(a+2b+2c, 2b+2c+d) \\ \min(2b+3c, b+2c+2d) \end{pmatrix}$$

if $a+2b+2c < 2b+2c+d$ this means $a < d$, which is not true (since $d \leq a$) therefore $2b+2c+d \leq a+2b+2c$, if $b+2c+d < 2b+3c$ this implies $d < b+c$ which is not true, so $2b+3c \leq b+2c+d$, hence we get

$$x(5) = \begin{pmatrix} 2b+2c+d \\ 2b+3c \end{pmatrix} = (b+c) \odot \begin{pmatrix} b+c+d \\ b+2c \end{pmatrix}$$

this implies $x(5) = x(3+2) = (b+c) \odot x(3)$, here $k = 3, q = b+c, p = 2$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{b+c}{2}$$

and

$$\begin{aligned} \text{tropical eigenvector} = x(2) \oplus \lambda \odot x(1) &= \begin{pmatrix} b+c \\ c+d \end{pmatrix} \oplus \begin{pmatrix} a+\frac{b+c}{2} \\ c+\frac{b+c}{2} \end{pmatrix} \\ &= \begin{pmatrix} \min(b+c, a+\frac{b+c}{2}) \\ \min(a+c, c+\frac{b+c}{2}) \end{pmatrix}, \end{aligned}$$

if $a+\frac{b+c}{2} < b+c$ this means $a < \frac{b+c}{2}$, which is not true, therefore $b+c \leq a+\frac{b+c}{2}$. If $a+c < c+\frac{b+c}{2}$ then again we get $a < \frac{b+c}{2}$ so $c+\frac{b+c}{2} \leq a+c$. Hence $v = \begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix}$.

Example 2.2.1 (case I). Let $A = \begin{pmatrix} \frac{5}{3} & \frac{-1}{2} \\ \frac{2}{3} & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = \frac{b+c}{2} = \frac{1}{12}$, and $v = \left(\begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix}$ is the tropical eigenvector.

Verification.

$$A \odot v = \begin{pmatrix} \frac{5}{3} & \frac{-1}{2} \\ \frac{2}{3} & 2 \end{pmatrix} \odot \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{5}{6} \end{pmatrix},$$

and

$$\lambda \odot v = \frac{1}{12} \odot \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{5}{6} \end{pmatrix}.$$

This implies $A \odot v = \lambda \odot v$.

Example 2.2.2 (case II). Let $A = \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = \frac{b+c}{2} = 1$, and $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is the tropical eigenvector.

Corollary 2.2. If in above theorem $b+c=0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix} 0 \\ c \end{pmatrix}$ is the tropical eigenvector, moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $b+c=0$ implies $c=-b$, also $b+c \leq a$ and $b+c \leq d$, so matrix A becomes $\begin{pmatrix} a & b \\ -b & d \end{pmatrix}$, then by Theorem 2.2 tropical eigenvalue $b+c$ is 0 and tropical eigenvector $\begin{pmatrix} 0 \\ -b \end{pmatrix}$. Now, we show that $X = r \odot v = \begin{pmatrix} r \\ r-b \end{pmatrix}$ is the fixed point of A,

$$A \odot X = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \odot \begin{pmatrix} r \\ r-b \end{pmatrix} = \begin{pmatrix} \min(a+r, b+r-b) \\ \min(r-b, r-b+d) \end{pmatrix} = \begin{pmatrix} r \\ r-b \end{pmatrix} = X.$$

Hence the required result.

Theorem 2.3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, if $2d \leq b+c$ and $d \leq a$ then d is the tropical eigenvalue and $\begin{pmatrix} b+c+(n-2)d \\ c+(n-1)d \end{pmatrix}$ where n is a natural number, is the corresponding tropical eigenvector of A (here $n=k$, where k is from Power Algorithm).

Proof. Let $x(0) = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number. Now

$$x(1) = A \odot x(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 0 \\ \infty \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$
$$x(2) = A \odot x(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \min(2a, b+c) \\ \min(c+a, c+d) \end{pmatrix},$$

here $c+a \leq c+d$, for $\min(2a, b+c)$ two cases arise.

Case I. If $b+c \leq 2a$ this implies

$$x(2) = \begin{pmatrix} b+c \\ c+d \end{pmatrix},$$
$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c \\ c+d \end{pmatrix} = \begin{pmatrix} \min(a+b+c, b+c+d) \\ \min(b+2c, c+2d) \end{pmatrix},$$

if $a+b+c < b+c+d$ this implies $a < d$, which is not true (since $d \leq a$), so $b+c+d \leq a+b+c$, if $b+2c < c+2d$ this means $b+c < 2d$, which is not true (since $2d \leq b+c$), therefore $c+2d \leq b+2c$, and we get

$$x(3) = \begin{pmatrix} b+c+d \\ c+2d \end{pmatrix},$$

$$x(3) = d \odot \begin{pmatrix} b+c \\ c+d \end{pmatrix},$$

this implies $x(3) = x(2+1) = d \odot x(2)$, here $k = 2, q = d, p = 1$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{d}{1} = d$$

and

$$\text{tropical eigenvector} = \begin{pmatrix} b+c \\ c+d \end{pmatrix}.$$

Case II. If $2a \leq b + c$ then

$$x(2) = \begin{pmatrix} 2a \\ c+d \end{pmatrix},$$

$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 2a \\ c+d \end{pmatrix} = \begin{pmatrix} \min(a+2a, b+c+d) \\ \min(c+2a, c+2d) \end{pmatrix}$$

here if we take $3a < b + c + d$ this implies

$$x(3) = \begin{pmatrix} 3a \\ c+2d \end{pmatrix},$$

$$x(4) = A \odot x(3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 3a \\ c+2d \end{pmatrix} = \begin{pmatrix} \min(a+3a, b+c+2d) \\ \min(c+3a, c+3d) \end{pmatrix},$$

since $d \leq a$ so proceeding as above we get some natural number n such that $b + c + (n-2)d \leq na$. Therefore, we have

$$x(n) = \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d) \end{pmatrix}$$

$$x(n+1) = A \odot x(n) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d) \end{pmatrix}$$

$$= \begin{pmatrix} \min(a+b+c+(n-2)d, b+c+(n-1)d) \\ \min(b+c+c+(n-2)d, c+(n-1)d)+d \end{pmatrix},$$

if $a + b + c + (n-2)d < b + c + (n-1)d$ this implies $a + b + c + (n-2)d < b + c + (n-2)d + d$, this means $a < d$ which is not true (since $d \leq a$), therefore $b + c + (n-1)d \leq a + b + c + (n-2)d$, if $b + c + c + (n-2)d < c + (n-1)d + d$ this implies $b + c + (n-2)d < (n-2)d + 2d$, this means $b + c < 2d$ which is not true, so $c + (n-1)d + d \leq b + c + c + (n-2)d$, hence we have

$$x(n+1) = \begin{pmatrix} b+c+(n-1)d) \\ c+(n-1)d+d \end{pmatrix} = d \odot \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d \end{pmatrix},$$

this implies $x(n+1) = d \odot x(n)$, here $k = n, q = d, p = 1$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{d}{1} = d$$

and

$$\text{tropical eigenvector} = \begin{pmatrix} b+c+(n-2)d \\ c+(n-1)d \end{pmatrix}.$$

Example 2.3.1 (case I). Let $A = \begin{pmatrix} \frac{11}{2} & \frac{6}{5} \\ \frac{9}{2} & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = d = 2$, and $v = \begin{pmatrix} b+c \\ c+d \end{pmatrix} = \begin{pmatrix} \frac{57}{10} \\ \frac{13}{2} \end{pmatrix}$ is the tropical eigenvector.

Example 2.3.2 (case II). Let $A = \begin{pmatrix} 5 & 9 \\ 7 & 3 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = d = 3$, here $b+c+(n-2)d \leq na$ implies $9+7+(n-2)3 \leq 5n$, solving this we get $5 \leq n$, let us take $n = 5$, therefore $v = \begin{pmatrix} b+c+3d \\ c+4d \end{pmatrix} = \begin{pmatrix} 25 \\ 19 \end{pmatrix}$ is the tropical eigenvector.

Corollary 2.3. If in Theorem 2.3 $d = 0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix} b+c \\ c \end{pmatrix}$ is the tropical eigenvector, moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $d = 0$, $0 \leq b+c$ and $0 \leq a$, so matrix A becomes $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, then by Theorem 2.3 tropical eigenvalue is zero and tropical eigenvector is $\begin{pmatrix} b+c \\ c \end{pmatrix}$. Now we show that $X = r \odot v = \begin{pmatrix} r+b+c \\ r+c \end{pmatrix}$ is the fixed point of A,

$$\begin{aligned} A \odot X &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \odot \begin{pmatrix} r+b+c \\ r+c \end{pmatrix} \\ &= \begin{pmatrix} \min(a+r+b+c, b+r+c) \\ \min(r+b+2c, r+c) \end{pmatrix} = \begin{pmatrix} b+r+c \\ r+c \end{pmatrix} = X. \end{aligned}$$

Hence the required result.

3. Conclusion

In this paper we first calculate the tropical eigenvalues and tropical eigenvectors of the group $GL(2,\mathbb{R})$. Then we show that if the tropical eigenvalue is zero then the tropical fixed points of elements of $GL(2,\mathbb{R})$ are the tropical scalar multiple of the tropical eigenvectors.

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Accepted: 10.04.2018

L^∞-asymptotic behavior for a finite element approximation to optimal control problems

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Abstract. In this paper, a system of parabolic quasi-variational inequalities relevant to the management of energy production is considered where a quasi-optimal of error estimate on uniform norm is proved, by using semi-implicit scheme combined with Galerkin method. Furthermore, an asymptotic behavior result in the same norm is given, taking into consideration the discrete stability properties.

Keywords: parabolic quasi variational inequalities, Galerkin method, L^∞-asymptotic behavior.

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1. Introduction

In this paper, we are concerned with the numerical approximation in the L^∞ norm for the following problem: find u such that

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \max\limits_{1 \leq i \leq J} \left(\mathcal{A}^i u - f^i \right) = 0 \text{ in } \mathbb{Q}_T := \Omega \times]0,\ T[, \\
u \Big|_{t=0} = u_0, \text{ in } \Omega, \\
u = 0 \text{ in } \sum_T :=]0,\ T[\times \Gamma, \\
u \geq 0,
\end{cases}
\tag{1.1}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary Γ, $\mathcal{A}^i$ are J-second-order, uniformly elliptic operators of the form

$$
\mathcal{A}^i = \sum_{j,k=1}^{d} a^i_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{d} b^i_j(x) \frac{\partial}{\partial x_j} + a^i_0(x),
\tag{1.2}
$$

f is a regular function satisfies

$$
f \in L^2(0, T, L^\infty(\Omega)) \cap C^1\left(0, T, H^{-1}(\Omega)\right).
\tag{1.3}
$$

It is known (see. [3-5], [9-12]) that the problem (1.1) can be approximated by the following weakly coupled system of parabolic quasi-variational inequalities (QVIs): find a vector $U = \left(u^1,\ u^2, \ldots,\ u^J\right) \in \left(L^2\left(0,\ T;\ H^1_0(\Omega)\right)\right)^J$ such that

$$
\begin{cases}
\dfrac{\partial}{\partial t}(u^i(t),\ v - u^i(t)) + a^i(u^i(t),\ v - u^i(t)) \geq (f^i,\ v - u^i(t)), \\
\qquad \forall v \in H^1_0(\Omega), \\
u^i \leq (MU)^i,\ v \leq (MU)^i,\ i = 1,\ 2, \ldots,\ J, \\
u^i \geq 0,
\end{cases}
\tag{1.4}
$$

where $a^i(.,.)$ are J-elliptic continuous and noncoercive bilinear forms associated $\mathcal{A}^i$ defined as:

$$
a^i(u,v) = \int_\Omega \left(\sum_{j,k=1}^{d} a^i_{jk}(x) \frac{\partial\ u}{\partial\ x_j} \frac{\partial\ v}{\partial\ x_k} + \sum_{k=1}^{d} b^i_k(x) \frac{\partial u}{\partial x_k} v + a^i_0(x)\, uv \right) dx,
\tag{1.5}
$$

where $\forall i = 1, \ldots, J,\ a^i_{jk}(.),\ b^i_j(.),\ a^i_0(.) \in C^2\left(\bar{\Omega}\right),\ x \in \bar{\Omega}, 1 \leq j,\ k \leq d$ are sufficiently smooth coefficients and satisfy the following conditions:

$$
\begin{cases}
a^i_{jk}(x) = a^i_{kj}(x), \\
a^i_0(x) \geq \beta > 0,\ \beta \text{ is a constant}
\end{cases}
\tag{1.6}
$$

and

$$\sum_{j,k=1}^{n} a_{jk}^{i}(x)\xi_j\xi_k \geq \gamma|\xi|^2;\ \xi \in \mathbb{R}^d,\ \gamma > 0,\ x \in \bar{\Omega}. \tag{1.7}$$

In the case studied here, $(MU)^i$ represents a "cost function" and the prototype encountered is

$$(MU)^i = \rho + \inf_{\mu \neq i} u^{\mu},\ i = 1, ...,\ J. \tag{1.8}$$

In (1.8), ρ represents the switching cost. It is positive when the unit is "turned on" and equal to zero when the unit is "turned off". Note also that operator M provides the coupling between the unknowns $u^1, \ldots, u^J$ (see. e.g. [1], [2] and the references therein).

In the stationary case M. Boulbrachen in [7] studied a particular class of problems related to the management of energy production problems and presented a study of the complete numerical analysis; his approach is based on the concept of subsolutions.

The aim of the present paper is to study the corresponding evolution case and to obtain a quasi-optimal L^{∞}-asymptotic behavior for a finite element approximation to parabolic quasi-variational inequalities.

The rest of the manuscript is structured as follows. In Section 2, we present the continuous problem. The discrete problem is proposed in Section 3. Then, in Section 4, we prove an error estimate on the uniform norm of the presented problem.

2. Statement of the continuous problem

2.1 The continuous system

2.1.1 Full discretization

In order to obtain a full discretization of (1.4), we consider a uniform mesh for the time variable t and define

$$t_n = n\ \Delta t,\ n = 0, 1, ..., \mathcal{N}, \tag{2.1}$$

$\Delta t > 0$ being the time-step, and $\mathcal{N} = \left[\frac{T}{\Delta t}\right]$, the integral part of $\frac{T}{\Delta t}$.

Next, we replace the time derivative by means of suitable difference quotients, thus constructing a sequence $u^{i,n} \in H_0^1(\Omega)$ that approaches $u^i(t_n,\ x)$.

For simplicity, we confine ourselves to the so-called semi-implicit scheme, which consists of replacing (1.4) by the following scheme: find a vector $U^n =$

$\left(u^{1,n},...,u^{J,n}\right)\in\left(H_0^1\left(\Omega\right)\right)^J$ such that

$$\text{(2.2)}\qquad \begin{cases} \dfrac{1}{\triangle t}(u^{i,n}-u^{i,n-1},\ v-u^{i,n})+a^i(u^{i,n},\ v-u^{i,n})\geq(f^{i,n},\ v-u^{i,n}),\\ \qquad \forall v\in H_0^1(\Omega),\\ u^{i,n}\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\ v\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\ n=1,...,\mathcal{N}-1,\\ u^i(0)=u_0^i, \end{cases}$$

where

$$\triangle t=\frac{T}{\mathcal{N}}.$$

By adding $(\frac{u^{i,n-1}}{\triangle t},\ v-u^{i,n})$ to both parties of the scheme (2.3), we get

$$\text{(2.3)}\qquad \begin{cases} a^i\left(u^{i,n},\ v-u^{i,n}\right)+\dfrac{1}{\triangle t}\left(u^{i,n},\ v-u^{i,n}\right)\\ \qquad\geq\left(f^{i,n}+\dfrac{1}{\triangle t}u^{i,n-1},\ v-u^{i,n}\right),\\ u^{i,n}\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\ v\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\\ u^i\left(0\right)=u_0^i. \end{cases}$$

The bilinear form $a^i\left(.,\ .\right),$ is a noncoercive in $H_0^1\left(\Omega\right),$ and satisfies the following condition: for all $\varphi\in H_0^1\left(\Omega\right)$ there exists $\gamma>0$, such that

$$\text{(2.4)}\qquad a^i\left(\varphi,\ \varphi\right)+\lambda\left\|\varphi\right\|_{L^2(\Omega)}^2\geq\gamma\left\|\varphi\right\|_{H_0^1(\Omega)}^2.$$

Set

$$\text{(2.5)}\qquad b^i\left(u,\ v\right)=a^i\left(u,\ v\right)+\lambda\left(u,\ v\right).$$

Thanks to [7] the bilinear $b^i\left(.,\ .\right)$ is strongly coercive and (2.3) can be transformed into the following continuous system of elliptic quasi-variational inequalities (QVIs): find a vector $U^n=\left(u^{1,n},...,u^{J,n}\right)\in\left(H_0^1\left(\Omega\right)\right)^J$ such that

$$\text{(2.6)}\qquad \begin{cases} b^i\left(u^{i,n},\ v-u^{i,n}\right)\geq\left(f^{i,n}+\lambda u^{i,n-1},\ v-u^{i,n}\right),\ \forall v\in H_0^1\left(\Omega\right),\\ u^{i,n}\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\ v\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,n},\ n=1,...,\mathcal{N}-1, \end{cases}$$

where

$$\text{(2.7)}\qquad \begin{cases} b^i\left(u^{i,n},\ v-u^{i,n}\right)=a^i\left(u^{i,n},\ v-u^{i,n}\right)+\lambda\left(u^{i,n},\ v-u^{i,n}\right),\\ \lambda=\dfrac{1}{\Delta t}>0. \end{cases}$$

2.2 Existence and uniqueness

Next, using the preceding assumptions, we shall prove the existence and uniqueness of a continuous solution for problem (2.6) by means of Banach's fixed point theorem.

Let $\mathbb{H}^+ = \prod_{i=1}^J L_+^\infty(\Omega) = \{W = (w^1, ..., w^J)$ such that $w^i \in L_+^\infty(\Omega)\}$, equipped with the norm

$$\|W\|_\infty = \max_{1\leq i\leq J} \left\|w^i\right\|_{L^\infty(\Omega)}, \tag{2.8}$$

where $L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$.

2.2.1 A fixed point mapping associated with the system (2.6)

We consider the following mapping:

$$\begin{aligned} \mathbb{T} \ : \ & \mathbb{H}^+ \longrightarrow \mathbb{H}^+, \\ & W \to \mathbb{T}W = \zeta^n = \left(\zeta^{1,n}, ..., \zeta^{J,n}\right), \end{aligned} \tag{2.9}$$

where $\zeta^{i,n} = \partial\left(f^{i,n},\ \rho\right) \in H_0^1(\Omega)$ is a solution to following continuous QVIs:

$$\begin{cases} b^i\left(\zeta^{i,n}, v - \zeta^{i,n}\right) \geq \left(f^i + \lambda w^i, v - \zeta^{i,n}\right), \\ \zeta^{i,n} \leq \rho + \inf\limits_{\mu\neq i} \zeta^{\mu,n},\ v \leq \rho + \inf\limits_{\mu\neq i} \zeta^{\mu,n}, \qquad n = 1, ..., \mathcal{N}-1. \end{cases} \tag{2.10}$$

The problem (2.10) being a coercive QVIs, thanks to [6], [13] has one and only one solution.

Theorem 1. *Under the preceding hypotheses and notations, the mapping $\mathbb{T}$ is a contraction in $\mathbb{H}^+$ with a contraction constant $\frac{1}{\beta\Delta t+1}$. Therefore, $\mathbb{T}$ admits a unique fixed point which coincides with the solution of problem* (2.6).

Proof. We adapt [4]. □

The mapping $\mathbb{T}$ generates the following continuous algorithm.

2.3 A continuous algorithm

Starting from $U^0 = U_0 = \left(u_0^1, ..., u^J\right)$ the solution of the following equation:

$$b^i\left(u_0^i, v\right) = \left(f^i + \lambda u_0^i, v\right), \qquad \forall v \in H_0^1(\Omega). \tag{2.11}$$

we define

$$u^{i,n} = \mathbb{T}u^{i,n-1},\ n = 1, ..., \mathcal{N}-1, \tag{2.12}$$

where $u^{i,n}$ is solution to (2.6).

Proposition 1. *Under the conditions of Theorem 1, we have:*

$$\max_{1\le i\le J}\left\|u^{i,n}-u^{i,\infty}\right\|_{\infty}\le\left(\frac{1}{\beta\Delta t+1}\right)^{n}\max_{1\le i\le J}\left\|u_0^{i}-u^{i,\infty}\right\|_{\infty},\tag{2.13}$$

where $u^{i,\infty}$ is the asymptotic solution of the continuous system of QVIs: find a vector $U^{\infty}=\left(u^{1,\infty},...,u^{J,\infty}\right)\in\left(H_0^1(\Omega)\right)^J$ such that

$$\begin{cases} b^i\left(u^{i,\infty},\ v-u^{i,\infty}\right)\ge\left(f^i+\lambda u^{i,\infty},\ v-u^{i,\infty}\right),\\ u^{i,\infty}\le\rho+\inf\limits_{\mu\ne i}u^{\mu,\infty},\ v\le\rho+\inf\limits_{\mu\ne i}u^{\mu,\infty}.\end{cases}\tag{23}$$

Proof. We adapt [4]. $\square$

3. Statement of the discrete problem

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h>0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider $\phi_l, l=1,2,...,m(h)$, the usual basis of affine functions defined by $\phi_l(M_s)=\delta_{l,\ s}$ where M_s is a vertex of the considered triangulation.

Let us $\mathbb{V}_h$ denote the standard piecewise linear finite element space such that

$$\mathbb{V}_h=\left\{\begin{array}{l} v_h\in C^0\left(\bar\Omega\right),\ v_h=0 \text{ on } \partial\Omega \text{ such that:}\\ v_h\left|_{K^i}\right.\in P_1,\ K\in\tau_h, v_h\le r_h\psi, v_h(.,0)=v_{0h} \text{ in } \Omega.\end{array}\right\}\tag{3.1}$$

Let also r_h be the usual interpolation operator defined by

$$v_h\in L^2\left([0,T];\ H_0^1(\Omega)\right)\cap C\left([0,T];\ H_0^1\left(\bar\Omega\right)\right),\ r_hv_h=\sum_{l=1}^{m(h)}v(M_l)\,\phi_l(x),\tag{3.2}$$

and $\mathbb{B}^i,\ 1\le i\le J$ be the matrix with generic entries

$$\left(\mathbb{B}^i\right)_{l,\ s}=b^i(\phi_l,\ \phi_s)=a^i(\phi_l,\ \phi_s)+\lambda\int_\Omega\phi_l\ \phi_s\ dx,\ 1\le l,s\le m(h).\tag{3.3}$$

In the sequel of the paper, we shall use the discrete maximum assumption (*d.m.p.*). In other words, we shall assume that the matrix $\mathbb{B}^i,\ 1\le i\le J$ is an M-matrix (cf. [14]).

Remark 1. Under the d.m.p., we shall achieve a similar study to that devoted to the continuous problem, therefore the qualitative properties and results stated in the continuous case are conserved in the discrete case.

3.1 The discrete system

As in the continuous situation, one can tackle the discrete system by considering the equivalent formulation: find a vector $U_h^n = (u_h^{1,n}, ..., u_h^{J,n}) \in (\mathbb{V}_h)^J$ such that

$$(3.4)\quad \begin{cases} b^i\left(u_h^{i,n},\ v_h - u_h^{i,n}\right) \geq \left(f^{i,n} + \lambda u_h^{i,n},\ v_h - u_h^{i,n}\right), \text{for all } v_h \in \mathbb{V}_h, \\ u_h^{i,n} \leq r_h\left(\rho + \inf\limits_{\mu \neq i} u_h^{\mu,n}\right),\ v_h \leq r_h\left(\rho + \inf\limits_{\mu \neq i} u_h^{\mu,n}\right),\ n = 1, ..., \mathcal{N}-1. \end{cases}$$

Existence and uniqueness of a solution of system (3.5) can be shown similar to that of the continuous case provided the discrete maximum principle is satisfied.

3.2 Existence and uniqueness

3.2.1 A fixed point mapping associated with discrete problem (3.5)

We consider the following mapping:

$$(3.5)\quad \begin{aligned} \mathbb{T}_h \ &: \ \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^J, \\ &W \mapsto \mathbb{T}_h W = \zeta_h^n = \left(\zeta_h^{1,n}, ..., \zeta_h^{J,n}\right), \end{aligned}$$

where $\zeta_h^{i,n} = \partial_h\left(f^{i,n},\ \rho\right) \in \mathbb{V}_h$ is a solution to following discrete coercive QVIs:

$$(3.6)\quad \begin{cases} b^i\left(\zeta_h^{i,n}, v_h - \zeta_h^{i,n}\right) \geq \left(f^{i,n} + \lambda w^i, v_h - \zeta_h^{i,n}\right),\ v_h \in \mathbb{V}_h, \\ \zeta_h^{i,n} \leq r_h\left(\rho + \inf\limits_{\mu \neq i} \zeta_h^{\mu,n}\right),\ v \leq r_h\left(\rho + \inf\limits_{\mu \neq i} \zeta_h^{\mu,n}\right). \end{cases}$$

Theorem 2. *Under the d.m.p and the preceding hypotheses and notation, the mapping $\mathbb{T}_h$ is a contraction in $\mathbb{H}^+$ with a contraction constant $\rho = \frac{1}{\beta \Delta t + 1}$. Therefore, $\mathbb{T}_h$ admits a unique fixed point which coincides with the solution of system* (3.5).

As in the continuous situation, one can define the following discrete iterative scheme.

3.3 A discrete algorithm

Starting from $U_h^0 = U_{0h} = \left(u_{0h}^1, ..., u_{0h}^J\right)$ solution of the following equation:

$$(3.7)\quad b^i\left(u_{0h}^i, v\right) = \left(f^i + \lambda u_{0h}^i, v\right),\ \forall v_h \in \mathbb{V}_h.$$

we define the sequences

$$(3.8)\quad u_h^{i,n} = \mathbb{T}_h u_h^{i,n-1},\ n = 1, ..., \mathcal{N}-1,$$

where $u_h^{i,n}$ is solution to (3.3).

Using the above result, we are able to establish the following geometric convergence of sequence U_h^n.

Proposition 2. *Under the d.m.p and Theorem 2, we have*

$$\max_{1\leq i\leq J}\left\|u_h^{i,n}-u_h^{i,\infty}\right\|_\infty \leq \left(\frac{1}{\beta\Delta t+1}\right)^n \max_{1\leq i\leq J}\left\|u_h^{i,0}-u_h^{i,\infty}\right\|_\infty . \tag{3.9}$$

where $u_h^{i,\infty}$ *is the asymptotic solution of the discrete system of QVIs: find a vector* $U_h^\infty=\left(u_h^{1,\infty},...,u_h^{J,\infty}\right)\in(\mathbb{V}_h)^J$ *such that*

$$\begin{cases} b^i\left(u_h^{i,\infty},\ v_h-u_h^{i,\infty}\right)\geq\left(f^{i,n}+\lambda u_h^{i,\infty},\ v_h-u_h^{i,\infty}\right), \\ u_h^{i,\infty}\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u_h^{\mu,\infty}\right),\ v_h\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u_h^{\mu,\infty}\right). \end{cases} \tag{3.10}$$

4. L^∞-asymptotic behavior

This section is devoted to estimating the error in the L^∞-norm between $U_h(T,.)$ the discrete solution calculated at the moment $T=n\,\Delta t$ and U^∞ the asymptotic solution of the continuous system of QVIs (2.14). To this end, we first recall some known L^∞-error estimates results, introduce an auxiliary discrete sequence and prove a fundamental Theorem.

Theorem 3 ([15, 16])**.** *Let* u_0^i (*respectively,* $u_{0,h}^i$)*, be the solution of problem (2.11),* (*respectively* (3.7))*. Then, there exists a constant* C *independent of* h*,* $\triangle t$ *and* n *such that*

$$\max_{1\leq i\leq J}\left\|u_{0,h}^i-u_0^i\right\|_\infty\leq Ch^2\left|\log h\right|^{\frac{3}{2}} . \tag{4.1}$$

We introduce the following auxiliary discrete sequences

$$\tilde{u}_h^{i,n}=\mathbb{T}_h u^{i,n-1},\ n=1,...,\mathcal{N}-1, \tag{4.2}$$

with $u_{0,h}^i$ is defined in (3.7) and for any $n=1,...,\mathcal{N}-1,\ \tilde{u}_h^{i,n}$ is a solution to following discrete system of variational inequality (V.I.):

$$\begin{cases} b^i\left(\tilde{u}_h^{i,n},\ v-\tilde{u}_h^{i,n}\right)\geq\left(f^{i,n}+\lambda u^{i,n-1},\ v-\tilde{u}_h^{i,n}\right),\ v\in H_0^1(\Omega), \\ \tilde{u}_h^{i,n}\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u^{\mu,n-1}\right),\ v\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u^{\mu,n-1}\right), \end{cases} \tag{4.3}$$

$U^n=\left(u^{1,n},...,u^{J,n}\right)$ is the solution of the continuous problem (2.6).

Remark 2. We notice that $\tilde{u}_h^{i,n}$ represents the standard finite element approximation of $u^{i,n}$.

Therefore, adapting [13], we have the following

Proposition 3. *There exists a constant* C *independent of* h*,* $\triangle t$ *and* n *such that*

$$\max_{1\leq i\leq J}\left\|\tilde{u}_h^{i,n}-u^{i,n}\right\|_\infty\leq Ch^2\left|\log h\right|^2 . \tag{4.4}$$

Next, by using the above result, we introduce the following:

Lemma 1 ([8]).

$$\max_{1\le i\le J}\left\|u_h^{i,n}-u^{i,n}\right\|_{\infty}\le\sum_{p=0}^{n}\max_{1\le i\le J}\left\|\tilde{u}_h^{i,p}-u^{i,p}\right\|_{\infty}. \tag{4.5}$$

Remark 3. Lemma 1 given above plays a crucial role in proving the following Theorem.

Theorem 4. *There exists a constant C independent of h, $\triangle t$ and n such that*

$$\max_{1\le i\le J}\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_{\infty}\le Ch^2\left|\log h\right|^3. \tag{4.6}$$

Proof. By combining estimates (2.13), (3.9), and (4.5), we get

$$\begin{aligned}
&\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_{\infty}=\left\|u_h^{i,\infty}-u^{i,n}+u^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left\|u_h^{i,\infty}-u^{i,n}\right\|_{\infty}+\left\|u^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left\|u_h^{i,\infty}-u_h^{i,n}+u_h^{i,n}-u^{i,n}\right\|_{\infty}+\left\|u^{i,n}-u_h^{i,n}+u_h^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left\|u_h^{i,\infty}-u_h^{i,n}\right\|_{\infty}+\left\|u_h^{i,n}-u^{i,n}\right\|_{\infty}+\left\|u^{i,n}-u_h^{i,n}\right\|_{\infty}+\left\|u_h^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left\|u_h^{i,\infty}-u_h^{i,n}\right\|_{\infty}+2\left\|u_h^{i,n}-u^{i,n}\right\|_{\infty}+\left\|u_h^{i,n}-u^{i,n}+u^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left\|u_h^{i,\infty}-u_h^{i,n}\right\|_{\infty}+3\left\|u_h^{i,n}-u^{i,n}\right\|_{\infty}+\left\|u^{i,n}-u^{i,\infty}\right\|_{\infty}.
\end{aligned}$$

Applying the previous results of Propositions 1, 2, Theorem 3 and Lemma 1, we get

$$\begin{aligned}
&\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_{\infty}\le\left\|u_h^{i,\infty}-u_h^{i,n}\right\|_{\infty}+3\left\|u_h^{i,n}-u^{i,n}\right\|_{\infty}+\left\|u^{i,n}-u^{i,\infty}\right\|_{\infty}\\
&\le\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u^{i,\infty}-u_0^i\right\|_{\infty}+\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u_h^{i,\infty}-u_{h0}^i\right\|_{\infty}+3\sum_{p=0}^{n}\left\|\bar{u}_h^{i,p}-u^{i,p}\right\|_{\infty}\\
&\le\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u^{i,\infty}-u_0^i\right\|_{\infty}+\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u_h^{i,\infty}-u_{h0}^i\right\|_{\infty}\\
&+3\left(\left\|\bar{u}_h^{i,0}-u^{i,0}\right\|_{\infty}+\sum_{p=1}^{n}\left\|\bar{u}_h^{i,p}-u^{i,p}\right\|_{\infty}\right)\\
&\le\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u^{i,\infty}-u_0^i\right\|_{\infty}+\left(\frac{1}{\beta\Delta t+1}\right)^n\left\|u_h^{i,\infty}-u_{h0}^i\right\|_{\infty}\\
&+Ch^2\left|\log h\right|^{\frac{3}{2}}+nCh^2\left|\log h\right|^2.
\end{aligned}$$

Finally, taking $h^2 = (\frac{1}{\beta\Delta t+1})^n$, we obtain

$$\max_{1\leq i\leq J}\left\|u_h^{i,\infty} - u^{i,\infty}\right\|_\infty \leq Ch^2 \left|\log h\right|^3,$$

which completes the proof. □

Remark 4. It should be noted that the same result was obtained in [8].

Now guided by Propositions 2, Theorem 4, we are in a position to prove the main result.

Theorem 5. *There exists a constant C independent of h, $\triangle t$ and n such that*

$$\left\|U_h\left(T,\ .\right) - U^\infty\left(.\right)\right\|_\infty \leq C\left(h^2\left|\log h\right|^3 + \left(\frac{1}{\beta\Delta t+1}\right)^N\right). \tag{4.7}$$

Proof. We have

$$u_h^{i,n}\left(x\right) = u_h^i\left(t,\ x\right) \text{ for all } t \in \left](n-1)\,.\Delta t,\ n.\Delta t\right[,$$

thus

$$u_h^{i,N}\left(x\right) = u_h^i\left(T,\ x\right)$$

So,

$$\begin{aligned}\left\|u_h^i\left(T,\ x\right) - u^{i,\infty}\left(x\right)\right\|_\infty &= \left\|u_h^{i,N}\left(x\right) - u^{i,\infty}\left(x\right)\right\|_\infty \\ &\leq \left\|u_h^{i,N} - u_h^{i,\infty}\right\|_\infty + \left\|u_h^{i,\infty} - u^{i,\infty}\right\|_\infty.\end{aligned}$$

Applying the previous results of Propositions 2 and Theorem 4, we get

$$\begin{aligned}\left\|u_h^i\left(T,\ .\right) - u^{i,\infty}\right\|_\infty &\leq \left(\frac{1}{\beta\Delta t+1}\right)^N\left\|u_h^{i,\infty} - u_{h0}^i\right\|_\infty + Ch^2\left|\log h\right|^3 \\ &\leq C\left(h^2\left|\log h\right|^3 + \left(\frac{1}{\beta\Delta t+1}\right)^N\right),\end{aligned}$$

which completes the proof. □

Acknowledgement

The authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions which helped them to improve the paper.

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Accepted: 2.07.2018

Rings with strongly algebraically closed lattices

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Abstract. In this article, we prove that if the central idempotents lattice of a Baer ring and the projection lattice of a $*$-Baer ring center and the set of all saturated subsets of a Noetherian regular ring are q'-compact, then they are strongly algebraically closed lattice. Also, for a commutative ring R, it is shown that if the set of idempotents of a Specker R-algebra is q'-compact, then it is a strongly algebraically closed lattice.

Keywords: strongly algebraically closed lattices, equationally Noetherian lattice, $*$-Baer rings, projection lattice.

1. Introduction

Universal algebraic geometry is a branch of mathematics and it deals with the solutions of systems of equations over an arbitrary algebraic structure (algebra for short). The main part of the investigations in this area are due to E. Daniyarova, A. Miasnikov, V. Remeslennikov, and the obtained results can be applied to algebraic geometry over an arbitrary algebra ([6]-[9]). In this paper, for a commutative ring R, we find some relationship among strongly algebraically closed lattices, Baer and $*$-Baer rings, and Specker R-algebra.

In section 2, we recall some basic notations and definitions from universal algebraic geometry. In section 3, it is proved that if central idempotents of a Baer ring and projection lattice of a $*$-Baer ring center are q'-compact, then the central idempotents and the projection lattice are strongly algebraically closed lattice. In addition, it is shown that the set of all saturated subsets of a Noetherian regular ring is also true. Finally, in section 4, we prove that if the set of idempotents of a Baer ring S is q'-compact, then it is a strongly algebraically closed lattice, which S is a Specker R-algebra.

2. Strongly algebraically closed lattices

Let S be a system of equations in an algebra A. The set of all logical consequences of S over A is the radical $Rad_A(S)$, which $V_A(S)$ is the sets of solutions of S in A. In other words, $Rad_A(S)$ is the set of all lattice equations $f \approx g$ such that $V_A(S) \subseteq V_A(f \approx g)$.

Definition 2.1. We say that two lattices A and B are geometrically equivalent, if for any system S, we have $Rad_A(S) = Rad_B(S)$. A lattice A is q'-compact, if it is geometrically equivalent to any of its elementary extensions.

The problem of geometric equivalence was posed in [15]. In [13] this problem was solved for equationally Noetherian groups. Now, in this section of paper we provide examples of geometric equivalence and q'-compact.

Example 2.2. For the first example of geometric equivalence, we have that two irreducible and faithful representations of finite groups over the same field are geometrically equivalent if and only if they are isomorphic. For the second example of geometric equivalence, we know that if two algebras are logically Noetherian, then they are geometrically equivalent if and only if they have the same quasi-identities. Also, consider a field and two its extensions F_1 and F_2. If both F_1 and F_2 are algebraically closed, then they are geometrically equivalent. Then they have the same equational theories. Actually, it is known that even their elementary theories coincide. Therefore, if two algebras are geometrically equivalent in universal logic, then they have the same universal theory.

Example 2.3. For example of q'-compact, it is clear that nontrivial lattices are geometrically equivalent. It is sufficient to prove that any nontrivial lattice L is geometrically equivalent to the two-element lattice $\{0, 1\}$ and finite lattice has no proper elementary extension.

By a Boolean lattice, we mean a complemented distributive lattice. By a Boolean algebra, we mean a Boolean lattice together with the unary operation of complementation (see [4]).

A lattice A is called algebraically closed, if any finite consistent system of equations with coefficients from A, has a solution in A. A system S with coefficients in A is called consistent, if there is an extension B, such that S has a solution in B.

Definition 2.4. A lattice A in a class of lattices is said to be strongly algebraically closed if every system (not necessarily finite) of equations with parameters in A which has a solution in some extension B of A in the class, has already a solution in A.

A lattice A is called equationally Noetherian, if any system of equations with coefficient in A is equivalent with a finite subsystem. If any system of equations over A is equivalent with a finite system then it is said weakly equationally Noetherian. Recall that, equationally and weak equationally Noetherian Boolean algebras (with coefficients) are characterized by Shevlyakov in [17]. Suppose $\mathcal{L}$ is an algebraic language and A is an algebra of type $\mathcal{L}$. If we attach the elements of A as constants to $\mathcal{L}$, then the new language will be denoted by $\mathcal{L}(A)$. We say that the algebra A is finitary equational Noetherian, if every finitary system of equations in the language $\mathcal{L}(A)$ is reducible over A to a finite system. A Boolean algebra A is complete if every subset B of A has a least upper bound $\bigvee B$ and a greatest lower bound $\bigwedge B$. Here, we state the following theorem:

Theorem 2.5 ([14])**.** *Let A be a complete Boolean lattice which is q'-compact. Then A is strongly algebraically closed in the class of distributive lattices.*

3. Baer and $*$-Baer rings

In this section, we present some basic notations and definitions that we use in this paper. For more detailed information, we refer the reader to [1] and [10]. Recall that the study of Baer rings has its roots in functional analysis and various authors have investigated properties of the star order (introduced by Drazin in 1978) on algebras of matrices and of bounded linear operators on a Hilbert space.

Definition 3.1. A commutative ring R is a Baer ring if the annihilator ideal of each subset of R is a principal ideal generated by an idempotent.

Theorem 3.2. *Let B be central idempotents of a Baer ring R. If B is q'-compact, then B is a strongly algebraically closed lattice.*

Proof. Let x and y be arbitrary elements of the idempotents of the center of R. We have that

$$x \cap y = xy, \quad x \cup y = x + y + xy, \quad x' = 1 - x,$$

form a Boolean algebra, which $x \leq y$ is defined by $xy = x$. Thus, B is Baer ring and Boolean algebra. Now, we prove that the center idempotents of a Baer ring form a complete lattice. For doing this work, assume that $S = \{x_i \mid i \in I\}$ subset of B and

$$S^r = \{x \in R \mid \forall s \in S\ (xs = 0)\}.$$

Since the center of a Baer ring is a Baer ring. So, there exists a central idempotent $y \in B$ such that $S^r = yR$. We set $z = 1 - y$ and claim that z is a supremum of the x_i. Suppose $i \in I$ and $t \in R$. If $x_i t = 0$, then $t \in yR$, $yt = t$, $zt = 0$. Now, assume that e is an arbitrary element of R that is idempotent, then $x_i(1-e) = 0$ and we conclude $z \leq e$. We observe that $z = \sup x_i$. For infimum suffices we set $1 - \sup(1 - x_i)$. So, B is a complete Boolean algebra. By applying theorem 2.3, since B is q'-compact and a complete Boolean algebra, then B a strongly algebraically closed lattice (see [1]). $\square$

Kaplansky's axiomatic approach for studying simultaneously the classical equivalence relations on projection lattices is developed in detail, culminating in the construction of a dimension function in that context.

Definition 3.3. An element p of a $*$-ring is called a projection if p is a self-adjoint ($p^* = p$) idempotent ($p^2 = p$).

An associative unital ring R is a $*$-ring (or ring with involution) if there exists an operation $* : R \longrightarrow R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$ for all $x, y \in R$.

Definition 3.4. A $*$-ring is called Baer $*$-ring if the right annihilator of every nonempty subset is generated by a projection.

Obviously, 0 and 1 are projections of any $*$-ring. Projection lattices naturally arise in the context of lattice packings. Now let us recall the following definition of Rickart $*$-ring of [3].

Definition 3.5. A Rickart $*$-ring is a $*$-ring such that the right annihilator of each element is the principal right ideal generated by a projection (a self-adjoint idempotent).

Theorem 3.6. *Let $P(Z)$ be the projection lattice of the center of a $*$-Baer ring. If $P(Z)$ is q'-compact, then $P(Z)$ is a strongly algebraically closed lattice.*

Proof. It is not hard to verify that any any Baer $*$-ring is a Rickart $*$-ring and a Baer ring. Recall that the projections of a Rickart $*$-ring form a lattice, with $e \cup f = f + RP[e(1 - f)]$, $e \cap f = e - LP[e(1 - f)]$ (see [12], Lemma 5.3). We observe that in a Rickart $*$-ring, every central idempotent is a projection. Because if u is a arbitrary idempotent element of center of R, then $u^* = u$. By ([11], Prop. 2.1), $uf = fR$ with a projection f, whence $u = fu = uf = f$. Since, we saw in Theorem 3.2, the central idempotents of a Baer ring form a complete Boolean algebra. So, the projection lattice of the center of $*$-Baer ring is a complete Boolean algebra. Here, $P(Z)$ is q'-compact. By theorem 2.1, $P(Z)$ is a strongly algebraically closed lattice (see [1]). □

A non empty subset F of a commutative ring R with identity is said to be saturated if for any x and y of R we have

$$xy \in F \Longleftrightarrow x, y \in F.$$

Now in the following Corollary we consider relationship between Noetherian regular rings and strongly algebraically closed lattices.

Corollary 3.7. *Suppose that $S(R)$ is the set of all saturated subsets of a Noetherian regular ring R. If $S(R)$ is q'-compact, then $S(R)$ is a strongly algebraically closed lattice.*

Proof. We know that if R is a Noetherian regular ring, then R is a direct sum of fields and will have

$$R = F_1 \oplus \ldots \oplus F_n.$$

Also, the prime ideals are

$$P_j = \Pi_{i \neq j} F_i, j = 1, 2, \ldots, n$$

and $S(R)$ is isomorphic to the Boolean algebra of subsets of $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$. Thus, $S(R)$ is a Boolean algebra. On the other hand, it is clear that the set of all saturated subsets of a commutative ring with identity form a complete lattice. Hence, $S(R)$ is a complete Boolean algebra and q'-compact. From theorem 2.3, we immediately obtain that is a strongly algebraically closed lattice. □

4. Specker R-algebra of a commutative ring

Throughout this section, R will be a commutative ring with 1. A R-algebra is a ring with identity together with a ring homomorphism $f : R \longrightarrow A$ such that the subring $f(R)$ of A is contained within the center of A. Let S be a commutative R-algebra and $Id(S)$ be the set of idempotents of S. We call a nonzero idempotent e of S faithful if for each $a \in R$, whenever $ae = 0$, then $a = 0$. Let B be a Boolean subalgebra of $Id(S)$ that generates S. We say that B is a faithful generating algebra of idempotents of S if each nonzero $e \in B$ is faithful. We recall that an R-algebra S is Specker R-algebra if S is a commutative R-algebra that has a faithful generating algebra of idempotents [5]. To build Specker R-algebras from Boolean algebras we introduce a construction which has its roots in the work of Bergman [2] and Rota [16].

Theorem 4.1. *Let S be a Specker R-algebra. If S is Baer and $Id(S)$ is q'-compact, then $Id(S)$ is a strongly algebraically closed lattice.*

Proof. We know that if S is a commutative R-algebra, then S is a commutative ring with 1, it is well known that the set $Id(S)$ of idempotents of S is a Boolean algebra via the operations

$$e \vee f = e + f - ef, \ \ e \wedge f = ef, \ \ \neg e = 1 - e.$$

In order to prove that $Id(S)$ is a strongly algebraically closed lattice, it remains to show that $Id(S)$ is a complete lattice. In the other words, we show that for every subset $E = \{e_i \,|\, i \in I\}$ of idempotents of S has a largest element. One can easily prove that if $K = \{1 - e_i \,|\, i \in I\}$, then $ann_S(1 - e_i) = e_i S$ and $ann_S(K) = \cap e_i S$. We have S is Baer, so that $ann_S(K) = eS$ for some $e \in Id(S)$. Now, we will prove $e = \wedge e_i$. It is easy to see $e \in ann_S(K)$, we have $ee_i = e$ and then $e \leq e_i$. But this says e is a lower bound of the e_i. First note that if $f \in Id(S)$ be a lower bound of the e_i, then $fe_i = f$, as a result $(1 - e_i)f = 0$. Therefore, $f \in ann_S(K) = eS$. This shows that $fe = f$, so $f \leq e$. Thus, $e = \wedge_i e_i$. It can be observe that, $Id(S)$ is a complete Boolean algebra and is q'-compact. By theorem 2.3, $Id(S)$ is a strongly algebraically closed lattice (see [1]). □

Corollary 4.2. *Let R be indecomposable and S be a Specker R-algebra. If S is Baer and $Id(S)$ is q'-compact, then $Id(S)$ is a strongly algebraically closed lattice.*

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Accepted: 6.07.2018

On the n^*- and γ_n^*- complete fuzzy hypergroups

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Abstract. We extend the fuzzy approach of algebraic hyperstructures to the context of complete fuzzy hypergroups. In this paper we introduce the classes of n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups which they are generalizations of two important classes of hypergroups, also we find some properties of them. Finally, we study 2^*- complete fuzzy hypergroups and give some properties and examples in this regard.

Keywords: fuzzy hypergroup, n^*- complete fuzzy hypergroup, γ_n^*- complete fuzzy hypergroup.

1. Introduction

The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures [3, 10]. In this paper we introduce two types of fuzzy hypergroups: n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups. They are generalizations of two important classes of hypergroups: n^*- complete hypergroups [4] and γ_n^*- complete hypergroups [5]. Notice in a fuzzy hypergroup the elements are combined by a fuzzy hyperoperations, while in a hypergroup, the elements are combined by a crisp hyperoperations. This idea was continuated

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by Sen, Ameri and Chowdhury in [10] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relations α^* and γ^* on fuzzy hypersemigroups are studied in [1] and [9]. In this paper, by using this relations, we introduce and study two types of fuzzy hypergroups and we present some properties of them. In the last section, we study in particular the 2^*- complete fuzzy hypergroups and we show that some classes of fuzzy hypergroups (fuzzy complete hypergroups, fuzzy join spaces, fuzzy canonical hypergroups, fuzzy steiner hypergroups) are 2^*- complete.

2. Preliminaries

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0,1]$ and $Supp(\mu) = \{x \in S : \mu(x) > 0\}$.We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S$, $(\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a, b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergroupoid*.

Definition 2.1 ([10])**.** A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S and all $r \in S$

$$(a \circ \mu)(r) = \begin{cases} \bigvee\limits_{t \in S} ((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee\limits_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

Definition 2.2. Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$. Then we define $\mu \circ \nu$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.

Definition 2.3 ([10])**.** A fuzzy hypersemigroup $(S, \circ)$ is called *fuzzy hypergroup* if $x \circ S = S \circ x = \chi_S$, for all $x \in S$, where χ_S is characteristic function of S.

Example 2.4. Consider a fuzzy hyperoperation $\circ$ on a non-empty set S by $a \circ b = \chi_{\{a,b\}}$, for all $a, b \in S$,. Then $(S, \circ)$ is a fuzzy hypersemigroup and fuzzy hypergroup as well.

Definition 2.5 ([10])**.** Let ρ be an equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, we define two relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ on $F^*(S)$ as follows: for

$\mu, \nu \in F^*(S)$; $\mu\overline{\rho}\nu$ if $\mu(a) > 0$ then there exists $b \in S$ such that $\nu(b) > 0$ and $a\rho b$, also if $\nu(x) > 0$ then there exists $y \in S$, such that $\mu(y) > 0$ and $x\rho y$. $\mu\overline{\overline{\rho}}\nu$ if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, $x\rho y$.

Definition 2.6. An equivalence relation ρ on a fuzzy hypersemigroup $(S, \circ)$ is said to be *(strongly) fuzzy regular* if $a\rho b, a'\rho b'$ implies $a \circ a' \ \overline{\rho} b \circ b' (a \circ a' \ \overline{\overline{\rho}} \ b \circ b')$.

Definition 2.7 ([1])**.** Let $(S, \circ)$ be a fuzzy hypergroup. We define the relation α on S in the following way: $\alpha = \cup_{n \geq 1} \alpha_n$, where $\alpha_1 = \{(x, x) \mid x \in S\}$ and for every $n > 1$ and $(a, b) \in S^2$

$a\alpha_n b \Longleftrightarrow \exists x_1, ..., x_n \in S (n \in \mathbb{N})$: $(x_1 \circ ... \circ x_n)(a) > 0$ and $(x_1 \circ ... \circ x_n)(b) > 0$.

It is clear that α is symetric and reflexive. We take α^* to be the transitive closure of α. Then α^* is an equivalence relation on S.

Theorem 2.8 ([1])**.** *The relation α^* is the smallest equivalence relation on a fuzzy hypergroup $(S, \circ)$ such that S/α^* is a group.*

Definition 2.9 ([9])**.** Let $(S, \circ)$ be a fuzzy hypergroup. The *commutative fundamental relation* on $(S, \circ)$ is the smallest equivalence relation ρ on S such that the quotient structure $(S/\rho, \oplus)$ is a commutative group.

Let $(S, \circ)$ be a fuzzy hypersemigroup. We define the relation γ on S in the following way, $\gamma = \bigcup_{n \geq 1} \gamma_n$ where $\gamma_1 = \{(s, s) : s \in S\}$ and for every $n \geq 2$, $a\gamma_n b$ if $\exists x_1, ..., x_n \in S (n \in \mathbb{N}), \exists \sigma \in S_n$: $(x_1 \circ \ldots \circ x_n)(a) > 0$ and $(x_{\sigma_1} \circ \ldots \circ x_{\sigma_n})(b) > 0$.

It is clear that γ is symetric and reflexive. We take γ^* to be the transitive closure of γ. Then γ^* is an equivalence relation on S.

Proposition 2.10 ([9])**.** *The relation γ^* is a strongly fuzzy regular relation.*

Corollary 2.11 ([9])**.** *Let S be a fuzzy hypersemigroup. Then the quotient S/γ^* is a commutative semigroup.*

3. n^*- and γ_n^*- complete fuzzy hypergroups

For every $n \in \mathbb{N}$ we will write α_n^* and γ_n^* to denote the transitive closure of the relations α_n and γ_n.

Definition 3.1. A fuzzy hypergroup S is said to be $n-$ complete if for every $z_1, ..., z_n, s, t \in S$, we have the following condition:

$$t \in Supp(\prod_{i=1}^{n} z_i), t\alpha s \Rightarrow s \in Supp(\prod_{i=1}^{n} z_i).$$

Proposition 3.2. (i) $\alpha_n \subseteq \alpha_{n+1}, \forall n \in \mathbb{N}$;

(ii) $\alpha_n^* \subseteq \alpha_{n+1}^*, \forall n \in \mathbb{N}$.

Proof. (i) $\forall(x,y)\in S^2, x\alpha_n y \Rightarrow \exists(z_1,...,z_n)\in S^n$:

$$(\prod_{i=1}^{n} z_i)(x) > 0 \text{ and } (\prod_{i=1}^{n} z_i)(y) > 0.$$

Since

$$0 < (\prod_{i=1}^{n} z_i)(x) = \bigvee_{r\in S}[(z_1...z_{n-1})(r)\wedge(rz_n)(x)],$$

there exists $r\in S$ such that $(z_1...z_{n-1})(r)>0$ and $(rz_n)(x)>0$. Also since S is a fuzzy hypergroup $\exists(t_1,t_2)\in S^2$ such that $(t_1t_2)(z_n)>0$. Now

$$(z_1....z_{n-1}t_1t_2)(x) = \bigvee_{p,q\in S}[(\prod_{i=1}^{n-1} z_i)(p)\wedge(t_1t_2)(q)\wedge(pq)(x)].$$

Let $p=r$ and $q=z_n$, then $(z_1....z_{n-1}t_1t_2)(x)>0$. In the similar way, we can show $(z_1....z_{n-1}t_1t_2)(y)>0$, Therefore $x\alpha_{n+1}y$ and $\alpha_n\subseteq\alpha_{n+1}$.

(ii) It follows from (i). □

Proposition 3.3. $\forall(a,b,x)\in S^3$, $a\alpha_n^*b \Rightarrow (ax)\overline{\overline{\alpha_{n+1}^*}}(bx), (xa)\overline{\overline{\alpha_{n+1}^*}}(xb)$.

Proof. If $a\alpha_n^*b$, then $\exists(z_1,...,z_m)\in S^m : a=z_0\alpha_n z_1\alpha_n....z_m\alpha_n z_{m+1}=b$. Thus $\forall j, 0\le j\le m, \exists(\alpha_1^j,\alpha_2^j,...,\alpha_n^j)\in S^n : \{z_j,z_{j+1}\}\subseteq Supp(\prod_{i=1}^n \alpha_i^j)$. Therefore $\forall j\in\{0,1,...,m\}, \forall t\in S$ such that $(z_jx)(t)>0$, since $(\alpha_1^j...\alpha_n^jx)(t) = \bigvee_{p\in S}[(\prod_{i=1}^n \alpha_i^j)(p)\wedge(px)(t)]$, we let $p=z_j$ and obtain $t\in Supp(\alpha_1^j...\alpha_n^jx)$. Also for every $s\in S$, if $(z_{j+1}x)(s)>0$, then $s\in Supp(\alpha_1^j...\alpha_n^jx)$. Therefore $(z_jx)\overline{\overline{\alpha_{n+1}}}(z_{j+1}x)$ and so $(ax)\overline{\overline{\alpha_{n+1}^*}}(bx)$. In a analogous way, we can prove the rest. □

Proposition 3.4. $\alpha_n^*=\alpha_{n+1}^* \Rightarrow \alpha_{n+1}^*=\alpha_{n+2}^*$.

Proof. It is sufficient to prove that $\alpha_{n+2}\subseteq\alpha_{n+1}^*$.

If $\{x,y\}\subseteq Supp(\prod_{i=1}^{n+2} z_i)$, then there exist $\{t_1,t_2\}\subseteq Supp(\prod_{i=1}^{n+1} z_i)$ such that $(t_1z_{n+2})(x)>0$ and $(t_2z_{n+2})(y)>0$. Obviously $t_1\alpha_{n+1}t_2$ and $t_1\alpha_n^*t_2$. Thus from 3.3, $(t_1z_{n+2})\overline{\overline{\alpha_{n+1}^*}}(t_2z_{n+2})$. It implies that $x\alpha_{n+1}^*y$. □

Proposition 3.5. *If there exists $n\in\mathbb{N}$ such that $\alpha_n^*=\alpha_{n+1}^*$ then $\alpha=\alpha_n^*$.*

Proof. It follows from 3.4 and Definition of the relation α. □

Definition 3.6. A fuzzy hypergroup S is said to be n^*- complete if there exists $n\in\mathbb{N}$ such that $\alpha_n^*=\alpha$ and $\alpha_n^*\neq\alpha_{n-1}^*$.

Remark 1. A fuzzy hypergroup S is n^*- complete if and only if $\alpha_{n+1}\subseteq\alpha_n^*\neq\alpha_{n-1}^*$.

Remark 2. S is n^*- complete if and only if n is the minimum integer such that S/α_n^* is a group.

Proposition 3.7. *Every finite fuzzy hypergroup is n^*- complete.*

Proof. It follows from $\alpha_1^* \subseteq \alpha_2^* \subseteq ... \subseteq \alpha_n^* \subseteq ....$ □

Remark 3. For every finite fuzzy hypergroup, there exists $m \in \mathbb{N}$ such that $\alpha = \alpha_m$.

Proposition 3.8. *If S is n-complete fuzzy hypergroup then $\exists m \leq n$ such that S is m^*- complete.*

Definition 3.9. A fuzzy hypergroup S is said to be γ_n- complete if for every $z_1, ..., z_n, s, t \in S$, and every $\sigma \in S_n$ the following condition holds:

$$t \in Supp(\prod_{i=1}^{n} z_{\sigma(i)}), t\gamma s \Rightarrow s \in Supp(\prod_{i=1}^{n} z_i).$$

Proposition 3.10. (i) $\gamma_n \subseteq \gamma_{n+1}, \forall n \in \mathbb{N}$;
(ii) $\gamma_n^* \subseteq \gamma_{n+1}^*, \forall n \in \mathbb{N}$.

Proof. (i) If $x\gamma_n y$, then $\exists(z_1, ..., z_n) \in S^n, \exists\sigma \in S_n$:

$$(\prod_{i=1}^{n} z_i)(x) > 0 \text{ and } (\prod_{i=1}^{n} z_{\sigma(i)})(y) > 0.$$

Since S is a fuzzy hypergroup, so $\exists(t_1, t_2) \in S^2$ such that $(t_1t_2)(z_n) > 0$. Let $z_i\prime = z_i$, for $1 \leq i \leq n-1$ and $z_n\prime = t_1, z_{n+1}\prime = t_2$. Thus $(\prod_{i=1}^{n+1} z_i\prime)(x) > 0$. Let $\sigma(k) = n$, now since $(z_{\sigma(1)}...z_{\sigma(k)}z_{\sigma(k+1)}...z_{\sigma(n)})(y) = \bigvee_{p,q\in S}[(z_{\sigma(1)}...z_{\sigma(k)})(p) \wedge (z_{\sigma(k+1)}...z_{\sigma(n)})(q) \wedge (pq)(y)]$, there exist $p, q \in S$ such that $(z_{\sigma(1)}...z_{\sigma(k)})(p) > 0$, $(z_{\sigma(k+1)}...z_{\sigma(n)})(q) > 0$ and $(pq)(y) > 0$. But $(z_{\sigma(1)}...z_{\sigma(k)})(p) > 0$ implies that there exists $r \in S$ such that $(z_{\sigma(1)}...z_{\sigma(k-1)})(r) > 0$ and $(rz_n)(p) > 0$. Now, $(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2z_{\sigma(k+1)}...z_{\sigma(n)})(y) = \bigvee_{p',q'\in S}[(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2)(p') \wedge (z_{\sigma(k+1)}...z_{\sigma(n)})(q') \wedge (p'q')(y)]$.

Let $p' = p$ and $q' = q$. Since

$$(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2)(p) = \bigvee_{r',s'\in S} [(z_{\sigma(1)}...z_{\sigma(k-1)})(r') \wedge (t_1t_2)(s') \wedge (r's')(p)].$$

Let $r' = r$ and $s' = z_n$. Therefore $(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2z_{\sigma(k+1)}...z_{\sigma(n)})(y) > 0$ and $x\gamma_{n+1}y$.

(ii) It follows from (i). □

Proposition 3.11. $\forall(a, b, x) \in S^3$,

$$a\gamma_n^* b \Rightarrow (ax)\overline{\overline{\gamma_{n+1}^*}}(bx) \text{ and } (xa)\overline{\overline{\gamma_{n+1}^*}}(xb).$$

Definition 3.12. A fuzzy hypergroup S is said to be γ_n^*- complete if there exists $n \in \mathbb{N}$ and n is the smallest integer such that $\gamma_n^* = \gamma$ and $\gamma_n^* \neq \gamma_{n-1}^*$.

We know $\alpha^* = \gamma^*$ in commutative fuzzy hypergroups, thus we obtain the following:

Proposition 3.13. *A commutative fuzzy hypergroup S is γ_n^*- complete if and only if S is n^*- complete fyzzy hypergroup.*

Proposition 3.14. *S is γ_n^*- complete fuzzy hypergroup if and only if S/γ_n^* is an abelian group.*

Proposition 3.15. *If S is γ_n- complete fuzzy hypergroup then $\exists m \leq n$ such that S is γ_m^*- complete.*

Proof. If S is γ_n- complete, then $\gamma_n = \gamma$, so $\gamma_n^* = \gamma$ and there exists $m \leq n$ such that $\gamma_m^* = \gamma$ and $\gamma_{m-1}^* \neq \gamma_m^*$. □

Proposition 3.16. *Every finite fuzzy hypergroup is γ_n^*- complete.*

Let $\phi : S \to S/\alpha^*$ be the canonical projection, then we denote $\omega_S = \phi^{-1}(1_{S/\alpha^*})$.

Proposition 3.17. *We have:*
(i) If $\forall (v, w) \in \omega_S^2$, $v\alpha_n w$ *then* $\alpha = \alpha_{n+1}$;
(ii) If $\forall (v, w) \in \omega_S^2$, $v\alpha_n^* w$ *then* $\alpha = \alpha_{n+1}^*$.

Proof. (i) If $x\alpha y$ then $\exists (v, w) \in \omega_S^2$ such that $(xv)(y) > 0$ and $(xw)(x) > 0$. But for the hypothesis $v\alpha_n w$ and so $(xv)\overline{\overline{\alpha_{n+1}}}(xw)$, whence $x\alpha_{n+1}y$, therefore $\alpha \subseteq \alpha_{n+1}$.
(ii) It follows from (i) and 3.3. □

Remark 4. Both of two parts of last proposition are verifiable, when we use γ instead of α.

Corollary 3.18. *If* $\forall (u, w) \in \omega_S^2$, $u\alpha_n^* w$ *and* $\exists (u', w') \in \omega_S^2$ *such that* $u' \notin \alpha_{n-1}^*(w')$, *then S is n^*-complete or $(n+1)^*$-complete.*

Remark 5. Both of the two possibilities of corollary are verifiable, as the following examples:

Example 3.19. Let $(S, \circ)$ be a fuzzy hypergroup, where is defined by:

$$\begin{aligned}
&(a \circ a)(a) = 0.3,\ \ (b \circ a)(b) = (a \circ b)(b) = 0.1,\\
&(a \circ c)(c) = (c \circ a)(c) = (c \circ b)(c) = 0.2,\\
&(b \circ b)(a) = (b \circ b)(b) = (b \circ c)(c) = 0.4,\ \ (c \circ c)(a) = (c \circ c)(b) = 0.5
\end{aligned}$$

The remaining binary products are zero. In this case it is easy to verify that S is n^*-complete.

Example 3.20. Let $(S' = \{a, b, c, d\}, \circ)$ be a fuzzy hypergroup with the hyperoperation is defined by:

$$\begin{aligned}
&(a \circ a)(a) = (a \circ b)(b) = (a \circ c)(c) = (a \circ d)(d) = 0.1 = (b \circ a)(b) \\
&\qquad = (c \circ a)(c) = (d \circ a)(d), \\
&(b \circ b)(a) = (b \circ c)(d) = (b \circ d)(c) = 0.2, \\
&(c \circ c)(a) = (c \circ c)(b) = (c \circ d)(a) = 0.4, \\
&(c \circ d)(b) = (c \circ b)(d) = (d \circ b)(c) = 0.3, \\
&(d \circ c)(a) = (d \circ c)(b) = (d \circ d)(a) = (d \circ d)(b) = 0.5.
\end{aligned}$$

It is easy to verify that S' is $(n+1)^*$-complete.

4. On 2^*-complete fuzzy hypergroup

Lemma 4.1. *If $\mathcal{A}$ denotes the family of the fuzzy hyperproducts of two elements of S and there exists a family $\mathcal{M} = \{M_1, M_2, \ldots, M_p\} \subseteq \mathcal{A}$ such that $M_i \wedge M_{i+1} \neq 0$ $(i \in \{1, 2, \ldots, p-1\})$ and $\bigvee_{i=1}^{p} M_i = \chi_S$ then S is 2^*-complete and $\omega_S = S$.*

Proof. It is sufficient to prove that $\alpha_3 \subseteq \alpha_2^*$. If $x\alpha_3 y$, then $\exists(z_1, z_2, z_3) \in S^3$ such that $(\prod_{i=1}^{3} z_i)(x) > 0$ and $(\prod_{i=1}^{3} z_i)(y) > 0$ since $(x, y) \in S^2$ and $\chi_S = \bigvee_{i=1}^{p} M_i$. Thus:

A: $\exists M_i$ such that $M_i(x) > 0$ and $M_i(y) > 0$ then $x\alpha_2^* y$.

B: $\exists M_i$ such that $M_i(x) > 0$ and $\exists M_{i+1}$ such that $M_{i+1}(y) > 0$, since $M_i \wedge M_{i+1} \neq 0 \rightarrow \exists t \in S$ such that $M_i(t) > 0$ and $M_{i+1}(t) > 0$ thus $x\alpha_2 t$ and $t\alpha_2 y$ then $x\alpha_2^* y$.

C: $\exists M_i$ such that $M_i(x) > 0$ and $\exists M_j$ $(j \neq i,\ j \neq i+1)$ such that $M_j(y) > 0$ since $M_i \wedge M_{i+1} \neq 0$ then there exists $t_1 \in S$; $M_i(t_1) > 0$ and $M_{i+1}(t_1) > 0$ then $x\alpha_2 t_1$ and since $M_{i+1} \wedge M_{i+2} \neq 0$ thus there exists $t_2 \in S$; $M_{i+1}(t_2) > 0$ and $M_{i+2}(t_2) > 0$ then $t_1\alpha_2 t_2$. So as a consequence one obtains $t_j\alpha_2 y$ and so $x\alpha_2^* y$. $\square$

Definition 4.2. An equivalence relation R on a fuzzy hypergroup S is called fuzzy feebly regular to the right if for every $x, y \in S$ and $\forall a \in S$, $\exists(u, v) \in S^2$ such that $(x \circ a)(u) > 0$ and $(y \circ a)(v) > 0$ and uRv. Analogously, we define the fuzzy feebly regularity to the left. An equivalence fuzzy feebly regular to the right and to the left is called fuzzy feebly regular.

Proposition 4.3. *Let $S = (S, \circ)$ be a fuzzy hypergroup. The following statements are equivalent:*
(i) *S is 2^*-complete;*
(ii) *α_2^* is fuzzy strongly regular;*

(*iii*) α_2^* *is fuzzy regular;*
(*iv*) α_2^* *is fuzzy feebly regular;*
(*v*) $\forall(x,y) \in S^2$, $x\alpha_2^*y \Longrightarrow \forall a \in S$, $[\alpha_2^*(x \circ a) \cap \alpha_2^*(y \circ a) \neq \emptyset]$ *and* $[\alpha_2^*(a \circ x) \cap \alpha_2^*(a \circ y) \neq \emptyset]$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate.

We show that $(iv) \Rightarrow (v)$. Since $\exists\alpha \in S$; $(x \circ a)(\alpha) > 0$ and $\exists\gamma \in S$; $(y \circ a)(\gamma) > 0$ and $\alpha\alpha_2^*\gamma$, it follows that $\alpha_2^*(\alpha) \subseteq \alpha_2^*(x \circ a)$ and $\alpha_2^*(\gamma) \subseteq \alpha_2^*(y \circ a)$. But $\beta_2^*(\alpha) = \beta_2^*(\gamma)$ then we obtain the implication.

We prove now $(v) \Rightarrow (ii)$. Let $x\alpha_2^*y$, certainly $\forall a \in S$, $\exists\alpha \in S$; $(x \circ a)(\alpha) > 0$ and $\exists\gamma \in S$; $(y \circ a)(\gamma) > 0$ and $\alpha\alpha_2^*\gamma$. Thus it follows that $\alpha_2^*(x \circ a) = \alpha_2^*(\alpha) = \alpha_2^*(\gamma) = \alpha_2^*(y \circ a)$ whence $\forall z \in S$; $(x \circ a)(z) > 0$ and $\forall t \in S$; $(y \circ a)(t) > 0$, $z\alpha_2^*t$. It shows that fuzzy strong regularity to the right. Similarly it proves fuzzy strong regularity to the left.

Finally, we prove that $(ii) \Rightarrow (i)$, being $\alpha_2^* \subseteq \alpha$ and considering that α is the smallest fuzzy strongly regular equivalence on a fuzzy hypergroup S. □

Corollary 4.4. *In every fuzzy hypergroup S, the following conditions, are equivalent:*
(*i*) S *is* 2^**-complete;*
(*ii*) $(S/\alpha_2^*, \otimes)$ *is a group;*
(*iii*) $(S/\alpha_2^*, \otimes)$ *is a hypergroup.*

Proposition 4.5. *In every fuzzy hypergroup $S = (S, \circ)$ we have*

$$\alpha_2^* = \alpha \Longleftrightarrow \forall(x,y) \in S^2, \exists z \in S; \alpha_2^*(x) \circ \alpha_2^*(y) \subseteq \alpha_2^*(z).$$

Proof. Let $\mathcal{C}(a)$ be the complete closure of a in S [1]. The right implication is a consequence of the fact that $\forall a \in S$, $\alpha_2^*(a) = \mathcal{C}(a)$. Now we prove that α_2^* is fuzzy strongly regular. If $x\alpha_2^*y$ then $\alpha_2^*(x) = \alpha_2^*(y)$ and so $\forall a \in S$, $\alpha_2^*(x) \circ \alpha_2^*(a) = \alpha_2^*(y) \circ \alpha_2^*(a)$. For the hypothesis, there exists an element $z \in S$ such that $\alpha_2^*(x) \circ \alpha_2^*(a) = \alpha_2^*(y) \circ \alpha_2^*(a) \subseteq \alpha_2^*(z)$. Thus $\forall t \in S$; $(x \circ a)(t) > 0$ and $\forall u \in S$; $(y \circ a)(u) > 0$. We obtain $\{t, u\} \subseteq \alpha_2^*(z)$ and finally $t\alpha_2^*u$. In the analogous way we can prove the strong regularity to the left. □

Definition 4.6. A fuzzy hypergroup $S = (S, \circ)$ is called 1-fuzzy hypergroups if w_S is a singleton.

Proposition 4.7. *If S is* 1*-fuzzy hypergroup, then S is* 2^**-complete.*

Proof. Let w_S be a singleton $\{e\}$, we have that the classes modulo α are the fuzzy hyperproducts $e \circ a$, $\forall a \in S$. It follows at once that $\alpha = \alpha_2 = \alpha_2^*$. □

Definition 4.8. A fuzzy hypergroup S is called fuzzy steiner hypergroup if $\forall(x,y) \in S^2$: $(x \circ y)(x) > 0$ and $(x \circ y)(y) > 0$.

Proposition 4.9. *Every fuzzy steiner hypergroup is* 2^**-complete.*

Proof. Since in such fuzzy hypergroup $\forall(x,y)\in S^2$, we have $(x\circ y)(x)>0$ and $(x\circ y)(y)>0$, thus $x\alpha_2 y$. Therefore $\alpha=\alpha_2=\alpha_2^*$. $\square$

Definition 4.10 ([7])**.** If $\circ: S\times S\longrightarrow F^*(S)$ is a fuzzy hypercomposition, then S is called mimic fuzzy hypergroup (fuzzy$_M$-hypergroup), if the following two axioms are valid:

i. $(a\circ b)\circ c=a\circ(b\circ c)$, $\forall(a,b,c)\in S^3$ (associativity)

ii. $a/b\neq 0_H$ and $a\setminus b\neq 0_H$, $(a,b)\in S^2$.

Proposition 4.11 ([7])**.** *In a fuzzy$_M$-hypergroup S, it holds that $a\circ b\neq 0_S$ ($\forall(a,b)\in S^2$).*

Example 4.12. Every fuzzy$_M$-hypergroup S is a fuzzy steiner hypergroup. Thus S is 2^*-complete.

Definition 4.13. A commutative fuzzy hypergroup S is called a fuzzy join space if $\forall(a,b,c,d)\in S^4$,

$$a/b\wedge c/d\neq 0\implies (a\circ d)\wedge(b\circ c)\neq 0.$$

Proposition 4.14. *Every fuzzy join space S is 2^*-complete.*

Proof. Suppose $S=(S,\circ)$ is a fuzzy join space which is not group. We prove that $\alpha_3\subseteq\alpha_2^*$. If $t\alpha_3 u$ then $\exists(z_1,z_2,z_3)\in S^3$ such that $(z_1\circ z_2\circ z_3)(t)>0$ and $(z_1\circ z_2\circ z_3)(u)>0$ and $z_1\circ z_2\circ z_3=z_1\circ z_3\circ z_2$. It follows that $\exists(a_1,a_2)\in S^2$, such that $(z_1\circ z_2)(a_1)>0$ and $(z_1\circ z_2)(a_2)>0$ and $\exists(b_1,b_2)\in S^2$; $(z_1\circ z_3)(b_1)>0$ and $(z_1\circ z_3)(b_2)>0$. Therefore $(a_1\circ z_3)(t)>0$ and $(b_1\circ z_2)(t)>0$ too $(a_2\circ z_3)(u)>0$ and $(b_2\circ z_2)(u)>0$. Therefore $a_1/z_2\wedge b_2/z_3\neq 0$ and since S is a fuzzy join space, one obtains that $(a_1\circ z_3)\wedge(b_2\circ z_2)\neq 0$. Therefore, $\exists w\in S$, such that $(a_1\circ z_3)(w)>0$ and $(b_2\circ z_2)(w)>0$ and since $(a_1\circ z_3)(t)>0$ and $(b_2\circ z_2)(u)>0$ whence $t\alpha_2^* u$. $\square$

Definition 4.15. A fuzzy hypergroup S is fuzzy regular if it has at least one identity and every elements has at least one inverse.

Definition 4.16. A fuzzy regular hypergroup is said to be fuzzy reversible if, denoting $\forall x\in S$ with $i(x)$ the set of the inverses of x, one has $\forall(a,b,x)\in S^3$

$$\begin{aligned}(b\circ x)(a)>0 &\Longrightarrow \exists x'\in i(x);\ (a\circ x')(b)>0,(x\circ b)(a)>0\\ &\Longrightarrow \exists x''\in i(x);(x''\circ a)(b)>0.\end{aligned}$$

Definition 4.17. A fuzzy commutative reversible hypergroup is called canonical if it has a scalar identity and $\forall x$, $i(x)$ is a singleton.

Corollary 4.18. *Canonical fuzzy hypergroups are 2^*-complete.*

Example 4.19. We give now a fuzzy commutative regular hypergroup S with scalar identity which is not reversible so is not 2^*-complete.

$$(d \circ d)(b) = (a \circ a)(a) = (a \circ b)(b) = (a \circ c)(c) = (a \circ d)(d) = 0.1$$
$$(d \circ c)(d) = (b \circ a)(b) = (c \circ a)(c) = (d \circ a)(d) = (b \circ b)(a) = 0.2$$
$$(d \circ c)(a) = (b \circ c)(d) = (b \circ d)(c) = (c \circ b)(d) = (d \circ b)(c) = 0.3$$
$$(d \circ d)(c) = (c \circ c)(b) = (c \circ c)(c) = 0.4$$
$$(c \circ d)(a) = (c \circ d)(d) = 0.5.$$

The remaining binary products are zero. S is not reversible, since $(c \circ d)(d) > 0$ and $(d \circ d^{-1})(c) = 0$, $(i(d) = \{a, d\})$, one has $\alpha_2^*(a) = \{a, d\}$, $\alpha_2^*(b) = \{b, c\}$ and so $\alpha_2^* \neq \alpha$. Thus S is not 2^*-complete.

Conclusions. We introduced the concepts of n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups by using fundamental relation and commutative fundamental relation of a fuzzy hypergroup and we determined some properties of them. We will study about these topics in fuzzy hyperrings.

Acknowledgements

The author is grateful to the referee(s) for reading the paper

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Accepted: 3.07.2018

Bifurcations of Liouville tori of generalized two-fixed center problem

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Abstract. We study the topological type of the level sets of generalized two-fixed center problem. Furthermore, all generic bifurcation of the level sets are presented. We determine the families of periodic solutions by giving the solution in terms of Jacobi's elliptic functions. Finally, the phase portrait is studied, and the singular points are classified.

Keywords: generalized two-fixed center, Hamilton-Jacobi's equations, bifurcations of Liouville tori, topology of the level sets, momentum maps, periodic solution, elliptic functions, phase portrait.

1. Introduction

The first one who investigated the problem of two fixed centers was Euler [11]. From that time, many authors were interested in studying the extensions and generalization of this problem. Lagrange [21] has extended the Euler's solution to the three-dimensional case of motion and made some generalization. Jacobi [15] generalized the three-dimensional problem of two fixed centers to the case of arbitrary number of additional attracting centers located at equal distances from each other on a single straight line with two basic centers of attraction, as well as to the case of presence of an additional force of arbitrary nature acting parallel to this straight line. He has also demonstrated the integrability of the problem under his generalizations.

Thereafter, a large number of papers have been written to generalize Euler problem such as Liouville [22] and Hiltebeitel [14]. In their papers, the integrable cases were determined in the restricted three-body problem with neglecting centrifugal and Coriolis forces in various combinations. In 1901 Darboux [7] presented another generalization of the Eulers problem in the case of a planar motion by introducing complex-conjugated masses and an imaginary distance

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between them. The potential of attraction always assumes real values in this case, and the solution of a problem is also reduced to quadratures. The generalized problem of two fixed centers has achieved a large amount of applications as follows, in [23], a brief of publications on the problem of two fixed centers was given, included its generalizations and astronomical applications. Darboux model was studied by Aksensov et al. [1] and they proved that with a convenient selection of free parameters, such a model can be used for constructing an analytical theory of satellite motion in the gravitational field of an oblate planet. The motion of a star in the stationary stellar system with an axisymmetric nucleus was studied by Kaisin for testing the motion of a spacecraft in the field of an oblate planet with regard to the thrust force of an engine, and to find the solutions to some other problems of astrodynamics [16, 17]. Koman [19, 20] applied the asymmetrical version of the three-dimensional problem of two fixed centers with real masses for studying the motion of artificial satellites of the Moon. Moreover, the model of three Newtonian fixed attracting centers with material masses was used by Arazov to approximate the gravitational potential of Jupiter [4]. Maciejewski and Maria Przybylska [24] studied the non-integrability of the generalized two fixed centers problem. The integrable spherical of the Darboux potential in the planar motion of a particle in the field of two and four fixed Newtonian centers was studied in [3]. Moreover, all results can be applied in theory of artificial Earth satellites. In [33], the bifurcation diagrams for planar motion were analyzed. The motion in 3-dimensions with arbitrary values of the angular motion was studied. Bifurcations in the topology of energy surfaces were discussed in terms of relative equilibria. They also calculated the monodromy matrices from an attempt to construct smooth actions from the natural ones. The quantum version of the two center problem was discussed in symmetric and asymmetric cases. New applications of the generalized two-fixed center problem are introduced in [18]. Thereafter, they used a symmetric version of the problem and the external field of gravitation is approximated. Varvoglis et al. [26] determined the trajectories according to an exhaustive scheme, comprising both periodic and quasi-periodic ones. They also identified the collision orbits and found that collision orbits are of complete measure in a 3-D sub manifold of the phase space while asymptotically collision orbits are of complete measure in the 4-D phase space.

The study of the bifurcation for the problem of two fixed centers was presented by [27],[28], [29]. She constructed the bifurcation set on the plane of values of integrals of motion, classification of domains of possible motion on the configurational space in spaces of constant curvature on a sphere and in Lobachevsky's space, while [30] studied the topological analysis of the two-center problem on the two-dimensional sphere. In [31],[32] the topology of isoenergy surfaces in the integrable problems of celestial mechanics in spaces of constant curvature was introduced, the topological invariants were constructed. El-Sabaa et al. [10] studied the complete description of the real phase topology of a two-fixed center problem.

We will give in this article the qualitative analysis of the generalized problem of two fixed centers by describing the bifurcation of the problem. To describe the real phase, we review the Liouville- Arnold theorem [5] which was stated that the phase space trajectories of a Hamiltonian system with n-degrees of freedom and possessing n-integrals of motion lie on n-dimensional manifold which is topologically equivalent to an n-torus and the regular tori (foliate) the bulk of $2n$-dimensional phase space and its $2n-1$ dimensional energy surface. Fomenko [12] proposed a new approach in the qualitative theory of integrable Hamiltonian system, given the separation of the system, the determination of critical values of the energy momentum map boils down to the analysis of the discriminant surface of a polynomial. Moreover, the hypersurfaces of the constant of energy in the space of the variables of separation can be determined and geometrically represented tori. In quantum mechanics the problem explains non-electron moving in the electric field of two nuclei may have the same charges (symmetric case) or different charges (an asymmetric case) [25].

The current paper is organized as follows, a short summary of the problem is given in Section 2. In Section 3, the topological analysis of the real invariant manifold of the system was studied by using Fomenko's theory [6]. Moreover, the bifurcation diagram of the problem is determined and the complete description of the topology of the level sets of the first integrals was given. The aim of Section 4 is to give the families of periodic solutions, this solution is given in terms of Jacobi's elliptic functions and when the bifurcations of Liouville tori take place, the level set becomes degenerate. The phase portrait is given in Section 5. A concluding remark is given in the last section.

2. Separation of the problem

The classical problem of two fixed centers consists as is known, in the study of the motion of a mass point under the attraction of two fixed mass points P_1 and P_2. Let the coordinate system be $Oxyz$ whose origin is at the center of mass P_1 and P_2 where the line P_1P_2 lies on the z-axis as shown in Figure 1. Then, the equations of motion of the mass points can be written in the form

$$\frac{d^2x}{dt^2} = \frac{\partial V}{\partial x},\quad \frac{d^2y}{dt^2} = \frac{\partial V}{\partial y},\quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}, \tag{1}$$

where the generalized potential function V is defined by

$$V = f(\frac{M_1}{r_1} + \frac{M_2}{r_2}), \tag{2}$$

where

$r_1 = \sqrt{x^2 + y^2 + (z - a_1)^2},\ r_2 = \sqrt{x^2 + y^2 + (z - a_2)^2}$,

M_1 and M_2 are the masses of P_1 and P_2, a_1 and a_2 are the distances of these points from the coordinate origin.

Introducing the quantity a which is represented the distance between P_1 and P_2 such that

$$a_1 = \frac{aM_2}{M_1 + M_2}, \quad a_2 = -\frac{aM_1}{M_1 + M_2}. \tag{3}$$

The inverse distances can be expanded in series of Legendre polynomials:

$$\frac{1}{r_1} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{a_1}{r})^n P_n(\frac{z}{r}), \quad \frac{1}{r_2} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{a_2}{r})^n P_n(\frac{z}{r}), \tag{4}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Then, the potential function become

$$V = \frac{fM}{r}[1 + \sum_{n=0}^{\infty}\frac{\gamma_n}{r^n}P_n(\frac{z}{r})], \tag{5}$$

where

$$\gamma_n = \frac{M_1 a_1^n + M_2 a_2^n}{M}, M = M_1 + M_2.$$

In order to, the potential function V which is given by (5) to be real, it is sufficient for the quantities M and γ_n to be real for any value of n. The function V is real in two case [2]:

1. In the first case: M_1, M_2, a_1 and a_2 are pairs of complex conjugate quantities, i.e.

$$M_1 = \frac{M}{2}(1 + i\sigma), \quad M_2 = \frac{M}{2}(1 - i\sigma), \quad a_1 = c(\sigma + i), \quad a_2 = c(\sigma - i), \tag{6}$$

where σ and c are real constants

2. In the second case: the constants M_1, M_2, a_1 and a_2 are real, i.e.

$$\begin{aligned} M_1 &= M(1-\gamma), \\ M_2 &= M\gamma, \\ a_1 &= a\gamma, \\ a_2 &= -a(1-\gamma), \end{aligned} \tag{7}$$

where γ is the ratio of M_2 to the total mass M.

We study the generalized problem of two fixed-center in the first case, so the potential function V has the form

$$V = \frac{fM}{2}[\frac{1+i\sigma}{r_1} + \frac{1-i\sigma}{r_2}], \tag{8}$$

where

$$\begin{aligned} r_1 &= \sqrt{x^2+y^2+[z-c(\sigma+i)]^2}, \\ r_2 &= \sqrt{x^2+y^2+[z-c(\sigma-i)]^2}. \end{aligned} \tag{9}$$

Expanding V in a series in Legendre polynomial,

$$V = \frac{fM}{r}[1 + \sum_{k=2}^{\infty} \frac{\gamma_k}{r^k} P_k(\frac{z}{2})], \tag{10}$$

where

$$\gamma_k = \frac{c^k}{2}[(1+i\sigma)(\sigma+i)^k + (1-i\sigma)(\sigma-i)^k]. \tag{11}$$

Introducing the new coordinates λ, μ, and ω such that

$$\begin{aligned} x &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\cos w, \\ y &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\sin w, \\ z &= c\sigma + c\lambda\mu, \end{aligned} \tag{12}$$

then equation (8) became

$$V = \frac{fM}{c}[\frac{\lambda - \sigma\mu}{\lambda^2+\mu^2}]. \tag{13}$$

The kinetic energy can be written as

$$T = \frac{c^2}{2}[(\frac{\dot{\lambda}^2}{1+\lambda^2} + \frac{\dot{\mu}^2}{1-\mu^2})I + \dot{w}^2(1+\lambda^2)(1-\mu^2)] \tag{14}$$

where

$$I = \lambda^2 + \mu^2, \tag{15}$$

then, the lagrange equations are given by

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{I\dot{\lambda}}{1+\lambda^2}\right) - 2\lambda(1-\mu^2)\dot{\omega}^2 = -\frac{1}{c^2}\frac{\partial V}{\partial \lambda};\\
&\frac{d}{dt}\left(\frac{I\dot{\mu}}{1+\mu^2}\right) + 2\mu(1+\lambda^2)\dot{\omega}^2 = -\frac{1}{c^2}\frac{\partial V}{\partial \mu};\\
&\frac{d}{dt}[\dot{w}(1+\lambda^2)(1-\mu^2)] = 0.
\end{aligned} \tag{16}
$$

The area integral is got from the last equation in the system (16)

$$\dot{w}(1+\lambda^2)(1-\mu^2) = c_1. \tag{17}$$

The Hamiltonian function takes the form

$$H=\frac{1}{2c^2}\left[\frac{\lambda^2+1}{\lambda^2+\mu^2}p_\lambda^2+\frac{1-\mu^2}{\lambda^2+\mu^2}p_\mu^2\right]-\frac{c^2c_1^2}{2(\lambda^2+1)(1-\mu^2)}+\frac{fM}{c}\left(\frac{\lambda-\sigma\mu}{\lambda^2+\mu^2}\right)=h, \tag{18}$$

and from (18) we have

$$F = (\lambda^2+1)p_\lambda^2 + 2fMc\lambda + \frac{c^4c_1^2}{1+\lambda^2} - 2c^2\lambda^2 h, \tag{19}$$

$$F = (1-\mu^2)p_\mu^2 - 2fMc\sigma\mu - \frac{c^4c_1^2}{1-\mu^2} - 2c^2\mu^2 h. \tag{20}$$

Now, by using Hamilton Jacobi method we have

$$
\begin{aligned}
&\frac{1}{2c^2}\left[(1+\lambda^2)\left(\frac{\partial W}{\partial \lambda}\right)^2 + (1-\mu^2)\left(\frac{\partial W}{\partial \mu}\right)^2\right] + h(\lambda^2+\mu^2)\\
&+\frac{fM}{c}(\lambda-\sigma\mu) - \frac{c^2c_1^2}{2}\left(\frac{1}{1-\mu^2} - \frac{1}{1+\lambda^2}\right) = 0.
\end{aligned} \tag{21}
$$

The complete integral of this equation is

$$W = W_1(\lambda) + W_2(\mu). \tag{22}$$

Therefore, the Hamilton-Jacobi equation (21) is satisfied if

$$(\lambda^2+1)\left(\frac{dW_1}{d\lambda}\right)^2 = \frac{2h}{c^2}\lambda^4 + \frac{2fM}{c^3}\lambda^3 + 2\left(c_2+\frac{h}{c^2}\right)\lambda^2 + \frac{2fM}{c^3}\lambda + (2c_2+c_1^2), \tag{23}$$

$$(1-\mu^2)\left(\frac{dW_2}{d\mu}\right)^2 = -\frac{2h}{c^2}\mu^4 + \frac{2fM\sigma}{c^3}\mu^3 + 2\left(c_2+\frac{h}{c^2}\right)\mu^2 - \frac{2fM\sigma}{c^3}\mu - (2c_2+c_1^2), \tag{24}$$

this yields that W takes the form

$$W = \int \frac{\sqrt{L(\lambda)}}{\lambda^2+1}d\lambda + \int \frac{\sqrt{M(\mu)}}{1-\mu^2}d\mu, \tag{25}$$

where,

$$L(\lambda) = \frac{2h}{c^2}(\lambda^2 + 1)(\lambda^2 + a\lambda + c_2c^2), \tag{26}$$

$$M(\mu) = \frac{2h}{c^2}(1 - \mu^2)(\mu^2 - a\sigma\mu - c_2c^2), \tag{27}$$

where $a = \frac{fM}{c}$, σ is a constant and c_2 is a constant of separation. Then,

$$\frac{\partial W}{\partial c_2} = \beta, \frac{\partial W}{\partial h} = t - t_0, \tag{28}$$

where β is a new arbitrary constant, then we have from (28) the following equations:

$$\begin{aligned} \int \frac{d\lambda}{\sqrt{L(\lambda)}} + \frac{d\mu}{\sqrt{M(\mu)}} &= \beta, \\ \int \frac{\lambda^2 d\lambda}{\sqrt{L(\lambda)}} + \frac{\mu^2 d\mu}{\sqrt{M(\mu)}} &= c^2(t - t_0). \end{aligned} \tag{29}$$

Introducing a new time defined by

$$d\tau = (\lambda^2 + \mu^2)dt. \tag{30}$$

Therefore, the differential equations satisfied by λ and μ are:

$$\begin{aligned} \int \frac{d\lambda}{\sqrt{L(\lambda)}} &= \tau - \tau_0, \\ \int \frac{d\mu}{\sqrt{M(\mu)}} &= \tau - \tau_0. \end{aligned} \tag{31}$$

3. Topological analysis

First, we give the following definitions as in [8]:

1. The smooth mapping
$$F : M^{2n} \to R^n,$$
where $F(x) = (f_1(x), ..., f_n(x))$ is said to be the momentum mapping, M^{2n} is a simplistic manifold in the integrable Hamiltonian system, and $f_1,\ f_2, ..., f_n$ its independent integrals.

2. If rank $dF(x) < n$, and its image $F(x)$ in R^n is a critical value, then the point $x \in M$ is a critical point of the momentum mapping.

3. If K is the set of all critical points of the momentum mapping such that $K \subset M$, then the set $\Sigma = F(k) \subset R^n$ is the bifurcation diagram, where the whole of Σ is the union of several pieces Σ^k.

The topology of the level sets is introduced as

$$L_S = (x, y, \dot{x}, \dot{y}) \in R^4 : H = h, F = c_2 \subset R^4. \tag{32}$$

The energy-momentum mapping is determined by getting the set of critical point

$$(x,\ y,\ \dot{x},\ \dot{y}) \to (H,\ F)$$

this means the bifurcation diagram Σ, where Σ is the discriminant of the polynomials $L(\lambda)$ and $M(\mu)$:

$$\Sigma=\Sigma_1\cup\Sigma_2=[(h,c_2)\in R^2/disc(L(\lambda))=0]\cup[(h,c_2)\in R^2/disc(M(\mu))=0]. \tag{33}$$

On the point (h, c_2), the topological type of L_S can be change. The set R^2/Σ consists of 8 connected parts as shown in Figure 2. So, in each connected portion of the set R^2/Σ, the topological type of L_S is similar.

The Arnold-Liouville's theorem [5] state that, for noncritical values of H and F the level set L_s is a limited union of low dimensional tori, whose number depends only on the number and the location of the allowed ovals on the Riemann surface connected to the elliptic curve Γ_1 and Γ_2 where

$$\Gamma_1 : \omega_1 = \sqrt{L(\lambda)} \text{ and} \Gamma_2 : \omega_2 = \sqrt{M(\mu)}.$$

In order to obtain the ovals of of Γ_1 and Γ_2 (see Table 2), the real roots of the polynomials $L(\lambda)$ and $M(\mu)$ must be studied which shown in Table 1.

The topological type of L_S is either a torus, two-tori $2T$, or empty as shown in Table 2 (see [13]).

For getting the generic bifurcations of the system (1) (see Table 3), we must use the bifurcation of the roots of the polynomials $L(\lambda)$ and $M(\mu)$ as shown in Figures (3−4).

4. The solution of the problem

In this section, the elliptic functions and the Jacobi elliptic function of motion are used. Through the study of topology on the problem, we found that there is a periodic solution on the curve C_2 where $1 - 4hc_2 = 0$, the torus T contracted to one axial circle S and then disappeared as shown in Figure 4. It found that $\lambda_{1,2} = b$ where b is a constant and the μ parameter takes values in the period $[-1, \mu_1]$.

Returning to the second equation of (31), the function $M(\mu)$ is a polynomial of fourth degree with four real roots $1,\ -1,\ \mu_1$ and μ_2 as shown in Table 4, such that

$$M(\mu) = (\mu + 1)(\mu - 1)(\mu_1 - \mu)(\mu - \mu_2). \tag{34}$$

If $h < 0$ and $c_2 < 0$ the real motion is bounded where $(-1 \leq\ \mu\ \leq \mu_1)$, let

$$\mu = \frac{-(1-\mu_1)+(1+\mu_1)\sin^2\phi}{(1-\mu_1)+(1+\mu_1)\sin^2\phi}, sin^2\phi = \frac{1-\mu_1}{1+\mu_1}\frac{\mu+1}{1-\mu}, \tag{35}$$

and

$$d\mu = \frac{4(1-\mu_1)(1+\mu_1)\sin\phi\cos\phi}{[(1-\mu_1)+(1+\mu_1)\sin^2\phi]^2}d\phi. \tag{36}$$

By substituting from (34-36) in the second equation of (31), we have

$$t - t_0 = \frac{1}{d}\int_0^{\phi}\frac{d\phi}{\sqrt{1-k_1^2\sin^2\phi}}, \tag{37}$$

then, the solution is

$$\mu(\tau) = \frac{-(1-\mu_1)+(1+\mu_1)sn^2[d(\tau-\tau_0),k_1]}{(1-\mu_1)+(1+\mu_1)sn^2[d(\tau-\tau_0),k_1]}, \tag{38}$$

and the period T of $\mu(\tau)$ is

$$T = \frac{1}{d}sn^{-1}(1,k_1) = \frac{1}{d}K(k_1), \tag{39}$$

where,

$$d = \frac{\sqrt{(1+\mu_2)(1-\mu_1)}}{2}, k_1^2 = \frac{(\mu_1+1)(1-\mu_2)}{(1+\mu_2)(\mu_1-1)}. \tag{40}$$

Similarly, we get the periodic solution on the curve C_3 where $1+4hc_2 = 0$, $\mu_{1,2} = e$ and the λ parameter takes values on the interval $[0, \lambda_2]$. By solving the first equation of (31) the function $L(\lambda)$ is a polynomial of fourth degree with two real roots λ_1 and λ_2 and two complex roots i, $-i$ such that

$$L(\lambda) = (\lambda^2+1)(\lambda-\lambda_1)(\lambda_2-\lambda). \tag{41}$$

If $h < 0$ and $c_2 > 0$ the real motion is bounded where $(\lambda_1 \leq \ \lambda \ \leq \lambda_2)$,

let

$$\lambda = (\frac{\lambda_2+\lambda_1}{2} - \frac{\lambda_2-\lambda_1}{2})(\frac{n-\cos\phi}{1-n\cos\phi}),\ \tan^2\frac{\phi}{2} = (\frac{\cos\sigma_1}{\cos\sigma_2})(\frac{\lambda_2-\lambda}{\lambda-\lambda_1}), \tag{42}$$

where

$$\begin{aligned} \tan\sigma_1 &= \lambda_2, \\ \tan\sigma_2 &= \lambda_1, \\ n &= \tan\frac{\sigma_1-\sigma_2}{2}\tan\frac{\sigma_1+\sigma_2}{2}, \end{aligned} \tag{43}$$

and

$$d\lambda = -\frac{(\lambda_2-\lambda_1)[1-(\lambda_2^2-\lambda_1^2)^2]}{[1-(\lambda_2^2-\lambda_1^2)\cos\phi]^2}\sin\phi d\phi. \tag{44}$$

By substituting from (41, 42, 44) in the first equation of (31), we have

$$t-t_0=\frac{1}{g}\int_0^{\phi}\frac{d\phi}{\sqrt{1-k_1^2\sin^2\phi}}, \tag{45}$$

then, the solution is

$$\lambda(\tau)=\frac{\lambda_1+\lambda_2}{2}-\frac{\lambda_1-\lambda_2}{2}\frac{n\ cn[g(\tau-\tau_0),k_2]}{1-n\ cn[g(\tau-\tau_0),k_2]}, \tag{46}$$

where

$$\begin{aligned} n&=\lambda_2^2-\lambda_1^2, g=-[(1+\lambda_1^2)(1+\lambda_2^2)]^{\frac{1}{4}},\\ k_2&=\frac{1}{2}[1-\frac{1-\lambda_1\lambda_2}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}], \end{aligned} \tag{47}$$

and the period T of $\lambda(\tau)$ is

$$T=\frac{1}{g}sn^{-1}(1,k_2)=\frac{1}{g}K(k_2). \tag{48}$$

5. Phase portrait of the separated functions

In this section, we use the phase portrait to find the topological translation of the path. El-Sabaa found the singular points and its types for separated functions of Kovaleveskaya top by using Kolsoff variables [9].

Consider the function

$$F_1=h(q_1^2-q_1^4)+\frac{1}{2c^2}(1-2q_1^2+q_1^4)p_1^2-k_3(q_1^2-q_1^3)-\alpha q_1^2-k_2, \tag{49}$$

where $k_2=2c_2+c_1^2$, $\alpha=c_2$ and $k_3=\frac{2fM}{c^3}$.

To construct the lines of constant F_1, we first study the singular points of F_1. These points can be found from the equations

$$\frac{\partial F_1}{\partial p_1}=\frac{1}{c^2}(1-2q_1^2+q_1^4)p_1=0, \tag{50}$$

$$\frac{\partial F_1}{\partial q_1}=2h(q_1-2q_1^3)+\frac{2}{c^2}p_1^2(q_1^3-q_1)-k_3(1-3q_1^2)-2\alpha q_1=0, \tag{51}$$

and hence we have the following where $p=0$ we get

$$-4hq_1^3+3q_1^2k_3+2(h-\alpha)q_1-k_3=0, \tag{52}$$

then from (50) we have $q_1 = \pm 1$,

we get the two equations

$$k_3 - h - \alpha = 0, \quad k_3 + h + \alpha = 0. \tag{53}$$

The positive regions of the functions

$$\begin{aligned} f_1 &= 128h^4 + 36h^2k_3^2 + 108k_3^4 - 384h^3\alpha \\ &\quad - 504k_3^2\alpha + 384h^2\alpha^2 + 36k_3^2\alpha^2 - 128h\alpha^3, \\ f_2 &= k_3 - h - \alpha, \\ f_3 &= k_3 + h + \alpha. \end{aligned} \tag{54}$$

are shown in Figure (5).

It is clear that the curve f_1 is tangent to the curves f_2 and f_3 at the points $\frac{1}{2}$ and $\frac{-1}{2}$.

We study the motion in domain $D_i (i = 1, 2, 3, ..., 8)$

1. The first region D_1: $f_1 < 0,\ f_2 < 0,\ f_3 > 0$.

At $p_1 = 0$ we have one point with q_1 coordinate

$$q_1^* = \frac{k_3}{4h} + \frac{-9k_3^2 - 24h(h-\alpha)}{[6\times 2^{\frac{2}{3}} h(216h^2k_3 - 54k_3^3 + 216hk_3\alpha + \sqrt{4(-9k_3^2 - 24h(h-\alpha))^3 + (216h^2k_3 - 54k_3^2 + 216hk_3\alpha)^2})]^{\frac{1}{3}}}$$
$$- \frac{(216h^2k_3 - 54k_3^3 + 216hk_3\alpha + \sqrt{4(-9k_3^2 - 24h(h-\alpha))^3 + (216h^2k_3 - 54k_3^2 + 216hk_3\alpha)^2})^{\frac{1}{3}}}{12\times 2^{\frac{1}{3}} h}.$$

Then, to get the type of this point, we put

$$q_1 = q_1^* + y, p_1 = x, \tag{55}$$

in the function F_1

$$\begin{aligned} F_1 = [&\tfrac{1}{4} + \tfrac{23}{512h^4} - \tfrac{5}{16h^3} + \tfrac{23}{96h^2} + \tfrac{1}{3h} - \tfrac{\rho}{6912h^4} + \tfrac{1115}{2^{\frac{2}{3}} u^{\frac{4}{3}}} + \tfrac{81}{128\times 2^{\frac{2}{3}} h^4 u^{\frac{4}{3}}} - \tfrac{27}{2^{\frac{2}{3}} h^3 u^{\frac{4}{3}}} + \\ &\tfrac{27}{4\times 2^{\frac{2}{3}} h^2 u^{\frac{4}{3}}} + \tfrac{54\times 2^{\frac{1}{3}}}{h^2 u^{\frac{4}{3}}} - \tfrac{246\times 2^{\frac{1}{3}}}{h u^{\frac{4}{3}}} - \tfrac{656\times 2^{\frac{1}{3}} h}{u^{\frac{4}{3}}} + \tfrac{408\times 2^{\frac{1}{3}} h^2}{u^{\frac{4}{3}}} - \tfrac{128\times 2^{\frac{1}{3}} h^3}{u^{\frac{4}{3}}} + \tfrac{16\times 2^{\frac{1}{3}} h^4}{u^{\frac{4}{3}}} + \tfrac{105}{u} + \\ &\tfrac{27}{64h^4u} - \tfrac{27}{4h^3u} + \tfrac{315}{8h^2u} + \tfrac{100}{hu} + \tfrac{48h}{u} - \tfrac{8h^2}{u} - \tfrac{3}{2^{\frac{4}{3}} u^{\frac{2}{3}}} + \tfrac{45}{128\times 2^{\frac{1}{3}} u^{\frac{2}{3}}} - \tfrac{9}{2\times 2^{\frac{1}{3}} h^3 u^{\frac{2}{3}}} + \tfrac{9}{8\times 2^{\frac{1}{3}} h^2 u^{\frac{2}{3}}} + \\ &\tfrac{9\times 2^{\frac{2}{3}}}{h^2 u^{\frac{2}{3}}} - \tfrac{41\times 2^{\frac{2}{3}}}{3hu^{\frac{2}{3}}} + \tfrac{8\times 2^{\frac{2}{3}} h}{u^{\frac{2}{3}}} - \tfrac{8\times 2^{\frac{2}{3}} h^2}{3u^{\frac{2}{3}}} + \tfrac{1}{2^{\frac{2}{3}} u^{\frac{1}{3}}} - \tfrac{3}{16\times 2^{\frac{2}{3}} h^4 u^{\frac{1}{3}}} + \tfrac{7}{4\times 2^{\frac{2}{3}} h^3 u^{\frac{1}{3}}} - \tfrac{1}{8\times 2^{\frac{2}{3}} h^2 u^{\frac{1}{3}}} - \tfrac{2^{\frac{4}{3}}}{h^2 u^{\frac{1}{3}}} - \\ &\tfrac{17u^{\frac{1}{3}}}{1536\times 2^{\frac{1}{3}} h^4} + \tfrac{3u^{\frac{1}{3}}}{64\times 2^{\frac{1}{3}} h^3} + \tfrac{3u^{\frac{1}{3}}}{128\times 2^{\frac{1}{3}} h^2} + \tfrac{\rho u^{\frac{1}{3}}}{82944\times 2^{\frac{1}{3}} h^4} + \tfrac{5u^{\frac{2}{3}}}{2304\times 2^{\frac{2}{3}} h^4} - \tfrac{u^{\frac{2}{3}}}{216\times 2^{\frac{2}{3}} h^3} - \tfrac{u^{\frac{2}{3}}}{216\times 2^{\frac{2}{3}} h^2}]x^2 + \\ &[+2 - \tfrac{3}{8h} - h + \tfrac{36\times 2^{\frac{2}{3}}}{u^{\frac{2}{3}}} - \tfrac{27}{4\times 2^{\frac{1}{3}} hu^{\frac{2}{3}}} - \tfrac{114\times 2^{\frac{2}{3}} h}{u^{\frac{2}{3}}} + \tfrac{96\times 2^{\frac{2}{3}} h^2}{u^{\frac{2}{3}}} - \tfrac{24\times 2^{\frac{2}{3}} h^3}{u^{\frac{2}{3}}} - \tfrac{u^{\frac{2}{3}}}{24\times 2^{2/3} h}]y^2 + A_0 \end{aligned}$$

where A_0 contains the zeros terms of x and y, $[\sqrt{4(-9 - 24(-2+h)h)^3 + (-54 + 432h + 216h^2)^2}] = \rho$ and $[(-54 + 432h + 216h^2) + \sqrt{4(-9 - 24(-2+h)h)^3 + (-54 + 432h + 216h^2)^2}] = u$.

The singular point is hyperbolic point, where

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{56}$$

In the same manner, we get the type of points in the domains D_i where, $i = 1, ..., 8$.

Table 5 shows the points and its types for all domains.

6. Summary-conclusions

In the current paper, the generalized two-fixed center problem provided us to get a complete picture of the dynamics of its potential:

1. The complete characterization of the real phase topology.
2. The periodic solution, where the variables of motion can be described through the Jacobi elliptic function which is a periodic function, the solution with Jacobi elliptic function has been explained in details in [30, 31].
3. Phase portrait: the singular points of the separated functions were determined. The type of these points is either elliptic or hyperbolic point. The elliptic points in the figures were stable while the hyperbolic points were unstable.

Table 1: Topological type of L_S and real roots of the polynomials $L(\lambda)$ and $M(\mu)$ for $(h, c_2) \in R^2/\Sigma$.

Domain	*Roots of* $L(\lambda)$	*Roots of* $M(\mu)$
1	$\lambda_2 < \lambda_1 < 0$	$-1 < \mu_2 < 0 < \mu_1 < 1$
2	$\lambda_2 < 0 < \lambda_1$	$-1 < 0 < \mu_2 < \mu_1 < 1$
3	$0 < \lambda_2 < \lambda_1$	$-1 < \mu_1 < 0 < \mu_2 < 1$
4	$\lambda_1 < 0 < \lambda_2$	$-1 < \mu_2 < \mu_1 < 0 < 1$
5	0	$-1 < \mu_2 < 0 < \mu_1 < 1$
6	$\lambda_2 < 0 < \lambda_1$	0
7	0	$-1 < \mu_1 < 0 < \mu_2 < 1$
8	$\lambda_1 < 0 < \lambda_2$	0

Table 2: Admissible ovals on diagram Σ.

Domain	$\lambda-plane\ \triangle_1$	$\mu-plane\ \triangle_2$	*Topological type*
1	$\emptyset$	$[-1, \mu_2]$	$\emptyset$
2	$[0, \lambda_1]$	$[-1, 0]$	T
3	$[\lambda_2, \lambda_1]$	$[-1, \mu_1]$	T
4	$[0, \lambda_2]$	$[\mu_2, \mu_1] \cup [-1, 0]$	$2T$
5	$\emptyset$	$[-1, \mu_2]$	$\emptyset$
6	$[0, \lambda_1]$	$\emptyset$	$\emptyset$
7	$\emptyset$	$[-1, \mu_1]$	$\emptyset$
8	$[0, \lambda_2]$	$\emptyset$	$\emptyset$

Table 3: Generic bifurcations of the level set L_S passing from domain i to domain j.

$2 \to 5$	$4 \to 5$	$2 \to 3$	$4 \to 2$
$2 \to 1$	$4 \to 1$		$4 \to 3$
$2 \to 6$	$4 \to 6$		
$2 \to 7$	$4 \to 7$		
$2 \to 8$	$4 \to 8$		
$3 \to 1$			
$3 \to 5$			
$3 \to 6$			
$3 \to 7$			
$3 \to 8$			
$T \to \emptyset$	$2T \to \emptyset$	$T \to T$	$2T \to T$

Table 4: Topological type of L_S for $(h, c_2) \in \Sigma$.

Domain	$\lambda - plane \triangle_1$	$\mu - plane \triangle_2$	*Topological type*
C_1	$[\lambda_2, \lambda_1]$	$[-1, 0]$	S
C_2	$[\lambda_2 = \lambda_1]$	$[-1, \mu_1]$	S
C_3	$[0, \lambda_2]$	$[-1, \mu_2 = \mu_1]$	$2S$
C_4	$[\lambda_1 = 0]$	$[-1, \mu_2 = 0]$	S
C_5	$[0, \lambda_2]$	$[-1, \mu_1]$	$S \times (S \wedge S)$

Table 5: The type of points in the domains D_i

Domain	The points	Types of points	*Figures*
$D_1 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 6
$D_2 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 7
$D_3 : f_1 > 0, f_2 < 0, f_3 < 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 8
$D_4 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 9
$D_5 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 10
$D_6 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 11
$D_7 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 12
$D_8 : f_1 > 0, f_2 > 0, f_3 < 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 13

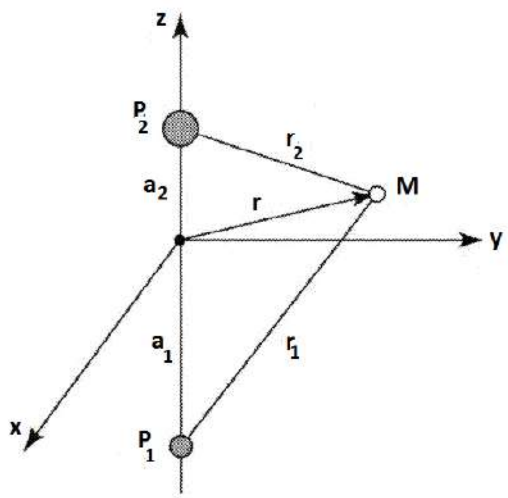

Figure 1: The description of the problem.

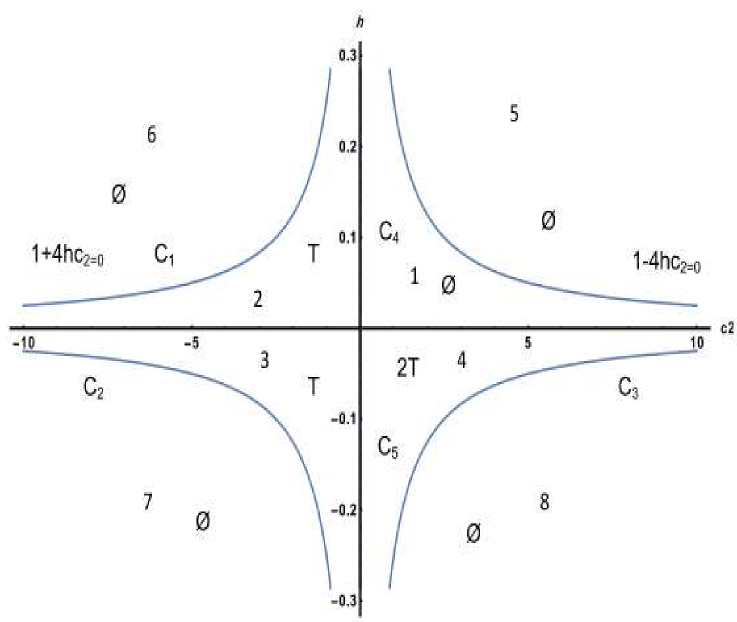

Figure 2: Diagram of bifurcation $\Sigma = \Sigma_1 \cup \Sigma_2$.

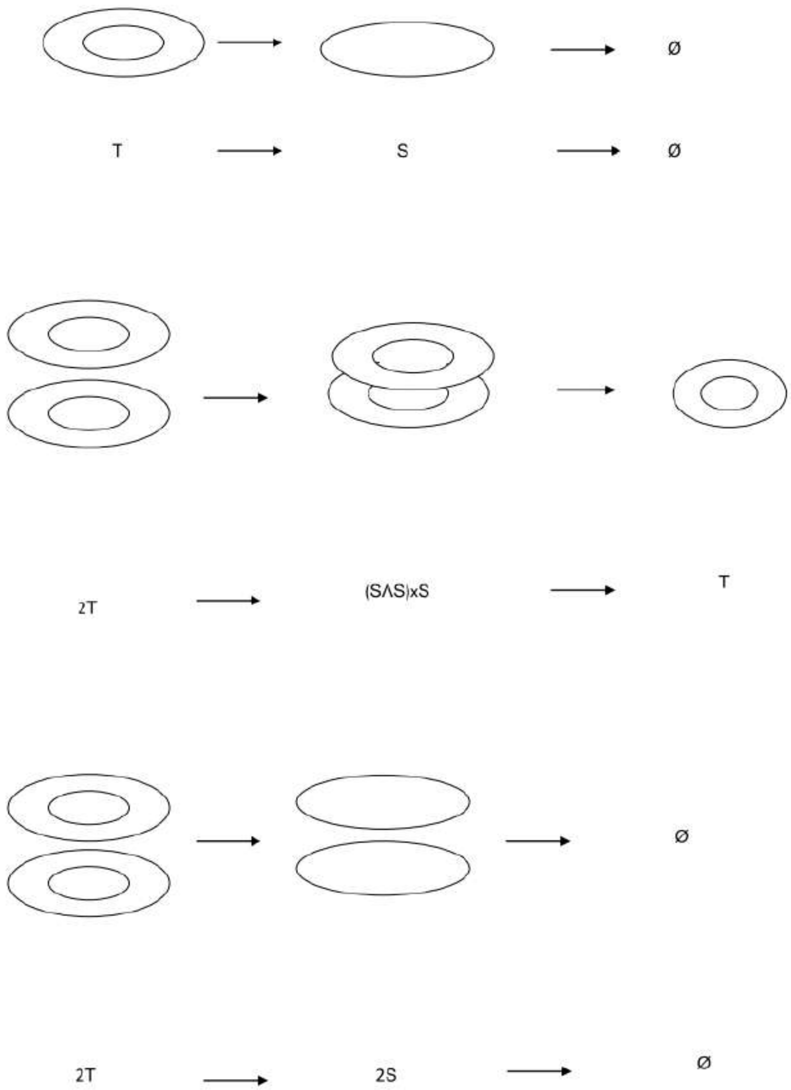

Figure 3: The bifurcation of liouvile tori, where a torus spirals twice around a torus and become twice, as a result, it is created $S \times (S \wedge S)$ which is a circle and two other circles that are above each other but not in the same level and have one common point.

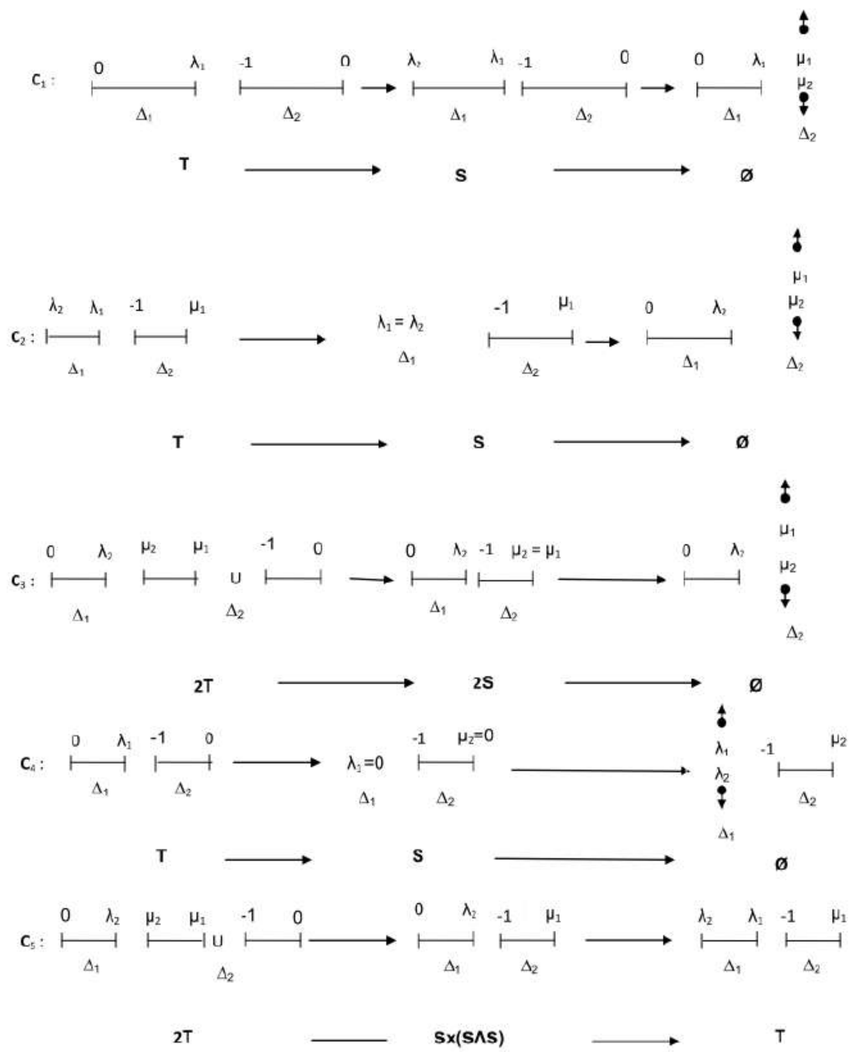

Figure 4: Correspondence between bifurcation of roots of polynomials $L(\lambda)$ *and* $M(\mu)$ and bifurcation of invariant Liouville tori.

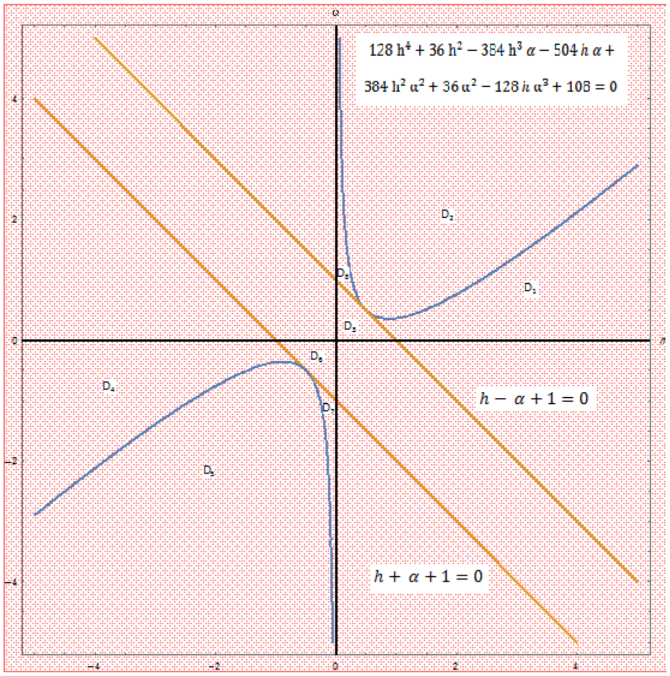

Figure 5: The regions D_i of the real motion on the (h,α) plane.

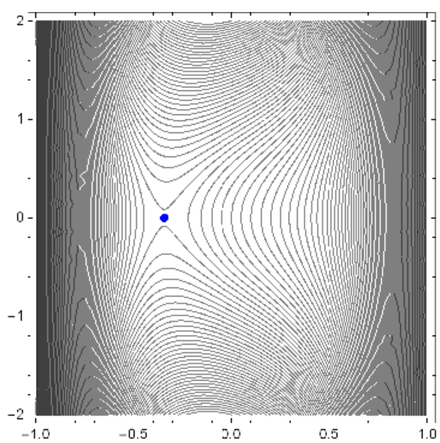

Figure 6: The one-hyperbolic point in domain D_1.

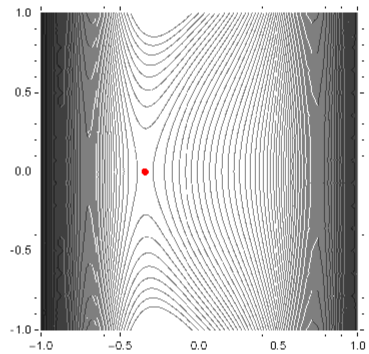

Figure 7: The one-hyperbolic point in domain D_2.

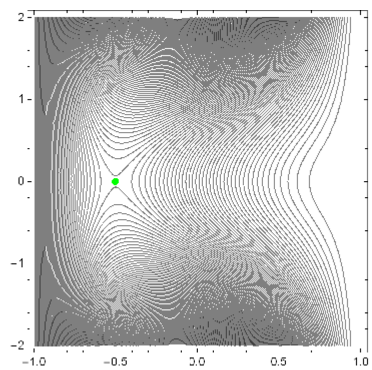

Figure 8: The one-hyperbolic point in domain D_3.

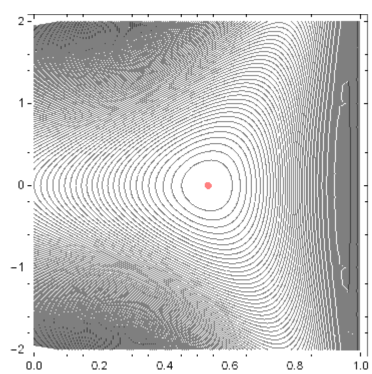

Figure 9: The one-elliptic point in domain D_4.

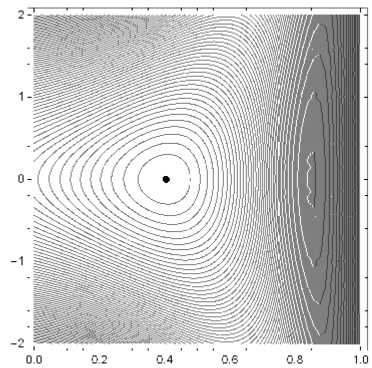

Figure 10: The one-elliptic point in domain D_5.

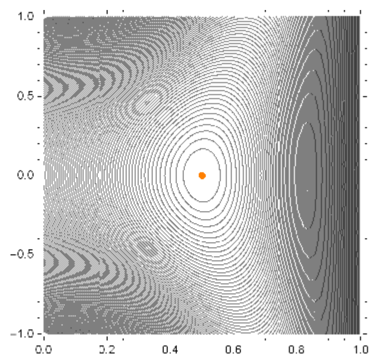

Figure 11: The one-elliptic point in domain D_6.

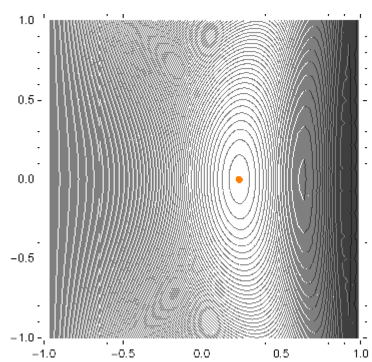

Figure 12: The one-hyperbolic point in domain D_7.

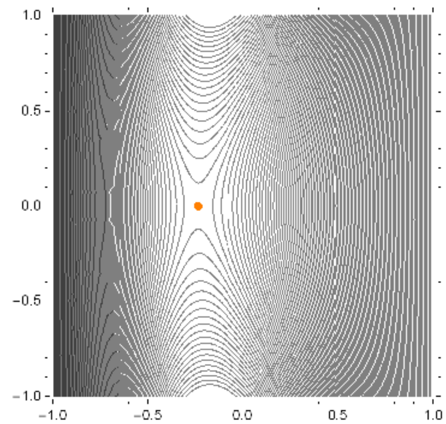

Figure 13: The one-hyperbolic point in domain D_8.

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Accepted: 28.07.2018

Optimization technique for solving fuzzy partial differential equations under strongly generalized differentiability

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Abstract. In this article, we develop and analyze the use of the combined Laplace transform-homotopy perturbation method C(LT-HPM) to find the exact and approximate solutions for fuzzy partial differential equations under strongly generalized differentiability. The C(LT-HPM) allows the solution of the fuzzy partial differential equation to be calculated in the form of an infinite series in which the components can be easily computed. The method is tested on some linear and nonlinear fuzzy partial differential equations with fuzzy initial conditions to show the effectiveness and accuracy of this method.

Keywords: fuzzy partial differential equation, fuzzy derivative, strongly generalized differentiability.

1. Introduction

The study of fuzzy partial differential equations (FPDEs) in both theoretical and numerical calculations of view has been growing in recent years. Generally, FPDEs in the fuzzy setting are a natural way to model dynamical systems when information about its behavior is inadequate. Some problems that lead to FPDEs are found in many applications of which some are mentioned in fields of physics and engineering where often have to solving those as numerical methods. Moreover, some researches present applications of FPDEs with fuzzy parameters that were obtained through fuzzy rule-based system, such as in [1] and [2]. Since L. Zadeh introduced the concept of fuzzy sets in [3] a great amount of research has been developed, including the studies on FPDEs as well as fuzzy set theory. The concept of FPDE was first introduced and investigated by J. Buckley and T. Feuring in [4].

The numerical solutions of FPDEs have been studied by several authors using different approaches. In [5] the author have used an explicit finite difference method (FDM) to solve linear FPDEs based on Seikkala derivative. In [6] also, the authors have developed optimal homotopy asymptotic method (OHAM) to find the approximate-analytical solution for linear partial differential equation involving a fuzzy heat equation based on Seikkala derivative. These approaches have drawbacks; it solves only linear FPDEs but does not discuss the nonlinear FPDEs. Hence, the fuzzy solution becomes fuzzier as time goes by [7, 8], and it behaves quite differently from the crisp solution. The strongly generalized differentiability was first introduced in [9] and studied in [7, 10, 11, 12, 16, 17, 33, 34]. This concept permits us to resolve the above-mentioned drawbacks. Indeed, the strongly generalized differentiability is defined for a larger class of fuzzy-valued functions which is a generalization of the Hukuhara derivative. The purpose of this article is to establish the approximate-analytical solutions for linear and nonlinear FPDE using the C(LT-HPM) under the assumption of strongly generalized differentiability which is as follows:

$$\varphi\left(D_t u(x,t)\right) = \Phi(D_x u(x,t)) + Nu(x,t),\ 0 < x < l,\ t > 0, \tag{1}$$

subject to the fuzzy initial conditions

$$u(x,0) = f(x;\alpha),\ u_t(x,0) = g(x;\alpha),\ 0 \leq x \leq l, \tag{2}$$

for all $\alpha \in [0,1]$, where the operators $\varphi\left(D_t\right)$ will be a polynomial with a constant coefficient in D_t, $\Phi(D_x)$ is a polynomial with a constant coefficient in D_x, N is the nonlinear operator, $f,\ g : [0,l] \to \mathbb{R}_F$ are continuous fuzzy-valued functions and $u : [0,l] \times (0,\infty) \to \mathbb{R}_F$ is a continuous fuzzy-valued function which is unknown function of independent variables x and t to be determined such that $\mathbb{R}_F$ is the set of fuzzy real numbers on $\mathbb{R}$.

Calculation of the solution of FPDEs is usually very difficult. We can find the exact solution only in a few extraordinary cases. When we are studying in fields of physics and engineering, we often meet the problems of FPDEs. Anyway, by using the parametric form of fuzzy numbers, we employ the C(LT-HPM) to find the exact and approximate solutions for FPDE (1)-(2). The C(LT-HPM) has the following characteristics; first, it is somewhat different from other approximate-analytical methods in that it gives extremely good results for even a large domain with minimal terms of the approximate series solution. The second advantage of this method is its ability to solve other mathematical, physical and engineering issues. Third, it is the first attempt in solving nonlinear FPDEs. Fourth, it is the first attempt gives two locally solutions under strongly generalized differentiability for linear and nonlinear FPDEs. More specifically, we investigate the solution of different types of FPDEs using C(LT-HPM).

This article is organized in six sections including the introduction. In Section 2, we present a few fundamental definitions and preliminary results from the fuzzy calculus theory, including concepts like fuzzy derivative and fuzzy solution. The procedure for converting fuzzy partial differential equation (1)-(2) under strongly generalized differentiability into two systems of crisp partial differential equations is presented in Section 3. In Section 4, the combined Laplace transform-homotopy perturbation technique is built and introduce. The numerical results are reported to illustrate the ability and superiority of the proposed method by considering three numerical examples in Section 5. Finally, the conclusion is drawn in Section 6 with a few concluding comments.

2. Preliminaries

This section present some basic definitions in fuzzy mathematics and introduce the necessary notations which can be used throughout the paper. Hereafter, we adopt strongly generalized differentiability which is a modification of the Hukuhara differentiability and has the advantage of dealing properly with FPDEs.

Definition 2.1 (see [18]). A mapping $w : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy number if the following properties are satisfied:

i) $w(\lambda s + (1-\lambda)t) \geq \min\{w(s), w(t)\}$ for each s, $t \in \mathbb{R}$ and $\lambda \in [0,1]$, which is called a convex property.

ii) $\exists s \in \mathbb{R}$ such that $w(s) = 1$, which is called a normal property.

iii) the set $\{s \in \mathbb{R} \mid w(s) > \alpha\}$ is closed for each $\alpha \in [0,1]$, which is called an upper semicontinuous property.

iv) the set $\overline{\{s \in \mathbb{R} \mid w(s) > 0\}}$ is compact, where $\overline{\{\cdot\}}$ is the closure of $\{\cdot\}$.

For $0 < \alpha \leq 1$, put $[w]_\alpha = \{s \in \mathbb{R} \mid w(s) \geq \alpha\}$, $[w]_0 = \overline{\{s \in \mathbb{R} \mid w(s) > 0\}}$, and $[w]_1 \neq \phi$ (see [15]). Thus, if w is a fuzzy number, then $[w]_\alpha = [\underline{w}(\alpha), \overline{w}(\alpha)]$, where $\underline{w}(\alpha) = \min\{s \mid s \in [w]_\alpha\}$ and $\overline{w}(\alpha) = \max\{s \mid s \in [w]_\alpha\}$ for each $\alpha \in [0,1]$. Hence, the notation $[w]_\alpha$ is called the α-cut representation or parametric form of a fuzzy number w.

Theorem 2.2 (see [15]). *A mapping $w : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy number with α-cut representation $[\underline{w}(\alpha), \overline{w}(\alpha)]$ if and only if the following conditions are satisfied:*

i) the function $\underline{w} : [0,1] \to \mathbb{R}$ is a bounded increasing.

ii) the function $\overline{w} : [0,1] \to \mathbb{R}$ is a bounded decreasing.

iii) for each $r \in (0,1]$, $\lim_{\alpha \to r^-} \underline{w}(\alpha) = \underline{w}(r)$ and $\lim_{\alpha \to r^-} \overline{w}(\alpha) = \overline{w}(r)$.

iv) for each $r \in (0,1]$, $\lim_{\alpha \to r^+} \underline{w}(\alpha) = \underline{w}(r)$ and $\lim_{\alpha \to r^+} \overline{w}(\alpha) = \overline{w}(r)$.

v) $\underline{w}(\alpha) \leq \overline{w}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 2.3 (see [32]). Let $H_d : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}^+ \cup \{0\}$. The Hausdorff metric H_d is a function defined by

$$H_d(w,z) = \sup_{\alpha \in [0,1]} \max\{|\underline{w}(\alpha) - \underline{z}(\alpha)|\,, |\overline{w}(\alpha) - \overline{z}(\alpha)|\}$$

for each w, $z \in \mathbb{R}_F$.

Theorem 2.4 (see [32]). *$(\mathbb{R}_F, H_d)$ is a complete fuzzy metric space.*

In what follows, we define some arithmetic operations in fuzzy mathematics as follows:

1. $[w]_\alpha + [z]_\alpha = [\underline{w}(\alpha) + \underline{z}(\alpha), \overline{w}(\alpha) + \overline{z}(\alpha)]$ for each $w, z \in \mathbb{R}_F$ and $0 \le \alpha \le 1$.
2. $[w]_\alpha \ominus [z]_\alpha = [\underline{w}(\alpha) - \overline{z}(\alpha), \overline{w}(\alpha) - \underline{z}(\alpha)]$, which is called the H-difference (Hukuhara difference) of w and z.
3. $\lambda^*[w]_\alpha = [\min\{\lambda^*\underline{w}(\alpha), \lambda^*\overline{w}(\alpha)\}, \max\{\lambda^*\underline{w}(\alpha), \lambda^*\overline{w}(\alpha)\}]$ for each $\lambda^* \in \mathbb{R}$.

Definition 2.5 (see [14]). Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function and $t_0 \in [a, b]$. We say u is strongly generalized differentiability at t_0, if there exists an element $u'(t_0) \in \mathbb{R}_F$ such that either:

i) for all $h > 0$ sufficiently near to 0, the H-differences $u(t_0 + h) \ominus u(t_0)$, $u(t_0) \ominus u(t_0 - h)$ exist and

$$u'(t_0) = \lim_{h \longrightarrow 0^+} \frac{u(t_0 + h) \ominus u(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{u(t_0) \ominus u(t_0 - h)}{h}.$$

In this part of the definition, we denote $u'(t_0)$ by $D_1^1 u(t_0)$, or

ii) for all $h < 0$ sufficiently near to 0, the H-differences $u(t_0 + h) \ominus u(t_0)$, $u(t_0) \ominus u(t_0 - h)$ exist and

$$u'(t_0) = \lim_{h \longrightarrow 0^-} \frac{u(t_0 + h) \ominus u(t_0)}{h} = \lim_{h \longrightarrow 0^-} \frac{u(t_0) \ominus u(t_0 - h)}{h}.$$

In this part of the definition, we denote $u'(t_0)$ by $D_2^1 u(t_0)$.

Definition 2.6 (see [10]). Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function. We say that u is (1)-differentiable on $[a, b]$ if u is differentiable in the first form (i) of Definition (2.5). Similarly, we say that u is (2)-differentiable on $[a, b]$ if u is differentiable in the second form (ii) of Definition (2.5).

Theorem 2.7 (see [11]). *Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function, where $[u(t)]_\alpha = [\underline{u}(t; \alpha), \overline{u}(t; \alpha)]$ for each $\alpha \in [0, 1]$.*

i) if u is (1)-differentiable, then $\underline{u}$ and $\overline{u}$ are differentiable functions and $[D_1^1 u(t)]_\alpha = [\underline{u}'(t; \alpha), \overline{u}'(t; \alpha)]$.

ii) if u is (2)-differentiable, then $\underline{u}$ and $\overline{u}$ are differentiable functions and $[D_2^1 u(t)]_\alpha = [\overline{u}'(t; \alpha), \underline{u}'(t; \alpha)]$.

Theorem 2.8 (see [13]). *Let $D_1^1 u : [a, b] \longrightarrow R_F$ or $D_2^1 u : [a, b] \longrightarrow R_F$ be fuzzy-valued functions, where $[u(t)]_\alpha = [\underline{u}(t; \alpha), \overline{u}(t; \alpha)]$ for each $\alpha \in [0, 1]$.*

i) if $D_1^1 u$ is (1)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\underline{u}''(t; \alpha), \overline{u}''(t; \alpha)]$.

ii) if $D_1^1 u$ is (2)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\overline{u}''(t; \alpha), \underline{u}''(t; \alpha)]$.

iii) if $D_2^1 u$ is (1)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\overline{u}''(t;\alpha), \underline{u}''(t;\alpha)]$.

iv) if $D_2^1 u$ is (2)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\underline{u}''(t;\alpha), \overline{u}''(t;\alpha)]$.

According to Theorem (2.8) that is showing us the way to translate FPDE (1)-(2) into two systems of crisp partial differential equations (PDEs), we might use the numerical method directly on the obtained crisp partial differential systems instead of rewriting this method for PDEs in a fuzzy setting as in the next Section.

3. The fuzzy partial differential equations

It is well known that all researchers in fuzzy mathematics transfer any fuzzy problem in mathematics to the system of PDEs or ODEs because there is no method to solve it without convert it to the system. Now, we study the FPDEs using the concept of a fuzzy derivative under strongly generalized differentiability in each step of differentiation.

Let $u : [0,l] \times (0,\infty) \longrightarrow \mathbb{R}_F$ be a continuous fuzzy-valued function such that $[u(x,t)]_\alpha = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$, and consequently the fuzzy functions f and g in Eq. (2) can be obtained via the Zadeh extension principle. If u satisfy FPDE (1)-(2), then we say that u is a fuzzy solution of FPDE (1)-(2). Now from Section 2 the defuzzification of FPDE (1)-(2) for all $\alpha \in [0,1]$ and to determine the lower and upper functions of the solution of FPDE (1)-(2), we discuss the following two cases:

Case 1. If we consider $\mu\,(D_x, D_t)\,u(x,t)$ by using the derivative in (1)-differentiable, then we have

$$[\mu\,(D_x, D_t)\,u(x,t)]_\alpha = [\mu\,(D_x, D_t)\,\underline{u}(x,t;\alpha), \mu\,(D_x, D_t)\,\overline{u}(x,t;\alpha)], \tag{3}$$

where μ is a polynomial with a constant coefficient in D_x and D_t, and we should solve the system of crisp PDEs

$$\varphi\,(D_t\underline{u}(x,t)) = \Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0, \tag{4}$$

$$\varphi\,(D_t\overline{u}(x,t)) = \Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0, \tag{5}$$

$$\underline{u}(x,0;\alpha) = \underline{f}(x;\alpha),\ \underline{u}_t(x,0;\alpha) = \underline{g}(x;\alpha),\ 0 \le x \le l, \tag{6}$$

$$\overline{u}(x,0;\alpha) = \overline{f}(x;\alpha),\ \overline{u}_t(x,0;\alpha) = \overline{g}(x;\alpha),\ 0 \le x \le l. \tag{7}$$

Case 2. If we consider $\mu\,(D_x, D_t)\,u(x,t)$ by using the derivative in (2)-differentiable, then we have

$$[\mu\,(D_x, D_t)\,u(x,t)]_\alpha = [\mu\,(D_x, D_t)\,\overline{u}(x,t;\alpha), \mu\,(D_x, D_t)\,\underline{u}(x,t;\alpha)], \tag{8}$$

and we should solve the system of crisp PDEs

(9) $\varphi\left(D_t\underline{u}(x,t)\right) = \Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$

(10) $\varphi\left(D_t\overline{u}(x,t)\right) = \Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$

(11) $\underline{u}(x,0;\alpha) = \underline{f}(x;\alpha),\ \underline{u}_t(x,0;\alpha) = \underline{g}(x;\alpha),\ 0 \le x \le l,$

(12) $\overline{u}(x,0;\alpha) = \overline{f}(x;\alpha),\ \overline{u}_t(x,0;\alpha) = \overline{g}(x;\alpha),\ 0 \le x \le l.$

Consequently, we use the strongly generalized differentiability in the present work. Under fitting conditions, the FPDE (1)-(2) under this interpretation has locally two solutions.

4. Analysis of the method

To illustrate the essential thoughts of the new method for solving linear and nonlinear FPDEs, we take the Laplace transform $\mathcal{L}$ on both sides of all equations in cases (1) and (2) as follows:
first system

(13) $\mathcal{L}\left\{\varphi\left(D_t\underline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

(14) $\mathcal{L}\left\{\varphi\left(D_t\overline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

second system

(15) $\mathcal{L}\left\{\varphi\left(D_t\underline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

(16) $\mathcal{L}\left\{\varphi\left(D_t\overline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0.$

Using the differentiation rule of the Laplace transform, we have $\underline{U}(x,s,\underline{f},\underline{g},\Phi\left(D_x\underline{u}\right),N\underline{u})$ and $\overline{U}(x,s,\overline{f},\overline{g},\Phi\left(D_x\overline{u}\right),N\overline{u})$ in the first system (13)-(14), and $\underline{U}(x,s,\underline{f},\underline{g},\Phi\left(D_x\overline{u}\right),N\overline{u})$ and $\overline{U}(x,s,\overline{f},\overline{g},\Phi\left(D_x\underline{u}\right),N\underline{u})$ in the second system (15)-(16). After that, taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides gives:

(17) $\underline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\Phi\left(D_x\underline{u}(x,s;\alpha)\right),N\underline{u}(x,s;\alpha))\right\},$

(18) $\overline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}\left(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\Phi\left(D_x\overline{u}(x,s;\alpha)\right),N\overline{u}(x,s;\alpha)\right)\right\},$

in the first system (13)-(14), and

(19) $\underline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\underline{U}\left(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\Phi\left(D_x\overline{u}(x,s;\alpha)\right),N\overline{u}(x,s;\alpha)\right)\right\},$

(20) $\overline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}\left(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\Phi\left(D_x\underline{u}(x,s;\alpha)\right),N\underline{u}(x,s;\alpha)\right)\right\},$

in the second system (15)-(16). Now, we construct a HPM to obtain approximate-analytical solutions of FPDE (1)-(2). Obviously, from Eqs. (3) and (8) we consider Eq. (1) as:

(21) $L(\underline{v}) = \underline{v} - N(\underline{v}) = 0,$

(22) $L(\overline{v}) = \overline{v} - N(\overline{v}) = 0,$

where L is a linear operator and N is a nonlinear operator with solutions $\underline{u}$ and $\overline{u}$, respectively. By the homotopy technique (see [20, 21, 22, 23, 25]), we construct a homotopy $\hat{H} : [0, l] \times (0, \infty) \times [0, 1] \to \mathbb{R}_F$ which satisfies

$$\hat{H}(\underline{v}, \lambda) = (1-\lambda)(L(\underline{v}) - L(\underline{u}_0)) + \lambda L(\underline{v}) = 0, \tag{23}$$

$$\hat{H}(\overline{v}, \lambda) = (1-\lambda)(L(\overline{v}) - L(\overline{u}_0)) + \lambda L(\overline{v}) = 0, \tag{24}$$

where $\lambda \in [0, 1]$ is an embedding parameter, $\underline{u}_0$ and $\overline{u}_0$ are initial approximations of Eqs. (21) and (22) which satisfies the fuzzy initial conditions. It is obvious that

$$\hat{H}(\underline{v}, 0) = L(\underline{v}) - L(\underline{u}_0) = 0, \tag{25}$$

$$\hat{H}(\overline{v}, 0) = L(\overline{v}) - L(\overline{u}_0) = 0, \tag{26}$$

$$\hat{H}(\underline{v}, 1) = L(\underline{v}) = 0, \tag{27}$$

$$\hat{H}(\overline{v}, 1) = L(\overline{v}) = 0, \tag{28}$$

the changing process of λ from zero to unity is just that of $\hat{H}(\underline{v}, \lambda)$ and $\hat{H}(\overline{v}, \lambda)$ from $\underline{u}_0(x, t; \alpha)$, $\overline{u}_0(x, t; \alpha)$ to $\underline{u}(x, t; \alpha)$, $\overline{u}(x, t; \alpha)$, respectively. The embedding parameter $\lambda \in [0, 1]$ can be considered as an expanding parameter (see [19, 20, 26]), and assume that the solutions of Eqs. (21) and (22) can be written as a power series in λ:

$$\underline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x, t; \alpha), \tag{29}$$

$$\overline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x, t; \alpha). \tag{30}$$

Definition 4.1 (see [21])**.** The He polynomials is defined as follows:

$$H_n(v_0, v_1, ..., v_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} N \left(\sum_{k=0}^{n} \lambda^k v_k \right), \; n = 0, 1, 2, ... \, .$$

If the terms $N\underline{v}(x, t; \alpha)$ and $N\overline{v}(x, t; \alpha)$ are nonlinear functions, then they can be decomposed as:

$$N\underline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n H_n \left(\underline{v}(x, t; \alpha) \right), \tag{31}$$

$$N\overline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n H_n \left(\overline{v}(x, t; \alpha) \right), \tag{32}$$

where the H_n are He's polynomials and are calculated by the last definition.

Theorem 4.2. Suppose that N is an increasing nonlinear fuzzy-valued function, and $v = [\sum_{k=0}^{\infty} \lambda^k \underline{v}_k, \sum_{k=0}^{\infty} \lambda^k \overline{v}_k]$, then for an embedding parameter $\lambda \in [0,1]$ we have

$$\frac{\partial^n}{\partial \lambda^n} N(v)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k \right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k \right)_{|\lambda=0} \right]. \tag{33}$$

Proof. Since N is an increasing function, then

$$\begin{aligned} N(v) &= \left[N\left(\sum_{k=0}^{\infty} \lambda^k \underline{v}_k \right), N\left(\sum_{k=0}^{\infty} \lambda^k \overline{v}_k \right) \right] \\ &= \left[N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \underline{v}_k \right), N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \overline{v}_k \right) \right], \end{aligned}$$

by using the derivative in (1)-differentiable, we have such result as following:

$$\begin{aligned} &\frac{\partial^n}{\partial \lambda^n} N(v)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k \underline{v}_k \right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k \overline{v}_k \right)_{|\lambda=0} \right] \\ &= \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \underline{v}_k \right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \overline{v}_k \right)_{|\lambda=0} \right] \\ &= \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k \right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k \right)_{|\lambda=0} \right]. \end{aligned}$$

So, the proof of the theorem is complete. □

Dependence on Theorem (4.2), if N is a decreasing nonlinear fuzzy-valued function, then

$$\frac{\partial^n}{\partial \lambda^n} N(v)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k \right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k \right)_{|\lambda=0} \right]. \tag{34}$$

Now, substituting Eqs. (29), (31) in (17) and (30), (32) in (18) respectively, we get the solution of the first system (13)-(14) as:

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,t;\alpha) = \mathcal{L}^{-1} \Big\{ &\underline{U}(x,s,\underline{f}(x;\alpha), \underline{g}(x;\alpha), \\ &\lambda \Phi(D_x \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,s;\alpha)), \lambda \sum_{n=0}^{\infty} \lambda^n H_n(\underline{v}(x,s;\alpha))) \Big\}, \end{aligned} \tag{35}$$

$$\sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\right.$$
$$\left.\lambda\Phi(D_x \sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,s;\alpha)), \lambda \sum_{n=0}^{\infty} \lambda^n H_n(\overline{v}(x,s;\alpha)))\right\}. \tag{36}$$

Similarly, substituting Eqs. (29), (30) and (32) in (19) and (29), (30) and (31) in (20) respectively, we get the solution of the second system (15)-(16) as:

$$\sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,t;\alpha) = \mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\right.$$
$$\left.\lambda\Phi(D_x \sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,s;\alpha)), \lambda \sum_{n=0}^{\infty} \lambda^n H_n(\overline{v}(x,s;\alpha)))\right\}, \tag{37}$$

$$\sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\right.$$
$$\left.\lambda\Phi(D_x \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,s;\alpha)), \lambda \sum_{n=0}^{\infty} \lambda^n H_n(\underline{v}(x,s;\alpha)))\right\}, \tag{38}$$

which is the combination of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of λ, the following approximations in Eqs. (35) and (36) are obtained

$$\begin{aligned}
\lambda^0 \quad &: \quad \underline{v}_0 = \mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha))\right\}, \\
&\quad\ \ \overline{v}_0 = \mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha))\right\}, \\
\lambda^1 \quad &: \quad \underline{v}_1 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_0(x,s;\alpha)), H_0(\underline{v}(x,s;\alpha)))\right\}, \\
&\quad\ \ \overline{v}_1 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_0(x,s;\alpha)), H_0(\overline{v}(x,s;\alpha)))\right\}, \\
\lambda^2 \quad &: \quad \underline{v}_2 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_1(x,s;\alpha)), H_1(\underline{v}(x,s;\alpha)))\right\}, \\
&\quad\ \ \overline{v}_2 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_1(x,s;\alpha)), H_1(\overline{v}(x,s;\alpha)))\right\}, \\
\lambda^3 \quad &: \quad \underline{v}_3 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_2(x,s;\alpha)), H_2(\underline{v}(x,s;\alpha)))\right\}, \\
&\quad\ \ \overline{v}_3 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_2(x,s;\alpha)), H_2(\overline{v}(x,s;\alpha)))\right\}, \\
&\quad\ \ \vdots
\end{aligned}$$

Furthermore, the following approximations in Eqs. (37) and (38) are obtained

$$\begin{aligned}
\lambda^0 \quad &: \quad \underline{v}_0 = \mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha))\right\}, \\
&\quad\ \ \overline{v}_0 = \mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha))\right\},
\end{aligned}$$

$$\begin{aligned}
\lambda^1 \quad &: \quad \underline{v}_1 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_0(x,s;\alpha)), H_0(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_1 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_0(x,s;\alpha)), H_0(\underline{v}(x,s;\alpha)))\right\},\\
\lambda^2 \quad &: \quad \underline{v}_2 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_1(x,s;\alpha)), H_1(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_2 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_1(x,s;\alpha)), H_1(\underline{v}(x,s;\alpha)))\right\},\\
\lambda^3 \quad &: \quad \underline{v}_3 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_2(x,s;\alpha)), H_2(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_3 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_2(x,s;\alpha)), H_2(\underline{v}(x,s;\alpha)))\right\},\\
&\quad\ \vdots
\end{aligned}$$

When $\lambda \to 1$, Eqs. (23) and (24) corresponds to Eqs. (21) and (22), and also Eqs. (29) and (30) becomes the approximate solutions of Eqs. (21) and (22), i.e.,

$$\underline{u}(x,t;\alpha) = \lim_{\lambda\to 1}\underline{v}(x,t;\alpha) = \sum_{n=0}^{\infty}\underline{v}_n(x,t;\alpha), \tag{39}$$

$$\overline{u}(x,t;\alpha) = \lim_{\lambda\to 1}\overline{v}(x,t;\alpha) = \sum_{n=0}^{\infty}\overline{v}_n(x,t;\alpha). \tag{40}$$

The series (39) and (40) are convergent for most cases, and as well the rate of convergence depends on $L(\underline{v})$ and $L(\overline{v})$ (see [23]). Now will be discuss the convergence on nonlinear operator N.

Theorem 4.3. *Suppose that (R_F, H_d) be a Banach space and $N : R_F \to R_F$ is a contraction nonlinear mapping with a contractivity $\rho \in (0,1)$, that is*

$$H_d\left(N(v), N(v^*)\right) \leq \rho H_d\left(v, v^*\right),$$

for all $v = [\underline{v}, \overline{v}]$, $v^* = [\underline{v}^*, \overline{v}^*] \in R_F$. The sequence generated by the homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}),\ V_{n-1} = \sum_{i=0}^{n-1} u_i,\ n = 1,2,3,...,$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u) = \{u^* \in R_F \mid H_d\left(u, u^*\right) < r\}$, then we have the following statements:

(i) $H_d\left(V_n, u\right) \leq \rho^n H_d\left(v_0, u\right)$.

(ii) $V_n \in B_r(u)$.

(iii) $\lim_{n\to\infty} V_n = u$.

Proof. (i) Since $(\mathbb{R}_F, H_d)$ is a complete metric space and N is a contraction on $(\mathbb{R}_F, H_d)$, then according to Banach's fixed point theorem the mapping N

has precisely one fixed point $u = [\underline{u}, \overline{u}] \in \mathbb{R}_F$ such that $N(u) = u$, that is, $[N(\underline{u}), N(\overline{u})] = [\underline{u}, \overline{u}]$. Now by the induction method, if $n = 1$ we have

$$H_d\left(V_1, u\right) = H_d\left(N(V_0), N(u)\right) = H_d\left(N(v_0), N(u)\right) \leq \rho H_d\left(v_0, u\right).$$

Again, if $n = 2$ we have

$$H_d\left(V_2, u\right) = H_d\left(N(V_1), N(u)\right) \leq \rho H_d\left(V_1, u\right) \leq \rho^2 H_d\left(v_0, u\right).$$

Assume that $H_d\left(V_{n-1}, u\right) \leq \rho^{n-1} H_d\left(v_0, u\right)$. In the same way, it is easy to see that

$$H_d\left(V_n, u\right) = H_d\left(N(V_{n-1}), N(u)\right) \leq \rho H_d\left(V_{n-1}, u\right) \leq \rho^n H_d\left(v_0, u\right).$$

(ii) Since $0 < \rho < 1$ and from (i), we have

$$H_d\left(V_n, u\right) \leq \rho^n H_d\left(v_0, u\right) < \rho^n r < r.$$

(iii) We know $\lim_{n\to\infty} \rho^n = 0$. Using (i), we have $\lim_{n\to\infty} H_d\left(V_n, u\right) = 0$, that is, $\lim\limits_{n\to\infty} V_n = u$. So, the proof of the theorem is complete. □

5. Numerical experiments

In this section, we provide three numerical examples to demonstrate the application of the C(LT-HPM) for solving the FPDEs. More precisely, in Examples (5.1) and (5.2), we give two locally solutions under strongly generalized differentiability for linear FPDEs unlike the previous papers (see [5, 6]) which give only one locally solution under Seikkala derivative. To test the C(LT-HPM) upon nonlinear FPDEs, we initially apply this method on two linear FPDEs.

Example 5.1. Consider the following linear fuzzy partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \ 0 < x < 1, \ t > 0, \tag{41}$$

subject to the fuzzy initial condition

$$u(x,0) = K(\alpha)\sin(\pi x), \ 0 \leq x \leq 1, \tag{42}$$

where $K(\alpha) = [\alpha - 1, 1 - \alpha]$ for all $\alpha \in [0,1]$.

According to Section 3, the FPDE (41)-(42) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial \underline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}, \ 0 < x < 1, \ t > 0, \tag{43}$$

$$\frac{\partial \overline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}, \ 0 < x < 1, \ t > 0, \tag{44}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (\alpha-1)\sin(\pi x),\ 0 \le x \le 1, \tag{45}$$

$$\overline{u}(x,0;\alpha) = (1-\alpha)\sin(\pi x),\ 0 \le x \le 1, \tag{46}$$

second system

$$\frac{\partial \underline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{47}$$

$$\frac{\partial \overline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{48}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (\alpha-1)\sin(\pi x),\ 0 \le x \le 1, \tag{49}$$

$$\overline{u}(x,0;\alpha) = (1-\alpha)\sin(\pi x),\ 0 \le x \le 1. \tag{50}$$

Firstly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (43) and (44):

$$\underline{U}(x,s;\alpha) = \frac{(\alpha-1)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0 < x < 1, \tag{51}$$

$$\overline{U}(x,s;\alpha) = \frac{(1-\alpha)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0 < x < 1. \tag{52}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (51) and (52) gives

$$\underline{u}(x,t;\alpha) = (\alpha-1)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\}, \tag{53}$$
$$0 < x < 1,\ t > 0,$$

$$\overline{u}(x,t;\alpha) = (1-\alpha)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\}, \tag{54}$$
$$0 < x < 1,\ t > 0.$$

Now, applying the HPM method

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,t;\alpha) &= (\alpha-1)\sin(\pi x)\\ &+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha)\right)\right\}\right\},\end{aligned} \tag{55}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,t;\alpha) &= (1-\alpha)\sin(\pi x)\\ &+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \overline{v}_n(x,t;\alpha)\right)\right\}\right\}.\end{aligned} \tag{56}$$

Comparing the coefficient of like powers of λ, the following approximations are obtained

$$\lambda^0: \quad \underline{v}_0(x,t;\alpha) = (\alpha-1)\sin(\pi x),\ \overline{v}_0(x,t;\alpha) = (1-\alpha)\sin(\pi x),$$

$$\lambda^1: \quad \begin{aligned} \underline{v}_1(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(\alpha-1)t\sin(\pi x),\\ \overline{v}_1(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(1-\alpha)t\sin(\pi x),\end{aligned}$$

$$\lambda^2: \quad \begin{aligned} \underline{v}_2(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x),\\ \overline{v}_2(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x),\end{aligned}$$

$$\lambda^3: \quad \begin{aligned} \underline{v}_3(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(\alpha-1)t^3}{3!}\sin(\pi x),\\ \overline{v}_3(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(1-\alpha)t^3}{3!}\sin(\pi x),\end{aligned}$$

$$\vdots$$

From Eqs. (39) and (40), the approximate solution of the first system is given by

$$\begin{aligned}\underline{u}(x,t;\alpha) &= (\alpha-1)\sin(\pi x) - \pi^2(\alpha-1)t\sin(\pi x) + \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x) - \ldots\\ &= (\alpha-1)\sin(\pi x)\left(1-\pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + \ldots\right),\end{aligned}$$

$$\begin{aligned}\overline{u}(x,t;\alpha) &= (1-\alpha)\sin(\pi x) - \pi^2(1-\alpha)t\sin(\pi x) + \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x) - \ldots\\ &= (1-\alpha)\sin(\pi x)\left(1-\pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + \ldots\right).\end{aligned}$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the first system is given by

$$u(x,t) = \left[(\alpha-1)e^{-\pi^2 t}\sin(\pi x), (1-\alpha)e^{-\pi^2 t}\sin(\pi x)\right].$$

Secondly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (47) and (48):

$$\underline{U}(x,s;\alpha) = \frac{(\alpha-1)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0<x<1, \tag{57}$$

$$\overline{U}(x,s;\alpha) = \frac{(1-\alpha)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0<x<1. \tag{58}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (57) and (58) gives

$$\begin{aligned}&\underline{u}(x,t;\alpha) = (\alpha-1)\sin(\pi x)\\ &+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\overline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\},\ 0<x<1,\ t>0,\end{aligned} \tag{59}$$

$$\begin{aligned}&\overline{u}(x,t;\alpha) = (1-\alpha)\sin(\pi x)\\ &+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\underline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\},\ 0<x<1,\ t>0.\end{aligned} \tag{60}$$

Now, applying the HPM method

$$\begin{aligned}&\sum_{n=0}^{\infty}\lambda^n\underline{v}_n(x,t;\alpha) = (\alpha-1)\sin(\pi x)\\ &+\lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)\right)\right\}\right\},\end{aligned} \tag{61}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha) = (1-\alpha)\sin(\pi x)\\ &+\lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n\underline{v}_n(x,t;\alpha)\right)\right\}\right\}.\end{aligned} \tag{62}$$

Comparing the coefficient of like powers of λ, the following approximations are obtained

$$\lambda^0:\qquad \underline{v}_0(x,t;\alpha) = (\alpha-1)\sin(\pi x),\ \overline{v}_0(x,t;\alpha) = (1-\alpha)\sin(\pi x),$$

$$\begin{aligned}\lambda^1:\quad &\underline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\overline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(1-\alpha)t\sin(\pi x),\\ &\overline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\underline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(\alpha-1)t\sin(\pi x),\end{aligned}$$

$$\begin{aligned}\lambda^2:\quad &\underline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\overline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x),\\ &\overline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\underline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x),\end{aligned}$$

$$\begin{aligned}\lambda^3:\quad &\underline{v}_3(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\overline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(1-\alpha)t^3}{3!}\sin(\pi x),\\ &\overline{v}_3(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2\underline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(\alpha-1)t^3}{3!}\sin(\pi x),\end{aligned}$$

$$\vdots$$

From Eqs. (39) and (40), the approximate solution of the second system is given by

$$\begin{aligned}\underline{u}(x,t;\alpha) &= (\alpha-1)\sin(\pi x) - \pi^2(1-\alpha)t\sin(\pi x) + \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x) - ...\\ &= (\alpha-1)\sin(\pi x)\left(1+\pi^2 t + \frac{\pi^4 t^2}{2!} + \frac{\pi^6 t^3}{3!} + ...\right),\end{aligned}$$

$$\begin{aligned}\overline{u}(x,t;\alpha) &= (1-\alpha)\sin(\pi x) - \pi^2(\alpha-1)t\sin(\pi x) + \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x) - ...\\ &= (1-\alpha)\sin(\pi x)\left(1+\pi^2 t + \frac{\pi^4 t^2}{2!} + \frac{\pi^6 t^3}{3!} + ...\right).\end{aligned}$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the second system is given by

$$u(x,t) = \left[(\alpha-1)e^{\pi^2 t}\sin(\pi x), (1-\alpha)e^{\pi^2 t}\sin(\pi x)\right].$$

Example 5.2. Consider the following linear fuzzy partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{63}$$

subject to the fuzzy initial conditions

$$u(x,0) = K(\alpha)\sin(\pi x),\ u_t(x,0) = 0,\ 0 \le x \le 1, \tag{64}$$

where $K(\alpha) = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ for all $\alpha \in [0,1]$.

According to Section 3, the FPDE (63)-(64) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{65}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{66}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x),\ \underline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1, \tag{67}$$

$$\overline{u}(x,0;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x),\ \overline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1, \tag{68}$$

second system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{69}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{70}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x),\ \underline{u}_t(x,0;\alpha) = 0,\ 0 \leq x \leq 1, \tag{71}$$

$$\overline{u}(x,0;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x),\ \overline{u}_t(x,0;\alpha) = 0,\ 0 \leq x \leq 1. \tag{72}$$

Again, by using the same procedure as mentioned in Section 4, the first few components in the first system are given by

$$\begin{aligned}
\lambda^0 \quad : \quad & \underline{v}_0(x,t;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x), \\
& \overline{v}_0(x,t;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x), \\
\lambda^1 \quad : \quad & \underline{v}_1(x,t;\alpha) = -2\pi^2(0.75 + 0.25\alpha)t^2\sin(\pi x), \\
& \overline{v}_1(x,t;\alpha) = -2\pi^2(1.25 - 0.25\alpha)t^2\sin(\pi x), \\
\lambda^2 \quad : \quad & \underline{v}_2(x,t;\alpha) = \frac{16\pi^4(0.75 + 0.25\alpha)t^4}{4!}\sin(\pi x), \\
& \overline{v}_2(x,t;\alpha) = \frac{16\pi^4(1.25 - 0.25\alpha)t^4}{4!}\sin(\pi x), \\
\lambda^3 \quad : \quad & \underline{v}_3(x,t;\alpha) = \frac{-64\pi^6(0.75 + 0.25\alpha)t^6}{6!}\sin(\pi x), \\
& \overline{v}_3(x,t;\alpha) = \frac{-64\pi^6(1.25 - 0.25\alpha)t^6}{6!}\sin(\pi x), \\
& \vdots
\end{aligned}$$

From Eqs. (39) and (40), the approximate solution of the first system is given by

$$\underline{u}(x,t;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x)\left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + \ldots\right),$$

$$\overline{u}(x,t;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x)\left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + \ldots\right).$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the first system is given by

$$u(x,t) = \left[(0.75 + 0.25\alpha)\sin(\pi x)\cos(2\pi t), (1.25 - 0.25\alpha)\sin(\pi x)\cos(2\pi t)\right].$$

After that, the first few components in the second system are given by

$$
\begin{aligned}
\lambda^0 \quad &: \quad \underline{v}_0(x,t;\alpha) = (0.75+0.25\alpha)\sin(\pi x),\\
&\quad\ \ \overline{v}_0(x,t;\alpha) = (1.25-0.25\alpha)\sin(\pi x),\\
\lambda^1 \quad &: \quad \underline{v}_1(x,t;\alpha) = -2(\pi t)^2(1.25-0.25\alpha)\sin(\pi x),\\
&\quad\ \ \overline{v}_1(x,t;\alpha) = -2(\pi t)^2(0.75+0.25\alpha)\sin(\pi x),\\
\lambda^2 \quad &: \quad \underline{v}_2(x,t;\alpha) = \frac{16(\pi t)^4(0.75+0.25\alpha)}{4!}\sin(\pi x),\\
&\quad\ \ \overline{v}_2(x,t;\alpha) = \frac{16(\pi t)^4(1.25-0.25\alpha)}{4!}\sin(\pi x),\\
\lambda^3 \quad &: \quad \underline{v}_3(x,t;\alpha) = \frac{-64(\pi t)^6(1.25-0.25\alpha)}{6!}\sin(\pi x),\\
&\quad\ \ \overline{v}_3(x,t;\alpha) = \frac{-64(\pi t)^6(0.75+0.25\alpha)}{6!}\sin(\pi x),\\
\lambda^4 \quad &: \quad \underline{v}_4(x,t;\alpha) = \frac{256(\pi t)^8(0.75+0.25\alpha)}{8!}\sin(\pi x),\\
&\quad\ \ \overline{v}_4(x,t;\alpha) = \frac{256(\pi t)^8(1.25-0.25\alpha)}{8!}\sin(\pi x),\\
\lambda^5 \quad &: \quad \underline{v}_5(x,t;\alpha) = \frac{-1024(\pi t)^{10}(1.25-0.25\alpha)}{10!}\sin(\pi x),\\
&\quad\ \ \overline{v}_5(x,t;\alpha) = \frac{-1024(\pi t)^{10}(0.75+0.25\alpha)}{10!}\sin(\pi x),\\
\lambda^6 \quad &: \quad \underline{v}_6(x,t;\alpha) = \frac{4096(\pi t)^{12}(0.75+0.25\alpha)}{12!}\sin(\pi x),\\
&\quad\ \ \overline{v}_6(x,t;\alpha) = \frac{4096(\pi t)^{12}(1.25-0.25\alpha)}{12!}\sin(\pi x),\\
\lambda^7 \quad &: \quad \underline{v}_7(x,t;\alpha) = \frac{-16\,384(\pi t)^{14}(1.25-0.25\alpha)}{14!}\sin(\pi x),\\
&\quad\ \ \overline{v}_7(x,t;\alpha) = \frac{-16\,384(\pi t)^{14}(0.75+0.25\alpha)}{14!}\sin(\pi x),\\
&\quad\ \ \vdots
\end{aligned}
$$

Therefore, the approximate solution of the second system is given by

$$
\begin{aligned}
\underline{u}(x,t;\alpha) &= (0.75+0.25\alpha)\sin(\pi x)\left(1+\frac{(2\pi t)^4}{4!}+\frac{(2\pi t)^8}{8!}+\frac{(2\pi t)^{12}}{12!}+...\right)\\
&-(1.25-0.25\alpha)\sin(\pi x)\left(\frac{(2\pi t)^2}{2!}+\frac{(2\pi t)^6}{6!}+\frac{(2\pi t)^{10}}{10!}+\frac{(2\pi t)^{14}}{14!}+...\right)\\
&=\left((0.75+0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}-(1.25-0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}\right)\sin(\pi x),
\end{aligned}
$$

$$\begin{aligned}\overline{u}(x,t;\alpha) &= (1.25-0.25\alpha)\sin(\pi x)\left(1+\frac{(2\pi t)^4}{4!}+\frac{(2\pi t)^8}{8!}+\frac{(2\pi t)^{12}}{12!}+...\right)\\ &-(0.75+0.25\alpha)\sin(\pi x)\left(\frac{(2\pi t)^2}{2!}+\frac{(2\pi t)^6}{6!}+\frac{(2\pi t)^{10}}{10!}+\frac{(2\pi t)^{14}}{14!}+...\right)\\ &=\left((1.25-0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}-(0.75+0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}\right)\sin(\pi x).\end{aligned}$$

By the ratio test, the series $\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}$ and $\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}$ are convergent for all $t\in(0,\infty)$.

To allow a clear overview of our work and to demonstrate the discussed method, we present the first attempt to solve nonlinear FPDEs under strongly generalized differentiability.

Example 5.3. Consider the following nonlinear fuzzy partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\sin(u(x,t))=\frac{\partial^2 u(x,t)}{\partial x^2},\ 0<x<1,\ t>0, \tag{73}$$

subject to the fuzzy initial conditions

$$u(x,0)=0,\ u_t(x,0)=K(\alpha)sechx,\ 0\le x\le 1, \tag{74}$$

where $K(\alpha)=[\alpha,2-\alpha]$ for all $\alpha\in[0,1]$.

It is noted here that $\sin x$ is a continuous increasing function on $(0,1)$. By using Zadeh's extension principle, we get $[\sin(u(x,t))]_\alpha=[\sin(\underline{u}(x,t)),\sin(\overline{u}(x,t))]$ for all $\alpha\in[0,1]$. According to Section 3, the FPDE (73)-(74) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2}+\sin(\underline{u}(x,t))=\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{75}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2}+\sin(\overline{u}(x,t))=\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{76}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha)=0,\ \underline{u}_t(x,0;\alpha)=\alpha sechx,\ 0\le x\le 1, \tag{77}$$

$$\overline{u}(x,0;\alpha)=0,\ \overline{u}_t(x,0;\alpha)=(2-\alpha)sechx,\ 0\le x\le 1, \tag{78}$$

second system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2}+\sin(\overline{u}(x,t))=\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{79}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} + \sin(\underline{u}(x,t)) = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{80}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = 0,\ \underline{u}_t(x,0;\alpha) = \alpha sechx,\ 0 \leq x \leq 1, \tag{81}$$

$$\overline{u}(x,0;\alpha) = 0,\ \overline{u}_t(x,0;\alpha) = (2-\alpha)sechx,\ 0 \leq x \leq 1. \tag{82}$$

Firstly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (75) and (76):

$$\underline{U}(x,s;\alpha) = \frac{\alpha sechx}{s^2} + \frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} - \sin(\underline{u}(x,t))\right\},\ 0 < x < 1, \tag{83}$$

$$\overline{U}(x,s;\alpha) = \frac{(2-\alpha)sechx}{s^2} + \frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2} - \sin(\overline{u}(x,t))\right\}, 0<x<1. \tag{84}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (83) and (84) gives

$$\underline{u}(x,t;\alpha) = \alpha t sechx + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} - \sin(\underline{u}(x,t))\right\}\right\}, \tag{85}$$

$$\overline{u}(x,t;\alpha) = (2-\alpha)t sechx + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2} - \sin(\overline{u}(x,t))\right\}\right\}. \tag{86}$$

Since

$$\begin{aligned}
\sin(\underline{u}(x,t)) &= \underline{u}(x,t) - \frac{(\underline{u}(x,t))^3}{3!} + ... + \frac{(-1)^n}{(2n+1)!}(\underline{u}(x,t))^{2n+1} + ...\\
&= \sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t) - \frac{1}{3!}\left(\sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t)\right)^3 + \frac{1}{5!}\left(\sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t)\right)^5 - ...\\
&= \sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t) - \frac{1}{3!}\sum_{n=0}^{\infty}\lambda^n A_n(\underline{u}(x,t)) + \frac{1}{5!}\sum_{n=0}^{\infty}\lambda^n B_n(\underline{u}(x,t)) - ...\\
&= \sum_{n=0}^{\infty}\lambda^n H_n(\underline{u}(x,t)),
\end{aligned} \tag{87}$$

where A_n, B_n are Adomain polynomials (see [23, 24]) and H_n are He's polynomials. From Section 4, then Eqs. (85) and (86) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha) = \alpha t sechx\\
&+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha)\right) - \sum_{n=0}^{\infty}\lambda^n H_n(\underline{v}(x,t;\alpha))\right\}\right\},
\end{aligned} \tag{88}$$

$$\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)=(2-\alpha)tsechx$$
$$+\lambda\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)\right)-\sum_{n=0}^{\infty}\lambda^n H_n(\overline{v}(x,t;\alpha))\right\}\right\}. \tag{89}$$

From Theorem (4.2) and Eq. (87), we have

$$\frac{\partial^n}{\partial\lambda^n}\left(A_0+\lambda A_1+...+\lambda^n A_n\right)_{|\lambda=0}$$
$$=\frac{\partial^n}{\partial\lambda^n}\left(\underline{u}_0(x,t)+\lambda\underline{u}_1(x,t)+...+\lambda^n\underline{u}_n(x,t)\right)^3_{|\lambda=0}, \tag{90}$$

$$\frac{\partial^n}{\partial\lambda^n}\left(B_0+\lambda B_1+...+\lambda^n B_n\right)_{|\lambda=0}$$
$$=\frac{\partial^n}{\partial\lambda^n}\left(\underline{u}_0(x,t)+\lambda\underline{u}_1(x,t)+...+\lambda^n\underline{u}_n(x,t)\right)^5_{|\lambda=0}. \tag{91}$$

Now, if $n=0$, we get

$$A_0=\underline{u}_0^3(x,t),\ B_0=\underline{u}_0^5(x,t), \tag{92}$$

if $n=1$, we get

$$A_1=3\underline{u}_0^2(x,t)\underline{u}_1(x,t),\ B_1=5\underline{u}_0^4(x,t)\underline{u}_1(x,t), \tag{93}$$

if $n=2$, we get

$$A_2=3\underline{u}_0^2(x,t)\underline{u}_2(x,t)+3\underline{u}_1^2(x,t)\underline{u}_0(x,t),$$
$$B_2=5\underline{u}_0^4(x,t)\underline{u}_2(x,t)+10\underline{u}_0^3(x,t)\underline{u}_1^2(x,t), \tag{94}$$

$$\vdots$$

From Eqs. (87), (92), (93) and (94), we have

$$H_0(\underline{u}(x,t))=\underline{u}_0(x,t)-\frac{\underline{u}_0^3(x,t)}{3!}+\frac{\underline{u}_0^5(x,t)}{5!}-..., \tag{95}$$

$$H_1(\underline{u}(x,t))=\underline{u}_1(x,t)-\frac{3\underline{u}_0^2(x,t)\underline{u}_1(x,t)}{3!}+\frac{5\underline{u}_0^4(x,t)\underline{u}_1(x,t)}{5!}-..., \tag{96}$$

$$H_2(\underline{u}(x,t))=\underline{u}_2(x,t)-\frac{3\underline{u}_0^2(x,t)\underline{u}_2(x,t)+3\underline{u}_1^2(x,t)\underline{u}_0(x,t)}{3!}$$
$$+\frac{5\underline{u}_0^4(x,t)\underline{u}_2(x,t)+10\underline{u}_0^3(x,t)\underline{u}_1^2(x,t)}{5!}-..., \tag{97}$$

$$\vdots$$

Similarly, we find $H_0(\overline{u}(x,t))$, $H_1(\overline{u}(x,t))$, $H_2(\overline{u}(x,t))$,... . The first few components in the first system are given by

$$
\begin{aligned}
\lambda^0 \quad : \quad & \underline{v}_0(x,t;\alpha) = \alpha t sechx, \\
& \overline{v}_0(x,t;\alpha) = (2-\alpha) t sechx, \\
\lambda^1 \quad : \quad & \underline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_0(x,t;\alpha)}{\partial x^2} - H_0(\underline{u}(x,t))\right\}\right\} \\
& = \frac{-\alpha t^3 sechx \tanh x}{3!} - \frac{\alpha t^3 sechx}{3!} + \frac{(\alpha sechx)^3 t^5}{5!} - \frac{(\alpha sechx)^5 t^7}{7!} + ..., \\
& \overline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_0(x,t;\alpha)}{\partial x^2} - H_0(\overline{u}(x,t))\right\}\right\} \\
& = \frac{-(2-\alpha) t^3 sechx \tanh x}{3!} - \frac{(2-\alpha) t^3 sechx}{3!} \\
& + \frac{((2-\alpha) sechx)^3 t^5}{5!} - \frac{((2-\alpha) sechx)^5 t^7}{7!} + ..., \\
\lambda^2 \quad : \quad & \underline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_1(x,t;\alpha)}{\partial x^2} - H_1(\underline{u}(x,t))\right\}\right\} \\
& = \frac{330(\alpha sechx)^9 t^{13}}{13!} - \frac{162(\alpha sechx)^7 t^{11}}{11!} \\
& + \frac{(\alpha sechx)^5 t^9}{9!}(35\tanh x + 62 - 30\tanh^2 x) \\
& + \frac{(\alpha sechx)^3 t^7}{7!}(12\tanh^2 x - 14 - 10\tanh x) \\
& + \frac{(\alpha sechx) t^5}{5!}(2 - 6\tanh^3 x - 2\tanh^2 x + 6\tanh x) + ..., \\
& \overline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_1(x,t;\alpha)}{\partial x^2} - H_1(\overline{u}(x,t))\right\}\right\} \\
& = \frac{330((2-\alpha) sechx)^9 t^{13}}{13!} - \frac{162((2-\alpha) sechx)^7 t^{11}}{11!} \\
& + \frac{((2-\alpha) sechx)^5 t^9}{9!}(35\tanh x + 62 - 30\tanh^2 x) \\
& + \frac{((2-\alpha) sechx)^3 t^7}{7!}(12\tanh^2 x - 14 - 10\tanh x) \\
& + \frac{((2-\alpha) sechx) t^5}{5!}(2 - 6\tanh^3 x - 2\tanh^2 x + 6\tanh x) + ..., \\
& \vdots
\end{aligned}
$$

Therefore, the approximate solution of the first system is given by

$$
\begin{aligned}
\underline{u}(x,t) \quad = \quad & \alpha t sechx - \frac{\alpha t^3 sechx \tanh x}{3!} - \frac{\alpha t^3 sechx}{3!} + \frac{(\alpha sechx)^3 t^5}{5!} \\
& - \frac{(\alpha sechx)^5 t^7}{7!} + \frac{330(\alpha sechx)^9 t^{13}}{13!} - \frac{162(\alpha sechx)^7 t^{11}}{11!}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\alpha sechx)^5t^9}{9!}(35\tanh x+62-30\tanh^2 x)\\
&+\frac{(\alpha sechx)^3t^7}{7!}(12\tanh^2 x-14-10\tanh x)\\
&+\frac{(\alpha sechx)t^5}{5!}(2-6\tanh^3 x-2\tanh^2 x+6\tanh x)+...,
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}(x,t)=&\,(2-\alpha)tsechx-\frac{(2-\alpha)t^3 sechx\tanh x}{3!}-\frac{(2-\alpha)t^3 sechx}{3!}\\
&+\frac{((2-\alpha)sechx)^3t^5}{5!}-\frac{((2-\alpha)sechx)^5t^7}{7!}+\frac{330((2-\alpha)sechx)^9t^{13}}{13!}\\
&-\frac{162((2-\alpha)sechx)^7t^{11}}{11!}+\frac{((2-\alpha)sechx)^5t^9}{9!}(35\tanh x+62-30\tanh^2 x)\\
&+\frac{((2-\alpha)sechx)^3t^7}{7!}(12\tanh^2 x-14-10\tanh x)\\
&+\frac{((2-\alpha)sechx)t^5}{5!}(2-6\tanh^3 x-2\tanh^2 x+6\tanh x)+\dots .
\end{aligned}
$$

The approximate fuzzy solution to the first system in parametric form for $\alpha = 0.5$, various x in $[0,1]$ and various t in $[0,5]$ is given in Figure 1. It is clear from the Figure 1 that, the numerical results obtained by C(LT-HPM) satisfy the convex symmetric triangular fuzzy number. Here, we use the first four terms in $\underline{u}(x,t)$ and $\overline{u}(x,t)$ to sketch the approximate fuzzy solution.

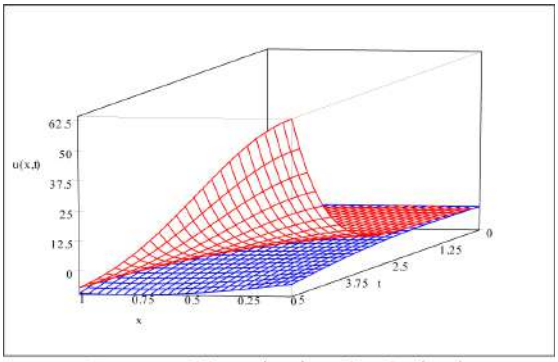

Figure 1: blue $\underline{u}(x,t)$ and red $\overline{u}(x,t)$

Secondly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (79) and (80):

$$(98)\qquad \underline{U}(x,s;\alpha)=\frac{\alpha sechx}{s^2}+\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\overline{u}(x,t;\alpha)}{\partial x^2}-\sin(\overline{u}(x,t))\right\},\ 0<x<1,$$

$$(99)\qquad \overline{U}(x,s;\alpha)=\frac{(2-\alpha)sechx}{s^2}+\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\underline{u}(x,t;\alpha)}{\partial x^2}-\sin(\underline{u}(x,t))\right\},0<x<1.$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (98) and (99) gives

$$(100)\qquad \underline{u}(x,t;\alpha)=\alpha tsechx+\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\overline{u}(x,t;\alpha)}{\partial x^2}-\sin(\overline{u}(x,t))\right\}\right\},$$

$$\overline{u}(x,t;\alpha) = (2-\alpha)tsechx + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} - \sin(\underline{u}(x,t))\right\}\right\}. \tag{101}$$

Again, using the same procedure in Section 4, the first few components in the second system are given by

$$\begin{aligned}
\lambda^0 \quad : \quad & \underline{v}_0(x,t;\alpha) = \alpha tsechx, \\
& \overline{v}_0(x,t;\alpha) = (2-\alpha)tsechx, \\
\lambda^1 \quad : \quad & \underline{v}_1(x,t;\alpha) = \frac{-2(2-\alpha)t^3sech^3x}{3!} + \frac{(2-\alpha)^3t^5sech^3x}{5!} \\
& -\frac{(2-\alpha)^5t^7sech^5x}{7!} + ..., \\
& \overline{v}_1(x,t;\alpha) = \frac{-2\alpha t^3sech^3x}{3!} + \frac{\alpha^3t^5sech^3x}{5!} - \frac{\alpha^5t^7sech^5x}{7!} + ..., \\
\lambda^2 \quad : \quad & \underline{v}_2(x,t;\alpha) = \left(\frac{\alpha t^5}{15} - \frac{\alpha^3t^7}{1260}\right)sech^3x + \frac{\alpha^5t^9sech^5x}{60\,480} \\
& +\left(\frac{\alpha^3t^7}{420} - \frac{\alpha t^5}{5}\right)\tanh^2 xsech^3x \\
& -\frac{\alpha^5t^9\tanh^2 xsech^5x}{12\,096} + \frac{1}{10\,080}\left(\frac{7\alpha^3(\alpha-2)^2t^9}{12} - 40\alpha(\alpha-2)^2t^7\right)sech^5x \\
& -\frac{\alpha^5(\alpha-2)^4t^{13}sech^9x}{18\,869\,760} \\
& +\left(\frac{\alpha^3(\alpha-2)^4t^{11}}{316\,800} - \frac{\alpha(\alpha-2)^4t^9}{5184} - \frac{\alpha^5(\alpha-2)^2t^{11}}{1108\,800}\right)sech^7x + ..., \\
& \overline{v}_2(x,t;\alpha) = \left(\frac{(2-\alpha)\,t^5}{15} - \frac{(2-\alpha)^3\,t^7}{1260}\right)sech^3x + \frac{(2-\alpha)^5\,t^9sech^5x}{60\,480} \\
& +\left(\frac{(2-\alpha)^3\,t^7}{420} - \frac{(2-\alpha)\,t^5}{5}\right)\tanh^2 xsech^3x - \frac{(2-\alpha)^5\,t^9\tanh^2 xsech^5x}{12\,096} \\
& +\frac{1}{10\,080}\left(\frac{7\alpha^2\,(2-\alpha)^3\,t^9}{12} - 40\alpha^2\,(2-\alpha)\,t^7\right)sech^5x - \frac{\alpha^4\,(2-\alpha)^5\,t^{13}sech^9x}{18\,869\,760} \\
& +\left(\frac{\alpha^4\,(2-\alpha)^3\,t^{11}}{316\,800} - \frac{\alpha^4\,(2-\alpha)\,t^9}{5184} - \frac{\alpha^2\,(2-\alpha)^5\,t^{11}}{1108\,800}\right)sech^7x + ..., \\
& \vdots
\end{aligned}$$

Therefore, the approximate solution of the second system is given by

$$\begin{aligned}
\underline{u}(x,t) = {} & \alpha tsechx - \frac{2(2-\alpha)}{3!}t^3sech^3x + \frac{(2-\alpha)^3}{5!}t^5sech^3x \\
& - \frac{(2-\alpha)^5}{7!}t^7sech^5x + \left(\frac{\alpha}{15}t^5 - \frac{\alpha^3}{1260}t^7\right)sech^3x + \frac{\alpha^5}{60\,480}t^9sech^5x
\end{aligned}$$

$$+\left(\frac{\alpha^3}{420}t^7-\frac{\alpha}{5}t^5\right)\tanh^2 xsech^3x-\frac{\alpha^5}{12\,096}t^9\tanh^2 xsech^5x$$
$$+\frac{1}{10\,080}\left(\frac{7}{12}\alpha^3\left(\alpha-2\right)^2t^9-40\alpha\left(\alpha-2\right)^2t^7\right)sech^5x-\frac{\alpha^5\left(\alpha-2\right)^4}{18\,869\,760}t^{13}sech^9x$$
$$+\left(\frac{\alpha^3\left(\alpha-2\right)^4}{316\,800}t^{11}-\frac{\alpha\left(\alpha-2\right)^4}{5184}t^9-\frac{\alpha^5\left(\alpha-2\right)^2}{1108\,800}t^{11}\right)sech^7x+\ldots .$$

$$\overline{u}(x,t)=(2-\alpha)tsechx-\frac{2\alpha}{3!}t^3sech^3x+\frac{\alpha^3}{5!}t^5sech^3x-\frac{\alpha^5}{7!}t^7sech^5x$$
$$+\left(\frac{(2-\alpha)}{15}t^5-\frac{(2-\alpha)^3}{1260}t^7\right)sech^3x+\frac{(2-\alpha)^5}{60\,480}t^9sech^5x$$
$$+\left(\frac{(2-\alpha)^3}{420}t^7-\frac{(2-\alpha)}{5}t^5\right)\tanh^2 xsech^3x-\frac{(2-\alpha)^5}{12\,096}t^9\tanh^2 xsech^5x$$
$$+\frac{1}{10\,080}\left(\frac{7}{12}\alpha^2(2-\alpha)^3t^9-40\alpha^2(2-\alpha)t^7\right)sech^5x-\frac{\alpha^4(2-\alpha)^5}{18\,869\,760}t^{13}sech^9x$$
$$+\left(\frac{\alpha^4(2-\alpha)^3}{316\,800}t^{11}-\frac{\alpha^4(2-\alpha)}{5184}t^9-\frac{\alpha^2(2-\alpha)^5}{1108\,800}t^{11}\right)sech^7x+...,$$

The approximate fuzzy solution to the second system in parametric form for $\alpha = 0.2$, various x in $[0,1]$ and various t in $[0,5]$ is given in Figure 2. It is clear from the Figure 2 that, the numerical results obtained by C(LT-HPM) satisfy the fuzzy numbers properties by taking the triangular fuzzy numbers shape. Here, we use the first seven terms in $\underline{u}(x,t)$ and $\overline{u}(x,t)$ to sketch the approximate fuzzy solution.

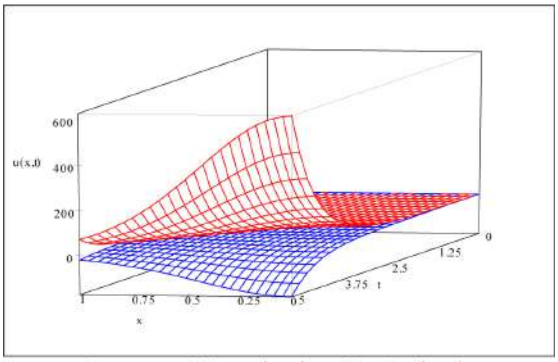

Figure 2: blue $\underline{u}(x,t)$ and red $\overline{u}(x,t)$

6. Conclusion

The primary objective of this paper is to determine an approximate-analytical solutions for the fuzzy partial differential equations. We have accomplished this objective by applying C(LT-HPM). The results are very encouraging, demonstrating the unwavering quality and proficiency of the proposed strategy with less computational work and time. This strategy is based on the definition of strongly generalized differentiability.

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Accepted: 7.08.2018

On some properties of certain subclasses of univalent functions

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Abstract. This study interested in two subclasses of analytic functions defined on the open unit disc of the complex plain, we discuss some neighborhood properties, integral means inequalities and some results concerning the partial sums of the functions belonging to these subclasses.
Keywords: analytic function, neighborhood, integral means, partial sums.

1. Introduction

Let T denoted to class of function of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0), \tag{1.1}$$

which are analytic function in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $T^*(\alpha)$ and $C(\alpha)$ the subclasses of starlike functions of order α, and convex functions of order α, respectively. Theses two subclases are defined by Silverman [11] as following:

$$T^*(\alpha) = \left\{ f \in T : \Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\}, \tag{1.2}$$

and

$$C(\alpha) = \left\{ f \in T : \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\}. \tag{1.3}$$

For $\mu > 0$ and $a, c \in \mathbb{C}$, are such that $\Re\{c - a\} \geq 0$, Raina and Sharma [9] (see also [3], [4]) defined the integral operator $J_{\mu}^{a,c} : T \longrightarrow T$, as following:

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(i) for $\Re\{c-a\}>0$ and $\Re\{a\}>-\mu$ by

$$J_\mu^{a,c}f(z)=\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f(zt^\mu)dt; \tag{1.4}$$

(ii) for $a=c$ by

$$J_\mu^{a,a}f(z)=f(z), \tag{1.5}$$

where Γ stands for Euler's Gamma function (which is valid for all complex numbers except the non-positive integers).

For $f(z)$ defined by (1.1), it is easily from (1.4) and (1.5) that:

$$J_\mu^{a,c}f(z)=z-\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_kz^k \quad (\mu>0,\Re\{c\}\geq\Re\{a\}>-\mu). \tag{1.6}$$

Let $M_\mu^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in T$ for which:

$$\frac{z(J_\mu^{a,c}f(z))'}{J_\mu^{a,c}f(z)}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha \qquad (-1\leq B<A\leq 1, 0\leq\alpha<1), \tag{1.7}$$

that is, that

$$M_\mu^{a,c}(\alpha;A,B)=\left\{f\in T:\left|\frac{\frac{z(J_\mu^{a,c}f(z))'}{J_\mu^{a,c}f(z)}-1}{B\frac{z(J_\mu^{a,c}f(z))'}{J_\mu^{a,c}f(z)}-[B+(A-B)(1-\alpha)]}\right|<1, z\in U\right\}. \tag{1.8}$$

Also, let $N_\mu^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in T$ for which:

$$1+\frac{z(J_\mu^{a,c}f(z))''}{(J_\mu^{a,c}f(z))'}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha,$$

form (1.7) and (1.8), it is clear that

$$f(z)\in N_\mu^{a,c}(\alpha;A,B)\iff zf'(z)\in M_\mu^{a,c}(\alpha;A,B). \tag{1.9}$$

It is easily to see that:

(i) $M_\mu^{a,a}(\alpha;A,B)=T^*(A,B,\alpha)$ and $N_\mu^{a,c}(\alpha;A,B)=C(A,B,\alpha)$, see [2, with $p=1$];

(ii) $M_\mu^{a,a}(\alpha;\beta,-\beta)=T^*(\alpha,\beta)$ and $N_\mu^{a,a}(\alpha;\beta,-\beta)=C(\alpha,\beta)$ the subclasses of starlike and convex of order $0\leq\alpha<1$ and type $0<\beta\leq1$, see [6];

(iii) $M_\mu^{a,a}(\alpha;1,-1)=T^*(\alpha)$ and $N_\mu^{a,a}(\alpha;1,-1)=C(\alpha)$ the subclasses of starlike and convex of order $0\leq\alpha<1$, see [11].

The object of the present paper is to determine the neighborhood properties for each of the subclasses $M_\mu^{a,c}(\alpha;A,B)$ and $N_\mu^{a,c}(\alpha;A,B)$. Moreover, investigate integral means inequalities, and some results concerning partial sums for functions belonging to the subclass $M_{\mu}^{a,c}(\alpha; A, B)$. We will make use of the following lemmas, also otherwise mentioned, we assume in the reminder of this paper that, $0 \le \alpha < 1, -1 \le B < A \le 1, \mu > 0$, $a, c \in \mathbb{R}, c > a > -\mu$ and $z \in U$.

Lemma 1 ([8]). *Let the function $f(z)$ be given by (1.1). Then $f \in M_{\mu}^{a,c}(\alpha; A, B)$, if and only if*

$$\sum_{k=2}^{\infty} [(1-B)(k-1) + (A-B)(1-\alpha)] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k \le (A-B)(1-\alpha)\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}. \tag{1.10}$$

Lemma 2 ([8]). *Let the function $f(z)$ be given by (1.1). Then $f \in N_{\mu}^{a,c}(\alpha; A, B)$, if and only if*

$$\sum_{k=2}^{\infty} [(1-B)(k-1) + (A-B)(1-\alpha)] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} k a_k \le (A-B)(1-\alpha)\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}. \tag{1.11}$$

2. Neighborhood results

Following the earlier investigations of Goodman [5] and Ruscheweyh [10], the $\delta-$ neighborhood is defined as following:

$$N_\delta(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \sum_{k=2}^{\infty} k\,|a_k - b_k| \le \delta \right\}. \tag{2.1}$$

For the identity function $e(z) = z$, we immediately have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \sum_{k=2}^{\infty} k\,|b_k| \le \delta \right\}, \tag{2.2}$$

where the function f is given by (1.1).

Theorem 1. *If the function $f(z)$ defined by* (1.1) *is in the subclass $M_{\mu}^{a,c}(\alpha; A, B)$.*

Then $M_{\mu}^{a,c}(\alpha; A, B) \subset N_\delta(e)$, where

$$\delta = \frac{2\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)}. \tag{2.3}$$

Proof. Since $f \in M_{\mu}^{a,c}(\alpha; A, B)$, by using Lemma 1 and from (1.10), we find

$$\frac{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)}{2\Gamma(a+\mu)\Gamma(c+2\mu)} \sum_{k=2}^{\infty} k a_k \le$$

$$\sum_{k=2}^{\infty}[(1-B)(k-1)+(A-B)(1-\alpha)]\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k \leq (A-B)(1-\alpha).$$

It is clear

$$\sum_{k=2}^{\infty}ka_k \leq \frac{2\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)} = \delta.$$

Corollary 1. *If $f \in T$ is in the class $T^*(A,B,\alpha)$. Then*

$$T^*(A,B,\alpha) \subset N(e),$$

where

$$\delta = \frac{2(A-B)(1-\alpha)}{(1-B)+(A-B)(1-\alpha)}.$$

Corollary 2. *If $f \in T$ is in the class $T^*(\alpha,\beta)$. Then*

$$T^*(\alpha,\beta) \subset N(e),$$

where

$$\delta = \frac{4\beta(1-\alpha)}{1+2\beta(2-\alpha)}.$$

Corollary 3. *If $f \in T$ is in the class $T^*(\alpha)$. Then*

$$T^*(\alpha) \subset N(e),$$

where

$$\delta = \frac{2(1-\alpha)}{2-\alpha}.$$

by similarly applying Lemma 2 instead of Lemma1, we can prove following.

Theorem 2. *If the function $f(z)$ defined by* (1.1) *is in the subclass $N_\mu^{a,c}(\alpha;A,B)$. Then $N_\mu^{a,c}(\alpha;A,B) \subset N_\delta(e)$,* where

$$\delta = \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)}. \tag{2.4}$$

Corollary 4. *If $f \in T$ is in the class $C(A,B,\alpha)$. Then*

$$C(A,B,\alpha) \subset N(e),$$

where

$$\delta = \frac{(A-B)(1-\alpha)}{(1-B)+(A-B)(1-\alpha)}.$$

Corollary 5. *If $f \in T$ is in the class $C(\alpha,\beta)$. Then*

$$C(\alpha,\beta) \subset N(e),$$

where

$$\delta = \frac{2\beta(1-\alpha)}{(1+\beta)+2\beta(1-\alpha)}.$$

Corollary 6. *If $f \in T$ is in the class $C(\alpha)$. Then*

$$C(\alpha) \subset N(e),$$

where

$$\delta = \frac{1-\alpha}{2-\alpha}.$$

We will determine the neighborhood properties for each of the following (slightly modified) function subclass $M_{\mu}^{a,c,\rho}(\alpha;A,B)$.

A functions $f \in T$ is said to be in the class $M_{\mu}^{a,c,\rho}(\alpha;A,B)$ if there exists a function $g \in M_{\mu}^{a,c}(\alpha;A,B)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\rho \qquad (z \in U; 0 \le \rho < 1). \tag{2.5}$$

The proofs of the following results involving the neighborhood properties for the subclass $M_{\mu}^{a,c,\rho}(\alpha;A,B)$, is similar to those given in [1].

Theorem 3. *If $g \in M_{\mu}^{a,c}(\alpha;A,B)$. Suppose also that*

$$\rho = 1 - \frac{\delta\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{2[\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)-\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)]}, \tag{2.6}$$

then

$$N_{\delta}(g) \subset M_{\mu}^{a,c,\rho}(\alpha;A,B).$$

Proof. let $f(z)$ be in $N_{\delta}(g)$. We then find from the definition (2.1) that

$$\sum_{k=2}^{\infty} k\,|a_k - b_k| \le \delta, \tag{2.7}$$

since $g \in M_{\mu}^{a,c}(\alpha;A,B)$, we have

$$\sum_{k=2}^{\infty} b_k \le \frac{\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=2}^{\infty}|a_k-b_k|}{1-\sum_{k=2}^{\infty} b_k} \\ &\le \frac{\delta}{2}\frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)-\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)} \\ &= 1-\rho, \end{aligned}$$

provided that ρ is given precisely by (2.6). Thus, by definition, $f \in M_{\mu}^{a,c,\rho}(\alpha; A, B)$ for ρ given by (2.6). This evidently completes our proof of Theorem 3.

A function $f \in T$ is said to be in the class $H_{\mu}^{a,c}(\alpha, \phi; A, B)$ if it satisfies the following non-homogeneous Cauchy-Euler differential equation:

$$z^2 \frac{d^2 f}{dz^2} + 2(\phi+1) z \frac{df(z)}{dz} + \phi(\phi+1) f(z) = (1+\phi)(2+\phi) g(z) \tag{2.8}$$

$$\big(g \in M_{\mu}^{a,c}(\alpha; A, B); \phi > -1\big)$$

Theorem 4. *If $f \in T$ is in the class $H_{\mu}^{a,c}(\alpha, \phi; A, B)$ then*

$$H_{\mu}^{a,c}(\alpha, \phi; A, B) \subset N_{\delta}(g), \tag{2.9}$$

where

$$\delta = \frac{4\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} \left(\frac{2+\phi}{3+\phi}\right) \tag{2.10}$$

Proof. Suppose that $f \in H_{\mu}^{a,c}(\alpha, \phi; A, B)$ and f is given by (1.1). From (2.8)

$$a_k = \frac{(1+\phi)(2+\phi)}{(k+\phi)(k+\phi+1)} b_k \quad (k \geq 2),$$

$$\sum_{k=2}^{\infty} k\,|b_k - a_k| \leq \sum_{k=2}^{\infty} k b_k + \sum_{k=2}^{\infty} k a_k \quad (a_k \geq 0, b_k \geq 0),$$

we obtain

$$\sum_{k=2}^{\infty} k\,|b_k - a_k| \leq \sum_{k=2}^{\infty} k b_k + \sum_{k=2}^{\infty} \frac{(1+\phi)(2+\phi)}{(k+\phi)(k+\phi+1)} k b_k. \tag{2.11}$$

Next, since $g \in M_{\mu}^{a,c}(\alpha; A, B)$, from (1.10) of the Lemma 1 yields

$$\sum_{k=2}^{\infty} k b_k \leq \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)}. \tag{2.12}$$

Finally, by making use of (2.11) on the right-hand side of (2.12), we find that

$$\begin{aligned}\sum_{k=2}^{\infty} k\,|b_k - a_k| &\leq \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)} \left(1 + \frac{(1+\phi)}{(3+\phi)}\right) \\ &= \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)} \left(\frac{2(2+\phi)}{3+\phi}\right) = \delta.\end{aligned}$$

Thus, by definition (2.1) with $g(z)$ interchanged by $f(z)$, $f \in N_{\delta}(g)$.This, evidently, completes the proof of Theorem 4.

3. Integral means inequalities

We shall need the concept of subordination theorem of Littlewood [7] in our investigation.

Lemma 3. (Littlewood's theory [7]). *If the functions $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$ then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \quad (\tau > 0; 0 < r < 1). \tag{3.1}$$

Theorem 5. *Let $f \in M_\mu^{a,c}(\alpha; A, B)$ and suppose that*

$$f_2(z) = z - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} z^2, \tag{3.2}$$

then for $\tau > 0, z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\tau d\theta \tag{3.3}$$

Proof. From (3.1), it would suffice to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} z.$$

By setting

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} w(z),$$

we find that

$$\begin{aligned} |w(z)| &= \left|\sum_{k=2}^{\infty} \frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)} a_k z^{k-1}\right| \\ &\le |z| \sum_{k=2}^{\infty} \frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)} a_k \\ &\le |z| \le 1, \end{aligned} \tag{3.4}$$

by using (1.10). Hence $f(z) \prec g(z)$ which readily yields the integral means inequality (3.3).

4. Partial sums

In this section we will study the ratio of a function of the form (1.1) to its sequence of partial sums defined by $f_m(z) = z$ and $f_m(z) = z - \sum_{k=2}^{m} a_k z^k$, when the coefficients of $f(z)$ are sufficiently small to satisfy the condition (1.9). We will determine sharp lower bounds, for $\Re \int (\frac{f(z)}{f_m(z)} \int), \Re \int (\frac{f_m(z)}{f(z)} \int), \Re \int (\frac{f'(z)}{f'_m(z)} \int)$ and $\Re \int (\frac{f'_m(z)}{f'(z)} \int)$.

In what follows, we will use the well known result

$$\Re \left(\frac{1 - w(z)}{1 + w(z)} \right) \qquad (z \in U)\,,$$

if and only if

$$w(z) = \sum_{k=1}^{\infty} D_k z^k,$$

satisfies the inequality $|w(z)| \leq |z|$.

Theorem 6. *Let* $f \in M_{\mu}^{a,c}(\alpha; A, B)$*, then*

$$\Re \left(\frac{f(z)}{f_m(z)} \right) \geq 1 - \frac{1}{D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.1}$$

and

$$\Re \left(\frac{f_m(z)}{f(z)} \right) \geq \frac{D_{m+1}}{1 + D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.2}$$

where

$$D_k = \frac{\Gamma(c+\mu)[(1-B)(k-1) + (A-B)(1-\alpha)]\Gamma(a+k\mu)}{\Gamma(a+\mu)\Gamma(c+k\mu)(A-B)(1-\alpha)}. \tag{4.3}$$

The estimates in (4.1) *and* (4.2) *are sharp.*

Proof. Employing the same technique used by Silverman [12]. The function $f \in M_{\mu}^{a,c}(\alpha; A, B)$ if and only if $\sum_{k=1}^{\infty} D_k z^k \leq 1$. It is easy to verify that $D_{k+1} > D_k > 1$. Thus

$$\sum_{k=1}^{m} a_k + D_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} D_k a_k < 1. \tag{4.4}$$

Now, setting

$$\begin{aligned} &D_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{D_{m+1}} \right) \right\} \\ &= \frac{1 - \sum\limits_{k=2}^{m} a_k z^{k-1} - D_{m+1} \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum\limits_{k=1}^{m} a_k z^{k-1}} = \frac{1 + E(z)}{1 + Y(z)}, \end{aligned}$$

and $\frac{1+E(z)}{1+Y(z)} = \frac{1-w(z)}{1+w(z)}$, then we have

$$w(z) = \frac{Y(z)-E(z)}{2+E(z)+Y(z)} = \frac{D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{2-2\sum\limits_{k=2}^{m} a_k z^{k-1} - D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}$$

which implies

$$|w(z)| \le \frac{D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k}{2-2\sum\limits_{k=2}^{m} a_k - D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k}.$$

Hence $|w(z)| \le 1$ if and only if

$$\sum_{k=2}^{m} a_k + D_{m+1}\sum_{k=m+1}^{\infty} a_k \le 1$$

which is true by (4.4). This readily yields (4.1).

Now consider the function

$$f(z) = 1 - \frac{z^{m+1}}{D_{m+1}} \tag{4.5}$$

Thus $\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{D_{m+1}}$. Letting $z \longrightarrow 1^-$, then $f(z) = 1 - \frac{1}{D_{m+1}}$. So $f(z)$ given by (4.5) satisfies the sharp result in (4.1). This shows that the bounds in (4.1) are best possible for each $m \in \mathbb{N}$.

Similarly, setting

$$\begin{aligned}(1+D_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{D_{m+1}}{1+D_{m+1}}\right\} &= \frac{1-\sum\limits_{k=2}^{m} a_k z^{k-1} + D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1-\sum\limits_{k=2}^{m} a_k z^{k-1}} \\ &\equiv \frac{1-w(z)}{1+w(z)},\end{aligned}$$

where

$$|w(z)| \le \frac{(1+D_{m+1})\sum\limits_{k=m+1}^{\infty} a_k}{2-2\sum\limits_{k=2}^{m} a_k + (1-D_{m+1})\sum\limits_{k=m+1}^{\infty} a_k}.$$

Now $|w(z)| \le 1$ if and only if

$$\sum_{k=2}^{m} a_k + D_{m+1}\sum_{k=m+1}^{\infty} a_k \le 1,$$

which readily implies the assertion (4.2). The estimate in (4.2) is sharp with the extremal function $f(z)$ given by (4.5). This completes the proof of the theorem.

Following similar steps to that followed in Theorem 6, we can state the following theorem

Theorem 7. *Let $f \in M_{\mu}^{a,c}(\alpha; A, B)$, then*

$$\Re\left(\frac{f'(z)}{f_m'(z)}\right) \geq 1 - \frac{m+1}{D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.6}$$

and

$$\Re\left(\frac{f_m'(z)}{f'(z)}\right) \geq \frac{D_{m+1}}{m+1+D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.7}$$

where $D_k, k \in \mathbb{N}$ is given by (4.3). The estimates in (4.6) and (4.7) are sharp with the extremal function $f(z)$ is as defined in (4.5).

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Accepted: 28.08.2018

On uniformly primary hyperideals and uniformly 2-absorbing primary hyperideals

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Abstract. Let R be a multiplicative hyperring. In this paper, we introduce the concepts of uniformly primary hyperideal and uniformly 2- absorbing primary hyperideal of R, which impose a certain boundedness condition on the usual notions of primary hyperideal and 2- absorbing primary hyperideal, respectively. We will show some properties of them.
Keywords: 2-absorbing primary hyperideal, uniformly primary hyperideal, uniformly 2-absorbing primary hyperideal, Noether strongly 2-absorbing primary hyperideal, special 2-absorbing primary hyperideal.

1. Introduction

The theory of algebraic hyperstructures was introduced in 1934 by Marty [8] during the 8^{th} Congress of Scandinavian Mathematicians. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups. Later on, many researchers have observed that the theory of hyperstructures also have many applications in both pure and applied sciences. A comprehensive review of this theory can be found in [3],[6],[9], [4] and [11]. The notion of multiplicative hyperring was introduced by R. Rota [10] in 1982. For example, applications of hyperstructures in chemistry and physics can be studied in Chapter 8, [6].

A triple $(R, +, \circ)$ is called a multiplicative hyperring if
(1)$(R, +)$ is an abelian group;
(2)$(R, \circ)$ is semihypergroup;
(3) for all $a, b, c \in R$, we have $a\circ(b+c) \subseteq a\circ b+a\circ c$ and $(b+c)\circ a \subseteq b\circ a+c\circ a$;
(4) for all $a, b \in R$, we have $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

For any two nonempty subsets A and B of R and $x \in R$, we define

$$A \circ B = \bigcup_{a\in A,\ b\in B} a \circ b, \qquad A \circ x = A \circ \{x\}$$

A non empty subset I of a multiplicative hyperring R is a hyperideal if

(1) If $a, b \in I$, then $a - b \in I$;

(2) If $x \in I$ and $r \in R$, then $r \circ x \subseteq I$.

The concept of 2-absorbing hyperideal was introduced in [7]. Really, it is a generalization of prime hyperideal. Precisely, a nonzero proper hyperideal I of a multiplicative hyperring R is called to be 2-absorbing if $x \circ y \circ z \subseteq I$ where $x, y, z \in R$, then $x \circ y \subseteq I$ or $y \circ z \subseteq I$ or $x \circ z \subseteq I$.

In this paper, we introduce the concepts of uniformly primary hyperideal and uniformly 2-absorbing primary hyperideal of R, which impose a certain boundedness condition on the usual notions of primary hyperideal and 2- absorbing primary hyperideal, respectively.

Among many results in this paper, it is shown (Theorem 3.6) that hyperideal Q of R is a uniformly P-primary hyperideal if and only if:

(1) Q is a P-primary hyperideal of R, and

(2) there exists a positive integer n such that $P = \{x \in R \mid x^n \subseteq Q\}$.

Moreover, $ord_H(Q) = k$ if and only if k is the smallest positive integer for which condition (2) holds. It is shown (Theorem 4.9) that if R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:

(1) If Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 with $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq 2\_ord_{H_{R_2}}(Q_2)$.

(2) If ϕ is an epimorphism and Q_1 is a uniformly 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2 with $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq 2\_ord_{H_{R_1}}(Q_1)$.

It is shown (Theorem 5.5) that if Q is a Noether strongly 2-absorbing primary hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q) \leq 2\_\mathfrak{e}_H(Q)$.

2. Preliminaries

Definition 2.1 ([5]). A nonzero proper hyperideal P of R is called a *prime hyperideal* if $x \circ y \subseteq P$ for $x, y \in R$ implies that $x \in P$ or $y \in P$. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by $r(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $r(I) = R$.

Definition 2.2 ([5]). Let $\mathbf{C}$ be the class of all finite products of elements of R i.e. $\mathbf{C} = \{r_1 \circ r_2 \circ ... \circ r_n \ : \ r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a $\mathbf{C}$-*hyperideal* of R if for any $A \in \mathbf{C}, A \cap I \neq \varnothing \Rightarrow A \subseteq I$.

Theorem 2.3 ([5], Proposition 3.2). *Let I be a hyperideal of R. Then, $D \subseteq r(I)$ where $D = \{r \in R : r^n \subseteq I \ for \ some \ n \in \mathbb{N}\}$. The equality holds when I is a* $\mathbf{C}$*-hyperideal of R.*

In this paper, we assume that all hyperideals are $\mathbf{C}$-hyperideal.

Definition 2.4 ([5]). A nonzero proper hyperideal Q of R is called a *primary hyperideal* if for any $x, y \in R$, $x \circ y \subseteq Q$ and $x \notin Q$, then $y^n \subseteq Q$ for some $n \in \mathbb{N}$.

Since $r(Q) = P$ is a prime hyperideal of R by Propodition 3.6 in [5], Q is referred to as a P-primary hyperideal of R.

Definition 2.5 ([1]). A nonzero proper hyperideal I of R is called *2-absorbing primary hyperideal* of R if $x \circ y \circ z \subseteq I$ for some $x, y, z \in R$, then $x \circ y \subseteq I$ or $x \circ z \subseteq r(I)$ or $y \circ z \subseteq r(I)$.

Theorem 2.6 ([1], Theorem 4.2). *Let I be a 2-absorbing primary hyperideal of R. Then $P = r(I)$ is a 2-absorbing hyperideal. We say that I is a P-2-absorbing primary hyperideal of R.*

Definition 2.7. Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be multiplicative hyperrings. A mapping from R_1 into R_2 is said to be a *good homomorphism* if for all $x, y \in R_1$, $\phi(x +_1 y) = \phi(x) +_2 \phi(y)$ and $\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y)$.

Definition 2.8. For $x \in R$, we define $(I :_R x) = \{r \in R \mid r \circ x \subseteq I\}$.

3. Uniformly primary hyperideals

Definition 3.1. Let Q be a proper hyperideal of R. Q is a *uniformly primary hyperideal* of R if there exists a positive integer n such that whenever $x, y \in R$ satisfy $x \circ y \subseteq Q$ and $x \notin Q$ then $y^n \subseteq Q$. If k is the smallest positive integer for which the above property holds ,then it is denoted by $ord_{H_R}(Q) = k$, or simply $ord_H(Q) = k$.

Definition 3.2. P-primary hyperideal Q of R is said to be a *Noether strongly primary hyperideal* if $P^n \subseteq Q$ for some positive integer n. If k is the smallest positive integer for which the above property holds ,then it is denoted by $\mathfrak{e}_{H_R}(Q) = k$, or simply $\mathfrak{e}_H(Q) = k$.

Example 3.3. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. We define the hyperoperation $a \circ b = \{2ab, 4ab\}$, for all $a, b \in \mathbb{Z}$. The hyperideal $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$ of the multiplicative hyperring $(\mathbb{Z}, +, \cdot)$ is a Noether strongly primary hyperideal.

Theorem 3.4. *If Q is a Noether strongly P-primary hyperideal of R then Q is a uniformly P-primary hyperideal of R. Also, $ord_H(Q) \leq \mathfrak{e}_H(Q)$.*

Proof. Assume that Q is a Noether strongly P-primary hyperideal of R. Let $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Thus $y \in P$ and so $y^{\mathfrak{e}_H(Q)} \subseteq P^{\mathfrak{e}_H(Q)} \subseteq Q$. Hence, Q is a uniformly P-primary hyperideal of R such that $ord_H(Q) \leq \mathfrak{e}_H(Q)$. □

Example 3.5. In Example 3.3, the hyperideal $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$ of the multiplicative hyperring $(\mathbb{Z}, +, \cdot)$ is a uniformly primary hyperideal.

Theorem 3.6. *Hyperideal Q of R is a uniformly P-primary hyperideal if and only if:*

(1) *Q is a P-primary hyperideal of R.*

(2) *there exists a positive integer n such that $P = \{x \in R \mid x^n \subseteq Q\}$.*

Moreover, $ord_H(Q) = k$ if and only if k is the smallest positive integer for which condition (2) holds.

Proof. $\Longrightarrow$ Let Q be a uniformly P-primary hyperideal of R with $ord_H(Q) = k$. Thus we have condition (1) clearly. Suppose that $x \in P$. So there exists some positive integer t with $x^{t-1} \circ x = x^t \subseteq Q$ such that $x^{t-1} \nsubseteq Q$. Since $ord_H(Q) = k$, we have $x^k \subseteq Q$. Thus the proof is completed.

$\Longleftarrow$ Assume that $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Then we have $y \in P$. On the other hand, by (2), there exists a positive integer n with $y^n \subseteq Q$ such that n is independent of y. Thus, Q is a uniformly primary hyperideal of R.

The "moreover" statement follows from the definition of $ord_H(Q)$. □

Theorem 3.7. *Let $Q_1 \subseteq Q_2$ be uniformly P-primary hyperideals of R. Then $ord_H(Q_1) \geq ord_H(Q_2)$.*

Proof. Put $k_1 = ord_(Q_1)$ and $k_2 = ord_H(Q_2)$. Then there exist elements $x, y \in R$ with $x \circ y \in Q_2$ such that $x \notin Q_2$, $y^n \subseteq Q_2$, and $y^{n-1} \nsubseteq Q_2$. Thus, we have $y \in P = r(Q_1)$ and so $y^{k_1} \subseteq Q_1 \subseteq Q_2$. Hence, $k_1 > k_2 - 1$, and then $k_1 \geq k_2$. □

Theorem 3.8. *Let $\{Q_i\}_{i \in I}$ be a collection of uniformly P-primary hyperideals of R such that $max_{i \in I}\{ord_H(Q_i)\} = n$, where n is a positive integer. Then $Q = \bigcap_{i \in I} Q_i$ is a uniformly P-primary hyperideal of R with $ord_H(Q) = n$.*

Proof. By Proposition 3.3 in [5], we have $r(Q) = r(\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} r(Q_i) = P$. Assume that $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Thus there exists some $j \in I$ such that $x \circ y \subseteq Qj$ and $x \notin Q_j$. It means $y \in P$ and hence $y^n \subseteq Q$. Thus Q is a uniformly P-primary hyperideal of R with $ord_H(Q_j) \leq n$. Assume that $Q_t \in \{Q_i\}_{i \in I}$ be a uniformly P-primary hyperideal of R with $ord_H(Q) = n$. Hence, by Theorem 3.6, n is the smallest positive integer with $P = \{x \in R \mid x^n \subseteq Q_t\}$. Hence, there exists $x \in P$ but $x^{n-1} \nsubseteq Q_t$, and so $x^{n-1} \nsubseteq Q$. Consequently $ord_H(Q) = n$. □

Theorem 3.9. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. If Q_2 is a uniformly P-primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly $\phi^{-1}(P)$-primary hyperideal of R_1 with $ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq ord_{H_{R_2}}(Q_2)$.*

Proof. The proof of the first statement is easy. Let Q_2 be a uniformly P-primary hyperideal of R_2 with $k = ord_{H_{R_2}}(Q_2)$. By Theorem 3.6, we have $P = \{y \in R_2 \mid y^k \subseteq Q_2\}$. Hence, $\phi^{-1}(P) = \{x \in R_1 \mid x^k \subseteq \phi^{-1}(Q_2)\}$.Therefore, we conclude that $\phi^{-1}(Q_2)$ is a uniformly $\phi^{-1}(P)$-primary hyperideal of R such that $ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq ord_{H_{R_2}}(Q_2)$. $\square$

Corollary 3.10. Let I and Q be hyperideals of R such that $I \subseteq Q$. Then Q is a uniformly P-primary hyperideal of R with $ord_{H_R}(Q) = k$ if and only if Q/I is a uniformly P/I -primary hyperideal of R/I with $ord_{H_{R/I}}(Q/I) = k$.

Proof. $\Longrightarrow$ It is straightforward.

$\Longleftarrow$ Assume that Q/I is a uniformly P/I -primary hyperideal of R/I with $ord_{H_{R/I}}(Q/I) = k$. By Theorem 3.9, Q is a uniformly P-primary hyperideal of R with $ord_{H_R}(Q) \leq k$. By Theorem 3.6, there exists $x + I \in P/I$ but $(x + I)^{k-1} \nsubseteq Q/I$. Hence, $x \in P$ and $x^{k-1} \nsubseteq Q$. Thus $ord_{H_R}(Q) = k$. $\square$

4. Uniformly 2-absorbing primary hyperideals

Definition 4.1. Let Q be a proper hyperideal of R. Q is a *uniformly 2-absorbing primary hyperidea*l of R if there exists a positive integer n such that whenever $x, y, z \in R$ satisfy $x \circ y \circ z \subseteq Q$, $x \circ y \nsubseteq Q$ and $x \circ z \nsubseteq r(Q)$, then $(y \circ z)^n \subseteq Q$. If k is the smallest positive integer for which the above property holds ,then it is denoted by $2\_ord_{H_R}(Q) = k$, or simply $2\_ord_H(Q) = k$.

Theorem 4.2. *If Q is a 2-absorbing hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal with $2\_ord_H(Q) = 1$.*

Proof. It is obvious. $\square$

Example 4.3. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. Corresponding to every subset $A \in P^*(\mathbb{Z}) = P(R)\backslash\{\varnothing\}$ $(|A| \geq 2)$, there exists a multiplicative hyperring $(\mathbb{Z}_A, +, \circ)$ where $\mathbb{Z}_{\mathbb{A}} = \mathbb{Z}$ and for any $x, y \in \mathbb{Z}_{\mathbb{A}}, x \circ y = \{x \cdot a \cdot y \mid a \in A\}$ [5]. In the multiplicative hyperring of integers Z_A with $A = \{5, 7\}$, the principal hyperideals $< 2 >$ and $< 3 >$ are prime hyperideals by Proposition 4.3 in [5]. Hence, hyperideal $< 2 > \cap < 3 >$ is a 2-absorbing hyperideal and so $< 2 > \cap < 3 >$ is a uniformly 2-absorbing primary hyperideal.

Theorem 4.4. *If Q is a uniformly 2-absorbing primary hyperideal of R, then Q is a 2-absorbing primary hyperideal of R with $2\_ord_H(Q) = 1$.*

Proof. It is clear. $\square$

Theorem 4.5. *Let Q be a proper hyperideal of R. If Q is a uniformly 2-absorbing primary hyperideal of R, then one of the following conditions must hold:*

(1) $r(Q) = P$ is a prime hyperideal.

(2) $r(Q) = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime hyperideals of R that are minimal over Q.

Proof. Apply Theorem 4.5 in [1]. $\square$

Theorem 4.6. *Let Q be a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *Q is uniformly 2-absorbing primary hyperideal.*

(2) *There exists a positive integer n such that for every $x, y \in R$ either $(xoy)^n \subseteq Q$ or $(Q :_R x \circ y) \subseteq (Q :_R x) \cup (r(Q) :_R y)$.*

(3) *There exists a positive integer n such that for every $x, y \in R$ either $(x \circ y)^n \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x) or (Q :_R x \circ y) \subseteq (r(Q) :_R y)$.*

(4) *There exists a positive integer n such that for every $x, y \in R$ and every hyperideal I of R, $x \circ y \circ I \subseteq Q$ implies that either $x \circ I \subseteq Q$ or $y \circ I \subseteq r(Q)$ or $(x \circ y)^n \subseteq Q$.*

Proof. (1) $\Rightarrow$ (2)Let Q be a uniformly 2-absorbing primary hyperideal of R such that $2\_ord_H(Q) = n$. Suppose that $(x \circ y)^n \nsubseteq Q$ for some $x, y \in R$. Let $a \in (Q :_R x \circ y)$. Hence $aoy \subseteq Q$. Therefore we have $a \circ x \subseteq Q$ or $a \circ y \subseteq r(Q)$. It means $a \in (Q :_R x)$ or $a \in (r(Q) :_R y)$. This implies that $(Q :_R x \circ y) \subseteq (Q :_R x) \cup (r(Q) :_R y)$.

(2)$\Rightarrow$ (3)If an hyperideal is a subset of the union of two hyperideals, then it is a subset of one of them.

(3)$\Rightarrow$ (4) Assume that n is a positive number such that for every $x, y \in R$ either $(x \circ y)^n \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x) or (Q :_R x \circ y) \subseteq (r(Q) :_R y)$. Assume that I is a hyperideal of R with $x \circ y \circ I \subseteq Q$ for some $x, y \in R$ such that $(x \circ y)^n \nsubseteq Q$. Therefore $I \subseteq (Q :_R x \circ y)$. Hence we have $I \subseteq (Q :_R x)$ or $I \subseteq (r(Q) :_R y)$. Thus $xoI \subseteq Q$ or $y \circ I \subseteq r(Q)$.

(4)$\Rightarrow$ (1) Straightforward. □

Theorem 4.7. *Let Q_1 be a uniformly P-primary hyperideal of R and Q_2 be a uniformly P-2-absorbing primary hyperideal of R such that $Q_1 \subseteq Q_2$. Then $2\_ord_H(Q_2) \leq ord_H(Q_1)$.*

Proof. Assume that $ord_H(Q_1) = k_1$ and $2\_ord_H(Q_2) = k_2$. Thus there are $x, y, z \in R$ with $x \circ y \circ z \subseteq Q_2$ such that $x \circ y \nsubseteq Q_2$, $x \circ z \nsubseteq r(Q_2)$, $(y \circ z)^{k_2} \subseteq Q_2$ and $(y \circ z)^{k_2 - 1} \nsubseteq Q_2$. Hence $y \circ z \subseteq r(Q_2) = r(Q_1)$. Thus $(y \circ z)^{k_1} \subseteq Q_1 \subseteq Q_2$ by Theorem 3.6. Thus $k_2 > k_1 - 1$. Then $k_2 \geq k_1$. □

Theorem 4.8. *Let $\{Q_i\}_{i \in I}$ be a chain of uniformly P-2-absorbing primary hyperideals of R such that $max_{i \in I}\{2\_ord_H(Q_i)\} = n$, where n is a positive integer. Then $Q = \bigcap_{i \in I} Q_i$ is a uniformly P-2-absorbing primary hyperideal of R with $2\_ord_H(Q) \leq n$.*

Proof. By Proposition 3.3 in [5], we have $r(Q) = r(\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} r(Q_i) = P$. Assume that $x \circ y \circ z \subseteq Q$ for some $x, y, z \in R$ such that $x \circ y \nsubseteq Q$ and $(y \circ z)^n \nsubseteq Q$. Since $\{Q_i\}_{i \in I}$ is a chain, there exists some $j \in I$ such that $xoy \nsubseteq Q_j$ and $(y \circ z)^n \nsubseteq Q_j$. Since Q_j is a uniformly 2-absorbing primary hyperideal of R with $2\_ord(Q_j) \leq n$, then $xoz \subseteq r(Q_i) = r(Q)$. Thus Q is a uniformly 2-absorbing primary hyperideal of R with $2\_ord_H(Q) \leq n$. □

Theorem 4.9. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:*

(1) *If Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 with $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq 2\_ord_{H_{R_2}}(Q_2)$.*

(2) *If ϕ is an epimorphism and Q_1 is a uniformly 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2 with $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq 2\_ord_{H_{R_1}}(Q_1)$.*

Proof. (1) Let $k = 2\_ord_{H_{R_2}}(Q_2)$ and $x \circ y \circ z \subseteq \phi^{-1}(Q_2)$ for some $x, y, z \in R_1$ such that $x \circ y \nsubseteq \phi^{-1}(Q_2)$ and $x \circ z \nsubseteq r(\phi^{-1}(Q_2))$. This implies that $\phi(x \circ y \circ z) = \phi(z) \circ \phi(y) \circ \phi(z) \subseteq Q_2$ such that $\phi(x \circ y) = \phi(x) \circ \phi(y) \nsubseteq Q_2$ and $\phi(x \circ z) = \phi(x) \circ \phi(z) \nsubseteq r(Q_2)$. Since Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, we have $\phi^k(y \circ z) \subseteq Q_2$. Thus $\phi((y \circ z)^k) \subseteq Q_2$. It means $(y \circ z)^k \subseteq \phi^{-1}(Q_2)$. Hence $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 such that $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq k = 2\_ord_{H_{R_2}}(Q_2)$.

(2) Let $k = 2\_ord_{H_{R_1}}(Q_1)$ and $x \circ y \circ z \subseteq \phi(Q_1)$ for some $x, y, z \in R_2$ such that $x \circ y \nsubseteq \phi(Q_1)$ and $x \circ z \nsubseteq r(\phi(Q_1))$. Since ϕ is an epimorphism, then there exist $a, b, c \in R_1$ with $\phi(a) = x$, $\phi(b) = y$ and $\phi(c) = z$. Thus $\phi(a \circ b \circ c) = x \circ y \circ z \subseteq \phi(Q_1)$ such that $\phi(a \circ b) = x \circ y \nsubseteq \phi(Q_1)$ and $\phi(a \circ c) = x \circ z \nsubseteq r(\phi(Q_1))$. Now take any $u \in a \circ b \circ c$. Then we get $\phi(u) \in \phi(a \circ b \circ c) \subseteq \phi(Q_1)$ and so $\phi(u) = \phi(w)$ for some $w \in Q_1$. This implies that $\phi(u - w) = 0 \in (0)$, that is, $u - w \in ker(\phi) \subseteq Q_1$ and so $u \in Q_1$. Since Q_1 is a **C**-hyperideal of R_1, then we conclude that $a \circ b \circ c \subseteq Q_1$. Since $\phi(r(Q_1)) \subseteq r(\phi(Q_1))$, then $a \circ b \nsubseteq Q_1$, and $a \circ c \nsubseteq r(Q_1)$, . Since Q_1 is a uniformly 2-absorbing primary hyperideal of R_1, then we have $(b \circ c)^k \subseteq Q_1$. Thus $\phi((b \circ c)^k) = (\phi(b) \circ \phi(c))^k = (y \circ z)^k \subseteq \phi(Q_1)$. Hence $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2. Also, $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq k = 2\_ord_{H_{R_1}}(Q_1)$. □

Corollary 4.10. Let Q be a hyperideal of R.

(1) If S is a subhyperring of R and Q is a uniformly 2-absorbing primary hyperideal of R, then $Q \cap S$ is a uniformly 2-absorbing primary hyperideal of S with $2\_ord_S(Q \cap S) \leq 2\_ord_R(Q)$.

(2) Let I be a hyperideal of R such that $I \subseteq Q$. Then Q is a uniformly 2-absorbing primary hyperideal of R if and only if Q/I is a uniformly 2-absorbing primary hyperideal of R/I.

Proof. It follows from Theorem 4.9. □

5. Noether strongly 2-absorbing primary hyperideals

Definition 5.1. P-2-absorbing primary hyperideal Q of R is said to be a *Noether strongly 2-absorbing primary hyperideal* if $P^n \subseteq Q$ for some positive integer n. If k be the smallest positive integer for which the above property holds ,then it is denoted by $2\_\mathfrak{e}_{H_R}(Q) = k$, or simply $2\_\mathfrak{e}_H(Q) = k$.

Example 5.2. Consider the ring $(\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}=\mathbb{Z}_6,\oplus,\odot)$ that for all $\bar{x},\bar{y}\in \mathbb{Z}_6$, $\bar{x}\oplus\bar{y}$ and $\bar{x}\odot\bar{y}$ are the remainder of $\frac{x+y}{6}$ and $\frac{x\cdot y}{6}$, respectively, which $+$ and $\cdot$ are ordinary addition and multiplication. We define the hyperoperation $x*y=\{\overline{xy},\overline{2xy},\overline{3xy},\overline{4xy},\overline{5xy}\}$, for all $\bar{x},\bar{y}\in\mathbb{Z}_6$. The hyperideal $\{\bar{0}\}$ of commutative multiplicative hyperring $(\mathbb{Z}_6,\oplus,\odot)$ is a Noether strongly 2-absorbing primary hyperideal.

Theorem 5.3. *Let Q_1 and Q_2 be Noether strongly primary hyperideals of R. Then, $Q_1\cap Q_2$ and $Q_1\circ Q_2$ are Noether strongly 2-absorbing primary hyperideals of R.*

Proof. Assume that Q_1 and Q_2 be primary hyperideals of R. By Theorem 4.6 in [1], $Q_1\cap Q_2$ and $Q_1\circ Q_2$ are 2-absorbing primary ideals of R, . $\square$

Theorem 5.4. *If Q is a 2-absorbing hyperideal of R, then Q is a Noether strongly 2-absorbing primary hyperideal with $2\_\mathfrak{e}_H(Q)\leq 2$.*

Proof. Since Q is a 2-absorbing hyperideal, we conclude that it is a 2-absorbing primary hyperideal and $r(Q)^2\subseteq Q$ by Theorem 4 in [7]. $\square$

Theorem 5.5. *If Q is a Noether strongly 2-absorbing primary hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q)\leq 2\_\mathfrak{e}_H(Q)$.*

Proof. Assume that Q be a Noether strongly 2-absorbing primary hyperideal of R. Let $x\circ y\circ z\subseteq Q$ for some $x,y,z\in R$ such that $x\circ y\nsubseteq Q$ and $x\circ z\nsubseteq r(Q)$. Since Q is a 2-absorbing primary hyperideal of R, we have $y\circ z\subseteq r(Q)$. Hence $(y\circ z)^{2\_\mathfrak{e}_H(Q)}\subseteq (r(Q))^{2\_\mathfrak{e}_H(Q)}\subseteq Q$. Thus Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q)\leq 2\_\mathfrak{e}_H(Q)$. $\square$

Theorem 5.6. *Let Q be a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *$r(Q)$ is a 2-absorbing hyperideal of R.*

(2) *For every $x,y,z\in R$, $x\circ y\circ x\subseteq Q$ implies that $xoy\subseteq r(Q)$ or $x\circ z\subseteq r(Q)$ or $y\circ z\subseteq r(Q)$.*

(3) *$r(Q)$ is a 2-absorbing primary hyperideal of R.*

(4) *$r(Q)$ is a Noether 2-absorbing primary hyperideal of R with $2\_\mathfrak{e}_H(r(Q))=1$.*

(5) *$r(Q)$ is a uniformly 2-absorbing primary hyperideal of R.*

Proof. (1)$\Longrightarrow$(2) It is evident. (2)$\Longleftarrow$(1) Assume that $a\circ b\circ c\subseteq r(Q)$ for some $a,b,c\in R$. Thus, there exists a positive integer n such that $(a\circ b\circ c)^n=a^n\circ b^n\circ c^n\subseteq Q$. Then, we have $a^n\circ b^n\subseteq r(Q)$ or $a^n\circ c^n\subseteq r(Q)$ or $b^n\circ c^n\subseteq r(Q)$, by the hypothesis in (2). Thus $a\circ b\subseteq r(Q)$ or $a\circ c\subseteq r(Q)$ or $b\circ c\subseteq r(Q)$. Therefore $r(Q)$ is a 2-absorbing hyperideal. (3)$\Longleftrightarrow$(4) and (1)$\Longleftrightarrow$(3) are obvious. (5)$\Longrightarrow$(3) Is straightforward. (4)$\Longrightarrow$(5) It follows by Theorem 5.5. $\square$

6. Special 2-absorbing primary hyperideals

Definition 6.1. Hyperideal Q of R is said to be a *special 2-absorbing primary hyperideal* if it is uniformly 2-absorbing primary hyperideal with $2\_ord_H(Q) = 1$.

Example 6.2. In Example 5.2, the hyperideal $\{\bar{0}\}$ of commutative multiplicative hyperring $(\mathbb{Z}_6, \oplus, \odot)$ is a a special 2-absorbing primary hyperideal.

Theorem 6.3. *Assume that Q is a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *Q is special 2-absorbing primary hyperideal.*

(2) *For every $x, y \in R$ either $x \circ y \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x)$ or $(Q :_R x \circ y) \subseteq (r(Q) :_R y)$.*

(3) *For every $x, y \in R$ and every hyperideal I of R, $x \circ y \circ I \subseteq Q$ implies that either $x \circ y \subseteq Q$ or $x \circ I \subseteq Q$ or $y \circ I \subseteq r(Q)$.*

(4) *For every $x \in R$ and every hyperideal I of R either $x \circ I \subseteq Q$ or $(Q :_R x \circ I) \subseteq (Q :_R x) \cup (r(Q) :_R I)$.*

(5) *For every $a \in R$ and every hyperideal I of R either $x \circ I \subseteq Q$ or $(Q :_R x \circ I) = (Q :_R x)$ or $(Q :_R x \circ I) \subseteq (r(Q) :_R I)$.*

(6) *For every $x \in R$ and every hyperideals I, J of R, $x \circ I \circ J \subseteq Q$ implies that either $x \circ I \subseteq Q$ or $I \circ J \subseteq r(Q)$ or $x \circ J \subseteq Q$.*

(7) *For every hyperideals I, J of R either $I \circ J \subseteq r(Q)$ or $(Q :_R I \circ J) \subseteq (Q :_R I) \cup (Q :_R J)$.*

(8) *For every hyperideals I, J of R either $I \circ J \subseteq r(Q)$ or $(Q :_R I \circ J) = (Q :_R I)$ or $(Q :_R I \circ J) = (Q :_R J)$.*

(9) *For every hyperideals I, J, K of R, $I \circ J \circ K \subseteq Q$ implies that either $I \circ J \subseteq r(Q)$ or $I \circ K \subseteq Q$ or $J \circ K \subseteq Q$.*

Proof. (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3) This follows by Theorem 4.6.

(3)$\Longleftrightarrow$(4) Suppose that I be a hyperideal of R and $x \in R$ such that $x \circ I \nsubseteq Q$. Assume that $a \in (Q :_R x \circ I)$. we have $x \circ a \circ I \subseteq Q$, and therefore $a \in (Q :_R x)$ or $a \in (r(Q) :_R I)$. Hence $(Q :_R x \circ I) \subseteq (Q :_R x) \cup (r(Q) :_R I)$.

The proof of other cases are straightforward. $\square$

Theorem 6.4. *Let Q be a special 2-absorbing primary hyperideal of R and $a \in R \setminus r(Q)$. The following conditions hold:*

(1) *$(Q :_R a) = (Q :_R a^n)$ for every $n \geq 2$;*

(2) *$(r(Q) :_R a) = r(Q :_R a)$.*

(3) *$(Q :_R a)$ is a special 2-absorbing primary hyperideal of R.*

Proof. (1) It is clear that $(Q :_R a) \subseteq (Q :_R a^n)$ for all $n \geq 2$. By induction on n, we show $(Q :_R a^n) \subseteq (Q :_R a)$. First, let $n = 2$ and $s \in (Q :_R a^2)$. We have $s \circ a^2 \subseteq Q$, and then either $s \circ a \subseteq Q$ or $a^2 \subseteq r(Q)$. The second case implies that $a \in r(Q)$ which is a contradiction. Hence $s \circ a \subseteq Q$ which means $s \in (Q :_R a)$. Thus $(Q :_R a) = (Q :_R a^2)$. Now, let $n > 2$. Assume that $(Q :_R a) = (Q :_R a^{n-1})$. Take $s \in (Q :_R a^n)$. We have $s \circ a^n \subseteq Q$. Since

$a \notin r(Q)$, we conclude that either $s \circ a^{n-1} \subseteq Q$ or $s \circ a \subseteq Q$. Both of them implies that $s \in (Q :_R a)$. Hence $(Q :_R a) = (Q :_R a^n)$.

(2) Clearly, $r(Q :_R x) \subseteq (r(Q) :_R a)$. Assume that $s \in (r(Q) :_R a)$. Thus there exists a positive integer t such that $(s \circ a)^t \subseteq Q$. Hence $s^t \subseteq (Q :_R a)$, by part (1). Therefore $s \in r(Q :_R a)$. Consequently $(r(Q) :_R a) = r(Q :_R a)$.

(3) Assume that $x \circ y \circ z \subseteq (Q :_R a)$ for some $x, y, z \in R$. Thus $x \circ a \circ (y \circ z) \subseteq Q$ and then $x \circ a \subseteq Q$ or $x \circ y \circ z \subseteq Q$ or $y \circ z \circ a \subseteq r(Q)$. If $x \circ a \subseteq Q$, then we have $x \circ y \subseteq (Q :_R a)$. If $x \circ y \circ z \subseteq Q$, then we have either $x \circ y \subseteq Q \subseteq (Q :_R a)$ or $x \circ z \subseteq Q \subseteq (Q :_R a)$ or $y \circ z \subseteq r(Q) \subseteq r(Q :_R a)$. If $y \circ z \circ a \subseteq r(Q)$, then by part (2) we get $y \circ z \subseteq (r(Q) :_R a) = r(Q :_R a)$. Hence $(Q :_R a)$ is a special 2-absorbing primary hyperideal of R. $\square$

Theorem 6.5. *Let Q be a special 2-absorbing primary hyperideal of R and P, P_1, P_2 be distinct prime hyperideals of R and $E_a = (Q :_R a)$.*

(1) If $r(Q) = P$, then $\{E_a \mid a \in R \setminus P\}$ is a totally ordered set.

(2) If $r(Q) = P_1 \cup P_2$, then $\{E_a \mid a \in R \setminus P_1 \cup P_2\}$ is a totally ordered set.

Proof. (1) Assume that $a, b \in R \setminus P$. We have $a \circ b \subseteq R \setminus P$. Clearly, $E_a \cup E_b \subseteq E_{a \circ b}$. Let $s \in E_{a \circ b}$. Thus $s \circ a \circ b \subseteq Q$. Since $a \circ b \nsubseteq r(Q)$ we have $s \circ a \subseteq Q$ or $s \circ b \subseteq Q$. Hence $E_{a \circ b} = E_a \cup E_b$. Therefore, we have either $E_{a \circ b} = E_a$ or $E_{a \circ b} = E_b$. Then either $E_b \subseteq E_a$ or $E_a \subseteq E_b$.

(2) It follows by using an argument to that in the proof of (1). $\square$

Theorem 6.6. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:*

(1) *If Q_2 is a special 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a special 2-absorbing primary hyperideal of R_1.*

(2) *If ϕ is an epimorphism and Q_1 is a special 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a special 2-absorbing primary hyperideal of R_2.*

Proof. It is similar to the proof of Theorem 4.9. $\square$

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Accepted: 18.09.2018

Classes of weighted tent function spaces and mixed norms with some applications

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Abstract. In this paper, some new definitions for weighted classes of analytic functions are introduced. Moreover, certain properties are presented for functions belonging to the defined classes in the unit disk. Besides, a class of weighted tent functions is also considered. Furthermore, some properties for identity operator are studied for the new tent function spaces.
Keywords: mixed norms, tent functions, Q_p-functions.

1. Introduction and preliminaries

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Assume that $H(\Delta)$ is the class of all holomorphic functions f on Δ.

An interesting class of analytic functions, which called Bloch space (see [10, 12]) and it is defined by:

$$\mathcal{B} = \{f : f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1 - |z|^2)|f'(z)| < \infty\}.$$

For more discussions on Bloch-type classes in $\mathbb{C}$, we may refer to [10,12,20,27,28,29] and the cited references therein.

Analytic Q_p-spaces are introduced by Aulaskari and Lappan (see [12]) as follows:

$$Q_p = \left\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_{\Delta} \left|f'(z)\right|^2 g^p(z,a)dA(z) < \infty\right\},$$

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where $0 < p < \infty$ and $dA(z) = dxdy$. Also, the function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ defines the Green's function in Δ. For more details on analytic Q_p spaces, we can refer to [22, 29].

In [30] Zhao defined $F(p,q,s)$ and $F_0(p,q,s)$ classes as follows:

Definition 1.1. *Let $f \in H(\Delta)$. Assume that $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If*

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

The classes $F(p,q,s)$ were discussed by Zhao in [30] and Rättyä in [24]. From ([30], Theorem 2.10), it is clear that, for $p \ge 1$, the classes $F(p,q,s)$ are Banach spaces using the norm

$$\|f\| = \|f\|_{F(p,q,s)} + |f(0)|.$$

The important various studies on $F(p,q,s)$ classes can be found in [20, 21, 30].

In the present article, ω stands for a reasonable function, which means that an analytic function satisfying some natural conditions.

Definition 1.2 ([25, 26])**.** *Let $\omega : (0,1] \to (0,\infty)$ and let $0 < \alpha < \infty$. Suppose that $f \in H(\Delta)$, then f is said to belong to the α,ω-Bloch class $\mathcal{B}^\alpha_\omega$ if*

$$\mathcal{B}^\alpha_\omega(f) = \|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\Delta} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| < \infty.$$

Definition 1.3 ([25, 26])**.** *Let $\omega : (0,1] \to (0,\infty)$. Suppose that $0 < q < \infty$ and $0 < p < \infty$. Suppose that $f \in H(\Delta)$, then*

$$\begin{aligned} f \in B^{p,q}_\omega \Longleftrightarrow B^{p,q}_\omega(f) &= \|f\|^q_{B^{p,q}_\omega} \\ &= \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-p} \frac{\left(1-|\varphi_a(z)|^2\right)^p}{\omega^q(1-|z|)} dA(z) < \infty. \end{aligned}$$

In the next definition, we clear one of the motivations for the present article.

Definition 1.4. *Let $\omega : (0,1] \to (0,\infty)$. Suppose that $0 < q < \infty$ and $0 < p < \infty$. The function $f \in H(\Delta)$ is said to belong to the $B^q(p,\omega,\varphi)$-class if*

$$\|f\|^q_{B^q(p,\omega,\varphi)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^q \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z) < \infty.$$

For some results in this article, we will use the following:

$$(1) \qquad \big(1-|\varphi_a(z)|^2\big) = \frac{\big(1-|a|^2\big)\big(1-|z|^2\big)}{|1-\bar{a}z|^2},$$

and

$$(2) \qquad 1-|z| \leq |1-\bar{a}z| \leq 1+|z| \quad \text{and also} \quad 1-|a| \leq |1-\bar{a}z| \leq 1+|a|.$$

Two equivalent quantities L_f and M_f, which are depending on $f \in H(\Delta)$, and we write $L_f \approx M_f$, is we get a constant $C > 0$, such that

$$\frac{1}{C} M_f \leq L_f \leq C\, M_f.$$

The symbole $A \lesssim B$ can be used instead of the inequality $A \leq C_1 B$, where C_1 is a constant and $C_1 > 0$.

2. Some weighted analytic functions

Certain important properties of analytic $B^{p,q}_\omega$ classes in Δ will be considered in this section. Essential relations connecting between mixed norms of weighted $B^{p,q}_\omega$-type classes and the norms of weighted $Q_{p,\omega}$-type classes are considered.

Proposition 2.1. *$f \in H(\Delta)$ and let $f \in \mathcal{B}_\omega$. Then for $0 \leq p < \infty$ and $0 < q < \infty$ with $0 \leq \frac{q-2}{2} < \infty$ and $\frac{q}{2} - 1 < p$, we have that*

$$\int_\Delta \big|f'(z)\big|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z) \leq 4\pi\lambda \mathcal{B}^q_\omega(f).$$

Proof. Since (see [25]),

$$\frac{(1-|z|^2)}{\omega(1-|z|)}\big|f'(z)\big| \leq \mathcal{B}_\omega(f).$$

Then,

$$\begin{aligned}
&\int_\Delta \big|f'(z)\big|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z) \\
&\leq \mathcal{B}^{p,q}_\omega(f) \int_\Delta \big(1-|z|^2\big)^{-p} \big(1-|\varphi_a(z)|^2\big)^p dA(z) \\
&= \mathcal{B}^{p,q}_\omega(f) \int_\Delta \big(1-|\varphi_a(z)|^2\big)^{-p} \big(1-|z|^2\big)^p \frac{\big(1-|a|^2\big)^2}{|1-\bar{a}z|^4} dA(z),
\end{aligned}$$

where, the Jacobian determinant is

$$(3) \qquad J_1 = \frac{\big(1-|a|^2\big)^2}{|1-\bar{a}z|^4}$$

stands for Jacobian determinant. For $0 \leq p < 2$, applying equality (1), we obtain that,

$$\int_{\Delta} |f'(z)|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z)$$
$$\leq \mathcal{B}^q_\omega(f) \int_{\Delta} \frac{\big(1-|a|^2\big)^{(2-p)}}{|1-\bar{a}ry|^{2(2-p)}} d\Gamma = 4\pi\lambda \mathcal{B}^q_\omega(f).$$

When $2 \leq p < \infty$, we can prove the Proposition 2.1, in view of the inequality

$$1-|a| \leq |1-\bar{a}r| \leq 1+|a|.$$

Hence, Proposition 2.1 is completely proved.

Now, in view of Proposition 2.1, we can give the following corollary:

Corollary 2.1. *For $0 \leq p < \infty$ and $0 < q < \infty$, we have*

$$\mathcal{B}_\omega \subset B^{p,q}_\omega.$$

Relationships between weighted $Q_{p_1,\omega}$ classes and $B^{p,q}_\omega$ classes are described in the following result.

Theorem 2.1. *Let $0 < q < 2$ and $2 < p_1 < 4-q$. Then,*

$$\cup_{p_1} Q_{p_1,\omega} \subset \cap_{p,q} B^{p,q}_\omega.$$

Proof. Let $f \in Q_{p_1,\omega}$ for $2 < p_1 < 4-q$ and $0 < q < 2$. By Hölder's inequality, we deduce

$$\int_{\Delta} |f'(z)|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z)$$
$$\leq \left\{ \int_{\Delta} \left[\frac{|f'(z)|^q}{\omega^q(1-|z|)} \big(1-|\varphi_a(z)|^2\big)^{\frac{qp_1}{2}} \right]^{\frac{2}{q}} dA(z) \right\}^{\frac{q}{2}}$$
$$\times \left\{ \int_{\Delta} \left[\big(1-|z|^2\big)^{q-p} \big(1-|\varphi_a(z)|^2\big)^{p-\frac{qp_1}{2}} \right]^{\frac{2}{2-q}} dA(z) \right\}^{\frac{2-q}{2}}$$
$$= \left\{ \int_{\Delta} |f'(z)|^2 \frac{\big(1-|\varphi_a(z)|^2\big)^{p_1}}{\omega^2(1-|z|)} dA(z) \right\}^{\frac{q}{2}}$$
$$\times \left\{ \int_{\Delta} \big(1-|z|^2\big)^{\frac{2(q-p)}{2-q}} \big(1-|\varphi_a(z)|^2\big)^{\frac{2p-qp_1}{2-q}} dA(z) \right\}^{\frac{2-q}{2}} \tag{4}$$

Because,

$$f \in Q_{p_1,\omega} \Longleftrightarrow \sup_{a\in\Delta} \int_{\Delta} |f'(z)|^2 \frac{\big(1-|\varphi_a(z)|^2\big)^{p_1}}{\omega^2(1-|z|)} dA(z) < \infty \quad \text{(see [25])}.$$

Then, by equality (1) and change z by $\varphi_a(z)$, we get

$$
\begin{aligned}
&\int_\Delta |f'(z)|^q (1-|z|^2)^{q-p} \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z) \\
&\le k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ \int_\Delta \frac{(1-|a|^2)^{2(\frac{q-p}{2-q}+1)} (1-|z|^2)^{2(\frac{q-p}{2-q})}}{|1-\bar{a}z|^{4+4\left(\frac{q-p}{2-q}\right)}} dA(z) \right\}^{\frac{2-q}{2}} \\
&\le k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ (1-|a|^2)^{(\frac{4-2p}{2-q})} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{4-2p}{2-q})}{n!\,\Gamma(\frac{4-2p}{2-q})} |a|^{2n} \right\}^{\frac{2-q}{2}} \\
&= k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ \frac{2-q}{q-2p+1} \right\}^{\frac{2-q}{2}}, \qquad (5)
\end{aligned}
$$

where the constant $k > 0$. Hence,

$$
\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} = \int_\Delta |f'(z)|^2 \frac{(1-|\varphi_a(z)|^2)^{p_1}}{\omega^2(1-|z|)} dA(z).
$$

$$
\Longrightarrow \int_\Delta |f'(z)|^q (1-|z|^2)^{q-p} \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z) \le k_1 \|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}},
$$

where the the constant $k_1 > 0$. Then,

$$
\| f \|_{B^{p,q}_\omega} \le \| f \|_{Q_{p_1,\omega}} < \infty.
$$

Therefore $f \in B^{p,q}_\omega$, where $0 < q < 2$ and $2 < p_1 < 4-q$. The proof of the theorem is therefore established.

3. Mixed norms

The mixed norm space $H_{p,q,\gamma}(\Delta)$, $0 < p, q < \infty$ and $-1 < \gamma < \infty$, consists of all $f \in H(\Delta)$ such that (see [18, 21])

$$
\|f\|^q_{p,q,\gamma} = \int_0^1 M^q_p(f,r)(1-r)^\gamma dr < \infty,
$$

where

$$
M_p(f,r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.
$$

Now, let $p, q, \alpha > 0$, $f \in H(\Delta)$, then

$$
f \in H(p,q,\alpha) \Longleftrightarrow \|f\|^q_{p,q,\alpha} = \frac{1}{2\pi} \int_0^1 (1-r)^{\alpha q-1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{q}{p}} dr < \infty
$$

The above definition appeared in (see [19]). $H(p,q,\alpha)$ class was studied and discussed by Flett (see [16, 17]). It should be mentioned that the class $H(p,q,\alpha)$

an interesting class of functions, which contains some known classes such analytic Hardy and analytic Bergman type classes.

One of our main aims in the present article is to introduce the following definition for the weighted mixed norm spaces, then we study some of its important properties.

Definition 3.1. *Suppose that* $\omega : (0,1] \to (0,\infty)$ *and* $0 < p, q, \alpha < \infty$, *and* $f \in H(\Delta)$, *then*

$$f \in H(p,q,\alpha,\omega) \Leftrightarrow \|f\|^q_{p,q,\alpha,\omega} = \frac{1}{2\pi}\int_0^1 \frac{(1-r)^{\alpha q-1}}{\omega(1-r)}\left(\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} dr < \infty.$$

Theorem 3.1. *Let* $0 < q < \infty$ *and* $0 \le p < \infty$, $f \in H(\Delta)$. *Suppose that* $f'(\varphi_a(w))$ *is a nondecreasing function. Let* $\omega : (0,1] \to (0,\infty)$, *then*

$$f \in B^{p,q}_{\omega} \Leftrightarrow \sup_{a\in\Delta}\int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{\big(g(z,a)\big)^p}{\omega^q(1-|z|)}dA(z) < \infty.$$

Proof. Let us consider the equivalence

$$\begin{aligned}&\int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)}dA(z)dA(z)\\ &\approx \int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{\big(g(z,a)\big)^p}{\omega^q(1-|z|)}dA(z),\end{aligned}$$

The change of variables $w = \varphi_a(z)$, resulting that

$$\begin{aligned}&\int_\Delta |f'(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{q-p}\frac{(1-|w|^2)^p}{\omega^q(1-|\varphi_a(w)|)}\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 dA(w)\\ &\approx \int_\Delta |f'(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{q-p}\frac{\log\left(\frac{1}{|w|}\right)}{\omega^q(1-|\varphi_a(w)|)}\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 dA(w).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Delta |f'(\varphi_a(w))|^q\frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\approx \int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w),\end{aligned}$$

we aim to evaluate constants $C_1(p)$ and $C_2(p)$ with

$$\begin{aligned}&C_1(p)\int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\le \int_\Delta |f'(\varphi_a(w))|^q\frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\le C_2(p)\int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w).\end{aligned}$$

Case 1. Let $C_2(p) = 2^p$. Using,

$$1-|a| \leq |1-\bar{a}w| \leq 1+|a| \quad \text{and} \quad 1-|w| \leq |1-\bar{a}w| \leq 1+|w|, \tag{6}$$

we obtain,

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&-2^p \int_\Delta |f'(\varphi_a(w))|^q \frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&= (1-|a|^2)^{2-p} \int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^q \; L(\log,|w|,p,q)}{|1-\bar{a}w|^{2q}\omega^q(1-|\varphi_a(w)|)} \, dA(w),
\end{aligned}$$

where

$$L(\log,|w|,p,q) = \frac{|1-\bar{a}w|^{2q}(1-|w|^2)^{-p}}{|1-\bar{a}w|^{q-p+4}} \Big[(1-|w|^2)^{-p} - 2^p \log\big(\frac{1}{|w|}\big)\Big].$$

hence, after some simple computation, we deduce

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&-2^p \int_\Delta |f'(\varphi_a(w))|^q \frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&\leq 2^{5+q-p}\pi \|f\|^q_{\mathcal{B}_\omega} (1-|a|^2)^{2-p} \int_0^1 (1-r)^{-(p+q)} \big[(1-r)^{2-p} + 2^p \log r\big] r dr
\end{aligned}$$

since, the last integral exists for all $0 \leq \frac{q-2}{2} < \infty$ and $\frac{q}{2} - 1 < p$.

Also, $f \in \mathcal{B}_\omega$, then

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&\leq C_1(p) \int_\Delta |f'(\varphi_a(w))|^q \frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w).
\end{aligned}$$

Case 2. Let $C_1(p) = \left(\frac{11}{100}\right)^p$. Then,

$$\begin{aligned}
I_2 = &\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\
&-C_2(p) \int_\Delta |f'(\varphi_a(w))|^q \frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w),
\end{aligned}$$

which implies that

$$\begin{aligned} I_2 &= \int_{\Delta} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &= \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &+ \int_{\Delta\setminus\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &= J_1 + J_2. \end{aligned}$$

where

$$G(\log,|w|,p) = \left\{(1-|w|^2)^{-p} + \left(\frac{11}{100}\right)^p \left(\frac{\log|w|}{(1-|w|^2)}\right)^p\right\}.$$

Since $G(\log,|w|,p) \le 0;\ \forall |w| \in [0,\frac{1}{10}]$, then using (2), we obtain that

$$\begin{aligned} J_1 &= \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\ &\ge k_2\ (1-|a|^2)^{q-p+4} \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)\ dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \end{aligned}$$

and

$$\begin{aligned} J_2 &= (1-|a|^2)^{q-p+2} \int_{\Delta\setminus\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\ &\ge k_3(1-|a|^2)^{q-p+4} \int_{\frac{1}{10}}^{1} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w), \end{aligned}$$

where the constants k_2 and k_3 are positive.

Since, $\left|f'(\varphi_a(w))\right|^q \ge 0$; and $G(\log,|w|,p) \le 0;\ \forall |w| = r \in [0,\frac{1}{10}]$.

Now, we want to compare the integral

$$k_2(1-|a|^2)^{q-p+2} \int_0^{\frac{1}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w),$$

and the integral $k_3(1-|a|^2)^{q-p+4} \int_{\frac{5}{10}}^{\frac{6}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w)$.

After simple calculation, we can obtain that

$$\begin{aligned} & k_2(1-|a|^2)^{q-p+2} \int_0^{\frac{1}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w), \\ & < k_3(1-|a|^2)^{q-p+4} \int_{\frac{5}{10}}^{\frac{6}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \end{aligned}$$

Since from the assumptions, we have that, $f'(\varphi_a(w))$ is a nondecreasing function, $\forall 0 \leq |w| < 1$. Thus,

$$I_2 = J_1 + J_2 \geq 0.$$

The proof of Theorem 3.1 is finished.

Theorem 3.2. *Let $f \in H(\Delta)$. Suppose that $\omega : (0,1] \to (0,\infty)$. Let $0 < p \leq \infty$, and $0 < \alpha, q < \infty$. Then, the function*

$$f(z) = \frac{\omega(1-|z|)}{(1-|z|)^{\gamma}}$$

belongs to the weighted mixed norm space

$$H(p,q,\alpha,\omega) \Longleftrightarrow \gamma < 1 + \alpha$$

and

$$f \in H(p,\infty,\alpha,\omega) \Longleftrightarrow \gamma \leq \frac{1}{p} + \alpha.$$

Proof. Using similar steps to the corresponding result in [11], with some simple modifications, we can easily establish the proof of Theorem 3.2.

4. Logarithmic tent spaces

In this section, we introduce weighted $(p,q;\ln,\omega)$-Carleson measures on the unit disk Δ. Then we study $(p,q;\ln,\omega)$-Carleson measures for the weighted classes of Bloch-type and $B^q(p,\ln,\omega,\varphi)$ type-spaces. Moreover, we define the conformally invariant Bloch space and the modified Möbius-invariant seminorms. Boundedness (resp., compactness) for the $B^q(p,\ln,\omega,\varphi)$ classes, which contained in the weighted tent-type space $\mathcal{T}_p^{\infty}(\mu_{\ln,\omega,p,q}, q)$ are also discussed.

For more studies about tent spaces, we refer to [14,15,23].

Definition 4.1. *For $0 \leq p < \infty,\ 0 < q < \infty$. Let $\omega : (0,1] \to (0,\infty)$, we will call that the weighted positive measure μ defined on Δ is a bounded weighted $(p,q;\ln,\omega)$-Carleson measure provided,*

$$\mu_{\ln,\omega,p,q}(S(I,\omega)) = O\left(\frac{|\, I \,|^p \ln(1+|I|)}{\omega^q(|I|)}\right), \tag{7}$$

for all subarcs I of $\partial\Delta$ where, $|\, I \,|$ denotes the arc length of $I \subset \partial\Delta$ and $S(I)$ defines the Carleson box, which is based on I, that is,

$$S(I) = \{z \in \Delta : z/|z| \in I,\ 1-|z| \leq |I|/2\pi\}.$$

Remark 4.1. When $p = 1, \omega \equiv 1$ and $|I| = e - 1$, then the usual standard definition of Carleson measure is obtained.

If $\omega \equiv 1$ and $|I| = e-1$, then the p-Carleson measure concept is also deduced.

Definition 4.2. *In Definition 4.1, if the right side of (7) becomes* $o(\frac{|I|^p \ln(1+|I|)}{\omega^q(|I|)})$ *as* $|I| \to 0$, *then we get the definition of weighted compact* $(p, q; \ln, \omega)$*-Carleson measure.*

Let $B^q(p, \ln, \omega, \varphi)$ be a class of all analytic functions f on Δ satisfying

$$\|f\|^q_{B^q(p,\ln,\omega,\varphi)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^q \frac{(1-|\varphi_a(z)|^2)^p \ \ln(2-|z|)}{\omega^q(1-|z|)} \, dA(z) < \infty,$$

where $0 < p < \infty$ and $0 < q < \infty$. Meanwhile, $\mathcal{T}_p^\infty(q, \mu_{\ln,\omega,p,q})$ denotes the weighted tent-type class of all $\mu_{\ln,\omega,p,q}$-measurable functions f on Δ satisfying

$$\|f\|^q_{\mathcal{T}_p^\infty(\mu_{\ln,\omega,p,q})} = \sup_{S(I)\subseteq\Delta} |I|^{-1} \int_{S(I)} |f|^q \ d\mu_{\ln,\omega,p,q} < \infty;$$

where

$$|I| = (2\pi)^{-1} \int_I |d\xi| \quad \text{and} \quad S(I) = \{r\xi \in \Delta : 1-|I| \le r < 1, \xi \in I\}$$

are the normalized length of the subarc I of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the Carleson square in Δ respectively. The norm of $f \in B^q(p, \ln, \omega, \varphi)$ is given by:

$$\sup_{S(I)\subseteq\Delta} |I|^{-p} \int_{S(I)} |f'(z)|^q \frac{(1-|z|^2)^p \ \ln(2-|z|)}{\omega^q(1-|z|)} \ dA(z).$$

Now, we give the following results:

Theorem 4.1. *Let* $\mu_{\ln,\omega,p,q}$ *be a nonnegative Borel measure on* Δ. *Then the identity operator* $I : B^q(p, \ln, \omega, \varphi) \longrightarrow \mathcal{T}_p^\infty(q, \mu_{\ln,\omega,p,q})$ *is bounded* $\Longleftrightarrow$

$$\|\mu_{\ln,\omega,p,q}\|^q_{\mathcal{LCM}_p} = \sup_{S(I)\subseteq\Delta} \frac{\mu_{\ln,\omega,p,q}(S(I))}{|I|^p(\log\frac{2}{|I|})^{-q}} < \infty.$$

Proof. Necessity. Given a subarc I of $\mathbb{T}$. If $f_{a,\ln,\omega}(z) = \frac{(1-\bar{a}z)\ln(2-|z|)}{\omega^q(1-|z|)}$ where $a = (1-|I|)\xi$ and ξ is the center of I, then

$$|f_{a,ln,\omega}(z)| \approx \frac{\ln(2|I|^{-p})\ln(1+|I|}{\omega^q(|I|^{-p})}, \qquad z \in S(I)$$

and

$$|I|^{-p} \int_{S(I)} |f_{a,\ln,\omega}|^q \ d\mu_{\ln,\omega,p,q} \le \|f_{a,\ln,\omega}\|^q_{B^q(p,\ln,\omega,\varphi)} \lesssim 1.$$

Accordingly, $\|\mu_{\ln,\omega,p,q}\|_{\mathcal{LCM}_p} \lesssim 1$.

Sufficiency. Assume that a nonnegative Borel measure $\mu_{\ln,\omega,p,q}$ on Δ is said to be weighted $(p,q;\ln,\omega)$-Carleson measures on the unit disk Δ for the weighted class $B^q(p,\ln,\omega,\varphi)$ of all $f \in \Delta$ which satisfying

$$\|f\|^q_{B^q(p,\ln,\omega,\varphi)} = \int_\Delta |f'(z)|^q \frac{(1-|z|^2)^p \ \ln(2-|z|)}{\omega^q(1-|z|)} \, dA(z) < \infty,$$

provided

$$\int_\Delta |f|^q \, d\mu_{\ln,\omega,p,q} \lesssim \|f\|^q_{B^q(p,\ln,\omega,\varphi)}.$$

This completes the proof.

Corollary 4.1. *Let $\mu_{\ln,\omega,p,q}$ be a nonnegative Borel measure on Δ. Then the identity operator $I_1 : B^q(p,\ln,\omega,\varphi) \longrightarrow \mathcal{T}_p^\infty(q,\mu_{\ln,\omega,p,q}) \Longleftrightarrow$*

$$\lim_{|I|\longrightarrow 0} \frac{\mu_{\ln,\omega,p,q}(S(I))}{|I|^p(\log \frac{2}{|I|})^{-q}} = 0.$$

Remark 4.2. An interesting and important question can be stated as follows:

Is the concept of analytic Tent function can be generalized using quaternion-functions?

For more information on several studied and various discussions using Clifford analysis, we can refer to the citations [1, 2, 3, 4, 5, 6, 7, 8, 9, 13] and others.

5. Conclusion

This paper starts with a concise overview of weighted function spaces in the sense of analytic functions.

New tools are used in studying some new weighted function spaces in Δ. Properties of weighted mixed normed spaces, which are generalizations of the so-called mixed normed spaces are introduced. Several inclusion/comparison results among these spaces are presented. Also we dealt with some properties of certain Carleson measures (which have introduced) and their relation with the Bloch-type spaces.

Acknowledgements

The authors are grateful to Taif University (KSA) for supporting this article under project 5094/437/1.

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Accepted: 20.09.2018

Toward strictly singular fractional operator restricted by Fredholm-Volterra in Sobolev space

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Abstract. In this paper, a reliable numerical technique is proposed for solving a class of singular fractional differential equations involving Fredholm and Volterra operators subjected to suitable three-point boundary conditions. The solution methodology is presented based on reproducing-kernel method (RKM), which is used directly without employing linearization and perturbation. However, a favorable Hilbert spaces are construcred, and then the orthonormal function system is generated by using Gram-Schmidt orthogonalization process. Error analysis is given in Sobolev space. Numerical example is tested to multipoint singular fractional differential problems with Fredholm and Volterra operators to show the theoretical statements of the RKHS method. The results obtained indicate that the RKHS method is easy to implement, reliability and capability with a great potential of such singular problems.
Keywords: singular integral operator, fractional differential equation, reproducing-kernel method, Caputo fractional derivative, Gram-Schmidt process.

1. Introduction

The multipoint singular boundary value problems (BVPs) arise in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures, and atomic calculations. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint singular BVP as in [1]. Many problems in the theory of elastic stability can be handled using multipoint singular BVPs as in [2]. In optimal bridge design, large size bridges are sometimes contrived with multipoint supports, which corresponds to a mul-

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tipoint singular BVP as in [3]. Therefore, it appears to be very important to develop numerical or analytical methods for solving such problems.

Most scientific problems and phenomenons in different fields of sciences and engineering occur nonlinearly with a set of finite singularity. To be more precisely, most of them can not be handled analytically. So these nonlinear singular equations should be solved using numerical methods or other analytical methods. Anyhow, when we use multipoint singular BVPs, the obtained numerical solutions could be not that required accurate outcomes or may even fail to converge due to singularity problem [4, 5, 6]; whilst analytical methods commonly used to solve nonlinear singular differential equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors. Thus, it is expected to have some restrictions to handle these kind of problems; because of two difficulties presenting in both nonlinearity of equations and the singularity case of BVPs.

In this paper, we aim to use appropriate theory for building the Hilbert spaces to develop IRKM algorithm for handling second-order singular ordinary differential equations with three-point boundary conditions. In particular, we provide the analytical-numerical solutions for the following singular differential-operator equation:

$$
\begin{aligned}
&D^{2\alpha}u\left(x\right)+P(x)D^{\alpha}\left(x\right)+Q(x)u\left(x\right)=F\left(x,Su\left(x\right),Tu\left(x\right)\right),x\in[0,1],\\
&Su\left(x\right)=\lambda_1\textstyle\int_0^1 k_1\left(x,\xi\right)u\left(\xi\right)d\xi,\\
&Tu\left(x\right)=\lambda_2\textstyle\int_0^x k_2\left(x,\xi\right)u\left(\xi\right)d\xi,
\end{aligned}
\tag{1}
$$

with the boundary conditions

$$
\begin{aligned}
&u\left(0\right)=0,\\
&u\left(1\right)-\alpha u\left(\eta\right)=0,\ 0<\eta<1,\ \alpha>0,\ \alpha\eta<1,
\end{aligned}
\tag{2}
$$

where $0<\alpha\leq 1$, λ_1 and λ_2 are constant parameters, $D^{\alpha}\left(x\right)$ is indicated to fractional derivative in the Caputo sense, $k_1\left(x,t\right),k_2\left(x,t\right)$ are continuous arbitrary kernel functions over $0<\xi<x<1$, $F\left(x,w_1,w_2\right)$ is continuous terms in $W_2^1\left[0,1\right]$ as $w_i=w_i\left(x\right)\in W_2^3\left[0,1\right]$, $-\infty<w_i<\infty$, $i=1,2$, $P\left(x\right)$ and $Q\left(x\right)$ are continuous real-valued functions and may be equal to zero at some $\{x_i\}_{i=1}^m\in[0,1]$, $u(x)$ is an unknown analytical function in $W_2^3\left[0,1\right]$ to be determined, and $W_2^1\left[0,1\right],W_2^3\left[0,1\right]$ are reproducing-kernel spaces. Here, we assume that Eq. (1) with conditions (2) has a unique smooth solution. Further, the Caputo fractional derivative of order $m-1<\alpha\leq m,m\in N$, can be defined as follows

$$
D^{\alpha}u\left(x\right)=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}u^{(m)}(t)dt,\ 0<t<x,
$$

Many authors have studied different kind of analysis about solvability of second-order, three point singular BVPs. In order to get more information about the properties and the applications have been studied, the reader can refer to [7, 8, 9, 10]. On the other hand, there is a few research papers about investigating second-order, three point singular BVPs restricted by Fredholm-Volterra operators numerically. Reproducing kernel theory has many applications in complex analysis, harmoni analysis, and quantum mechanics [11, 12, 13, 14, 15, 16]. Recently, a lot of research work has been carried out to apply of the RKHS algorithm for wide classes of stochastic and deterministic problems involving operator equations, differential equations, integral equations, and integro-differential equations. The RKHS algorithm was successfully used by many authors to investigate many scientific applications side by side with their theories. To understand the properties and the fundamentals of reproducing kernel Hilbert spaces, the reader can return to the references [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Fractional DEs are a type of differential equations that involving an unknown function with fractional derivatives. These types of equations are used to formulate problems involving functions of single or several variables and to aid a solution of many physical phenomena in terms of fractional DEs. The Caputo's definition will be used, taking the feature of Caputo's approach that the initial conditions of fractional DEs with Caputo's derivatives take the traditional form. For more details, we refer to [27, 28, 29, 30, 31, 32].

The outline of the paper is as follows. In section 2, two appropriate inner product spaces are constructed to apply RKHS method to solve the presented BVP with Fredholm Volterra operator. In section 3, Gram-Schmidt orthogonalization process is used to obtain the orthonormal basis. Meanwhile, the efficiency of the method is proposed in section 3 by proving that the numerical solution converges to the analytical solution uniformly. After all, numerical algorithm with numerical example are presented to show how the process does work in section 4. Finally, we summarize up the process with some concluding remarks in section 5.

2. Toward to reproducing-kernel function

In this section, a method for constructing a reproducing kernel function that satisfying the two-point boundary conditions $v(0) = 0$ and $v(1) = 0$ is presented. By applying some good properties of the reproducing kernel space, a very simple numerical method is provided for obtaining approximation to the solution of Eqs. (1) and (2). Here, $L^2[0,1] = \{v \mid \int_0^1 v^2(x)\,dx < \infty\}$ and $l^2 = \{A \mid \sum_{i=1}^{\infty} (A_i)^2 < \infty\}$.

Definition 2.1 ([33]). *Let Π be a Hilbert space of function $\theta : \Omega \to \Pi$ on a set Ω. A function $\Gamma : \Omega \times \Omega \to \mathbb{C}$ is a reproducing kernel of Π if the following conditions are satisfied. Firstly, $\Gamma(\cdot, x) \in \Pi$ for each $x \in \Omega$. Secondly, $\langle \theta(\cdot), \Gamma(\cdot, x) \rangle = \theta(x)$ for each $\theta \in \Pi$ and each $x \in \Omega$.*

To solve Eqs. (1) and (2) using RKHS algorithm, we first define and construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the two-point boundary conditions $v(0) = 0$ and $v(1) = 0$. After that, we utilize a reproducing kernel space $W_2^1[0,1]$.

Definition 2.2. *The space* $W_2^3[0,1]$ *is defined as* $W_2^3[0,1] = \{v \mid v, v', v''$ *are absolutely continuous on* $[0,1]$, $v, v', v'', v''' \in L^2[0,1]$, *and* $v(0) = 0, v(1) = 0\}$. *On the other hand, the inner product and the norm in* $W_2^3[0,1]$ *are defined, respectively, by*

$$\langle v(x), w(x)\rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0)\, w^{(i)}(0) + \int_0^1 v'''(x)w'''(x)dx, \tag{3}$$

and $||v||_{W_2^3} = \sqrt{\langle v(x), v(x)\rangle_{W_2^3}}$, *where* $v, w \in W_2^3[0,1]$.

It is easy to see that $\langle u(x), v(x)\rangle_{W_2^3}$ satisfies all the requirements of the inner product as follows; first, $\langle u(x), u(x)\rangle_{W_2^3} \geq 0$; second, $\langle u(x), v(x)\rangle_{W_2^3} = \langle v(x), u(x)\rangle_{W_2^3}$; third, $\langle \gamma u(x), v(x)\rangle_{W_2^3} = \gamma\langle u(x), v(x)\rangle_{W_2^3}$; fourth, $\langle u(x) + w(x), v(x)\rangle_{W_2^3} = \langle u(x), v(x)\rangle_{W_2^3} + \langle w(x), v(x)\rangle_{W_2^3}$, where $u, v, w \in W_2^3[0,1]$. Indeed, it is obvious that when $u(x) = 0$, then $\langle u(x), u(x)\rangle_{W_2^3} = 0$, while on the other aspect as well, if $\langle u(x), u(x)\rangle_{W_2^3} = 0$, then by Eq. (3); $\langle u(x), u(x)\rangle_{W_2^3} = \sum_{i=0}^{2}(u^{(i)}(0))^2 + \int_0^1 (u'''(x))^2 dx = 0$, therefore, $u(0) = u'(0) = u''(0) = 0$ and $u'''(x) = 0$. Thus, one can obtain $u(x) = 0$.

The Hilbert space $W_2^3[0,1]$ is called a reproducing kernel if for each fixed $x \in [0,1]$, there exist $R^{\{1\}}(x,y) \in W_2^3[0,1]$ (simply $R_x^{\{1\}}(y)$) such that $\langle v(y), R_x^{\{1\}}(y)\rangle_{W_2^3} = v(x)$ for any $v(y) \in W_2^3[0,1]$ and $y \in [0,1]$.

Theorem 2.1 ([34])**.** *The Hilbert space* $W_2^3[0,1]$ *is a complete reproducing kernel with reproducing kernel function*

$$R_x^{\{1\}}(y) = \begin{cases} a_1(x) + a_2(x)y + a_3(x)y^2 + a_4(x)y^3 \\ \qquad +a_5(x)y^4 + a_6(x)y^5, y \leq x, \\ b_1(x) + b_2(x)y + b_3(x)y^2 + b_4(x)y^3 \\ \qquad +b_5(x)y^4 + b_6(x)y^5, y > x, \end{cases} \tag{4}$$

where $a_i(x)$ *and* $b_i(x)$, $i = 1, 2, ..., 6$, *are unknown coefficients of* $R_x^{\{1\}}(y)$ *and are given as*

a_i*'s coefficients*	b_i*'s coefficients*
$a_1(x) = 0,$	$b_1(x) = \frac{1}{120}x^5,$
$a_2(x) = -\frac{1}{156}x(-36 + 30x + 10x^2 - 5x^3 + x^4),$	$b_2(x) = -\frac{1}{312}x(-72 + 60x + 20x^2 + 3x^3 + 2x^4),$
$a_3(x) = -\frac{1}{624}x(120 - 126x + 10x^2 - 5x^3 + x^4),$	$b_3(x) = -\frac{1}{624}x(120 - 126x - 42x^2 - 5x^3 + x^4),$
$a_4(x) = -\frac{1}{1872}x(120 - 126x + 10x^2 - 5x^3 + x^4),$	$b_4(x) = -\frac{1}{1872}x(120 + 30x + 10x^2 - 5x^3 + x^4),$
$a_5(x) = \frac{1}{3744}x(-36 + 30x + 10x^2 - 5x^3 + x^4),$	$b_5(x) = \frac{1}{3744}x(120 + 30x + 10x^2 - 5x^3 + x^4),$
$a_6(x) = \frac{1}{18720}(156 - 120x - 30x^2 - 10x^3 + 5x^4 - x^5),$	$b_6(x) = -\frac{1}{18720}x(120 + 30x + 10x^2 - 5x^3 + x^4).$

Definition 2.3 ([17]). *The space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{v \mid v$ is absolutely continuous on $[0,1]$ and $v' \in L^2[0,1]\}$. On the other hand, the inner product and the norm in $W_2^1[0,1]$ are defined, respectively, by*

$$\langle v(x), w(x)\rangle_{W_2^1} = v(0)\, w(0) + \int_0^1 v'(x) w'(x) dx,$$

and $||v||_{W_2^1} = \sqrt{\langle v(x), v(x)\rangle_{W_2^1}}$, where $v, w \in W_2^1[0,1]$.

Theorem 2.2 ([17]). *The Hilbert space $W_2^1[0,1]$ is a complete reproducing kernel with reproducing kernel function*

$$R_x^{\{2\}}(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$$

3. Structure of the method

Here, the formulation of a differential linear operator is presented in $W_2^3[0,1]$. After that, we use the Gram-Schmidt orthogonalization process on the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ and normalizing them on $W_2^3[0,1]$ to obtain the required orthogonalization coefficients in order to obtain the analytical-numerical solutions of Eqs. (1) and (2) using RKHS algorithm.

Let us consider the differential operator $L : W_2^3[0,1] \to W_2^1[0,1]$ such that $Lv(x) = D^{2\alpha}v(x) + P(x) D^{\alpha}v(x) + Q(x) v(x)$, $v(x) = u(x) - \phi(x)$, where $\phi(x)$ satisfies $\phi(0) = 0$ and $\phi(1) = \gamma$, that is, $\phi(x) = \gamma x$ and $v(x) = u(x) - \gamma x$. Thus, Eqs. (1) and (2) can be equivalently converted into the form:

$$Lv(x) = F(x, (v+\phi)(x), S(v+\phi)(x), T(v+\phi)(x)) - \left(\phi' P + \phi Q\right)(x), \tag{5}$$

with respect to the two-point boundary conditions

$$v(0) = 0,\ v(1) = 0. \tag{6}$$

Theorem 3.1. *The operator $L : W_2^3[0,1] \to W_2^1[0,1]$ is bounded and linear.*

Proof. Clearly, $\|Lv\|^2_{W_2^1} \le M\|v\|^2_{W_2^3}$, where $M > 0$. From the definition of $W_2^1[0,1]$, we have $\|Lv\|^2_{W_2^1} = \langle Lv(x), Lv(x)\rangle_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lv)'(x)]^2 dx$. By the Schwarz inequality and reproducing properties $v(x) = \langle v(y), R_x^{\{1\}}(y)\rangle_{W_2^3}$, $(Lv)(x) = \langle v(y), (LR_x^{\{1\}})(y)\rangle_{W_2^3}$, and $(Lv)'(x) = \langle v(y), (LR_x^{\{1\}})'(y)\rangle_{W_2^3}$, we get

$$|(Lv)(x)| = \left|\left\langle v(x), (LR_x^{\{1\}})(x)\right\rangle_{W_2^3}\right| \le \left\|LR_x^{\{1\}}\right\|_{W_2^3} \|v\|_{W_2^3} = M_1 \|v\|_{W_2^3},$$

$$\left|(Lv)'(x)\right| = \left|\left\langle v(x), (LR_x^{\{1\}})'(x)\right\rangle_{W_2^3}\right| \le \left\|(LR_x^{\{1\}})'\right\|_{W_2^3} \|v\|_{W_2^3} = M_2 \|v\|_{W_2^3},$$

where $M_i > 0$, $i = 1, 2$. Thus, $\|Lv\|^2_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lv)'(x)]^2 dx \le (M_1^2 + M_2^2)\|v\|^2_{W_2^3}$. The linearity part is obvious. □

To construct an orthogonal function system of $W_2^3[0,1]$; put $\varphi_i(x) = R_{x_i}^{\{2\}}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and L^* is the adjoint operator of L. In other words, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i)$, $i = 1, 2, \ldots$. The orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{7}$$

where $\beta_{ij} = \frac{1}{\|\psi_1\|_{W_2^3}}$ for $i = j = 1$, $\beta_{ij} = \frac{1}{\sqrt{\|\psi_i\|^2_{W_2^3} - \sum_{k=1}^{i-1}(\langle\psi_i(x), \overline{\psi}_k(x)\rangle_{W_2^3})^2}}$ for $i = j \neq 1$, and $\beta_{ij} = -\frac{1}{\sqrt{\|\psi_i\|^2_{W_2^3} - \sum_{k=1}^{i-1}(c_{ik})^2}} \times \sum_{k=j}^{i-1}\langle\psi_i(x), \overline{\psi}_k(x)\rangle_{W_2^3}\beta_{kj}$ for $i > j$.

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of the space $W_2^3[0,1]$.*

Proof. Clearly, $\psi_i(x) = L^*\varphi_i(x) = \langle L^*\varphi_i(x), R_x^{\{1\}}(y)\rangle_{W_2^3} = \langle\varphi_i(x), L_y R_x^{\{1\}}(y)\rangle_{W_2^1} = L_y R_x^{\{1\}}(y)|_{y=x_i} \in W_2^3[0,1]$, so, $\psi_i(x) = .L_y R_x^{\{1\}}(y)|_{y=x_i}$. For each fixed $v \in W_2^3[0,1]$, let $\langle v(x), \psi_i(x)\rangle_{W_2^3} = 0$, so, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i) = 0$. But since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore $Lv(x) = 0$. It follows that $v(x) = 0$ from the existence of L^{-1}. □

Theorem 3.3. *For each $v \in W_2^3[0,1]$, $\sum_{i=1}^{\infty}\langle v(x), \bar{\psi}_i(x)\rangle\bar{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^3[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the analytical solution of Eqs. (5) and (6) is*

$$\begin{aligned} v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[&F(x_k, (v+\phi)(x_k), S(v+\phi)(x_k), T(v+\phi)(x_k)) \\ &- (\phi' P + \phi Q)(x_k)]\bar{\psi}_i(x). \end{aligned} \tag{8}$$

Proof. Using Eq. (7), it easy to see that

$$\begin{aligned} v(x) &= L^{-1}F(x, (v+\phi)(x), S(v+\phi)(x_k), T(v+\phi)(x)) - (\phi' P + \phi Q)(x) \\ &= \sum_{i=1}^{\infty}\langle v(x), \bar{\psi}_i(x)\rangle_{W_2^3}\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle v(x), \psi_k(x)\rangle_{W_2^3}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle v(x), L^*\varphi_k(x)\rangle_{W_2^3}\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lv(x), \varphi_k(x)\rangle_{W_2^1}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle F(x, (v+\phi)(x), S(v+\phi)(x), T(v+\phi)(x)) \\ &\quad - (\phi' P + \phi Q)(x), \varphi_k(x)\rangle_{W_2^1}\bar{\psi}_i(x) \end{aligned}$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[F(x_k,(v+\phi)(x_k),S(v+\phi)(x_k),T(v+\phi)(x_k)) - (\phi'P+\phi Q)(x_k)]\bar{\psi}_i(x).$$

Hence, Eq. (8) is the analytical solution of Eqs. (5) and (6). □

Let $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ be the normal orthogonal system derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$, then according to Eq. (8), the analytical solution of Eqs. (5) and (6) can be denoted by

$$v(x) = \sum_{i=1}^{\infty} B_i\bar{\psi}_i(x), \tag{9}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}[F(x_k,(v_{k-1}+\phi)(x_k),S(v_{k-1}+\phi)(x_k),T(v_{k-1}+\phi)(x_k)) - (\phi'P+\phi Q)(x_k)]$. In fact, B_i in Eq. (9) are unknown, we will approximate B_i using known A_i. For a numerical computations, we define the initial function $v_0(x_1) = 0$, put $v_0(x_1) = v(x_1)$, and define the n-term approximation $v_n(x)$ to $v(x)$ as

$$v_n(x) = \sum_{i=1}^{n} A_i\bar{\psi}_i(x), \tag{10}$$

$$A_i = \sum_{k=1}^{i}\beta_{ik}\big[F(x_k,(v_{k-1}+\phi)(x_k),S(v_{k-1}+\phi)(x_k),T(v_{k-1}+\phi)(x_k)) - (\phi'P+\phi Q)(x_k)\big]. \tag{11}$$

Theorem 3.4. *If $||v_n||_{W_2^3}$ is bounded and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term numerical solution $v_n(x)$ in the iterative formula of Eq. (10) converges to the analytical solution $v(x)$ of Eqs. (5) and (6) in the space $W_2^3[0,1]$ and $v(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i(x)$, where A_i is given by Eq. (11).*

Proof. The proof is straightforward. □

If $\delta_n = ||v - v_n||_{W_2^3}$, where $v(x)$ and $v_n(x)$ are given by Eqs. (9) and (10), respectively, then $\delta_n^2 = \|\sum_{i=n+1}^{\infty} A_i\bar{\psi}_i\|_{W_2^3}^2 = \sum_{i=n+1}^{\infty}(A_i)^2$ and $\delta_{n-1}^2 = \|\sum_{i=n}^{\infty} A_i\bar{\psi}_i\|_{W_2^3}^2 = \sum_{i=n}^{\infty}(A_i)^2$. Thus, $\delta_{n-1} \geq \delta_n$, and consequently $\{\delta_n\}$ are monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$. By Theorem 3.3, $\sum_{i=1}^{\infty} A_i\bar{\psi}_i(x)$ is convergent, so, $\delta_n^2 = \sum_{i=n+1}^{\infty}(A_i)^2 \to 0$ or $\delta_n \to 0$ as $n \to \infty$.

4. Numerical example

In order to solve multipoint singular BVPs restricted by Fredholm-Volterra operators numerically and to show behavior, properties, efficiency, and applicability of the present RKHS algorithm, four multipoint singular BVPs restricted by

Fredholm-Volterra operators will be solved numerically in this section. Here, all the symbolic and numerical computations were performed by using MAPLE 13 software package.

Using RKHS algorithm, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, applying $R_x^{\{1\}}(y)$ and $R_x^{\{2\}}(y)$ on $[0, 1]$. Some tabulate data are presented and discussed quantitatively at some selected grid points on $[0, 1]$ to illustrate the numerical solutions for the following multipoint singular BVPs restricted by the given Fredholm-Volterra operators.

Example 1. Consider the singularities at two endpoint of $[0, 1]$:

$$
\begin{aligned}
&D^{2\alpha}u(x) + \frac{1}{\sin(x)}D^{\alpha}u(x) - \frac{1}{x(x-1)}u(x) = [Tu](x) + f(x),\\
&[Tu](x) = \int_0^1 x^2 t u(t)dt + \int_0^x (x+1)tu(t)dt,
\end{aligned}
$$

subject to the three-point boundary conditions

$$
\begin{aligned}
&u(0) = 0,\\
&u(1) - 4u\left(\tfrac{1}{9}\right) = 0,
\end{aligned}
$$

where $0 < t < x < 1$. The analytical solution at $\alpha = 1$ is $u(x) = x(x-1)(x - \frac{1}{9})\cos(x)$.

Example 2. Consider the singularities at two endpoint of $[0, 1]$:

$$
\begin{aligned}
&D^{2\alpha}u(x) - \frac{1}{x^2(1-x)^2}D^{\alpha}u(x) + \frac{1}{\sinh(x)}u(x)\\
&= u^2(x) + \sinh^{-1}(u(x)) + [Tu](x) + f(x),\\
&[Tu](x) = \int_0^1 xtu^3(t)\,dt + \int_0^x (x-t)u^2(t)\,dt,
\end{aligned}
$$

subject to the three-point boundary conditions

$$
\begin{aligned}
&u(0) = 0,\\
&u(1) - u\left(\tfrac{1}{2}\right) = 0,
\end{aligned}
$$

where $0 < t < x < 1$. The analytical solution at $\alpha = 1$ is $u(x) = (x - \frac{1}{2})^2(x - 1)^2\sinh(x)$.

Our next goal is to illustrate some numerical results of the RKHS solutions of the aforementioned examples in numeric values. In fact, results from numerical analysis are an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. Next, the agreement between the analytical-numerical solutions is investigated for Examples 1 and 2 at $\alpha = 1$

and various x in $[0, 1]$ by computing the absolute errors and the relative errors of numerically approximating their analytical solutions for the corresponding equivalent equations as shown in Table 1, and Table 2, respectively. Anyhow, it is clear from the tables that, the numerical solutions are in close agreement with the analytical solutions for all examples, while the accuracy is in advanced by using only few tens of the RKHS iterations. Indeed, we can conclude that higher accuracy can be achieved by computing further RKHS iterations.

Table 1. The analytical-numerical solutions and errors at $\alpha = 1$ for Example 1.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.00648674140330012	−0.00648836153953347	1.62014×10^{-6}	2.49761×10^{-4}
0.32	−0.04314675763472329	−0.04314611857352801	6.39061×10^{-7}	1.48113×10^{-5}
0.48	−0.08166976184992833	−0.08166610925390749	3.65260×10^{-6}	4.47240×10^{-5}
0.64	−0.09774018067274835	−0.09773862645916864	1.55421×10^{-6}	1.59015×10^{-5}
0.80	−0.07679256174137644	−0.07679374066742663	1.17893×10^{-6}	1.53521×10^{-5}
0.96	−0.01869522215933257	−0.01869699234994678	1.77019×10^{-6}	9.46868×10^{-5}

Table 2. The analytical-numerical solutions and errors at $\alpha = 1$ for Example 2.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	0.01310653223586577	0.01310629969713650	2.32539×10^{-7}	1.77422×10^{-5}
0.32	0.00487640352847436	0.00487621848205233	1.85046×10^{-7}	3.79473×10^{-5}
0.48	0.00005393349784172	0.00005386640494187	6.70929×10^{-8}	1.24399×10^{-3}
0.64	0.00173897887325904	0.00173902449076913	4.56175×10^{-8}	2.62324×10^{-5}
0.80	0.00319718153587544	0.00319727450018004	9.29643×10^{-8}	2.90770×10^{-5}
0.96	0.00037729187128320	0.00037738952445887	9.76532×10^{-8}	2.58827×10^{-4}

5. Concluding remarks

In this work, we have used the reproducing kernel algorithm for solving linear and nonlinear second-order, three-point singular BVPs restricted by Fredholm-Volterra operators. In the meantime, we employed our algorithm and its conjugate operator to construct the complete orthonormal basis in the reproducing kernel space $W_2^3[0, 1]$. By separating the multipoint boundary conditions and adding the initial and boundary conditions to the reproducing kernel space that satisfying these points, we obtain the analytical-numerical solutions of the problem. The algorithm is applied in a direct way without using linearization, perturbation, or any restrictive assumptions. It may be concluded that RKHS algorithm is very powerful and efficient in finding the analytical-numerical solutions for a wide class of multipoint singular BVPs. It is worth mentioning here that the algorithm is capable of reducing the volume of the computational work and complexity while still maintaining the high accuracy of the numerical results.

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Accepted: 6.10.2018

Nonlinear left $*$-Lie triple mappings of standard operator algebras

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Abstract. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. For $A, B \in \mathcal{A}$, define by ${}_*[A, B] = AB - B^*A$ the left $*$-Lie product of A and B. In this paper, we prove that a mapping $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies $\phi({}_*[A, {}_*[B, C]]) = {}_*[\phi(A), {}_*[B, C]] + {}_*[A, {}_*[\phi(B), C]] + {}_*[A, {}_*[B, \phi(C)]]$, for all $A, B, C \in \mathcal{A}$ is automatically linear. Moreover, ϕ is an inner $*$-derivation.

Keywords: left $*$-Lie triple product, derivation, standard operator algebras.

1. Introduction

Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *nonlinear Lie derivation* if $\phi([A, B]) = [\phi(A), B] + [A, \phi(B)]$ holds true for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product. Furthermore, if $\mathcal{A}$ is an algebra with involution, a mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *nonlinear $*$-Lie derivation* if for any $A, B \in \mathcal{A}$, $\phi([A, B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$, where $[A, B]_* = AB - BA^*$ is the skew Lie product of A and B. Note that for both cases no additivity is assumed on ϕ. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $\phi(AB) = \phi(A)B + A\phi(B)$, for all $A, B \in \mathcal{A}$. ϕ is a $*$-*derivation* provided that

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$\phi(A^*) = \phi(A)^*$, for all $A \in \mathcal{A}$. A derivation on $\mathcal{A}$ is inner if there exists $T \in \mathcal{A}$ such that $\phi(A) = AT - TA$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *Jordan derivation* if $\phi(A^2) = \phi(A)A + A\phi(A)$, for all $A \in \mathcal{A}$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *Jordan left $*$-derivation* if $\phi(A^2) = \phi(A)A + A^*\phi(A)$ holds true for any $A \in \mathcal{A}$.

Concerning Lie product, Lu and Liu [6] proved that every Lie derivation on $\mathcal{B}(\mathcal{X})$ can be expressed as the sum of an additive derivation of $\mathcal{B}(\mathcal{X})$ into itself and a central mapping on $\mathcal{B}(\mathcal{X})$ vanishing on each commutator. This result was generalized to the case of Lie derivation on prime rings in [3]. The skew Lie product is found playing an important role in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [8, 9, 10]) and in the problem of characterizing ideals (see, for example, [1, 7]). In [13] Yu and Zhang showed that every nonlinear $*$-Lie derivation from a factor von Neumann algebra on an infinite dimensional complex Hilbert space into itself is an additive $*$-derivation. In [5], Li, Lu and Fang arrived the same conclusion on von Neumann algebra without central abelian projections. Recently, Jing [4] proved that every nonlinear $*$-Lie derivation of standard operator algebra on complex Hilbert space is an inner $*$-derivation.

In this paper, we define left $*$-Lie product by ${}_*[A, B] = AB - B^*A$, for all $A, B \in \mathcal{A}$, in fact, it have a close relationship to Jordan left $*$-derivation [11]. And we call a nonlinear mapping ϕ is a nonlinear left $*$-Lie triple mapping if it satisfying $\phi({}_*[A, {}_*[B, C]]) = {}_*[\phi(A), {}_*[B, C]] + {}_*[A, {}_*[\phi(B), C]] + {}_*[A, {}_*[B, \phi(C)]]$ for all $A, B, C \in \mathcal{A}$. We shall show every nonlinear left $*$-Lie triple mapping of standard operator algebras which are closed under adjoint operation on infinite dimensional complex Hilbert space is automatically linear. Moreover it is an inner $*$-derivation.

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field, $\mathcal{B}(\mathcal{H})$ will represent the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We will denote by $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ the subalgebra of all bounded finite rank operators. We call a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ a *standard operator algebra* if it contain $\mathcal{F}(\mathcal{H})$. Note that, different from von Neumann algebra which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra $\mathcal{A}$ is *prime* if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projection provided $P^* = P$ and $P^2 = P$. It is well known that every standard operator algebra is prime and its commutant is $\mathbb{C}I$.

2. The main result and its proof

The main result in this paper is as follows.

Theorem 2.1. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. If $\mathcal{A}$ is* *closed under the adjoint operation and* $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *satisfies*

$$\phi({}_*[A,\ {}_*[B,C]]) = {}_*[\phi(A),\ {}_*[B,C]] + {}_*[A,\ {}_*[\phi(B),C]] + {}_*[A,\ {}_*[B,\phi(C)]],$$

for all $A, B, C \in \mathcal{A}$*, then* ϕ *is a linear* $*$*-derivation. Moreover, there exists an operator* $T \in \mathcal{B}(\mathcal{H})$ *satisfying* $T + T^* = 0$ *such that* $\phi(A) = AT - TA$*, for all* $A \in \mathcal{A}$*, that is,* ϕ *is inner.*

To complete the proof of the main theorem, we begin with the following lemmas.

Lemma 2.1. *Let* $\mathcal{A}$ *be a standard operator algebra containing identity* I *on a complex Hilbert space which is closed under adjoint operation. If* $AB = B^*A$ *holds true for all* $A \in \mathcal{A}$*, then* $B \in \mathbb{R}I$*.*

Proof. In fact, take $A = I$, then $B = B^*$. Thus the condition becomes $AB = BA$. It follows that $B \in \mathbb{C}I$, the center of $\mathcal{A}$, and so $B \in \mathbb{R}I$. □

Lemma 2.2. $\phi(0) = 0$*.*

Proof. It follows from the following:

$$\phi(0) = \phi({}_*[0,\ {}_*[0,0]]) = {}_*[\phi(0),\ {}_*[0,0]] +_* [0,\ {}_*[\phi(0),0]] + {}_*[0,\ {}_*[0,\phi(0)]] = 0.$$

□

Lemma 2.3. $\phi(\mathbb{R}I) \subseteq \mathbb{R}I, \phi(\mathbb{C}I) \subseteq \mathbb{C}I$*. For any* $A \in \mathcal{A}$ *with* $A = A^*$*,* $\phi(A^*) = \phi(A)^*$*.*

Proof. For any $\lambda \in \mathbb{R}$, we consider

$$\begin{aligned} 0 &= \phi({}_*[I,\ {}_*[A,\lambda I]]) \\ &= {}_*[\phi(I),\ {}_*[A,\lambda I]] + {}_*[I,\ {}_*[\phi(A),\lambda I]] + {}_*[I,\ {}_*[A,\phi(\lambda I)]] \\ &= {}_*[I,\ {}_*[A,\phi(\lambda I)]] \\ &= (A + A^*)\phi(\lambda I) - \phi(\lambda I)^*(A + A^*). \end{aligned}$$

This gives us $(A+A^*)\phi(\lambda I) = \phi(\lambda I)^*(A+A^*)$ holds true for all $A \in \mathcal{A}$. That is, $B\phi(\lambda I) = \phi(\lambda I)^*B$ holds true for all $B = B^* \in \mathcal{A}$. Since every element in $\mathcal{A}$ is a linear span of two self-adjoint operators, it follows that $B\phi(\lambda I) = \phi(\lambda I)^*B$ holds true for all $B \in \mathcal{A}$. By Lemma 2.1, we have $\phi(\lambda I) \in \mathbb{R}I$. Hence $\phi(\mathbb{R}I) \subseteq \mathbb{R}I$. Let $A = A^* \in \mathcal{A}$. Since $\phi(I) \in \mathbb{R}I$, we have that

$$\begin{aligned} 0 &= \phi({}_*[I,\ {}_*[I,A]]) \\ &= {}_*[\phi(I),\ {}_*[I,A]] + {}_*[I,\ {}_*[\phi(I),A]] + {}_*[I,\ {}_*[I,\phi(A)]] \\ &= {}_*[I,\ {}_*[I,\phi(A)]] \\ &= 2\phi(A) - 2\Phi(A)^*. \end{aligned}$$

Hence $\phi(A) = \phi(A)^*$. For any $\lambda \in \mathbb{C}$ and $A \in \mathcal{A}$ with $A = A^* \in \mathcal{A}$, applying above results, we see that

$$\begin{aligned}
0 &= \phi({}_*[C,\ {}_*[\lambda I, A]]) \\
&= {}_*[\phi(C),\ {}_*[\lambda I, A]] + {}_*[C,\ {}_*[\phi(\lambda I), A]] + {}_*[C,\ {}_*[\lambda I, \phi(A)]] \\
&= {}_*[C,\ {}_*[\phi(\lambda I), A]]
\end{aligned}$$

holds true for all $C \in \mathcal{A}$. It follows from Lemma 2.1 that ${}_*[\phi(\lambda I), A] \in \mathbb{R}I$. This yields that $[\phi(\lambda I), A] \in \mathbb{R}I$, for all $A \in \mathcal{A}$ with $A = A^*$. By the Kleinecke-Shirokov theorem (cf. [2, Problem 230]) , we get $[\phi(\lambda I), A] = 0$, that is, $\phi(\lambda I)A = A\phi(\lambda I)$, for all $A \in \mathcal{A}$ with $A = A^*$. It follows that $\phi(\lambda I)A = A\phi(\lambda I)$ for any $A \in \mathcal{A}$, and so $\phi(\lambda I) \in \mathbb{C}I$. Therefore, $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$. □

Lemma 2.4. *$\phi(\frac{1}{2}I) = \phi(\frac{1}{2}iI) = 0$ and $\phi(iA) = i\phi(A)$, for all $A \in \mathcal{A}$, where i is the imaginary unit.*

Proof. We compute

$$\begin{aligned}
0 &= \phi({}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}iI, -\frac{1}{2}iI]]) \\
&= {}_*[\phi(-\frac{1}{2}I),\ {}_*[-\frac{1}{2}iI, -\frac{1}{2}iI]] + {}_*[-\frac{1}{2}I,\ {}_*[\phi(-\frac{1}{2}iI), -\frac{1}{2}iI]] \\
&+ {}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}iI, \phi(-\frac{1}{2}iI)]] \\
&= {}_*[\phi(-\frac{1}{2}I), -\frac{1}{2}I] + {}_*[-\frac{1}{2}I, -i\phi(-\frac{1}{2}iI)] + {}_*[-\frac{1}{2}I, -\frac{1}{2}i(\phi(-\frac{1}{2}iI) - \phi(-\frac{1}{2}iI)^*)] \\
&= i\phi(-\frac{1}{2}iI) - i\phi(-\frac{1}{2}iI)^*.
\end{aligned}$$

It follows that $\phi(-\frac{1}{2}iI) = -\phi(-\frac{1}{2}iI)^*$. Similarly, by the equality $0 = {}_*[\frac{1}{2}I,\ {}_*[\frac{1}{2}iI, \frac{1}{2}iI]]$, we can get $\phi(\frac{1}{2}iI) = -\phi(\frac{1}{2}iI)^*$. We may also compute

$$\begin{aligned}
\phi(-\frac{1}{2}iI) &= \phi({}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\
&= {}_*[\phi(-\frac{1}{2}I), -\frac{1}{2}iI] + {}_*[-\frac{1}{2}I, -i\phi(\frac{1}{2}iI)] + {}_*[-\frac{1}{2}I, -\phi(-\frac{1}{2}iI)] \\
&= 2i\phi(-\frac{1}{2}I) + \phi(-\frac{1}{2}iI).
\end{aligned}$$

It follows that $\phi(-\frac{1}{2}I) = 0$. The equality $-\frac{1}{2}I = {}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$ implies

$$\begin{aligned}
0 = \phi(-\frac{1}{2}I) &= \phi({}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\
&= {}_*[\phi(\frac{1}{2}iI),\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]] + 0 + {}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \phi(-\frac{1}{2}iI)]] \\
&= i\phi(\frac{1}{2}iI) - i\phi(-\frac{1}{2}iI).
\end{aligned}$$

Hence

$$\phi(\frac{1}{2}iI) = \phi(-\frac{1}{2}iI). \tag{1}$$

Since the equality $\frac{1}{2}iI = {}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$ hold true, we have

$$\begin{aligned}
\phi(\frac{1}{2}iI) &= \phi({}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\
&= {}_*[\phi(\frac{1}{2}I),\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]] + 0 + {}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, \phi(-\frac{1}{2}iI)]] \\
&= {}_*[\phi(\frac{1}{2}I), \frac{1}{2}iI] + {}_*[\frac{1}{2}I, -\phi(-\frac{1}{2}iI)] \\
&= i\phi(\frac{1}{2}I) - \phi(-\frac{1}{2}iI).
\end{aligned}$$

It follows that

$$\phi(\frac{1}{2}iI) + \phi(-\frac{1}{2}iI) = i\phi(\frac{1}{2}I). \tag{2}$$

Finally, by the equality $\frac{1}{2}I = {}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$, we can get

$$\begin{aligned}
\phi(\frac{1}{2}I) &= \phi({}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \frac{1}{2}iI]]) \\
&= {}_*[\phi(-\frac{1}{2}iI),\ {}_*[-\frac{1}{2}I, \frac{1}{2}iI]] + 0 + {}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \phi(\frac{1}{2}iI)]] \\
&= {}_*[\phi(-\frac{1}{2}iI), \frac{1}{2}I] + {}_*[-\frac{1}{2}iI, -\phi(-\frac{1}{2}iI)] \\
&= i\phi(-\frac{1}{2}iI) + i\phi(-\frac{1}{2}iI) = 2i\phi(-\frac{1}{2}iI).
\end{aligned}$$

It follows that

$$2\phi(-\frac{1}{2}iI) = -i\phi(\frac{1}{2}I). \tag{3}$$

Hence by Eq. (1), Eq. (2) and Eq. (3), we have $\phi(\frac{1}{2}iI) = \phi(-\frac{1}{2}iI) = 0$. For every $A \in \mathcal{A}$, it follows from $iA = {}_*[A,\ {}_*[\frac{1}{2}I, \frac{1}{2}iI]]$ that

$$\phi(iA) = \phi({}_*[A,\ {}_*[\tfrac{1}{2}I, \tfrac{1}{2}iI,]]) = {}_*[\phi(A),\ {}_*[\tfrac{1}{2}I, \tfrac{1}{2}iI]] = i\phi(A). \qquad \square$$

We now choose a nontrivial projection $P_1 \in \mathcal{A}$ and let $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j, i, j = 1, 2,$. Then we have the Peirce decomposition of $\mathcal{A}$ as $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. Note that any operator $A \in \mathcal{A}$ can be expressed as $A = A_{11} + A_{12} + A_{21} + A_{22}$, and $A_{ij}^* \in \mathcal{A}_{ji}$ for any $A_{ij} \in \mathcal{A}_{ij}$.

Lemma 2.5. *For any $A \in \mathcal{A}$,*
(1) ${}_*[A,\ {}_*[I, i(P_2 - P_1)]] = 0$ *implies* $A_{11} = A_{22} = 0$,
(2) ${}_*[I,\ {}_*[P_1, A]] = 0$ *implies* $A_{12} = 0$,
(3) ${}_*[I,\ {}_*[P_2, A]] = 0$ *implies* $A_{21} = 0$,
(4) ${}_*[A,\ {}_*[I, iP_1]] = 0$ *implies* $A_{11} = A_{12} = A_{21} = 0$,
(5) ${}_*[A,\ {}_*[I, iP_2]] = 0$ *implies* $A_{22} = A_{12} = A_{21} = 0$.

Proof. We only show (1). The proofs of (2), (3), (4) and (5) go similarly. We compute

$$\begin{aligned} 0 &= {}_*[A, {}_*[I, i(P_2 - P_1)]] = {}_*[A, 2i(P_2 - P_1)] \\ &= 2i(A(P_2 - P_1) + (P_2 - P_1)A) \\ &= 4i(A_{22} - A_{11}), \end{aligned}$$

which leads to $A_{22} = A_{11} = 0$. □

Lemma 2.6. *For any $A_{12} \in \mathcal{A}_{12}$ and $B_{21} \in \mathcal{A}_{21}$, we have*

$$\phi(A_{12} + B_{21}) = \phi(A_{12}) + \Phi(B_{21}).$$

Proof. Let $M = \phi(A_{12} + B_{21}) - \phi(A_{12}) - \phi(B_{21})$. We now show that $M = 0$. On one hand, since ${}_*[A_{12}, {}_*[I, i(P_2 - P_1)]] = {}_*[B_{21}, {}_*[I, i(P_2 - P_1)]] = 0$, we have

$$\begin{aligned} 0 &= \phi({}_*[A_{12} + B_{21}, {}_*[I, i(P_2 - P_1)]]) \\ &= {}_*[\phi(A_{12} + B_{21}), {}_*[I, i(P_2 - P_1]] + {}_*[A_{12} + B_{21}, {}_*[\phi(I), i(P_2 - P_1)]] \\ &\quad + {}_*[A_{12} + B_{21}, {}_*[I, \phi(i(P_2 - P_1))]]. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \phi({}_*[A_{12}, {}_*[I, i(P_2 - P_1)]]) + \phi({}_*[B_{21}, {}_*[I, i(P_2 - P_1)]]) \\ &= {}_*[\phi(A_{12}) + \phi(B_{21}), {}_*[I, i(i(P_2 - P_1))]] + {}_*[A_{12} + B_{21}, {}_*[\phi(I), i(P_2 - P_1)]] \\ &\quad + {}_*[A_{12} + B_{21}, {}_*[I, \phi(i(P_2 - P_1))]]. \end{aligned}$$

Comparing the above two equalities, we arrive at ${}_*[M, {}_*[I, i(P_2 - P_1)]] = 0$. It follows from Lemma 2.5 (1), that $M_{11} = M_{22} = 0$.

Since ${}_*[I, {}_*[P_1, B_{21}]] = 0$, we have that

$$\begin{aligned} &{}_*[\phi(I), {}_*[P_1, A_{12} + B_{21}]] + {}_*[I, {}_*[\phi(P_1), A_{12} + B_{21}]] + {}_*[I, {}_*[P_1, \phi(A_{12} + B_{21})]] \\ &= \phi({}_*[I, {}_*[P_1, A_{12} + B_{21}]]) \\ &= \phi({}_*[I, {}_*[P_1, A_{12}]]) + \phi({}_*[I, {}_*[P_1, B_{21}]]) \\ &= {}_*[\phi(I), {}_*[P_1, A_{12} + B_{21}]] + {}_*[I, {}_*[\phi(P_1), A_{12} + B_{21}]] + {}_*[I, {}_*[P_1, \phi(A_{12} + B_{21})]]. \end{aligned}$$

Hence ${}_*[I, {}_*[P_1, M]] = 0$. By Lemma 2.5 (2), we get that $M_{12} = 0$. Similarly, by using the fact ${}_*[I, {}_*[P_2, A_{12}]] = 0$, one can show $M_{21} = 0$. □

Lemma 2.7. *For any $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$,*

(1) $\phi(A_{11} + B_{12} + C_{21}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21})$.

(2) $\phi(B_{12} + C_{21} + D_{22}) = \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22})$.

Proof. (1) Since ${}_*[A_{11},\ {}_*[I, iP_2]] = 0$, by Lemma 2.6, we obtain

$$
\begin{aligned}
&{}_*[\phi(A_{11} + B_{12} + C_{21}),\ {}_*[I, iP_2]] + {}_*[A_{11} + B_{12} + C_{21},\ {}_*[\phi(I), iP_2]] \\
&+ {}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, \phi(iP_2)]] \\
&= \phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, iP_2)]]) \\
&= \phi({}_*[A_{11},\ {}_*[I, iP_2]]) + \phi({}_*[B_{12} + C_{21},\ {}_*[I, P_2]]) \\
&= {}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}),\ {}_*[I, iP_2]] + {}_*[A_{11} + B_{12} + C_{21},\ {}_*[\phi(I), iP_2]] \\
&+ {}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i\phi(P_2)]].
\end{aligned}
$$

Letting $M = \phi(A_{11}+B_{12}+C_{21})-\phi(A_{11})-\phi(B_{12})-\phi(C_{21})$, we get ${}_*[M,\ {}_*[I, iP_2]] = 0$. It follows from Lemma 2.5 (5) that $M_{12} = M_{21} = M_{22} = 0$.

We now show that $M_{11} = 0$. By noting ${}_*[B_{12}, [I,\ {}_*i(P_2-P_1)]] = {}_*[C_{21},\ {}_*[I, i(P_2-P_1)]] = 0$, we have

$$
\begin{aligned}
&\phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i(P_2 - P_1)]]) \\
&= \phi({}_*[A_{11},\ {}_*[I, i(P_2 - P_1), I]]) + \phi({}_*[B_{12},\ {}_*[I, i(P_2 - P_1)]]) \\
&+ \phi({}_*[C_{21},\ {}_*[I, i(P_2 - P_1)]]).
\end{aligned}
$$

By using the similar argument, we can get ${}_*[M,\ {}_*[I, i(P_2-P_1)]] = 0$. Therefore, $M_{11} = 0$ by Lemma 2.5 (3).

(2) Considering ${}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}),\ {}_*[I, iP_1]]$ and $\phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i(P_2 - P_1)]])$, with the same argument as in (1), one can get $\phi(B_{12} + C_{21} + D_{22}) = \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22})$. □

Lemma 2.8. *For any $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$,*

$$
\phi(A_{11} + B_{12} + C_{21} + D_{22}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}).
$$

Proof. Let $M = \phi(A_{11} + B_{12} + C_{21} + D_{22}) - \phi(A_{11} - \phi(B_{12}) - \phi(C_{21}) - \phi(D_{22})$. Noticing that ${}_*[D_{22},\ {}_*[I, iP_1]] = 0$ and applying (1) in Lemma 2.7, we have

$$
\begin{aligned}
&{}_*[\phi(A_{11} + B_{12} + C_{21} + D_{22}),\ {}_*[I, iP_1]] + {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[\phi(I), iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, \phi(iP_1)]] \\
&= \phi({}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, iP_1]]) \\
&= \phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, iP_1]]) + \phi({}_*[D_{22},\ {}_*[I, iP_1]]) \\
&= {}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}),\ {}_*[I, iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[\phi(I), iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, \phi(iP_1)]].
\end{aligned}
$$

It follows that ${}_*[M,\ {}_*[I, iP_1]] = 0$, so $M_{11} = M_{12} = M_{21} = 0$ by Lemma 2.5. Using the fact that ${}_*[A_{11},\ {}_*[I, iP_2]] = 0$ and the similar argument above, we can get ${}_*[M,\ {}_*[I, iP_2]] = 0$ which leads $M_{22} = 0$, completing the proof. □

Lemma 2.9. *For any $A_{jk}, B_{jk} \in \mathcal{A}_{jk}$, where $1 \leq j \neq k \leq 2$, we have*

$$\phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk}).$$

Proof. On one hand, by Lemma 2.7,

$$\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^* B_{jk}) = \phi(iA_{jk} + iB_{jk}) + \phi(i(A_{jk}^*) + \phi(i(A_{jk}^* B_{jk}).$$

On the other hand, since

$${}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] = i(A_{jk} + B_{jk}) + i(A_{jk}^*) + i(A_{jk}^* B_{jk}),$$

using Lemma2.8 again,

$$\begin{aligned}
&\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^* B_{jk}) \\
&= \phi({}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \frac{i}{2}I]]) \\
&= {}_*[\phi(P_j + B_{jk}),\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] + {}_*[P_j + B_{jk},\ {}_*[\phi(P_k + A_{jk}), \frac{i}{2}I]] \\
&+ {}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \phi(\frac{i}{2}I)]] \\
&= {}_*[\phi(P_j) + \phi(B_{jk}),\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] + {}_*[P_j + B_{jk},\ {}_*[\phi(P_k) + \phi(A_{jk}), \frac{i}{2}I]] \\
&+ {}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \phi(\frac{i}{2}I)]] \\
&= \phi({}_*[P_j,\ {}_*[P_k, \frac{i}{2}I]]) + \phi({}_*[B_{jk},\ {}_*[P_j, \frac{i}{2}I]]) + \phi({}_*[P_j,\ {}_*[A_{jk}, \frac{i}{2}I]]) \\
&+ \phi({}_*[B_{jk},\ {}_*[A_{jk}, \frac{i}{2}I]]) \\
&= \phi(iB_{jk}) + \phi(iA_{jk} + iA_{jk}^*) + \phi(iA_{jk}^* B_{jk}) \\
&= \phi(iB_{jk}) + \phi(iA_{jk}) + \phi(iA_{jk}^*) + \phi(iA_{jk}^* B_{jk}).
\end{aligned}$$

Note that in the last identity above, we are using Lemma 2.6. We now can conclude that $\phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk})$ by Lemma 2.4. $\square$

Lemma 2.10. *For any $A_{jj}, B_{jj} \in \mathcal{A}_{jj}$, where $1 \leq j \leq 2$, we have*

$$\phi(A_{jj} + B_{jj}) = \phi(A_{jj}) + \phi(B_{jj}).$$

Proof. Let $k \in \{1, 2\}$, with $k \neq j$. We compute

$$\begin{aligned}
&{}_*[\phi(A_{jj} + B_{jj}),\ {}_*[I, iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[\phi(I), iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[I, \phi(iP_k)]] \\
&= \phi({}_*[A_{jj} + B_{jj},\ {}_*[I, iP_k]]) = 0 \\
&= \phi({}_*[A_{jj},\ {}_*[I, \phi(iP_k)]]) + \phi({}_*[B_{jj}, [I,\ {}_*\phi(iP_k)]]) \\
&= {}_*[\phi(A_{jj}) + \phi(B_{jj}),\ {}_*[I, iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[\phi(I), iP_k]] \\
&+ {}_*[A_{jj} + B_{jj},\ {}_*[I, \phi(iP_k)]].
\end{aligned}$$

Write $M = \phi(A_{jj} + B_{jj}) - \phi(A_{jj}) - \phi(B_{jj})$. The above computation yields that ${}_*[M,\ {}_*[I, iP_k]] = 0$. By Lemma 2.4, we have $M_{jk} = M_{kj} = M_{kk} = 0$. We now show that $M_{jj} = 0$. For any $C_{jk} \in \mathcal{A}_{jk}$, by Lemma 2.7,

$$\begin{aligned}
&{}_*[\phi(C_{jk}),\ {}_*[A_{jj} + B_{jj}, \frac{1}{2}iP_j)]] + {}_*[C_{jk},\ {}_*[\phi(A_{jj} + B_{jj}), \frac{1}{2}iP_j]] \\
&+ {}_*[C_{jk},\ {}_*[A_{jj} + B_{jj}, \phi(\frac{1}{2}iP_j)]] = \phi({}_*[C_{jk},\ {}_*[A_{jj} + B_{jj}, \frac{1}{2}iP_j]]) \\
&= \phi({}_*[C_{jk},\ {}_*[A_{jj}, \frac{1}{2}iP_j]]) + \phi({}_*[C_{jk},\ {}_*[B_{jj}, \frac{1}{2}iP_j]]) \\
&= {}_*[\phi(C_{jk}),\ {}_*[A_{jj} + B_{jj}, \frac{1}{2}iP_j)]] + {}_*[C_{jk},\ {}_*[\phi(A_{jj} + B_{jj}), \frac{1}{2}iP_j]] \\
&+ {}_*[C_{jk},\ {}_*[A_{jj} + B_{jj}, \phi(\frac{1}{2}iP_j)]].
\end{aligned}$$

Therefore, ${}_*[C_{jk},\ {}_*[M, \frac{1}{2}iP_j]] = 0$ which leads to $M_{jj}^* C_{jk} = 0$, for all $C_{jk} \in \mathcal{A}_{jk}$. Since $\mathcal{A}$ is prime, we see that $M_{jj} = 0$. □

Lemma 2.11. *ϕ is an additive derivation with $\phi(A^*) = \phi(A)^*$, for all $A \in \mathcal{A}$.*

Proof. We first show that ϕ is additive. For arbitrary $A, B \in \mathcal{A}$, we write $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$. By Lemma 2.8, Lemma 2.9 and Lemma 2.10, we obtain

$$\begin{aligned}
\phi(A + B) &= \phi(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}) = \sum_{i,j=1}^{2} \Phi(A_{ij} + B_{ij}) \\
&= \sum_{i,j=1}^{2} \phi(A_{ij}) + \sum_{i,j=1}^{2} \phi(B_{ij}) = \phi(\sum_{i,j=1}^{2} A_{ij}) + \phi(\sum_{i,j=1}^{2} B_{ij}) \\
&= \phi(A) + \phi(B).
\end{aligned}$$

We now show $\phi(A^*) = \phi(A)^*$. For every $A \in \mathcal{A}$, we write $A = A_1 + iA_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2i}$ are self-adjoint elements. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\phi(A^*) &= \phi(A_1 - iA_2) = \phi(A_1) - \phi(iA_2) \\
&= \phi(A_1) - i\phi(A_2) = \phi(A_1)^* - i\phi(A_2)^* \\
&= \phi(A_1)^* + (i\phi(A_2))^* = \phi(A_1 + iA_2)^* = \phi(A)^*.
\end{aligned}$$

To complete the proof, we need to show that ϕ is a derivation. By the additivity of ϕ and Lemma 2.5, we have $\phi(iI) = 2\phi(\frac{1}{2}iI) = 0$. Note that ${}_*[A,\ {}_*[B, iI]] = 2i(AB + B^*A)$. We compute

$$\begin{aligned}
2i\phi(AB + B^*A) &= \phi(2i(AB + B^*A)) \\
&= \phi({}_*[A,\ {}_*[B, iI]]) \\
&= {}_*[\phi(A),\ {}_*[B, iI]] + {}_*[A, {}_*[\phi(B), iI]] + {}_*[A,\ {}_*[B, \phi(iI)]] \\
&= 2i(\phi(A)B + B^*\phi(A)) + A\phi(B) + \phi(B)^*A).
\end{aligned}$$

It follows that

$$\phi(AB + B^*A) = \phi(A)B + B^*\phi(A) + A\phi(B) + \phi(B)^*A.$$

Replacing B by iB in the above equality, we get

$$\phi(AB - B^*A) = \phi(A)B - B^*\phi(A) + A\phi(B) - \phi(B)^*A.$$

Thus $\phi(AB) = \phi(A)B + A\phi(B)$, it is a derivation. □

The proof of the main theorem. By Lemma 2.11, we see that ϕ is an additive derivation with $\phi(A^*) = \phi(A)^*$. It follows from [12, Theorem 2.3] that ϕ is an linear inner derivation, that is, there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = AS - SA$, for all $A \in \mathcal{A}$. Since $\phi(A^*) = \phi(A)^*$, we have

$$A^*S - SA^* = \phi(A^*) = \phi(A)^* = S^*A^* - A^*S^*$$

for any $A \in \mathcal{A}$. This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Letting $T = S - \frac{1}{2}\lambda I$, one can check that $T + T^* = 0$ and $\phi(A) = AT - TA$, for all $A \in \mathcal{A}$.

Corollary 2.1. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is nonlinear left ∗-Lie triple mapping, then ϕ is an inner ∗-derivation, that is, there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\phi(A) = AT - TA$, for all $A \in \mathcal{A}$.*

Acknowledgments

The authors wish to thank anonymous reviewers for their constructive and valuable suggestions which have considerably improved the presentation of the paper. This work was supported by the National Natural Science Foundation of China (No.11471199, No.11601010) and the Postdoctoral Science Foundation of China (No. 2018M633450). The first author is supported by Foundation of Educational Commission (No. KY[2017]092) and of Science and Technology department (No. [2018]1001) of Guizhou Province of China.

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Accepted: 9.10.2018

A common fixed point theorem without continuity under weak compatible mappings in uniform convex Banach spaces

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Abstract. In this paper, we prove a common fixed point theorem for four mappings under the condition of weak compatibility on a closed subset of a uniformly convex Banach space without taking under consideration the continuity of mappings. We provide an example in support of our result.
Keywords: common fixed point, uniform convex Banach space, weakly compatible mappings, closed subset.

1. Introduction

Imdad et al. [3] obtained some results on common fixed points for three mappings defined on a closed subset of a uniformly convex Banach space. Their results extended and refined some results of Husain and Sehgal [2] and Khan and Imdad [10]. Rashwan [11] extended results of Imdad et al. [3] by employing four compatible mappings of type (A) instead of three weakly commuting mappings and by using one continuous mapping as opposed to two. In this paper, we improve the result of Rashwan[11] by removing the condition of continuity and using weak compatible mappings. For preliminaries and definitions we refer to ([1], [4], [5], [6],[7],[8],[9],[12]).

2. Preliminaries

Throughout the paper, X stands for a uniformly convex Banach space. Let R^+ denote the set of all non negative real numbers and F be the family of mappings f from $(R^+)^5$ into R^+ such that f is upper semicontinuous, non-decreasing in each coordinate variable. The modulus of convexity of X is a function δ from $(0, 2]$ into $(0, 1]$ defined by

$$\delta(\epsilon) = inf\{1 - \frac{1}{2}\|x - y\|, x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}.$$

Moreover, if X is uniformly convex, then δ is strictly increasing, $\delta(\epsilon) \to 0$ as $\epsilon \to 0, \delta(2) = 1, \eta(t) < 2$ when $t < 1$ and η is the inverse of δ.

For our main theorem we need the following lemma:

Lemma 2.1 ([1])**.** *Let X be uniformly convex Banach space and B_r, the closed ball X centred at the origin with radius $r > 0$. If $x_1, x_2, x_3 \in B_r$ satisfy*

$$\|x_1 - x_2\| \geq \|x_2 - x_3\| \geq d > 0 \textit{ and } \textit{ if } \|x_2\| \geq \left(1 - \frac{1}{2}\delta\left(\frac{d}{l}\right)\right)l,$$

then

$$\|x_1 - x_3\| \leq \eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{l}\right)\right)\|x_1 - x_2\|.$$

Now, we shall give some definitions;

Definition 2.1 ([12])**.** Let S and T be self commuting on X. Then $\{S, T\}$ is called a weakly commuting pair on X if $\|STx - TSx\| \geq \|Sx - Tx\|$ for all $x \in X$.

Definition 2.2 ([4])**.** Let $S, T : X \to X$ be mappings. S and T are said to be compatible if $\lim_{n\to\infty} \|STx_n - TSx_n\| = 0$, whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, examples are given in [4], [5], [6], and [12] to show neither of the above implications are reversible.

Definition 2.3 ([8])**.** A pair of mappings S and T is called weakly compatible pair in fuzzy metric space if they commute at coincidence points; i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$. It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.

Definition 2.4 ([9])**.** Let $S, T : X \to X$ be mappings. S and T are said to be compatible of type (A) if

$$\lim_{n\to\infty} \|STx_n - SSx_n\| = 0, \lim_{n\to\infty} \|STx_n - TTx_n\| = 0.$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Imdad et al. [3] proved the following:

Theorem 2.1. *Let X be uniformly convex Banach space and K a non empty closed subset of X. Let A, S and T be three self mappings of K satisfying the following conditions:*

$$S \textit{ and} T \textit{ are continuous}, \ AK \subset SK \cap TK, \tag{2.1}$$

$$\{A, S\} \textit{ and } \{A, T\} \textit{ are weakly commuting pairs on } K, \tag{2.2}$$

there exists a function $f \in F$ such that for every $x, y \in K$:

$$\|Ax - Ay\| \le f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - Ay\|, \|Ty - Ax\|, \|Ty - Ay\|), \tag{2.3}$$

where f has the additional requirements:

$$\begin{aligned} &for\, t > 0, f(t, t, 0, \alpha t, t) \le \beta t \, and \, f(t, t, \alpha t, 0, t) \le \beta t \\ &being\ \beta < 1\, for\, \alpha < 2 \end{aligned} \tag{2.4}$$

and $\beta = 1\, for\, \alpha = 2, \alpha, \beta \in R^+$,

$$f(t, 0, t, t, 0) < t\, for\, t > 0; \tag{2.5}$$

Then, there exists a point u in K such that:

(i) u is the common fixed point of $A, S\, and\, T$.

(ii) For any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by

$$Tx_{2n} = Ax_{2n-1}, Sx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...,$$

converges strongly to u.

Rashwan [11] proved a following common fixed point theorem for four compatible mappings of type (A) which extends and improves Theorem (2.1).

Theorem 2.2. *Let X and K be as in Theorem* (2.1). *Let A, B, S and T be four self mappings of K satisfying the following conditions:*

$$One\, of\, A, B, S\, and\, T\, are\, continuous\, and\, AK \subseteq TK\, and\, BK \subseteq SK, \tag{2.6}$$

$$\{A, S\}\, and\, \{B, T\}\, are\, compatible\, of\, type\, (A), \tag{2.7}$$

there exists a function $f \in F$ such that for every $x, y \in K$:

$$\|Ax - By\| \le f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - By\|, \|Ty - Ax\|, \|Ty - By\|), \tag{2.8}$$

where f satisfies the condition (2.4) *and* (2.5) *as in Theorem* (2.1).

Then there exists a point u in K such that:

(i) u is the common fixed point of $A, B, S,\, and\, T$;

(ii) For any $x_0 \in K$, the sequence $\{y_n\}$ defined by

$$y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...$$

converges strongly to u.

3. Main results

Theorem 3.1. *Let X be uniformly convex Banach space and K a non empty closed subset of X. Let $A, B, S,$ and T be four mappings of K satisfying the following conditions:*

$$AK \subset TK \, and \, BK \subset SK, \tag{3.1}$$

$$\{A, S\} \, and \, \{B, T\} \, are \, weakly \, compatible, \tag{3.2}$$

there exists a function $f \in F$ such that for every $x, y \in K$:

$$\begin{aligned} &\|Ax - By\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - By\|, \\ &\|Ty - Ax\|, \|Ty - By\|), \end{aligned} \tag{3.3}$$

where f has the additional requirements:

$$\begin{aligned} &for \, t > 0, f(t, t, 0, \alpha t, t) \leq \beta t \, and \, f(t, t, \alpha t, 0, t) \\ &\leq \beta t \ being \ \beta < 1 \, for \, \alpha < 2 \end{aligned} \tag{3.4}$$

$and \, \beta = 1 \, for \, \alpha = 2, \alpha, \beta \in R^+$,

$$f(t, 0, t, t, 0) < t \, for \, t > 0, \tag{3.5}$$

Then there exists a point z in K such that:

(i) z is the common fixed point of $A, B, S,$ and T.

(ii) For any $x_0 \in K$, the sequence $\{y_n\}$ defined by

$$y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...,$$

converges strongly to z.

Proof. Let $x_0 \in K$. Since $AK \subset TK$,$BK \subset SK$ we can always define a sequence $\{y_n\}$ as $y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...$ converges strongly to z..

Let $d_n = \|y_n - y_{n+1}\|, n = 0, 1, 2, ...$ $\lim_{n\to\infty} d_n = 0$. Now, for any even integer n, we have

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|AX_n - BX_{n-1}\| \\ &\leq f\big(\|Sx_n - Tx_{n-1}\|, \|Sx_n - Ax_n\|, \|Sx_n - Bx_{n-1}\|, \\ &\quad \|Tx_{n-1} - Ax_n\|, \|Tx_{n-1} - Bx_{n-1}\|\big), \end{aligned} \tag{3.6}$$

which implies $d_n = f(d_{n-1}, d_n, 0, d_{n-1} + d_n, d_{n-1})$.

Similarly for an odd n, we obtain

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|AX_n - BX_{n-1}\| \\ &\leq f\Big[\|Sx_n - Tx_{n-1}\|, \|Sx_n - Ax_n\|, \|Sx_n - Bx_{n-1}\|, \\ &\quad \|Tx_{n-1} - Ax_n\|, \|Tx_{n-1} - Bx_{n-1}\|\Big], \end{aligned} \tag{3.7}$$

which gives $d_n = f(d_{n-1}, d_{n-1}, 0, d_{n-1} + d_n, d_n)$. If $d_n > d_{n-1}$ for some $n \geq 1$, then $d_{n-1} + d_n = \alpha d_n$ with $\alpha < 2, \alpha \in R$.

Since f is non decreasing in each coordinate variable.

$$d_n = \begin{cases} f(d_n, d_n, 0, \alpha d_n, d_n), \text{ if } n \text{ is even,} \\ f(d_n, d_n, \alpha d_n, 0, d_n), \text{ if } n \text{ is odd.} \end{cases}$$

In both cases , by (3.4), we get $d_n \leq \beta d_n < d_n$, for some $\beta < 1, \beta \in R^+$, a contradiction. Thus $d_n \leq d_{n-1}$ for $n = 1, 2, 3, ...$ Suppose $d > 0$. Without loss of generality, we can pastulate that the origin of $X \in K$,

$$\lim_{n \to \infty} \sup \|y_n\| = l' > 0.$$

Let $l \in R^+$ be chosen in such a way that $l' < 1$ and $\eta[1 - (\frac{1}{2})\delta(\frac{d}{l})] < l'$, then there exists a sequence $\{n(k)\}, k = 0, 1, 2, ..., n(0) > 1$ of positive integers such that $\|y_{n(k)}\| \geq [(1 - \frac{1}{2})\delta(\frac{d}{l}))]$, where as it is $\|y_n\| \leq l$ for any $n \geq n(0)$. Since $d_{n(k)-1} \geq d_{n(k)} \geq d > 0, k = 0, 1, 2, ...$. From Lemma (2.1) it follows that

$$\|y_{n(k)-1} - y_{n(k)+1}\| \leq \eta(\frac{l'}{l}) d_{n(k)-1}. \tag{3.8}$$

where $\eta(\frac{l'}{l}) < 2$ being $(\frac{l'}{l}) < 1$. Then by (3.6),(3.7) and (3.8), we have

$$d_{n(k)} = \begin{cases} f(d_{n(k)-1}, d_{n(k)-1}, 0, \eta(\frac{l'}{l}) d_{n(k)-1}, d_{n(k)-1}), \text{ if } n \text{ is even,} \\ f(d_{n(k)-1}, d_{n(k)-1}, 0, \eta(\frac{l'}{l}) d_{n(k)-1}, d_{n(k)-1}), \text{ if } n \text{ is odd,} \end{cases}$$

In both cases, (3.4) implies $d_{n(k)} \leq \beta d_{n(k)-1}$, for some $\beta < 1$. Observing that β does not depend on K, the foregoing inequality gives as $n \to \infty$ that $d \leq \beta d < d$, a contradiction. This means that $d = 0$. Now, we'll prove that $\{y_n\}$ is a cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$, it is sufficient to show that the sequence $\{y_{2n}\}$ is a cauchy sequence. If not, then there is an $\epsilon > 0$ such that for every even integer 2k, k=0,1,2,..., there exists two sequences $\{2n(k)\}, \{2m(k)\}$ with $2k \leq 2n(k) \leq 2m(k)$ for which

$$\|y_{n(k)} - y_{m(k)}\| > \epsilon, \tag{3.9}$$

for each even integer 2k, let 2m(k) be the least even integer exceeding n(k) and satisfying (3.9) . Then $\|y_{2n(k)} - y_{2m(k)-2}\| \leq \epsilon$ and $\|y_{2n(k)} - y_{2m(k)}\| > \epsilon$, for each k=0,1,2,..., we have

$$\begin{aligned} \epsilon &\leq \|y_{2n(k)} - y_{2m(k)}\| \\ &\leq \epsilon \|y_{2n(k)} - y_{2m(k)-2}\| + \|y_{2m(k-2)} - y_{2m(k)-1}\| + \|y_{2m(k)-1} - y_{2m(k)}\| \\ &\leq \epsilon + d_{2m(k)-2} + d_{2m(k)-1}, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \|y_{2n(k)} - y_{2m(k)}\| = \epsilon \tag{3.10}$$

Further, from triangular inequality , it follows that

$$| \left\|y_{2n(k)} - y_{2m(k)-1}\right\| - \left\|y_{2n(k)} - y_{2m(k)}\right\| | \leq d_{2m(k)-1}$$

and

$$| \left\|y_{2n(k)+1} - y_{2m(k)-1}\right\| - \left\|y_{2n(k)} - y_{2m(k)}\right\| | \leq d_{2m(k)-1} + d_{2n(k)}.$$

Hence for $k \to \infty$, we find by (3.10) that

$$\left\|y_{2n(k)} - y_{2m(k)-1}\right\| \to \epsilon \, and \, \left\|y_{2n(k)} - y_{2m(k)-1}\right\| \to \epsilon. \tag{3.11}$$

On the other hand ,using (3.3) we deduce that

$$\begin{aligned} \left\|y_{2n(k)} - y_{2m(k)}\right\| &\leq d_{2n(k)} + \left\|y_{2n(k)+1} - y_{2m(k)}\right\| \\ &\leq d_{2n(k)} + f\big(\left\|y_{2m(k)-1} - y_{2n(k)}\right\|, d_{2n(k)}, \\ &\left\|y_{2m(k)-1} - y_{2n(k)+1}\right\|, \left\|y_{2n(k)} - y_{2m(k)}\right\|, d_{2n(k)}\big) \end{aligned} \tag{3.12}$$

by (3.10), (3.11), the upper-semicontinuity and non-decreasing properties of f, and condition (3.5), we have from (3.12) for $k \to \infty$, $\epsilon \leq f(\epsilon, 0, \epsilon, \epsilon, 0) \leq \epsilon$, which is a contradiction. Therefore $\{y_{2n}\}$ is a cauchy sequence in K and so is $\{y_n.\}$

But K is a closed subset of Banach space X, therefore $\{y_n\}$ converges to a point z in K. On the other hand , the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to z. Since $BK \subset SK$, there exists a point u in K such that $Su = z$.

By using (3.3) we write $\|Au - Bx_{2n+1}\| \leq f(\|Su - Tx_{2n+1}\|, \|Su - Au\|, \|Su - Bx_{2n+1}\|, \|Tx_{2n+1} - Au\|, \|Tx_{2n+1} - Bx_{2n+1}\|)$.

Taking n tends to ∞ , we get $\|Au - z\| \leq f(\|z - z\|, \|z - Au\|, \|z - z\|, \|z - Au\|, \|z - z\|)$.

This gives $\|Au - z\| \leq f(\|Au - z\|)$, which is a contradiction. Therefore we have $z = Au$. Thus $Au = Su = z$. Since $AK \subset TK$, there exists a point $v \in K$ such thst $Tv = z$. Then using (3.8), we have $\|Ax_{2n} - Bv\| \leq f(\|Sx_{2n} - Tv\|, \|Sx_{2n} - Ax_{2n}\|, \|Sx_{2n} - Bv\|, \|Tv - Ax_{2n}\|, \|Tv - Bv\|)$. Letting n tends to ∞, we get

$$\begin{aligned} &\|z - Bv\| \leq f(\|z - Tv\|, \|z - z\|, \|z - Bv\|, \|Tv - z\|, \|Tv - Bv\|), \\ &\|z - Bv\| \leq f(\|z - z\|, \|z - z\|, \|z - Bv\|, \|z - z\|, \|z - Bv\|). \end{aligned}$$

This yields $\|z - Bv\| \leq f(\|z - Bv\|)$. which is a contradiction. Thus $z = Bv$. Therefore $z = Bv = Tv$. Hence $Au = Su = Bv = Tv = z$. Since A and S are weakly compatible, therefore A and S commute at their coincidence points i.e. $ASu = SAu$ or Az =Sz. Similarly $BTv = TBv$ or $Bz = Tz$.

Now we prove $Az = z$ by using (3.8), we have

$$\begin{aligned} \|Az - Bx_{2n+1}\| \leq f(&\|Sz - Tx_{2n+1}\|, \|Sz - Az\|, \|Sz - Bx_{2n+1}\|, \\ &\|Tx_{2n+1} - Az\|, \|Tx_{2n+1} - Bx_{2n+1}\|). \end{aligned}$$

Letting n tends to ∞, we have $\|Az - z\| \leq f(\|Sz - z\|, \|Sz - Az\|, \|Sz - z\|, \|z - Az\|, \|z - z\|)$, $\|Az - z\| \leq f(\|Az - z\|, \|Az - Az\|, \|Az - z\|, \|z - Az\|, \|z - z\|)$. This yields $\|Az - z\| \leq f(\|Az - z\|)$, which is a contradiction. Thus $Az = z$. Therefore $Az = Sz = z$. Similarly, we can show that $Bz = Tz = z$. This means that z is a fixed point of A, B, S and T.

For uniqueness of common fixed point , let $w \neq z$ be another common fixed point of A, B, S, and T. Then by (3.8), we have $\|Az - Bw\| \leq f(\|Sz - Tw\|, \|Sz - Az\|, \|Sz - Bw\|, \|Tw - Az\|, \|Tw - Bw\|)$, $\|z - w\| \leq f(\|z-w\|, \|z-z\|, \|z - w\|, \|w - z\|, \|w - w\|)$.

This gives $\|z - w\| \leq f(\|z - w\|)$, which is a contradiction. This $z = w$.

This completes the proof of the theorem.

Example 3.1. Let $X = K = [0, 2]$ with the Euclidean norm $\|.\|$. Define $A, B, S, T : K \to K$ by

$$Ax = \begin{cases} 0, & \text{if } x = 0, \\ 0.15, & \text{if } x > 0. \end{cases}$$

$$Bx = \begin{cases} 0, & \text{if } x = 0, \\ 0.35, & \text{if } x > 0. \end{cases}$$

$$Sx = \begin{cases} 0, \text{if } x = 0, \\ 0.35, & \text{if } 0 < x < 0.5, \\ x - 0.35, & \text{if } x \geq 0.5. \end{cases}$$

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ 0.15, & \text{if } 0 < x < 0.5, \\ x - 0.15, & \text{if } x \geq 0.5. \end{cases}$$

We see that A, B, S and T satisfy all the conditions of Theorem (3.1) and have a unique common fixed point of $0 \in X$. It may be noted in this example that the mappings A and S commute at coincidence point $0 \in X$. So A and S are weakly compatible maps. Similarly B and T are weakly compatible maps. To see the pairs $\{A, S\}$ and $\{B, T\}$ are non compatible, let us consider a decreasing sequence $\{x_n\}$ such that $x_n \to 0.5$. Then $\{Ax_n\} \to 0.15$, $\{Sx_n\} \to 0.15$, but $\lim_{n\to\infty} \|ASx_n - SAx_n\| \neq 0$. So the pair $\{A, S\}$ is noncompatible. Also $Bx_n \to 0.35, Tx_n \to 0.35$, but $\lim_{n\to\infty} \|BTx_n - TBx_n\| \neq 0$. So the pair {B,T} is non compatible. All the mappings involved in this example are discontinuous at the common fixed point.

Acknowledgement

The author would like to offer her sincere gratitude to the editor and the anonymous referees for their valuable comments and suggestions.

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Accepted: 8.10.2018

A further study on the hyperideals of ordered semihypergroups

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Abstract. In this paper, we first introduce the concepts of prime, weakly prime and semiprime hyperideals in ordered semihypergroups, and give some characterizations of them. Furthermore, we consider the extensions of hyperideals in commutative ordered semihypergroups. As a generalization of the concept of prime hyperideals of ordered semihypergroups, the concept of n-prime hyperideals of ordered semihypergroups is introduced, and related properties are discussed. In particular, we prove that every $(n-1)$-prime hyperideal of ordered semihypergroups is n-prime for any positive integer $n \geq 3$. Moreover, we investigate the relationship between n-prime hyperideals and extensions of hyperideals, and prove that a hyperideal I of a commutative ordered semihypergroup is n-prime if and only if any extension of I is $(n-1)$-prime ($n \geq 3$). Finally, we prove that if I is a semiprime hyperideal of a commutative ordered semihypergroup S, then I is the intersection of all extensions of I. Especially, if I is also n-prime ($n \geq 3$), then I can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it.

Keywords: ordered semihypergroup, prime hyperideal, semiprime hyperideal, n-prime hyperideal, extension of hyperideal.

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1. Introduction

The algebraic hyperstructure is a natural generalization of the classical algebraic structures which was first introduced by Marty [18] in 1934. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. After the pioneering work of Marty, algebraic hyperstructures have been intensively studied, both from the theoretical point of view and especially for their applications in other fields such as Euclidean and non-Euclidean geometries, graphs and hypergraphs, fuzzy sets, automata, cryptography, artificial intelligence, codes, probabilities, lattices and so on (see [4]). Recently, algebraic hyperstructures have been developed by many researchers. A lot of papers and several books have been written on algebraic hyperstructure theory, see [7, 8, 10, 14, 19, 20]. There are some books on the general theory of algebraic hyperstructures: one by Corsini [3] on the basic theory of hypergroups, another by Vougiouklis [27], mostly on representations of hypergroups and on H_v-structures, which are hyperstructures satisfying conditions weaker than the classic ones.

Semihypergroups have been found useful for dealing with problems in different areas of algebraic hyperstructures. Many authors studied different aspects of semihypergroups, for instance, Anvariyeh et al. [1], Davvaz [5], Davvaz and Poursalavati [9], Hasankhani [12], Hila and Abdullah [15] and Leoreanu [17], also see [11, 21, 31]. It is now natural to investigate the existing subsystems of other algebraic hyperstructures. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics and error-correcting codes. There are several results which have been added to the theory of ordered semigroups by Kehayopulu, Davvaz, Satyanarayana, Xie, and many other researchers. For more details, the reader is referred to [16, 23, 28, 29]. A theory of hyperstructures on ordered semigroups can be developed. In [13], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. In particular, they defined and studied the hyperideals of an ordered semihypergroup. Also see [2, 6, 24]. In [30], Yaqoob et al. also defined the partially ordered left almost semihypergroups, and studied related properties.

It is well known that hyperideals of a semihypergroup with special properties always play an important role in the study of semihypergroups structure. Motivated by the study of hyperideals in hyperrings and semihypergroups, and also motivated by Davvaz's works in ordered hyperstructures, we attempt in the present paper to study hyperideals of ordered semihypergroups in detail. The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we introduce the concepts of prime, weakly prime and semiprime hyperideals in ordered semihypergroups, and give some characterizations of them. In Section 4, we consider the extensions of hyperideals in commutative ordered semihypergroups. In addition, we define n-prime hyperideals and n-semiprime hyperideals of ordered semihypergroups, and investigate their related properties. In particular, we show that for any positive integer $n \geq 2$, n-prime hyperideals of an ordered semihypergroup are a generalization of prime hyperideals. We also prove that every $(n-1)$-prime hyperideal of ordered semihypergroups is n-prime for any positive integer $n \geq 3$. Moreover, we investigate the relationship between extensions of hyperideals and n-prime hyperideals, and prove that a hyperideal I of a commutative ordered semihypergroup is n-prime if and only if any extension of I is $(n-1)$-prime $(n \geq 3)$. Especially, we prove that a semiprime, n-prime hyperideal $(n \geq 3)$ of a commutative ordered semihypergroup S can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it. As an application of the results of this paper, the corresponding results in ordinary semihypergroups can be also obtained by moderate modification.

2. Preliminaries and some notations

Recall that a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all the nonempty subsets of S. The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Also $A \circ x$ is used for $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [3]). A semihypergroup $(S, \circ)$ is called a *hypersemilattice* if $x \in x \circ x$ and $x \circ y = y \circ x$ for all $x, y \in S$ (see [22]).

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. In the following, we shall extend the concept of ordered semigroups to the hyper version, and introduce the concept of ordered semihypergroups from [13].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [13]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Definition 2.2. An element e in an ordered semihypergroup $(S, \circ, \leq)$ is called *identity* if $a \in a \circ e \cap e \circ a$ for any $a \in S$.

Definition 2.3. Let $(S, \circ, \leq)$ be ordered semihypergroup. Then S is called *commutative* if $a \circ b = b \circ a$ for any $a, b \in S$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. In the following we give two examples of ordered semihypergroups.

Example 2.4. Let $(S, \leq)$ be a partially ordered set. If for every $x, y \in S$, we define $x \circ y = \{x, y\}$, then $(S, \circ, \leq)$ is a commutative ordered semihypergroup.

Example 2.5 ([13]). Let $(S, \cdot, \leq)$ be an ordered semigroup. If for every $x, y \in S$, we define $x \circ y =< x, y >$, where $< x, y >$ is the ideal of S generated by $\{x, y\}$, then $(S, \circ, \leq)$ is an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a (*two-sided*) *hyperideal* of S (see [13]). We denote by $L(A)$ (resp. $R(A), I(A)$) the left (resp. right, two-sided) hyperideal of S generated by A ($\emptyset \neq A \subseteq S$). One can easily prove that $L(A) = (A \cup S \circ A]$, $R(A) = (A \cup A \circ S]$ and $I(A) = (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S]$. In particular, if $A = \{a\}$, then we write $L(a), R(a), I(a)$ instead of $L(\{a\}), R(\{a\}), I(\{a\})$, respectively. If S is commutative, then $I(a) = (a \cup S \circ a] = (a \cup a \circ S]$.

Lemma 2.6. *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) $A \subseteq (A]$, $\forall A \subseteq S$.

(2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*

(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B]$, $\forall A, B \subseteq S$.

(4) $((A]] = (A]$, $\forall A \subseteq S$.

(5) *For every hyperideal T of S, we have $(T] = T$.*

(6) *If A, B are hyperideals of S, then $(A \circ B]$ is a hyperideal of S.*

(7) *For every $a \in S$, $(S \circ a \circ S]$ is a hyperideal of S.*

(8) *If T is a hyperideal of S and A, B are two nonempty subsets of S such that $A \preceq B \subseteq T$, then $A \subseteq T$.*

(9) *For any two nonempty subsets A, B of S such that $A \preceq B$, we have $C \circ A \preceq C \circ B$ and $A \circ C \preceq B \circ C$ for any nonempty subset C of S.*

Proof. Straightforward.

Lemma 2.7. *Let S be an ordered semihypergroup and $\{A_i \mid i \in I\}$ a family of hyperideals of S. Then $\bigcup_{i \in I} A_i$ is a hyperideal of S and $\bigcap_{i \in I} A_i$ is also a hyperideal of S if $\bigcap_{i \in I} A_i \neq \emptyset$.*

Proof. Straightforward.

For the sake of simplicity, throughout this paper, we denote $A^n = A \circ A \circ \cdots \circ A$ (n-copies).

Lemma 2.8. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*
(1) $(A^2] = A$ *for every hyperideal A of S.*
(2) $A \cap B = (A \circ B]$ *for all hyperideals A, B of S.*
(3) $I(a) \cap I(b) = (I(a) \circ I(b)]$ *for any $a, b \in S$.*
(4) $I(a) = ((I(a))^2]$ *for all $a \in S$.*
(5) $a \in (S \circ a \circ S \circ a \circ S]$ *for all $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let A and B be hyperideals of S. Then, by Lemma 2.6, $(A \circ B] \subseteq (A \circ S] \subseteq (A] = A$ and $(A \circ B] \subseteq (S \circ B] \subseteq (B] = B$, from which we can conclude that $(A \circ B] \subseteq A \cap B$. On the other hand, by Lemma 2.7, $A \cap B$ is a hyperideal of S. Then, by (1), we have

$$A \cap B = ((A \cap B)^2] = ((A \cap B) \circ (A \cap B)] \subseteq (A \circ B].$$

Thus $A \cap B = (A \circ B]$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (5). Let $a \in S$. By hypothesis and Lemma 2.6, we have

$$(I(a))^2 = ((I(a))^2] \circ I(a) = ((I(a))^2] \circ (I(a)] \subseteq ((I(a))^3].$$

Then, we have

$$(I(a))^3 = (I(a))^2 \circ I(a) = (I(a))^2 \circ (I(a)] \subseteq ((I(a))^3] \circ (I(a)] \subseteq ((I(a))^4].$$

Further, it can be shown that $(I(a))^4 \subseteq ((I(a))^5]$. Thus

$$\begin{aligned} I(a) &= ((I(a))^2] \subseteq (((I(a))^3]] = ((I(a))^3] \subseteq (((I(a))^4]] = ((I(a))^4] \\ &\subseteq (((I(a))^5]] = ((I(a))^5] \subseteq (S \circ I(a)] \subseteq (I(a)] = I(a), \end{aligned}$$

which implies that $I(a) = ((I(a))^5]$. On the other hand, we have

$$\begin{aligned} (I(a))^3 &= (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]^3 \\ &\subseteq ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)^2] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq ((S \circ a \cup S \circ a \circ S) \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)] \subseteq (S \circ a \circ S]. \end{aligned}$$

Then,

$$\begin{aligned} (I(a))^4 &\subseteq (S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \circ S \circ a \cup S \circ a \circ S \circ a \circ S], \end{aligned}$$

and we have

$$\begin{aligned}(I(a))^5 &\subseteq (S\circ a\circ S\circ a\cup S\circ a\circ S\circ a\circ S]\circ(a\cup S\circ a\cup a\circ S\cup S\circ a\circ S]\\ &\subseteq (S\circ a\circ S\circ a\circ S].\end{aligned}$$

Therefore, $a\in I(a)=((I(a))^5]\subseteq((S\circ a\circ S\circ a\circ S]]=(S\circ a\circ S\circ a\circ S]$.

(5) $\Longrightarrow$ (1). Let A be a hyperideal of S. Then $(A^2]=(A\circ A]\subseteq(A\circ S]\subseteq(A]=A$. Conversely, let $x\in A$. Then, by (5) and Lemma 2.6, we have

$$\begin{aligned}x &\in (S\circ x\circ S\circ x\circ S]\subseteq(S\circ A\circ S\circ A\circ S]\\ &= ((S\circ A)\circ S\circ(A\circ S)]\subseteq(A\circ S\circ A]\subseteq(A\circ A]=(A^2],\end{aligned}$$

which means that $A\subseteq(A^2]$. This completes the proof.

The reader is referred to [4, 28] for notation and terminology not defined in this paper.

3. Prime hyperideals of ordered semihypergroups

In this section we introduce and characterize the prime, weakly prime and semiprime hyperideals in ordered semihypergroups. Some properties of them are investigated.

Definition 3.1. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *prime* if for all nonempty subsets A, B of S such that $A\circ B\subseteq T$, we have $A\subseteq T$ or $B\subseteq T$. Equivalently, if for any element a, b of S such that $a\circ b\subseteq T$, we have $a\in T$ or $b\in T$.

Definition 3.2. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *weakly prime* if for all hyperideals A, B of S such that $A\circ B\subseteq T$, we have $A\subseteq T$ or $B\subseteq T$.

Definition 3.3. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *semiprime* if for any nonempty subset A of S such that $A\circ A\subseteq T$, we have $A\subseteq T$. Equivalently, if for any element a of S such that $a\circ a\subseteq T$, we have $a\in T$.

One can easily observe that the prime subsets of an ordered semihypergroup are weakly prime and semiprime. However, the converse is not true, in general, as shown in the following example.

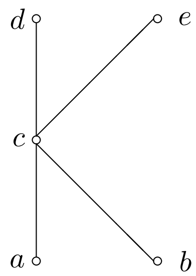
Example 3.4. We consider a set $S:=\{a,b,c,d,e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$$\begin{aligned}\leq := \{&(a,a),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),\\&(c,d),(c,e),(d,d),(e,e)\}.\end{aligned}$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,c),(b,c),(c,d),(c,e)\}.$$



Then $(S,\circ,\leq)$ is an ordered semihypergroup ([26]), and the sets $\{a,b\}$, $\{a,b,c,e\}$ and S are all hyperideals of S. We can easily verify that the nonempty subset $\{a,b,c\}$ of S is weakly prime and semiprime, but it is not prime. In fact, since $e \circ d = \{c\} \subseteq \{a,b,c\}$, but $e \notin \{a,b,c\}$ and $d \notin \{a,b,c\}$.

In the following we shall characterize the prime, weakly prime and semiprime hyperideals of ordered semihypergroups.

Theorem 3.5. *Let S be an ordered semihypergroup and T a hyperideal of S. Then the following statements are equivalent:*

(1) *T is prime.*

(2) *If A is a left hyperideal, B a right hyperideal of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(3) *If $a,b \in S$ such that $L(a) \circ R(b) \subseteq T$, then $a \in T$ or $b \in T$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1). Let $a,b \in S$ be such that $a \circ b \subseteq T$. Then, since T is a hyperideal of S, we have $L(a) \circ R(b) = (a \cup S \circ a] \circ (b \cup b \circ S] \subseteq (a \circ b \cup S \circ a \circ b \cup a \circ b \circ S \cup S \circ a \circ b \circ S] \subseteq (T \cup S \circ T \cup T \circ S \cup S \circ T \circ S] \subseteq (T] = T$. Thus, by hypothesis, $a \in T$ or $b \in T$. Hence T is prime.

Theorem 3.6. *Let S be an ordered semihypergroup and T a hyperideal of S. Then the following statements are equivalent:*

(1) *T is weakly prime.*

(2) *If $a,b \in S$ such that $(a \circ S \circ b] \subseteq T$, then $a \in T$ or $b \in T$.*

(3) *If $a,b \in S$ such that $I(a) \circ I(b) \subseteq T$, then $a \in T$ or $b \in T$.*

(4) *If A,B are two right hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(5) *If A,B are two left hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(6) *If A is a right hyperideal, B a left hyperideal of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

Proof. (1) $\Rightarrow$ (2). Let $a, b \in S$ be such that $(a \circ S \circ b] \subseteq T$. Then, by Lemma 2.6, we have

$$\begin{aligned}(S \circ a \circ S] \circ (S \circ b \circ S] &= (S \circ a \circ S \circ S \circ b \circ S] \subseteq (S \circ (a \circ S \circ b) \circ S] \\ &\subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

Since $(S \circ a \circ S], (S \circ b \circ S]$ are hyperideals of S and T is weakly prime, we have $(S \circ a \circ S] \subseteq T$ or $(S \circ b \circ S] \subseteq T$. Say $(S \circ a \circ S] \subseteq T$, then, by Lemma 2.6 and the proof of Lemma 2.8, we have

$$\begin{aligned}((I(a))^2] \circ I(a) &= ((I(a))^2] \circ (I(a)] \subseteq ((I(a))^3] \\ &\subseteq ((S \circ a \circ S]] = (S \circ a \circ S] \subseteq (T] = T.\end{aligned}$$

Since T is weakly prime and $((I(a))^2]$ is a hyperideal of S, we have $((I(a))^2] \subseteq T$ or $I(a) \subseteq T$. If $I(a) \subseteq T$, then $a \in I(a) \subseteq T$. Let $((I(a))^2] \subseteq T$. Then, by Lemma 2.6(1), $(I(a))^2 \subseteq T$. Since T is weakly prime, we have $I(a) \subseteq T$ and $a \in T$. Similarly, say $(S \circ b \circ S] \subseteq T$, we have $b \in T$.

(2) $\Rightarrow$ (3). Let $a, b \in S$ be such that $I(a) \circ I(b) \subseteq T$. Then, by Lemma 2.6, we have

$$(a \circ S \circ b] \subseteq ((a] \circ (S \circ b]] \subseteq (I(a) \circ I(b)] \subseteq (T] = T.$$

By (2), we have $a \in T$ or $b \in T$.

(3) $\Rightarrow$ (4). Suppose that A, B are right hyperideals of S, $A \circ B \subseteq T$ and $A \not\subseteq T$. Then we prove that $B \subseteq T$. In fact, let $a \in A, a \notin T$ and $b \in B$. Then, we have:

$$I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \subseteq (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S] = (A \cup S \circ A],$$

$$I(b) = (b \cup S \circ b \cup b \circ S \cup S \circ b \circ S] \subseteq (B \cup S \circ B \cup B \circ S \cup S \circ B \circ S] = (B \cup S \circ B].$$

Thus we have

$$\begin{aligned}I(a) \circ I(b) &\subseteq (A \cup S \circ A] \circ (B \cup S \circ B] \subseteq ((A \cup S \circ A) \circ (B \cup S \circ B)] \\ &\subseteq (A \circ B \cup S \circ A \circ B \cup A \circ S \circ B \cup S \circ A \circ S \circ B] \\ &= (A \circ B \cup S \circ A \circ B] \subseteq (T \cup S \circ T] = (T] = T.\end{aligned}$$

Since $a \notin T$, by (3), we have $b \in T$. Hence $B \subseteq T$.

(3) $\Rightarrow$ (5). Similar to the proof of (3) $\Rightarrow$ (4), we omit it.

(3) $\Rightarrow$ (6). Let A be a right hyperideal, B a left hyperideal of S such that $A \circ B \subseteq T$ and $A \not\subseteq T$. Then we wish to show that $B \subseteq T$. To do this, let $a \in A, a \notin T$ and $b \in B$. Since $I(a) \subseteq (A \cup S \circ A], I(b) \subseteq (B \cup B \circ S]$, we have

$$\begin{aligned}I(a) \circ I(b) &\subseteq (A \cup S \circ A] \circ (B \cup B \circ S] \subseteq ((A \cup S \circ A) \circ (B \cup B \circ S)] \\ &\subseteq (A \circ B \cup S \circ A \circ B \cup A \circ B \circ S \cup S \circ A \circ B \circ S] \\ &\subseteq (T \cup S \circ T \cup T \circ S \cup S \circ T \circ S] = (T] = T.\end{aligned}$$

By (3), $a \in T$ or $b \in T$. Since $a \notin T$, we have $b \in T$. Thus $B \subseteq T$.

(4) $\Rightarrow$ (1), (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) are clear. This completes the proof.

Theorem 3.7. *Let S be an ordered semihypergroup and T a hyperideal of S. Then T is weakly prime if and only if for all hyperideals A, B of S such that $(A \circ B] \cap (B \circ A] \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.*

Proof. Let T be weakly prime, A, B hyperideals of S and $(A \circ B] \cap (B \circ A] \subseteq T$. By Lemma 2.6(6), $(A \circ B]$ and $(B \circ A]$ are hyperideals of S, and we have

$$(A \circ B] \circ (B \circ A] \subseteq (A \circ B] \cap (B \circ A] \subseteq T,$$

Since T is weakly prime, we have $(A \circ B] \subseteq T$ or $(B \circ A] \subseteq T$. Say $(A \circ B] \subseteq T$, then by Lemma 2.6(1) we have $A \circ B \subseteq T$, and we deduce that $A \subseteq T$ or $B \subseteq T$. Similarly, say $(B \circ A] \subseteq T$, we have $B \subseteq T$ or $A \subseteq T$.

Conversely, let A, B be hyperideals of S such that $A \circ B \subseteq T$. Then $(A \circ B] \cap (B \circ A] \subseteq (A \circ B] \subseteq (T] = T$. Thus, by hypothesis, we have $A \subseteq T$ or $B \subseteq T$. Therefore, T is weakly prime.

Theorem 3.8. *Let S be an ordered semihypergroup. Then the hyperideals of S are weakly prime if and only if they form a chain under inclusion and one of the five equivalent conditions of Lemma 2.8 holds in S.*

Proof. Suppose that the hyperideals of S are weakly prime. Let A, B be hyperideals of S. Then, by hypothesis and Lemma 2.6(6), $(A \circ B]$ is a weakly prime hyperideal of S. Since $A \circ B \subseteq (A \circ B]$, we have

$$A \subseteq (A \circ B] \subseteq (S \circ B] \subseteq (B] = B$$

or

$$B \subseteq (A \circ B] \subseteq (A \circ S] \subseteq (A] = A.$$

Also, since $A^2 \subseteq (A^2]$ and $(A^2]$ is a hyperideal of S, we have $A \subseteq (A^2]$. On the other hand, by Lemma 2.6, $(A^2] = (A \circ A] \subseteq (A \circ S] \subseteq (A] = A$. Thus $(A^2] = A$.

Conversely, let A, B, T be hyperideals of S such that $A \circ B \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. Say $A \subseteq B$, then, by Lemma 2.8, $A = A \cap B = (A \circ B] \subseteq (T] = T$. Similarly, say $B \subseteq A$, we have $B \subseteq T$. Therefore, T is weakly prime.

Definition 3.9. Let $(S, \circ, \leq)$ be an ordered semihypergroup. S is called *intra-regular* if, for each element a of S, there exist $x, y \in S$ such that $a \preceq x \circ a \circ a \circ y$. Equivalently, $a \in (S \circ a \circ a \circ S], \forall a \in S$.

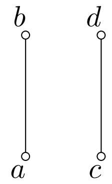
Example 3.10. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$	S
b	$\{b\}$	$\{b\}$	$\{b, d\}$	$\{b, d\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (c, d)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. Moreover, S is intra-regular. Indeed, for any $x \in S$, we have $x \in x^2 = x^4 \subseteq S \circ x \circ x \circ S$, which implies that $x \in (S \circ x \circ x \circ S]$.

In order to characterize the prime hyperideals of an ordered semihypergroup, we need the following lemmas.

Lemma 3.11. *If the hyperideals of an ordered semihypergroup S are semiprime, then the following statements hold:*

(1) *$I(x) = (S \circ x \circ S]$ for any $x \in S$.*

(2) *For any $x, y \in S$, $I(x) \cap I(y) = I(z)$ for some $z \in x \circ y$.*

Proof. (1) Let $x \in S$. Since $x^2 \circ x^2 = x^2 \circ x \circ x \subseteq (S \circ x \circ S]$ and $(S \circ x \circ S]$ is a hyperideal of S, by hypothesis, we have $x^2 \subseteq (S \circ x \circ S]$ and $x \in (S \circ x \circ S]$. Hence $I(x) \subseteq (S \circ x \circ S]$. The reverse inclusion is immediate. Thus $I(x) = (S \circ x \circ S]$.

(2) Let $x, y, z \in S$ such that $z \in x \circ y$. Then, we have

$$z \in x \circ y \subseteq I(x) \circ S \subseteq I(x)$$

and

$$z \in x \circ y \subseteq S \circ I(y) \subseteq I(y),$$

from which we can conclude that $I(z) \subseteq I(x), I(z) \subseteq I(y)$, and we have $I(z) \subseteq I(x) \cap I(y)$. To prove the inverse inclusion, let $t \in I(x) \cap I(y)$. By (1), $t \in (S \circ x \circ S]$ and $t \in (S \circ y \circ S]$. Then there exist $a, b, c, d \in S$ such that $t \preceq a \circ x \circ b$ and $t \preceq c \circ y \circ d$. Thus, by Lemma 2.6(9), $t \circ t \preceq c \circ y \circ d \circ a \circ x \circ b$. On the other hand, $y \circ d \circ a \circ x \subseteq (S \circ (x \circ y) \circ S]$. Indeed, by Lemma 2.6(1) we have

$$(y \circ d \circ a \circ x)^2 = y \circ d \circ a \circ x \circ y \circ d \circ a \circ x \subseteq S \circ (x \circ y) \circ S \subseteq (S \circ (x \circ y) \circ S].$$

Since the hyperideal $(S \circ (x \circ y) \circ S]$ is semiprime, we have $y \circ d \circ a \circ x \subseteq (S \circ (x \circ y) \circ S]$. Furthermore, since $(S \circ (x \circ y) \circ S]$ is a hyperideal of S, we have $c \circ y \circ d \circ a \circ x \circ b \subseteq (S \circ (x \circ y) \circ S]$. Thus, by Lemma 2.6(8), we have $t \circ t \subseteq (S \circ (x \circ y) \circ S]$. Also, since $(S \circ (x \circ y) \circ S]$ is semiprime, we have $t \in (S \circ (x \circ y) \circ S]$, and there exists $z \in x \circ y$ such that $t \in (S \circ z \circ S] = I(z)$. It implies that $I(x) \cap I(y) \subseteq I(z)$. Therefore, $I(x) \cap I(y) = I(z)$ for some $z \in x \circ y$.

Lemma 3.12. *Let S be an ordered semihypergroup. Then S is intra-regular if and only if the hyperideals of S are semiprime.*

Proof. Suppose that S is intra-regular. Let T be a hyperideal of S and $a \in S$ such that $a \circ a \subseteq T$. Then, since S is intra-regular, we have

$$a \in (S \circ a \circ a \circ S] \subseteq (S \circ T \circ S] \subseteq (T] = T,$$

which means that T is semiprime.

Conversely, assume that the hyperideals of S are semiprime. Let $a \in S$. We denote by $I(a^2)$ the hyperideal of S generated by a^2. Since $a^2 \subseteq I(a^2)$, by hypothesis we have

$$a \in I(a^2) = (a^2 \cup S \circ a^2 \cup a^2 \circ S \cup S \circ a^2 \circ S].$$

Then $a \leq t$ for some $t \in a^2 \cup S \circ a^2 \cup a^2 \circ S \cup S \circ a^2 \circ S$. If $t \in a^2$, then $a \in (a^2] \subseteq ((a^2] \circ (a^2]] = (a^4] \subseteq (S \circ a^2 \circ S]$. If $t \in S \circ a^2$, then $a \in (S \circ a^2] \subseteq (S \circ (S \circ a^2] \circ a] \subseteq ((S] \circ (S \circ a^2] \circ (a]] = (S \circ (S \circ a^2) \circ a] \subseteq (S \circ a^2 \circ S]$. If $t \in a^2 \circ S$, then $a \in (a^2 \circ S] \subseteq (a \circ (a^2 \circ S] \circ S] \subseteq ((a] \circ (a^2 \circ S] \circ (S]] = (a \circ (a^2 \circ S) \circ S] \subseteq (S \circ a^2 \circ S]$. If $t \in S \circ a^2 \circ S$, then $a \in (S \circ a^2 \circ S]$. Therefore, S is intra-regular.

Theorem 3.13. *Let S be an ordered semihypergroup. Then the hyperideals of S are prime if and only if they form a chain under inclusion and S is intra-regular.*

Proof. Let all hyperideals of S be prime. Then they are weakly prime. By Theorem 3.8, they form a chain with respect to the inclusion relation. By hypothesis, the hyperideals of S are also semiprime. Thus S is intra-regular by Lemma 3.12.

Conversely, suppose that S is an intra-regular ordered semihypergroup and the hyperideals of S form a chain. We prove that the hyperideals of S are prime. In fact, let T be a hyperideal of S and $a, b \in S$ such that $a \circ b \subseteq T$. By Lemma 3.12, the hyperideals of S are semiprime. Then, by Lemma 3.11(2), there exists $c \in a \circ b \subseteq T$ such that $I(a) \cap I(b) = I(c)$. By hypothesis, we have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a \in I(a) = I(a) \cap I(b) = I(c) \subseteq T$, i.e., $a \in T$. If $I(b) \subseteq I(a)$, then $b \in I(b) = I(a) \cap I(b) = I(c) \subseteq T$, i.e., $b \in T$. We have thus shown that T is prime.

In the following we shall investigate the relationships among the prime hyperideals, weakly prime hyperideals and semiprime hyperideals in ordered semihypergroups.

Theorem 3.14. *Let S be an ordered semihypergroup and T a hyperideal of S. Then T is prime if and only if T is weakly prime and semiprime. In a commutative ordered semihypergroup the prime and weakly prime hyperideals coincide.*

Proof. Let S be an ordered semihypergroup and T a prime hyperideal of S. Clearly T is weakly prime and semiprime. Conversely, assume that T is weakly prime and semiprime and let $a, b \in S$ such that $a \circ b \subseteq T$. Then, we have

$$\begin{aligned}(b \circ S \circ a] \circ (b \circ S \circ a] &\subseteq (b \circ S \circ a \circ b \circ S \circ a] \subseteq (S \circ (a \circ b) \circ S] \\ &\subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

Since T is semiprime, we have $(b \circ S \circ a] \subseteq T$. Thus, by Lemma 2.6, we have

$$\begin{aligned}(S \circ b \circ S] \circ (S \circ a \circ S] &\subseteq (S \circ b \circ S \circ S \circ a \circ S] \subseteq (S \circ (b \circ S \circ a) \circ S] \\ &\subseteq (S \circ (b \circ S \circ a] \circ S] \subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

Since $(S \circ b \circ S], (S \circ a \circ S]$ are hyperideals of S, and T is weakly prime, we have $(S \circ b \circ S] \subseteq T$ or $(S \circ a \circ S] \subseteq T$. Similar to the proof of (1)$\Longrightarrow$(2) in Theorem 3.6, we have $a \in T$ or $b \in T$. Therefore, T is prime.

In particular, let S be a commutative ordered semihypergroup. Then every weakly prime hyperideal of S is prime. Indeed, let T be a weakly prime hyperideal of S and $a, b \in S$ such that $a \circ b \subseteq T$. Then, we have

$$\begin{aligned}I(a) \circ I(b) &\subseteq (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \circ (b \cup S \circ b \cup b \circ S \cup S \circ b \circ S] \\ &\subseteq ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S) \circ (b \cup S \circ b \cup b \circ S \cup S \circ b \circ S)] \\ &= (a \circ b \cup S \circ a \circ b] \subseteq (T \cup S \circ T] = (T] = T.\end{aligned}$$

Since T is weakly prime, by Theorem 3.6 we have $a \in T$ or $b \in T$. Thus T is prime.

Theorem 3.15. *Let S be an ordered semihypergroup and $\{T_i \mid i \in I\}$ a family of prime hyperideals of S. Then $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S if $\bigcap_{i \in I} T_i \neq \emptyset$.*

Proof. Let T_i be a prime hyperideal of S for any $i \in I$. Assume that $\bigcap_{i \in I} T_i \neq \emptyset$. Then, by Lemma 2.7, $\bigcap_{i \in I} T_i$ is a hyperideal of S. Moreover, we can show that $\bigcap_{i \in I} T_i$ is semiprime. In fact, let $a \in S$ be such that $a \circ a \subseteq \bigcap_{i \in I} T_i$. Then $a \circ a \subseteq T_i$ for every $i \in I$. Hence, by hypothesis, $a \in T_i$ for every $i \in I$. It thus follows that $a \in \bigcap_{i \in I} T_i$. Therefore, $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S.

In the above theorem we have shown that every nonempty intersection of prime hyperideals of an ordered semihypergroup S is semiprime. But the nonempty intersection of prime hyperideals of S is not necessarily a prime hyperideal of S. We can illustrate it by the following example.

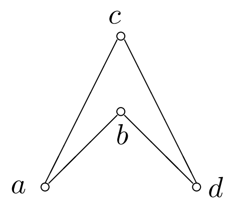
Example 3.16. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (a, c), (d, b), (d, c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup ([25]). We can easily verify that $T_1 = \{a, b, d\}$, $T_2 = \{a, c, d\}$ are prime hyperideals of S. But $T_1 \cap T_2 = \{a, d\}$ is not a prime hyperideal of S. In fact, since $b \circ c = \{a, d\} \subseteq \{a, d\}$, but $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

Let S be an ordered semihypergroup. A hyperideal A of S is called *proper* if $A \neq S$. A proper hyperideal T of S is called *maximal* if A is a hyperideal of S such that $T \subset A$, we have $A = S$. Equivalently, if for any proper hyperideal A of S such that $T \subseteq A$, we have $A = T$.

Theorem 3.17. *If S is an ordered semihypergroup satisfying $S = (S^2]$, then every maximal hyperideal of S is weakly prime.*

Proof. Let M be a maximal hyperideal of S and A, B hyperideals of S such that $A \circ B \subseteq M$. Then $A \subseteq M$ or $B \subseteq M$. Indeed, suppose that $A \nsubseteq M$ and $B \nsubseteq M$. Then $M \subset M \cup A, M \subset M \cup B$. By Lemma 2.7, $M \cup A$ and $M \cup B$ are two hyperideals of S. Since M is maximal, we have $M \cup A = S$ and $M \cup B = S$. Then

$$\begin{aligned} S &= (S^2] = ((M \cup A) \circ (M \cup B)] \\ &= (M \circ M \cup M \circ B \cup A \circ M \cup A \circ B] \subseteq (M] = M, \end{aligned}$$

from which we deduce that $M = S$. It contradicts the fact that M is maximal. Thus M is weakly prime.

Corollary 3.18. *If S is an intra-regular ordered semihypergroup, then every maximal hyperideal of S is weakly prime.*

Proof. Suppose that S is an intra-regular ordered semihypergroup and $a \in S$. Then, we have $a \in (S \circ a \circ a \circ S] \subseteq (S^2]$. It implies that $S \subseteq (S^2]$, and thus $S = (S^2]$. Consequently, by Theorem 3.17, every maximal hyperideal of S is weakly prime.

Corollary 3.19. *If S is an ordered semihypergroup with an identity, then every maximal hyperideal of S is weakly prime.*

Proof. Let S be an ordered semihypergroup containing an identity e. Then, we have $S = e \circ S \subseteq S \circ S = S^2 \subseteq (S^2] \subseteq S$, which implies that $S = (S^2]$. Therefore, every maximal hyperideal of S is weakly prime by Theorem 3.17.

The following is an immediate corollary of Theorem 3.14 and Corollary 3.19.

Corollary 3.20. *If S is a commutative ordered semihypergroup with an identity, then every maximal hyperideal of S is prime.*

4. Hyperideal extensions of ordered semihypergroups

In the current section we consider the extensions of hyperideals in commutative ordered semihypergroups. Moreover, we define n-prime hyperideals and n-semiprime hyperideals of ordered semihypergroups, and investigate the relationship between extensions of hyperideals and n-prime hyperideals.

Definition 4.1. Let I be a hyperideal of S, $x \in I$. The set

$$< x, I >:= \{a \in S \mid x \circ a \subseteq I\}$$

is called the *extension* of I by x.

Proposition 4.2. *Let I be a hyperideal of a commutative ordered semihypergroup S, $x \in S$. Then the following statements hold:*

(1) *$< x, I >$ is a hyperideal of S.*
(2) *$I \subseteq < x, I > \subseteq < y, I >$, for any $y \in x^2$.*
(3) *If $x \in I$, then $< x, I >= S$.*
(4) *I is prime if and only if $< x, I >= I$ for any $x \in S \backslash I$.*

Proof. (1) Let $a \in < x, I >, S \ni b \leq a$. Then $b \in < x, I >$. In fact, since $x \circ a \subseteq I, S \supseteq x \circ b \preceq x \circ a$, by Lemma 2.6(8) we have $x \circ b \subseteq I$, i.e., $b \in < x, I >$. Furthermore, let $a \in < x, I >$ and $b \in S$. Then $x \circ a \subseteq I$, and, for any $c \in a \circ b$, we have

$$x \circ c \subseteq x \circ (a \circ b) = (x \circ a) \circ b \subseteq I \circ S \subseteq I.$$

It implies that $c \in < x, I >$, and we have $a \circ b \subseteq < x, I >$. Hence $< x, I >$ is a right hyperideal of S. Since S is commutative, we obtain the requested result.

(2) If $a \in I$, then $x \circ a \subseteq S \circ I \subseteq I$, i.e., $a \in < x, I >$. It implies that $I \subseteq < x, I >$. Furthermore, let $a \in < x, I >$. Then $x \circ a \subseteq I$, and, for any $y \in x^2$, we have

$$y \circ a \subseteq x^2 \circ a = x \circ (x \circ a) \subseteq S \circ I \subseteq I.$$

Thus it can be shown that $< x, I > \subseteq < y, I >$ for any $y \in x^2$.

(3) Let $a \in S, x \in I$. Then $x \circ a \subseteq I \circ S \subseteq I$, and we have $a \in < x, I >$. It thus follows that $< x, I >= S$.

(4) Assume that I is a prime hyperideal of S. Let $x \in S \backslash I$ and $a \in < x, I >$. Then $x \circ a \subseteq I$, and, by hypothesis, we have $a \in I$. It implies that $< x, I > \subseteq I$. By (2), the inverse inclusion holds. Hence $< x, I >= I$ for any $x \in S \backslash I$.

Conversely, let $x, y \in S$ be such that $x \circ y \subseteq I$. Then $y \in < x, I >$. We claim that $x \in I$ or $y \in I$. If $x \notin I$, then, by hypothesis, we have $y \in < x, I >= I$. Thus I is prime.

Proposition 4.3. *Let I be a hyperideal of a commutative ordered semihypergroup S and $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ a family of prime hyperideal of S. If $I = \bigcap_{\alpha \in \mathcal{A}} I_\alpha$, then, for any $x \in S$, $< x, I >$ is a semiprime hyperideal of S whenever $I \neq \emptyset$.*

Proof. Let $x \in S$. We first show that

$$< x, I >=< x, \bigcap_{\alpha \in \mathcal{A}} I_\alpha >= \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha > .$$

In fact,

$$\begin{aligned} a \in < x, \bigcap_{\alpha \in \mathcal{A}} I_\alpha > &\iff x \circ a \subseteq \bigcap_{\alpha \in \mathcal{A}} I_\alpha \\ &\iff x \circ a \subseteq I_\alpha, \forall \alpha \in \mathcal{A} \\ &\iff a \in < x, I_\alpha >, \forall \alpha \in \mathcal{A} \\ &\iff a \in \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha > . \end{aligned}$$

Now we consider the following cases:

Case 1. If $x \in I_\alpha$ for any $\alpha \in \mathcal{A}$, then, by Proposition 4.2(3), we have $< x, I_\alpha >= S$. Then $< x, I >= \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha >= S$, and $< x, I >$ is a semiprime hyperideal of S.

Case 2. Let $x \in S \backslash I_\alpha$ for some $\alpha \in \mathcal{A}$. Then, by Proposition 4.2(4), $< x, I_\alpha >= I_\alpha$. Let $\mathcal{B} := \{\alpha \in \mathcal{A} \mid x \notin I_\alpha\}$. Then $\mathcal{B} \neq \emptyset$ and we have

$$< x, I >= \bigcap_{\alpha \in \mathcal{B}} I_\alpha .$$

It thus follows from Theorem 3.15 that $< x, I >$ is semiprime.

Proposition 4.4. *Let S be a commutative ordered semihypergroup containing an identity e, and $x, y \in S$. Then $I(x) \subseteq I(y)$ if and only if, for any hyperideal J of S, we have $< x, J > \supseteq < y, J >$.*

Proof. Let J be a hyperideal of S and $a \in < y, J >$. Then $y \circ a \subseteq J$. Since S is a commutative ordered semihypergroup containing an identity e and $I(x) \subseteq I(y)$, we have $x \in I(y) = (y \cup S \circ y] \subseteq (e \circ y \cup S \circ y] = (S \circ y]$, that is, $x \preceq z \circ y$ for some $z \in S$. Then, we have

$$x \circ a \preceq (z \circ y) \circ a = z \circ (y \circ a) \subseteq S \circ J \subseteq J.$$

It thus follows from Lemma 2.6(9) that $x \circ a \subseteq J$. Hence $a \in < x, J >$.

Conversely, since $I(y)$ is a hyperideal of S, by hypothesis, we have $< x, I(y) > \supseteq < y, I(y) >$. Since $y \in I(y)$, by Proposition 4.2(3), $< y, I(y) >= S$. Then we can deduce that $< x, I(y) >= S$. Hence $e \in < x, I(y) >$, and $x \in x \circ e \subseteq I(y)$. Thus it can be obtained that $I(x) \subseteq I(y)$.

In order to characterize the hyperideal extensions of ordered semihypergroups, we need introduce the concept of n-prime hyperideals of ordered semihypergroups.

Let n be any positive integer such that $n \geq 2$. For any $x_i \in S$ $(i = 1, 2, \cdots, n)$, and j being a positive integer such that $2 \leq j \leq n-1$, we define

$$\begin{aligned} I_{1,n} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-1} \circ x_n, \\ I_{j,n} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n, \\ I_{n,n} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1}. \end{aligned}$$

Definition 4.5. Let I be a hyperideal of an ordered semihypergroup S. I is called *n-prime* if for any $x_i \in S$ $(i = 1, 2, \cdots, n)$, $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$ implies there exists a positive integer i $(1 \leq i \leq n)$ such that

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n,n} \subseteq I.$$

Definition 4.6. Let I be a hyperideal of an ordered semihypergroup S. I is called *n-semiprime* if for any $x_1, x_2, \cdots, x_n \in S$ with $x_1 = x_2 = \cdots = x_n$, $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$ implies $I_{n,n} \subseteq I$.

Theorem 4.7. *Let S be an ordered semihypergroup. Then the following statements are true:*

(1) *Every n-prime hyperideal of S is n-semiprime.*
(2) *The prime hyperideals and 2-prime hyperideals of S coincide.*
(3) *The semiprime hyperideals and 2-semiprime hyperideals of S coincide.*

Proof. Straightforward.

Theorem 4.8. *Let S be an ordered semihypergroup. Then every $(n-1)$-prime hyperideal of S is an n-prime hyperideal of S for all positive integers $n \geq 3$.*

Proof. Suppose that I is an $(n-1)$-prime hyperideal of S. Let $x_1, x_2, \cdots, x_n \in S$ be such that $x_1 \circ x_2 \circ x_3 \circ x_4 \circ \cdots \circ x_{n-3} \circ x_{n-2} \circ x_{n-1} \circ x_n \subseteq I$. Then, for any $z \in x_{n-1} \circ x_n$, $x_1 \circ x_2 \circ x_3 \circ x_4 \circ \cdots \circ x_{n-3} \circ x_{n-2} \circ z \subseteq I$. We define:

$$\begin{aligned}
J_{1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ z, \\
J_{2,n-1} &:= x_1 \circ x_3 \circ \cdots \circ x_{n-2} \circ z, \\
&\vdots \\
J_{n-2,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ z, \\
J_{n-1,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}.
\end{aligned}$$

By hypothesis, there exists a positive integer i $(1 \leq i \leq n-1)$ such that

$$J_{1,n-1}, J_{2,n-1}, \cdots, J_{i-1,n-1}, J_{i+1,n-1}, \cdots, J_{n-1,n-1} \subseteq I.$$

We consider the following two cases:

Case 1. Let $J_{n-1,n-1} \nsubseteq I$. Then $J_{1,n-1}, J_{2,n-1}, \cdots, J_{n-2,n-1} \subseteq I$. Thus, by the arbitrariness of z, $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n} \subseteq I$. By $I_{1,n} \subseteq I$, we have $x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1} \circ x_n \subseteq I$. We define

$$\begin{aligned}
K_{1,n-1} &:= x_3 \circ x_4 \circ \cdots \circ x_{n-1} \circ x_n, \\
K_{2,n-1} &:= x_2 \circ x_4 \circ \cdots \circ x_{n-1} \circ x_n, \\
&\vdots \\
K_{n-1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1}.
\end{aligned}$$

By hypothesis, there exists a positive integer j $(1 \leq j \leq n-1)$ such that

$$K_{1,n-1}, K_{2,n-1}, \cdots, K_{j-1,n-1}, K_{j+1,n-1}, \cdots, K_{n-1,n-1} \subseteq I.$$

Then, we have

$$K_{n-2,n-1} = x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_n \subseteq I$$

or

$$K_{n-1,n-1} = x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq I.$$

Thus, since I is a hyperideal of S, we have

$$I_{n-1,n} = x_1 \circ K_{n-2,n-1} \subseteq I$$

or

$$I_{n,n} = x_1 \circ K_{n-1,n-1} \subseteq I.$$

Hence $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n-1,n} \subseteq I$ or $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n,n} \subseteq I$.

Case 2. Let $J_{n-1,n-1} \subseteq I$. Then there exists a positive integer k ($1 \leq k \leq n-2$) such that

$$J_{1,n-1}, J_{2,n-1}, \cdots, J_{k-1,n-1}, J_{k+1,n-1}, \cdots, J_{n-2,n-1} \subseteq I.$$

Thus, since z is an arbitrary element of $x_{n-1} \circ x_n$, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{k-1,n}, I_{k+1,n}, \cdots, I_{n-2,n} \subseteq I.$$

Since I is a hyperideal of S and $J_{n-1,n-1} \subseteq I$, we have

$$I_{n-1,n} = J_{n-1,n-1} \circ x_n \subseteq I$$

and

$$I_{n,n} = J_{n-1,n-1} \circ x_{n-1} \subseteq I.$$

Hence, in this case, $I_{1,n}, I_{2,n}, \cdots, I_{j-1,n}, I_{j+1,n}, \cdots, I_{n-1,n}, I_{n,n} \subseteq I$ for some positive integer k ($1 \leq k \leq n-2$).

Therefore, I is an n-prime hyperideal of S.

By the above theorem, we immediately obtain the following corollary:

Corollary 4.9. *Let S be an ordered semihypergroup. Then every prime hyperideal of S is an n-prime hyperideal of S for all positive integers $n \geq 2$.*

The converse of Theorem 4.8 is not true in general. We can illustrate it by the following example:

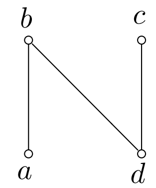
Example 4.10. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{b,d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a,a), (a,b), (b,b), (c,c), (d,b), (d,c), (d,d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec := \{(a,b), (d,b), (d,c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. One can easily show that $I = \{d\}$ is a 3-prime hyperideal of S, but it is not 2-prime. Indeed, since $b \circ c = \{d\} \subseteq I$, but $b \notin I$ and $c \notin I$.

Now we give a relationship between extensions of hyperideals and n-prime hyperideals in commutative ordered semihypergroups.

Theorem 4.11. *Let S be a commutative ordered semihypergroup and I a hyperideal of S. Then I is n-prime if and only if any extension of I is $(n-1)$-prime for all positive integers $n \geq 3$.*

Proof. Let I be an n-prime hyperideal of S. For any $x \in S$, let $x_1, x_2, \cdots, x_{n-2}, x_{n-1} \in S$ be such that $x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq < x, I >$. Then $x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq I$. We define

$$
\begin{aligned}
&J_{1,n} := x_1 \circ x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1},\\
&J_{2,n} := x \circ x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1},\\
&\vdots\\
&J_{n-1,n} := x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-1},\\
&J_{n,n} := x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}.
\end{aligned}
$$

Since I is an n-prime hyperideal of S, there exists a positive integer i $(1 \leq i \leq n)$ such that

$$J_{1,n}, J_{2,n}, \cdots, J_{i-1,n}, J_{i+1,n}, \cdots, J_{n,n} \subseteq I.$$

Thus, there exists a positive integer j $(2 \leq j \leq n)$ such that

$$J_{2,n}, J_{3,n}, \cdots, J_{j-1,n}, J_{j+1,n}, \cdots, J_{n,n} \subseteq I.$$

It implies that there exists a positive integer $k = j - 1$ $(1 \leq k \leq n-1)$ such that

$$x \circ I_{1,n-1}, x \circ I_{2,n-1}, \cdots, x \circ I_{k-1,n-1}, x \circ I_{k+1,n-1}, \cdots, x \circ I_{n-1,n-1} \subseteq I,$$

where

$$
\begin{aligned}
&I_{1,n-1} := x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1},\\
&I_{l,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{l-1} \circ x_{l+1} \circ \cdots \circ x_{n-2} \circ x_{n-1} \ (2 \leq l \leq n-2),\\
&I_{n-1,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}.
\end{aligned}
$$

Hence, it can be easily shown that $I_{1,n-1}, I_{2,n-1}, \cdots, I_{k-1,n-1}, I_{k+1,n-1}, \cdots, I_{n-1,n-1} \subseteq < x, I >$. In other words, $< x, I >$ is indeed an $(n-1)$-prime hyperideal of S.

Conversely, suppose that any extension of I is an $(n-1)$-prime hyperideal of S. Let $x_1, x_2, \cdots, x_n \in S$ be such that $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$. Then it can be easily shown that $x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq < x_n, I >$. Thus, by hypothesis, there exists a positive integer i $(1 \leq i \leq n-1)$ such that

$$(*) \qquad I_{1,n-1}, I_{2,n-1}, \cdots, I_{i-1,n-1}, I_{i+1,n-1}, \cdots, I_{n-1,n-1} \subseteq < x_n, I > .$$

To prove that I is n-prime, we consider the following two cases:

Case 1. If $I_{i,n-1} \subseteq < x_n, I >$, then we have

$$I_{1,n-1}, I_{2,n-1}, \cdots, I_{n-2,n-1}, I_{n-1,n-1} \subseteq < x_n, I > .$$

It thus follows that $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n-1,n} \subseteq I$. Consequently, I is n-prime.

Case 2. Let $I_{i,n-1} \not\subseteq < x_n, I >$. Then, by $(*)$, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n-1,n} \subseteq I.$$

We can prove that $I_{n,n} \subseteq I$. In fact, take a positive integer j such that $1 \leq j \leq n-1$ and $j \neq i$. Since $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$, we have

$$x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n \subseteq < x_j, I > .$$

Now, we define

$$\begin{aligned}
&K_{1,n-1} := x_2 \circ x_3 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n,\\
&K_{2,n-1} := x_1 \circ x_3 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n,\\
&\vdots\\
&K_{j-1,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{j-2} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n,\\
&K_{j,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+2} \circ \cdots \circ x_{n-1} \circ x_n,\\
&\vdots\\
&K_{n-2,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-2} \circ x_n,\\
&K_{n-1,n-1} := x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-2} \circ x_{n-1}.
\end{aligned}$$

By hypothesis, $< x_j, I >$ is an $(n-1)$-prime hyperideal of S, and there exists a positive integer k $(1 \leq k \leq n-1)$ such that

$$K_{1,n-1}, K_{2,n-1}, \cdots, K_{k-1,n-1}, K_{k+1,n-1}, \cdots, K_{n-1,n-1} \subseteq < x_j, I > .$$

It thus follows that there exists a positive integer l $(1 \leq l \leq n)$ and $l \neq j$ (assume $l < j$) such that

$$I_{1,n}, I_{2,n}, \cdots, I_{l-1,n}, I_{l+1,n}, \cdots, I_{j-1,n}, I_{j+1,n}, \cdots, I_{n,n} \subseteq I.$$

Since $j \neq i$ and $j \neq n$, we have $I_{i,n} \subseteq I$ or $I_{n,n} \subseteq I$. Again since $I_{i,n-1} \not\subseteq < x_n, I >$, we have $I_{i,n} \not\subseteq I$, and we deduce that $I_{n,n} \subseteq I$. Thus, in this case, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n-1,n}, I_{n,n} \subseteq I,$$

and I is an n-prime hyperideal of S. This completes the proof.

Theorem 4.12. *Let S be a commutative ordered semihypergroup containing an identity e. Then the n-prime hyperideals and the $(n-1)$-prime hyperideals of S coincide for all positive integers $n \geq 3$.*

Proof. Let I be an n-prime hyperideal of S. By Theorem 4.11, $< e, I >$ is an $(n-1)$-prime hyperideal of S. Let $a \in < e, I >$. Then $a \in e \circ a \subseteq I$. Thus $< e, I > \subseteq I$. By Proposition 4.2(2), $< e, I > = I$. Hence I is $(n-1)$-prime. Consequently, by Theorem 4.8, the proof is completed.

Lemma 4.13. *Let S be a commutative ordered semihypergroup and I a semiprime hyperideal of S. Then $I = \bigcap_{x \in S} < x, I >$.*

Proof. By Proposition 4.2(2), $I \subseteq < x, I >$ for any $x \in S$. Then $I \subseteq \bigcap_{x \in S} < x, I >$. To prove the inverse inclusion, let $a \in \bigcap_{x \in S} < x, I >$. Then $a \in < a, I >$, and we have $a \circ a \subseteq I$. Since I is a semiprime hyperideal of S, we have $a \in I$. Therefore, we obtain the requested result.

Theorem 4.14. *Let I be a semiprime and n-prime hyperideal of a commutative ordered semihypergroup S, $n \geq 3$. Let*

$$\mathcal{P} := \{T \mid T \textit{ is an } (n-1)\textit{-prime hyperideal of } S \textit{ and } I \subseteq T\}.$$

Then $I = \bigcap_{T \in \mathcal{P}} T$.

Proof. Obviously, $I \subseteq \bigcap_{T \in \mathcal{P}} T$. On the other hand, since I is a semiprime hyperideal of S, by Lemma 4.13, $I = \bigcap_{x \in S} < x, I >$. Furthermore, since I is also n-prime, by Proposition 4.2(2) and Theorem 4.11, $< x, I >$ is an $(n-1)$-prime hyperideal of S for any $x \in S$ and $I \subseteq < x, I >$. Thus, for any $x \in S$, $< x, I > \in \mathcal{P}$. Hence $\bigcap_{T \in \mathcal{P}} T \subseteq \bigcap_{x \in S} < x, I > = I$. Therefore, $I = \bigcap_{T \in \mathcal{P}} T$.

The hypothesis that I is semiprime cannot be removed in the above theorem. Otherwise, Theorem 4.14 does not hold in general. We can illustrate it by the following example.

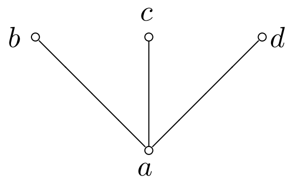
Example 4.15. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, d\}$

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec := \{(a, b), (a, c), (a, d)\}.$$



Then $(S, \circ, \leq)$ is a commutative ordered semihypergroup and $\{a\}$ is a hyperideal of S. We can easily prove that $\{a\}$ is 3-prime, but it is not semiprime. In addition, with a small amount of effort one can verify that the sets $\{a, b, c\}$ and S are all prime hyperideals of S containing a, while $\{a, b, c\} \cap S = \{a, b, c\} \neq \{a\}$.

Corollary 4.16. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If $(S, \circ)$ is a hypersemilattice, then every n-prime hyperideal $(n \geq 3)$ of S can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it.*

Acknowledgments

This work was supported by the Natural Science Foundation of China (No. 11801081, 11701504), the Demonstration Project of Grass-roots Teaching and Research Section in Anhui Province (No. 2018jyssf053), Anhui Provincial Excellent Youth Talent Foundation (No. gxyqZD2019043) and the University Natural Science Project of Anhui Province (No. KJ2018A0329).

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Accepted: 10.10.2018

Finite group with coincide automizer and central automorphism of subgroups

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Abstract. In this paper, we characterize the finite group G such that $Aut_G(H) = Aut_c(H)$ for every (abelian, non-abelian) subgroup H of G, where $Aut_G(H)$ and $Aut_c(H)$ are automizer and central automorphism of H in G.

Keywords: nilpotent group, automizer, central automorphism.

1. Introduction

All groups considered in this paper are finite.

Let G be a group and H a subgroup of G. The automizer $Aut_G(H)$ of H in G is defined as the group of automorphisms of H induced by conjugation of elements of $N_G(H)$,

$$Aut_G(H) \cong N_G(H)/C_G(H)$$

and we obviously have $Inn(H) \leq Aut_G(H) \leq Aut(H)$. $Aut_G(H)$ is small if $Aut_G(H) = Inn(H)$ and large if $Aut_G(H) = Aut(H)$.

Automizers of some special subgroups had the strong influence toward the group, the best example is probably the well-known Frobenius criterion for p-nilpotency: a finite group is p-nilpotent if and only if the automizers of all its p-subgroups are p-groups. Brandl and Deaconescu [2] gave the structure of finite SANS-groups (Small Automizers for Non-abelian Subgroups) in which the automizers of all non-abelian subgroups are small. Also, Deaconescu and Mazurov obtained the finite groups with large automizers for their non-abelian subgroups in [5], which is called LANS-groups (Large Automizers for Non-abelian Subgroups).

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An automorphism α of G is called a central if $x^{-1}x^{\alpha} \in Z(G)$ for each $x \in G$. The set of all central automorphisms of G, denoted by $Aut_c(G)$, is a normal subgroup of $Aut(G)$. It is easy to see that

$$Aut_c(G) = C_{Aut(G)}(Inn(G)).$$

There have been a number of results on the central automorphism of a finite group, for example: Curran and McCaughan [3, 4] characterized finite p-groups G for which $Aut_c(G) = Inn(G)$ and $Aut_c(G) = Z(Inn(G))$.

From the results above, we found that automizers and central automorphisms have good relationship with Inner automorphism in the group theory. Therefore, it is an interesting topic to investigate the relationship between automizers and central automorphisms in finite groups. In this paper, we focus on the finite group G such that $Aut_G(H) = Aut_c(H)$ for every (abelian, non-abelian) subgroup H of G.

2. Notations and lemmas

The terminology and the notation in this paper are standard as in [6]. For a prime p, $\{a^p | a \in G\}$ will be denoted by $\mho_1(G)$. We use $c(G)$ to denote the nilpotency class of a group G. Following lemmas will be used in the sequel.

Lemma 2.1 ([3])**.** If G is a finite p-group, then $Aut_c(G) = Inn(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.

Lemma 2.2 ([1])**.** If finite group G with large automizers of abelian subgroups, then G is isomorphic to either S_3, for $n \leq 3$ or to Q_8.

3. Main theorem

It is well-known that S_3 is the non-abelian group of minimal order and Q_8, D_8 are the non-abelian 2-groups of minimal order. It is easy to see that $S_3 = Aut_{S_3}(S_3) \neq Aut_c(S_3) = 1$ and $K_4 \cong Aut_{Q_8}(Q_8) = Aut_c(Q_8) = Aut_c(D_8)$. However, it holds that $Aut_G(H) = Aut_c(H)$ for every abelian subgroup H of S_3. And it is true that $Aut_G(H) = Aut_c(H)$ for every subgroup H of Q_8. But the result above is not true for D_8. Since D_8 contains an elementary abelian 2-group of order 4. Let H be an elementary abelian 2-group of order 4. Then

$$Aut_{D_8}(H) = N_{D_8}(H)/C_{D_8}(H) = C_{Aut(H)}(Inn(H)) = Aut(H) = S_3,$$

a contradiction. Hence it is interesting to investigate the finite group G such that $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G.

Theorem 3.1. Let G be a non-abelian group. Then $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G if and only if

(i) G is a p-group, $G' = Z(G)$ and $Z(G)$ is cyclic.

(ii) $G = P \times C_2$, where P is a p-group such that $p \neq 2$, $P' = Z(P)$ and $Z(P)$ is cyclic.

Proof. It is easy to check that $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G if G satisfying (i), (ii). Conversely, proof can proceed by following steps.

Step 1. $P' = Z(P)$ and $Z(P)$ is cyclic for $P \in Syl_p(G)$.

It is easy to see that $Inn(G) = Aut_G(G) = Aut_c(G)$ for non-abelian group G, then $Inn(G)$ is abelian and G is nilpotent group of class 2.

Since G is a non-abelian group, there exists at least one non-abelian Sylow subgroup of G. For $p \in \pi(G)$. Let $P \in Syl_p(G)$ and P is non-abelian. Since G is nilpotent, P is normal in G. Then

$(*)$

$$Inn(P) = P/C_P(P) \cong G/C_G(P) = Aut_G(P) = Aut_c(P) = C_{Aut(P)}(Inn(P))$$

By Lemma 2.1, $P' = Z(P)$ and $Z(P)$ is cyclic.

Case 1. If G is a p-group, then $G = P$, $P' = Z(P)$ and $Z(P)$ is cyclic, as required.

Case 2. If G is not a p-group, there exists prime $q \neq p \in \pi(G)$. Let $Q \in Syl_q(G)$.

Step 2. Q is abelian.

If Q is non-abelian, then $|Q| \geq q^3$. There exists normal subgroup Q_1 of order q in Q, which is also normal in G. Then

$$\begin{aligned} Inn(P) &= P/Z(P) \cong (P \times Q_1)/(Z(P) \times Z(Q_1)) \\ &\cong G/C_G(P \times Q_1) = Aut_G(P \times Q_1) \\ &= Aut_c(P \times Q_1) = C_{Aut(P \times Q_1)}(Inn(P \times Q_1)) \\ &= C_{Aut(P) \times Aut(Q_1)}(Inn(P)). \end{aligned}$$

By $(*)$, $Aut(Q_1) = 1$, so $Q_1 = C_2$, and Q is a 2-group.

Now we choose a normal subgroup P_1 of order p in P, which is also normal in G. We consider non-abelian group $P_1 \times Q$, by the similar argument,

$$\begin{aligned} Inn(Q) &= Q/Z(Q) \cong (Q \times P_1)/(Z(Q) \times Z(P_1)) \cong G/C_G(Q \times P_1) \\ &= Aut_G(Q \times P_1) = Aut_c(Q \times P_1) = C_{Aut(Q \times P_1)}(Inn(Q \times P_1)) \\ &= C_{Aut(Q) \times Aut(P_1)}(Inn(Q)). \end{aligned}$$

By $(*)$, $Aut(P_1) = 1$, so $P_1 = C_2$, and P is a 2-group, a contradiction.

Step 3. $p \neq 2$, and $G = P \times C_2$.

Since Q is abelian, we consider non-abelian subgroup $H = P \times Q$. Hence

$$\begin{aligned} Inn(P) &= P/Z(P) \cong (P \times Q)/(Z(P) \times Z(Q)) \\ &\cong G/C_G(H) = Aut_G(H) = Aut_c(H) \\ &= C_{Aut(H)}(Inn(H)) = C_{Aut(P \times Q)}(Inn(P \times Q)) = C_{Aut(P \times Q)}(Inn(P)) \\ &= C_{Aut(P) \times Aut(Q)}(Inn(P)). \end{aligned}$$

By $(*)$, $Aut(Q) = 1$, so $Q = 1$ or $Q = C_2$.

If $p = 2$, then $G = P$, a contradiction. Hence $p \neq 2$, and $G = P \times C_2$, $P' = Z(P)$ and $Z(P)$ is cyclic, as required.

Zassenhaus proved in [7] that a finite group G is abelian if and only if $N_G(H) = C_G(H)$ for all abelian subgroups $H$of G. Translated into automizer terminology, the elegant result: a finite group is abelian if and only if the automizers of all its abelian subgroups are small. Later on, Bechtell, Deaconescu and Silberberg [1] classified the finite group with large automizers of abelian subgroups, which is called LAAS-groups.

It is easy to see that $Aut_c(H) = Aut(H)$ for every abelian subgroup H of G, that is, LAAS-groups is equivalent to $Aut_G(H) = Aut(H)$ for every abelian subgroup H of G. By Lemma 2.2, we can get the following theorem.

Theorem 3.2. Let G be a group. Then $Aut_G(H) = Aut_c(H)$ for every abelian subgroup H of G if and only if $G = S_n$, for $n \leq 3$ or Q_8.

Finally, we can easy to see that $G = C_2$, or Q_8 if $Aut_G(H) = Aut_c(H)$ for every abelian and non-abelian subgroups H of G by Theorem 3.1 and 3.2 above. Independent on the result above, we classify the finite group G such that $Aut_G(H) = Aut_c(H)$ for every subgroup H of G by the elementary way as follows.

Theorem 3.3. Let G be a group. Then $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if and only if $G = C_2$, or Q_8.

Proof. It is easy to check that $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if $G = C_2$, or $G = Q_8$. Conversely, proof can proceed by following steps.

Step 1. G is nilpotent and $c(G) = 2$.

Since $Aut_G(H) = Aut_c(H)$ for every subgroup H of G,

$$N_G(H)/C_G(H) = C_{Aut(H)}(Inn(H)).$$

Then it is clear that $Inn(H)$ is abelian by $Inn(H) \leq Aut_G(H)$. Hence $H' \leq Z(H)$ for every subgroup H of G, so G is nilpotent and $c(G) = 2$.

Step 2. $Z(G) = C_2$.

Consider $H = Z(G)$. Then $Aut_G(Z(G)) = Aut_c(Z(G))$, it is easy to see that

$$1 = N_G(Z(G))/C_G(Z(G)) = C_{Aut(Z(G))}(Inn(Z(G))) = Aut(Z(G)).$$

Hence $Z(G) = C_2$ by G is nilpotent.

Step 3. $G' = 1$ or $G' = C_2$.

Let $H = G'$. Then $Aut_G(G') = Aut_c(G')$, that is,

$$N_G(G')/C_G(G') = C_{Aut(G')}(Inn(G')).$$

Since $G' \leq Z(G)$, $1 = N_G(G')/C_G(G') = C_{Aut(G')}(Inn(G')) = Aut(G')$. Hence $G' = 1$ or $G' = C_2$.

Step 4. G is a 2-group.

If $G' = 1$, then $Z(G) = C_2 = G$.

If $G' = C_2 = Z(G)$, then we assume that G is not a 2-group. Let $P_2 \in Syl_2(G)$. Since G is nilpotent, P_2 is normal in G. Then $G = P_2 \times K$, and K is Hall $2'$-subgroup of G. By $G' = P_2' \times K' = C_2$, then $P_2' = C_2$, and K is abelian. Then $Z(G) = Z(P_2) \times Z(K) = Z(P_2) \times K$. By

$$Aut_G(Z(P_2)) = Aut_c(Z(P_2)),$$

we have $Z(P_2) = C_2$. Then $Z(G) = Z(P_2) = C_2$, a contradiction. Hence G is a 2-group.

Step 5. If $G' = C_2 = Z(G)$, then $\Phi(G) = C_2$ and G is a extra-special 2-group.

For any $a, b \in G$, $[a^2, b] = [a, b]^2 = 1$ by $Z(G) = C_2 = G'$. Then $a^2 \in Z(G)$, and so $\mho_1(G) \leq Z(G)$. Hence $\Phi(G) = G'\mho_1(G) = Z(G) = C_2$, and so G is a extra-special 2-group.

Step 6. If G is a extra-special 2-group, then $G = Q_8$.

It is well-known that G is a central product of D_8's or a central product of D_8's and a single Q_8 if G is a extra-special 2-group. It is easy to see that $Inn(D_8) \leq Aut(D_8) = D_8$ and $Inn(D_8)$ is elementary abelian 2-group of order 4.

If G contains D_8, then G contains an elementary abelian 2-group of order 4. Let H be an elementary abelian 2-group of order 4. Then

$$N_G(H)/C_G(H) = C_{Aut(H)}(Inn(H)) = Aut(H) = S_3.$$

It is obviously that $N_G(H)/C_G(H)$ is a 2-group, a contradiction. That is, G does not contain D_8, so $G = Q_8$, as required.

Corollary 3.4. Let G be a non-abelian group. Then $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if and only if $G = Q_8$.

Acknowledgements

The research of the work was supported by the National Natural Science Foundation of China (11601245).

The authors would like to thank Professor A. Ballester-Bolinches of Valencia University for his valuable suggestions and useful comments contributed to this paper.

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Accepted: 13.10.2018

Ricci semi-symmetric normal complex contact metric manifolds

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Abstract. In this paper, we obtain the necessary and sufficient conditions a complex almost contact metric manifold to be normal. In addition, we give some new identities for the Riemann curvature and the Ricci curvatures of normal complex contact metric manifolds. Furthermore, we show that a Ricci semi-symmetric normal complex contact metric manifold is Einstein.

Keywords: normal complex contact metric manifold, Ricci semi-symmetric, curvature.

1. Introduction

Kobayashi [18] started studies on complex contact manifolds in 1959. After this Boothby [7], [8] and Wolf [22] presented new results about complex contact manifolds. Further research started again in the early 1980's by Ishihara and Konishi [15], [16] and [17]. They introduced a concept of normality which is called IK-normality in literature [16]. But according to their normality condition complex structure is Kähler. In 1996, Foreman investigated special metrics on complex contact manifolds by studying critical condition of various Riemanian functionals on particular classes of Riemanian metrics called the associated metrics [11]. He studied on classification of three-dimensional complex homogeneous complex contact manifolds, strict normal complex contact manifolds and the Boothby-Wang fibration on complex contact manifolds [12], [13], [14]. In 2000 Korkmaz gave a weaker version of normality in [19] , which must not to be Kähler, and defined the $\mathcal{GH}-$sectional curvature. Blair and present author

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studied energy and corrected energy of vertical distribution for normal complex contact metric manifolds in [5], [21]. Fetcu studied an adapted connection on a strict complex contact manifolds and harmonic maps between complex Sasakian manifolds in [9], [10].

Blair and Molina [2] studied conformal and Bochner curvature tensor of normal complex contact metric manifolds. Korkmaz [19] proved that normality is invariant under $\mathcal{H}$-homothetic deformations and such normal complex contact metric manifolds is called complex $(\kappa,\mu)-$space. In 2012, Blair and Mihai studied on complex $(\kappa,\mu)-$space and they studied on locally symmetric condition of normal complex contact metric manifolds [3], [4].

Our paper is organized as following. In Section 2, some fundamental tools and basic facts are given. Some results on the Riemann curvature are presented in Section 3. Also a new theorem obtained in same section which gives the necessary and sufficient conditions for normality, contains ∇G and ∇H. In addition some results on the Ricci curvature are given in Section 4 and, we proved that a Ricci semi-symmetric normal complex contact metric manifold is Einstein.

2. Preliminaries

Let M be a complex manifold of odd complex dimension $2m+1$ covered by an open covering $\mathcal{A}=\{\mathcal{U}_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form ω_i in each $\mathcal{U}_i\in\mathcal{A}$ in such a way that for any $\mathcal{U}_i,\mathcal{U}_j\in\mathcal{A}$

$$\omega_i\wedge(d\omega_j)^n\neq 0 \text{ in } \mathcal{U}_i,\quad \text{and}\quad \omega_i=f_{ij}\omega_j,\ \mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset,$$

where f_{ij} is a holomorphic function on $\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$ then ω_i is called the complex contact form in $\mathcal{U}_i$, and (M,ω_i) is called a complex contact manifold [18].

$\omega_i=0$ defines a $2m$-dimensional complex vector subspace $\mathcal{H}_x$ of T_xM [18]. Let $\mathcal{H}$ be the vector bundle over M with fibres $\mathcal{H}_x$ and $\mathcal{V}$ be the line bundle $TM/\mathcal{H}$, from Whitney sum we have $TM=\mathcal{H}\oplus\mathcal{V}$. $\mathcal{H}$ and $\mathcal{V}$ are called horizontal subbundle and vertical subbundle, respectively.

Ishihara and Konishi [15], [16], [17] and Shibuya [20] proved existence of complex almost contact metric structure.

Definition 2.1. *Let (M,J,g) be a Hermitian manifold and $\mathcal{A}=\{\mathcal{U}_i\}$ be open covering of M with coordinate neighbourhoods $\{\mathcal{U}_i\}$. M is called a complex almost contact metric manifold if following two conditions are satisfied:*

1. In each $\mathcal{U}_i$ there exist 1-forms u_i and $v_i=u_i\circ J$, with dual vector fields U_i and $V_i=-JU_i$ and $(1,1)$ tensor fields G_i and $H_i=G_iJ$ such that

$$\begin{aligned}&H_i^2=G_i^2=-I+u_i\otimes U_i+v_i\otimes V_i,\\ &G_iJ=-JG_i,\ GU_i=0,\quad g(X,G_iY)=-g(G_iX,Y).\end{aligned}$$

2.On $\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$ we have

$$u_j=au_i-bv_i, v_j=bu_i+av_i,\quad G_j=aG_i-bH_i, H_j=bG_i+aH_i,$$

where a and b are functions on $\mathcal{U}_i \cap \mathcal{U}_j$ with $a^2 + b^2 = 1$.

Also from the second condition on $\mathcal{U}_i \cap \mathcal{U}_j$, we have $U_j = aU_i - bV_i$ and $V_j = bU_i + aV_i$. Since $a^2 + b^2 = 1$ we get $U_i \wedge V_j = U \wedge V$. Thus we have a global vertical distribution $\mathcal{V}$ which is spanned by U and V. In general is assumed that $\mathcal{V}$ is integrable.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form ω is $u - iv$ to within a non-vanishing complex-valued function multiple and the local fields G and H are related to du and dv by

$$du(W,T) = g(W,GT) + (\sigma \wedge v)(W,T), \quad dv(W,T) = g(W,HT) - (\sigma \wedge u)(W,T)$$

where $\sigma(W) = g(\nabla_W U, V)$ [16]. With these conditions M is called a *complex almost contact metric manifold.*

Ishihara and Konishi [15], [16] defined two tensor fields $\mathcal{S}$ and $\mathcal{T}$ given by

$$\begin{aligned} \mathcal{S}(W,T) &= [G,G](W,T) + 2g(W,GT)U - 2g(W,HT)V \\ &\quad +2(v(T)HW - v(W)HT) + \sigma(GT)HW \\ &\quad -\sigma(GW)HT + \sigma(W)GHT - \sigma(T)GHW, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathcal{T}(W,T) &= [H,H](W,T) - 2g(W,GT)U + 2g(W,HT)V \\ &\quad +2(u(T)GW - u(W)GT) + \sigma(HW)GT \\ &\quad -\sigma(HT)GW + \sigma(W)GHT - \sigma(T)GHW. \end{aligned} \tag{2.2}$$

Here $[G,G]$ is the Nijenhuis torsion of G which is defined following:

$$[G,G](W,T) = (\nabla_{GW}G)T - (\nabla_{GT}G)W - G(\nabla_W G)T + G(\nabla_T G)W.$$

According to Ishihara and Konishi's definition M is normal if $\mathcal{S}(W,T) = \mathcal{T}(W,T) = 0$ for arbitrary vector fields W,T on M. Such manifolds are called IK-normal and an IK-normal complex contact manifold is Kähler. The complex Heisenberg group has not Kähler structure. So it is not IK-normal. For this reason Korkmaz [19] gave an extended definition for normality;

Definition 2.2. *A complex almost contact metric manifold M is called normal if the following conditions are satisfied:*

1. $\mathcal{S}(W,T) = \mathcal{T}(W,T) = 0$, *for* $W,T \in \mathcal{H}$,
2. $\mathcal{S}(W,U) = \mathcal{T}(W,V) = 0$, *for* $W \in TM$.

Korkmaz obtained some results on normal complex contact metric manifolds which we list here. For details we refer to reader [1] and [19].

Proposition 2.3. *A complex contact metric manifold is normal if and only if*

$$\begin{aligned} g((\nabla_W G)Z,T) &= \sigma(W)g(HZ,T) + v(W)d\sigma(GT,GZ) - 2v(W)g(HGZ,T) \\ &\quad - u(Z)g(W,T) - v(Z)g(JW,T) + u(T)g(W,Z) \\ &\quad + v(T)g(JW,Z), \end{aligned} \tag{2.3}$$

$$\begin{aligned} g((\nabla_W H)Z,T) &= -\sigma(W)g(GZ,T) - u(W)d\sigma(HT,HZ) - 2u(W)g(HGZ,T) \\ &\quad + u(Z)g(JW,T) - v(Z)g(W,T) - u(T)g(JW,Z) \\ &\quad + v(T)g(W,Z), \end{aligned} \tag{2.4}$$

for arbitrary vector fields W,T on M.

Also from above proposition we have

$$\begin{aligned} g((\nabla_W J)Z,T) &= u(W)(d\sigma(T,GZ) - 2g(HZ,T)) \\ &\quad + v(W)(d\sigma(T,HZ) + 2g(GZ,T)). \end{aligned} \tag{2.5}$$

For W and T horizontal vector fields we have the followings [19];

$$\nabla_W U = -GW + \sigma(W)V, \quad \nabla_W V = -HW - \sigma(W)U, \tag{2.6}$$

$$\nabla_U U = \sigma(U)V, \nabla_U V = -\sigma(U)U, \nabla_V U = \sigma(V)V, \nabla_V V = -\sigma(V)U, \tag{2.7}$$

$$d\sigma(GW,GT) = d\sigma(HW,HT) = d\sigma(T,W) - 2u \wedge v(T,W)d\sigma(U,V), \tag{2.8}$$

$$d\sigma(U,W) = v(W)d\sigma(U,V), \ d\sigma(V,W) = -u(W)d\sigma(U,V). \tag{2.9}$$

$$R(U,V,V,U) = R(V,U,U,V) = -2d\sigma(U,V) \tag{2.10}$$

$$R(W,U)U = W, \quad R(W,V)V = W, \ R(U,V)W = JW \tag{2.11}$$

$$R(W,T)U = 2(g(W,JT) + d\sigma(W,T))V \tag{2.12}$$

$$R(W,T)V = -2(g(W,JT) + d\sigma(W,T))U \tag{2.13}$$

$$R(W,U)V = \sigma(U)GW + (\triangledown_U H)W - JW \tag{2.14}$$

$$R(W,V)U = -\sigma(V)HW + (\nabla_V G)W + JW \tag{2.15}$$

$$R(W,U)T = -g(W,T)U - g(JW,T)V + d\sigma(T,W)V, \tag{2.16}$$

$$R(W,V)T = -g(W,T)V + g(JW,T)U - d\sigma(T,W)U \tag{2.17}$$

$$\begin{aligned} &g(R(GW,GT)GZ,GY) = g(R(W,T)Z,W) - 2g(JZ,Y)d\sigma(W,T) \\ &\quad + 2g(HW,T)d\sigma(GZ,Y) + 2g(JW,T)d\sigma(Z,Y) \\ &\quad - 2g(HZ,W)d\sigma(GY,T), \end{aligned} \tag{2.18}$$

$$\begin{aligned} &g(R(HW,HT)HZ,HY) = g(R(W,T)Z,Y) - 2g(JZ,W)d\sigma(W,T) \\ &\quad - 2g(GW,T)d\sigma(HZ,Y) + 2g(JW,T)d\sigma(Z,Y) \\ &\quad + 2g(GZ,Y)d\sigma(HW,T). \end{aligned} \tag{2.19}$$

On the other hand, in [11] we have

$$d\sigma(W,T) = 2g(JW,T) + g((\triangledown_U J)GW,T). \tag{2.20}$$

For an arbitrary vector field W on M we can write

$$W = W_0 + u(W)U + v(W)V, \quad W_0 \in \mathcal{H}. \tag{2.21}$$

3. Curvature properties and normality

In this section we obtain some results on the Riemannian curvature of normal complex contact metric manifolds. In addition we give a new expression for normality by covariant derivation of G and H structure tensors. Firstly we have some useful results which are listed following.

Theorem 3.1. *Let M be a normal complex contact metric manifold. For X, Y, Z, T horizontal vector fields we have*

$$g(R(GX, GY)GZ, GT) = g(R(HX, HY)HZ, HT) = g(R(X, Y)Z, T). \tag{3.1}$$

Proof. By using (2.20) in (2.18) we have

$$\begin{aligned}
&-2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+ 2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y)\\
&= -2g(JZ, T)\,(2g(JX, Y) + g((\nabla_U J)GX, Y))\\
&+ 2g(HX, Y)\,\big(2g(JGZ, T) + g((\nabla_U J)G^2 Z, T)\big)\\
&+ 2g(JX, Y)\,(2g(JZ, T) + g((\nabla_U J)GZ, T))\\
&- 2g(HZ, T)(2g(JGX, Y) + g((\nabla_U J)G^2 X, Y)).
\end{aligned}$$

Since $JG = -H$ and for X horizontal vector field $G^2X = -X$ we have

$$\begin{aligned}
&-2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+ 2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y))\\
&= -2g(JZ, T)g((\nabla_U J)GX, Y) - 2g(HX, Y)g((\nabla_U J)GZ, T)\\
&+ 2g(JX, Y)g((\nabla_U J)GZ, T) + 2g(HZ, T)g((\nabla_U J)GX, Y)).
\end{aligned}$$

From (2.5) and by simply computation we get

$$\begin{aligned}
&-2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y) = 0.
\end{aligned}$$

Considering (2.18) from last equation we obtain (3.1). By the same way we can easily show that $g(R(HX, HY)HZ, HT) = g(R(X, Y)Z, T)$. So, the proof is completed. □

Curvature identities for normal complex contact metric manifolds were computed by Korkmaz [19] for horizontal vector fields. From (2.21) and by direct computation we obtain the following Lemma which presents the curvature identities for general vector fields.

Lemma 3.2. *Let M be a normal complex contact metric manifold and W, T be two arbitrary vector fields on M. Then the Riemannian curvature of a normal* *complex contact metric manifold satisfies following equations.*

$$
\begin{aligned}
&(3.2) && R(W,U)U = W_0 - 2d\sigma(U,V)v(W)V,\\
& && R(W,V)V = W_0 - 2d\sigma(U,V)u(W)U,\\
&(3.3) && R(W,U)V = \sigma(U)GW_0 + (\nabla_U H)W_0 - JW_0 + 2d\sigma(U,V)v(W)U,\\
&(3.4) && R(W,V)U = -\sigma(V)HW_0 + (\nabla_V G)W_0 + JW_0 + 2d\sigma(U,V)u(W)V,\\
&(3.5) && R(U,V)W = JW_0 + (u(W)V - v(W)U)2d\sigma(U,V),\\
&(3.6) && R(W,T)U = -u(W)T_0 + v(W)(\sigma(V)HT_0 + (\nabla_V G)T_0 + JT_0)\\
& && \quad + u(T)W_0 + v(T)(-\sigma(V)HW_0 + (\nabla_V G)W_0 + JW_0)\\
& && \quad + [2(g(W_0,JT_0) + d\sigma(W_0,T_0)) + 2d\sigma(U,V)u\wedge v(W,T)]V,\\
&(3.7) && R(W,T)V = -u(W)(\sigma(U)GT_0 + (\nabla_U H)T_0 - JT_0) - v(W)T_0\\
& && \quad u(T)(-\sigma(U)GW_0 + (\nabla_U H)W_0 - JW_0) + v(T)W_0\\
& && \quad + [-2(g(W_0,JT_0) + d\sigma(W_0,T_0)) - 2d\sigma(U,V)u\wedge v(W,T)]U,\\
&(3.8) && R(W,U)T = u(T)W_0 - v(W)JT_0 + v(T)(\sigma(U)GW_0 + (\nabla_U H)W_0\\
& && \quad - JW_0) + [-g(W_0,T_0) - 2d\sigma(U,V)v(W)v(T)]U\\
& && \quad + [d\sigma(T_0,W_0) - g(JW_0,T_0) - 2d\sigma(U,V)v(W)u(T))]V,\\
&(3.9) && R(W,V)T = u(W)JT_0 + v(T)W_0 + u(T)(-\sigma(U)HW_0 + (\nabla_V G)W_0\\
& && \quad + JW_0) + [-g(W_0,T_0) + u(W)u(T)2d\sigma(U,V)]V\\
& && \quad [-d\sigma(T_0,W_0) + g(JW_0,T_0) - 2d\sigma(U,V)u(W)v(T)]U.
\end{aligned}
$$

Proposition 3.3. *Let M be a normal complex contact metric manifold . Then for arbitrary vector fields Z and T on M we have*

$$d\sigma(Z,T) = 2g(JZ_0,T_0) + g((\nabla_U J)GZ_0,T_0) + d\sigma(U,V)u\wedge v(Z,T). \tag{3.10}$$

Proof. For vector fields Z and T we have

$$
\begin{aligned}
2d\sigma(Z,T) &= Zg(\nabla_T U,V) - Tg(\nabla_Z U,V) - g\left(\nabla_{[Z,T]}U,V\right)\\
&= g(\nabla_Z\nabla_T U,V) + g(\nabla_T U,\nabla_Z V) - g(\nabla_T\nabla_Z U,V)\\
&\quad -g(\nabla_Z U,\nabla_T V) - g\left(\nabla_{[Z,T]}U,V\right)\\
&= g(R(Z,T)U,V) + g(\nabla_T U,\nabla_Z V) - g(\nabla_Z U,\nabla_T V)
\end{aligned}
$$

from (2.6) and since $HG = -GH = J + u\otimes V - v\otimes U$ we have

$$2d\sigma(Z,T) = g(R(Z,T)U,V) + 2g(JZ,T) + 2u\wedge v(Z,T).$$

In addition from (3.6) we get

$$g(R(Z,T)U,V) = 2(g(Z_0,JT_0) + d\sigma(Z_0,T_0)) + 2d\sigma(U,V)u\wedge v(Z,T)$$

and since $g(JZ,T) = g(JZ_0,T_0) - u\wedge v(Z,T)$ we obtain (3.10). □

Ishihara and Konishi gave an expression of covariant derivative of G and H for IK-normal complex contact metric manifolds. Korkmaz [19] gave a weaker definition for normality of a complex contact metric manifold and obtained Proposition 2.1. In following theorem by using this result we give a new expression for the covariant derivative of G and H such as Ishihara and Konishi's result. Our result is necessary and sufficient condition for normality of complex contact metric manifolds in the sense of Korkmaz's definition.

Theorem 3.4. *A complex contact metric manifold is normal M if and only if*

$$\begin{aligned}(\nabla_W G)T &= \sigma(W)HT - 2v(W)JT - u(T)W \\ &\quad - v(T)JW + v(W)\left(2JT_0 - (\nabla_U J)\,GT_0\right) \\ &\quad + g(W,T)U + g(JW,T)V - d\sigma(U,V)v(W)\left(u(T)V - v(T)U\right),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}(\nabla_W H)T &= -\sigma(W)GT + 2u(W)JT + u(T)JW \\ &\quad - v(T)W + u(W)\left(-2JT_0 - (\nabla_U J)\,GT_0\right) \\ &\quad - g(JW,T)U + g(W,T)V + d\sigma(U,V)u(W)\left(u(T)V - v(T)U\right).\end{aligned} \tag{3.12}$$

Proof. Suppose that M is a normal complex contact metric manifold. Then from (2.3) and (2.4) we have

$$\begin{aligned}g\left((\nabla_W G)T, Z\right) &= g\left(\sigma(W)HT - 2v(W)JT - u(T)W - v(T)JW\right. \\ &\quad \left.+g(W,T)U + g(W,JT)V, Z\right) + v(W)\,d\sigma(GZ,GT).\end{aligned}$$

Since $u \wedge v(T,Z) = g\left(u(T)V - v(T)U, Z\right)$ and from (2.8) we get

$$\begin{aligned}g\left((\nabla_W G)T, Z\right) &= g\left(\sigma(W)HT - 2v(W)JT - u(T)W - v(T)JW\right. \\ &\quad \left.+g(W,T)U + g(W,JT)V, Z\right) \\ &\quad v(W)\left[d\sigma(T,Z) - 2d\sigma(U,V)\,g\left(u(T)V - v(T)U, Z\right)\right].\end{aligned}$$

From (3.10) we can write

$$d\sigma(T,Z) = g(2JT_0 + (\triangledown_U J)\,GT_0 + d\sigma(U,V)\left(u(T)V - v(T)U\right), Z).$$

By using this equation we obtain (3.11). Similarly one can get (3.12).

Conversely suppose that (3.11) and (3.12) hold. For arbitrary vector field W and from (2.1), (2.2) we have

$$\begin{aligned}&\mathcal{S}(W,U) = (\nabla_{GW}G)U - G(\nabla_W G)U + G(\nabla_U G)W - \sigma(U)\,GHW, \\ &\mathcal{T}(W,V) = (\nabla_{HW}H)V - H(\nabla_W H)V + H(\nabla_V H)W - \sigma(V)\,GHW.\end{aligned}$$

From (3.11) and (3.12) we get $S(W,U) = T(W,V) = 0$.

Now let W and T be two horizontal vector fields. Then from (2.1) and (2.2) we have

$$\begin{aligned}\mathcal{S}(W,T) &= (\nabla_{GW}G)T - (\nabla_{GT}G)W - G(\nabla_W G)T + G(\nabla_T G)W \\ &\quad +2g(W,GT)U - 2g(W,HT)V + \sigma(GT)HW \\ &\quad -\sigma(GW)HT + \sigma(W)GHT - \sigma(T)GHW, \\ \mathcal{T}(W,T) &= (\nabla_{HW}H)T - (\nabla_{HT}H)W - H(\nabla_W H)T + H(\nabla_T H)W \\ &\quad -2g(W,GT)U + 2g(W,HT)V + \sigma(HW)GT \\ &\quad -\sigma(HT)GW + \sigma(W)GHT - \sigma(T)GHW.\end{aligned}$$

By applying (3.11) and (3.12) we get

$$\begin{aligned}\mathcal{S}(W,T) &= \sigma(GW)HT - \sigma(GT)HW - 2g(W,GT)U \\ &\quad +2g(W,HT)V - \sigma(W)GHT + \sigma(T)GHW \\ &\quad +2g(W,GT)U - 2g(W,HT)V + \sigma(GT)HW \\ &\quad -\sigma(GW)HT + \sigma(W)GHT - \sigma(T)GHW = 0, \\ \mathcal{T}(W,T) &= -\sigma(HW)GT + \sigma(HT)GW + \sigma(W)HGT \\ &\quad -\sigma(T)HGW + 2g(W,GT)U - 2g(W,HT)V \\ &\quad -2g(W,GT)U + 2g(W,HT)V + \sigma(HW)GT \\ &\quad -\sigma(HT)GW + \sigma(W)GHT - \sigma(T)GHW = 0.\end{aligned}$$

Therefore M is normal. □

By using (2.5), (3.10), (3.11) and (3.12) following corollary is obtained.

Corollary 3.5. *Let M be a normal complex contact metric manifold and W, T be two arbitrary vector fields on M. Then we have*

$$\begin{aligned}(\nabla_W J)T &= -2u(W)HT + 2v(W)GT + u(W)\left(2HT_0 + (\nabla_U J)T_0\right) \\ &\quad +v(W)\left(-2GT_0 + (\nabla_U J)JT_0\right).\end{aligned}$$

4. Ricci semi-symmetric normal complex contact metric manifold

In this section we studied the Ricci curvature of normal complex contact metric manifolds. We obtain some useful results for future works and apply all curvature results to complex Heisenberg group. Finally we examine the Ricci semi-symmetric normal complex contact metric manifolds.

Let us choose a local orthonormal basis of the form $\{E_i, GE_i, HE_i, JE_i, U, V : 1 \leq i \leq n\}$ for a $(2n+1)-$ complex dimensional normal complex contact metric manifold M. Then the Ricci curvature of M has the form

$$\begin{aligned}Ric(W,T) &= \sum_{i=1}^{n}[g(R(E_iW)T, E_i) + g(R(GE_i, W)T, GE_i) \\ &\quad +g(R(HW_i, W)T, HE_i) + g(R(JE_i, W)T, JE_i)] \\ &\quad +g(R(U,W)T,U) + g(R(V,W)T,V).\end{aligned} \tag{4.1}$$

We obtain useful relations for Ricci curvature at the next results.

Lemma 4.1. *Let M be a normal complex contact metric manifold and W, T be horizontal vector fields on M. Then we have*

$$\begin{aligned} &Ric\,(GW, GT) = Ric\,(HW, HT) = Ric\,(W, T) \\ &Ric\,(GW, T) = -Ric\,(W, GT)\,, \quad Ric\,(HW, T) = -Ric\,(W, HT)\,. \end{aligned}$$

Proof. By (4.1) we can write

$$\begin{aligned} Ric(GW, GT) \;=\; &\sum_{i=1}^{n}[g(R(E_iGW)GT, E_i) + g(R(GE_i, GW)GT, GE_i) \\ &+g(R(HE_i, GW)GT, HE_i) + g(R(JE_i, GW)GT, JE_i)] \\ &+g(R(U, GW)GT, U) + g(R(V, GW)GT, V). \end{aligned} \tag{4.2}$$

From (3.1) we have

$$\begin{aligned} &g(R(W_i, GW)GT, E_i) = g(R(GE_iGGW)GGT, GE_i) = (g(R(GE_iW)T, GE_i), \\ &g(R(GE_i, GW)GT, GE_i) = g(R(E_i, W)T, E_i), \\ &g(R(HE_i, GW)GT, HE_i) = g(R(GJE_i, GW)GT, GJE_i) = g(R(JE_i, W)T, JE_i), \\ &g(R(JE_i, GW)GT, JE_i) = g(R(-GHE_i, GW)GT, -GHE_i) \\ &= g(R(HE_i, W)T, HW_i). \end{aligned}$$

From (2.11) we have $g(R(U, GW)GT, U) = g(R(U, W)T, U), g(R(V, GW)GT, V) = g(W, T) = g(R(V, W)T, V)$. Using these equations in (4.2) we get $Ric(GW, GT) = Ric(W, T)$. Similarly, it can be easily show that $Ric(HW, HT) = Ric(W, T)$. In addition from (2.11) , (3.1) and (4.1) we obtain $Ric\,(GW, T) = -Ric\,(W, GT)$. Similarly, one can show that $Ric\,(HW, T) = -Ric\,(W, HT)$. □

Lemma 4.2. *Let M be a normal complex contact metric manifold. For any W horizontal vector filed on M the Ricci curvature tensor satisfies*

$$Ric(W, U) = Ric(W, V) = 0, \tag{4.3}$$

$$Ric(U, U) = Ric(V, V) = 4n - 2d\sigma(U, V), \quad Ric(U, V) = 0. \tag{4.4}$$

Proof. Since E_i, W are horizontal vector fields on M from (2.12) and (2.13) we get

$$\begin{aligned} g(R(E_iW)U, E_i) \;&=\; g(2(g(E_i, JW) + d\sigma(E_i, W))V, E_i) \\ &=\; 2(g(E_i, JW) + d\sigma(E_i, W))g(V, E_i) = 0. \end{aligned}$$

By the same way, we get

$$g(R(GE_i, W)U, GE_i) = g(R(HE_i, W)U, HE_i) = g(R(JE_i, W)U, JE_i) = 0.$$

On the other hand using (2.11),(2.14) and (2.15) we have

$$g(R(U,W)U,U) = -g(R(W,U)U,U) = -g(W,U) = 0,$$
$$g(R(V,W)U,V) = -g(-\sigma(V)HW + (\nabla_V G)W + JW, V) = 0.$$

Thus from (4.1) we get (4.3) and by following same steps we obtain (4.4). □

Also by using curvature properties and from (2.21) we get following corollaries.

Corollary 4.3. *For arbitrary W vector field on a normal complex contact metric manifold M we have*

$$Ric(W,U){=}(4n-2d\sigma(U,V))u(W),\, Ric(W,V){=}(4n-2d\sigma(U,V))v(W). \tag{4.5}$$

Corollary 4.4. *Let M be a normal complex contact metric manifold. Assume that W and T be two arbitrary vector fields on M provided W_0 and T_0 are the horizontal part of W and T, respectively. Then the Ricci curvature tensor satisfies*

$$Ric(W,T) = Ric(W_0,T_0) + (4n-2d\sigma(U,V))(u(W)u(T)+v(W)v(T)). \tag{4.6}$$

Corollary 4.5. *Let M be a normal complex contact metric manifold and W,T be two arbitrary vector fields on M. Then the Ricci curvature tensor satisfies*

$$\begin{aligned} Ric(W,T) &= Ric(GW,GT) + (4n-2d\sigma(U,V))\left(u(W)u(T)+v(W)v(T)\right), \\ Ric(W,T) &= Ric(HW,HT) + (4n-2d\sigma(U,V))\left(u(W)u(T)+v(W)v(T)\right). \end{aligned}$$

Corollary 4.6. *On a normal complex contact metric manifold M, for Q the Ricci operator we have $QG = GQ$, $QH = HQ$.*

The well known example of complex contact metric manifolds is Iwasawa manifold. We compute the Riemann, Ricci and scalar curvatures of Iwasawa manifold.

Example 4.7. The closed subgroup $H_{\mathbb{C}}$ of $GL(3,\mathbb{C})$ is presented by

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} : b_{12}, b_{13}, b_{23} \in \mathbb{C} \right\} \simeq \mathbb{C}^3$$

is called the complex Heisenberg group. Baikoussis, Blair and Gouli-Andreou defined the following complex contact metric structure on $H_{\mathbb{C}}$ in [6]. Let z_1, z_2, z_3 be the coordinates on $H_{\mathbb{C}} \simeq \mathbb{C}^3$ defined by $z_1(B) = b_{23}, z_2(B) = b_{12}, z_3(B) = b_{13}$ for B in $H_{\mathbb{C}}$. Here $H_{\mathbb{C}} \simeq \mathbb{C}^3$ and $\theta = \frac{1}{2}(dz_3 - z_2 dz_1)$ is global, so the structure tensors may be taken globally. With J denoting the standard almost complex structure on $\mathbb{C}^3$, we may give a complex almost contact structure to $H_{\mathbb{C}}$ as follows. Since θ is holomorphic, set $\theta = u + iv$, $v = u \circ J$; also set $4\frac{\partial}{\partial z_3} =$

$U+iV$. Then $u(W)=g(U,W)$ and $v(W)=g(V,W)$. Since we will work in real coordinates, G and H are given by

$$G=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & y_2 & -x_2 & 0 & 0 \end{bmatrix},$$

$$H=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \end{bmatrix}.$$

Then relative to the coordinates $(x_1,y_1,x_2,y_2,x_3,y_3)$ the Hermitian metric (matrix) is given by

$$g=\frac{1}{4}\begin{bmatrix} 1+x_2^2+y_2^2 & 0 & 0 & 0 & x_2 & y_2 \\ 0 & 1+x_2^2+y_2^2 & 0 & 0 & y_2 & x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ x_2 & y_2 & 0 & 0 & 1 & 0 \\ y_2 & x_2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\{e_1;e_1^*;e_2;e_2^*;e_3;e_3^*\}$ be an orthonormal basis where

$$\begin{aligned} &e_1=2\left(\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_3}+y_2\frac{\partial}{\partial y_3}\right),\quad e_1^*=2\left(\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial y_3}\right),\\ &e_2=2\frac{\partial}{\partial x_2},\quad e_2^*=2\frac{\partial}{\partial y_2},\quad e_3=U=2\frac{\partial}{\partial x_3},\quad e_3^*=V=2\frac{\partial}{\partial y_3}. \end{aligned} \tag{4.7}$$

Furthermore we have [21]

$$\begin{aligned} Ge_1 &= -e_2,\ Ge_1^*=e_2^*,\ Ge_2=e_1,\ Ge_2^*=-e_1^*,\\ He_1 &= -e_2^*,\ He_1^*=-e_2,\ He_2=e_1^*,\ He_2^*=-e_1,\\ Je_1 &= -e_1^*,\ Je_1^*=e_1,\ Je_2=-e_2^*,\ Je_2^*=-e_2. \end{aligned}$$

Let $\bigtriangledown$ be the Levi-Civita connection with respect to metric g. Then from (4.7) we have

$$[e_1,e_2]=-2e_3,\ [e_1,e_2^*]=-2e_3^*,\ [e_1^*,e_2]=-2e_3^*,\ [e_1^*,e_2^*]=2e_3 \tag{4.8}$$

and the other Lie brackets are zero [21]. In addition we have

$$2g(\bigtriangledown_{e_i}e_j,e_k)=g\left[e_i,e_j\right],e_k+g\left(\left[e_k,e_i\right],e_j\right)-g\left(\left[e_j,e_k\right],e_i\right)$$

and from that we obtain

$$\nabla_{e_j} e_j = \nabla_{e_j} e_j = \nabla_{e_j} e_{j^*} = \nabla_{e_j^*} e_j^* = 0, \tag{4.9}$$

where $j = 1, 2, 3$. From (4.8) and (4.9) we need only list following

$$\begin{aligned} \nabla_{e_2} e_3 &= \nabla_{e_2^*} e_3^* = -e_1, & \nabla_{e_2^*} e_3 = -\nabla_{e_2} e_3^* = e_1^*, \\ \nabla_{e_1} e_3 &= \nabla_{e_1^*} e_3^* = e_2, & \nabla_{e_1} e_3^* = -\nabla_{e_1^*} e_3 = e_2^*, \\ -\nabla_{e_1} e_2 &= \nabla_{e_1^*} e_2^* = e_3, & \nabla_{e_1} e_2^* = \nabla_{e_1^*} e_2 = -e_3^*. \end{aligned}$$

Now, let

$$\Gamma = \left\{ \left(\begin{array}{ccc} 1 & \gamma_2 & \gamma_3 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{array} \right) \middle| \gamma_k = m_k + i n_k,\ m_k,\ n_k \in \mathbb{Z} \right\}.$$

Γ is subgroup of $H_{\mathbb{C}} \simeq \mathbb{C}^3$, the 1-form $dz_3 - z_2 dz_1$ is invariant under the action on Γ and with $\xi = U \wedge V$, hence the quotient $H_{\mathbb{C}}/\Gamma$ is a compact complex contact manifold with a global complex contact form. $H_{\mathbb{C}}/\Gamma$ is known the *Iwasawa manifold.*

It is known that with the help of the above results , it can be easily verified that

$$\begin{aligned} &R(e_1, e_1^*)e_1 = 0, R(e_1, e_1^*)e_1^* = 0, R(e_1, e_1^*)e_2 = -2e_2^*, R(e_1, e_1^*)e_2^* = 2e_2, \\ &R(e_1, e_2)e_1 = 3e_2, R(e_1, e_2)e_1^* = -e_2^*, R(e_1, e_2)e_2 = -3e_1, R(e_1, e_2)e_2^* = e_1^*, \\ &R(e_1, e_2^*)e_1 = 3e_2, R(e_1, e_2^*)e_1^* = 0, R(e_1, e_2^*)e_2 = -e_2^*, R(e_1, e_2^*)e_2^* = -3e_1, \\ &R(e_1^*, e_2)e_1 = e_2^*, R(e_1^*, e_2)e_1^* = 3e_2, R(e_1^*, e_2)e_2 = -3e_1^*, R(e_1^*, e_2)e_2^* = 3e_1 \\ &R(e_1^*, e_2^*)e_1 = -e_2, R(e_1^*, e_2^*)e_1^* = 3e_2^*, R(e_1^*, e_2^*)e_2 = e_1, R(e_1^*, e_2^*)e_2^* = -3e_1^*, \\ &R(e_2, e_2^*)e_1 = -2e_1^*, R(e_2, e_2^*)e_1^* = 2e_1, R(e_2, e_2^*)e_2 = 0, R(e_2, e_2^*)e_2^* = 0\ . \end{aligned}$$

From (2.7) and since $\sigma = 0$ [19] $R(e_3, e_3^*)e_3^* = 0$ and we have $R(W, U)U = W$ and $R(W, V)V = W$ for $W \in \mathcal{H}$. Similarly from (2.14) and (2.15) we get

$$\begin{aligned} &R(e_1, e_3)e_3^* = -e_1^*, R(e_1^*, e_3^*)e_3 = 3e_1, R(e_2, e_3)e_3^* = -e_2^*, R(e_2^*, e_3^*)e_3 = e_2, \\ &R(e_1, e_3^*)e_3 = e_1^*, R(e_1^*, e_3^*)e_3 = -e_1, R(e_2, e_3^*)e_3 = e_2^*, R(e_2^*, e_3^*)e_3 = -3e_2, \end{aligned}$$

and from (2.12), (2.13) we get

$$\begin{aligned} R(e_1, e_1^*)e_3 = R(e_1, e_2)e_3 = R(e_1, e_2^*)e_3 = 0 \\ R(e_1^*, e_2)e_3 = R(e_1^*, e_2^*)e_3 = R(e_2, e_2^*)e_3 = 0 \end{aligned}$$

and

$$\begin{aligned} R(e_1, e_1^*)e_3^* = R(e_1, e_2)e_3^* = R(e_1, e_2^*)e_3^* = 0 \\ R(e_1^*, e_2)e_3^* = R(e_1^*, e_2^*)e_3^* = R(e_2, e_2^*)e_3^* = 0 \quad . \end{aligned}$$

Using these equations and from (4.3) and (4.4) the Ricci curvature is obtained

$$Ric\,(e_i, e_i) = Ric\,(e_i^*, e_i^*) = 4,\ i = 1,2 \ \text{ ve } \ Ric\,(e_3, e_3) = Ric\,(e_3^*, e_3^*)$$
$$Ric\,(e_i, e_j) = Ric\left(e_i^*, e_j^*\right) = 0,\ j = 1,2,3.$$

By direct computation the scalar curvature of Iwasava manifold is obtained as $\tau = -8$. Furthermore from curvature equalities the sectional curvature is

$$\begin{aligned} k(e_1, e_3) &= k(e_1^*, e_3) = k(e_2, e_3) = k(e_2^*, e_3) = 1, \\ k(e_1, e_3^*) &= k(e_1^*, e_3^*) = k(e_2, e_3^*) = k(e_2^*, e_3^*) = 1 \end{aligned}$$

and since $\sigma = 0$ we get $k(e_3, e_3^*) = 0$. In addition can be easily verified that

$$\begin{aligned} k(e_1, e_1^*) &= k(e_1, e_2^*) = k(e_1^*, e_2) = k(e_2, e_2^*) = 0 \\ k(e_1, e_2) &= 3 \text{ and } k(e_1^*, e_2^*) = 1. \end{aligned}$$

Definition 4.8. *A complex contact metric manifold M is said to be Ricci semi-symmetric if it satisfies the condition*

$$R(X,Y).Ric = 0 \tag{4.10}$$

for all X, Y vector fields on M.

Theorem 4.9. *A Ricci semi-symmetric normal complex contact metric manifold is Einstein.*

Proof. Let us consider a semi-symmetric complex contact metric manifold. Then for arbitrary vector fields X, Y, W and T on M and from (4.10) we have

$$Ric(R(X,Y)W, T) + Ric(W, R(X,Y)T) = 0. \tag{4.11}$$

By setting $Y = W = U$ and $X = X_0$, $T = T_0$, $X_0,\ T_0 \in \mathcal{H}$ in (4.11) we have

$$Ric(R(X_0,U)U, T_0) + Ric(U, R(X_0,U)T_0) = 0.$$

From (480) and (2.16) we get

$$Ric(X_0, T_0) + Ric(U, -g(X_0,T_0)U - g(JX_0,T_0)V + d\sigma(T_0,X_0)V) = 0$$

and

$$Ric(X_0,T_0) - g(X_0,T_0)Ric(U,U) - g(JX_0,T_0)Ric(U,V) + d\sigma(T_0,X_0)Ric(U,V) = 0.$$

From (4.4) we obtain

$$Ric(X_0, T_0) = (4n - 2d\sigma(U,V))\, g(X_0, T_0). \tag{4.12}$$

By similar way taking $Y = W = V$ and $X = X_0,\ T = T_0$, $X_0,\ T_0 \in \mathcal{H}$ in (4.11) and from (2.11) and (2.17) we have

$$Ric(X_0,T_0) - g(X_0,T_0)Ric(V,V) + g(JX_0,T_0)Ric(U,V) - d\sigma(T_0,X_0)Ric(U,V) = 0.$$

By using (4.4) we get (4.12). So, the manifold is Einstein. $\square$

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Accepted: 17.10.2018

The path graph of the amalgamated graph of C_3 and C_n at an edge or at a vertex

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Abstract. Path graphs were proposed as a generalization of line graphs. The 2-path graph denoted by $P_2(G)$, of a graph G has vertex set the set of all paths of length two. Two such vertices are adjacent in the new graph if their union is a path of length three or a cycle of length three. In this paper we will introduce the path graph of the amalgamated graph of C_3 and C_n at an edge and at a vertex. Also, some new properties of these graphs will be given such as the independence number, domination number and matching number.

Keywords: path graphs, amalgamated graph, independence number, domination number, matching number.

1. Introduction

For any graph G, as a generalization of the line graph Broesma and Hoede, see [3], defind the k-path graphs of G denoted by $P_k(G)$. They studied some properties of these graphs.

Definition 1.1. *The k-path graph of a graph G denoted by $P_k(G)$ has a vertex set the set of all paths of length k in G. Two such vertices are adjacent in $P_k(G)$ if their union is a path or a cycle of length $k + 1$.*

In this paper, we will focus our study on the graph $P_2(G)$.

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A characterization of 2-path graphs has been given by Broesma, Hoede and by Huaien Li, see [3] and [4]. Later on, Prisner gave a new characterization of k-path graphs, see [7]. Diameters, centers and distance in path graphs were studied in [2], [5] and [6]. Isomorphisms of path graphs were studied in [1] and [8]. Paths of length 2 in G as well as vertices of $P_2(G)$ will be represented by triples abc, where b is the middle vertex of the path of length 2 in G from a to c and $abc = cba$.

The following two examples explain the definition of path graphs, see [3].

Example 1.1. Let G be the graph obtained from $K_{1,3}$ by subdividing all of its edges once, this graph is denoted by $S(K_{1,3})$. Observe that $P_2(S(K_{1,3})) = C_6$.

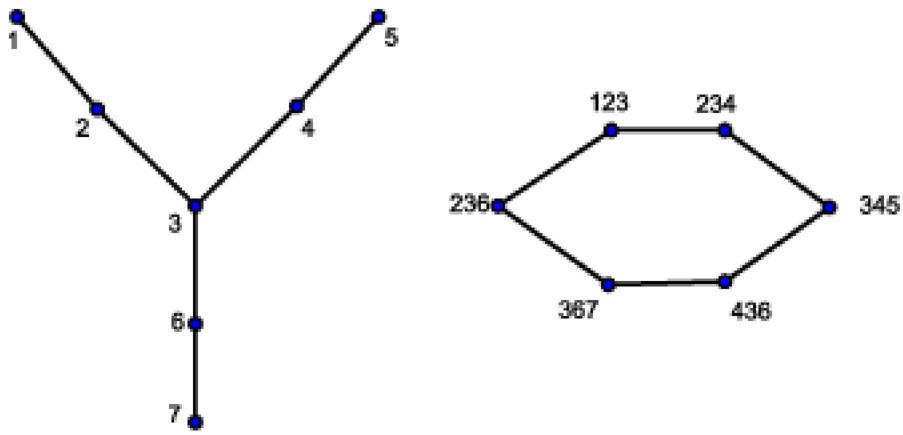

Figure 1. The graph $S(K_{1,3})$ and $P_2(S(K_{1,3}))$

Example 1.2. The graph $S(K_{1,3}) - s$, where s is an end vertex, is denoted by Y. Observe that $P_2(Y) = P_5$. Figure 2 shows the graph Y and $P_2(Y)$.

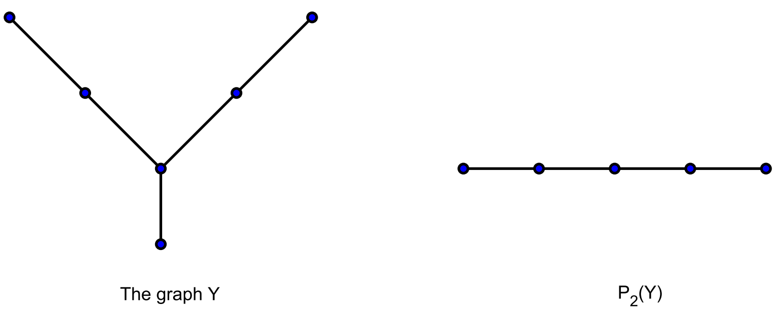

Figure 2. The graph Y and $P_2(Y)$.

Definition 1.2. *Any two graphs G and H are said to be amalgamated at an edge or at a vertex if G and H have exactly one edge or one vertex in common respectively.*

We need the following result about path graphs, see [3].

Theorem 1.1. *For a vertex abc of* $P_2(G)$, *deg(abc) = deg(a) + deg(c) -2. Note that deg(a) and deg(c) are degrees in G, wheareas deg(abc) is the degree of the vertex abc of* $P_2(G)$.

2. The path graph of the amalgamated graph of C_3 and C_n at an edge

First, we give some examples to show how the path graph of the amalgamated graph of C_3 and C_i at an edge look like, for i=3, 4, 5 . Then from these examples we will deduce the path graph of the amalgamated graph of C_3 and C_n at an edge and some of its properties for any n.

Example 2.1. Consider the graph G_1, the amalgamated graph of C_3 and C_3 at an edge. We represent this graph and its path graph in Figure 3.

We have $V(P_2(G_1)) = 1, 2, 3, ..., 8$ and $E(P_2(G_1))=\{e_1, e_2, e_3, ..., e_{12}\}$.

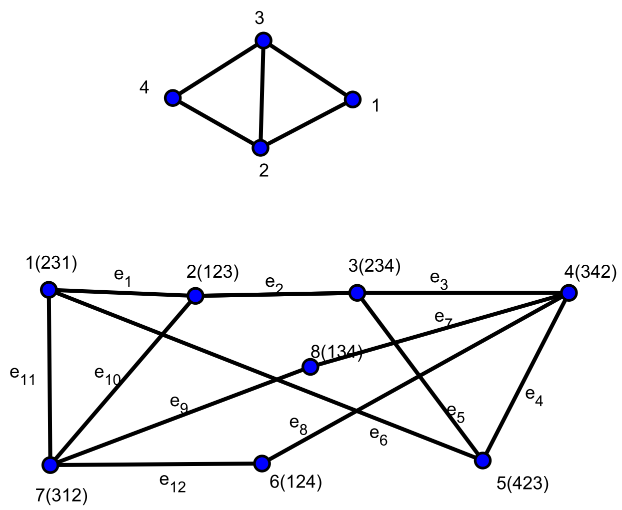

Figure 3. G_1 and $P_2(G_1)$.

Notice that $|V(G_1)|=4$ and $|V(P_2(G_1)|=8$. Also we can see that $|E(P_2(G_1)|=$ 12.

To find the domination number of the path graph of G_1. Observe that S=\{4, 7\} is a minimum dominating set and thus $\gamma(P_2(G_1))=2$.

To find the independent number, observe that the sets X={1, 3, 6, 8}, Y={2, 4} and Z= {5, 7} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_1))=|X|=4$.

In the graph $P_2(G_1)$, we have the following maximal matching sets, A={e_1, e_3, e_9}, B={e_2, e_6, e_7, e_{12}}, C={e_5, e_8, e_{10}} and D={e_2, e_4, e_{11}}. Observe that the matching number of $P_2(G_1)$ equals 4.

Notice that $P_2(G_1)$ has a Hamiltonion path which is 6, 7, 8, 4, 3, 2, 1, 5.

Example 2.2. Consider the graph G_2, the amalgamated graph of C_3 and C_4 at an edge. We represent this graph and its path graph in Figure 4. Observe that $V(P_2(G_2))$={ 1, 2, 3, ..., 8, 9} and $E(P_2(G_2))$={e_1, e_2, e_3, ..., e_{12}, e_{13}}.

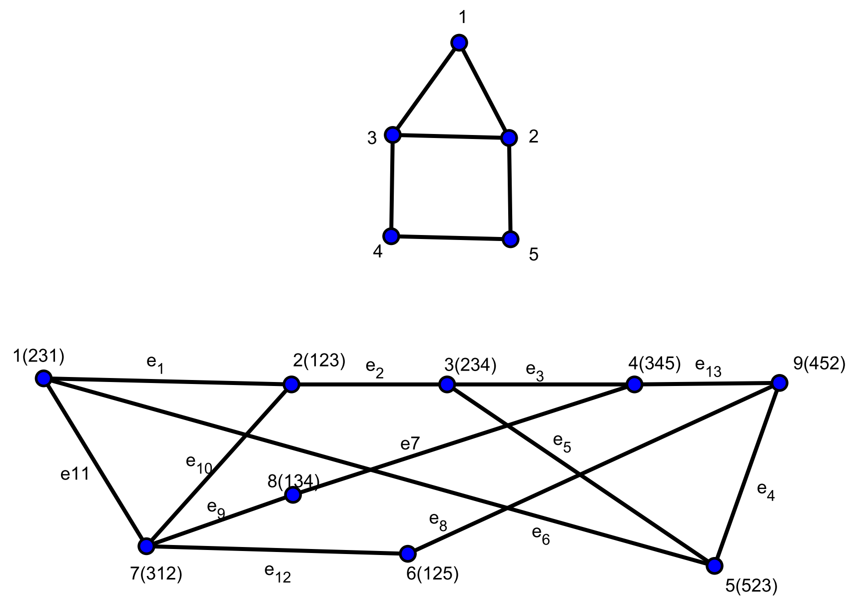

Figure 4. G_2 and $P_2(G_2)$.

Notice that $|V(G_2)|=5$ and $|V(P_2(G_2))|=9$. Also we can see that $|E(P_2(G_2))|=$ 13.

One can easily check that the set {4, 5, 7} is a minimum dominating set and hence $\gamma(P_2(G_2))=3$.

The sets X={1, 3, 8, 9}, Y={2, 4, 5, 6} and Z= {3, 7, 9} are maximal independent sets. Observe that X is a maximum independent set and so $\alpha(P_2(G_1))=|X|=4$.

In the graph $P_2(G_2)$, we have the following maximal matching sets, A={e_1, e_3, e_4, e_9}, B={e_4, e_7, e_{10}}, C={e_2, e_6, e_{12}, e_{13}} and D={e_5, e_7, e_8, e_{11}}. Observe that the matching number equals 4.

Notice that $P_2(G_2)$ has a Hamiltonion path which is 6, 9, 5, 3, 2, 1, 7, 8, 4.

Example 2.3. Consider the graph G_3, *the amalgamated graph of C_3 and C_5 at an edge. We represent this graph and its path graph in Figure* 5.

We have V(P_2(G_3)) = {1, 2, 3, ..., 14} and E(P_2(G_3)) = {e_1, e_2, e_3, ..., e_{14}}.

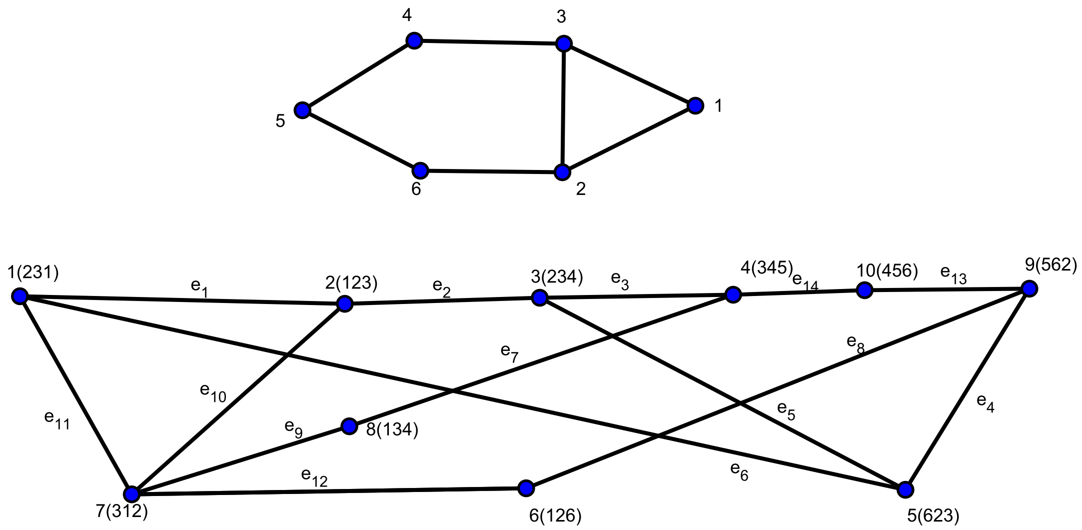

Figure 5. G_3 and $P_2(G_3)$.

Notice that |V(G_3)|=6 and |V(P_2(G_3))|=10. Also we can see that |E(P_2(G_3))|= 14.

One can check that S={4, 5, 7, 10} is a minimum dominating set and thus γ(P_2(G_3))=4.

Observe that the sets X={1, 3, 6, 8, 10}, Y={2, 4, 9} and Z= {4, 5, 7} are maximal independent sets. The set X is a maximum independent set. So, α(P_2(G_3))=|X|=5.

In the graph P_2(G_3), we have the following maximal matching sets, A={e_1, e_3, e_4, e_9}, B={e_2, e_6, e_7, e_{12}, e_{13}}, C={e_5, e_8, e_{11}, e_{14}} and D={e_4, e_{10}, e_{14}}. Observe that the matching number of P_2(G_3) equals 5.

Notice that P_2(G_3) has a Hamiltonion path which is 6, 9, 5, 3, 2, 1, 7, 8, 4, 10.

We follow the same way as given in the last three examples to get the general form of the path graph of the amalgamated graph of C_3 and C_n. Now we give the following theorem that gives the number of edges and vertices of the path graph of the amalgamated graph of C_3 and C_n at an edge. This graph shown in Figure 6.

Theorem 2.1. *Let G be the amalgamated graph of C_3 and C_n at an edge and $|V(G)| = n + 1$. Then $|V(P_2(G)| = n + 1 + 4$ and $|E(P_2(G) | = n + 1 + 8$.*

Proof. The graph G has exactly two vertices of degree three each one of them give rise to three vertices of P_2(G). The remaining $(n + 1 - 2)$ vertices of G are of degree 2. Each one of these gives rise to one vertex of P_2 (G). Thus $|V(P_2(G)|=6+n+1-2=n+1+4$.

The graph G has only one vertex of degree two that is adjacent to two vertices of degree three. This gives two edges in P_2(G). The graph G has two vertices of degree three and each one of these two vertices is adjacent to two vertices of degree two. Each one of these vertices contributes four to E(P_2(G)). There are two vertices of degree two and each one of them is adjacent to a vertex of degree three and a vertex of degree two.This contributes three to the edges of P_2(G). The remaining (n+1 − 5) vertices of G are of degree two and each one of them is adjacent to two vertices of degree 2. Thus every vertex of the remaining (n+1 − 5) vertices increase $|E(P_2(G))|$ by one. Hence $|E(P_2(G))| = 2+4+4+3+(n+1-5) = n+1+8$. □

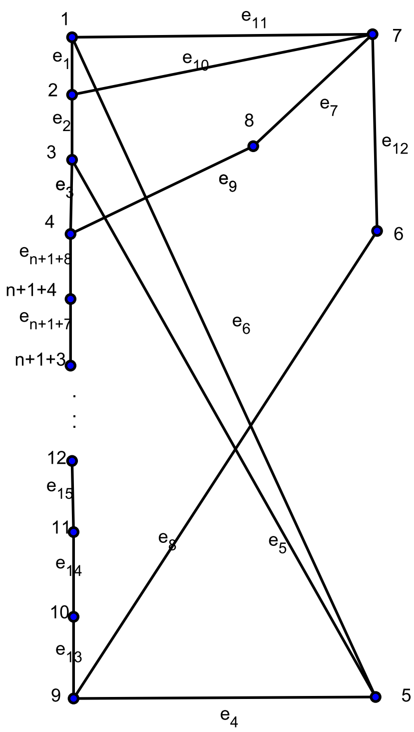

Figure 6. The path graph of the amalgamated graph of C_3 and C_n at an edge.

Let G be the amalgamated graph of C_3 and C_n for n≥6.

From the sketch of the graph P_2(G), we can get the following results

1. A minimum dominating set of P_2(G) is S ={4, 5, 7, 10, 12, 14, ..., n+1+3} if n+1 is odd, and S={ 4, 5, 7, 11, 13, 15, 17, ..., n+1+3} if n+1 is even. Hence $\gamma(G)=|S| = \lfloor \frac{|V(P_2(G))|}{2} \rfloor - 2$.

2. P_2(G) has a Hamiltonion path which is 10, 11, 12, ..., n+1+4, 4, 8, 7, 1, 2, 3, 5, 9, 6.

3. To find the independence number of $P_2(G)$, we have two cases to consider. If n+1 is even, then we have the following maximal Independent sets, X={1, 3, 6, 8, 10, 12, 14, ..., n+1+2, n+1+4}, Y={2, 4, 9, 11, 13, ..., n+1-2, n, n+1+1, n+1+3} and Z={4, 5, 7, 10, 12 14, 16, ..., n+1, n+1+2}. The set X is a maximum independent set. So,

$$\alpha(P_2(G)) = |X| = \lfloor \frac{V(P_2(G))}{2} \rfloor$$

If n+1 is odd, then we have the following maximal independent sets, X={1, 3, 8, 9, 11, 13, ..., n+1+2, n+1+4}, Y={2, 4, 5, 6, 10, 12, 14, ..., n, n+1+1, n+1+3} and Z={4, 5, 7, 10, 12, 14, 16, ..., n+1+1, n+1+3}. The set Y is a maximum independent set. So,

$$\alpha(P_2(G)) = |Y| = \lfloor \frac{|V(P_2(G))|}{2} \rfloor.$$

4. In the graph $P_2(G)$, if n+1 is even we have the following maximal matching sets, A={e_1, e_3, e_{12}, e_{13}, e_{15}, e_{17}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}, B={e_2, e_4, e_{11}, e_{14}, e_{16}, e_{18}, ..., e_{n+1+2}, e_{n+1+4}, e_{n+1+6}, e_{n+1+8}}, C={e_2,e_6,e_7, e_8, e_{14}, e_{16}, e_{18}, ..., e_{n+1+4}, e_{n+1+6}, e_{n+1+8}} and D={e_1, e_5, e_9, e_{13}, e_{15}, e_{17}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}.

If n+1 is odd, then we have the following maximal matching sets, A={e_1, e_3, e_{12}, e_{13}, e_{15}, e_{17}, ..., e_{n+1+4}, e_{n+1+6}}, B={ e_2, e_4, e_9, e_{11}, e_{14}, e_{16}, e_{18}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}, C={e_2, e_6, e_7, e_8, e_{14}, e_{16}, e_{18}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}} and D={e_1, e_5, e_9, e_{13}, e_{15}, e_{17}, ..., e_{n+1+2}, e_{n+1+4}, e_{n+1+6}}.

Observe that the matching number of $P_2(G)$ equals $\lceil\frac{|E(P_2(G))|-10}{2}\rceil$+3.

3. The path graph of the amalgamated graph of C_3 and C_n at a vertex

In this section, we will introduce the path graph of the amalgamated graph of C_3 and C_n at a vertex. Then some properties of this graph will be studied.

First we give some examples to show how the path graph of the amalgamated graph of C_3 and C_i at a vertex look like, for i=3, 4 and 5.

Example 3.1. Consider the graph G_3^*, the amalgamated graph of C_3 and C_3 at a vertex. This graph and its path graph are represented in Figure 7.

We have $V(P_2(G_3^*))$ = {a, b, c, d, e, f, g, h, i, k} and $E(P_2(G_3^*))$ ={e_1, e_2, e_3, ..., e_{13}}.

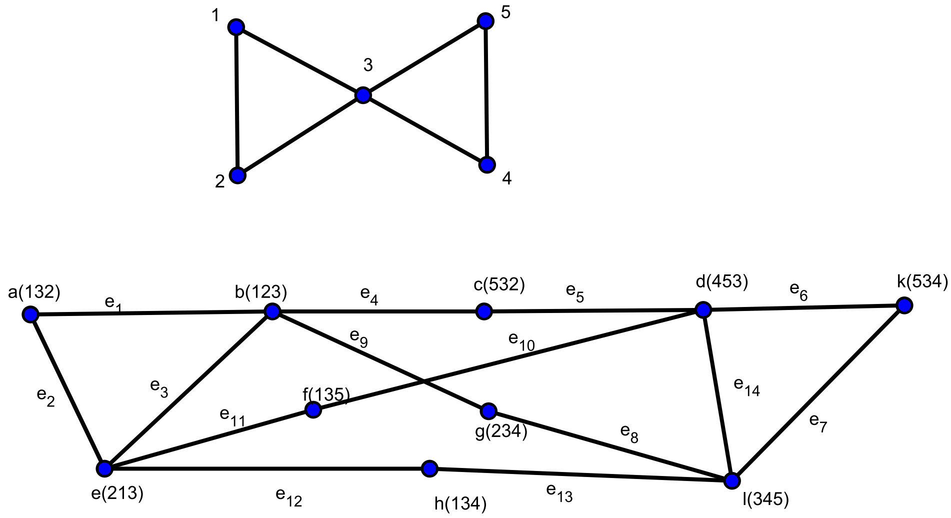

Figure 7. The graph of G_3^* and $P_2(G_3^*)$ at an edge.

Notice that $|V(G_3^*)|=5$ and $|V(P_2(G_3^*)|=10$. Also we can see that $|E(P_2(G_3^*)|=14$.

The set S={e, l, d} is a minimum dominating set of $P_2(G_3^*)$ and thus $\gamma(P_2(G_3^*))=3$.

Observe that the sets X={a, c, k, g, f, h}, Y={b, d, h} and Z= {e, l, c} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_3^*))=|X|=6$.

In the graph $P_2(G_3^*)$, we have the following maximal matching sets, A={e_1, e_5, e_7, e_{11}}, B={ e_2, e_4, e_6, e_8}, C={e_5, e_7, e_9, e_{12}} and D={e_3, e_{10}, e_{13}}. Observe that the matching number of $P_2(G_3^*)$ equals 4.

Example 3.2. Consider the graph G_4^*, the amalgamated graph of $C_3$and C_4 *at a vertex. This graph and its path graph are represented in Figure* 8.

We have $V(P_2(G_4^*))$ = {a, b, c, d, e, f, g, h, i, k, i_1} and $E(P_2(G_4^*))$ ={e_1, e_2, e_3, ...,e_{13}, $\acute{e}_1$, $\acute{e}_2$}.

Notice that $|V(G_4^*)|=6$ and $|V(P_2(G_4^*)|=11$. Also we can see that $|E(P_2(G_4^*)|=15$.

The set S={e, l, d} is a minimum dominating set of $P_2(G_4^*)$ and thus $\gamma(P_2(G_4^*))=3$.

Observe that the sets X={a, c, k, g, f, h, i_1}, Y={b, d, h} and Z= {e, l, c} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_4^*))=|X|=7$.

In the graph $P_2(G_4^*)$ we have the following maximal matching sets, A={e_1, e_5, e_7, e_{11}}, B={ e_2, e_4, e_6, e_8}, C={e_5, e_7, e_9, e_{12}} and D={e_3, e_{10}, e_{13}}. Observe that the matching number of $P_2(G_4^*)$ equals 4.

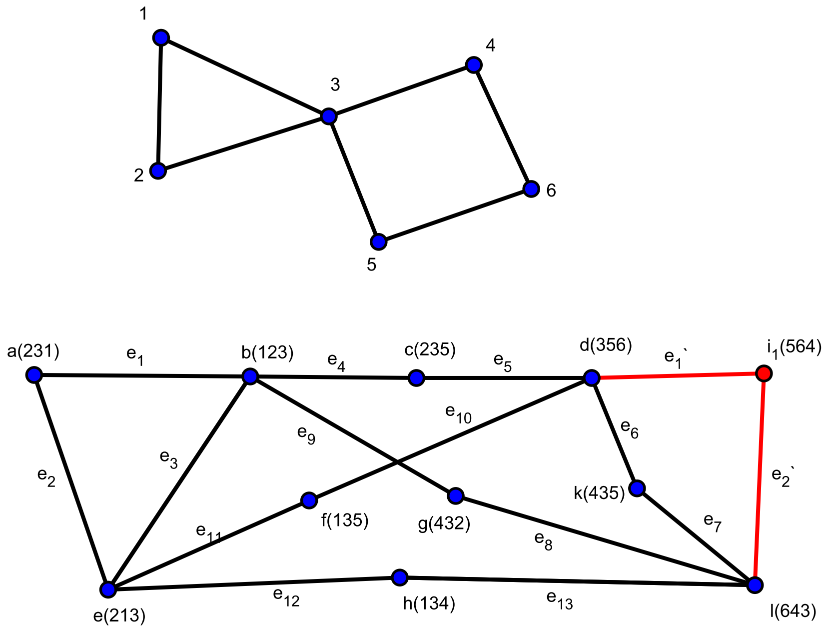

Figure 8. The graph of G_4^* and $P_2(G_4^*)$ at an edge.

Example 3.3. Consider the graph G_5^*, the amalgamated graph of $C_3$and C_5 at a vertex. We represent this graph and its path graph in Figure 9.

We have $V(P_2(G_5^*))$ = {a, b, c, d, e, f, g, h, i, k, i_1, i_2} and $E(P_2(G_5^*))$ ={e_1, e_2, e_3, ...,e_{13}, $\acute{e}_1$, $\acute{e}_2$, $\acute{e}_3$}.

Notice that $|V(G_5^*)|$=7 and $|V(P_2(G_5^*)|$=12. Also we can see that $|E(P_2(G_5^*)|$= 16.

The set S={ e, l, d} is a minimum dominating set of $P_2(G_5^*)$ and thus $\gamma(P_2(G_5^*))$=3.

Observe that the sets X={a, c, k, g, f, h, i_1}, Y={b, d, h, i_2} and Z= {e, l, c, i_1} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_5^*))$=|X|=7.

In the graph $P_2(G_5^*)$ we have the following maximal matching sets, A={e_1, $\acute{e}_2$, e_5, e_7, e_{11}}, B={ e_2, $\acute{e}_2$, e_4, e_6, e_8}, C={$\acute{e}_2$, e_5, e_7, e_9, e_{12}} and D={$\acute{e}_2$, e_3, e_{10}, e_{13}}. Observe that matching number of $P_2(G_5^*)$ equals 5.

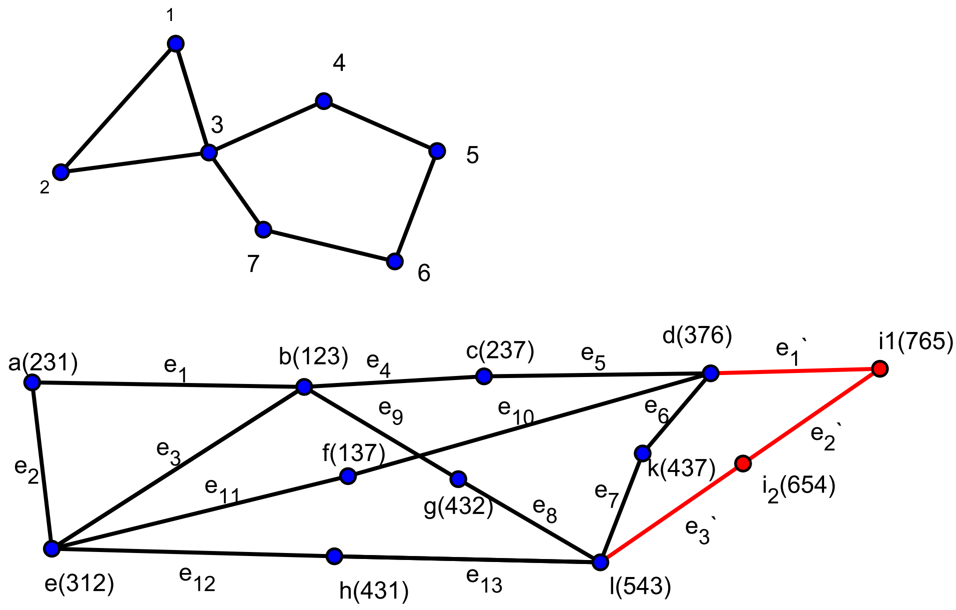

Figure 9. The graph of G_5^* and $P_2(G_5^*)$ at an edge.

We follow the same way as the last three examples to get the general form of the amalgamated graph of C_3 and C_n at a vertex. We denote the amalgamated graph of C_3 and C_n at a vertex by G_n^*. This graph is shown in Figure 10.

Notice that if $|V(G_n^*)|$=n+2 then, $|V(P_2(G_n^*)|$=n+7, also we can see that for all n≥4, $|E(P_2(G_i^*)|= n+11$.

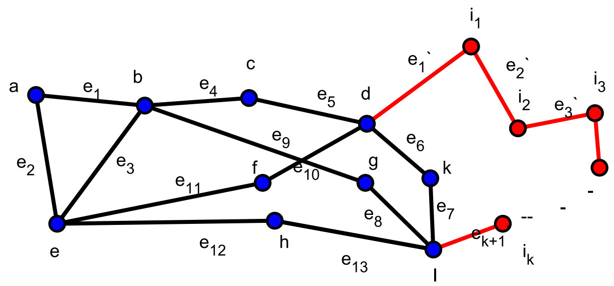

Figure 10. The path graph of the amalgamated graph of C_3 and C_n at a vertex.

From the sketch of the graph $P_2(G_n^*)$, we can get the following results

A minimum dominating set of $P_2(G_n^*)$ is S={e, l, d}∪{i_3, i_5, i_7, ..., i_{k-3}, i_{k-1}}. Hence $\gamma(P_2(G_n^*))=3+\gamma(P_{k-2})=3+\lceil \frac{k-2}{3} \rceil$, ($P_{k-2}$ is the path i_2, i_3, ..., i_{k-1}).

Now, we want to find independent number of the graph $P_2(G_n{}^*)$. First denote the path $i_1,i_2,i_3,...,i_k$ by P_k. Observe that if k is odd, then the sets X={a, c, k, g, f, h, i_1, i_3,..., i_k}, Y={b, d, h, i_2, i_4,...,i_{k-1}} and Z={e, l, c, i_1, i_3, ...,i_{k-2}} are maximal independent sets. The set X is a maximum independent set. If k is even, then the sets X={a, c, k, g, f, h, i_1, i_3,..., i_{k-1}}, Y={b, d, h, i_2, i_4,...,i_k} and Z={e, l, c, i_1, i_3, ..., i_{k-1}} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_n^*))=6+\alpha(P_k)=6+\lceil \frac{K}{2} \rceil$.

In the graph $P_2(G_n^*)$ if k is even we have the following maximal matching sets, A={e_1, e_5, e_{11}, e_7, $é_2$, $é_4$,....,$é_k$}, B={ e_4, e_2, e_8, e_6,$é_2$, $é_4$,....,$é_k$}, C={e_9, e_{12}, e_5, e_7, $é_2$, $é_4$, ..., $é_k$} and D={e_3, e_{13}, e_{10}, $é_2$, $é_4$, ...$é_k$}. If k is odd, then $P_2(G_n^*)$ has the following maximal matching sets, A={e_1, e_5, e_{11}, e_7, $é_2$, $é_4$,...., $é_{k-1}$}, B={ e_4, e_2, e_8, e_6,$é_2$, $é_4$,....,$é_{k-1}$}, C={e_9, e_{12}, e_5, e_7, $é_2$, $é_4$, ..., $é_{k-1}$} and D={e_3, e_{13}, e_{10}, $é_2$, $é_4$, ...$é_{k-1\}}$. Observe that the matching number of $P_2(G_n^*)$=|A|=4+$\lceil \frac{k-1}{3} \rceil$.

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Accepted: 10.11.2018

A characterization of some alternating group by its order and special conjugacy class sizes

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Abstract. Let G be a group and $N(G)$ be the set of the sizes of conjugacy class of G. Let $m_p(G)$ be the number from $N(G)$ which is not divisible by p and let A_n be the alternating group of degree n. The alternating groups A_5, A_6, A_7, A_8, and A_9 are characterized by their orders and special conjugacy class sizes. So in generality, are the alternating groups characterized by their orders and some special conjugacy class size(s)? In this paper, we show that G is a finite group such that $m_p(G) = m_p(A_n)$ and $m_2(G) = m_2(A_n)$ where $n \in \{p, p+1, p+2\}$, then G is isomorphic to A_n.

Keywords: element order, alternating group, Thompson's problem, conjugacy classes sizes.

1. Introduction

All groups in this paper are finite, and simple groups are non-abelian. For a group G, let $\pi(G)$ denote the set of prime divisors of $|G|$. The prime graph of G is a graph $GK(G)$ with vertex set $\pi(G)$ and two distinct vertices q and q are adjacent by an edge if G has an element of order pq. We denote by $s(G)$ the number of connected components of $GK(G)$. Let $\pi_i = \pi_i(G)$, $i = 1, 2, \cdots, s(G)$, be the connected components of $GK(G)$. For an even order group, let $2 \in \pi_1(G)$. Then $|G|$ can be expressed as a product of $m_1, m_2, \cdots, m_{s(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. These m_i's are called the order components of G. Obviously, m_i's are odd components of G with $i \geq 2$. Using [14] and [23], we list the order components for non-abelian finite simple groups L in Tables 1, 2 and 3. This information is used to prove our main theorem. Let A_n

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be the alternating group of degree n. For alternating group A_p with p prime, $p \in \{m_i, i \geq 2\}$. In 1987, J. G. Thompson put forward the following conjecture.

Conjecture 1 ([19, Problem 12.38])**.** *Let G be a group with trivial center. If L is a simple group satisfying that $N(G) = N(L)$, then $G \cong L$.*

Some authors proved that Thompson's conjecture is valid for groups: $L_n(q)$ [1, 5, 13], $D_n(q)$ [2], ${}^2D_n(q)$ [3], $E_7(q)$ [25], A_{p+3} [18], for simple groups with $s(G) \geq 2$ [7, 8], A_{p+4} [27], all almost sporadic simple groups [21], A_{10} [12], A_{22} [24], A_{26} [17]. Recently, G. Chen and J. Li contributed their interests on the Thompson's conjecture under a weak condition. They successfully characterized some sporadic simple groups and simple K_3-groups (A finite simple group G is called a simple K_n-group if G is simple and $n = |\pi(G)|$) in [16]. Chen et al in [26] showed that simple K_4-groups are also characterized by its order and one special conjugacy class size. For convenience, we denote by $m_p(G)$ the p'-number from $N(G)$. As the development of this topic, we will prove the following.

Main Theorem 1.2. Let G be a finite group and let $n \in \{p, p+1, p+2\}$ where $5 \leq p$ is a prime. Then $G \cong A_n$ if and only if $|G| = |A_n|$, $m_p(G) = m_p(A_n)$ and $m_2(G) = m_2(A_n)$.

We introduce some notation which will be needed in the proof of the main theorem. Let $a \cdot b$ denote the products of an integer a by an integer b. Let G be a group and r a prime. Then we denote the number of the Sylow r-subgroup G_r of G by $n_r(G)$ or n_r. Let S_n be the symmetric group of degree n. Let $\omega(G)$ be the set of element orders of G. Let $x \in \omega(G)$. Let x^G denote the conjugacy classes of G containing x. The other symbols are standard (see [9], for instance).

2. Some preliminary results

In this section, we give some lemmas used to prove the main theorem.

Let $\exp(n, r) = a$ denote that $r^a \mid n$ but $r^{a+1} \nmid n$.

Lemma 1. *Let A_{p+k} be an alternating group of degree $p+k$ where $p+i$ is composite, $i = 1, \cdots, k$, and p is a prime. Then the following hold.*

(1) $\exp(|A_{p+k}|, 2) = \sum\limits_{i=1}^{\infty}[\frac{p+k}{2^i}] - 1$. *In particular,* $\exp(|A_{p+k}|, 2) \leq p+k-1$.

(2) $\exp(|A_{p+k}|, r) = \sum\limits_{i=1}^{\infty}[\frac{p+k}{r^i}]$ *for each* $r \in \pi(A_{p+k})\backslash\{2\}$. *Furthermore,* $\exp(|A_{p+k}|, r) < \frac{p+k}{2}$, *where* $3 \leq r \in \pi(A_{p+k})$. *In particular, if* $r > [\frac{p+k}{2}]$, *then* $\exp(|A_{p+k}|, r) = 1$.

Proof. (1) By the definition of Gaussian integer function, we have that

$$
\begin{aligned}
\exp(|A_{p+k}|, 2) &= \sum_{i=1}^{\infty}[\frac{p+k}{2^i}] - 1 \\
&= ([\frac{p+k}{2}] + [\frac{p+k}{2^2}] + [\frac{p+k}{2^3}] + \cdots) - 1 \\
&\leq (\frac{p+k}{2} + \frac{p+k}{2^2} + \frac{p+k}{2^3} + \cdots) - 1 \\
&= (p+k)(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots) - 1 \\
&= p+k-1.
\end{aligned}
$$

(2) Similarly as (1), we have that

$$
\begin{aligned}
\exp(|A_{p+k}|, r) &\leq (p+k)(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots) \\
&= \frac{p+k}{r-1} \\
&\leq \frac{p+k}{2}
\end{aligned}
$$

for an odd prime $r \in \pi(A_{p+k})$. If $r > [\frac{p+k}{2}]$, $\exp(|A_{p+k}|, r) = 1$.

The proof is complete. □

Let G be a group whose order is divisible by prime p. The group G is called a C_{pp}-group if the centralizers of a p-element are p-groups (see [4], for instance).

Lemma 2. *Let A_n be an alternating group of degree n, where $n = p, p+1, p+2$. Then the following hold.*

(1) $m_p(A_n) = \begin{cases} \dfrac{(p-1)!}{2}, & n = p; \\ \dfrac{(p-1)! \cdot (p+1)}{2}, & n = p+1; \\ \dfrac{(p-1)! \cdot (p+1) \cdot (p+2)}{2}, & n = p+2. \end{cases}$

(2) *Let $p = 4k+1$. Then* $m_2(A_n) = \begin{cases} \dfrac{p!}{2^{2k} \cdot (2k)!}, & n = p; \\ \dfrac{(p+1)!}{2 \cdot 2^{2k} \cdot (2k)!}, & n = p+1; \\ \dfrac{p!}{2^{2k} \cdot (2k)! \cdot 3!}, & n = p+2. \end{cases}$

Let $p = 4k+3$. Then $m_2(A_n) = \begin{cases} \dfrac{p!}{2^{2k} 3! \cdot (2k)!}, & n = p; \\ \dfrac{(p+1)!}{2 \cdot 2^{2(k+1)} \cdot (2(k+1))!}, & n = p+1; \\ \dfrac{(p+2)!}{2^{2(k+1)} \cdot (2(k+1))!}, & n = p+2. \end{cases}$

Proof. We knew that A_n is a C_{pp}-group. Let's say the cycle type has c_1 1-cycles, c_2 2-cycles, and so on, up to c_k k-cycles, where $1c_1 + 2c_2 + \cdots + kc_k = n$. The number of permutations in the conjugacy classes described by the c_i's is

$$\frac{n!}{\prod_{i=1}^{k} i^{c_i} \prod_{i=1}^{k} c_i!}.$$

Conjugacy classes of permutation in S_n stay the same size in A_n for all cycle types except those cycle type consists of parts that are all odd and distinct.

Table 1. The order components of finite simple groups L with $s(L) = 2$

L	Restrictions of L	m_1	m_2
A_n	$6 < n = p, p+1, p+2$ one of $n, n-2$ is not a prime	$n!/2p$	p
$A_{p-1}(q)$	$(p, q) \neq (3, 2), (3, 4)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1) \mid (p+1)$ $(p, q) \neq (3, 3), (5, 2)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$		$2^6 \cdot 3^4$	5
$B_n(q)$	$n = 2^m \geq 4$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p + 1) \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geq 2$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2, 3$	$q^{p^2}(q^p + 1) \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geq 5, q = 2, 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2, 3$	$q^{p(p+1)}(q^p + 1)(q^{p+1} - 1)$ $\prod_{i=1}^{p-1}(q^{2i} - 1)/(2, p-1)$	$\frac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m + 1 \geq 5$	$2^{n(n-1)}(2^n + 1)(2^{n-1} - 1)$ $\prod_{i=1}^{n-2}(2^{2i} - 1)$	$2^{n-1} + 1$
${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$	$9 \leq 2^m + 1 \neq p$	$3^{n(n-1)}(3^n + 1)(3^{n-1} - 1)$ $\prod_{i=1}^{n-1}(3^{2i} - 1)/2$	$\frac{3^{n-1}}{2}$
$G_2(q)$	$2 < q \equiv \epsilon \mod 3, \epsilon = \pm 1$	$q^6(q^3 - \epsilon)(q^2 - 1)(q + \epsilon)$	$q^2 - \epsilon q + 1$
${}^3D_4(q)$		$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 = 1$
$F_4(q)$	q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2(q^4 - 1)$	$q^4 - q^2 + 1$
${}^2F_4(2)'$		$2^{11} \cdot 3^3 \cdot 5^2$	13
$E_6(q)$		$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)$ $(q^5 - 1)(q^3 - 1)(q^2 - 1)$	$(q^6 + q^3 + 1)/(3, q-1)$
${}^2E_6(q)$	$q > 2$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)$ $(q^5 + 1)(q^3 + 1)(q^2 - 1)$	$(q^6 - q^3 + 1)/(3, q+1)$
M_{12}		$2^6 \cdot 3^3 \cdot 5$	5
J_2		$2^7 \cdot 3^3 \cdot 5^2$	7
Ru		$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29
He		$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17
McL		$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11
Co_1		$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23
Co_3		$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23
Fi_{22}		$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13
HN		$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19

Case 1. $n = p$

In this case, $n = p = p \cdot 1$ and so the cycle types are odd and so by [22], the conjugacy classes of a p-element of A_p split into two classes and so $m_p(A_p) = \frac{p!}{2p} = \frac{(p-1)!}{2}$.

Case 2. $n = p + 1$.

In this case, $n = p + 1 = 1 \cdot 1 + p \cdot 1$ and so the cycle types are odd and so by [22], the conjugacy classes of a p-element of A_{p+1} split into two classes and so $m_p(A_{p+1}) = \frac{p!(p+1)}{2p} = \frac{(p-1)! \cdot (p+1)}{2}$.

Case 3. $n = p + 2$.

Then $n = p+2 = 1\cdot 2+p\cdot 1$ and so $m_p(A_{p+2}) = \frac{p!(p+1)(p+2)}{2p} = \frac{(p-1)!\cdot(p+1)\cdot(p+2)}{2}$.
Similar as the case $m_p(A_n)$, we can compute the $m_2(A_n)$. $\square$

Lemma 3 ([11, pp. 85, Theorem 80]). *For any prime p, $(p-1)! \equiv -1 (mod\, p)$.*

Table 2. The order components of finite simple groups L with $s(L) = 3$

L	Restrictions of L	m_1	m_2	m_3
A_n	$6 < n = p, p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$4 \mid q+1$	$q+1$	q	$\frac{q-1}{2}$
$A_1(q)$	$4 \mid q-1$	$q-1$	q	$\frac{q+1}{2}$
$A_1(q)$	$2 \mid q$	q	$q+1$	$q-1$
$A_2(2)$		8	3	7
${}^2A_5(2)$		$2^{15}\cdot 3^6\cdot 5$	7	11
${}^2D_p(3)$	$5 \le p = 2^m+1$	$2\cdot 3^{p(p-1)}(3^{p-1}+1)$ $\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}}{2}$	$\frac{3^p+1}{4}$
${}^2D_{p+1}(2)$	$n \ge 2, p = 2^m-1$	$2^{p(p-1)}(2^p-1)$ $\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q \equiv 0 \mod 3$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$q = 2^{2m+1} > 2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+$ $q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$
$E_7(2)$		$2^{36}\cdot 3^{11}\cdot 5^2\cdot 7^3\cdot 11$ $13\cdot 17\cdot 19\cdot 43$	73	127
$E_7(3)$		$2^{23}\cdot 3^{63}\cdot 5^2\cdot 7^3\cdot 11^2$ $13^2\cdot 19\cdot 37\cdot 41\cdot 61\cdot 73\cdot 547$	757	1093
M_{11}		$2^4\cdot 3^2$	5	11
M_{23}		$2^7\cdot 3^2\cdot 5\cdot 7$	11	23
M_{24}		$2^{10}\cdot 3^3\cdot 5\cdot 7$	11	13
J_3		$2^7\cdot 3^5\cdot 5$	17	19
HS		$2^9\cdot 3^2\cdot 5^3$	7	11
Suz		$2^{13}\cdot 3^7\cdot 5^2\cdot 7$	11	13
Co_2		$2^{18}\cdot 3^6\cdot 5^3\cdot 7$	11	23
Fi_{23}		$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13$	17	23
F_3		$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13$	19	31
F_2		$2^{24}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot$ $11\cdot 13\cdot 17\cdot 19\cdot 23$	31	47

Table 3. The order components of finite simple groups L with $s(L) > 3$

L	Restrictions of L	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}+1$	$q^2+\sqrt{2q}+1$		
${}^2E_6(2)$		$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11$	13	17	19		
$E_8(q)$	$q \equiv 2, 3 \mod 5$	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1		
M_{22}		$2^7\cdot 3^2$	5	7	11		
J_1		$2^3\cdot 3\cdot 5$	7	11	19		
ON		$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
LyS		$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Fi'_{24}		$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
F_1		$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3$ $17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 47$	41	59	71		
$E_8(q)$	$q \equiv 0, 1, 4 \mod 5$	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\frac{q^{10}+1}{q^2+1}$	
J_4		$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43

Lemma 4 ([10, Theorem 9.3.1]). *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1}\cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

(1) *$q_i^{\beta_i} \equiv 1 (mod\, p_j)$ for some p_j.*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

Lemma 5 ([3, Lemma 1.2] and [20, Lemma 7]). *Let $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$. Then*

(1) $C_G(xy) = C_G(x) \cap C_G(y)$;

(2) $|x^G|$ *divides* $|(xy)^G|$;

(3) *If* $|x^G| = |(xy)^G|$, *then* $C_G(x) \leq C_G(y)$

Lemma 6 ([15, Lemma 1]). *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here*

(1) $s(n) = 6$ *for* $n \geq 48$;
(2) $s(n) = 5$ *for* $42 \leq n \leq 47$;
(3) $s(n) = 4$ *for* $38 \leq n \leq 41$;
(4) $s(n) = 3$ *for* $18 \leq n \leq 37$;
(5) $s(n) = 2$ *for* $14 \leq n \leq 17$;
(6) $s(n) = 1$ *for* $6 \leq n \leq 13$.

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n-1$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Lemma 7 ([15, Lemmas 3 and 6]). *Let P be a finite simple group. Then the following results hold.*

(1) *If $GK(P)$ is disconnected graph $GK(P)$. Then $m_i(P) = 1$ for $2 < i < t(P)$. Let n_i stand for the only element of $m_i(P)$ for $i > 1$. Then P, and m_i for $2 < i < t(P)$ are such as in Tables 1-3, where p is an odd prime number.*

(2) *If P is not isomorphic to ${}^2G_2(q)$, then, for every i, there is at most one prime number $s \in \pi_i(P)$ such that $(r+1)/2 < s < r$.*

(3) *If P is isomorphic to ${}^2G_2(q)$, then there are at most three prime numbers $s \in \pi(P)$ such that $(r+1)/2 < s < r$.*

(4) *For every prime number s satisfying the inequality $(r+1)/2 < s < r$, the order of the factor group $Aut(P)/P$ is not divisible by s.*

A finite group G is 2-Frobenius group if G has a normal series $1 \leq H \leq K \leq G$ such that K and G/K are Frobenius groups with Kernels H and K/H, respectively.

Lemma 8 ([23]). *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(1) *G is a Frobenius group or 2-Frobenius group;*

(2) *G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Lemma 9 ([6]). *Let G be a Frobenius group of even order with kernel K and complement H. Then $s(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) K *is nilpotent;*
(2) $|K| \equiv 1(mod\,|H|)$.

3. The proof of Main Theorem

In this section, we give the main theorem's proof.

Proof of the main theorem.

Proof. By [16] and [26], the alternating groups A_5, A_7, A_8, A_9 and A_{10} are valid for the main theorem. So in the following, we assume that $p \geq 11$.

We will prove the theorem by the following lemmas.

Lemma 10. *G is insoluble.*

Proof. By Lemma 1 and hypotheses, we have that $|G_p| = |A_n|_p = p$. It's known that p is the greatest prime divisor of $|A_n|$. Assume that G is soluble. We consider three cases.

Case 1. $n = p$.

By Lemma 2, $m_p(A_p) = \frac{(p-1)!}{2}$. On the other hand, $|G| = p \cdot m_p(G) = \frac{p!}{2}$ and so G has a maximal subgroup of $m_p(A_p)$ (actually, the maximal subgroup is a Hall $\pi(G)\backslash\{p\}$-subgroup). Then by Lemma 4, $\frac{(p-1)!}{2} \equiv 1(\text{mod } p)$. On the other hand, by Lemma 3 $(p-1)! \equiv -1(\text{mod } p)$. It follows that $p = 3$ contradicting $p \geq 11$.

Case 2. $n = p+1$.

By Lemma 2, $m_p(A_{p+1}) = \frac{(p-1)!\cdot(p+1)}{2}$. Note that $|G| = p{\cdot}m_p(A_{p+1}) = \frac{(p+1)!}{2}$. So G has a maximal subgroup of order $m_p(A_{p+1})$ (actually, the maximal subgroup is a Hall $\pi(G)\backslash\{p\}$-subgroup). Then by Lemma 4, $\frac{(p-1)!\cdot(p+1)}{2} \equiv \frac{(p-1)!}{2} \equiv 1(\text{mod } p)$. But by Lemma 3 $(p-1)! \equiv -1(\text{mod } p)$. It follows that $p = 3$ contradicting $p \geq 11$.

Case 3. $n = p+2$.

By Lemma 2, $m_p(A_{p+2}) = \frac{(p-1)!\cdot(p+1)(p+2)}{2}$. Similarly as the proof of Case 1 or 2, we have $\frac{(p-1)!\cdot(p+1)(p+2)}{2} \equiv (p-1)!(p+2) \equiv 1(\text{mod } p)$. It follows from Lemma 3, that $(p+2) \equiv -1(\text{mod } p)$ so $p = 3$, a contradiction. $\square$

Lemma 11. *p is the odd component of G. In particular $s(G) \geq 2$.*

Proof. We knew that A_n with $n \in \{p, p+1, p+2\}$ is C_{pp}-group. In fact, we show that G is also a C_{pp}-group. Assume the contrary, then there exists an element x of G of order r such that $r \cdot p \mid |C_G(x)|$. Let y be an element of $C_G(x)$ having order p. Then $xy = yx$. Since $|G_p| = p$, then G_p ia abellian and $|y^G|$ is a p'-number. But by Lemma 5, $|y^G| \mid |(xy)^G|$. It follows that $[|x^G|, |y^G|] \mid |(xy)^G|$. But $m_p(G)$ is the only maximal p'-number of conjugacy classes sizes of G since $|G| = n!$ and $|G| = p \cdot m_p(G)$. Thus p divides $|x^G|$. Hence $pr \cdot m_p(G)$ divides $|(xy)^G|$ Therefore $|(xy)^G| \geq prm_p(G) \geq 2|G|$, a contradiction. $\square$

By Lemma 11, $s(G) \geq 2$, and so, by Lemma 8, G has one of the following structures:

(1) G is a Frobenius group or 2-Frobenius group;

(2) G has a normal series $1 \leq K \leq H \leq G$ such that $\pi(G/H) \cup \pi(K) \subseteq \pi_1$ and H/K is a non-abelian simple group. In particular, K is nilpotent, $G/H \lesssim \mathrm{Out}(H/K)$ and the odd order components of G are the odd order components of H/K.

So in the following, we consider case by case.

Lemma 12. *G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Suppose the contrary. we first consider when G is a Frobenius group with kernel K and complement H. By Lemma 9, $\{\pi(K), \pi(H)\} = \{\pi((p-1)!), \{p\}\}$. The following two cases are considered:

(1) If $\pi(H) = \{p\}$, then $p \geq 11$ and $\pi(K) = \pi((p-1)!)$. By Lemma 6, there is a prime r such that $\frac{p+1}{2} < r < p$. Since K is nilpotent and G_r is of order r, then Lemma 9(2) implies $p = |H| \mid |G_r| - 1 = r - 1 < p - 1$, a contradiction.

(2) If $\pi(K) = \{p\}$, then $\pi(H) = \pi((p-1)!)$. Lemma 6, there is a prime r with that $\frac{p+1}{2} < r < p$ and $|H_r| = r$. Hence $[K]H_r$ is a Frobenius group and so, $|H_r| \mid |K| - 1 < p - 1$. It follows that $r < \frac{p-1}{2}$, a contradiction.

Let G be a 2-Frobenius group. Then G is soluble contradicting to Lemma 10. □

Lemma 13. *Let G be a finite group and $r \in \pi(G)$. If $r^2 \nmid |G|$, then G has a normal series $1 \leq K \leq H \leq G$, such that H/K is a simple group and $r \in \pi(H/K)$.*

Proof. Since G is a finite group, G has a chief series. So let $G_0 \leq G_1 \leq G_2 \cdots \leq G_l = G$ be a chief series of G. There exists some t, such that $1 \leq t \leq l$ and $r \in \pi(G_t) \setminus \pi(G_{t-1})$. Let $H = G_t$ and $K = G_{t-1}$, then $1 \leq K \leq H \leq G$ is a normal series of G and H/K is a chief factor of G. Therefore H/K is a minimal normal subgroup of G/K. We know that the chief factors are characteristically simple. Also every characteristically simple group is a simple group or a product of isomorphic simple groups. So H/K is a simple group or a product of isomorphic simple groups. If $r^2 \nmid |G|$, then by Lemma 1, $|G_r| = r$ and $r > [\frac{n}{2}]$. By Lemma 7(4), $r \nmid \mathrm{Out}(H/K)$. So we have $r \mid |K|$. By our assumption, $p \geq 11$ and so there is a Hall $\{p, r\}$-subgroup L with $p \neq r$. So L is cyclic and hence there is an element of order $p \cdot r$, contradicting Lemma 11. It follows that $r \in \pi(H/K)$. □

Lemma 14. *H/K is not isomorphic to any sporadic simple groups.*

Proof. Suppose the contrary. By Lemma 13, for any prime r such that $\frac{p+1}{2} < r < p$, then $r \in \pi(H/K)$. By Tables 1, 2 and 3, H/K is not isomorphic to one of the following groups: M_{12}, Ru, He, McL, Co_1, Co_3, HN, M_{11}, M_{23}, M_{24}, J_3, Co_2, Fi_{23}, F_2, F_3, J_1, ON, LyS, $Fi_{24}^{'}$, F_1 and J_4. If $K/H \cong J_2$, then $|K/H| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, so $p = 7$ and hence, $5^2 \mid |H/K| \nmid |A_n|$ contradicting Lemma 1, If K/H is isomorphic to McL or HS, then $p = 11$ and hence, $5^3 \mid |H/K| \nmid |A_n|$, a contradiction. If H/K is isomorphic to Szu or Fi_{22}, then $p = 13$ and hence, $|H/K| \nmid 2|A_n|$, contradiction. Finally, $H/K \cong M_{22}$, then $p = 11$. Since $|G| = |K||H/K||G/H|$, then $|H|_3 = 3^3$. By Lemma 11, $3 \cdot p \notin \omega(G)$. It follows that the Sylow 11-subgroup acts fixed freely on the set of elements of order 3 and so $11 \mid 3^2 - 1$, a contradiction. □

Lemma 15. *H/K is not isomorphic to any finite simple groups of Lie type.*

Proof. By Lemma 10 G is insoluble. If $r \in \pi(G)$ and $r^2 \nmid |G|$, then by Lemma 13, $r \mid |H/K|$. By Lemma 11, p is the odd component of G and so is of H/K. By Lemma 7

(1) If P is not isomorphic to ${}^2G_2(q)$, then, for every i, there is at most one prime number $s \in \pi_i(P)$ such that $(r+1)/2 < s < r$.

(2) If P is isomorphic to ${}^2G_2(q)$, then there are at most three prime numbers $s \in \pi(P)$ such that $(r+1)/2 < s < r$.

(3) $s(G) \geq 2$ and $p \in \{m_i\}$ for $i \geq 2$

In the following, we consider three cases.

(1) $s(G) = 2$.

1.1. $H/K \cong A_{p'-1}(q')$ with $(p', q') \neq (3,2), (3,4)$. We have

$$p = \frac{q'^{p'} - 1}{(q'-1)(p', q'-1)}.$$

By Lemma 6, $p \leq 13$ and so $(p', q') = (3,3)$. Thus $|A_2(3)| = 3^3 \cdot 2^3 \cdot 2 \cdot 13$, which contradicting Lemma 13.

1.2. $H/K \cong A_{p'}(q')$ with $q' - 1 \mid p' + 1$. Then $p = \frac{q'^{p'} - 1}{q'-1}$. By Lemma 6, $p \leq 13$ and and so $(p', q') = (3,3)$. Thus $|A_3(3)| = 3^3 \cdot 2^4 \cdot 5 \cdot 2^3 \cdot 2 \cdot 13$, which contradicting Lemma 13.

1.3. $H/K \cong^2 A_{p'-1}(q')$. Then

$$p = \frac{q'^{p'} + 1}{(q'+1)(q', p'+1)}.$$

Similarly, $p \leq 13$ and so $q' = 4, p' = 3$. Hence $|{}^2A_2(4)| = 2^{12} \cdot 3 \cdot 5 \cdot 3 \cdot 13$. But $11 \mid |H/K|$, a contradiction.

1.4. $H/K \cong^2 A_3(2)$. Then $p = 5$ and so $2^6 \nmid 2|A_n|$ by Lemma 1, a contradiction.

1.5. $H/K \cong B_n(q')$ with $n = 2^m \geq 4$, q' odd. Then $p = (q'^n + 1)/2$. Thus $q' = 5$ and $n = 2 \ngeq 4$, contradicting Lemma 1.

1.6. $H/K \cong B_{p'}(3)$. We have $p = \frac{3^{p'}-1}{2}$ and so $p = 3$. It follows that $3^9 \mid |A_n|$, a contradictino.

1.7. $H/K \cong C_n(q')$ with $n = 2^m \geq 2$, q' odd. Then $p = \frac{q'^n+1}{(2,q-1)}$ and so $q' = 5, n = 2$. It follows that $5^4 \mid |A_n|$, a contradiction.

1.8. $H/K \cong C_{p'}(q')$ with $q' = 2, 3$. Then $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and so $p' = 3, q' = 3$. Whence, $3^9 \mid |A_n|$, a contradiction.

1.9. $H/K \cong D_{p'}(q')$ with $p' \geq 5, q = 2, 3, 5$. Then $p = \frac{q'^{p'}-1}{q'-1}$ and so $p' = 3 \ngeq 5$, a contradiction.

1.10. $H/K \cong D_{p'+1}(q')$ with $q' = 2, 3$. $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and so $p' = 3, q' = 3$. Whence, $3^{12} \mid |A_n|$, a contradiction.

1.11. $H/K \cong^2 D_n(q')$ with $n = 2^m \geq 4$. Then $p = \frac{q'^n+1}{(2,q'+1)}$. Then $p = \frac{q'^n+1}{(2,q'+1)}$ and so $q' = 5, n = 2$. Since $|^2D_2(5)| = 5^2 \cdot 2^2 \cdot 13$, then $11 \nmid |A_n|$ contradicting Lemma 13.

1.12. $H/K \cong^2 D_n(2)$ with $n = 2^m + 1 \geq 5$, $H/K \cong^2 D_n(3)$ with $9 \leq 2^m + 1 \neq p'$ and $H/K \cong F_4(q')$ with q' odd. There is no prime number r such that $7 \leq r = 2^{n-1} + 1 \leq 13$, $7 \leq r = \frac{3^{n-1}+1}{2} \leq 13$ and $7 \leq r = q'^4 - q'^2 + 1 \leq 13$.

1.13. $H/H \cong^2 D_p(3)$ with $5 \leq p' \neq 2^m = 1$. Since $p = \frac{3^{p'}+1}{4}$, then $p' = 3$. Therefore $3^6 \mid |A_n|$, a contradiction.

1.14. $H/K \cong^3 D_4(q')$. Then $p = q'^4 - q'^2 + 1$ and so $q' = 2$ since $7 \leq p \leq 13$. But $11 \nmid |^3D_4(2)| = 2^{12} \cdot 3^2 \cdot 7 \cdot 3 \cdot 3 \cdot 7 \cdot 13$ contradicting Lemma 13.

1.15. $H/K \cong G_2(q')$ with $2 < q' \equiv \epsilon \mod 3, \epsilon = \pm 1$. Then $p = q'^2 - \varepsilon q' + 1$ and so $q' = 4, \varepsilon = 1$. It follows that $2^{24} \mid 2|A_n|$, a contradiction.

1.16. $H/K \cong^2 F_4(2)'$. Then $p = 13$ and $11 \mid |^2F_4(2)'|$, a contradiction.

1.17. $H/K \cong E_6(q')$. Then $p = \frac{q'^6+q'^3+1}{(3,q'-1)} > 13$ and so we rule out this case.

(2) $s(G) = 3$.

2.1. $H/K \cong A_1(q')$ with $4 \mid q' + 1$. By Lemmas 7(2) and 13, $7 \leq p \leq 13$. Thus $p = q'$ or $p = \frac{q'-1}{2}$.

If the former, then $p = q' = 7$ and $p = q' = 11$. If $p = q' = 7$, then $5 \nmid |A_n|$, a contradiction. If $p = q' = 11$, then $7 \nmid |A_n|$, a contradiction.

If the latter, then $q' = 23 \nleq 13$, a contradiction.

2.2. $H/K \cong A_1(q')$ with $4 \mid q-1$. Then $p = q'$ or $p = \frac{q'+1}{2}$. If the former, then $p = q' = 13$ and so $7 \nmid |A_n|$, a contradiction. If the latter, then $p = 13, q' = 25$ and so $7, 11 \nmid |A_n|$, a contradiction.

2.2. $H/K \cong A_1(q')$ with $2 \mid q'$. Then $p = q'+1$ or $p = q'-1$. If the former, there is no solution since $7 \leq p \leq 13$. If the latter, $q' = 8$ and so $5 \nmid |A_n|$, a contradiction.

2.3. $H/K \cong A_2(2)$. Then $p = 7$ and so $5 \nmid |A_2(2)|$, a contradiction.

2.4. $H/K \cong E_7(2)$ or $H/K \cong E_7(3)$. Then the primes are larger than 13 and so we rule out these cases.

2.5. $H/K \cong^2 A_5(2)$. Then $p = 11$. In this case, $2^{15} \nmid 2|A_n|$, a contradiction.

2.6. $H/K \cong^2 D_{p'}(3)$ with $5 \leq p' = 2^m + 1$. Then $p = \frac{3^{p'-1}+1}{2}$ or $\frac{3^{p'}+1}{4}$. By Lemmas 7(2) and 13, $7 \leq p \leq 13$. Thus the equations have no solution. Similar, we can rule out "$H/K \cong^2 D_{p'+1}(2)$ with $m \geq 2, p' = 2^m - 1$".

2.7. $H/K \cong G_2(q')$ with $q' \equiv 0 \mod 3$. Then $p = q'^2 - q' + 1$ or $p = q'^2 + q' + 1$ and so $q' = 3$. But $5 \nmid |A_n|$, a contradiction.

2.8. $H/K \cong^2 G_2(q')$ with $q' = 3^{2m+1} > 3$. Then $p = q' - \sqrt{3q} + 1$ or $p = q' + \sqrt{3q} + 1$. By Lemmas 7(3) and 13, $p \leq 37$. It follows that $q' = 27$. Thus $11, 31 \nmid |A_n|$, a contradiction.

2.9. $H/K \cong^2 F_4(q')$ with $q' = 2^{2m+1} > 2$ and $H/K \cong F_4(q')$ with q' even. In both cases, there is no solution.

(3) $s(G) > 3$.

3.1. $H/K \cong A_2(4)$. Then $|K/H| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and so $p = 7$, $m_7(A_2(4)) = 2^6 \cdot 3^2 \cdot 5$.

If $n = 7$, then $2^6 \mid 7!$, a contradiction.

If $n = 8$, then by [9, pp. 24], $m_2(A_2(4)) = 3^2 \cdot 5 \cdot 7$. Since in this case, $|H/K| = |A_8|$, then $|G| = |A_8|$. But $m_2(A_8) = 3 \cdot 5 \cdot 7$, a contradiction by Lemma 2.4 of [1].

If $n = 9$, then $|G| = |A_9| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$ by hypotheses, and so $m_2(A_9) = 3^3 \cdot 5 \cdot 7$, $m_7(A_9) = 2^6 \cdot 3^4 \cdot 5$. By Lemma 8(2) and [9], $A_2(4) \leq G/K \leq \mathrm{Aut}(A_2(4))$. If $G/K \cong A_2(4)$, then $|K| = 3^2$ and $Z(G) = 3^2$. It follows that there is an element of order $3 \cdot 7$, contradicting to Lemma 11. If $G/K \cong \mathrm{Aut}(A_2(4))$, then since $|\mathrm{Out}(A_2(4))| = |2 \times S_3| = 2^2 \cdot 3$, order consideration rules out.

3.2. $H/K \cong^2 E_6(2)$. Then $p = 19 \nleq 13$, a contradiction.

3.2. $H/K \cong^2 B_2(q')$ with $q' = 2^{2m+1} > 2$. Then $p = q' - 1, q' - \sqrt{2q'} + 1$ or $q' + \sqrt{2q'} + 1$ and so, $q' = 8$. But $11 \nmid |A_n|$, a contradiction.

3.3. $H/K \cong E_8(q')$. In these cases, $p > 13$ and so , we rule out these cases.

This completes the proof of the lemma. □

Lemma 16. *G is isomorphic to A_n with $n \in \{p, p+1, p+2\}$.*

Proof. By Lemmas 12, 14 and 15, $H/K \cong A_n$. Order consideration, we have that $n = p, p+1, p+2$.

If $n = p$, then $A_p \leq H/K \leq S_p$. If $H/K \cong A_p$, then order consideration gets the desired results. If $H/K \cong S_p$, then we rule out this case by group order.

Similarly, we can conclude that $G \cong A_{p+1}$ if $n = p+1$; $G \cong A_{p+2}$ if $n = p+2$.

The Lemma is proved. □

This completes the proof of Main Theorem. □

Acknowledgements

The project was supported by the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (Grant Nos: 2017QZJ01 and 2018QYJ01) and Postgraduate Research and Practice Innovation Program of Jiangsu Province (Grant No: KYCX17-2012), and NSFC(Grant No: 11871360). The authors are very grateful for the helpful suggestions of the referee.

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Accepted: 8.12.2018

Invo-clean rings associated with central polynomials

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Abstract. Let R be an associative ring with identity and let $C(R)$ be the center of a ring R and let $g(x)$ be a fixed polynomial in $C(R)[x]$. We defined R to be $g(x)$-invo clean if every element in R can be written as a sum of an involution and a root of $g(x)$. In this paper, we investigate conditions on a ring to be $g(x)$-invo clean ring. Some properties and several examples are given.

Keywords: clean rings, $g(x)$-invo clean rings, invo clean rings.

1. Introduction

Let R be an associative ring with identity. Following [6], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is clean. Clean rings were first introduced by Nicholson [6] as a class of exchange rings.

The invo-clean rings was introduced by Danchev [2]. He defined and completely described the structure of invo-clean rings having identity.

Camillo and Simon [1] , defined $g(x)$-clean rings. An element $r \in R$ is called $g(x)$-clean if $r = s + u$ where $g(s) = 0$ and u is a unit of R and R is a $g(x)$-clean ring if every element is $g(x)$-clean. The $(x^2 - x)$-clean rings are precisely the clean rings. In Fan and Yang [3], authors studied more properties of g(x)-clean rings. Among many conclusions, they proved that if $g(x) \in (x-a)(x-b)C(R)[x]$. where $a, b \in C(R)$ with $(b - a)$ unit in R, then R is a clean ring if and only if R is $(x - a)(x - b)$-clean. For the study of clean rings and their generalizations, we refer to [4], [5], [7].

In this paper, we introduce the notion of $g(x)$-invo clean ring. A ring R is said to be $g(x)$-invo clean ring if any element in R can be written as a sum of

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involution and a root of $g(x)$. Clearly, invo-clean rings are (x^2-x)-invo clean rings.

Throughout this paper, we assume that all rings are associative with identity and all modules are unitary. As usual, $U(R)$ denotes the set of all units of R, $Inv(R)$ the subset of $U(R)$ consisting of all involutions (i.e.; $v \in Inv(R)$ then $v^2 = 1$) of $R, Id(R)$ the set of all idempotents of R and $Nil(R)$ the set of all nilpotents, $C(R)$ denotes the center of R and $g(x)$ be a fixed polynomial with coefficients in $C(R)$.

2. $g(x)$-Invo clean rings

In this section, we define $g(x)$ -invo clean rings, we give some properties of $g(x)$-invo clean ring and present several examples.

Definition 2.1. Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-invo clean if $r = v + s$ where $g(s) = 0$ and v is an involution of R i.e., $v^2 = 1$. We say that R is $g(x)$-invo clean if every element in R is $g(x)$-invo clean.

Clearly, Every $(x^2 - x)$-invo clean ring is invo clean.

Example 2.2. $\mathbb{Z}_7$ is ($x^6 - 1$)-invo clean ring which is not invo-clean ring**.**

Example 2.3. The ring $M_2(\mathbb{Z}_2)$ is $\left(x^3 - x\right)$ -invo clean ring.

Proposition 2.4. *Every $g(x)$ -invo clean ring is $g(x)$ -clean ring.*

Proof. Suppose R is a $g(x)$ -invo clean ring and let $r \in R$. Then $r = v + s$ where v is involution and $g(s) = 0$. But every involution is unit. Thus, R is $g(x)$ -clean ring.
The converse of Proposition 2.4 is not true in general. For example, we can see that $M_2(\mathbb{Z}_2)$ is $\left(x^6-1\right)$ -clean ring which not $\left(x^6-1\right)$-invo clean, since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ cannot be written as a sum of involution and a root of $\left(x^6-1\right)$. $\square$

Let R and S be rings and $\Psi : C(R) \to C(S)$ be a ring epimorphism with $\Psi(1_R) = 1_S$. For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, we let $g^*(x) = \sum_{i=0}^{n} \Psi(a_i)x^i \in C(S)[x]$. In particular, If $g(x) \in \mathbb{Z}[x]$, then $g^*(x) = g(x)$.

Proposition 2.5. *Let $\theta : R \to S$ be a ring epimorphism. If R is $g(x)$ -invo clean, then S is $g^*(x)$ -invo clean.*

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ and consider $g^*(x) = \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$. For every $\beta \in S$, there exist $r \in R$ such that $\theta(r) = \beta$. Since R is $g(x)$ -invo clean, there exists $s \in R$ and $v \in Inv(R)$ such that $r = v + s$ and $g(s) = 0$. Then $\beta = \theta(r) = \theta(v + s) = \theta(v) + \theta(s)$ with $\theta(v) \in Inv(S)$, and $g^*(\theta(s)) = \sum_{i=0}^{n} \theta(a_i)(\theta(s))^i = \sum_{i=0}^{n} \theta(a_i)\theta(s^i) = \sum_{i=0}^{n} \theta(a_i s^i) = \theta(\sum_{i=0}^{n} a_i s^i)$ $= \theta(g(s)) = \theta(0) = 0$. Therefore, S is $g^*(x)$ -invo clean. $\square$

Proposition 2.6. *Let R be an $g(x)$-nil clean with $n^2 = -2n$ for every $n \in Nil(R)$. Then R is $g(x)$ -invo clean.*

Proof. Suppose R is a $g(x)$-nil clean and let $r \in R$. Then $r - 1 = n + s$ where $n \in Nil(R)$ and $g(s) = 0$. Thus $r = (1 + n) + s$. Indeed $1 + n$ is an involution. Therefore R is $g(x)$ -invo clean. □

Proposition 2.7. *If R an $g(x)$-invo clean ring and I is an ideal of R, then $R = R/I$ is $g^*(x)$ -invo clean.*

Proof. Let R be an $g(x)$ -invo clean ring and $\theta : R \to R/I$ defined by $\theta(r) = r = r + I$. Then θ is an epimorphism. By Proposition 2.5 R/I is $g(x)$-invo clean. □

Proposition 2.8. *Let $R_1, R_2, \ldots, R_k$ be rings and $g(x) \in \mathbb{Z}[x]$. Then $R = \prod_{i=1}^{k} R_i$ is $g(x)$-invo clean if and only if R_i is $g(x)$-invo clean for all $i \in \{1, 2, \ldots, n\}$.*

Proof. $\Rightarrow)$: For each $i \in \{1, 2, \ldots, n\}$, R_i is a homomorphic image of $\prod_{i=1}^{k} R_i$ under the projection homomorphism. Hence, R_i is $g(x)$ -invo clean by Proposition 2.5.
$\Leftarrow$) : Let $(x_1, x_2, \ldots, x_k) \in \prod_{i=1}^{k} R_i$. For each i, write $x_i = v_i + s_i$ where $v_i \in Inv(R_i)$, $g(s_i) = 0$. Let $v = (v_1, v_2, \ldots, v_k)$ and $s = (s_1, s_2, \ldots, s_k)$. Then, it is clear that $v \in Inv(R)$ and $g(s) = 0$. Therefore, R is $g(x)$-invo clean. □

Theorem 2.9. *Let R be a ring and let $R[t]$ be the rings of polynomial in an indeterminate t with coefficients in R and let $f(t) = a_0 + a_1 t + \ldots + a_n t^n \in R[t]$. If $f(t)$ is an involution then a_0 is an involution in R and $a_1, \ldots, a_n$ are nilpotents.*

Proof. Assume $f(t)$ is a unit then a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotents. Since $Inv(R) \subseteq U(R)$, the statement holds. □

Proposition 2.10. *Let R be a commutative ring, then the ring of polynomials $R[t]$ is not invo clean (not $\left(x^2 - x\right)$ -invo clean).*

Proof. Let t be an invo clean, then we may write $t = a_0 + a_1 t + \ldots + a_n t^n + e$ where $e \in Id(R[t]) = Id(R)$ and $a_0 \in Inv(R), a_1, \ldots, a_n \in Nil(R)$. Hence, $1 = a_1 \in J(R)$. Which a contradiction. Hence, $R[t]$ is not invo-clean. □

Let R be a commutative ring and M an R -module. The idealization $R(M)$ of R and M is the ring $R(M) = R \oplus M$ with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. Note that if $(r, m) \in R(M)$, then $(r, m)^k = \left(r^k, k r^{k-1} m\right)$ for any $k \in \mathbb{N}$.

Lemma 2.11. *Let R be a commutative ring with $char(R) = 2$ and M an R -module. Then (v, m) is an involution in $R(M)$ if and only if v is involution in R.*

Proof. $\Rightarrow$) : Let $(v,m) \in R(M)$ then $(v,m)^2 = \left(v^2, 2vm\right) = (1,0)$. So, $v^2 = 1$. Thus, v is involution.
$\Leftarrow$) : Let v be an involution, and $(v,m) \in R(M)$. Then $(v,m)^2 = (1,0)$. Hence, (v,m) is an involution of $R(M)$. $\square$

We recall that R logically embeds into $R(M)$ via $r \to (r,0)$. Therefore any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^{n} (a_i, 0) x^i \in R(M)[x]$ and conversely.

Proposition 2.12. *Suppose R is a commutative ring with $Char(R) = 2$ and M an R -module. So the idealization $R(M)$ of R and M is $g(x)$-invo clean if and only if R is $g(x)$-invo clean.*

Proof. $\Rightarrow$) : Since $R \simeq R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence R is $g(x)$ -invo clean by Proposition 2.5.
$\Leftarrow$) : Let $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Write $r = v + s$ where $v \in Inv(R)$ and $g(s) = 0$. Then for $m \in M$, $(r,m) = (v,m) + (s,0)$ where $(v,m) \in Inv(R(M))$ and

$g(s,0) = a_0(1,0) + a_1(s,0) + a_2(s,0)^2 + \ldots + a_n(s,0)^n$
$= a_0(1,0) + a_1(s,0) + a_2\left(s^2,0\right) + \ldots + a_n(s^n,0)$
$= \left(a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n, 0\right) = (g(s),0) = (0,0)$. Therefore, $R(M)$ is $g(x)$-invo clean. $\square$

3. $(x^2 + cx + d)$-invo clean rings

We consider some types of $\left(x^2 + cx + d\right)$ -invo clean rings.

Theorem 3.1. *Let R be a ring and $a, b \in C(R)$ and $g(x) \in (x-a)(x-b)$ where $b - a \in Inv(R)$. Then R is invo-clean if and only if R is $(x-a)(x-b)$-invo clean.*

Proof. $\Rightarrow$) : Since R is invo-clean and $r \in R$ then $\frac{r-a}{b-a} = v + e$ where $v \in Inv(R)$ and $e \in Id(R)$ then $r = v(b-a) + e(b-a) + a$, $b - a \in C(R)$ and $C(R)$ is a subring of R. Since $(e(b-a)+a-a)(e(b-a)+a-b) = (eb-ea)(eb-ea+a-b) =$
$e^2b^2 - e^2ba + eab - eb^2 - e^2ab + e^2a^2 - e^2a^2 - ea^2 + eab = 0$, it follows $e(b-a)+a$ is root of $(x-a)(x-b)$. Since $v(b-a) \in Inv(R)$ by $(v(b-a))^2 = v(b-a)v(b-a) = v^2(b-a)^2 = 1.1 = 1$, it follow that $v(b-a) \in Inv(R)$. Then R is $(x-a)(x-b)$-invo clean.
$\Leftarrow$) : Let $r \in R$. Since R is $(x-a)(x-b)$-invo clean, $r(b-a)+a = v+e$ where e is root of $(x-a)(x-b)$ and $v \in Inv(R)$. Thus, $r = \frac{e-a}{b-a} + \frac{v}{b-a}$. Clearly, $\frac{v}{b-a} \in Inv(R)$ and $\frac{e-a}{b-a}$ is an idempotent since $\left(\frac{e-a}{b-a}\right)^2 = \frac{e-a}{b-a}$. Hence R is invo-clean. $\square$

Corollary 3.2. *Let R be a ring. Then R is invo-clean if and only if R is $\left(x^2 + x\right)$-invo clean.*

Proof. In the previous Theorem 3.1 but $a = 0$ and $b = -1$. $\square$

Proposition 3.3. *Let R be a ring with $2 \in Inv\,(R)$ and $k \in N$. Then the following are equivalent:*

(1) R is invo clean
(2) R is $\left(x^2 - 2x\right)$-invo clean
(3) R is $\left(x^2 + 2x\right)$-invo clean
(4) R is $\left(x^2 - 2^{2k}x\right)$-invo clean
(5) R is $\left(x^2 + 2^{2k}x\right)$-invo clean
(6) R is $\left(x^2 - 1\right)$-invo clean
(7) R is For every $r \in R$, r can be expressed as $r = v + s$ with $v, s \in Inv\,(R)$.

Proof. (1) $\Rightarrow$ (2) Since R is invo clean and $r \in R$, $\frac{r}{2} = v + s$ with $v \in Inv\,(R)$ and $s^2 = s$, then $r = 2v + 2s$ with $2v \in Inv\,(R)$ and $(2s)^2 - 2\,(2s) = 4s^2 - 4s = 0$. Hence, R is $\left(x^2 - 2x\right)$-invo clean.

(2) $\Rightarrow$ (1) Since R is $\left(x^2 - 2x\right)$-invo clean, $2r = v + s$ where $v \in Inv\,(R)$ and s is a root of $\left(x^2 - 2x\right)$. Then, $r = \frac{v}{2} + \frac{s}{2}$, where $\frac{v}{2}$ is an invo of R and $\left(\frac{s}{2}\right)^2 = \frac{(s)(s-2+2)}{(2)^2} = \frac{s.2}{(2)^2} = \frac{s}{2}$. So, R is invo clean. Correspondingly, we may prove (3) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3) R is $\left(x^2 - 2x\right)$-invo clean and let $r \in R$, $-r = v + s$ such that $v \in Inv\,(R)$ and $s^2 - 2s = 0$. Then, $r = (-v) + (-s)$ with $-v \in Inv\,(R)$ and $(-s)^2 + 2\,(-s) = s^2 - 2s = 0$. Thus, R is $\left(x^2 + 2x\right)$-invo clean.

(1) $\Leftrightarrow$ (4) By Theorem 3.1, let $a = 0$ and $b = 2^{2k}$, Then, R is $\left(x^2 - 2^{2k}x\right)$-invo clean.

(1) $\Leftrightarrow$ (5) Can be proved by (1) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (6) Since R is invo clean and $r \in R$ then $r = v + s$ where $v, s \in Inv\,(R)$ and $s^2 = s$. Then s is a root of $x^2 - 1$ by (7). Then $\left(x^2 - 1\right)$-invo clean.

(7) $\Rightarrow$ (6) Let $r \in R$ we write $r = v + s$ with $v, s \in Inv\,(R)$ and $s^2 = 1$, then s is a root of $x^2 - 1$ and $v \in Inv\,(R)$. Then, $\left(x^2 - 1\right)$ is invo clean ring.

(6) $\Rightarrow$ (7) If R is $\left(x^2 - 1\right)$-invo clean, then for every $r \in R$ there exist $v, s \in Inv\,(R)$ such that $r = v + s$. $\square$

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Accepted: 9.12.2018

On developing an optimal Jarratt-like class for solving nonlinear equations

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Abstract. It is attempted to derive an optimal class of methods without memory from Ozban's method [A. Y. Ozban, Some New Variants of Newton's Method, Appl. Math. Lett. 17 (2004) 677-682]. To this end, we try to introduce a weight function in the second step of the method and to find some suitable conditions, so that the modified method is optimal in the sense of Kung and Traub's conjecture. Also, convergence analysis along with numerical implementations are included to verify both theoretical and practical aspects of the proposed optimal class of methods without memory.
Keywords: nonlinear equations, Kung and Traub's conjecture, iterative method, optimal method, convergence analysis.

1. Introduction

The main objective of this work is to derive an optimal class of methods without memory for approximating a simple root of a nonlinear equation. For this purpose, we consider a non-optimal method without memory developed by Ozban in [8]. Although this method is one of the most cited works in the literature, it is not optimal in the sense of Kung and Traub's conjecture. Based on this conjecture, any two-step method without memory is optimal if it has convergence order four using three functional evaluations per iteration [4, 12], while

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the pointed method uses three functional evaluations per iteration and has convergence order three, see Theorem 4.1 in [8].

There are so many two-step optimal methods without memory which we recall some of them here. To the best of our knowledge, there are three general kinds of optimal methods without memory: Jarratt-, Ostrowski- and Steffensen-type methods. Jarratt's method [2] is given by:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.1}$$

where its error equation is $e_{n+1} = (c_2^3 - c_2c_3 + \frac{c_4}{9}c)e_n^4 + O(e_n^5)$, with $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k = 2, 3, ...,$ and α is a simple zero of $f(x) = 0$, i.e., $f'(\alpha) \neq 0 = f(\alpha)$. Jarratt's method (1.1) uses three functional evaluations per iteration and has convergence order four so it is optimal. In other words, it uses functional evaluation of its derivation in two points, say $f'(x_n)$ and $f'(y_n)$, and one functional evaluation of the given function, says $f(x_n)$, in each iteration. Such methods in which, one uses two evaluations of the derivatives of the given functions and one evaluation of the given function are called Jarratt-type methods. Soleymani et al. [10] suggested the following optimization of Jarratt-type method:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + (\dfrac{f(x_n)}{f'(x_n)})^3)(2 - \dfrac{7}{4}s + \dfrac{3}{4}s^2)\dfrac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \tag{1.2}$$

Another optimal method of this type is considered by Lotfi [5]

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (2 - \dfrac{7}{4}s + \dfrac{3}{4}s^2)\dfrac{2f(x_n)}{f'(x_n)+f'(y_n)}. \end{cases} \tag{1.3}$$

Some other optimal Jarratt-type methods and different anomalies in a Jarratt family can be found in the literature [5, 10]. Similar to Jarratt-type methods, there is another set of methods in which they use derivative of the function in each iteration. However, these kinds use two function evaluations and one derivative evaluation, say $f(x_n)$, $f(y_n)$ and $f'(x_n)$. We call these kinds of iterative methods Ostrowski-type methods. Indeed, Ostrowski's method is given by [3]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)}{f(x_n)-2f(y_n)}\dfrac{f(y_n)}{f'(x_n)}, \end{cases} \tag{1.4}$$

with the following error equation $e_{n+1} = (c_2^3 - c_2c_3)e_n^4 + O(e_n^5)$.

It is worth noting that Ostrowski's method (1.4) is a special case, b=0, of King's family [3] which is defined as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n) + bf(y_n)}{f(x_n) + (b-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

Also, we consider the first two-step iterative method by Kung and Traub [4] as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^2 f(y_n)}{f'(x_n)(f(x_n) - f(y_n))^2}. \end{cases}$$

There is another kind of the Ostrowski-type method which can be obtained via Hermit-interpolation as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, x_n, x_n](y_n - x_n)}. \end{cases} \tag{1.6}$$

Finally, there is another kind of optimal two-step methods without memory in which, one does not use derivatives. We call them Steffensen-type method. In what follows, we recall two of them. First, we consider the first two-step derivative-free version of Kung and Traub [4] given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, \omega_n]}, \qquad \omega_n = x_n + f(x_n), \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n]} \dfrac{f(\omega_n)}{(f(\omega_n) - f(y_n))}. \end{cases} \tag{1.7}$$

Bi et al. [9] and Zheng et al. [14] simultaneously derived the following method based on Newton interpolation

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, \omega_n]}, \qquad \omega_n = x_n + \gamma f(x_n), \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f[x_n, y_n] + f[y_n, x_n, \omega_n](y_n - x_n)}. \end{cases} \tag{1.8}$$

All of the mentioned methods can be considered as a special case of the optimal class of two-step methods without memory. Detailed description, convergence and analysis of these methods may be found in [1, 3, 5, 6, 7, 12] and references therein .

This work is organized as follows: Section 2 is devoted to extracting optimal method from non-optimal method by Ozban [8]. Furthermore, we discuss the convergence analysis of the developed method in this section, and also some concrete functions are given based on the developed method. Section 3 represents numerical implimentations and comparisons. Finally, Section 4 concludes this work. For some given methods in this work, we append their Mathematica codes, too.

2. Method and result

In this section, we deal with developing a new optimal class of Jarratt-type methods to approximate a simple zero of $f(x) = 0$. Also, we discuss a theoretical aspect of the developed class, namely convergence analysis. We recall the following method by Ozban [8]

$$
\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{(f'(x_n) + f'(y_n))}{2f'(y_n)} \dfrac{f(x_n)}{f'(x_n)}, \qquad (n = 0, 1, ...). \end{cases} \tag{2.1}
$$

Theorem 4.1. in [8] considers the error analysis of this method. The following self-explanatory Mathematica code decodes and deciphers the same results quickly. We introduce the following abbreviations used in this program.

$\mathtt{c_k} = \mathtt{f}^{(\mathtt{k})}(\alpha)/(\mathtt{k!f'}(\alpha))$, $\mathtt{e} = \mathtt{x_n} - \alpha$, $\mathtt{ey1} = \mathtt{y_n} - \alpha$, $\mathtt{ey} = \mathtt{x_{n+1}} - \alpha$, $\mathtt{f[e]} = \mathtt{f(e)}$, $\mathtt{f1a} = \mathtt{f}'(\alpha)$.

Program 1. Mathematica code:
$f[\mathtt{e}] = \mathtt{f1a}(\mathtt{e} + \mathtt{c_2 e^2} + \mathtt{c_3 e^3} + \mathtt{c_4 e^4})$;
$\mathtt{ey1} = e - \mathtt{Series}\left[\frac{f[e]}{f'[e]}, \{e, 0, 3\}\right]$ //Simplify;
$\mathtt{ey} = e - \mathtt{Series}\left[\frac{f[e](f'[e]+f'[\mathrm{ey1}])}{2f'[e]f'[\mathrm{ey1}]}, \{e, 0, 3\}\right]$ //Simplify
$\mathtt{Out[ey]} = \frac{c_3 e^3}{2} + \mathtt{O[e]}^4$

Remark 1. As can be seen, this method is not optimal based on Kung and Traub's conjecture. It uses three functional evaluations per iteration while it has convergence order three. Here, our aim is to modify method (2.1) in such a way that it becomes optimal. More details are given in what follows.

Let us consider the following changes to (2.1). The first step of Ozban's method, namely Newton's method, is exchanged with the first step of Jarratt's method, namely weighted Newton's method. Then in the second step of Ozban's method, we introduce a weight function, say $h(t)$, as follows

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - h(t_n)\dfrac{f(x_n)}{f'(x_n)}\dfrac{(f'(x_n)+f'(y_n))}{2f'(y_n)}, \end{cases} \tag{2.2}$$

where $t_n = \frac{f'(y_n)}{f'(x_n)}$. Now, it is tried to optimize this new method. To this end, we impose some conditions on $h(t)$ so that we achieve an optimal class of Jarratt-type methods. Instead of using pencil-paper method to discuss the mentioned aim, we prefer to use the Mathematica approach. We think this technique has several advantages: it is fast, it saves space of the paper, and it avoids involving tedious and cumbersome calculations with using many terms of Taylor's series.

We reuse the symbols introduced before in giving the error equation for the method (2.1), also the rest of the abbreviations used are introduced as follows
$\mathtt{h} = h(0)$, $\mathtt{h1} = h'(0)$, $\mathtt{h2} = h''(0)$.

Program 2. Mathematica code:
$f[\mathsf{e}] = \mathtt{f1a}(\mathsf{e} + \mathsf{c}_2\mathsf{e}^2 + \mathsf{c}_3\mathsf{e}^3 + \mathsf{c}_4\mathsf{e}^4)$;
$\mathtt{ey1} = e - \mathtt{Series}\left[\frac{2*f[e]}{3f'[e]}, \{e,0,8\}\right]$;
$\mathtt{t} = \frac{\mathtt{f'[ey1]}}{f'[e]}$;
$h[\mathtt{t}] = \mathtt{h} + \mathtt{h1}t + \frac{\mathtt{h2}t^2}{2}$;
$\mathtt{ey} = \mathtt{e} - \frac{\mathtt{f[e](f'[e]+f'[ey1])*h[t]}}{\mathtt{2f'[e]f'[ey1]}}$//FullSimplify;
$\mathtt{a1} = \mathtt{Coefficient}[\mathtt{ey}, e]$//Simplify
$\mathtt{a2} = \mathtt{Coefficient}[\mathtt{ey}, e^2]$//Simplify
$\mathtt{a3} = \mathtt{Coefficient}[\mathtt{ey}, e^3]$//Simplify
$\mathtt{a4} = \mathtt{Coefficient}[\mathtt{ey}, e^4]$//Simplify

$\mathtt{Out[a1]=1\text{-}h} - \mathtt{h1} - \frac{\mathtt{h2}}{2}$
$\mathtt{Out[a2]} = \frac{1}{6}(\mathtt{2\ h\text{+}10h1} + \mathtt{9h2})c_2$
$\mathtt{Out[a3]} = \frac{1}{9}(-(\mathtt{2\ h\text{+}42h1} + \mathtt{49h2})c_2^2 + \mathtt{3\ (2\ h\text{+}10h1} + \mathtt{9h2})c_3)$
$\mathtt{Out[a4]} = \frac{1}{54}((\mathtt{20\ h\text{+}684h1} + \mathtt{978h2})c_2^3 - 3(\mathtt{22\ h\text{+}294h1} + \mathtt{347h2})c_2c_3$
$+ (\mathtt{58\ h\text{+}266h1} + \mathtt{237h2})c_4))$

To obtain an optimal class, the coefficients of e, e^2, and e^3 in the error equation of the new class (2.1) need to vanish. By solving the above system of equations simultaneously, equations $Out[a1]$, $Out[a2]$ and $Out[a3]$, the desired results are obtained and we have $\left\{\left\{\mathtt{h} \to \frac{7}{4}, \mathtt{h1} \to -\frac{5}{4}, \mathtt{h2} \to 1\right\}\right\}$. In other words, to provide the fourth order of convergence of the proposed method, it is necessary to choose $h = \frac{7}{4}$, $h1 = \frac{-5}{4}$ and $h2 = 1$. Therefore, we have established the following theorem about the convergence order of the optimal class (2.2).

Theorem 1. Let α be a simple zero of $f(x) = 0$ and function $h(t)$ is chosen so that the conditions $h(0) = \frac{7}{4}$, $h'(0) = \frac{-5}{4}$, and $h''(0) = 1$ hold. If an initial approximation is sufficiently close to α, then the equation (2.2) has the order of convergence four with the following error equation

$$e_{n+1} = (\frac{79}{27}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5). \tag{2.3}$$

The function $h(t)$ can take many forms satisfying the conditions of Theorem 1, examples of which are:

$$h_1(t) = \frac{7}{4} - \frac{5}{4}t + \frac{1}{2}t^2, \qquad h_2(t) = \frac{1}{\frac{4}{7} + \frac{20}{49}t + \frac{44}{343}t^2}, \qquad h_3(t) = \frac{\frac{7}{4} - \frac{1}{2}t - \frac{3}{4}t^2}{1+t}.$$

Accordingly, we can consider the following optimal method as a typical example of our proposed class

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (\frac{7}{4} - \frac{5}{4}t_n + \frac{1}{2}t_n^2)\frac{f(x_n)}{f'(x_n)}\frac{(f'(x_n) + f'(y_n))}{2f'(y_n)}. \end{cases} \tag{2.4}$$

3. Numerical implimentations

To verify the applicability of the derived method (2.2) of the optimal class of Jarratt-type methods we give two examples. Also, we report the results of the other methods given in this work for comparison. The implementations were ran in Mathematica. In Tables 1 and 2, the values of the computational order of convergence are computed by the following approximate formula (see Weerakoon and Fernando [13])

$$coc = \frac{ln(|x_{n+1} - \alpha|/|x_n - \alpha|)}{ln(|x_n - \alpha|/|x_{n-1} - \alpha|)},$$

where $|x_n - \alpha|$ denots absolute errors of approximations and $a(-b)$ means $a \times 10^{-b}$.

Example 1. Consider the following nonlinear equation

$$f(x) = e^{2+x-x^2} - \cos(1+x) + x^3 + 1, \qquad \alpha = -1,$$

with the initial approximation $x_0 = -0.7$.

Example 2. Consider the following nonlinear equation

$$f(x) = ln(1+x^2) + e^{-3x+x^2}\sin(x), \qquad \alpha = 0,$$

with the initial approximation $x_0 = 0.35$.

We have reported the obtained numerical results in Table 1 and 2. These results

Table 1: Numerical results of Example 1 in the first three iterations

	Absolute Error			
Two-point methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	coc
New Method (2.4)	0.2189(−3)	0.1566(−15)	0.4104(−64)	4
Method (1.4)	0.4557(−3)	0.2790(−14)	0.3925(−59)	4
Method (1.7)	0.4357(−1)	0.1170(−5)	0.5534(−23)	4
Method (1.1)	0.6543(−3)	0.1411(−13)	0.3056(−56)	4
Method (1.2)	0.6113(−2)	0.1490(−8)	0.5245(−35)	4

Table 2: Numerical results of Example 2 in the first three iterations

	Absolute Error			
Two-point methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	coc
New Method (2.4)	0.1965(−2)	0.2035(−9)	0.2326(−37)	4
Method (1.4)	0.5733(−2)	0.2999(−8)	0.2158(−33)	4
Method (1.7)	0.8517(−2)	0.1282(−6)	0.5757(−26)	4
Method (1.3)	0.1990(−2)	0.3071(−9)	0.1734(−36)	4
Method (1.2)	0.1948(−1)	0.3038(−5)	0.1747(−20)	4

confirm the theoretical prediction, which has been proved in the previous section. Moreover, it can be concluded that the proposed method (2.4) generates slightly better results in comparison with the other numerical methods mentioned in this paper.

4. Conclusion

In this research, a new optimal fourth order method based on Ozban's method has been developed for solving simple roots of nonlinear equations. The presented method has the convergence order four. It supports the Kung and Traub's conjecture requiring only three function evaluations per iteration and it has the efficiency index $4^{1/3} \approx 1.587$, which is better than Ozban's method $3^{1/3} \approx 1.390$ (for the definition of efficiency index see [11]).

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Accepted: 2.12.2018

Some spectral inclusion for strongly continuous semigroups operators

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Abstract. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. In this paper, we show that if there exists $t_0 > 0$ such that $T(t_0)$ is a pseudo B-Fredholm operator, then $T(t)$ is pseudo B-Fredholm for all $t \geq 0$, which is equivalent that $T(t)$ is generalized Drazin invertible for all $t \geq 0$. Also we prove that the spectral inclusion of strongly continuous semigroup hold for pseudo Fredholm, generalized Drazin and pseudo B-Fredholm spectra.

Keywords: C_0-semigroups, direct decomposition, pseudo Fredholm spectrum, generalized Drazin spectrum, pseudo B-Fredholm spectrum.

1. Introduction

Throughout, X denotes a complex Banach space, let us denote by $B(X)$ the algebra of bounded linear operators on X, let A be a closed linear operator on X with domain $D(A) \subseteq X$, we denote by A^*, $N(A)$, $R(A)$, $R^\infty(A) = \bigcap_{n\geq 0} R(A^n)$, $N^\infty(A) = \bigcup_{n\geq 0} N(A^n)$, $K(A)$, $H_0(T)$, $\rho(A)$, $\sigma(A)$, respectively the adjoint, the null space, the range, the hyper-range, the hyper-kernel, the analytic core, the quasi-nilpotent part, the resolvent set and the spectrum of A.

A closed operator A is said to be semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$, see [11]. A closed linear operator A is said to be upper semi-Fredholm if

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$R(A)$ is closed and $dimN(A) < \infty$, and A is lower semi-Fredholm if $codimR(A) < \infty$. If $dimN(A)$ and $codimR(A)$ are both finite then A is called Fredholm operator.

A closed operator A admits a generalized Kato decomposition (GKD) if there exist M, N two closed subspaces of X, A-invariant such that $X = M \oplus N$ and $A = A_{|N} \oplus A_{|M}$, with $A_{|N}$ is a quasi-nilpotent operator and $A_{|M}$ is a semi-regular operator, in this case A is called a pseudo-Fredholm operator (see [9, Definition 1]). The pseudo-Fredholm spectrum is defined by

$$\sigma_{pF}(A) = \{\lambda \in \mathbb{C} \ : A - \lambda \text{ is not pseudo-Fredholm}\}.$$

An operator A is called a pseudo B-Fredholm operator [1], if $A_{|M}$ is a Fredholm operator and $A_{|N}$ is a quasi-nilpotent operator. If $A_{|M}$ is an upper semi Fredholm operator, A is called upper pseudo B-Fredholm. Also if $A_{|M}$ is a lower semi Fredholm operator, A is called lower pseudo B-Fredholm [17].

The pseudo B-Fredholm spectrum, the upper pseudo B-Fredholm spectrum and the lower pseudo B-Fredholm spectrum are defined respectively by:

$$\sigma_{pBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not pseudo B-Fredholm}\},$$
$$\sigma_{upBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not upper pseudo B-Fredholm}\},$$
$$\sigma_{lpBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not lower pseudo B-Fredholm}\}.$$

The concept of generalized Drazin invertible operator has been defined by Koliha. A closed operator A is said to be generalized Drazin invertible, if there exists an operator $S \in B(X)$, $R(S) \subset D(A)$, $R(I - AS) \subset D(A)$, and $SA = AS$, $SAS = S$, $\sigma(A(I - SA)) = \{0\}$, this is equivalent that $A = A_1 \oplus A_2$ where A_1 is an invertible operator and A_2 is a quasi-nilpotent operator [8].

Let E be a subset of X. E is said T-invariant if $T(E) \subseteq E$. If E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$, we say that T is completely reduced by the pair (E, F) and it is denoted by $(E, F) \in Red(T)$. In this case we write $T = T_{|E} \oplus T_{|F}$ and say that T is the direct sum of $T_{|E}$ and $T_{|F}$.

In [3], M D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is bounded below and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.6]. An operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ and $(K(T), N) \in Red(T)$ which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is surjective and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.7].

The generalized Drazin invertible spectrum, generalized Drazin bounded below and surjective of $T \in \mathcal{B}(X)$ are defined respectively by

$\sigma_{gD}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not generalized Drazin } \}$,
$\sigma_{gD\mathcal{M}}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin bounded below}\}$;
$\sigma_{gD\mathcal{Q}}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin surjective}\}$.

We have:

$$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T) \cup \sigma_{gD\mathcal{Q}}(T).$$

A family $(T(t))_{t\geq 0}$ of operators on X is called a strongly continuous semigroup of operators if:

1. $T(0) := I$,
2. $T(s+t) := T(s)T(t) \quad$ for all $s, t \geq 0$
3. $\lim_{t\downarrow 0} T(t)x := x$, for every $x \in X$.

The linear operator A defined in the domain:

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} exists\}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \,|_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$, we note that the domain of A is dense in X and A is a closed operator.

In [2], [5] and [12], the authors proved that: $e^{t\sigma(A)} \subset \sigma(T(t))$ and $e^{t\nu(A)} \subseteq \nu(T(t)) \subseteq e^{t\nu(A)} \cup \{0\}$ where $\nu \in \{\sigma_p, \sigma_r\}$, point spectrum and residual spectrum.

After than Engle et al. [5] give a condition for a strongly continuous semigroup that satisfies this equality for spectrum and approximative spectrum, they proved that:

$$\sigma_{ap}(T(t)) \setminus \{0\} = e^{t\sigma_{ap}(A)}, t \geq 0,$$

and

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, t \geq 0,$$

where $T(t)$ is a eventually norm-continuous semigroup.

A. Elkoutri and M. A. Taoudi [4] proved that:

$$e^{t\nu(A)} \subseteq \nu(T(t)), \text{ for all } t \geq 0,$$

where $\nu(.) \in \{\sigma_\gamma(.); \sigma_{\gamma e}(.); \sigma_\pi(.); \sigma_F(.)\}$ the semi regular spectrum, essentially semi regular spectrum, upper semi-Fredholm and Fredholm spectrum, respectively.

In [14] we gave conditions of a strongly continuous semigroup that satisfies:

$$e^{t\sigma_\nu(A)} \subseteq \sigma_\nu(T(t)) \subseteq e^{t\sigma_\nu(A)} \cup \{0\},$$

for $\sigma_\nu(A)$ the semi regular spectrum, essentially semi regular spectrum, upper semi-Fredholm and Fredholm spectrum and proved that the first inclusion is true for B-Fredholm spectrum. In the same direction we proved that this inclusion is hold for Drazin invertible spectrum and quasi-Fredholm spectrum see [15]. The main objective of this article is to continue in the same direction and development of spectral theory for a C_0-semigroup and its generator. In section 2 we will give some proposition for the decomposition of strongly continuous semigroup and we prove that if there exists t_0 such that $T(t_0)$ is upper pseudo B-Fredholm (res.lower pseudo B-Fredholm, pseudo B-Fredholm) operator then $T(t)$ is upper pseudo B-Fredholm (resp.lower pseudo B-Fredholm, pseudo B-Fredholm) for all $t \geq 0$, same thing for left and right generalized Drazin invertible, B-Fredholm operator and for Drazin invertible.

In section 3 we prove that the spectral inclusion of strongly continuous semigroup hold for the pseudo-Fredholm spectrum, pseudo-B-Fredholm and generalized Drazin spectrum. Also, we will prove under the condition of a C_0-semigroup, that The following assertions are equivalents:
(i) A is pseudo-Fredholm;
(ii) A is generalized Drazin invertible;
(iii) A is pseudo B-Fredholm.

2. Decomposition of strongly continuous semigroup.

Let $T(t)$ be a strongly continuous semigroup and A its infinitesimal generator. In the first we will gives the following definition and some properties necessary for proof the subsequent results.

Subspace semigroups [5]. If Y is a closed subspace of X such that $T(t)Y \subseteq Y$ for all $t \geq 0$, (i.e., if Y is $(T(t))_{t\geq 0}$-invariant), then the restrictions $T(t)_| := T(t)_{|Y}$ form a strongly continuous semigroup $(T(t)_|)_{t\geq 0}$, called the subspace semigroup, on the Banach space Y.

The part of A in Y is the operator $A_|$ defined by

$$A_{|y} := Ay$$

with domain

$$D(A_|) := \{y \in D(A) \cap Y : Ay \in Y\}.$$

In other words, $A_|$ is the "maximal" operator induced by A on Y and, as will be seen, coincides with the generator of the semigroup $(T(t)_|)_{t\geq 0}$ on Y.

Proposition 2.1 ([5]). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X and assume that the restricted semigroup $(T(t)_|)_{t\geq 0}$ is strongly continuous on some $(T(t))_{t\geq 0}$-invariant Banach space $Y \hookrightarrow X$. Then the generator of $(T(t)|)_{t\geq 0}$ is the part $(A_|, D(A_|))$ of A in Y.*

Lemma 2.1 ([6, Lemma 332]). *If A is a closed linear operator $(X \to X')$ with $\beta(A) < \infty$, then A has closed range.*

It is clear that, if $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ two C_0-semigroups with generators A and B respectively, then for all $t \geq 0$, $R(t) = T(t) \oplus S(t)$ is a C_0-semigroups its generator is $R = A \oplus B$ [16], in the following proposition we prove the converse.

Now we denote by $T(t)_{|X_s}$ the restrictions of $T(t)$ on X_s and $T(t)_{|X_u}$ the restrictions of $T(t)_{t\geq 0}$ on X_u.

Proposition 2.2. *Let$(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. If there exist X_s, X_u two closed $(T(t))_{t\geq 0}$-invariants subspaces of X, such that $X = X_s \oplus X_u$ then $T(t)_{|X_s}$ and $T(t)_{|X_u}$ are strongly continuous semigroups, furthermore the generator of a strongly continuous semigroup $T(t) = T(t)_{|X_s} \oplus T(t)_{|X_u}$ is $A = A_{|X_s \cap D(A)} \oplus A_{|X_u \cap D(A)}$ defined in $D(A) = D(A) \cap X_s \oplus D(A) \cap X_u$.*

Proof. According to the definition of subspace semigroup and X_s, X_u are a closed $(T(t))_{t\geq 0}$-invariants subspaces of X, then X_s and X_u are a Banach spaces therefore the strongly continuity of $(T(t)_{|X_s})_{t\geq 0}$ and $(T(t)_{|X_u})_{t\geq 0}$ are automatic.

Moreover the existence of

$$y = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x_s - x_s) \in X,$$

for some $x_s \in X_s$ implies that $y \in X_s$, therefore the generator of $(T(t)_{|X_s})_{t\geq 0}$ is $A_{|X_s \cap D(A)}$ with domain $D(A) \cap X_s$, the same for the generator of $(T(t)_{|X_u})_{t\geq 0}$ is $A_{|X_u \cap D(A)}$ with domain $D(A) \cap X_u$ and $A_{|X_s \cap D(A)} \oplus A_{|X_u \cap D(A)}$ is a generator of a strongly continuous semigroup $T(t) = T(t)_{|X_s} \oplus T(t)_{|X_u}$ with domain $D(A) = D(A) \cap X_s \oplus D(A) \cap X_u = D(A) \cap (X_s \oplus X_u) = D(A) \cap X$. $\square$

Remark 1. We recall that the C_0-semigroup $(T(t))_{t\geq 0}$ is nilpotent if there exists $t_0 > 0$, such that $T(t) = 0$ for $t \geq t_0$. It is clear that if there exists $t_0 > 0$, such that $T(t_0)$ is nilpotent operator then $T(t)$ is nilpotent for all $t \geq 0$.

Also, we recall that the C_0-semigroup $(T(t))_{t\geq 0}$ is quasi-nilpotent if $\{0\} = \sigma(T(t))$, then if there exists $t_0 > 0$, such that $T(t_0)$ is quasi-nilpotent operator then $T(t)$ is quasi-nilpotent for all $t \geq 0$.

Now we will proof the following property that depends of the decomposition of strongly continuous semigroup.

Proposition 2.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup.*

1. *If there exists $t_0 > 0$ such that $T(t_0)$ is upper pseudo B-Fredholm then $T(t)$ is upper pseudo B-Fredholm for all $t \geq 0$.*

2. *If there exists $t_0 > 0$ such that $T(t_0)$ is lower pseudo B-Fredholm then $T(t)$ is lower pseudo B-Fredholm for all $t \geq 0$.*

3. *If there exists $t_0 > 0$ such that $T(t_0)$ is pseudo B-Fredholm then $T(t)$ is pseudo B-Fredholm for all $t \geq 0$.*

Proof. 1. If there exists $t_0 > 0$ such that $T(t_0)$ is upper pseudo-B-Fredholm then there exist two closed $T(t_0)$-invariants subspaces $X_1, X_2 \subset X$ such that $T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}, T(t_0)_{|X_1}$ is upper semi Fredholm and $T(t_0)_{|X_2}$ is quasi-nilpotent. Since $T(t_0)_{|X_1}$ is upper semi Fredholm then $\alpha(T(t_0)_{|X_1}) < \infty$ and $R(T(t_0)_{|X_1})$ is closed. We show that $\alpha(T(t)_{|X_1}) < \infty$ and $R(T(t)_{|X_1})$ is closed for all $t \geq 0$. Since $\alpha(T(t_0)_{|X_1}) < \infty$ then 0 is an eigenvalue with finite multiplicity of $T(t_0)$. As proof [12, Theorem 6.6], let $x \in X_1, x \neq 0$ be an eigenvector associated to 0. Putting $t_1 = t_0/2$, then $T(t_0)x = T(t_1)T(t_1)x = 0$, hence 0 is an eigenvalue of $T(t_1)$. Proceeding by induction, we define a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $T(t_n)$, for all $n \in \mathbb{N}$.

For $n \geq 0$, we define the sets

$$\digamma_n = N(T(t_n)_{|X_1}) \bigcap \{x \in X_1 : ||x|| = 1\}.$$

Clearly, the inclusion $N(T(s)_{|X_1}) \subseteq N(T(t)_{|X_1})$, for $s \geq t$ implies that $(\digamma_n)_n$ is a decreasing sequence (in the sense of the inclusion) of nonempty compact subsets of X_1. Thus $\bigcap_{n=0}^{\infty} \digamma_n \neq \emptyset$. If $x \in \bigcap_{n=0}^{\infty} \digamma_n$ then

$$(**) \qquad \|T(t_n)x - x\| = \|x\| = 1 \text{ for all } n \geq 1$$

Since $t_n \to 0$ as $n \to \infty$, (**) contradicts the strong continuity of $(T(t)_{|X_1})_{t\geq 0}$.

This shows that $N(T(t_0)_{|X_1}) = \{0\}$, that is, $(T(t_0)_{|X_1})$ is injective and $\alpha(T(t_0)_{|X_1}) = 0$. Let $0 < t \leq t_0$. The inclusion $N(T(t)_{|X_1}) \subseteq N(T(t_0)_{|X_1})$ implies that $\alpha(T(t)_{|X_1}) = 0$. Assume now that $t > t_0$ and $x \in N(T(t)_{|X_1})$, then there exists an integer n such that $nt_0 > t$ and therefore $T(nt_0)x = T(nt_0 - t)T(t)x = 0$. Hence, we have $x = 0$ and consequently $N(T(t)_{|X_1}) = \{0\}$ for all $t > t_0$, therefore $(T(t)_{|X_1})$ is injective and $\alpha(T(t)_{|X_1}) = 0$ for all $t \geq 0$.

It remains to show that $R(T(t)_{|X_1})$ is closed for all $t \geq 0$. Assume that $T(t_0)_{|X_1}$ is upper semi Fredholm , then $\alpha(T(t_0)_{|X_1}) < \infty$ and $\beta(T(t_0)_{|X_1}) = \infty$ (if $\beta(T(t_0)_{|X_1}) < \infty$, as proof (2) then $\beta(T(t)_{|X_1}) < \infty$ for all $t \geq 0$ according to lemma 2.1 $R(T(t_0)_{|X_1})$ is closed). Let $T^*(t_0)$ be the dual operator of $T(t_0)$. Obviously, $(T^*(t_0)_{|X_1^*})$ is lower semi Fredholm and consequently $\beta(T^*(t_0)_{|X_1^*}) < \infty$. Hence $\beta(T^*(t)_{|X_1^*}) < \infty$ for all $t \geq 0$. Now applying lemma 2.1 we infer that $R(T^*(t))$ is closed in X_1^*, for all $t \geq 0$. This together with the closed graph theorem of Banach [19, page 205] implies that $R(T(t))$ is closed in X_1 for all $t \geq 0$. Therefore $T(t)_{|X_1}$ is upper semi Fredholm for all $t \geq 0$. Also we have $T(t_0)_{|X_2}$ is quasi-nilpotent implies that $T(t)_{|X_1}$ is quasi-nilpotent for all $t \geq 0$, therefore $T(t)$ is upper pseudo B-Fredholm for all $t \geq 0$.

2. To prove this item, we will proceed by duality. Let $(T^*(t))_{t\geq 0}$ be the dual semigroup of $(T(t))_{t\geq 0}$. Since $\beta(T(t)_{|X_1}) = \alpha(T^*(t)_{|X_1^*})$, then it suffices to show that $\alpha(T^*(t)_{|X_1^*}) = 0$ for all $t \geq 0$. By hypothesis, we have $\alpha(T^*(t_0)_{|X_1^*}) <$

∞. Let x^* be an element of $N(T^*(t_0)_{|X_1^*})$. Arguing as above, we construct a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $T^*(t_n)$, for all $n \in \mathbb{N}$ and we define the sets

$$\daleth_n = N(T^*(t_n)_{|X_1^*}) \bigcap \{x^* \in X_1^* : ||x^*|| \neq 1\}.$$

Clearly, the inclusion $N(T^*(s)_{|X_1^*}) \subseteq N(T^*(t)_{|X_1^*})$, for $s \geq t$, imply that $(\daleth_n)_n$ is a decreasing sequence (in the sense of the inclusion) of nonempty compact subsets of X_1^*. Thus

$$\bigcap_{n=0}^{\infty} \daleth_n \neq \emptyset.$$

If $x^* \in \bigcap_{n=0}^{\infty} \daleth_n$ then

$$(***) \qquad | < T^*(t_n)x^* - x^*, x > | = | < x^*, x > | \neq 0 \ \ \forall n \geq 1, \text{ for all } x \in X_1.$$

Using the fact that $(T^*(t))_{t\geq 0}$ is continuous in the *weak** topology at $t = 0$, we conclude that

$$(****) \qquad \lim_{t\to 0} | < T^*(t_n)x^* - x^*, x > | = 0, \qquad \text{for all } x \in X_1.$$

Combining (***) and (****), we obtain $< x^*, x >= 0$ for all $x \in X_1$. This shows that $x^* = 0$ and therefore $\alpha(T^*(t_0)) = 0$. By the same argument as above, we show that $\alpha(T^*(t)_{|X_1^*}) = 0$ for all $t \geq 0$.

Assume now that $T(t_0)_{|X_1}$ is lower semi Fredholm, then $\beta(T(t_0)) < \infty$ and $\alpha(T(t_0)_{|X_1}) = \infty$ (if $\alpha(T(t_0)_{|X_1}) < \infty$ the proof is contained in (1)). It follows From the above that $\beta(T(t)_{|X_1}) < \infty$ for all $t \geq 0$. Again using 2.1 we see that $R(T(t)_{|X_1})$ is closed in X_1 for all $t \geq 0$, which completes the proof of (2).

3. It follows from (1) and (2). □

The proof of the following Theorem produces directly from proof of Proposition 2.3.

Note that $(T(t))_{t\geq 0}$ is upper(lower) pseudo B-Fredholm if $(T(t))_{t\geq 0}$ is upper(lower) pseudo B-Fredholm for all $t \geq 0$.

Theorem 2.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup.*

1. *A C_0-semigroup $T(t)$ is upper pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin bounded below.*

2. *A C_0-semigroup $T(t)$ is a lower pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin surjective.*

3. *A C_0-semigroup $T(t)$ is pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin invertible.*

Proposition 2.4. *Let $t_0 > 0$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X.*

1. *If $T(t_0)$ is a B-Fredholm operator, then $T(t)$ is a B-Fredholm operator for all $t \geq 0$.*
2. *If $T(t_0)$ is Drazin invertible, then $T(t)$ is Drazin invertible for all $t \geq 0$.*
3. *If $T(t_0)$ is a generalized Drazin invertible operator, then $T(t)$ is a generalized Drazin invertible operator for all $t \geq 0$.*

Proof. 1. Suppose that $T(t_0)$ is a B-Fredholm operator, then there exist two closed subspaces $X_1, X_2 \subset X$ $T(t)$-invariants, such that

$$X = X_1 \oplus X_2, \; T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}.$$

$T(t_0)_{|X_1}$ is a Fredholm operator and $T(t_0)_{|X_2}$ is nilpotent. Moreover as a C_0-semigroup $T(t_0)_{|X_1}$ is a Fredholm operator, then according to proof (3) of Proposition 2.3, we have $T(t)_{|X_1}$ is a Fredholm operator for all $t \geq 0$ and also from remake 1 $T(t)_{|X_2}$ is nilpotent for all $t \geq 0$. This show that $T(t)$ is a B-Fredholm operator, for all $t \geq 0$.

2. Suppose that $T(t_0)$ is Drazin invertible, then there exist two closed subspaces $X_1, X_2 \subset X$ $T(t)$-invariants, such that

$$X = X_1 \oplus X_2, \; T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}.$$

$T(t_0)_{|X_1}$ is an invertible operator and $T(t_0)_{|X_2}$ is nilpotent. As a C_0-semigroup $T(t_0)_{|X_1}$ is an invertible operator, according to [5, Proposition page 80], we have $T(t)_{|X_1}$ is an invertible operator for all $t \geq 0$ and also $T(t)_{|X_2}$ is nilpotent for all $t \geq 0$. This show that $T(t)$ is a Drazin invertible operator, for all $t \geq 0$.

3. By the same argument of (2) □

3. Spectrum inclusion for C_0-semigroup

To continue the development of a spectral theory for semigroups and their generators, we will give a technique to prove that the inclusion spectral is holds for $\sigma_{pF}, \sigma_{lgD}, \sigma_{rgD}, \sigma_{gD}$ and σ_{pBF}. For this we begin with proved the following result which will be used to prove the following theorem.

Proposition 3.1. *Let $(T(t))_{t\geq}$ a C_0-semigroup and A its generator. If $e^{\lambda t} - T(t)$ is quasi-nilpotent for some $\lambda \in \mathbb{C}$, then $\lambda - A$ is quasi-nilpotent.*

Proof. We have $e^{\lambda t} - T(t)$ is quasi-nilpotent for some $\lambda \in \mathbb{C}$, then $\sigma(e^{\lambda t} - T(t)) = \{0\}$, since $e^{t\sigma(A)} \subseteq \sigma(T(t)) = \{e^{\lambda t}\}$, this implies that $\sigma(A) \subseteq \{\lambda\}$ therefore $\sigma(\lambda - A) \subseteq \{0\}$. Hence $\lambda - A$ is quasi-nilpotent. □

Theorem 3.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{pF}(.); \sigma_{pBF}(.)\}$.

Proof. Pseudo-Fredholm spectrum. Let $t_0 > 0$ be fixed and suppose that $(e^{\lambda t_0} - T(t_0))$ is pseudo-Fredholm, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then there exist two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces X_1, X_2 of X such that $X = X_1 \oplus X_2$, $(e^{\lambda t_0} - T(t_0))_{|X_1}$ is a semi regular operator and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent.

From [4, Theorem 2.1] this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is a semi regular operator and according to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is pseudo-Fredholm.

Pseudo B-Fredholm. Let $t_0 > 0$ be fixed and suppose that $(e^{\lambda t_0} - T(t_0))$ is pseudo B-Fredholm, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then there exist X_1, X_2 two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is a Fredholm operator and $(e^{\lambda t_0} - T(t_0)\big)_{|X_2}$ is quasi-nilpotent. From [13] this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is a Fredholm operator and according to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is pseudo B- Fredholm. □

By the same argument we can proof the following theorem.

Theorem 3.2. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{upBF}(.); \sigma_{lpBF}(.)\}$.

Theorem 3.3. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{gD\mathcal{M}}(.); \sigma_{gD\mathcal{Q}}(.); \sigma_{gD}(.)\}$.

Proof. Generalized Drazin bounded below:

Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin bounded below, then, there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - (T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is bounded below, and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent. From [5], this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is bounded below, and according to proposition 3.1, we have $(\lambda - A_{|(D(A)\cap X_2)})$ is quasi-nilpotent, then $(\lambda - A)$ is generalized Drazin bounded below.

Generalized Drazin surjective. Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin surjective, then there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is surjective and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent. As we have

$$X_1 = R(e^{\lambda t_0} - T(t_0))_{|X_1}) \subseteq R((\lambda - A)_{|(D(A)\cap X_1)}),$$

then $(\lambda - A)_{|(D(A)\cap X_1)}$ is surjective. According to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is right generalized Drazin inverse.

Generalized Drazin inverse. Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin inverse then there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is invertible and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent.

As X_1, X_2 two subspaces closed of X then X_1, X_2 are a Banach spaces and from [5, 18] and [12, Theorem 2.3], we have $(e^{\lambda t_0} - T(t_0))_{|X_1}$ is invertible this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is invertible, and according to proposition 3.1, $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is generalized Drazin inverse. □

In the end of this paper we prove the following theorem.

Theorem 3.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$.*

If $\lim_{t\to\infty} \frac{1}{t^n}\|T(t)\| = 0$, for some $n \in \mathbb{N}$, the following assertions are equivalents:

1. *A is pseudo-Fredholm;*
2. *A is generalized Drazin invertible;*
3. *A is pseudo B-Fredholm.*

Proof. $(1) \Rightarrow (2)$: Since A is pseudo-Fredholm then there exist $(X_1 \cap D(A), X_2 \cap D(A))$ two closed A-invariant subspaces of $D(A)$, such that

$$D(A) = X_1 \cap D(A) \oplus X_2 \cap D(A); \quad A = (A_{|D(A)\cap X_1}) \oplus (A_{|D(A)\cap X_2}),$$

$(A_{|(D(A)\cap X_2)})$ is quasi-nilpotent and $(A_{|(D(A)\cap X_1)})$ is a semi regular operator.

Since $A_{|(D(A)\cap X_1)}$ is a semi regular operator, therefore $R(A_{|(X_1\cap D(A))})$ is closed and $N(A_{|(X_1\cap D(A))}) \subseteq R^\infty(A_{|(X_1\cap D(A))}) \subseteq R^\infty(A)$.

Let $y \in N(A_{|(X_1 \cap D(A))})$ then there exists $x \in (X_1 \cap D(A^n))$ such that $y = A^n x$.

We integrate by parts in the formal :

$$T(t)x - x = \int_0^t T(s)Axds, \qquad \text{for all } x \in (X_1 \cap D(A^n)), \text{ and for all } t \geq 0.$$

We obtain,

$$T(t)x = x + tAx + \frac{t^2}{2!}A^2x + \int_0^t \frac{(t-s)^2}{2!}T(s)A^3xds.$$

We repeat these operations we obtain:

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^nxds.$$

Hence,

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + y\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds.$$

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \frac{t^n}{n!}y.$$

Dividing by $t^n > 0$:

$$\frac{1}{t^n}T(t)x = \frac{1}{t^n}\sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \frac{1}{n!}y.$$

As $\lim_{t\to\infty} \frac{1}{t^n}\|T(t)\| = 0$ and $\lim_{t\to\infty} \frac{1}{t^n}\sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx = 0$ for all $0 \leq k \leq n-1$, then $y = 0$, yields $N(A_{|X_1 \cap D(A)}) = \{0\}$.

On the other hand, let $(T(t)')_{t\geq 0}$ with generator A' the adjoint semigroup of $(T(t))_{t\geq 0}$. Since $A_{|(X_1 \cap D(A))}$ is semi regular, then $A'_{|(X_1' \cap D(A'))}$ is also semi regular see [10, Proposition 1.6]. By using the formula [18, Proposition 1.2.2],

$$T(t)'x' - x' = weak^* \int_0^t T(s)'A'x'ds, \text{ for all } x' \in (X_1' \cap D(A')), \text{ and for all } t \geq 0.$$

In the same manner as above we can show that: $N(A'_{|(X_1' \cap D(A'))}) = \{0\}$. This is equivalent to $\overline{R(A_{|(X_1 \cap D(A))})} = (X_1 \cap D(A))$.

Since $R(A_{|(X_1 \cap D(A))})$ is closed therefore $R(A_{|(X_1 \bigcap D(A))}) = (X_1 \cap D(A))$. Then $A_{|(X_1 \cap D(A))}$ is surjective then $A_{|(X_1 \cap D(A))}$ invertible and as $A_{|(X_2 \cap D(A))}$ is quasi-nilpotent consequently A is generalized Drazin invertible.

(2) $\Rightarrow$ (1): Obvious.

Since the class of generalized Drazin invertible operator is a subclass of pseudo B-Fredholm operator and the class of pseudo B-Fredholm operator is a subclass of pseudo-Fredholm operator, hence (1) and (2) and (3) are equivalent. $\square$

Acknowledgement

The authors thank the referees for their suggestions, comments thorough reading of the manuscript.

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Accepted: 12.12.2018

A reliable non-standard finite difference scheme for solving nonlinear biochemical reaction model

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Abstract. In this paper, we present an efficient and accurate numerical scheme for the solution of a model biochemical reaction. The non–standard finite difference scheme based on Adomian decomposition method does not need to linearized or non-locally linearized for the nonlinear term of differential equation. The decomposition method is adopted to construct the numerical solutions. The results demonstrate reliability and efficiency of the algorithm developed.

Keywords: non-standard finite deference schemes, Adomian decomposition method, Biochemical reaction model.

1. Introduction

In recent times, the non-standard finite difference schemes by Mickens [1, 2, 3, 4, 5, 6] (in short NSFD) has developed as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, biological models, chaotic systems [5]. The technique has many advantages over the classical techniques [16], and provides an efficient numerical solution.

The well-known Michalis-Menten biochemical reaction model [14]

$$E + A \rightleftharpoons Y \longrightarrow E + X, \tag{1}$$

where E is the enzyme, A the substrate, Y the intermediate complex and X the product. The time evolution of scheme 26 can be determined from the solution

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of the system of coupled nonlinear $ODEs$[15]

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -k_1EA + k_{-1}Y, \tag{2}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -k_1EA + (k_{-1} + k_2)Y, \tag{3}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = k_1EA - (k_{-1} + k_2)Y, \tag{4}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = k_2Y, \tag{5}$$

where the initial conditions

$$A(0) = A_0, \quad E(0) = E_0, \quad Y(0) = 0, \quad X(0) = 0, \tag{6}$$

and the parameters k_1, k_{-1} and k_2 are positive rate constants for each reaction.

The NSFD solution [8] for such differential equation of the form

$$\frac{dy}{dt} = f(t, y(t)), \tag{7}$$

where $f(t, y(t))$ called the nonlinear term in the differential equation. Using finite difference method we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y_{k+1} - \psi(h)y_k}{\phi(h)}, \tag{8}$$

where ϕ and ψ are functions of the step size $h = \Delta t$. The ψ and ϕ have the following properties

$$\psi = 1 + o(h), \tag{9}$$

$$\phi(h, \lambda) = h + o(h^2), \tag{10}$$

$h \to 0$ and $\lambda =$ fixed, the numerator function ψ is usually equal to one [13] unless the system has dissipation.

Examples of functions $\phi(h, \lambda)$ that satisfy the previous condition are $\phi(h) = h$, $\sin(h)$, $\sinh(h)$, $e^h - 1$, $\frac{1-e^{-\lambda h}}{\lambda}$, etc.

Non linear terms can be in general be replaced by nonlocal discrete representations. For example, $y^2 \approx y_k y_{k+1}$, $y^3 \approx (\frac{y_{k+1}+y_{k-1}}{2})y_k^2$. Set $h = T/N$, $t_n = nh$, $n = 0, 1, ..., N \in Z^+$. Then Eq.(7) can be discretized as follows,

$$y(t_{n+1}) = \psi(h)y(t_n) + \phi(h)f(t_{n+1}, y(t_{n+1}), y(t_n)), \tag{11}$$

where $f(t_n, y(t_{n+1}), y(t_n))$ is come from the non-locally linearized of $f(t_{n+1}, y(t_{n+1}))$.

If we do not use the non-locally linearized the system the differential equation, a somewhat better method is chosen–the Newton iteration method–to numerically solve the algebraic equation. This requires that $f(t, y(t))$ be smooth and that the inverse of the derivative operator f_y exists. For a system of equations, Newton method often needs a lot of time so it is not economical. Luckily, Adomian decomposition method (ADM) [10, 11, 12] can be used to solve this problem effectively.

2. Adomian decomposition

Consider the general nonlinear equation [9]

$$u = N(u) + f,$$

where N is a nonlinear operator, and where f is supposed to be known. The decomposition method consists in looking for a solution having the series form

$$u = \sum_{i=0}^{\infty} u_i.$$

The nonlinear operator N is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n's are called Adomian's polynomials. In the first approach given by Adomian, A_n's are abtaind from the following equalities

$$\begin{aligned} v &= \sum_{i=0}^{\infty} \lambda^i u_i, \\ N(v) &= N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) = \sum_{n=0}^{\infty} \lambda^i A_n \end{aligned}$$

we remark that A_n's are formally obtained from the relationship

$$A_n = \frac{1}{n!}\frac{\mathrm{d^n}}{\mathrm{d}\lambda^{\mathrm{n}}}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}.$$

The above process leads to the equality

$$\sum_{i=0}^{\infty} u_i = \sum_{n=0}^{\infty} A_n + f,$$

and the Adomian method consists in identifying u_i by means of the formula below

$$\begin{aligned} u_0 &= f, \\ u_1 &= A_0(u_0), \\ u_2 &= A_1(u_0, u_1), \\ &\vdots \\ u_n &= A_{n-1}(u_0, u_1 \cdots u_{n-1}). \end{aligned}$$

The solution u can be written as a series of functions u_i i.e.,

$$\sum_{i=0}^{\infty} |u_i| < +\infty.$$

3. The numerical scheme

Consider the nonlinear differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y(t)), \tag{12}$$

using the first derivatives of Mickens gives

$$\frac{y_{k+1} - y_k}{\phi(h)} = f(t_{k+1}, y_{k+1}), \tag{13}$$

solving Eq.13 for y_{k+1} gives

$$y_{k+1} = y_k + \phi(h) f(t_{k+1}, y_{k+1}). \tag{14}$$

For the nonlinear difference algebraic equations (14) using ADM to solve this kind of problems.

Suppose

$$y_{k+1} = \sum_{i=0}^{\infty} u_i, \tag{15}$$

where

$$u_0 = y_k, \tag{16}$$
$$u_1 = \phi(h) f(A_0), \tag{17}$$
$$u_2 = \phi(h) f(A_1), \tag{18}$$
$$\vdots$$
$$u_n = \phi(h) f(A_{n-1}). \tag{19}$$

For the n-term of the ADM solution we have

$$y_{k+1} = \sum_{i=0}^{n-1} u_i. \tag{20}$$

4. Application and results

Systems (2)–(5) can be written as [15]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + (\beta - \alpha) y + xy, \tag{21}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\epsilon}(x - \beta y - xy), \tag{22}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0, \tag{23}$$

(24)

where α, β and ϵ are dimensionless parameters. We will illustrate the ANSFD scheme to solve the system (21) and (22).

$$\frac{x_{k+1}-x_k}{\phi(h)} = -x_{k+1}+(\beta-\alpha)y_{k+1}+x_{k+1}y_{k+1}, \tag{25}$$

$$\frac{y_{k+1}-y_k}{\phi(h)} = \frac{1}{\epsilon}(x_{k+1}-\beta y_{k+1}-x_{k+1}y_{k+1}), \tag{26}$$

where $\phi(h)=\sin(h)$. Solving (25) and (26) for x_{k+1} and y_{k+1} gives

$$x_{k+1} = x_k+\phi(h)(-x_{k+1}+(\beta-\alpha)y_{k+1}+x_{k+1}y_{k+1}), \tag{27}$$

$$y_{k+1} = y_k+\frac{\phi(h)}{\epsilon}(x_{k+1}-\beta y_{k+1}-x_{k+1}y_{k+1}). \tag{28}$$

Using the ADM to solve (27) and (28)

$$x_{k+1} = \sum_{i=0}^{n-1} v_i, \quad y_{k+1}=\sum_{i=0}^{n-1} u_i, \tag{29}$$

where

$$v_0 = x_k, \quad u_0 = y_k, \tag{30}$$

$$v_1 = \phi(h)[-v_0+(\beta-\alpha)u_0+v_0u_0], \quad v_1=\frac{\phi(h)}{\epsilon}[v_0-\beta u_0-v_0u_0], \tag{31}$$

$$\vdots$$

$$v_n = \phi(h)[-v_{n-1}+(\beta-\alpha)u_{n-1}+\sum_{i=0}^{n-1} v_j u_{n-1-j}], \tag{32}$$

$$u_n = \frac{\phi(h)}{\epsilon}[v_{n-1}-\beta u_{n-1}-\sum_{j=0}^{n-1} v_j u_{n-1-j}]. \tag{33}$$

The biochemical reaction model (21) and (22) were numerically integrated using the ANSFD scheme is coded in the computer algebra package Maple and we employ the Maple's built-in fourth-order Runge-kutta procedure RK4. The Maple environment variable Digits controlling the number of significant Digits is set to 35 in all the calculations done in this paper we have set the dimensionless parameters $\alpha = 0.375$, $\beta = 1.0$ and $\epsilon = 0.1$ with initial conditions $x(0) = 1$, $y(0) = 0$. It is observed that the 4–term ANSFD solutions agree very well with the RK4 solution for time range $t \in [0, 20]$.

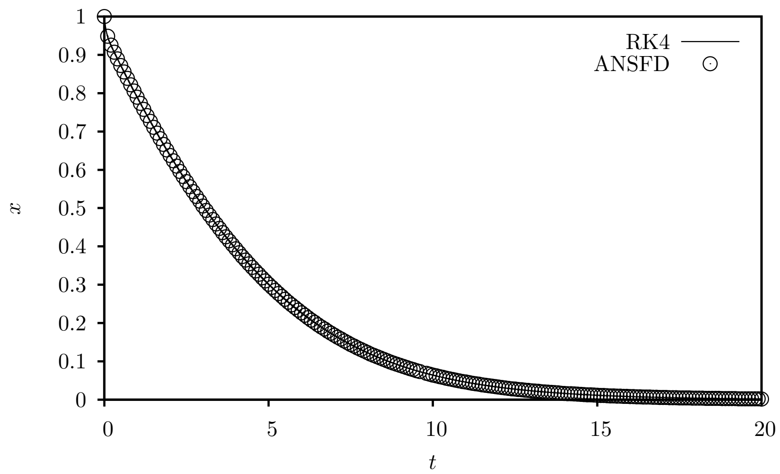
In Table 1 we present the absolute errors between RK4 solutions at time step $\Delta t = 0.01$ and ANSFD solutions at time step $\Delta t = 0.01$ and $\Delta t = 0.01$. Fig. 1 a and b show the solution for x and y, respectively, obtained by the 4-term ANSFD with $\Delta t = 0.01$. It is observed that the ANSFD solutions agree very well with the RK4 solutions for t upto $t = 20$.

5. Conclusion

In this paper, we derive a reliable algorithm based on Adomian decomposition for differential equations to solve a biochemical reaction model. The results obtained are in excellent agreement with those by (RK4) .

Table 1: Differences between 4-term ANSFD and RK4 solutions for the biochemical reaction model.

t	$\Delta = \lvert \mathrm{RK4}_{0.001} - \mathrm{ANSFD}_{0.01} \rvert$		$\Delta = \lvert \mathrm{RK4}_{0.001} - \mathrm{ANSFD}_{0.001} \rvert$	
	Δx	Δy	Δx	Δy
2	1.804E-04	6.406E-05	1.761E-05	6.249E-06
4	2.96E-04	1.493E-04	2.899E-05	1.462E-05
6	3.338E-04	2.213E-04	3.280E-05	2.174E-05
8	2.993E-04	2.4E-04	2.947E-05	2.364E-05
10	2.283E-04	2.06E-04	2.251E-05	2.032E-05
12	1.557E-04	1.502E-04	1.537E-05	1.483E-05
14	9.863E-05	9.858E-05	9.735E-06	9.731E-06
16	5.944E-05	6.050E-05	5.867E-06	5.972E-06
18	3.462E-05	3.556E-05	3.417E-06	3.510E-06
20	1.969E-05	2.032E-05	1.943E-06	2.005E-06



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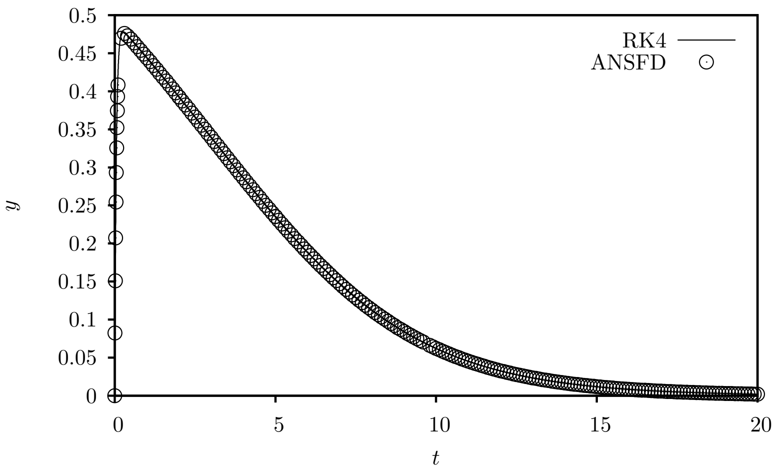

Figure 1: Solution of x and y using 4-term ANSFD ($h = 0.01$) and RK4 ($h = 0.001$)

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Accepted: 1.12.2018

Numerical simulations for reactive nitrogen compounds pollution measurements in a stream using Saulyev method

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Abstract. Nutrient pollution is one of most harmful environmental problems, and is caused by surplus nitrogen in water. This nitrogen concentration occurring in water can take several forms, such as organic nitrogen, ammonia, nitrite, nitrate, and dissolved nitrogen gas. Pollution levels can be measured via data collection; however, this is a rather difficult and complex process, and the results obtained widely deviate in term of measurement. A mathematical model can be used in complicated water-quality measurement. The advection-dispersion-reaction model provides a pollutant concentration field. In this research, there are five numerical models for nitrogen pollutant concentration measurement in a stream proposed: a total nitrogen dispersion model, an organic nitrogen dispersion model, an ammonia dispersion model, a nitrite dispersion model, and a nitrate dispersion model. The traditional Forward Time Central Space finite difference technique and the unconditionally explicit Saulyev technique are employed to obtain five approximated types of organic and inorganic nitrogen pollutant concentrations in each time and place. This paper proposes five forms of nitrogen pollutant measurement model for the unconditionally stable Saulyev method, so as to make it more accurate without incurring any significant loss of computational efficiency. The

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five approximated forms of pollutant concentrations obtained indicate that all models improve the nutrient pollution measurement process.

Keywords: numerical simulations, nitrogen compound, Saulyev technique.

1. Introduction

Water is a main factor in the survival of life on Earth. Water pollution is a global problem caused by global population growth and economic growth. It is an important problem affecting society and the environment. The main causes of this problem are human settlements, industries, and agriculture. One of the leading water pollution causes is nutrient pollution. Nutrients, such as nitrogen and phosphorus, are chemicals that flow into natural water, such as rivers, lakes, and coastal oceans, coming from either point or nonpoint sources. Nutrients are necessary for aquatic and coastal ecosystems such as plant growth. Major sources of nutrients are transportation, industry, urban activity, fertilizer, animal wastes, plant debris, and nitrogen transported from excess or effluent communities in soil to water sources [1].

Nitrogen is one of the essential contaminants of water. Nitrogen concentrations occurring in natural waters can be modified into a variety of organic and inorganic forms, five being organic nitrogen, ammonia, nitrite, nitrate, and dissolved nitrogen gas [2]. Due to dissolved nitrogen gas having no biological effect, this is ignored. Excess nitrogen in the water can occur in pollution in general, such as low oxygen concentrations in natural waters caused by nitrogen compound oxidation, or toxic substances found in aquatic animals and affecting human health. Nitrogen is seen as one of the key nutrients for eutrophication, public health matters, and aquatic and the other problems.

Mathematical models are widely used in explaining environmental processes such as chemical, biology, physical, and other processes; parabolic equations can be reasonably explained in mathematical models [3]. The advection and diffusion equation is a form of PDE parabolic equation which plays an important role in describing transport processes and has that obtained popularity in solving various problems, including environmental problems such as water quality measurement [4], [5], and [6], air quality measurement, and others. For the measurement of water quality, many researches have used these equations to measure concentrations occurring in natural water sources, such as rivers, and to compare computed values with real values; their compared yields are similar [5] and [6]. The finite difference method is one of the most efficient methods of problem solving in advection and diffusion equations, etc. They can be classified into two forms: explicit and implicit schemes [3]. Explicit schemes are simple to calculate, and have been used to present stability and accuracy properties. In [4], a water quality assessment of a non-uniform flow stream was conducted by using the Crank-Nicolson method, and the explicit finite difference schemes were proposed respectively. The forward Time Central Space (FTCS) and Saulyev schemes were used to determine water quality concentration. In [7], they were implemented to measure air pollutant concentrations in an area under a sky train; three dimensional advection and diffusion equations were solved by using the explicit FTCS method, such that this method was used in two cases of wind inflow as the x direction and the $x-y$ directions. [8], described the measurement of water pollutant concentration control in a connected-pond reservoir that were connected two ponds using a hydrodynamic model and a steady-state pollutant dispersion model through a three different finite difference scheme as backward, forward, and central-in-space, with a steady-state pollutant dispersion model that formed a steady-state advection-diffusion equation, which was then used to perform water pollutant concentration level control and cost optimization.

Thus, the simple finite different methods are represented here as simple explicit schemes in the forms of the FTCS scheme and the Saulyev scheme [4]. Both schemes are used in calculating mathematical model as water quality models. When compared, two comparisons have shown that the Saulyev scheme is more efficient [4].

For implementation, we analyze nitrogen pollutant models from the advection-dispersion-reaction equation to estimate pollutant concentrations in terms of total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate concentrations. We take two numerical methods, the FTCS and the Saulyev methods, to compare with the analytical solution that forms the governing equation. We compute nitrogen pollutant models with two numerical methods to compare efficient solvability. We solve previous models with the Saulyev method by comparing different right boundary conditions.

2. Dispersion models

2.1 Advectiondiffusionreaction equation

In a stream water pollutant concentration measurement model, the governing equation is a one-dimensional advection-diffusion-reaction equation. A simplified representation, averaging the equation over the depths, as shown in [6], is

$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2} - f\left(C\right), \quad 0 \le x \le L, 0 < t \le T, \tag{2.1}$$

with the initial condition :

$$C\left(x,0\right) = k\left(x\right), \quad 0 \le x \le L, \tag{2.2}$$

and the boundary conditions :

$$\begin{aligned} C\left(0,t\right) &= g\left(t\right), \quad 0 < t \le T, \\ C\left(1,t\right) &= h\left(t\right), \quad 0 < t \le T, \end{aligned} \tag{2.3}$$

where $C\left(x,t\right)$ is the concentration at the point x and at time t, D is the diffusion coefficient of nitrogen pollution, u is the velocity component, $f\left(C\right)$ is reaction to a sink or source terms, and $k\left(x\right), g\left(t\right)$ and $h\left(t\right)$ are given.

2.2 Nitrogen dispersion models

We consider the nitrogen pollutant concentration models in surface water by using a modified model for approximate concentrations of some nitrogen pollutants: the general nitrogen forms, total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate which play roles in nitrogen processes are described by the general knowledge of nitrogen [1] and [10].

$$\text{ON} \xrightarrow{\text{ammonification}} \text{NH}_3 \xrightarrow{\text{nitrification}} \text{NO}_2 \xrightarrow{\text{nitrification}} \text{NO}_3 \tag{2.4}$$

Nitrogen from wastewater is discharged from many sources, such as human, animal, industry, agriculture, and other sources. General discharged raw wastewater contains around $40-45$ % organic nitrogen; and about $55-60\%$ ammonia, with the sum of nitrates and nitrites making up about $0-5\%$ of the total nitrogen [12].

Therefore, different nitrogen pollutant concentrations are analyzed under reaction terms using five different nitrogen dispersion models, which were established by (2.1), and associated with (2.2) and (2.3). We consider these models to approximate nitrogen pollutant concentration behaviors that occur in natural water sources to be total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate. These models are described by the different dispersion models, such as the total nitrogen concentration model, the organic nitrogen concentration model, the ammonia concentration model, the nitrite concentration model, and the nitrate concentration model.

2.2.1 Total nitrogen dispersion model

The total nitrogen (TN) pollutant concentration measurement in a stream is described by the one-dimensional advection-diffusion-reaction equation.

$$\frac{\partial C_1}{\partial t} = -u\frac{\partial C_1}{\partial x} + D_1\frac{\partial^2 C_1}{\partial x^2} - R_1\left(C_1\right) + Q, \quad 0 \le x \le 1, 0 < t \le 1, \tag{2.5}$$

the initial condition

$$C_1\left(x,0\right) = k_1\left(x\right), \quad 0 \le x \le 1, \tag{2.6}$$

and the boundary conditions :

$$\begin{aligned} C_1\left(0,t\right) &= g_1\left(t\right), \quad 0 < t \le 1, \\ C_1\left(1,t\right) &= h_1\left(t\right), \quad 0 < t \le 1, \end{aligned} \tag{2.7}$$

where u is water flow velocity, $C_1(x,t)$ is the total nitrogen concentration at the point x and time t, D_1 is the total nitrogen diffusion coefficient, R_1 is the reaction rate due to the degradation, Q is the inlet total nitrogen concentration due to sources, $k_1(x)$ is the potential total nitrogen concentration function along the stream, $g_1(t)$ is the total nitrogen concentration function at the discharge point, and $h_1(t)$ is the rate of change of the total nitrogen concentration with respect to distance at the end of the stream.

2.2.2 Organic nitrogen dispersion model

Most organic nitrogen (ON) pollutants are dissolved in living and nonliving forms from domestic wastes such as urea, uric acid, polypeptides and amino acids [11]. This model describes the organic nitrogen pollutant concentration behavior where the performance measurement concentration occurs under reaction term by the degradation of the organic nitrogen concentration in water, with consideration of the total nitrogen concentration.

The organic nitrogen (ON) pollutant concentration measurement in a stream is described by the one-dimensional advection-diffusion-reaction equation.

$$\frac{\partial C_2}{\partial t} = -u\frac{\partial C_2}{\partial x} + D_2\frac{\partial^2 C_2}{\partial x^2} + R_2\left(R_1 C_1\right), \quad 0 \le x \le 1, 0 < t \le 1, \tag{2.8}$$

the initial condition

$$C_2\left(x,0\right) = k_2\left(x\right), \quad 0 \le x \le 1, \tag{2.9}$$

and the boundary conditions :

$$\begin{aligned} \frac{\partial C_2\left(0,t\right)}{\partial t} &= g_2\left(t\right), \quad 0 < t \le 1, \\ \frac{\partial C_2\left(1,t\right)}{\partial t} &= h_2\left(t\right), \quad 0 < t \le 1, \end{aligned} \tag{2.10}$$

where u is water flow velocity, $C_2(x,t)$ is the organic nitrogen concentration at the point x and time t, D_2 is the organic nitrogen diffusion coefficient, R_2 is the reaction rate due to the degradation, $k_2(x)$ is the potential organic nitrogen concentration function along the stream, $g_2(t)$ is the organic nitrogen concentration function at the discharge point, and $h_2(t)$ is the rate of change of the organic nitrogen concentration with respect to distance at the end of the stream.

2.2.3 Ammonia dispersion model

Ammonia (NH_3) occurs in organic nitrogen by the ammonification process, which gives ammonia (NH_3) and ammonium (NH_4^+). The mass of ammonia (NH_3) and ammonium (NH_4^+) are considered in terms of pH and temperature. Ammonia (NH_3) is toxic to aquatic life, whereas ammonium (NH_4^+), supports algae and aquatic plant growth. However, ammonia and ammonium are similar [11]. This model describes ammonia pollutant concentration behavior where the performance measurement concentration occurs under reaction term by the degradation of the ammonia pollutant concentration in water, with consideration of the total nitrogen concentration.

The ammonia (NH_3) pollutant concentration measurement in a stream is described by the one-dimensional advection-diffusion-reaction equation.

$$\frac{\partial C_3}{\partial t} = -u\frac{\partial C_3}{\partial x} + D_3\frac{\partial^2 C_3}{\partial x^2} + R_3\left(R_1 C_1\right), \quad 0 \le x \le 1, 0 < t \le 1, \tag{2.11}$$

the initial condition

$$C_3(x,0) = k_3(x), \quad 0 \le x \le 1, \tag{2.12}$$

and the boundary conditions :

$$\begin{aligned} \frac{\partial C_3(0,t)}{\partial t} &= g_3(t), \quad 0 < t \le 1, \\ \frac{\partial C_3(1,t)}{\partial t} &= h_3(t), \quad 0 < t \le 1, \end{aligned} \tag{2.13}$$

where u is water flow velocity, $C_3(x,t)$ is the ammonia concentration at the point x and time t, D_3 is the ammonia diffusion coefficient, R_3 is the reaction rate due to the degradation, $k_3(x)$ is the potential ammonia concentration function along the stream, $g_3(t)$ is the ammonia concentration function at the discharge point, and $h_3(t)$ is the rate of change of the ammonia concentration with respect to distance at the end of the stream.

2.2.4 Nitrite dispersion model

The nitrite (NO_2) pollutant is oxidized by the nitrification process, such as to convert ammonia to nitrite. Nitrite is toxic to infants when at excessive levels [11]. This model describes nitrite pollutant concentration behavior where the performance measurement concentration occurs under reaction term by the degradation of the nitrite pollutant concentration in water, with consideration of the total nitrogen concentration.

The nitrite (NO_2) pollutant concentration measurement in a stream is described by the one-dimensional advection-diffusion-reaction equation.

$$\frac{\partial C_4}{\partial t} = -u\frac{\partial C_4}{\partial x} + D_4\frac{\partial^2 C_4}{\partial x^2} + R_4(R_1 C_1), \quad 0 \le x \le 1, 0 < t \le 1, \tag{2.14}$$

the initial condition

$$C_4(x,0) = k_4(x), \quad 0 \le x \le 1, \tag{2.15}$$

and the boundary conditions :

$$\begin{aligned} \frac{\partial C_4(0,t)}{\partial t} &= g_4(t), \quad 0 < t \le 1, \\ \frac{\partial C_4(1,t)}{\partial t} &= h_4(t), \quad 0 < t \le 1, \end{aligned} \tag{2.16}$$

where u is water flow velocity, $C_4(x,t)$ is the nitrite concentration at the point x and time t, D_4 is the nitrite diffusion coefficient, R_4 is the reaction rate due to the degradation, $k_4(x)$ is the potential nitrite concentration function along the stream, $g_4(t)$ is the nitrite concentration function at the discharge point, and $h_4(t)$ is the rate of change of the nitrite concentration with respect to distance at the end of the stream.

2.2.5 Nitrate dispersion model

The nitrate (NO_3) pollutant is transformed from nitrite by the nitrification process and can convert to nitrite [11], which is toxic and hazardous to infants. This model describes nitrate pollutant concentration behavior where the performance measurement concentration occurs under reaction term by the degradation of the nitrate pollutant concentration in water, with consideration of the total nitrogen concentration.

The nitrate (NO_3) pollutant concentration measurement in a stream is described by the one-dimensional advection-diffusion-reaction equation.

$$\frac{\partial C_5}{\partial t} = -u\frac{\partial C_5}{\partial x} + D_5\frac{\partial^2 C_5}{\partial x^2} + R_5\left(R_1 C_1\right), \quad 0 \le x \le 1, 0 < t \le 1, \tag{2.17}$$

the initial condition

$$C_5\left(x, 0\right) = k_5\left(x\right), \quad 0 \le x \le 1, \tag{2.18}$$

and the boundary conditions :

$$\begin{aligned} \frac{\partial C_5\left(0, t\right)}{\partial t} &= g_5\left(t\right), \quad 0 < t \le 1, \\ \frac{\partial C_5\left(1, t\right)}{\partial t} &= h_5\left(t\right), \quad 0 < t \le 1, \end{aligned} \tag{2.19}$$

where u is water flow velocity, $C_5(x,t)$ is the nitrate concentration at the point x and time t, D_5 is the nitrate diffusion coefficient, R_5 is the reaction rate due to the degradation, $k_5(x)$ is the potential nitrate concentration function along the stream, $g_5(t)$ is the nitrate concentration function at the discharge point, and $h_5(t)$ is the rate of change of the nitrate concentration with respect to distance at the end of the stream.

3. Numerical techniques for nitrogen dispersion models

This section presents the two numerical schemes of finite difference methods, the FTCS scheme and the Saulyev scheme, which are explicit schemes.

We can solve $C(x_i, t_n)$ or C_i^n at grid point (x_i, t_n) where $0 \le i \le M$ and $0 \le n \le N$ such that i and n are positive integers. Each node of x_i and t_n is separated by an equal grid on the x-range and t-range as the column of space and time, respectively, where $x_i = i\Delta x$, $i = 0, 1, 2, ..., M$, $t_n = n\Delta t$, $n = 0, 1, 2, ..., N$ and the values of $\Delta x > 0$ and $\Delta t > 0$ are the grid of space and time increments, respectively [4].

3.1 Forward Time Central Space technique applied to five forms of nitrogen measurement model

The FTCS scheme can be written as the discretization of time and space derivative term as follows [4]

$$
\begin{aligned}
&\frac{\partial C}{\partial t} = \frac{C_i^{n+1} - C_i^n}{\Delta t}, \\
&\frac{\partial C}{\partial x} = \frac{C_{i+1}^n - C_{i-1}^{n+1}}{2\Delta x}, \\
\text{and} \quad &\frac{\partial^2 C}{\partial x^2} = \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^{n+1}}{(\Delta x)^2}.
\end{aligned}
\tag{3.1}
$$

Substituting (3.1) into nitrogen dispersion models (the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate measurement models) results in (2.5), (2.8), (2.11), (2.14), and (2.17), respectively.

Moreover, the FTCS has numerical stability that depends on the condition of the diffusion number (λ) and the advection number (γ) , as follows

$$
\begin{aligned}
&\lambda = \frac{D\Delta t}{(\Delta x)^2} < \frac{1}{2}, \\
\text{and} \quad &\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x} < 1.
\end{aligned}
\tag{3.2}
$$

These values depend on choosing a suitable grid time increment, as Δt is not of much high width.

3.1.1 Forward Time Central Space technique for the total nitrogen dispersion model

Taking (3.1) into (2.5), we obtain the discretization of total nitrogen dispersion model as

$$
\begin{aligned}
\frac{(C_1)_i^{n+1} - (C_1)_i^n}{\Delta t} = &- u_i^n \left(\frac{(C_1)_{i+1}^n - (C_1)_{i-1}^n}{2\Delta x} \right) \\
&+ D_1 \left(\frac{(C_1)_{i+1}^n - 2(C_1)_i^n + (C_1)_{i-1}^n}{(\Delta x)^2} \right) - R_1 (C_1)_i^n + Q.
\end{aligned}
\tag{3.3}
$$

Rearranging (3.3) leads to (3.4) for the FTCS solution, represented as

$$
\begin{aligned}
(C_1)_i^{n+1} = &\left(\frac{1}{2}\gamma_i^n + \lambda \right)(C_1)_{i-1}^n + (1 - R_1\Delta t - 2\lambda)(C_1)_i^n \\
&+ \left(\lambda - \frac{1}{2}\gamma_i^n \right)(C_1)_{i+1}^n + Q\Delta t,
\end{aligned}
\tag{3.4}
$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

Substituting (3.1) into (2.7) and rearranging on the right-bound of the boundary condition, which is the derivative equation, we get (3.5).

The left boundary condition; $i = 0$,

$$C_1(0,t) = 1 = g_1(t),$$

and the right boundary condition; $i = M$,

$$\begin{aligned} \frac{\partial C_1(M,t)}{\partial x} &= \frac{(C_1)^n_{M+1} - (C_1)^n_{M-1}}{\Delta x} = h_1(t), \\ (C_1)^n_{M+1} &= 2h_1(t)\Delta x + (C_1)^n_{M-1}. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) leads to (3.6) of the right side equation, given as

$$\begin{aligned} (C_1)^{n+1}_M =& 2\lambda(C_1)^n_{M-1} + 2h_1(t)\Delta x\left(\lambda - \frac{1}{2}\gamma^n_M\right) \\ &+ (1 - R_1\Delta t - 2\lambda)(C_1)^n_M + Q\Delta t. \end{aligned} \tag{3.6}$$

3.1.2 Forward Time Central Space technique for the organic nitrogen dispersion model

Likewise, substituting (3.1) into (2.8) leads to (3.7), written as

$$\begin{aligned} \frac{(C_2)^{n+1}_i - (C_2)^n_i}{\Delta t} =& -u^n_i\left(\frac{(C_2)^n_{i+1} - (C_2)^n_{i-1}}{2\Delta x}\right) \\ &+ D_2\left(\frac{(C_2)^n_{i+1} - 2(C_2)^n_i + (C_2)^n_{i-1}}{(\Delta x)^2}\right) + R_2R_1(C_1)^n_i. \end{aligned} \tag{3.7}$$

Rearranging (3.7) leads to (3.8) for the FTCS solution, represented as

$$\begin{aligned} (C_2)^{n+1}_i =& \left(\frac{1}{2}\gamma^n_i + \lambda\right)(C_2)^n_{i-1} + (1-2\lambda)(C_2)^n_i \\ &+ \left(\lambda - \frac{1}{2}\gamma^n_i\right)(C_2)^n_{i+1} + R_2R_1(C_1)^n_i\Delta t, \end{aligned} \tag{3.8}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma^n_i = \frac{u^n_i\Delta t}{\Delta x}$.

Substituting (3.1) into (2.10) and rearranging on the left and right-bound of the boundary condition of this model which are the derivative equations, we get (3.9).

The left boundary condition; $i = 0$,

$$\begin{aligned} \frac{\partial C_2(0,t)}{\partial x} =& \frac{(C_2)^n_1 - (C_2)^n_{-1}}{2\Delta x} = g_2(t), \\ (C_2)^n_{-1} =& (C_2)^n_1 - 2g_2(t)\Delta x, \end{aligned}$$

and the right boundary condition; $i = M$,

$$\begin{aligned} \frac{\partial C_2(M,t)}{\partial x} =& \frac{(C_2)^n_{M+1} - (C_2)^n_{M-1}}{2\Delta x} = h_2(t), \\ (C_2)^n_{M+1} =& 2h_2(t)\Delta x + (C_2)^n_{M-1}. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) leads to the left and right side (3.10) and (3.11), respectively, given as follows

$$\begin{aligned}(C_2)_0^{n+1} =& 2\lambda(C_2)_1^n - 2g_2(t)\Delta x\left(\lambda + \frac{1}{2}\gamma_0^n\right) + (1-2\lambda)(C_2)_0^n \\ &+ R_2R_1(C_1)_0^n\Delta t\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(C_2)_M^{n+1} =& 2\lambda(C_2)_{M-1}^n + 2h_2(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right) + (1-2\lambda)(C_2)_M^n \\ &+ R_2R_1(C_1)_M^n\Delta t.\end{aligned} \tag{3.11}$$

3.1.3 Forward Time Central Space technique for the ammonia dispersion model

Likewise, substituting (3.1) into (2.11) leads to (3.12), written as

$$\begin{aligned}\frac{(C_3)_i^{n+1} - (C_3)_i^n}{\Delta t} =& -u_i^n\left(\frac{(C_3)_{i+1}^n - (C_3)_{i-1}^n}{2\Delta x}\right) \\ &+ D_3\left(\frac{(C_3)_{i+1}^n - 2(C_3)_i^n + (C_3)_{i-1}^n}{(\Delta x)^2}\right) + R_3R_1(C_1)_i^n.\end{aligned} \tag{3.12}$$

Rearranging (3.12) leads to (3.13), represented as

$$\begin{aligned}(C_3)_i^{n+1} =& \left(\frac{1}{2}\gamma_i^n + \lambda\right)(C_3)_{i-1}^n + (1-2\lambda)(C_3)_i^n + \left(\lambda - \frac{1}{2}\gamma_i^n\right)(C_3)_{i+1}^n \\ &+ R_3R_1(C_1)_i^n\Delta t,\end{aligned} \tag{3.13}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n\Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.14) for the left and right sides, as below, from substituting (3.1) into (2.13) and rearranging these equations.

The left boundary condition; $i = 0$,

$$\begin{aligned}\frac{\partial C_3(0,t)}{\partial x} =& \frac{(C_3)_1^n - (C_3)_{-1}^n}{2\Delta x} = g_3(t), \\ (C_3)_{-1}^n =& (C_3)_1^n - 2g_3(t)\Delta x,\end{aligned}$$

and the right boundary condition; $i = M$,

$$\begin{aligned}\frac{\partial C_3(M,t)}{\partial x} =& \frac{(C_3)_{M+1}^n - (C_3)_{M-1}^n}{2\Delta x} = h_3(t), \\ (C_3)_{M+1}^n =& 2h_3(t)\Delta x + (C_3)_{M-1}^n.\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13) which leads to the left and right side, (3.15) and (3.16), respectively, given as follows

$$\begin{aligned}(C_3)_0^{n+1} =& 2\lambda(C_3)_1^n - 2g_3(t)\Delta x\left(\lambda + \frac{1}{2}\gamma_0^n\right) + (1-2\lambda)(C_3)_0^n \\ &+ R_3R_1(C_1)_0^n\Delta t\end{aligned} \tag{3.15}$$

and

$$(C_3)_M^{n+1} = 2\lambda(C_3)_{M-1}^n + 2h_3(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right) + (1-2\lambda)(C_3)_M^n + R_3R_1(C_1)_M^n\Delta t. \tag{3.16}$$

3.1.4 Forward Time Central Space technique for the nitrite dispersion model

Likewise, substituting (3.1) into (2.14) leads to (3.17), written as

$$\frac{(C_4)_i^{n+1} - (C_4)_i^n}{\Delta t} = -u_i^n\left(\frac{(C_4)_{i+1}^n - (C_4)_{i-1}^n}{2\Delta x}\right) + D_4\left(\frac{(C_4)_{i+1}^n - 2(C_4)_i^n + (C_4)_{i-1}^n}{(\Delta x)^2}\right) + R_4R_1(C_1)_i^n. \tag{3.17}$$

Rearranging (3.17) leads to (3.18), represented as

$$(C_4)_i^{n+1} = \left(\frac{1}{2}\gamma_i^n + \lambda\right)(C_4)_{i-1}^n + (1-2\lambda)(C_4)_i^n + \left(\lambda - \frac{1}{2}\gamma_i^n\right)(C_4)_{i+1}^n + R_4R_1(C_1)_i^n\Delta t, \tag{3.18}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n\Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.19) for the left and right sides, as below, from substituting (3.1) into (2.16) and rearranging these equations.

The left boundary condition; $i = 0$,

$$\frac{\partial C_4(0,t)}{\partial x} = \frac{(C_4)_1^n - (C_4)_{-1}^n}{2\Delta x} = g_4(t),$$
$$(C_4)_{-1}^n = (C_4)_1^n - 2g_4(t)\Delta x,$$

and the right boundary condition; $i = M$,

$$\frac{\partial C_4(M,t)}{\partial x} = \frac{(C_4)_{M+1}^n - (C_4)_{M-1}^n}{2\Delta x} = h_4(t),$$
$$(C_4)_{M+1}^n = 2h_4(t)\Delta x + (C_4)_{M-1}^n. \tag{3.19}$$

Substituting (3.19) into (3.18) leads to the left and right side, (3.20) and (3.21), respectively, given as follows

$$(C_4)_0^{n+1} = 2\lambda(C_4)_1^n - 2g_4(t)\Delta x\left(\lambda + \frac{1}{2}\gamma_0^n\right) + (1-2\lambda)(C_4)_0^n + R_4R_1(C_1)_0^n\Delta t \tag{3.20}$$

and

$$(C_4)_M^{n+1} = 2\lambda(C_4)_{M-1}^n + 2h_4(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right) + (1-2\lambda)(C_4)_M^n + R_4R_1(C_1)_M^n\Delta t. \tag{3.21}$$

3.1.5 Forward Time Central Space technique for the nitrate dispersion model

Likewise, substituting (3.1) into (2.17) leads to (3.22), written as

$$\begin{aligned}\frac{(C_5)_i^{n+1}-(C_5)_i^n}{\Delta t} =& - u_i^n\left(\frac{(C_5)_{i+1}^n-(C_5)_{i-1}^n}{2\Delta x}\right)\\ &+ D_5\left(\frac{(C_5)_{i+1}^n-2(C_5)_i^n+(C_5)_{i-1}^n}{(\Delta x)^2}\right)+R_5R_1(C_1)_i^n.\end{aligned}\tag{3.22}$$

Rearranging (3.22) leads to (3.23), represented as

$$\begin{aligned}(C_5)_i^{n+1} =& \left(\frac{1}{2}\gamma_i^n+\lambda\right)(C_5)_{i-1}^n+(1-2\lambda)(C_5)_i^n\\ &+\left(\lambda-\frac{1}{2}\gamma_i^n\right)(C_5)_{i+1}^n+R_5R_1(C_1)_i^n\Delta t,\end{aligned}\tag{3.23}$$

where $\lambda=\frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n=\frac{u_i^n\Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.24) for the left and right sides, as below, from substituting (3.1) into (2.19) and rearranging these equations.

The left boundary condition; $i=0$,

$$\begin{aligned}\frac{\partial C_5(0,t)}{\partial x} =& \frac{(C_5)_1^n-(C_5)_{-1}^n}{2\Delta x}=g_5(t),\\ (C_5)_{-1}^n =& (C_5)_1^n-2g_5(t)\Delta x,\end{aligned}$$

and the right boundary condition; $i=M$,

$$\begin{aligned}\frac{\partial C_5(M,t)}{\partial x} =& \frac{(C_5)_{M+1}^n-(C_5)_{M-1}^n}{2\Delta x}=h_5(t),\\ (C_5)_{M+1}^n =& 2h_5(t)\Delta x+(C_5)_{M-1}^n.\end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.23) leads to the left and right side, (3.25) and (3.26), respectively, given as follows

$$\begin{aligned}(C_5)_0^{n+1} =& 2\lambda(C_5)_1^n-2g_5(t)\Delta x\left(\lambda+\frac{1}{2}\gamma_0^n\right)+(1-2\lambda)(C_5)_0^n\\ &+R_5R_1(C_1)_0^n\Delta t\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}(C_5)_M^{n+1} =& 2\lambda(C_5)_{M-1}^n+2h_5(t)\Delta x\left(\lambda-\frac{1}{2}\gamma_M^n\right)+(1-2\lambda)(C_5)_M^n\\ &+R_5R_1(C_1)_M^n\Delta t.\end{aligned}\tag{3.26}$$

3.2 Unconditionally stable Saulyev technique applied to five forms of nitrogen dispersion measurement model

Saulyev (1964) introduced asymmetric approximations for the parabolic equations that are widely used in environmental solving, such as advection and diffusion equations, etc. The solution is given to be an approximation that is explicit and has unconditional stability [13].

The Saulyev scheme can be written as the discretization of time, and space derivative term, as per [4]
where let $C(x,t)$ and u denote C_i^n and u_i^n then

$$\begin{aligned} \frac{\partial C}{\partial t} &= \frac{C_i^{n+1} - C_i^n}{\Delta t}, \\ \frac{\partial C}{\partial x} &= \frac{C_{i+1}^n - C_{i-1}^{n+1}}{2\Delta x}, \\ \text{and} \qquad \frac{\partial^2 C}{\partial x^2} &= \frac{C_{i+1}^n - C_i^n - C_i^{n+1} + C_{i-1}^{n+1}}{(\Delta x)^2}. \end{aligned} \tag{3.27}$$

Substituting (3.27) into nitrogen dispersion models (the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate measurement models) results in (2.5), (2.8), (2.11), (2.14), and (2.17), respectively.

3.2.1 Saulyev technique for the total nitrogen dispersion model

Likewise, substituting (3.27) into (2.5) leads to (3.28), written as

$$\begin{aligned} \frac{(C_1)_i^{n+1} - (C_1)_i^n}{\Delta t} = &- u_i^n \left(\frac{(C_1)_{i+1}^n - (C_1)_{i-1}^{n+1}}{2\Delta x} \right) \\ &+ D_1 \left(\frac{(C_1)_{i+1}^n - (C_1)_i^n - (C_1)_i^{n+1} + (C_1)_{i-1}^{n+1}}{(\Delta x)^2} \right) \\ &- R_1 (C_1)_i^n + Q. \end{aligned} \tag{3.28}$$

Rearranging (3.28) leads to (3.29), represented as

$$\begin{aligned} (C_1)_i^{n+1} = &\frac{1}{(1+\lambda)} \Bigg(\left(\frac{1}{2}\gamma_i^n + \lambda \right) (C_1)_{i-1}^{n+1} + (1 - \lambda - R_1\Delta t)(C_1)_i^n \\ &+ \left(\lambda - \frac{1}{2}\gamma_i^n \right) (C_1)_{i+1}^n + Q\Delta t \Bigg), \end{aligned} \tag{3.29}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

Substituting (3.27) into (2.7) and rearranging on the right-bound of the boundary condition, which is the derivative equation, we get (3.30).

The left boundary condition; $i = 0$,

$$C_1(0,t) = 1,$$

and the right boundary condition; $i = M$,

$$\begin{aligned}\frac{\partial C_1(M,t)}{\partial x} &= \frac{(C_1)^n_{M+1} - (C_1)^n_{M-1}}{2\Delta x} = h_1(t),\\ (C_1)^n_{M+1} &= 2h_1(t)\Delta x + (C_1)^n_{M-1}.\end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29) leads to (3.31) of the right side equation, given as

$$\begin{aligned}(C_1)^{n+1}_i =& \frac{1}{(1+\lambda)}\Bigg(2\lambda(C_1)^{n+1}_{M-1} + (1-\lambda-R_1\Delta t)(C_1)^n_M + 2h_1(t)\Delta x\\ &\left(\lambda - \frac{1}{2}\gamma^n_M\right) + Q\Delta t\Bigg).\end{aligned} \tag{3.31}$$

3.2.2 Saulyev technique for the organic nitrogen dispersion model

Likewise, substituting (3.27) into (2.8) leads to (3.32), written as

$$\begin{aligned}\frac{(C_2)^{n+1}_i - (C_2)^n_i}{\Delta t} =& -u^n_i\left(\frac{(C_2)^n_{i+1} - (C_2)^{n+1}_{i-1}}{2\Delta x}\right)\\ &+ D_2\left(\frac{(C_2)^n_{i+1} - (C_2)^n_i - (C_2)^{n+1}_i + (C_2)^{n+1}_{i-1}}{(\Delta x)^2}\right)\\ &+ R_2R_1(C_1)^n_i.\end{aligned} \tag{3.32}$$

Rearranging (3.32) leads to (3.33), represented as

$$\begin{aligned}(C_2)^{n+1}_i =& \frac{1}{(1+\lambda)}\Bigg(\left(\frac{1}{2}\gamma^n_i + \lambda\right)(C_2)^{n+1}_{i-1} + (1-\lambda)(C_2)^n_i\\ &+ \left(\lambda - \frac{1}{2}\gamma^n_i\right)(C_2)^n_{i+1} + R_2R_1(C_1)^n_i\Delta t\Bigg),\end{aligned} \tag{3.33}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma^n_i = \frac{u^n_i\Delta t}{\Delta x}$.

Substituting (3.27) into (2.10) and rearranging on the left and right-bound of the boundary condition of this model, which are the derivative equation, we get (3.34).

The left boundary condition; $i = 0$,

$$\begin{aligned}\frac{\partial C_2(0,t)}{\partial x} =& \frac{(C_2)_1^n - (C_2)_{-1}^{n+1}}{2\Delta x} = g_2(t),\\ (C_2)_{-1}^n =& (C_2)_1^{n+1} - 2g_2(t)\Delta x,\end{aligned}$$

and the right boundary condition; $i = M$,

$$\begin{aligned}\frac{\partial C_2(M,t)}{\partial x} =& \frac{(C_2)_{M+1}^n - (C_2)_{M-1}^{n+1}}{2\Delta x} = h_2(t),\\ (C_2)_{M+1}^n =& 2h_2(t)\Delta x + (C_2)_{M-1}^{n+1}.\end{aligned} \tag{3.34}$$

Substituting (3.34) into (3.33) leads to the left and right side (3.35) and (3.36), respectively, given as follows

$$\begin{aligned}(C_2)_0^{n+1} =& \frac{1}{(1+\lambda)}\Bigg(2\lambda(C_2)_1^{n+1} + (1-\lambda)(C_2)_0^n - 2g_2(t)\Delta x\\ &\left(\lambda + \frac{1}{2}\gamma_0^n\right) + R_2R_1(C_1)_0^n\Delta t\Bigg)\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}(C_2)_M^{n+1} =& \frac{1}{(1+\lambda)}\Bigg((2\lambda(C_2)_{M-1}^{n+1} + (1-\lambda)(C_2)_M^n + 2h_2(t)\Delta x\\ &\left(\lambda - \frac{1}{2}\gamma_M^n\right) + R_2R_1(C_1)_M^n\Delta t\Bigg).\end{aligned} \tag{3.36}$$

3.2.3 Saulyev technique for the ammonia dispersion model

Likewise, substituting (3.27) into (2.11) leads to (3.37), written as

$$\begin{aligned}\frac{(C_3)_i^{n+1} - (C_3)_i^n}{\Delta t} =& -u_i^n\left(\frac{(C_3)_{i+1}^n - (C_3)_{i-1}^{n+1}}{2\Delta x}\right)\\ &+ D_3\left(\frac{(C_3)_{i+1}^n - (C_3)_i^n - (C_3)_i^{n+1} + (C_3)_{i-1}^{n+1}}{(\Delta x)^2}\right)\\ &+ R_3R_1(C_1)_i^n.\end{aligned} \tag{3.37}$$

Rearranging (3.37) leads to (3.38), represented as

$$\begin{aligned}(C_3)_i^{n+1} =& \frac{1}{(1+\lambda)}\Bigg(\left(\frac{1}{2}\gamma_i^n + \lambda\right)(C_3)_{i-1}^{n+1} + (1-\lambda)(C_3)_i^n\\ &+ \left(\lambda - \frac{1}{2}\gamma_i^n\right)(C_3)_{i+1}^n + R_3R_1(C_1)_i^n\Delta t\Bigg),\end{aligned} \tag{3.38}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.39) for the left and right sides, as below, from substituting (3.27) into (2.13) and rearranging these equations.

The left boundary condition; $i = 0$,

$$\begin{aligned}\frac{\partial C_3(0,t)}{\partial x} =& \frac{(C_3)_1^n - (C_3)_{-1}^{n+1}}{2\Delta x} = g_3(t),\\ (C_3)_{-1}^n =& (C_3)_1^{n+1} - 2g_3(t)\Delta x,\end{aligned}$$

and the right boundary condition; $i = M$,

$$\begin{aligned}\frac{\partial C_3(M,t)}{\partial x} =& \frac{(C_3)_{M+1}^n - (C_3)_{M-1}^{n+1}}{2\Delta x} = h_3(t),\\ (C_3)_{M+1}^n =& 2h_3(t)\Delta x + (C_3)_{M-1}^{n+1}.\end{aligned} \tag{3.39}$$

Substituting (3.39) into (3.38) which leads to the left and right side (3.40) and (3.41), respectively, given as follows

$$\begin{aligned}(C_3)_0^{n+1} =& \frac{1}{(1+\lambda)}\left(2\lambda (C_3)_1^{n+1} + (1-\lambda)(C_3)_0^n - 2g_3(t)\Delta x\left(\lambda + \frac{1}{2}\gamma_0^n\right)\right.\\ &\left. + R_3 R_1 (C_1)_0^n \Delta t\right)\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}(C_3)_M^{n+1} =& \frac{1}{(1+\lambda)}\left(2\lambda (C_3)_{M-1}^{n+1} + (1-\lambda)(C_3)_M^n + 2h_3(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right)\right.\\ &\left. + R_3 R_1 (C_1)_M^n \Delta t\right).\end{aligned} \tag{3.41}$$

3.2.4 Saulyev technique for the nitrite dispersion model

Likewise, substituting (3.27) into (2.14) leads to (3.42), written as

$$\begin{aligned}\frac{(C_4)_i^{n+1} - (C_4)_i^n}{\Delta t} =& -u_i^n\left(\frac{(C_4)_{i+1}^n - (C_4)_{i-1}^{n+1}}{2\Delta x}\right)\\ &+ D_4\left(\frac{(C_4)_{i+1}^n - (C_4)_i^n - (C_4)_i^{n+1} + (C_4)_{i-1}^{n+1}}{(\Delta x)^2}\right)\\ &+ R_4 R_1 (C_1)_i^n.\end{aligned} \tag{3.42}$$

Rearranging (3.42) which leads to (3.43), represented as

$$
\begin{aligned}
(C_4)_i^{n+1} =& \frac{1}{(1+\lambda)}\Bigg(\left(\frac{1}{2}\gamma_i^n + \lambda\right)(C_4)_{i-1}^{n+1} + (1-\lambda)(C_4)_i^n \\
&+ \left(\lambda - \frac{1}{2}\gamma_i^n\right)(C_4)_{i+1}^n + R_4R_1(C_1)_i^n\Delta t\Bigg),
\end{aligned} \tag{3.43}
$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.44) for the left and right sides, as below, from substituting (3.27) into (2.16) and rearranging these equations.

The left boundary condition; $i = 0$,

$$
\begin{aligned}
\frac{\partial C_4(0,t)}{\partial x} =& \frac{(C_4)_1^n - (C_4)_{-1}^{n+1}}{2\Delta x} = g_4(t), \\
(C_4)_{-1}^n =& (C_4)_1^{n+1} - 2g_4(t)\Delta x,
\end{aligned}
$$

and the right boundary condition; $i = M$,

$$
\begin{aligned}
\frac{\partial C_4(M,t)}{\partial x} =& \frac{(C_4)_{M+1}^n - (C_4)_{M-1}^{n+1}}{2\Delta x} = h_4(t), \\
(C_4)_{M+1}^n =& 2h_4(t)\Delta x + (C_4)_{M-1}^{n+1}.
\end{aligned} \tag{3.44}
$$

Substituting (3.44) into (3.43) leads to the left and right side (3.45) and (3.46), respectively, given as follows

$$
\begin{aligned}
(C_4)_0^{n+1} =& \frac{1}{(1+\lambda)}\Bigg(2\lambda(C_4)_1^{n+1} + (1-\lambda)(C_4)_0^n - 2g_4(t)\Delta x\left(\lambda + \frac{1}{2}\gamma_0^n\right) \\
&+ R_4R_1(C_1)_0^n\Delta t\Bigg)
\end{aligned} \tag{3.45}
$$

and

$$
\begin{aligned}
(C_4)_M^{n+1} =& \frac{1}{(1+\lambda)}\Bigg(2\lambda(C_4)_{M-1}^{n+1} + (1-\lambda)(C_4)_M^n + 2h_4(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right) \\
&+ R_4R_1(C_1)_M^n\Delta t\Bigg).
\end{aligned} \tag{3.46}
$$

3.2.5 Saulyev technique for the nitrate dispersion model

Likewise, substituting (3.27) into (2.17) leads to (3.47), written as

$$\begin{aligned}\frac{(C_5)_i^{n+1}-(C_5)_i^n}{\Delta t} =& - u_i^n\left(\frac{(C_5)_{i+1}^n-(C_5)_{i-1}^{n+1}}{2\Delta x}\right)\\ &+ D_5\left(\frac{(C_5)_{i+1}^n-(C_5)_i^n-(C_5)_i^{n+1}+(C_5)_{i-1}^{n+1}}{(\Delta x)^2}\right)\\ &+ R_5R_1(C_1)_i^n.\end{aligned} \tag{3.47}$$

Rearranging (3.47) leads to (3.48), represented as

$$\begin{aligned}(C_5)_i^{n+1} =&\frac{1}{(1+\lambda)}\left(\left(\frac{1}{2}\gamma_i^n+\lambda\right)(C_5)_{i-1}^{n+1}+(1-\lambda)(C_5)_i^n\right.\\ &\left.+\left(\lambda-\frac{1}{2}\gamma_i^n\right)(C_5)_{i+1}^n+R_5R_1(C_1)_i^n\Delta t\right),\end{aligned} \tag{3.48}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n\Delta t}{\Delta x}$.

The left and right-bounds of the boundary conditions of this model are the derivative equations. Likewise, we can get (3.49) for the left and right sides, as below, from substituting (3.27) into (2.19) and rearranging these equations.

The left boundary condition; $i = 0$,

$$\begin{aligned}\frac{\partial C_5(0,t)}{\partial x} =&\frac{(C_5)_1^n-(C_5)_{-1}^{n+1}}{2\Delta x} = g_5(t),\\ (C_5)_{-1}^n =&(C_5)_1^{n+1}-2g_5(t)\Delta x,\end{aligned}$$

and the right boundary condition; $i = M$,

$$\begin{aligned}\frac{\partial C_5(M,t)}{\partial x} =&\frac{(C_5)_{M+1}^n-(C_5)_{M-1}^{n+1}}{2\Delta x} = h_5(t),\\ (C_5)_{M+1}^n =&2h_5(t)\Delta x+(C_5)_{M-1}^{n+1}.\end{aligned} \tag{3.49}$$

Substituting (3.49) into (3.48) leads to the left and right side (3.50) and (3.51), respectively, given as follows

$$\begin{aligned}(C_5)_0^{n+1} =&\frac{1}{(1+\lambda)}\left(2\lambda(C_5)_1^{n+1}+(1-\lambda)(C_5)_0^n-2g_5(t)\Delta x\left(\lambda+\frac{1}{2}\gamma_0^n\right)\right.\\ &\left.+R_5R_1(C_1)_0^n\Delta t\right)\end{aligned} \tag{3.50}$$

and

$$(C_5)_M^{n+1} = \frac{1}{(1+\lambda)}\Bigg(2\lambda(C_5)_{M-1}^{n+1} + (1-\lambda)(C_5)_M^n + 2h_5(t)\Delta x\left(\lambda - \frac{1}{2}\gamma_M^n\right) + R_5R_1(C_1)_M^n\Delta t\Bigg). \tag{3.51}$$

4. Numerical experiments

In the section, we implement analytical, FTCS, and Saulyev solutions for approximation in three experiments. The first experiment is an accuracy comparison of the numerical methods with an analytical solution. The second experiment is an efficiency comparison between the FTCS and Saulyev solutions of the nitrogen dispersion models. The last part is a performance simulation of the nitrogen dispersion models with the Saulyev method.

4.1 Numerical simulation of an ideal pollutant dispersion measurement

We compare effective methods between the analytical method and numerical solutions using the FTCS and Saulyev methods. We consider an analytical solution to the simplest case of the governing equation with defined initial and boundary conditions, taken from [9], as an example to compare with the two numerical solutions.

We perform all three cases by computing (2.1) without the term of reactive pollutant ($f(C) = 0$) with the analytical, FTCS, and Saulyev methods. All three cases are easily solvable where it is assumed that the stream length is 1 km, the performance over the entire time interval is [0,1], the velocity component (u) is 1 m/s, the diffusion coefficient (D) is 0.01 m^2/s, and the grid step size of space (Δx) and time (Δt) are 0.05 and 0.0025.

The initial and boundary conditions are given to follow (2.2) and (2.3), and are, respectively [9];

$$\begin{aligned} C(x,0) &= k(x) = exp\left(-\frac{(x+0.5)^2}{0.00125}\right), \quad 0 \le x \le 1, \\ C(0,t) &= g(t) = \frac{0.025}{\sqrt{0.000625+0.02t}}exp\left(-\frac{(0.5-t)^2}{(0.00125+0.04t)}\right), \\ &\quad 0 < t \le 1, \\ C(1,t) &= h(t) = \frac{0.025}{\sqrt{0.000625+0.02t}}exp\left(-\frac{(1.5-t)^2}{(0.00125+0.04t)}\right), \\ &\quad 0 < t \le 1. \end{aligned} \tag{4.1}$$

Performing (2.1) without the term of reactive pollutant leads to (4.2) for analytical solution, represented as follows [9]

$$C(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} exp\left(-\frac{(x+0.5-t)^2}{0.00125+0.04t}\right), \quad 0 \le x \le 1, 0 < t \le 1. \tag{4.2}$$

Conducting (2.1) without the term of reactive pollutant by using the FTCS scheme from (3.1), we get and rearrange (4.3), which leads to (4.4)

$$\frac{C_i^{n+1} - C_i^n}{\Delta t} = -u_i^n \left(\frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}\right) + D\left(\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2}\right), \tag{4.3}$$

and

$$C_i^{n+1} = \left(\frac{1}{2}\gamma_i^n + \lambda\right) C_{i-1}^n + (1-2\lambda)C_i^n + \left(\lambda - \frac{1}{2}\gamma_i^n\right) C_{i+1}^n, \tag{4.4}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

Performing (2.1) without the term of reactive pollutant by using the Saulyev scheme from (3.27), we get and rearrange (4.5), which leads to (4.6).

$$\frac{C_i^{n+1} - C_i^n}{\Delta t} = -u_i^n \left(\frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}\right) + D\left(\frac{C_{i+1}^n - C_i^n - C_i^{n+1} + C_{i-1}^{n+1}}{(\Delta x)^2}\right), \tag{4.5}$$

and

$$C_i^{n+1} = \frac{1}{(1+\lambda)}\left(\left(\lambda + \frac{1}{2}\gamma_i^n\right) C_{i-1}^n + (1-\lambda)C_i^n + \left(\lambda - \frac{1}{2}\gamma_i^n\right) C_{i+1}^n\right), \tag{4.6}$$

where $\lambda = \frac{D\Delta t}{(\Delta x)^2}$ and $\gamma_i^n = \frac{u_i^n \Delta t}{\Delta x}$.

Consider that (4.2), (4.4), and (4.6) associate with the initial and boundary condition from (4.1). We get the approximate values for all three cases as analytical, FTCS, and Saulyev solutions, respectively.

Hence, these solutions can be shown by graph in order to compare the accuracy assessment of the two numerical methods. Fig. 1 shows the result of two numerical solutions for comparison with the analytical solution at $C(0.5, t)$, which describe the approximate values of two numerical solutions near fixed points of the analytic solution, showing that the numerical solutions are best.

4.2 Numerical simulations of five forms of nitrogen pollutant concentration measurements

We consider here the experiment for comparing two numerical methods, the FTCS and Saulyev methods, with the application of nitrogen dispersion models in section 2.2. It is possible to use numerical methods to solve each case of the defined grid space (Δx) and time (Δt) increments.

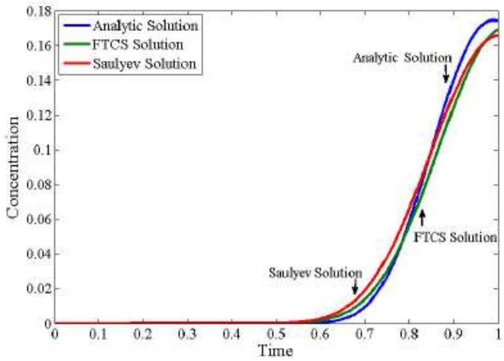


Figure 1: Comparison of concentrations (kg/m^3) of analytic, FTCS and Saulyev solutions at $C(0.5, t)$.

Table 1: Comparison Δx and Δt for computing nitrogen pollutant models of the total nitrogen with two numerical methods which are possible in the solving.

Δx	Δt	λ	γ	FTCS	SAULYEV
0.2	0.1	0.25	0.05	stable	stable
	0.05	0.125	0.025	stable	stable
	0.025	0.0625	0.0125	stable	stable
	0.0125	0.0313	0.00625	stable	stable
0.1	0.1	1	0.1	unstable	stable
	0.05	0.5	0.05	stable	stable
	0.025	0.25	0.025	stable	stable
	0.0125	0.125	0.0125	stable	stable
0.05	0.1	4	0.2	unstable	stable
	0.05	2	0.1	unstable	stable
	0.025	1	0.05	unstable	stable
	0.0125	0.5	0.025	stable	stable
0.025	0.1	16	0.4	unstable	stable
	0.05	8	0.2	unstable	stable
	0.025	4	0.1	unstable	stable
	0.0125	2	0.05	unstable	stable

For example, considering the concentration measurement of the nitrogen pollutant concentration in a stream at time t of total nitrogen (C_1), organic nitrogen (C_2), ammonia (C_3), nitrite (C_4), and nitrate (C_5) concentrations, assume that the stream reach is about 1 km, which is considered over the entire time interval [0,1]. This stream is surrounded by residential components. Wastewater is discharged into the stream, such that the wastewater contains nitrogen as the total nitrogen concentration form, which is everywhere throughout the stream, is described by a interpolated function $C_1(x,0) = k_1(x) = 1 + x(1-x)$ kg/m^3 initially, this discharged pollutant concentration at the left side of the stream $(x=0)$ is $C_1(0,t) = g_1(t) = 1$ kg/m^3 for all time and at $t=0$, and the right side of the stream $(x=1)$ is the rate of change of the total nitrogen concentration for releasing out $\frac{\partial C_1(1,t)}{\partial x} = h_1(t) = -0.001$ for all time and at $t=0$. Organic nitrogen, ammonia, nitrite, and nitrate pollutant concentrations depend on the total nitrogen concentration, such that they are defined by the same values as there are pollutant concentrations everywhere throughout the stream, and are kept at $C_2(x,0) = C_3(x,0) = C_4(x,0) = C_5(x,0) = 0$ kg/m^3 initially, at the left side of the stream $(x=0)$ is the rate of change of nitrogen pollutant concentration for releasing in as $\frac{\partial C_2(0,t)}{\partial x} = \frac{\partial C_3(0,t)}{\partial x} = \frac{\partial C_4(0,t)}{\partial x} = \frac{\partial C_5(0,t)}{\partial x} = 0$ for all time and at $t=0$, and the right side of the stream is the rate of change of nitrogen pollutant concentration for releasing out as $\frac{\partial C_2(1,t)}{\partial x} = \frac{\partial C_3(1,t)}{\partial x} = \frac{\partial C_4(1,t)}{\partial x} = \frac{\partial C_5(1,t)}{\partial x} = -0.001$ for all time and at $t=0$. The component of velocity is a constant as $u = 0.1$ m/s. This stream has approximate diffusion coefficients of total nitrogen (D_1), organic nitrogen (D_2), ammonia (D_3), nitrite (D_4), and nitrate (D_5) concentrations of 0.1, $0.85D_1$, $0.65D_1$, $0.45D_1$, and $0.3D_1$ m^2/s, respectively. The rates of degradation of total nitrogen (R_1), organic nitrogen (R_2), ammonia (R_3), nitrite (R_4), and nitrate (R_5) concentrations are 0.1, $0.85R_1$, $0.65R_1$, $0.45R_1$, and $0.3R_1$ s^{-1}, respectively. The inlet flow of the total nitrogen concentration into the stream is 0.001 m^2/s.

Start by computing the numerical solution equations (the FTCS and Saulyev methods). For the FTCS method, implement solution equatios as the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate in section 3.1, which associate with setting parameter values. For the Saulyev method, implement solution equations as the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate in section 3.2, which associate with setting parameter values.

Perform both methods for each Δx and Δt to compare each output of the numerical method that can be possible to measure for each case, as shown in Table 1.

From Table 1, we observe that it is possible to use the test to solve all of the cases for the Saulyev solutions. The FTCS solutions represent divergent yields and cannot be used in some cases. Therefore, the Saulyev method shows that is a consistent method for all cases under unconditional stability.

The FTCS method is impossible to compute if Δt has such large increases to not be satisfied for stability conditions which depend on λ and γ.

4.3 Numerical simulations of water-quality measurement in a stream with nitrogen pollutant concentration measurement using Saulyev method

According to the numerical computation of the nitrogen dispersion models, the FTCS method is impossible to use for the majority of cases. In the current section, we implement the Saulyev method with the nitrogen dispersion models.

This presents a comparative computation of nitrogen pollutant concentration for the different rates of change observed in them at the right boundary conditions, in order to explain the different nitrogen pollutant concentration behaviors.

Similarly, from the previous section, the same implementation of the Saulyev method is done with the concentration measurement of the nitrogen pollutant concentration in the stream at time t as the total nitrogen(C_1), organic nitrogen(C_2), ammonia (C_3), nitrite (C_4), and nitrate (C_5). Assume that the stream reach is about 1 km, which is considered over the entire time interval [0,1], some stream physical characteristics, initial and boundary conditions are similar, but some stream physical characteristics are different, such as the component of velocity depending on distance x, described by a function $u = 0.1+x(1-x)(0.1)$ m/s. This stream has an approximate diffusion coefficient of the total nitrogen concentration (D_1) with a function $D_1 = 0.1+(1+x(1-x))(0.1)$ m^2/s, depending on distances. Furthermore, the grid space (Δx) and time (Δt) increments are defined by 0.00625 and 0.01, respectively.

We perform solution equations for the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate with the Saulyev method in section 3.2, all five cases which associate with setting parameter values above and the previous section, by determining the different rates of change of nitrogen pollutant concentrations at right boundary conditions of -0.001, -0.002, -0.003, -0.004 and -0.005. Table 2 and 3 show each nitrogen pollutant concentration where the rates of change of nitrogen pollutant concentrations at the right boundary conditions of -0.001 and -0.005, respectively.

We observed that each measurement of all five cases gives each output of concentration in the same direction of the approximated value and each output of concentration in the cases of organic nitrogen, ammonia, nitrite, and nitrate obtained depend on the different defined rates of change of nitrogen pollutant concentrations at right boundary conditions.

5. Discussion

Consider that the given five graphs in Fig. 2 are examples from the nitrogen pollutant models in the case of the rate of change of nitrogen pollutant concentrations at the right boundary condition of -0.001. The graphs clearly show the effects of the increase or decrease in nitrogen pollutant concentrations in the stream, such that the total nitrogen pollutant concentration decreased continuously in Fig.2 (a), while the organic nitrogen, ammonia, nitrite and nitrate pollutant concentrations increased continuously in Fig. 2 (b)-(e). It can be explained that discharged wastewater consists of nitrogen pollutant concentrations, as the total nitrogen concentration, which is the origination concentration, can measure four different nitrogen pollutant concentrations (organic nitrogen, ammonia, nitrite, and nitrate) at many space points at various times. Fig. 3 compares each nitrogen pollutant concentration (organic nitrogen, ammonia, nitrite, and nitrate pollutant concentrations) from the total nitrogen concentra-

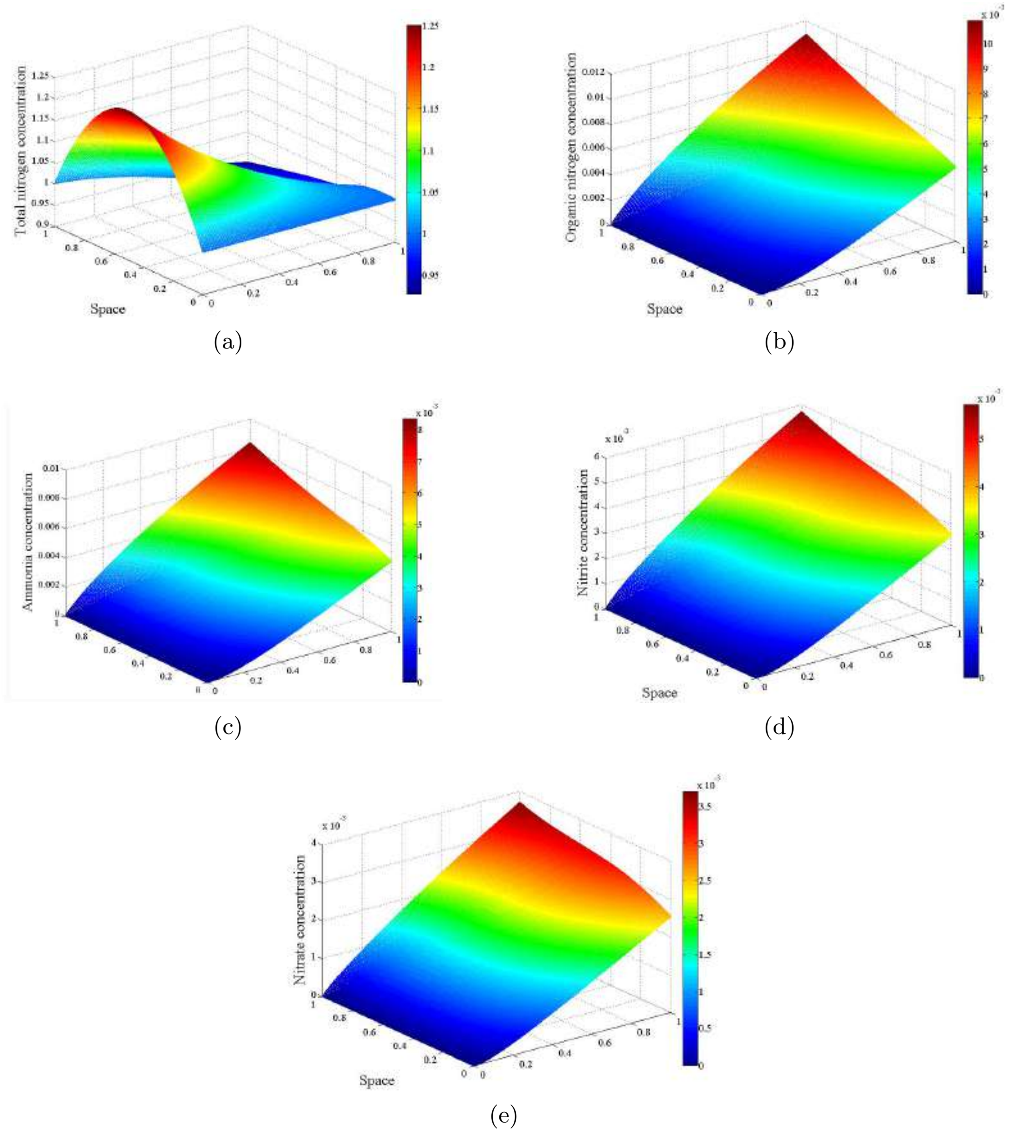
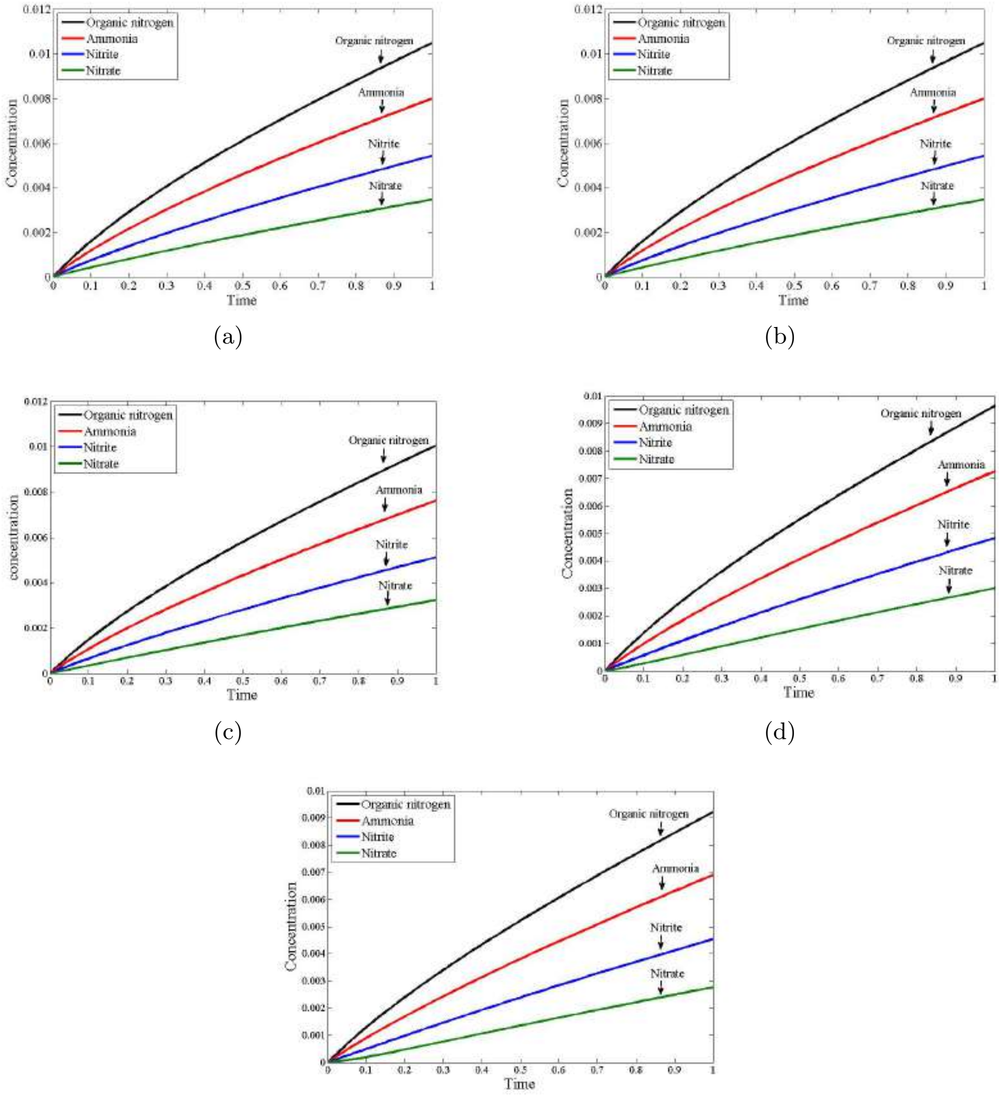


Figure 2: (a)The total nitrogen, (b)organic nitrogen, (c)ammonia, (d) nitrite, and (e)nitrate (kg/m^3) where the rate of change at the right boundary condition is -0.001.

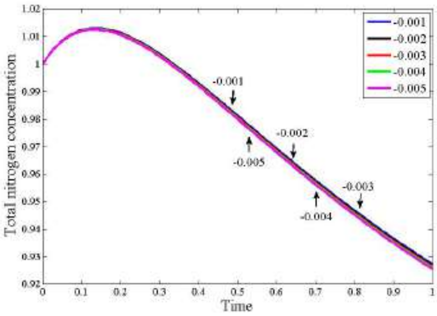
tions by considering the different rates of change of nitrogen pollutant concentrations at right boundary conditions of -0.001, -0.002, -0.003, -0.004, and, -0.005, such that the different rates of change of nitrogen pollutant concentrations at the right boundary condition is similar, the organic nitrogen pollutant concentration is at a higher level, and the other pollutant concentrations (ammonia, nitrite, and nitrate) have high concentrations, respectively, following nature. Fig. 4



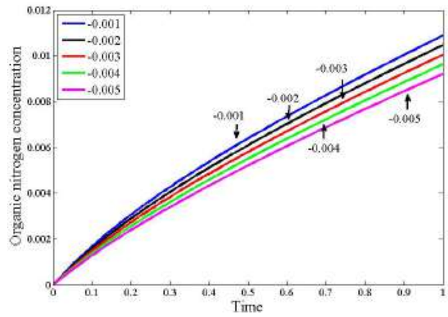
(a) (b) (c) (d) (e)

Figure 3: The organic nitrogen, ammonia, nitrite, and nitrate concentrations (kg/m^3) from the total nitrogen concentration in cases of the rates of change at right boundary condition of (a)-0.001, (b)-0.002, (c)-0.003, (d)-0.004, and (e)-0.005 at $C(1,t)$.

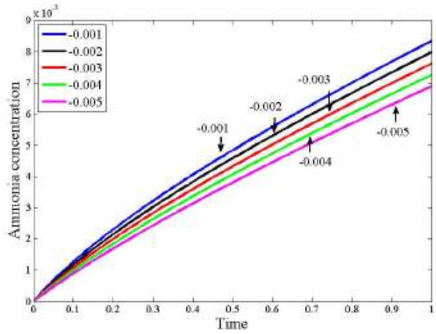
compares the different rates of change of nitrogen pollutant concentrations at right boundary conditions of -0.001, -0.002, -0.003, -0.004, and -0.005 such that they affect the pollutant concentration levels of each nitrogen pollutant concentration (the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate pollutant concentrations). The difference of each nitrogen pollutant concentra-



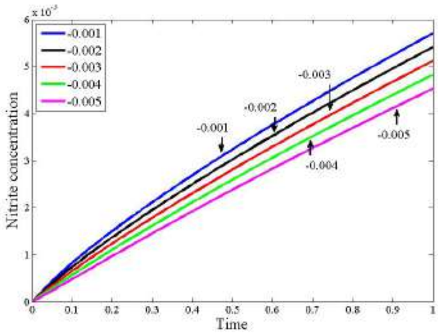
(a)



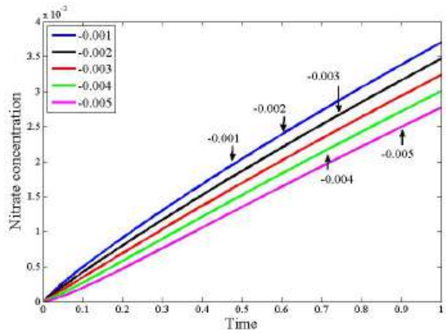
(b)



(c)



(d)



(e)

Figure 4: Comparison (a) the total nitrogen, (b) organic nitrogen, (c) ammonia, (d) nitrite, and (e) nitrate (kg/m^3) when the rates of change at the right boundary condition are -0.001, -0.002, -0.003, -0.004, and -0.005 at $C(1,t)$.

tion between the rates of change at right boundary conditions at $C(1,t)$ have equal pollutant concentration values, which shown in Table 4. Observe that if the rates of change of nitrogen pollutant concentrations have low values to high values, the nitrogen pollutant concentration levels will also have low values to high values respectively.

(a)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
0	1	1.1600	1.2400	1.2400	1.1600	1
0.2	1	1.1112	1.1620	1.1465	1.0616	1.0112
0.4	1	1.0734	1.0991	1.0689	1.0058	0.9930
0.6	1	1.0429	1.0470	1.0034	0.9860	0.9693
0.8	1	1.0177	1.0031	0.9766	0.9635	0.9467
1	1	0.9965	0.9702	0.9582	0.9430	0.9273

(b)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
			$(\times 10^{-3})$			
0	0	0	0	0	0	0
0.2	0.5422	1.1541	1.5746	1.8371	1.9462	3.0696
0.4	1.6457	2.5138	3.1438	3.5373	3.9721	5.3720
0.6	2.9711	3.9580	4.6862	5.1254	6.0974	7.3600
0.8	4.3962	5.4335	6.1916	6.9160	7.9936	9.1780
1	5.8637	6.9113	7.6932	8.6747	9.7592	10.8905

(c)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
			$(\times 10^{-3})$			
0	0	0	0	0	0	0
0.2	0.4918	1.0041	1.3282	1.5092	1.5594	2.3043
0.4	1.4213	2.1222	2.605	2.8789	3.1126	4.0544
0.6	2.4970	3.2809	3.8376	4.1442	4.7102	5.5914
0.8	3.6337	4.4507	5.0295	5.4828	6.1653	7.0068
1	4.7942	5.6159	6.1984	6.812	7.5319	8.3428

(d)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
			$(\times 10^{-3})$			
0	0	0	0	0	0	0
0.2	0.4172	0.8025	1.0171	1.1166	1.1187	1.5024
0.4	1.1281	1.6371	1.9618	2.1187	2.1853	2.6907
0.6	1.9180	2.4801	2.8581	3.0372	3.2616	3.763
0.8	2.7389	3.3215	3.7170	3.9480	4.2688	4.7613
1	3.5708	4.1553	4.5481	4.8565	5.2251	5.7072

(e)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
			$(\times 10^{-3})$			
0	0	0	0	0	0	0
0.2	0.3335	0.6017	0.7295	0.7752	0.7561	0.9103
0.4	0.8445	1.1932	1.3956	1.4732	1.4590	1.6760
0.6	1.3949	1.7786	2.0202	2.1125	2.1570	2.3872
0.8	1.9602	2.3576	2.6145	2.723	2.8245	3.0587
1	2.5303	2.9292	3.1857	3.3267	3.4648	3.6994

Table 2. (a) The total nitrogen, (b) organic nitrogen, (c) ammonia, (d) nitrite, and (e) nitrate, (kg/m^3) where the rate of change at the right boundary condition is -0.001 such that there are Δx=0.00625 and Δt=0.01.

6. Conclusion

In this study, the nitrogen pollutant concentration models in a stream, the total nitrogen, organic nitrogen, ammonia, nitrite, and nitrate concentrations, are considered. These models show that each nitrogen pollutant concentration behavior, such as the total nitrogen concentration, affects the measurement of various pollutant concentrations of nitrogen, such as the organic nitrogen,

(a)

$t \setminus x$	0	0.2	0.4	0.6	0.8	1
0	1	1.1600	1.2400	1.2400	1.1600	1
0.2	1	1.1112	1.1620	1.1465	1.0616	1.0105
0.4	1	1.0734	1.0991	1.0689	1.0056	0.9919
0.6	1	1.0429	1.0470	1.0034	0.9855	0.9679
0.8	1	1.0177	1.0031	0.9764	0.9628	0.9451
1	1	0.9965	0.9702	0.9578	0.9421	0.9255

(b)

$t \setminus x$	0	0.2	0.4 ($\times 10^{-3}$)	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0.5422	1.1541	1.5746	1.8371	1.9462	2.3966
0.4	1.6457	2.5138	3.1438	3.5373	3.8069	4.3332
0.6	2.9711	3.9580	4.6862	5.1254	5.6310	6.064
0.8	4.3962	5.4335	6.1916	6.7568	7.3045	7.6775
1	5.8637	6.9113	7.6716	8.3508	8.8875	9.2172

(c)

$t \setminus x$	0	0.2	0.4 ($\times 10^{-3}$)	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0.4918	1.0041	1.3282	1.5092	1.5594	1.6906
0.4	1.4213	2.1222	2.6050	2.8789	2.9834	3.1357
0.6	2.4970	3.2809	3.8376	4.1442	4.3456	4.4593
0.8	3.6337	4.4507	5.0295	5.3777	5.6233	5.7045
1	4.7942	5.6159	6.1871	6.5895	6.8423	6.8966

(d)

$t \setminus x$	0	0.2	0.4 ($\times 10^{-3}$)	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0.4172	0.8025	1.0171	1.1166	1.1187	0.9729
0.4	1.1281	1.6371	1.9618	2.1187	2.1017	1.9249
0.6	1.9180	2.4801	2.8581	3.0372	3.0172	2.8312
0.8	2.7389	3.3215	3.717	3.8967	3.8967	3.6969
1	3.5708	4.1553	4.5447	4.7367	4.7447	4.5310

(e)

$t \setminus x$	0	0.2	0.4 ($\times 10^{-3}$)	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0.3335	0.6017	0.7295	0.7752	0.7561	0.4709
0.4	0.8445	1.1932	1.3956	1.4732	1.4151	1.0574
0.6	1.3949	1.7786	2.0202	2.1125	2.0141	1.6426
0.8	1.9602	2.3576	2.6145	2.7048	2.5964	2.2139
1	2.5303	2.9292	3.1852	3.2761	3.1626	2.7709

Table 3. (a) The total nitrogen, (b) organic nitrogen, (c) ammonia, (d) nitrite, and (e) nitrate, (kg/m^3) where the rate of change at the right boundary condition is -0.005 such that there are Δx=0.00625 and Δt=0.01.

ammonia, nitrite, and nitrate. The solution performance of these models can be shown by using the numerical methods (the FTCS and Saulyev methods), which are explicit schemes. Both methods are in good agreement with the analytic solution but they have differences under conditions, such that the FTCS method can solve problems under a stability condition, while the Saulyev method be able to solve many scenarios without the limitation of stability conditions. The Saulyev method gives good agreement approximated solutions without stability

(a)

$t \setminus rates$	-0.001 and -0.002	-0.002 and -0.003	-0.003 and -0.004	-0.004 and -0.005
		($\times 10^{-3}$)		
0	0	0	0	0
0.2	1.7636	1.7636	1.7636	1.7636
0.4	2.7637	2.7637	2.7637	2.7637
0.6	3.4577	3.4577	3.4577	3.4577
0.8	3.9962	3.9962	3.9962	3.9962
1	4.4401	4.4401	4.4401	4.4401

(b)

$t \setminus rates$	-0.001 and -0.002	-0.002 and -0.003	-0.003 and -0.004	-0.004 and -0.005
		($\times 10^{-4}$)		
0	0	0	0	0
0.2	1.6824	1.6824	1.6824	1.6824
0.4	2.5970	2.5970	2.5970	2.5970
0.6	3.2400	3.2400	3.2400	3.2400
0.8	3.7513	3.7513	3.7513	3.7513
1	4.1833	4.1833	4.1833	4.1833

(c)

$t \setminus rates$	-0.001 and -0.002	-0.002 and -0.003	-0.003 and -0.004	-0.004 and -0.005
		($\times 10^{-4}$)		
0	0	0	0	0
0.2	1.5341	1.5341	1.5341	1.5341
0.4	2.2966	2.2966	2.2966	2.2966
0.6	2.8301	2.8301	2.8301	2.8301
0.8	3.2557	3.2557	3.2557	3.2557
1	3.6156	3.6156	3.6156	3.6156

(d)

$t \setminus rates$	-0.001 and -0.002	-0.002 and -0.003	-0.003 and -0.004	-0.004 and -0.005
		($\times 10^{-4}$)		
0	0	0	0	0
0.2	1.3237	1.3237	1.3237	1.3237
0.4	1.9145	1.9145	1.9145	1.9145
0.6	2.3296	2.3296	2.3296	2.3296
0.8	2.6610	2.6610	2.6610	2.6610
1	2.9404	2.9404	2.9404	2.9404

(e)

$t \setminus rates$	-0.001 and -0.002	-0.002 and -0.003	-0.003 and -0.004	-0.004 and -0.005
		($\times 10^{-4}$)		
0	0	0	0	0
0.2	1.0985	1.0985	1.0985	1.0985
0.4	1.5463	1.5463	1.5463	1.5463
0.6	1.8617	1.8617	1.8617	1.8617
0.8	2.1119	2.1119	2.1119	2.1119
1	2.3214	2.3214	2.3214	2.3214

Table 4. Comparison each two rates of change at $C(1,t)$ right boundary condition at of (a) the total nitrogen, (b) organic nitrogen, (c) ammonia, (d) nitrite, and (e) nitrate concentrations (kg/m^3) such that there are Δx=0.00625 and Δt=0.01.

limitation. It is a good method for several realistic scenarios. In the simulation, we can see that the total nitrogen pollutant controlling gives better overall water-quality levels than another nitrogen pollutant compounds controlling.

Acknowledgements

This paper is supported by the Centre of Excellence in Mathematics Program of the Commission on Higher Education (CEM), Thailand. The authors greatly appreciate valuable comments received from the reviewers.

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Accepted: 19.12.2018

Some results on K-frames

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Abstract. In this paper we present some results on K-frames when $K \in B(H)$ is an injective closed range operator. Also we give a condition on K-frames $\{f_n\}_{n\in N}$ and $\{g_n\}_{n\in N}$ so that $\{f_n + g_n\}_{n\in N}$ is again a K-frame for H. Finally, Schatten class operators are also discussed in terms of K-frames.

Keywords: K-frames, Schatten class operators.

1. Introduction

Frames in Hilbert spaces were introduced by R.J. Duffin and A.C. Schaffer. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. Frames were introduced as a better replacement to orthonormal basis. We refer [2] for an introduction to frame theory.

K-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. K-frames are more general than classical frames. In K-frames the lower bound only holds for the elements in the range of K.

Some basic definitions and results related to frames and K-frames are contained in section 2. In section 3 we have included some new results on K-frames. Section 4 contains our main results relating K-frames and operators in Schatten classes.

Throughout this paper, H is a separable Hilbert space and we denote by $B(H)$, the space of all linear bounded operators on H. For $K \in B(H)$, we denote $R(K)$ the range of K. Also, $GL(H)$ denote the set of all bounded linear operators which have bounded inverses.

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2. Preliminaries

For a separable Hilbert space H, a sequence $\{f_n\}_{n\in N} \subset H$ is said to be a **frame** ([2]) for H if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.If $A = B$, we say that $\{f_n\}_{n\in N}$ is a tight frame in H. Let $K \in B(H)$. We say that $\{f_n\}_{n\in N} \subset H$ is a K**-frame** ([3]) for H if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.

If $\{f_n\}_{n\in N} \subset H$ is an ordinary frame for H, then $\{Kf_n\}_{n\in N}$ is a K-frame for H ([5]). If $T \in B(H)$ and $\{f_n\}_{n\in N}$ is a K-frame for H, then $\{Tf_n\}_{n\in N}$ is a TK-frame for H ([5]). If $\{f_n\}_{n\in N} \subset H$ is a K-frame for H, then $\{K^N f_n\}_{n\in N}$ is a K^N-frame for H where $N \geq 1$ is a fixed integer ([5]). $\mathcal{F}_K(H) \subset \mathcal{F}_M(H)$ if and only if $R(K) \supset R(M)$ where $\mathcal{F}_K(H), \mathcal{F}_M(H)$ denote the set of all K-frames and M-frames on H ([4]). Also, we use the result: $T \in B(H)$ is an injective and closed range operator if and only if there exists a constant $c > 0$ such that $c\|x\|^2 \leq \|Tx\|^2$, for all $x \in H$ ([6]),in the proof of our main results.

3. K-frames

In this section we present our results on K-frames.

Theorem 3.1. *Let $K \in B(H)$ be an injective and closed range operator.If $\{f_n\}_{n\in N}$ is a frame for $R(K)$, then $\{K^*f_n\}_{n\in N}$ is a frame for H and hence $\{KK^*f_n\}_{n\in N}$ is a K-frame for H.*

Proof. Let $\{f_n\}_{n\in N}$ be a frame for $R(K)$.Then there exist constants $A, B > 0$ such that, for all $x \in R(K)$,

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2.$$

Also,by our assumption, there exists $c > 0$ such that $c\|x\|^2 \leq \|Kx\|^2$, for all $x \in H$. For $x \in H, Kx \in R(K)$, and we get

$$A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2.$$

Therefore,

$$Ac\|x\|^2 \leq A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2 \leq B\alpha^2\|x\|^2,$$

for all $x \in H$ and for some $\alpha > 0$, i.e.

$$E\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, K^* f_n\rangle|^2 \leq F\|x\|^2,$$

for all $x \in H$ where $E = Ac > 0, F = B\alpha^2 > 0$. Therefore, $\{K^* f_n\}_{n\in N}$ is a frame for H and hence $\{KK^* f_n\}_{n\in N}$ is a K-frame for H. □

Corollary 3.2. *Let $K \in B(H)$ be an injective and closed range operator and $\{f_n\}_{n\in N} \subset H$ be such that $\{(K^{-1})^* f_n\}_{n\in N}$ is a frame for $R(K)$. Then $\{f_n\}_{n\in N}$ is a frame for H.*

Theorem 3.3. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then there exist constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2,$$

for all $x \in H$.

Proof. Since $\{f_n\}_{n\in N}$ is a K-frame for H, there exist constants $C, D > 0$ such that

$$C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2,$$

for all $x \in H$. Since $K^* \in B(H)$ is an injective and closed range operator, there exist $d > 0$ such that

$$d\|x\|^2 \leq \|K^*x\|^2,$$

for all $x \in H$. Therefore, for all $x \in H$,

$$C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2 \leq (D/d)\|K^*x\|^2,$$

for all $x \in H$ there exist $A = C, B = D/d > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2.$$

□

Corollary 3.4. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then $\{f_n\}_{n\in N}$ is a frame for H.*

Definition 3.5. *A sequence $\{f_n\}_{n\in N} \subset H$ is said to be a $2K$-frame for H if there exist $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2,$$

for all $x \in H$.

Theorem 3.6. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with bounds A_1, B_1 and $\{g_n\}_{n\in N}$ be a $2K$-frame for H with bounds A_2, B_2 such that $0 < B_2 < A_1$. Then $\{f_n+g_n\}_{n\in N}$ is a K-frame for H with frame bounds $A_1 - B_2$ and $B_1 + B_2\|K^*\|^2$.*

Proof. By definition of K-frame and $2K$-frame, we have

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B_1\|x\|^2$$

and

$$A_2\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \leq B_2\|K^*x\|^2,$$

for all $x \in H$. Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \tag{1}$$

$$\leq B_1\|x\|^2 + B_2\|K^*x\|^2 \tag{2}$$

$$\leq (B_1 + B_2\|K^*\|^2)\|x\|^2, \tag{3}$$

for all $x \in H$.

Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 = \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n - g_n\rangle|^2 \tag{4}$$

$$\leq \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2. \tag{5}$$

This implies that,

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 + B_2\|K^*x\|^2$$

$$i.e. \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 \geq (A_1 - B_2)\|K^*x\|^2$$

where $A_1 - B_2 > 0$.This completes the proof. □

4. K-frames and operators in Schatten classes

Definition 4.1 ([7]). *Let T be a compact operator on a separable Hilbert space H. Given $0 < p < \infty$, we define the **Schatten** p **- class** of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p.$S_p(H)$ is a two sided ideal in $B(H)$.*

Following two theorems by H. Bingyang, L.H. Khoi and K. Zhu gives a characterization for Schatten p-class operators in terms of frames.

Theorem 4.2 ([1]). *Suppose T is a compact operator on H and $2 \leq p < \infty$.Then the following conditions are equivalent:*

(a) *$T \in S_p$;*

(b) *$\{\|Te_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H;*

(c) *$\{\|Tf_n\|\}_{n\in N} \in l^p$ for every frame $\{f_n\}_{n\in N}$ in H.*

Theorem 4.3 ([1]). *Suppose T is a compact operator on H and $0 \leq p \leq 2$.Then the following conditions are equivalent:*

(a) *$T \in S_p$;*

(b) *$\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N}$ in H;*

(c) *$\{\|Tf_n\|\}_{n\in N} \in l^p$ for some frame $\{f_n\}_{n\in N}$ in H.*

At first we focus on the case where $2 \leq p < \infty$.

Theorem 4.4. *Suppose T is a compact operator on H and $K \in B(H)$. If T is in the Schatten class S_p, then $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, where $2 \leq p < \infty$.*

Proof. Suppose $T \in S_p$,$2 \leq p < \infty$.

Let $\{f_n\}_{n\in N}$ be a K-frame for H and $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{h_n\}_{n\in N} = \{f_n\}_{n\in N} \bigcup \{e_n\}_{n\in N}$ is a frame for H and $\{\|Th_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$. Therefore $\{\|Tf_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$ and the result is proved. □

Theorem 4.5. *Suppose T is a compact operator on H and $K \in B(H)$.If $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H, where $2 \leq p < \infty$.*

Proof. Let $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{Ke_n\}_{n\in N}$ is a K-frame for H. Therefore by our assumption $\{\|TKe_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$.Hence $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. □

Theorem 4.6. *Suppose T is a compact operator on H and $K \in GL(H)$ and $2 \leq p < \infty$.Then the following are equivalent:*

(*a*) *T is in the Schatten class S_p;*

(*b*) $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for every K-frame* $\{f_n\}_{n\in N}$ *in H.*

Proof. Clearly, (a) implies (b) holds by Theorem 4.4. Now suppose (b) holds. Then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. This implies that $TK \in S_p$. Using the fact that S_p is a two- sided ideal in $B(H)$, $TKK^{-1} \in S_p$, i.e. $T \in S_p$. This completes the proof. □

Now we move onto the case where $0 < p \leq 2$.

Theorem 4.7. *Let T be a compact operator on H and $K \in B(H)$. Suppose $\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N} \subset H$. Then $\{\|Tf_n\|\}_{n\in N} \in l^p$ for some K-frame $\{f_n\}_{n\in N}$ for H, where $0 < p \leq 2$.*

Proof. Suppose $\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N} \subset H$. Then $T \in S_p$, which implies that $TK \in S_p$ for any $K \in B(H)$. By Theorem 4.3, $\{\|TKe_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N}$ in H. Now take $f_n = Ke_n$, so that $\{f_n\}_{n\in N}$ is a K-frame for H and hence the theorem holds. □

Theorem 4.8. *Let T be a compact operator on H and $K \in B(H)$,where K^* is an injective closed range operator. If $\{\|Tf_n\|\}_{n\in N} \in l^p$ for some K-frame $\{f_n\}_{n\in N}$ for H,then $T \in S_p$, where $0 < p \leq 2$.*

Proof. By Corollary 3.4,if K^* is an injective closed range operator,then every K-frame is a frame and then applying Theorem 4.3, we get $T \in S_p$. □

5. Acknowledgement

The first author acknowledges the financial support of University Grants Commission.

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Accepted: 20.12.2018

Analysis and implementation of kidney stones detection by applying segmentation techniques on computerized tomography scans

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Abstract. Kidney stone disease is one of the risks for life throughout the world and majority of people with stone formation in kidney at the initial stage do not notice it as disease and it damages the organ slowly. Current estimation is that there are 30 million people suffering by this disease. There are different imaging techniques for diagnosing kidney diseases, such as CT images, X-rays, and Ultrasound imaging. In this study we explored the deployment of three segmentation techniques using matlab to examine the kidney area, and to enhance kidney stone detection. The segmentation techniques under investigation are: threshold based segmentation, watershed based segmentation, and edge based segmentation.
Keywords: image processing, segmentation, computerized tomography, kidney stones.

1. Introduction

There are various imaging techniques that can be used in the medical practice such as computed tomography (CT) scan, X-rays, and magnetic resonance imaging (MRI). CT scans are one of the most widely available imaging techniques. These techniques are used for diagnosis and follow-up of different kidney abnormalities. These abnormalities comprise stone disease, kidney cysts, hydronephrosis (blockage of urine), congenital anomalies, as well as urinary tract tumors. Moreover, successful management and surgical treatment of such conditions depends largely on accurate identification of such anatomical details. The detection of kidney stones using ultrasound imaging is a highly challenging task and is largely operator dependent as they are of low contrast and contain speckle

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noise. This challenge is overcome by employing other imaging techniques. In fact, non-contrast enhanced CT scan of the urinary tract is considered the gold standard of stone disease (Petrik, et al., 2016). The performance of CT scan is not only superior to all other imaging modalities but also provides various vital information that play a crucial role in patient counseling in treatment decision such as stone density, structure of the stone and skin-to-stone distance (Viswanath and Gunasundari, 2015).

The images obtained by X-ray or CT in adverse conditions may be contaminated with noise that can affect the detection of diseases, abnormality or cancerous cells. A large number of image processing techniques (filters) have been proposed to remove noise. These techniques depend on the type of noise present in the image (Sanchez et al., 2012).

The operation of producing the estimated clean, original images out from noisy/corrupt images is called image restoration. There are many forms of corruption on images, such as: motion blur, camera noise, and camera misfocus (Banham and Katsaggelos, 1996). The goal of restorative image investigation is to procure helpful data about the body organs and the medical courses of action needed. This can be done by utilizing outer and inside wellsprings of vitality, kidney image analysis performed using CT scans. CT scans are preferred method to take images of the kidney because of its straightforwardness, convenience and cost viability (Ebrahimi and Mariano, 2015).

An important source of information for kidney imaging is the presence and distribution of micro calcifications in the kidney, this anatomical information can be obtained with high resolution technology using CT scans, As yet there is no comprehensive imaging modality for all CT scan applications and needs, despite the fact that the capacity to mechanize and examine restorative pictures gives a capable intends to help doctors; subsequently computer projects, handling techniques that get the information and data from medicinal imaging scanners must be painstakingly created to save and improve the most imperative clinical data as opposed to presenting extra curios. The ability to improve diagnostic information from medical images can be further enhanced by designing computer processing algorithms, application and software intelligently, that is why we proposed an application based on matlab software to clarified boundaries for stones in kidney (Ebrahimi and Mariano, 2015).

The compelling question is: why do physicians treat kidney stones? First of all, this disease is usually a painful condition where pain medication is a temporary measure and a definitive measure should be undertaken i.e stone removal. Moreover, pain medications are not effective to alleviate pain. Interesting, results from historical landmark observational studies (Blandy and Singh, 1976). In 1970s reported a 10-year mortality rate of up to 30% in patients with large kidney stone (staghorn stone) who were treated conservatively (without surgical removal). Additionally, one quarter of these patients suffer from severe urinary tract infections that have significant impact on kidney function. (Rous and Turner, 1977). In this paper we aim to provide physicians with accurate results related to the existence of kidney stones on the CT images, using efficient segmentation techniques.

2. Literature review

X-rays and Ultrasound imaging are considered low cost imaging techniques. While CT are considered high cost technique. The researchers in (Vasanthselvakumar, et al., 2017) has focused their studies on applying image processing techniques on Ultrasonic images, they have applied an automatic detection and classification of various diseases such as stone, cyst and cancer masses present in the pelvic region of the kidney.

While other researchers have focused on analyzing the images resulted by using MRI systems (Youngwoo, et al., 2016), and (Hamed and Fei, 2013). The authors in (Youngwoo, et al., 2016), have developed an automated method for segmentation and a method to provide the measurements of kidneys from magnetic resonance images in patients with autosomal dominant polycystic kidney disease, the authors also assessed the performance of the automated method with the reference manual segmentation method. In (Hamed and Fei, 2013), the authors have provided an automatic segmentation of the kidney in 3D MR images, by extracting texture features and statistical matching of geometrical shape of the kidney. The researchers in (Altintas, et al., 2010) have used x-ray images to detect kidney stones. The authors in (Tulin and Das, 2017) and (Natarajan, et al., 2013) have applied image processing techniques on CT scan images.

Image segmentation is the process of partitioning a digital image into multiple segments. The goal of segmentation is to simplify or change the representation of an image into something that is more meaningful and easier to analyze. Several image segmentation techniques have been developed by the researchers in order to make images smooth and easy to evaluate. In (Tulin and Das, 2017), the authors aim to remove the Gaussian noise from the abdominal CT scan images and segment the kidney region from this abdominal area to make a size measurement and to characterize the difference between healthy and diseased kidney. In (Natarajan, et al., 2013), the authors have developed an approach for extracting kidney in abdominal CT scan images. The images have been divided into two stages. In the first stage a template evaluation method has been developed for extracting the desired region in an image on the basis of properties of an organ, which helps in processing to a confined region and is an automated process. The second stage uses the concept of intensity values of a pixel and separates the desired region from the original image on the basis of a computed threshold range. In addition to this, the authors have used a set of morphological operations for fine coarse kidney segmentation and various filters for removing noise from an image. In (Ebrahimi and Mariano, 2015), the authors have provided a development of a semi-automated program that used image processing techniques and geometry principles to define the boundary, and segmentation of the kidney area, and to enhance kidney stone detection. It marked detected kidney stones and provided an output that identifies the size and location of the kidney based on pixel count. The program was tested on standard Kidney Urine Belly (KUB) CT scan slides. The KUB CT scan slides who were divided into two groups based on the presence and absence of kidney stones in their hospital records.

3. Data and methodology

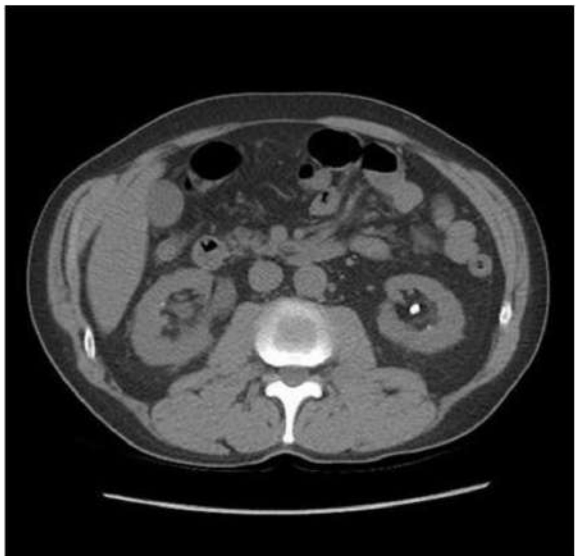

Figure 1: The Original Image

Matlab is a high-performance language for education and research as it integrates computation, visualization and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation and also it has toolboxes for signal processing, neural network, image processing, database etc. Matlab software was used to implement the algorithm, since Matlab Image Processing Toolbox is a collection of functions that extend the capability of the Matlab numeric-computing environment. The toolbox supports a wide range of image processing operations, such as Image analysis and enhancement. Region of interest operations, linear filtering and filter design (Beucher, et al., 1990). All the functions and equations used in this study are from Matlab image processing toolbox.

The CT scan images are taken from (Kidney Stones, 2018), the original image was segmented by threshold, edge-based segmentation, and watershed segmentation, the original image is shown in Fig.1. There are many segmentation techniques as shown in Fig.3. In this paper we have applied three segmentation techniques on the gray scale image after applied Gaussian filter twice on

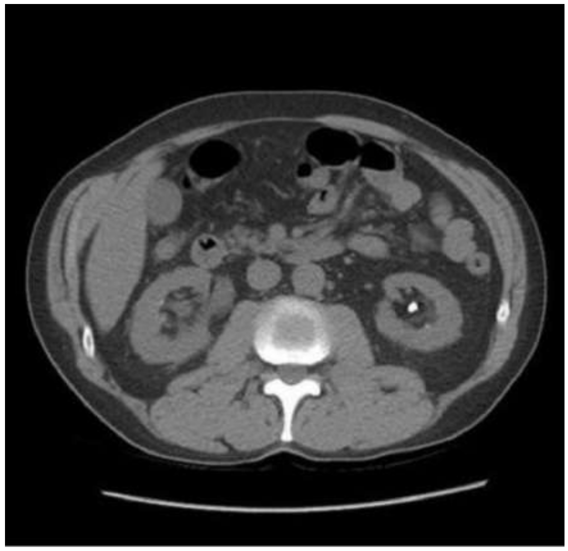

Figure 2: Gray Scale Image

original image shown in Fig.2. These segmentation techniques are Edge based segmentation, Watershed segmentation, and Threshold based segmentation.

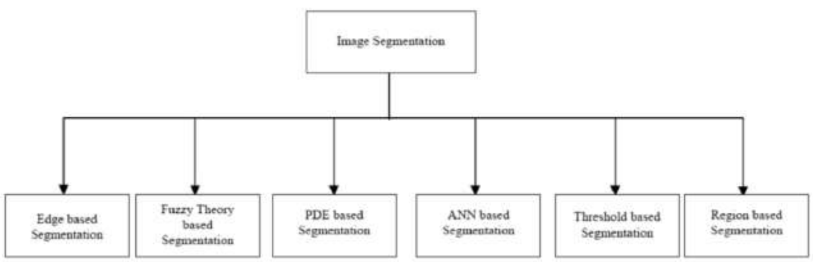

Figure 3: Image Segmentation Techniques

3.1 Edge based segmentation

All images produced in this paper were processed using matlab software. As shown in Fig.4 illustrate steps edge based segmentation method started by reading image, then detecting the entire cell, two cells are presented in this image, but only one cell can be seen in its entirety, this cell will be detected. Another word for object detection is segmentation. The object to be segmented differs greatly in contrast from the background image. Changes in contrast can be detected by operators that calculate the gradient of an image. The gradient image can be calculated and a threshold can be applied to create a binary mask containing the segmented cell. First, we use edge and the Sobel operator to calculate the threshold value. We then tune the threshold value and use edge again to obtain a binary mask that contains the segmented cell. The next step is to dilate the image. The binary gradient mask shows lines of high contrast in the image. These lines do not quite delineate the outline of the object of interest. Compared to the original image, you can see gaps in the lines surrounding the object in the gradient mask. These linear gaps will disappear if the Sobel image is dilated using linear structuring elements, which we can create with the strel function. The binary gradient mask is dilated using the vertical structuring element followed by the horizontal structuring element. The imdilate function dilates the image. Filling the interior gaps is the next step. The dilated gradient mask shows the outline of the cell quite nicely, but there are still holes in the interior of the cell. To fill these holes, we use the imfill function. Then the connected objects in border should be removed. The cell of interest has been successfully segmented, but it is not the only object that has been found. Any objects that are connected to the border of the image can be removed using the imclearborder function. The connectivity in the imclearborder function was set to 4 to remove diagonal connections. The next step is smoothening the object. Finally, in order to make the segmented object look natural, we smoothen the object by eroding the image twice with a diamond structuring element. We create the diamond structuring element using the strel function. An alternate method for displaying the segmented object would be to place an outline around the segmented cell. The outline is created by the bwperim function.

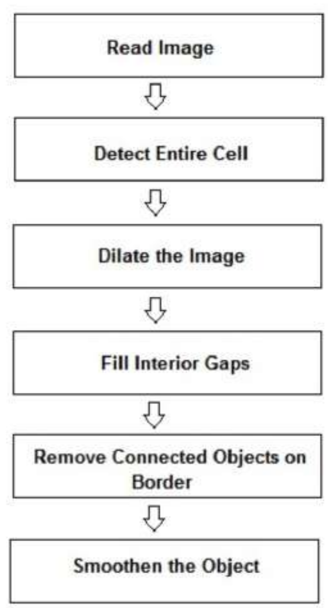

Figure 4: Edge Based Segmentation

3.2 Watershed based segmentation

There are many category watershed segmentation techniques. In this paper; marker controlled watershed segmentation has been applied sequential steps on the original 3D image as Illustrate in Fig.5 Watershed segmentation has been used to separate touching objects in an image. The watershed transform finds "catchment basins" and "watershed ridge lines" in an image by treating it as a surface where light pixels are high and dark pixels are low. Segmentation using the watershed transform works better if you can identify, or "mark," foreground objects and background locations. Marker-controlled watershed segmentation follows the following procedure: compute a segmentation function; where the image whose dark regions are the objects you are trying to segment. The next step is to compute foreground markers, there are connected blobs of pixels within each of the objects. Then we should compute background markers to remove the pixels that are not part of any object. Applying the segmentation function so that it only has minima at the foreground and background marker locations. The last step is to compute the watershed transform of the modified segmentation function.

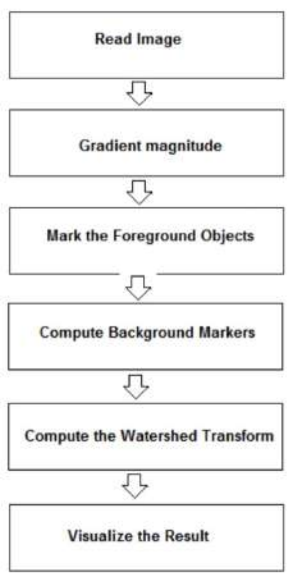

Figure 5: Marker Controlled Watershed Segmentation

3.3 Threshold based segmentation

Thresholding is the simplest method of image segmentation. From a gray scale image, thresholding can be used to create binary images. Binary images are produced from color images by segmentation. Segmentation is the process of assigning each pixel in the source image to two or more classes. If there are more than two classes, then the usual result is several binary images. In image processing, thresholding is used to split an image into smaller segments, or junks, using at least one color or gray scale value to define their boundary. The advantage of obtaining first a binary image is that it reduces the complexity of the data and simplifies the process of recognition and classification. The most common way to convert a gray level image to a binary image is to select a single threshold value. The input to a thresholding operation is typically a gray scale or color image. In the simplest implementation, the output is a binary image representing the segmentation. Black pixels correspond to background and white pixels correspond to foreground (or vice versa). This method of segmentation applies a single fixed criterion to all pixels in the image simultaneously. There are three types of thresholding algorithms.

- Global thresholding
- Local thresholding
- Adaptive thresholding

4. The results

In many cases kidney CT scan images cannot give enough information to the physicians; these information includes: the size, position of stones and other issues. In this paper, the experiments focused on detecting and clarifying stones in CT scan images using three methods. The first step is applying enhancement on image as we mentioned before by using double Gaussian filter, and convert the original image to gray scale; this approach is used in edge base segmentation and thresholding segmentation. Segmentation methods are based on gray intensity on image while all medical images are gray scale.

4.1 Edge based segmentation

In edge-base segmentation technique; a connected pixel that is found on the boundary of the region is called an edge, so these pixels on the edge are known as edge points. Edge can be calculated by finding the derivative of an image function. Some edges are very easy to find. These are: Ramp edge, Step edge, Roof edge, Spike edge. Step edge is an abrupt change in intensity level. Ramp edges a gradual change in intensity. Spike edges a quick change in intensity and after that returns immediately to an original intensity. Roof edge is not instantaneous over a short distance. Edge based image segmentation method falls under structural techniques. As show in in the figures below, the result from Dilated Gradient Image and Binary Gradient Image is better than thresholding image. Kidney stone is cleared and surrounded by edge, edge-base segmentation is also fit for two dimension images.

Figure 6: Dilated Gradient Image

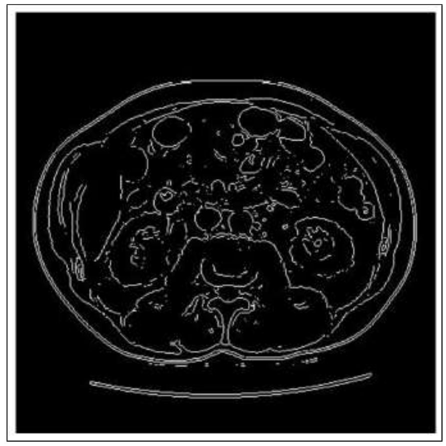

Figure 7: Binary Gradient Image

4.2 Marker-controlled watershed segmentation

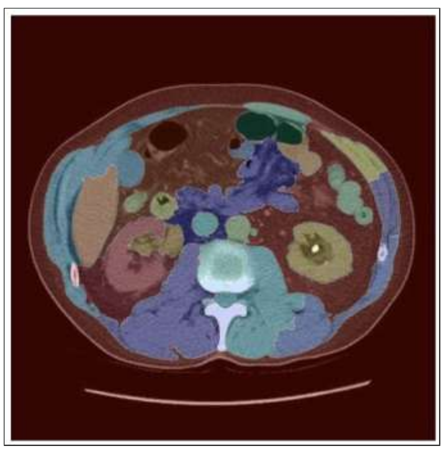

Figure 8: Superimposed Image

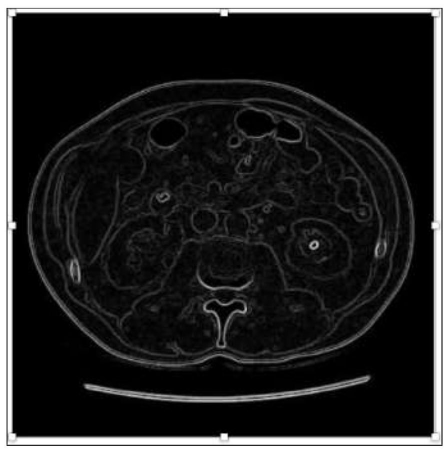

Figure 9: Gradient Magnitude Image

In watershed Segmentation technique after applying the double Gaussian filter of the three dimension image (topography surface) and then start filling images by water, this is done to create Dam. This technique give us notifications about topography of images (high and low places), this technique will colored segments with different color, its helpful for physician to detect and allocate stones in kidney as show in Figure 13.

4.3 Threshold segmentation

In threshold based segmentation: images will change pixels color that convey value of intensity to 1or 0 based in thresholding variable T, as given in the equation 1. As seen in Fig. 15, the result after applying thresholding segmentation

Figure 10: Markers and Object Boundaries Superimposed on Original Image

Figure 11: Modified Regional Maxima Superimposed on Original Image

Figure 12: Gradient Magnitude Image

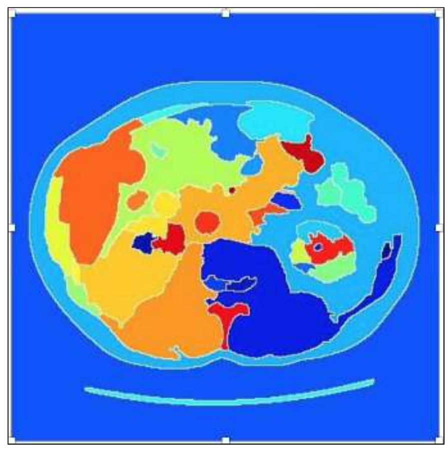

Figure 13: Colored Watershed Matrix Image

technique on gray scale filter. Fig. 15 displays stones in kidney but its not clear when its compared with other techniques. Thresholding is segmentation technique suitable for two-dimension images.

$$q(x,y) = \begin{cases} 1, & \text{if } p(x,y) > T \\ 0, & \text{if } \; p(x,y) \le T. \end{cases} \tag{1}$$

Figure 14: Gray Scale Image

Figure 15: Threshold Segmentation Image

5. Conclusions and future work

In this work we investigated the application of different segmentation techniques in detecting kidney stones. The segmentation techniques under investigation are: edge based segmentation, watershed based segmentation and threshold based segmentation. Furthermore, depending on our experiments, we nominate watershed algorithms for detecting kidney stones. Future work may focus on applying other segmentation techniques to detect kidney stones, and improving the quality and efficiency of a reasonable analysis algorithm. Another interesting work is to expand the analysis region to apply different segmentation techniques to be used for diagnosis and follow-up of different kidney abnormalities. These abnormalities comprise stone disease, kidney cysts, hydronephrosis (blockage of urine), congenital anomalies, as well as urinary tract tumors.

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Accepted: 20.12.2018

On conjugate trigonometrically ρ-convex functions

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Abstract. The aim of this article is to introduce a definition of conjugate trigonometrically ρ-convex functions by using Young's inequality which plays an important role in linking the concept of duality between trigonometrically ρ-convex functions, rather the definition given by Fenchel. Furthermore, we show that the integration of any increasing function is trigonometrically ρ-convex.

Keywords: integral inequalities, supporting functions, trigonometrically ρ-convex functions.

1. Introduction

In 1908, Phragmén and Lindelöf (see for example [13]) presented that if $F(z)$ is an analytic function inside an angle $D = \{z = re^{\imath\theta} : u < \theta < v\}$, then the function

$$h(\theta) = h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r^\rho},$$

is called the indicator function of $F(z)$ with respect to the order $0 < \rho < \infty$, and has the property:

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If $0 < \rho(v-u) < \pi$, and $M(\theta)$ is the function defined by,

$$M(\theta) := A\cos\rho\theta + B\sin\rho\theta,$$

(such functions are called sinusoidal or ρ-trigonometric) which has the same value of $h(\theta)$ at α and at β, then for $u \leq \theta \leq v$. We have

$$h(\theta) \leq M(\theta).$$

This property is called a trigonometric ρ-convexity.

In [5], Beckenbach and Bing ([4] and [19]) introduced a generalization of the classical convexity by replacing linear functions with another family of continuous functions such that for each pair of points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ of the plane there exists exactly one member of the family with a graph joining these points.

In fact, the topic of conjugate convex functions really originate in a paper of Young [23]. This topic attracted some interests [12], [18] and [20], after the work of Fenchel. In [9], [10] Fenchel greatly generalized the whole idea and applied it to the programming problem. Conjugate convex functions have numerous applications mentioned in [11], [17] and [22]. More precisely, in 2014 Gardiner et al. [11] modified an algorithm for computing the convex (Legendre-Fenchel) conjugate of convex piecewise linear-quadratic functions of two variables, to compute its partial conjugate i.e. the conjugate with respect to one of the variables. The structure of the original algorithm is preserved including its time complexity (linear time with some approximation and log-linear time without approximation). Applying twice the partial conjugate (and a variable switching operator) recovers the full conjugate. They presented our partial conjugate algorithm, which was more flexible and simpler than the original full conjugate algorithm. They emphasized the difference with the full conjugate algorithm and illustrate results by computing partial conjugates, partial Moreau envelopes, and partial proximal averages. In 2017, Notarnicola and Notarstefano [17] proposed that a class of distributed optimization algorithms based on proximal gradient methods applied to the dual problem. They showed that, by choosing suitable primal variable copies, the dual problem is itself separable when written in terms of conjugate functions, and the dual variables can be stacked into non-overlapping blocks associated to the computing nodes. In 2018, Rodrigues [22] proposed that a unified optimal control framework that can be used to formulate and solve aircraft performance problems, such as maximum endurance and maximum range, for both propeller-driven airplanes and jet-propelled aircraft. It was proved that such problems have a common mathematical formulation and, under strict convexity assumptions, they had a unique feedback solution for the speed as a function of weight. The feedback solution yields an analytic expression for the optimal speed. For maximum endurance, the solution corresponds to the minimization of the rate of fuel consumption per unit time. For maximum range, the rate of fuel consumption per unit distance is minimized. Moreover, the optimal solution for maximum range was interpreted geometrically using the concept of convex conjugate function and Legendre transformation.

In this paper, we deal with the generalized convex functions in the notion of Beckenbach. For particular choices of the two parameter family $\{M(x)\}$, we consider the following class of generalized convex functions $\{M(x) = A\cos\rho x + B\sin\rho x\}$. This class is called trigonometrically ρ-convex functions (see for examples [2]-[3], [7]-[8] and [13]) which have interesting applications in the design of cavitation-free hydrofoils ([1] and [16]) and in the extremum property [2].

The objective of the present paper is to define a conjugate trigonometrically ρ-convex functions defined on the real line $\mathbb{R}$. We shall be interested in real finite functions on a finite or infinite interval I such that $I \subset \mathbb{R}$ and an interior I^o of I.

2. Definitions and preliminary results

In this section, we present the basic definitions and results which are used later, see for details [2], [13]-[15] and [21].

Definition 2.1 ([14], (see for example [2], [13], [15])). *A function $f : I \to \mathbb{R}$ is said to be **Trigonometrically ρ-Convex Function** if for any arbitrary closed subinterval $[u, v]$ of I such that $0 < \rho(v-u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the ρ-trigonometric function, determined by the equation*

$$M(x) = M(x; u, v, f) = A\cos\rho x + B\sin\rho x,$$

where A and B are chosen such that $M(u) = f(u)$, and $M(v) = f(v)$.

Equivalently, if for all $x \in [u, v]$

$$f(x) \le M(x) = \frac{f(u)\sin\rho(v-x) + f(v)\sin\rho(x-u)}{\sin\rho(v-u)}. \tag{2.1}$$

The trigonometrically ρ-convex functions possess a number of properties analogous to those of convex functions.

For example: If $f : I \to \mathbb{R}$ is trigonometrically ρ-convex function, then for any $u, v \in I$ such that $0 < \rho(v-u) < \pi$, the inequality

$$f(x) \ge M(x; u, v, f),$$

holds outside the interval $[u, v]$.

Definition 2.2 ([4], [5]). *A function $T_u(x) = A\cos\rho x + B\sin\rho x$, is said to be **supporting function** for $f(x)$ at the point $u \in I$, if*

$$T_u(u) = f(u), \text{ and } T_u(x) \le f(x), \quad \forall x \in I. \tag{2.2}$$

Theorem 2.1 ([2]). *A function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex function on I if and only if there exists a supporting function for $f(x)$ at each point $x \in I$.*

Remark 2.1. [[2]] If $f : I \to \mathbb{R}$ is differentiable trigonometrically ρ-convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form

$$T_u(x) = f(u)\cos\rho(x-u) + \frac{f'(u)}{\rho}\sin\rho(x-u), \quad \forall x \in I. \tag{2.3}$$

Remark 2.2. [[2]] For a trigonometrically ρ-convex function $f : I \to \mathbb{R}$, if $f(x)$ is not differentiable at the point u then the supporting function has the form

$$T_u(x) = f(u)\cos\rho(x-u) + K_{u,f}\sin\rho(x-u), \quad \forall x \in I, \tag{2.4}$$

where $K_{u,f} \in [\frac{f'_-(u)}{\rho}, \frac{f'_+(u)}{\rho}]$.

Theorem 2.2 ([13]-[15]). *A trigonometrically ρ-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$, $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$.*

Theorem 2.3 ([14]). *Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function. Then f is trigonometrically ρ-convex on I if and only if $f''(x) + \rho^2 f(x) \geq 0$, $\forall x \in I$.*

Property 2.1 ([2]). *A necessary and sufficient condition for the function $f(x)$ to be a trigonometrically ρ-convex in I is that the function*

$$\varphi(x) = f'(x) + \rho^2 \int_w^x f(t)dt, \quad w \in I$$

is non-decreasing in I.

Property 2.2 ([15], [21]). *If a trigonometrically ρ-convex function $f : I \to \mathbb{R}$ is bounded, i.e, $|f(x)| < k$ for $x \in I$, then it is a continuous function of $x \in I$, and in each closed subinterval J of I, it satisfies a Lipschitz condition, that is*

$$|f(x) - f(y)| \leq k|x-y|, \ for\ some\ k\ and\ \forall x,\ y \in J. \tag{2.5}$$

The relationship between a convex function and its conjugate is at the heart of much recent research. The basic idea can be traced back to Young's Inequality

Theorem 2.4 ([6], Young's Inequality). *Suppose that $g : [0,\infty) \to [0,\infty)$ be strictly increasing and continuous function with $g(0) = 0$ and $g(t) \to \infty$ as $t \to \infty$ (under these circumstances, g has an inverse function g^{-1}, which has the same properties as g). Then, for any $x \geq 0$, $y \geq 0$*

$$xy \leq \int_0^x g(t)dt + \int_0^y g^{-1}(t)dt. \tag{2.6}$$

We can now state the main result on the operation of conjugacy:

Theorem 2.5 ([21])**.** *Let $f : I \to \mathbb{R}$ be a convex and closed function, then $f^* : I^* \to \mathbb{R}$ is denote the conjugate function and defined by*

$$f^*(y) = \sup_{x \in I}[xy - f(x)],$$

it is convex and closed with the domain $I^* = \{y \in \mathbb{R} : f^*(y) < \infty\}$ and

(a) $xy \leq f(x) + f^*(y)$ *for all* $x \in I$ $y \in I^*$,

(b) $xy = f(x) + f^*(y)$ *if and only if* $y \in \partial f(x)$,

(c) $\partial(f^*) = (\partial f)^{-1}$,

(d) $f^{**} = f$,

where $\partial f(x) = \{y \in \mathbb{R}; y \text{ is the slope of a support line for } f \text{ at } x\}$.

3. Main results

The purpose of the present section is to show that the integration of increasing function is trigonometrically ρ-convex. Moreover, we show Young's inequality in the class of trigonometrically ρ-convex. Furthermore, we introduce a definition of conjugate trigonometrically ρ-convex. The relationship between trigonometrically ρ-convex and its conjugate is revealed through Theorem 3.2.

Theorem 3.1. *Let $g : I \to [0, \infty)$ be an increasing function, and $c \in I^o$. Then, $\int_c^x g(t)dt$ is trigonometrically ρ-convex function for all $x \in I$.*

Proof. Put,

$$f(x) = \int_c^x g(t)dt$$

let $u, v \in I$ such that $0 < \rho(v - u) < \pi$, $x = \lambda u + \mu v$, where $\lambda + \mu = 1$ and $\lambda, \mu \in [0, 1]$,

$$\begin{aligned}
f(x) &= \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x + x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x)\cos\rho(x - u) + \cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)} \int_c^x g(t)dt + \frac{\cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x)\cos\rho(x - u)}{\sin\rho(v - u)} [\int_c^u g(t)dt + \int_u^x g(t)dt] \\
&\quad + \frac{\cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)} [\int_c^v g(t)dt - \int_x^v g(t)dt].
\end{aligned}$$

Since

$$\cos\rho(x-u) \le 1,\ \cos\rho(v-x) \le 1, \tag{3.1}$$

we get that

$$\begin{aligned} f(x) \le & \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\ & + \frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)}\int_u^x g(t)dt - \frac{\cos\rho(v-x)\sin\rho(x-u)}{\sin\rho(v-u)}\int_x^v g(t)dt. \end{aligned}$$

Take

$$\lambda = \frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)} \ \ \& \ \ \mu = \frac{\cos\rho(v-x)\sin\rho(x-u)}{\sin\rho(v-u)}.$$

We obtain

$$f(x) \le \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} + \lambda\int_u^x g(t)dt - \mu\int_x^v g(t)dt.$$

Since g is increasing, then $g(t) \le g(x)$ for all $t \in [u,x]$, and $-g(t) \le -g(x)$ for all $t \in [x,v]$, we conclude that

$$\begin{aligned} f(x) \ \le \ & \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\ & + \lambda\int_u^x g(x)dt - \mu\int_x^v g(x)dt \\ = \ & \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\ & + \lambda g(x)[x-u] - \mu g(x)[v-x] \\ = \ & \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\ & + g(x)[(\lambda+\mu)x - (\lambda u + \mu v)] \\ = \ & \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)}. \end{aligned}$$

Applying Definition 2.1, then $\int_c^x g(t)dt$ is trigonometrically ρ-convex function $\forall x \in I$. □

Example 3.1. Let $g(x) = \sin\rho x$, $\forall x \in [0, \frac{\pi}{2\rho}]$. Then, $f(x) = \int_0^x \sin\rho t dt$ is trigonometrically ρ-convex function $\forall x \in [0, \frac{\pi}{2\rho}]$.

As

$$f(x) = \int_0^x \sin\rho t dt = \frac{1}{\rho}(1-\cos\rho x), \quad \forall x \in [0, \frac{\pi}{2\rho}]. \tag{3.2}$$

Differentiate $f(x) = \int_0^x \sin \rho t dt$ with respect to x, implies

$$f^{'}(x) = \sin \rho x,$$
$$f^{''}(x) = \rho \cos \rho x. \tag{3.3}$$

From equations (3.2), (3.3) implies

$$f^{''}(x) + \rho^2 f(x) = \rho \cos \rho x + \rho^2 \frac{1}{\rho}(1 - \cos \rho x) = \rho,$$

since $0 < \rho < \infty$, then $f^{''}(x) + \rho^2 f(x) = \rho > 0$, $\forall x \in [0, \frac{\pi}{2\rho}]$. By using Theorem 2.3, then $f(x)$ trigonometrically ρ-convex function.

Proposition 3.1. *Suppose that $g : [0, \frac{\pi}{2\rho}] \to [0, \infty)$ be strictly increasing and continuous function with $g(0) = 0$, $g(x) \to \infty$ as $x \to \infty$.*

If we take,

$$f(x) = \int_0^x g(t)dt, \quad f^*(m) = \int_0^m g^{-1}(t)dt. \tag{3.4}$$

Then f and f^* are both trigonometrically ρ-convex functions for all $x \in [0, \frac{\pi}{2\rho}]$, $m \in [0, \infty)$, and satisfy

$$m \sin(\rho x) \leq f(x) + f^*(m), \ \ \forall x \in [0, \frac{\pi}{2\rho}]. \tag{3.5}$$

Proof. Since g is strictly increasing and continuous function with $g(0) = 0$, then g has an inverse function g^{-1}, which has the same properties as g for all $x \in [0, \frac{\pi}{2\rho}]$, and by using Theorem 3.1, then f *and* f^* are both trigonometrically ρ-convex functions for any $x \in [0, \frac{\pi}{2\rho}]$, $m \in [0, \infty)$.

Now, we prove the inequality (3.5).

Since $\sin \rho x$ is trigonometrically ρ-convex function and from Property 2.2, $|\sin \rho x - \sin \rho 0| \leq k|\rho x - 0|$. Take $k = \frac{1}{\rho}$, then

$$\sin \rho x \leq x, \ \ \forall x \in [0, \frac{\pi}{2\rho}]. \tag{3.6}$$

From inequality (3.6) and Theorem 2.4, implies

$$m \sin \rho x \leq mx \leq \int_0^x g(t)dt + \int_0^m g^{-1}(t)dt, \ \ \forall x \in [0, \frac{\pi}{2\rho}].$$

Then, $m \sin(\rho x) \leq f(x) + f^*(m)$, $\forall x \in [0, \frac{\pi}{2\rho}]$. □

Example 3.2. Let $g(x) = \sin \rho x$ for all $x \in [0, \frac{\pi}{2\rho}]$. Then, $f(x) = \int_0^x g(t)dt$ and $f^*(m) = \int_0^m g^{-1}(t)dt$ are trigonometrically ρ-convex functions, and satisfy

$$m \sin(\rho x) \leq f(x) + f^*(m), \ \ \forall x \in [0, \frac{\pi}{2\rho}].$$

As from Example 3.1, $f(x)$ is trigonometrically ρ-convex function. Since $g(x) = \sin\rho x$, we observe that $g^{-1}(m) = \frac{1}{\rho}\arcsin m, \ \ \forall m \in [0,1]$, then

$$\begin{aligned} f^*(m) &= \frac{1}{\rho}\int_0^m \arcsin t dt \\ &= \frac{1}{\rho}[m\arcsin m + \sqrt{1-m^2} - 1] \qquad (3.7) \\ f^{*'}(m) &= \frac{1}{\rho}\arcsin m \\ f^{*''}(m) &= \frac{1}{\rho}\frac{1}{\sqrt{1-t^2}}, \end{aligned}$$

using Theorem 2.3, we observe that

$$f^{*''}(m) + \rho^2 f^*(m) = \frac{1}{\rho}\frac{1}{\sqrt{1-t^2}} + \rho\int_0^m \arcsin t dt \geq 0,$$

then $f^*(m)$ is trigonometrically ρ-convex function for all $m \in [0,1]$.
Now, we Check the inequality (3.5).

Define

$$h(x,m) = \int_0^x \sin\rho t dt + \int_0^m \frac{1}{\rho}\arcsin t dt - xm,$$

for $x \in [0, \frac{\pi}{2\rho}]$ and m in $[0,1]$,

$$\begin{aligned} h(x,m) - h(x,\sin\rho x) &= \int_0^m \frac{1}{\rho}\arcsin t dt - \int_0^{\sin\rho x}\frac{1}{\rho}\arcsin t dt - xm + x\sin\rho x \\ &= \int_{\sin\rho x}^m [\frac{1}{\rho}\arcsin t - x]dt \\ &= \int_m^{\sin\rho x}[x - \frac{1}{\rho}\arcsin t]dt. \end{aligned}$$

The first case is $m \geq \sin\rho x$, we have $\arcsin m \geq \arcsin t \geq \arcsin\sin\rho x = \rho x$, $\forall t \in [\sin\rho x, m]$.

Consequently,

$$h(x,m) - h(x,\sin\rho x) = \int_{\sin\rho x}^m [\frac{1}{\rho}\arcsin t - x]dt \geq 0. \tag{3.8}$$

The second case is $m \leq \sin\rho x$, we have $\arcsin m \leq \arcsin t \leq \arcsin\sin\rho x = \rho x$ $\forall t \in [m, \sin\rho x]$.
Consequently,

$$h(x,m) - h(x,\sin\rho x) = \int_m^{\sin\rho x}[x - \frac{1}{\rho}\arcsin t]dt \geq 0. \tag{3.9}$$

Using equations (3.2) and (3.7), we obtain that

$$\begin{aligned} h(x, \sin\rho x) = \frac{1}{\rho}[1-\cos\rho x] + \frac{1}{\rho}[\sin\rho x \arcsin\sin\rho x + \sqrt{1-\sin^2\rho x} - 1] \\ - x\sin\rho x = 0. \end{aligned} \tag{3.10}$$

From equations (3.8), (3.9), (3.10) and by using inequality (3.6), we conclude that $f(x) + f^*(m) \geq mx \geq m\sin\rho x,\ \ \forall x \in [0, \frac{\pi}{2\rho}]$.

Lemma 3.1. *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of trigonometrically ρ-convex functions and*

$$f(x) = \sup_\alpha(f_\alpha(x)) \tag{3.11}$$

if $J = \{x \in I : f(x) < \infty\}$ is nonempty, then $f : J \to \mathbb{R}$ is trigonometrically ρ-convex function.

Proof. Let $x \in [a, b] \subseteq J \subseteq I$ such that $0 < \rho(b-a) < \pi$, since $f_\alpha(x)$ is trigonometrically ρ-convex function for all α and equation (3.11). Then,

$$\begin{aligned} f(x) &= \sup_\alpha(f_\alpha(x)) \\ &\leq \sup_\alpha[\frac{f_\alpha(a)\sin\rho(b-x) + f_\alpha(b)\sin\rho(x-a)}{\sin\rho(b-a)}] \\ &\leq \frac{\sup_\alpha[f_\alpha(a)]\sin\rho(b-x) + \sup_\alpha[f_\alpha(b)]\sin\rho(x-a)}{\sin\rho(b-a)} \\ &= \frac{f(a)\sin\rho(b-x) + f(b)\sin\rho(x-a)}{\sin\rho(b-a)}. \end{aligned}$$

From Definition 2.1, implies $f(x) = \sup_\alpha(f_\alpha(x))$ is trigonometrically ρ-convex function. □

Definition 3.1. *If $f : [0, \frac{\pi}{\rho}] \to \mathbb{R}$ is trigonometrically ρ-convex function, then $f^* : I^* \to \mathbb{R}$ is the* ***conjugate of trigonometrically ρ-convex function*** *and defined by*

$$f^*(m) := \sup_x[m\sin\rho x - f(x)], \tag{3.12}$$

with domain $I^* = \{m \in \mathbb{R} : f^*(m) < \infty\}$ such that $m\sin\rho x \geq f(x)$, for all m in I^*.

Example 3.3. Let $f(x) = \sin\rho x\ \ \ \forall x \in [0, \frac{\pi}{\rho}]$ be trigonometrically ρ-convex function such that $m \geq 1$. Then, its conjugate $f^*(m) = m - 1$, is trigonometrically ρ-convex function

As from Definition 3.1,

$$f^*(m) = \sup_x [m \sin \rho x - \sin \rho x], = (m-1) \sup_x [\sin \rho x], = m-1,$$

then $f^{*'}(m) = 1, f^{*''}(m) = 0$. From $m \geq 1$ and $0 < \rho < \infty$. Then

$$f^{*''}(m) + \rho^2 f^*(m) = \rho^2 (m-1) \geq 0.$$

By using Theorem 2.3, then $f^*(m) = m-1$ is trigonometrically ρ-convex function.

Theorem 3.2. *If $f : [0, \frac{\pi}{\rho}] \to \mathbb{R}$ is trigonometrically ρ-convex function. Its conjugate $f^* : I^* \to \mathbb{R}$ such that $m \sin \rho x \geq f(x), \ \forall m \in I^*$. Then*

(c1) *$f^*(m)$ is trigonometrically ρ-convex function.*

(c2) *$m \sin \rho x \leq f(x) + f^*(m)$.*

(c3) *If f is differentiable then*
$m \sin \rho x = f(x) + f^*(m)$ if and only if $\rho m \cos \rho x = f'(x)$.

(c4) *For every*

$$g : [0, \frac{\pi}{\rho}] \to \mathbb{R}, m \sin \rho x \geq g(x), \ \forall m \in I^*, \text{ and } f \leq g \text{ on } [0, \frac{\pi}{\rho}],$$
$$\text{implies } f^* \geq g^* \text{ on } I^*. \tag{3.13}$$

Proof. We first prove that $I^* \neq \emptyset$. For if I is single point x_o, $f(x)$ is trigonometrically ρ-convex function and from Theorem 2.1, Remark 2.1, then $T_{x_o}(x) = f(x_o) \cos \rho(x - x_o) + K_{x_o,f} \sin \rho(x - x_o)$ supports f for each $K_{x_o,f} \in \mathbb{R}$. Otherwise, we choose any interior point x_o, choose $K_{x_o,f} \in \frac{1}{\rho}[f'_-(x_o), f'_+(x_o)]$, and again from Theorem 2.1, Remark 2.2 then also $T_{x_o}(x)$ is supporting function for $f(x)$ at x_o. In either case then choose $K_{x_o,f}$ such that $T_{x_o}(x) \leq f(x), \ \forall x \in [0, \frac{\pi}{\rho}]$,

$$f(x_o) \cos \rho(x - x_o) + K_{x_o,f} \sin \rho(x - x_o) \leq f(x), \tag{3.14}$$

implies

$$\begin{aligned} &f(x_o) \cos \rho x \cos \rho x_o + f(x_o) \sin \rho x \sin \rho x_o + K_{x_o,f} \sin \rho x \cos \rho x_o \\ &- K_{x_o,f} \cos \rho x \sin \rho x_o \leq f(x), \end{aligned}$$

hence

$$\begin{aligned} &(K_{x_o,f} \cos \rho x_o + f(x_o) \sin \rho x_o) \sin \rho x - f(x) \leq (K_{x_o,f} \sin \rho x_o \\ &- f(x_o) \cos \rho x_o) \cos \rho x, \quad \forall x \in [0, \frac{\pi}{\rho}]. \end{aligned} \tag{3.15}$$

Let $m = K_{x_o,f}\cos\rho x_o + f(x_o)\sin\rho x_o$, $A = K_{x_o,f}\sin\rho x_o - f(x_o)\cos\rho x_o$, implies $m\sin\rho x - f(x) \le A\cos\rho x$. If A positive, then we get

$$m\sin\rho x - f(x) \le A \quad : \cos\rho x \le 1 \;\; \forall x \in [0, \frac{\pi}{\rho}].$$

Otherwise, if A negative, then we get

$$m\sin\rho x - f(x) \le -A \quad : \cos\rho x \ge -1 \;\; \forall x \in [0, \frac{\pi}{\rho}].$$

In either case, we have $f^*(m) = \sup_x[m\sin\rho x - f(x)] < \infty$.

Then $I^* \neq \emptyset$.

(c1) Let $g_x(m) = m\sin\rho x - f(x)$, suppose that $u, v \in I^* : 0 < \rho(v-u) < \pi$ and let $m \in (u,v)$, $m = \lambda u + \mu v : \lambda + \mu = 1$ and $\lambda, \mu \in [0,1]$,

$$\begin{aligned} g_x(m) &= (\lambda u + \mu v)\sin\rho x - f(x)(\lambda+\mu) \\ &= \lambda(u\sin\rho x - f(x)) + \mu(v\sin\rho x - f(x)). \end{aligned}$$

Take

$$\lambda = \frac{\sin\rho(v-m)\cos\rho(m-u)}{\sin\rho(v-u)}, \quad \mu = \frac{\sin\rho(m-u)\cos\rho(v-m)}{\sin\rho(v-u)},$$

hence

$$\begin{aligned} g_x(m) = &\frac{\sin\rho(v-m)\cos\rho(m-u)}{\sin\rho(v-u)}(u\sin\rho x - f(x)) \\ &+ \frac{\sin\rho(m-u)\cos\rho(v-m)}{\sin\rho(v-u)}(v\sin\rho x - f(x)). \end{aligned}$$

Since $u, v \in I^*$ implies $u\sin\rho x - f(x) \ge 0$ and $v\sin\rho x - f(x) \ge 0$, and from $\cos\rho(m-u) \le 1$ and $\cos\rho(v-m) \le 1$. Then

$$\begin{aligned} g_x(m) &\le \frac{\sin\rho(v-m)}{\sin\rho(v-u)}(u\sin\rho x - f(x)) + \frac{\sin\rho(m-u)}{\sin\rho(v-u)}(v\sin\rho x - f(x)) \\ &= \frac{(u\sin\rho x - f(x))\sin\rho(v-m) + (v\sin\rho x - f(x))\sin\rho(m-u)}{\sin\rho(v-u)} \\ &= \frac{g_x(u)\sin\rho(v-m) + g_x(v)\sin\rho(m-u)}{\sin\rho(v-u)}. \end{aligned}$$

By using Definition 2.1, then $g_x(x)$ is a trigonometrically ρ-convex function, and by the Lemma 3.1, $f^*(m) = \sup_x(g_x(m))$ is a trigonometrically ρ-convex function.

(c2) Since equation (3.12), implies

$$m\sin\rho x - f(x) \le f^*(m), \tag{3.16}$$

then $m \sin \rho x \leq f(x) + f^*(m)$.

(c3) To prove the necessity, by differentiate $m \sin \rho x = f(x) + f^*(m)$ with respect to x implies, $\rho m \cos \rho x = f^{'}(x)$.

The sufficiency, let

$$\rho m \cos \rho x = f^{'}(x), \tag{3.17}$$

implies $\int_0^x \rho m \cos \rho t dt = \int_0^x f^{'}(t) dt$. Hence,

$$m \sin \rho x = f(x) - f(0), \tag{3.18}$$

since $m \in I^*$, then

$$-f(0) = m \sin \rho x - f(x) \geq 0. \tag{3.19}$$

From Remark 2.1, then the supporting function for $f(x)$ at the point $0 \in [0, \frac{\pi}{\rho}]$, has the form

$$T_0(x) = f(0) \cos \rho x + \frac{f^{'}(0)}{\rho} \sin \rho x \leq f(x), \quad \forall x \in [0, \frac{\pi}{\rho}]$$

from equation(3.17) at $x = 0$ implies $m = \frac{f^{'}(0)}{\rho}$, and we get $m \sin \rho x - f(x) \leq -f(0) \cos \rho x$, hence

$$\begin{aligned} \sup_x [m \sin \rho x - f(x)] &\leq -f(0) \cos \rho x \\ f^*(m) = \sup_x [m \sin \rho x - f(x)] &\leq -f(0) \cos \rho x \\ f^*(m) &\leq -f(0) \cos \rho x. \end{aligned} \tag{3.20}$$

From equation (3.19), inequality (3.20) and $\cos \rho x \leq 1$, implies

$$f^*(m) \leq m \sin \rho x - f(x) \tag{3.21}$$

from inequalities(3.16), and(3.21), then

$$m \sin \rho x - f(x) = f^*(m). \tag{3.22}$$

(c4) Since $f \leq g$ implies $f(x) \leq g(x), \forall x \in [0, \frac{\pi}{\rho}]$, then

$$\begin{aligned} -g(x) &\leq -f(x) \\ m \sin \rho x - g(x) &\leq m \sin \rho x - f(x) \\ \sup_x [m \sin \rho x - g(x)] &\leq m \sin \rho x - f(x), \end{aligned} \tag{3.23}$$

from Definition 3.1, $m \sin \rho x \geq g(x)$ and inequality (3.23) implies

$$\begin{aligned} g^*(m) &= \sup_x [m \sin \rho x - g(x)] \\ &\leq m \sin \rho x - f(x) \\ &\leq \sup_x [m \sin \rho x - f(x)], \end{aligned}$$

from Definition 3.1, and $m \sin \rho x \geq f(x)$ then

$$g^*(m) \leq f^*(m), \quad \forall m \in I^*, \tag{3.24}$$

then $f^* \geq g^*$ *on* I^*.

□

Remark 3.1. For a trigonometrically ρ-convex function $f : [0, \frac{\pi}{2\rho}] \to \mathbb{R}$ if $\inf f(x) \neq -\infty$, then the domain of its conjugate $I^* = \mathbb{R}$, where

$$f^*(m) = \sup_x [m \sin \rho x - f(x)] \leq m - \inf f(x).$$

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Accepted: 22.12.2018

On decomposable MS-algebras

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Abstract. In this paper we give some results on the direct product, subalgebras and homomorphisms of decomposable MS-algebras. We Show how direct products and canonical projections are related. Also, we study homomorphic images of subalgebras of decomposable MS-algebras.

Keywords: direct product, MS-algebra, decomposable MS-algebra, subalgebra, homomorphism.

1. Introduction

MS-algebras were initiated by T.S. Blyth and J.C. Varlet, see [6], as a generalization of both de Morgan and Stone algebras. In [8], T.S. Blyth and J.C. Varlet described the lattice $\Lambda(\mathbf{MS})$ of subclasses of the class $\mathbf{MS}$ of all MS-algebras. In [3], S. El-Assar and A. Badawy studied many properties of homomorphisms and subalgebras of MS-algebras from the subclass $\mathbf{K_2}$. In [1], A. Badawy, D. Guffova and M. Haviar introduced and characterized decomposable MS-algebras by means of decomposable MS-triples. In [2], A. Badawy and R. El-Fawal studied many properties of decomposable MS-algebras in terms of decomposable MS-triples as homomorphisms and subalgebras. Also, they solved

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some fill in problems concerning homomorphisms and subalgebras of decomposable MS-algebras.

In this paper we study many properties related to the direct product and subalgebras of decomposable MS-algebras. Also, we reveal the connection between homomorphisms and direct products. We finish with some results on homomorphic images of subalgebras of decomposable MS-algebras.

2. Preliminaries

In this section, we present definitions and main results which are needed through this paper. For basic facts about MS-algebras and related structures we refer the reader to [5], [6], [7], [8], [9] and [10].

An MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^\circ$ satisfies:

$$x \leq x^{\circ\circ}, (x \wedge y)^\circ = x^\circ \vee y^\circ, 1^\circ = 0.$$

The following theorem gives the basic properties of MS-algebras.

Theorem 2.1 ([6], [9])**.** *For any two elements a, b of an MS-algebra L, we have:*

(1) $0^\circ = 1$,

(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,

(3) $a^{\circ\circ\circ} = a^\circ$,

(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,

(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,

(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Lemma 2.2 ([1], [6])**.** *Let L be an MS-algebra. Then:*

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ *is a de Morgan subalgebra of L,*

(2) $D(L) = \{x \in L : x^\circ = 0\}$ *is a filter (filter of dense elements) of L.*

Definition 2.3 ([4])**.** Let $L = (L; \vee, \wedge, 0_L, 1_L)$ and $L_1 = (L_1; \vee, \wedge, 0_{L_1}, 1_{L_1})$ be bounded lattices. The map $f : L \to L_1$ is called a (0,1)-lattice homomorphism if:

(1) $f(0_L) = 0_{L_1}$ and $f(1_L) = 1_{L_1}$,

(2) f preserves joins, that is, $f(x \vee y) = f(x) \vee f(y)$ for every $x, y \in L$,

(3) f preserves meets, that is, $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in L$.

Definition 2.4 ([4])**.** A (0,1)-lattice homomorphism $f : L \to L_1$ of an MS-algebra L into an MS-algebra L_1 is called a homomorphism if $f(x^\circ) = (f(x))^\circ$ for all $x \in L$.

Definition 2.5 ([1]). An MS-algebra L is called decomposable MS-algebra if for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{\circ\circ} \wedge d$.

Definition 2.6 ([2]). A bounded sublattice of a decomposable MS-algebra L is called a subalgebra of L if:

(1) $x^{\circ} \in A, \forall x \in A$,

(2) For every $x \in A$, there exists $d \in D(A)$ such that $x = x^{\circ\circ} \wedge d$.

Definition 2.7 ([2]). A subalgebra of a decomposable MS-algebra L is called a K_2-subalgebra of L if for every $x, y \in A$, the following holds:

(1) $x \wedge x^{\circ} = x^{\circ} \wedge x^{\circ\circ}$,

(2) $x \wedge x^{\circ} \leqslant y \vee y^{\circ}$.

Definition 2.8 ([2]). A subalgebra of a decomposable MS-algebra L is called a Stone subalgebra of L if for every $x \in A$, $x^{\circ} \vee x^{\circ\circ} = 1$

3. Direct products and subalgebras of decomposable *MS*-algebras

We begin by recalling the definition of direct product of MS-algebras.

Definition 3.1. Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then, the direct product $\prod_{i=1}^{n} L_i$ is defined as $\prod_{i=1}^{n} L_i = \{(x_1, x_2, ..., x_n), x_i \in L_i, i \in\ I_n\}$ where the operations $\vee, \wedge$ are defined componentwise and $(x_1, x_2, ..., x_n)^{\circ} = (x_1^{\circ}, x_2^{\circ}, ..., x_n^{\circ})$.

The proof of the following lemma is straightforward.

Lemma 3.2. *Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then:*

1. $(\prod_{i=1}^{n} L_i)^{\circ\circ} = \prod_{i=1}^{n} L_i^{\circ\circ}$,
2. $D(\prod_{i=1}^{n} L_i) = \prod_{i=1}^{n} D(L_i)$.

Theorem 3.3. *Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then, $\prod_{i=1}^{n} L_i$ is decomposable if and only if L_i is decomposable for each $i \in\ I_n$.*

Proof. Suppose that $\prod_{i=1}^{n} L_i$ is decomposable. Let $x_i \in\ L_i,\ i \in\ I_n$. Then,

$$
\begin{aligned}
& (x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} L_i \\
\Rightarrow\quad & (x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n)^{\circ\circ} \wedge (d_1, d_2, ..., d_n),\ d_i \in\ D(L_i),\ i \in\ I_n \\
\Rightarrow\quad & (x_1, x_2, ..., x_n) = (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n) \\
\Rightarrow\quad & (x_1, x_2, ..., x_n) = (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n) \\
\Rightarrow\quad & x_i = x_i^{\circ\circ} \wedge d_i,\ d_i \in\ D(L_i), \forall\ i \in\ I_n, \\
\Rightarrow\quad & L_i \text{ is decomposable}, \forall\ i \in\ I_n.
\end{aligned}
$$

Conversely, suppose that L_i is decomposable, $\forall i \in I_n$, and $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n L_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n) &= (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n), d_i \in D(L_i) \\ &= (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n) \\ &= (x_1, x_2, ..., x_n)^{\circ\circ} \wedge (d_1, d_2, ..., d_n)\end{aligned}$$

Since $(d_1, d_2, ..., d_n) \in \prod_{i=1}^n D(L_i) = D(\prod_{i=1}^n L_i)$, then $\prod_{i=1}^n L_i$ is decomposable. □

Theorem 3.4. *Let A_i be a subalgebra of a decomposable MS-algebra $L_i, i \in I_n$. Then, $\prod_{i=1}^n A_i$ is a subalgebra of $\prod_{i=1}^n L_i$.*

Proof. Clearly, $\prod_{i=1}^n A_i$ is a bounded sublattice of $\prod_{i=1}^n L_i$. Let $(x_1, x_2, ...x_n) \in \prod_{i=1}^n A_i$. Then, $(x_1, x_2, ..., x_n)^\circ = (x_1^\circ, x_2^\circ, ..., x_n^\circ) \in \prod_{i=1}^n A_i$ (as $x_i^\circ \in A_i$). Assuming that $x_i = x_i^{\circ\circ} \wedge d_i$, $d_i \in D(A_i)$, we get

$$(x_1, x_2, ..., x_n) = (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n) = (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n).$$

Since $(d_1, d_2, ..., d_n) \in D(\prod_{i=1}^n A_i)$, then $\prod_{i=1}^n A_i$ is a subalgebra of $\prod_{i=1}^n L_i$. □

Corollary 3.5. $(\prod_{i=1}^n L_i)^{\circ\circ}$ *is a subalgebra of* $\prod_{i=1}^n L_i$.

Proof. Since $(\prod_{i=1}^n L_i)^{\circ\circ} = \prod_{i=1}^n L_i^{\circ\circ}$ and $L_i^{\circ\circ}$ is a subalgebra of L_i, then $(\prod_{i=1}^n L_i)^{\circ\circ}$ is a subalgebra of $\prod_{i=1}^n L_i$. □

Lemma 3.6. *Let A_i be a K_2-subalgebra of a decomposable MS-algebra L_i, $i \in I_n$. Then, $\prod_{i=1}^n A_i$ is a K_2-subalgebra of $\prod_{i=1}^n L_i$.*

Proof. By Theorem 3.4, $\prod_{i=1}^n A_i$ is a subalgebra of $\prod_{i=1}^n L_i$.
Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n A_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n) \wedge (x_1, x_2, ..., x_n)^\circ &= (x_1 \wedge x_1^\circ, x_2 \wedge x_2^\circ, ..., x_n \wedge x_n^\circ) \\ &= (x_1^\circ \wedge x_1^{\circ\circ}, x_2^\circ \wedge x_2^{\circ\circ}, ..., x_n^\circ \wedge x_n^{\circ\circ}) \\ &= (x_1, x_2, ..., x_n)^\circ \wedge (x_1, x_2, ..., x_n)^{\circ\circ}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\quad (x_1, x_2, ..., x_n) \wedge (x_1, x_2, ..., x_n)^\circ \\ &= (x_1 \wedge x_1^\circ, x_2 \wedge x_2^\circ, ..., x_n \wedge x_n^\circ) \\ &\leq (y_1 \vee y_1^\circ, y_2 \vee y_2^\circ, ..., y_n \vee y_n^\circ), \quad \forall\, y_i \in A_i \\ &= (y_1, y_2, ..., y_n) \vee (y_1, y_2, ..., y_n)^\circ, \quad \forall\, (y_1, y_2, ..., y_n) \in \prod_{i=1}^n A_i.\end{aligned}$$

Hence, $\prod_{i=1}^n A_i$ is a K_2-subalgebra of $\prod_{i=1}^n L_i$. □

Lemma 3.7. *Let S_i be a Stone subalgebra of a decomposable MS-algebra L_i, $i \in I_n$. Then, $\prod_{i=1}^n S_i$ is a Stone subalgebra of $\prod_{i=1}^n A_i$.*

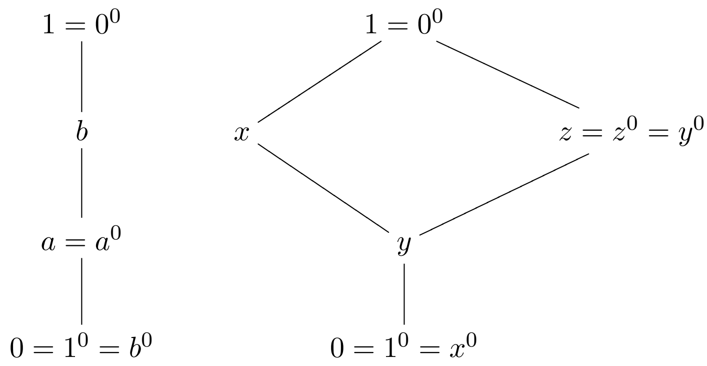
Proof. We need to verify the Stone identity. Namely, $z^\circ \vee z^{\circ\circ} = 1$, $\forall z \in \prod_{i=1}^n S_i$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n S_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n)^\circ \vee (x_1, x_2, ..., x_n)^{\circ\circ} &= (x_1^\circ \vee x_1^{\circ\circ}, x_2^\circ \vee x_2^{\circ\circ}, ..., x_n^\circ \vee x_n^{\circ\circ}) \\ &= (1_1, 1_2, ..., 1_n),\end{aligned}$$

where 1_i is the greatest element of S_i. Thus, $\prod_{i=1}^n S_i$ is a Stone subalgebra of $\prod_{i=1}^n L_i$. $\square$

The following example shows that the converse of Theorem 3.4, lemma 3.6 and lemma 3.7 is not true, respectively.

Example 3.8. *Consider the following two decomposable MS-algebras:*



L_1 L_2

$A = \{(1,1), (0,0)\}$ *is a subalgebra* (*respectively a K_2-subalgebra, a Stone subalgebra*) *of $L_1 \times L_2$ while it can not be written as a product of two subalgebras* (*respectively K_2-subalgebras, Stone subalgebras*) *of L_1 and L_2.*

Lemma 3.9. *Let $\{A_i,\ i \in I_n\}$ be a family of subalgebras of a decomposable MS-algebra L. Then:*

1. *$\bigcap_{i=1}^n A_i$ is a subalgebra of L,*
2. *$\bigcup_{i=1}^n A_i$ is not necessarily a subalgebra of L.*

Proof. 1. Clearly, $\bigcap_{i=1}^n A_i$ is a bounded sublattice of L. Let $x \in \bigcap_{i=1}^n A_i$. Then, $x \in A_i$, $\forall i \in I_n$. Consequently, $x^\circ \in A_i$, $\forall i \in I_n$. Hence, $x^\circ \in \bigcap_{i=1}^n A_i$. Moreover, we have $x = x^{\circ\circ} \wedge d_i$, $d_i \in D(A_i)$, $i \in I_n$. As $d_i \in A_i$, then $\bigvee_{i=1}^n d_i \in A_i$, $\forall i \in I_n$. Also, $(\bigvee_{i=1}^n d_i)^\circ = \bigwedge_{i=1}^n d_i^\circ = 0$. Then, $\bigvee_{i=1}^n d_i \in \bigcap_{i=1}^n D(A_i) = D(\bigcap_{i=1}^n A_i)$. Now, we can write $x = x^{\circ\circ} \vee d$ where $d = \bigvee_{i=1}^n d_i \in D(\bigcap_{i=1}^n A_i)$. Hence, $\bigcap_{i=1}^n A_i$ is a subalgebra of L.

2. Consider L_2 of example 3.8, we observe that $A_1 = \{1, 0, x\}$ and $A_2 = \{1, 0, z\}$ are subalgebras of L_2 while $A_1 \cup A_2 = \{1, 0, x, z\}$ is not a subalgebra of L_2 (as $x \wedge z = y \notin A_1 \cup A_2$). $\square$

4. Direct products and homomorphisms of decomposable MS-algebras

Theorem 4.1. *Let $\{\varphi_i : A_i \to B_i, i \in I_n\}$ be a family of homomorphisms between MS-algebras. Define $\varphi : \prod_{i=1}^n A_i \to \prod_{i=1}^n B_i$, by $\varphi(a_1, a_2, ..., a_n) = (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))$. Then:*

1. *φ is a homomorphism,*
2. *φ is one to one if and only if each φ_i is one to one,*
3. *φ is onto if and only if each φ_i is onto,*
4. *$ker\varphi = \prod_{i=1}^n ker\varphi_i$,*
5. *$\varphi(\prod_{i=1}^n A_i) = \prod_{i=1}^n \varphi_i(A_i)$.*

Proof.

(1) Let $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n A_i$. Then,

$$\begin{aligned}
& \varphi\big((a_1, a_2, ..., a_n) \vee (b_1, b_2, ..., b_n)\big) \\
= \; & \varphi(a_1 \vee b_1, a_2 \vee b_2, ..., a_n \vee b_n) \\
= \; & (\varphi_1(a_1 \vee b_1), \varphi_2(a_2 \vee b_2), ..., \varphi_n(a_n \vee b_n)) \\
= \; & (\varphi_1(a_1) \vee \varphi_1(b_1), \varphi_2(a_2) \vee \varphi_2(b_2), ..., \varphi_n(a_n) \vee \varphi_n(b_n)) \\
= \; & (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) \vee (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n)) \\
= \; & \varphi(a_1, a_2, ..., a_n) \vee \varphi(b_1, b_2, ..., b_n).
\end{aligned}$$

Similarly, we can show that

$$\varphi\big((a_1, a_2, ..., a_n) \wedge (b_1, b_2, ..., b_n)\big) = \varphi(a_1, a_2, ..., a_n) \wedge \varphi(b_1, b_2, ..., b_n).$$

Moreover,

$$\begin{aligned}
\varphi(a_1, a_2, ..., a_n)^\circ &= \varphi(a_1^\circ, a_2^\circ, ..., a_n^\circ) \\
&= (\varphi_1(a_1^\circ), \varphi_2(a_2^\circ), ..., \varphi_n(a_n^\circ)) \\
&= (\varphi_1(a_1)^\circ, \varphi_2(a_2)^\circ, ..., \varphi_n(a_n)^\circ) \\
&= (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))^\circ \\
&= (\varphi(a_1, a_2, ..., a_n))^\circ.
\end{aligned}$$

Hence, φ is a homomorphism from $\prod_{i=1}^n A_i$ into $\prod_{i=1}^n B_i$.

(2) Let φ be one to one and suppose that $\varphi_i(a_i) = \varphi_i(b_i), i \in I_n$. Then,

$$\begin{aligned}\varphi(a_1, a_2, ..., a_n) &= (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) \\ &= (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n)) \\ &= \varphi(b_1, b_2, ..., b_n).\end{aligned}$$

This gives $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$. So, $a_i = b_i$, $\forall i \in I_n$. Hence, each φ_i is one to one. Conversely, assume φ_i is one to one for each i and $\varphi(a_1, a_2, ..., a_n) = \varphi(b_1, b_2, ..., b_n)$. Then, $(\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n))$. Thus, $\varphi_i(a_i) = \varphi_i(b_i)$ $\forall i$. Hence, φ is one to one.

(3) Let φ be onto and $b_i \in B_i$, $\forall i$. Then, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n B_i$. As φ is onto, there exists $(a_1, a_2, ..., a_n) \in \prod_{i=1}^n A_i$ such that $\varphi(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$. Equivalently, $(\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (b_1, b_2, ..., b_n)$. That is, $\varphi_i(a_i) = b_i$, $\forall i$. Hence, each φ_i is onto. Conversely, let φ_i be onto for each i and $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n B_i$. Since $b_i \in B_i$ and φ_i is onto, then there exists $a_i \in A_i$ such that $b_i = \varphi_i(a_i)$, $\forall i$. So, $(b_1, b_2, ..., b_n) = (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))$. Consequently, φ is onto.

(4)
$$\begin{aligned}(a_1, a_2, ..., a_n) \in ker\varphi &\Leftrightarrow \varphi(a_1, a_2, ..., a_n) = (0_1, 0_2, ..., 0_n) \\ &\Leftrightarrow (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (0_1, 0_2, ..., 0_n) \\ &\Leftrightarrow \varphi_i(a_i) = 0_i, \ \forall i \in I_n \\ &\Leftrightarrow a_i \in ker\varphi_i \ \forall i \in I_n \\ &\Leftrightarrow (a_1, a_2, ..., a_n) \in \prod_{i=1}^{n} ker\varphi_i.\end{aligned}$$

(5)
$$\begin{aligned}&(b_1, b_2, ..., b_n) \in \varphi(\prod_{n}^{i=1} A_i) \\ &\Leftrightarrow (b_1, b_2, ..., b_n) = \varphi((a_1, a_2, ..., a_n)), (a_1, a_2, ..., a_n) \in \prod_{n}^{i=1} A_i \\ &\Leftrightarrow \varphi_i(a_i) = b_i, a_i \in A_i \\ &\Leftrightarrow (b_1, b_2, ..., b_n) \in \prod_{i=1}^{n} \varphi_i(A_i).\end{aligned}$$

□

Theorem 4.2. *Let $\{A_i, i \in I_n\}$ be a family of MS-algebras. Then, the map $\varphi_k : \prod_{i=1}^n A_i \to A_k$ defined by $\varphi_k(a_1, a_2, ..., a_k, ..., a_n) = a_k$ is an epimorphism for each $k \in I_n$.*

Proof. Let $(a_1, a_2, ..., a_k, ..., a_n) = (b_1, b_2, ..., b_k, ..., b_n)$. Then, $a_i = b_i$ $\forall i \in I_n$. Therefore, $\varphi_k(a_1, a_2, ..., a_k, ..., a_n) = a_k = b_k = \varphi_k(b_1, b_2, ..., b_k, ..., b_n)$. So, φ_k is well defined, $\forall k \in I_n$. Now, suppose that $(a_1, a_2, ..., a_k, ..., a_n)$, $(b_1, b_2, ..., b_k, ..., b_n)$ $\in \prod_{i=1}^{n} A_i$. Then, $\varphi_k\big((a_1, a_2, ..., a_k, ..., a_n) \vee (b_1, b_2, ..., b_k, ..., b_n)\big) = a_k \vee b_k = \varphi_k(a_1, a_2, ..., a_k, ..., a_n) \vee \varphi_k(b_1, b_2, ..., b_k, ..., b_n)$. Similarly, φ_k preserves the meet operation. Besides,

$$\varphi_k(a_1, a_2, ..., a_k, ..., a_n)^\circ = a_k^\circ = (\varphi_k(a_1, a_2, ..., a_k, ..., a_n))^\circ$$

Finally, if $c_k \in A_K$, then $(0_1, 0_2, ..., c_k, ..., 0_n) \in \prod_{i=1}^{n} A_i$ with $\varphi_k(0_1, 0_2, ..., c_k, ..., 0_n) = c_k$. Thus, φ_k is onto and hence φ_k is an epimorphism . □

The previous maps ($\varphi_k s$) are called the canonical projections of the direct product.

Theorem 4.3. *Let $\{L_i, i \in I_n\}$ be a family of MS-algebras. Then there exists a unique (up to isomorphism) MS-algebra L, together with a family of homomorphisms $\{\varphi_i : L \to L_i, i \in I_n\}$, with the following property:*

For any MS-algebra M and any family of homomorphisms $\{f_i : M \to L_i, i \in I_n\}$, there exists a unique homomorphism $f : M \to L$ such that $\varphi_i \circ f = f_i$, $\forall i \in I_n$.

Proof. Let $L = \prod_{i=1}^{n} L_i$ and $\{\varphi_i : L \to L_i, i \in I_n\}$ be the family of canonical projections. Define $f : M \to L$ by $f(a) = (f_1(a), f_2(a), ..., f_n(a)), \forall a \in M$. For any $a, b \in M$, we have

$$\begin{aligned} f\big(a \vee b\big) &= (f_1(a \vee b), f_2(a \vee b), ..., f_n(a \vee b)) \\ &= (f_1(a) \vee f_1(b), f_2(a) \vee f_2(b), ..., f_n(a) \vee f_n(b)) \\ &= (f_1(a), f_2(a), ..., f_n(a)) \vee (f_1(b), f_2(b), ..., f_n(b)) \\ &= f(a) \vee f(b). \end{aligned}$$

Similarly, $f\big(a \wedge b\big) = f(a) \wedge f(b)$. Also,

$$f(a^\circ) = (f_1(a^\circ), f_2(a^\circ), ..., f_n(a^\circ)) = ((f_1(a))^\circ, (f_2(a))^\circ, ..., (f_n(a))^\circ) = ((f(a))^\circ.$$

Thus, f is a homomorphism. Moreover,

$$(\varphi_i \circ f)(a) = \varphi_i(f(a)) = \varphi_i(f_1(a), f_2(a), ..., f_n(a)) = f_i(a), \ \forall a \in M.$$

Hence, $\varphi_i \circ f = f_i$, $\forall i \in I_n$. To prove the uniqueness of f, let $g : M \to L$ be another homomorphism such that $\varphi_i \circ g = f_i$, $\forall i \in I_n$. This implies that $(\varphi_i \circ f)(a) = f_i(a) = (\varphi_i \circ g)(a)$, $\forall a \in M$, Assume that $g(a) = (a_1, a_2, ..., a_n)$, $\forall a \in M$. Then,

$$\begin{aligned} a_i &= \varphi_i\big(a_1, a_2, ..., a_n\big) = \varphi_i(g(a)) \\ &= \varphi_i(f(a)) = \varphi_i\big(f_1(a), f_2(a), ..., f_n(a)\big) = f_i(a) \ \forall i \in I_n. \end{aligned}$$

Therefore, $f(a) = (a_1, a_2, ..., a_n) = g(a)$, $\forall a \in M$. So, $f = g$ and f is unique. It remains to prove the uniqueness of L. Suppose that L_1 is an MS-algebra which has the same property as L with the family of homomorphisms $\{\psi_i : L_1 \to L_i,\ i \in I_n\}$. Apply the property to L and L_1, we get unique homomorphisms $\alpha : L_1 \to L$ and $\beta : L \to L_1$ with $\varphi_i \circ \alpha = \psi_i$ and $\psi_i \circ \beta = \varphi_i,\ \forall i \in I_n$. Consequently, $\alpha \circ \beta : L \to L_1$ is a unique homomorphism with $\varphi_i \circ (\alpha \circ \beta) = \varphi_i\ \forall i \in I_n$. Since the identity map $id_L : L \to L$ is also a homomorphism with $\varphi_i \circ id_L = \varphi_i\ \forall i \in I_n$, then $\alpha \circ \beta = id_L$. Similarly, $\beta \circ \alpha = id_{L_1}$. This shows that β is an isomorphism and L is unique up to isomorphism. □

Noting that the proofs of the previous three theorems do not rely on the decomposability of the MS-algebras, we conclude that they hold for decomposable MS-algebras.

Theorem 4.4. *Let $\varphi : L_1 \to L_2$ be a homomorphism between decomposable MS-algebras L_1 and L_2. If A is a subalgebra of L_1, then $\varphi(A)$ is a subalgebra of L_2.*

Proof. Let $b_1, b_2 \in \varphi(A)$. Then, there exist $a_1, a_2 \in A$ with $\varphi(a_1) = b_1$, $\varphi(a_2) = b_2$. So, $\varphi(a_1 \vee a_2) = b_1 \vee b_2$. As $a_1 \vee a_2 \in A$, then $b_1 \vee b_2 \in \varphi(A)$. A similar argument shows that $b_1 \wedge b_2 \in \varphi(A)$. Now, let $b \in \varphi(A)$. Then, $b = \varphi(a)$, for some $a \in A$. So, $b^\circ = \varphi(a^\circ)$. Since $a^\circ \in A$, then $b^\circ \in \varphi(A)$. Writing $a = a^{\circ\circ} \wedge d, d \in D(A)$, we get

$$b = \varphi(a) = \varphi(a^{\circ\circ} \wedge d) = \varphi(a^{\circ\circ}) \wedge \varphi(d) = (\varphi(a))^{\circ\circ} \wedge \varphi(d) = b^{\circ\circ} \wedge \varphi(d).$$

We note that $(\varphi(d))^\circ = \varphi(d^\circ) = \varphi(0_1) = 0_2$. So, $\varphi(d) \in D(\varphi(A))$. Hence, $\varphi(A)$ is a subalgebra of L_2. □

Theorem 4.5. *Let $\varphi : L_1 \to L_2$ be a monomorphism. If B is a subalgebra of L_2, then $\varphi^{-1}(B)$ is a subalgebra of L_1.*

Proof. Let $a_1, a_2 \in \varphi^{-1}(B)$. Then, there exist $b_1, b_2 \in B$ with $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$. So, $\varphi(a_1 \vee a_2) = b_1 \vee b_2$ and $\varphi(a_1 \wedge a_2) = b_1 \wedge b_2$. As $b_1 \vee b_2, b_1 \wedge b_2 \in B$, then $a_1 \vee a_2, a_1 \wedge a_2 \in \varphi^{-1}(B)$. Now, let $a \in \varphi^{-1}(B)$, then $a = \varphi^{-1}(b)$ for some $b \in B$. So, $\varphi(a) = b$. Then, $\varphi(a^\circ) = b^\circ$. As $b^\circ \in B$, then $a^\circ \in \varphi^{-1}(B)$. Assuming that $b = b^{\circ\circ} \wedge e,\ e \in D(B)$, we get

$$\begin{aligned} a \;=\; \varphi^{-1}(b) &= \varphi^{-1}(b^{\circ\circ} \wedge e),\ e \in D(B). \\ &= \varphi^{-1}(b^{\circ\circ}) \wedge \varphi^{-1}(e) \\ &= (\varphi^{-1}(b))^{\circ\circ} \wedge \varphi^{-1}(e)) = a^{\circ\circ} \wedge \varphi^{-1}(e). \end{aligned}$$

Now, we prove that $\varphi^{-1}(e) \in D(\varphi^{-1}(B))$. Let $d = \varphi^{-1}(e)$. Then, $\varphi(d) = e$. This gives $\varphi(d^\circ) = e^\circ = 0_2$. Therefore, $d^\circ = \varphi^{-1}(0_2) = 0_1$. So, $d \in D(\varphi^{-1}(B))$. Hence, $a = a^{\circ\circ} \wedge d$, $d \in D(\varphi^{-1}(B))$. Hence, $\varphi^{-1}(B)$ is a subalgebra of L_1. □

Theorem 4.6. *Let L_1 and L_2 be two MS-algebras. Then, L_1 can be embedded into $L_1 \times L_2$ if and only if there exists a homomorphism from L_1 to L_2.*

Proof. Assume that L_1 can be embedded into $L_1 \times L_2$. Then, there exists a monomorphism $\varphi : L_1 \to L_1 \times L_2$. Let $\varphi(a) = (a_1, a_2)$, $\forall a \in L_1$. Define $f : L_1 \to L_2$ by $f(a) = a_2$. Then, $f(a \vee b) = a_2 \vee b_2 = f(a) \vee f(b)$. Similarly, $f(a \wedge b) = f(a) \wedge f(b)$ Also, $f(a^\circ) = a_2^\circ = (f(a))^\circ$. Hence, f is a homomorphism.

Conversely, assume that there exists a homomorphism $f : L_1 \to L_2$. Define $\phi : L_1 \to L_1 \times L_2$ by $\phi(a) = (a, f(a))$. Then,

$$\begin{aligned} \phi(a \vee b) &= (a \vee b, f(a \vee b)) = (a \vee b, f(a) \vee f(b)) = (a, f(a)) \vee (b, f(b)) \\ &= \phi(a) \vee \phi(b), \ \forall a, b \in L_1. \end{aligned}$$

Analogously, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. Also, $\phi(a^\circ) = (a^\circ, f(a^\circ)) = (a, f(a))^\circ = (\phi(a))^\circ$. Assume $\phi(a) = \phi(b)$, then $(a, f(a)) = (b, f(b))$. This gives $a = b$. Hence, ϕ is an embedding. □

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Accepted: 22.12.2018

Common fuzzy fixed points of α-fuzzy mappings

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Abstract. The aim of this paper is to obtain the common fuzzy fixed points of α-fuzzy mappings satisfying generalized almost Θ-*contraction* in the setting of complete metric space. In this way, we generalize several well known recent and classical results. Finally, we provide an example to show the significance of the investigation of this paper.

Keywords: complete metric space, almost Θ-contractions, fixed point, fuzzy mappings.

1. Introduction and preliminaries

In 1922, Banach [11] presented a revolutionary contraction principle (namely called Banach contraction principle) in which Picard iteration process was used for the evaluation of a fixed point. This principle guarantees the existence and uniqueness of fixed points of certain self-mappings of metric spaces, and provides a constructive method to find those fixed points. The Banach contraction principle was also used to establish the existence of a unique solution for a nonlinear integral equation [22]. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in compu-

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tational mathematics. Because of its importance and usefulness for mathematical theory, it has become a very popular tool in solving existence problems in many directions. Several authors have obtained various extensions and generalizations of Banach's theorem by defining a variety of contractive type conditions for self and non-self mappings on metric spaces.

In [12, 13] Berinde studied many kinds of contraction mappings and gave the concept of almost contraction in following way.

Definition 1 ([12]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is called an almost contraction if there exists a constant $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

He also generalized the above almost contraction in this way.

Definition 2 ([13]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be generalized almost contraction if there exists a constant $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Very recently, Jleli and Samet [20] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 3. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if

$$\lim_{n\to\infty} (\alpha_n) = 0;$$

(Θ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $T : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \Longrightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^k. \tag{1.1}$$

Theorem 4 ([20]). *Let (X, d) be a complete metric space and $T : X \to X$ be a Θ-contraction, then T has a unique fixed point.*

Later on Hancer et al.[17] modified the above definitions by adding a general condition (Θ_4) which is given in this way:

(Θ_4) $\Theta(\inf A) = \inf \Theta(A)$, for all $A \subset (0, \infty)$ with $\inf A > 0$.

Following Hancer et al. [17], we represent the set of all continuous functions $\Theta : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\Theta_1) - (\Theta_4)$ conditions by Ω.

For more details on Θ-contractions, we refer the reader to [2, 4, 19, 23, 25, 30].

Following the Banach contraction principle Nadler [24] introduced the concept of multi-valued contractions using the Hausdorff metric and established that a multi-valued contraction possesses a fixed point in a complete metric space. In 1981, Heilpern [18] used the concept of fuzzy set to introduce a class of fuzzy mappings, which is a generalization of the set-valued mappings, and proved a fixed point theorem for fuzzy contraction mappings in metric linear space. It is worth noting that the result announced by Heilpern [18] is a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied existence of fixed points of fuzzy mappings, for example, Azam et [8, 9], Bose et al. [14], Chang et al. [15], Cho et al. [16], Qiu et al. [26], Rashwan et al. [27], Shi-sheng [29].

In the following we always suppose that (X, d) is a complete metric space. Moreover, we shall use the following notations which have been recorded from [1, 5, 10, 28, 31]:

Let $CB(X)$ be the family of nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

where

$$d(x, A) = \inf_{y \in A} d(x, y).$$

A fuzzy set in X is a function with domain X and values in $[0, 1]$, I^X is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α -level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1],$$
$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

Here $\overline{B}$ denotes the closure of the set B. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in a metric space X. For $A, B \in \mathcal{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. We denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless otherwise is stated, where $\chi_{\{x\}}$ is the characteristic function of the crisp set A. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in CB(X)$, then define

$$p_\alpha(A, B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x, y),$$

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha).$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0,1]$, then define

$$p(A,B) = \sup_\alpha p_\alpha(A,B),$$

$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

We write $p(x,B)$ instead of $p(\{x\},B)$. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup_{x\in V} A(x) = 1$. The collection of all approximate quantities in V is denoted by $W(V)$. Let X be an arbitrary set, Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into $\mathcal{F}(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 5. *Let S,T be fuzzy mappings from X into $\mathcal{F}(X)$. A point $u \in X$ is called an α- fuzzy fixed point of T if there exists $\alpha \in [0,1]$ such that $u \in [Tu]_\alpha$. The point $u \in X$ is called a common α- fuzzy fixed point of S and T if there exists $\alpha \in [0,1]$ such that $u \in [Su]_\alpha \cap [Tu]_\alpha$. When $\alpha = 1$, it is called a common fixed point of fuzzy mappings.*

For the sake of convenience, we first state some known results for subsequent use in the next section.

Lemma 6. *Let (X,d) be a metric space and $A,B \in CB(X)$. Then for each $a \in A$,*

$$d(a,B) \leq H(A,B).$$

Lemma 7 ([5])**.** *Let V be a metric linear space, $T : X \to W(V)$ be a fuzzy mapping and $x_0 \in V$. Then there exists $x_1 \in V$ such that $\{x_1\} \subset T(x_0)$.*

2. Main results

Theorem 8. *Let (X,d) be a complete metric space and let S,T be fuzzy mappings from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}$, $[Ty]_{\alpha_T(x)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\right)\right) \leq \Theta(d\,(x,y)\,,)^k + LM(x,y), \tag{2.1}$$

for all $x,y \in X$ with $H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) > 0$, where

$$\begin{aligned} M(x,y) = \min\Big\{ & d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Ty]_{\alpha_T(y)}\right), d\left(x, [Ty]_{\alpha_T(y)}\right), \\ & d\left(y, [Sx]_{\alpha_S(x)}\right)\Big\}. \end{aligned} \tag{2.2}$$

Then there exists some $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof. Let x_0 be an arbitrary point in X, then by hypotheses there exists $\alpha_S(x_0) \in (0,1]$ such that $[Sx_0]_{\alpha_S(x_0)}$ is a nonempty, closed and bounded subset of X. For convenience, we denote $\alpha_S(x_0)$ by α_1. Let $x_1 \in [Sx_0]_{\alpha_S(x_0)}$. For this x_1, there exists $\alpha_T(x_1) \in (0,1]$ such that $[Tx_1]_{\alpha_T(x_1)}$ is a nonempty, closed and bounded subset of X. By Lemma 6, (Θ_1) and (2.1), we have

$$\begin{aligned} \Theta\left(d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right)\right) &\leq \Theta(H\left([Sx_0]_{\alpha_S(x_0)}, [Tx_1]_{\alpha_T(x_1)}\right) \\ &\leq \Theta(d(x_0,x_1))^k + LM(x_0,x_1), \end{aligned}$$

where

$$M(x_0,x_1) = \min\left\{ \begin{array}{c} d\left(x_0, [Sx_0]_{\alpha_S(x_0)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_0, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_0]_{\alpha_S(x_0)}\right) \end{array} \right\}.$$

From (Θ_4), we know that

$$\Theta\left(d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right)\right) = \inf_{y \in [Tx_1]_{\alpha_T(x_1)}} \Theta(d(x_1,y)).$$

Thus

$$\begin{aligned} \inf_{y \in [Tx_1]_{\alpha_T(x_1)}} \Theta(d(x_1,y)) \leq [\Theta(d(x_0,x_1)]^k \\ + L \min\left\{ \begin{array}{c} d\left(x_0, [Sx_0]_{\alpha_S(x_0)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_0, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_0]_{\alpha_S(x_0)}\right) \end{array} \right\}. \end{aligned}$$

Then, from above there exists $x_2 \in [Tx_1]_{\alpha_T(x_1)}$ such that

$$\begin{aligned} \Theta(d(x_1,x_2)) &\leq [\Theta(d(x_0,x_1)]^k \\ &+ L \min\left\{d(x_0,x_1), d(x_1,x_2), d(x_0,x_2), d(x_1,x_1)\right\} \\ &= [\Theta(d(x_0,x_1)]^k. \end{aligned} \tag{2.3}$$

For this x_2 there exists $\alpha_S(x_2) \in (0,1]$ such that $[Sx_2]_{\alpha_S(x_2)}$ is a nonempty, closed and bounded subset of X. By Lemma 6, (Θ_1) and (2.1), we have

$$\begin{aligned} \Theta\left(d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right)\right) &\leq \Theta\left(H\left([Tx_1]_{\alpha_T(x_1)}, [Sx_2]_{\alpha_S(x_2)}\right)\right) \\ &= \Theta\left(H\left([Sx_2]_{\alpha_S(x_2)}, [Tx_1]_{\alpha_T(x_1)}\right)\right) \\ &\leq \Theta(d(x_2,x_1))^k + +LM(x_2,x_1) \end{aligned}$$

where

$$M(x_2,x_1) = \min\left\{ \begin{array}{c} d(x_2,x_1), d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_2, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_2]_{\alpha_S(x_2)}\right) \end{array} \right\}.$$

From (Θ_4), we know that

$$\Theta\left[d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right)\right] = \inf_{y_1 \in [Sx_2]_{\alpha_S(x_2)}} \Theta(d(x_2, y_1)).$$

Thus

$$\begin{aligned}
&\inf_{y_1 \in [Sx_2]_{\alpha_S(x_2)}} \Theta(d(x_2, y_1)) \leq \Theta\left[d(x_1, x_2)\right]^k \\
&+ L \min \left\{ \begin{array}{c} d(x_2, x_1), d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_2, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_2]_{\alpha_S(x_2)}\right) \end{array} \right\}.
\end{aligned}$$

Then, from above there exists $x_3 \in [Sx_2]_{\alpha_S(x_2)}$ such that

$$\begin{aligned}
\Theta(d(x_2, x_3)) &\leq [\Theta(d(x_1, x_2)]^k \\
&\quad + L \min \left\{d(x_2, x_1), d\left(x_2, x_3\right), d\left(x_1, x_2\right), d\left(x_2, x_2\right), d\left(x_1, x_3\right)\right\} \\
&= [\Theta(d(x_1, x_2)]^k. \qquad (2.4)
\end{aligned}$$

So, continuing recursively, we obtain a sequence $\{x_n\}$ in X such that

$$x_{2n+1} \in [Sx_{2n}]_{\alpha_S(x_{2n})} \text{ and } \quad x_{2n+2} \in [Tx_{2n+1}]_{\alpha_T(x_{2n+1})} \qquad (2.5)$$

with

$$\Theta(d(x_{2n+1}, x_{2n+2})) \leq [\Theta(d(x_{2n}, x_{2n+1})]^k \qquad (2.6)$$

and

$$\Theta(d(x_{2n+2}, x_{2n+3})) \leq [\Theta(d(x_{2n+1}, x_{2n+2})]^k, \qquad (2.7)$$

for all $n \in \mathbb{N}$. From (2.6) and (2.7), we have

$$\Theta(d(x_n, x_{n+1})) \leq [\Theta(d(x_{n-1}, x_n)]^k \qquad (2.8)$$

which further implies that

$$\begin{aligned}
\Theta(d(x_n, x_{n+1})) &\leq [\Theta(d(x_{n-1}, x_n)]^k \leq [\Theta(d(x_{n-2}, x_{n-1})]^{k^2} \\
&\leq ... \leq [\Theta(d(x_0, x_1)]^{k^n}, \qquad (2.9)
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, by taking limit as $n \to \infty$ in (2.9) we have,

$$\lim_{n \to \infty} \Theta(d(x_n, x_{n+1})) = 1 \qquad (2.10)$$

which implies that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \qquad (2.11)$$

by (Θ_2). From the condition (Θ_3), there exist $0 < r < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} = l. \tag{2.12}$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} - l| \leq B,$$

for all $n > n_0$. This implies that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} \geq l - B = \frac{l}{2} = B,$$

for all $n > n_0$. Then

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1], \tag{2.13}$$

for all $n > n_0$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r},$$

for all $n > n_0$. This implies that

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1],$$

for all $n > n_0$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1], \tag{2.14}$$

for all $n > n_0$. Thus by (2.9) and (2.14), we get

$$nd(x_n, x_{n+1})^r \leq An([(\Theta d(x_0, x_1))]^{r^n} - 1). \tag{2.15}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nd(x_n, x_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{2.16}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \tag{2.17}$$

Since, $0 < r < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in (X, d). The completeness of (X, d) ensures that there exists $u \in X$ such that, $\lim_{n\to\infty} x_n \to u$. Now, we prove that $u \in [Tu]_{\alpha_T(u)}$. We suppose on the contrary that $u \notin [Tu]_{\alpha_T(u)}$, then there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{2n_k+1}, [Tu]_{\alpha_T(u)}) > 0$, for all $n_k \geq n_0$. Since $d(x_{2n_k+1}, [Tu]_{\alpha_T(u)}) > 0$, for all $n_k \geq n_0$, so by (Θ_1), we have

$$\begin{aligned}
&\Theta\left[d(x_{2n_k+1}, [Tu]_{\alpha_T(u)})\right] \leq \Theta\left[H([Sx_{2n_k}]_{\alpha_S(x_{2n_k})}, [Tu]_{\alpha_T(u)})\right] \\
&\leq [\Theta(d(x_{2n_k}, u))]^k \\
&+ L \min \left\{ \begin{array}{c} d\left(x_{2n_k}, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right), d\left(u, [Tu]_{\alpha_T(u)}\right), \\ d\left(x_{2n_k}, [Tu]_{\alpha_T(u)}\right), d\left(u, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right) \end{array} \right\} \\
&\leq [\Theta(d(x_{2n_k}, u))]^k + L \min \left\{ \begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Tu]_{\alpha_T(u)}\right), \\ d\left(x_{2n_k}, [Tu]_{\alpha_T(u)}\right), d\left(u, x_{2n_k+1}\right) \end{array} \right\}.
\end{aligned}$$

Letting $k \to \infty$, in the above inequality and using the continuity of Θ, we have

$$\Theta\left[d(u, [Tu]_{\alpha_T(u)})\right] \leq 0.$$

Hence $u \in [Tu]_{\alpha_T(u)}$. Similarly, one can easily prove that $u \in [Su]_{\alpha_S(u)}$. Thus $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$. □

Corollary 9. *Let (X, d) be a complete metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}$, $[Ty]_{\alpha_T(x)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\right)\right) \leq \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) > 0$. Then there exists some $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof. Taking $L = 0$ in Theorem 8. □

Corollary 10. *Let (X, d) be a complete metric space and let S be fuzzy mapping from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that*

$[Sx]_{\alpha_S(x)}$, $[Sy]_{\alpha_S(y)}$ *are nonempty, closed and bounded subsets of* X. *Assume that there exist some* $\Theta \in \Omega$, $k \in (0,1)$ *and* $L \geq 0$ *such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right)\right) \leq \Theta(d(x,y))^k + LM(x,y)$$

for all $x, y \in X$ *with* $H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right) > 0$, *where*

$$M(x,y) = \min\left\{d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Sy]_{\alpha_S(y)}\right), d\left(x, [Sy]_{\alpha_S(y)}\right), d\left(y, [Sx]_{\alpha_S(x)}\right)\right\}.$$

Then there exists some $u \in X$ *such that* $u \in [Su]_{\alpha_S(u)}$.

Proof. Taking $S = T$ in Theorem 8. □

Corollary 11. *Let* (X,d) *be a complete metric space and let* S *be a fuzzy mapping from* X *into* $\mathcal{F}(X)$ *and for each* $x \in X$, *there exist* $\alpha_S(x), \alpha_T(y) \in (0,1]$ *such that* $[Sx]_{\alpha_S(x)}$, $[Sy]_{\alpha_S(y)}$ *are nonempty, closed and bounded subsets of* X. *Assume that there exist some* $\Theta \in \Omega$ *and* $k \in (0,1)$ *such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_T(y)}\right)\right) \leq \Theta(d(x,y))^k,$$

for all $x, y \in X$ *with* $H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_T(y)}\right) > 0$. *Then there exists some* $u \in [Su]_{\alpha_S(u)}$.

Proof. Taking $S = T$ and $L = 0$ in Theorem 8. □

Now we state a common fixed point result for two multivalued mappings.

Theorem 12. *Let* (X,d) *be a complete metric space and let* $F, G : X \to CB(X)$ *be multivalued mappings. Assume that there exist some* $\Theta \in \Omega$, $k \in (0,1)$ *and* $L \geq 0$ *such that*

$$\Theta(H(Fx, Gy)) \leq \Theta(d(x,y))^k + LM(x,y),$$

for all $x, y \in X$ *with* $H(Fx, Gy) > 0$, *where*

$$M(x,y) = \min\{d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)\}.$$

Then there exists some $u \in Fu \cap Gu$.

Proof. Consider a mapping $\alpha : X \to (0,1]$ and a pair of fuzzy mappings $S, T : X \to \mathcal{F}(X)$ defined by

$$S(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Fx, \\ 0, & \text{if } t \notin Fx \end{cases}$$

and

$$T(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Gx, \\ 0, & \text{if } t \notin Gx. \end{cases}$$

Then$[Sx]_{\alpha(x)} = \{t : S(x)(t) \geq \alpha(x)\} = Fx$ and $[Tx]_{\alpha(x)} = \{t : T(x)(t) \geq \alpha(x)\} = Gx$. Thus, Theorem 8 can be applied to obtain $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)} = Fu \cap Gu$. □

Corollary 13. *Let (X, d) be a complete metric space and let $F, G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(H(Fx, Gy)) \leq \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H(Fx, Gy) > 0$. Then there exists some $u \in Fu \cap Gu$.

Proof. Taking $L = 0$ in Theorem 12. □

Corollary 14. *Let (X, d) be a complete metric space and let $G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $L \geq 0$ such that*

$$\Theta(H(Gx, Gy)) \leq \Theta(d(x, y))^k + LM(x, y),$$

for all $x, y \in X$ with $H(Gx, Gy) > 0$, where

$$M(x, y) = \min\{d(x, Gx), d(y, Gy), d(x, Gy), d(y, Gx)\}.$$

Then there exists some $u \in X$ such that $u \in Gu$.

Proof. Taking $F = G$ in Theorem 12. □

Corollary 15. *Let (X, d) be a complete metric space and let $G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(H(Gx, Gy)) \leq \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H(Gx, Gy) > 0$. Then there exists some $u \in X$ such that $u \in Gu$.

Proof. Taking $F = G$ and $L = 0$ in Theorem 12. □

Theorem 16. *Let (X, d) be a complete metric linear space and let $S, T : X \to W(X)$ be fuzzy mappings. Suppose that there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $L \geq 0$ such that*

$$\Theta(d_\infty(S(x), T(y))) \leq \Theta(p(x, y))^k + LM(x, y),$$

for all $x, y \in X$ with $d_\infty(S(x), T(y)) > 0$, where

$$M(x, y) = \min\{p(x, S(x)), p(y, T(y)), p(x, T(y)), p(y, S(x))\}.$$

Then there exists some $u \in X$ such that $\{u\} \subset S(u)$ and $\{u\} \subset T(u)$.

Proof. Let $x \in X$, then by Lemma 6 there exists $y \in X$ such that $y \in [Sx]_1$. Similarly, we can find $z \in X$ such that $z \in [Tx]_1$. It follows that for each $x \in X$, $[Sx]_{\alpha(x)}$, $[Tx]_{\alpha(x)}$ are nonempty, closed and bounded subsets of X. As $\alpha(x) = \alpha(y) = 1$, by the definition of a d_∞-metric for fuzzy sets, we have

$$H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(x)}\right) \leq d_\infty(S(x), T(y)),$$

for all $x, y \in X$. From (Θ_1), we have

$$\begin{aligned}\Theta\left(H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(x)}\right)\right) &\leq \Theta\left(d_\infty(S(x), T(y))\right) \\ &\leq \left[\Theta\left(p\left(x, y\right)\right)\right]^k + LM(x, y)\end{aligned}$$

where

$$M(x, y) = \min\left\{p\left(x, S(x)\right), p\left(y, T(y)\right), p\left(x, T(y)\right), p\left(y, S(x)\right)\right\},$$

for all $x, y \in X$. Since $[Sx]_1 \subseteq [Sx]_\alpha$ for each $\alpha \in (0, 1]$. Therefore $d(x, [Sx]_\alpha) \leq d(x, [Sx]_1)$ for each $\alpha \in (0, 1]$. It implies that $p(x, S(x)) \leq d(x, [Sx]_1)$. Similarly, $p(x, T(x)) \leq d(x, [Tx]_1)$. Furthermore this implies that for all $x, y \in X$,

$$\Theta\left(H\left([Sx]_1, [Ty]_1\right)\right) \leq \left[\Theta\left(d\left(x, y\right)\right)\right]^k + LM(x, y)$$

where

$$M(x, y) = \min\left\{d\left(x, [Sx]_1\right), d\left(y, [Ty]_1\right), d\left(x, [Ty]_1\right), d\left(y, [Sx]_1\right)\right\}.$$

Now, by Theorem 8, we obtain $u \in X$ such that $u \in [Su]_1 \cap [Tu]_1$, i.e., $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$. □

Corollary 17. *Let (X, d) be a complete metric linear space and let $S, T : X \to W(X)$ be fuzzy mappings. Suppose that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(d_\infty\left(S(x), T(y)\right)) \leq \Theta(p(x, y))^k,$$

for all $x, y \in X$ with $d_\infty\left(S(x), T(y)\right) > 0$. Then there exists some $u \in X$ such that $\{u\} \subset S(u)$ and $\{u\} \subset T(u)$.

In the following, we suppose that $\widehat{T}$ (for details, see [[28], [29]]) is the set-valued mapping induced by fuzzy mappings $T : X \to \mathcal{F}(X)$, i.e.,

$$\widehat{T}x = \left\{y : T(x)(t) = \max_{t \in X} T(x)(t)\right\}.$$

Proof. Taking $L = 0$ in Theorem 16. □

Corollary 18. *Let (X,d) be a complete metric space and let $S,T:X\to\mathcal{F}(X)$ be fuzzy mappings such that for all $x\in X$, $\widehat{S}(x),\widehat{T}(x)$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta\in\Omega$, $k\in(0,1)$ and $L\geq 0$ such that*

$$\Theta\left(H\left(\widehat{S}(x),\widehat{T}(y)\right)\right)\leq\Theta(d(x,y))^k+LM(x,y),$$

for all $x,y\in X$ with $H(\widehat{S}(x),\widehat{T}(y))>0$, where

$$M(x,y)=\min\left\{d\left(x,\widehat{S}(x)\right),d\left(y,\widehat{T}(y)\right),d\left(x,\widehat{T}(y)\right),d\left(y,\widehat{S}(x)\right)\right\}.$$

Then there exists a point $x^*\in X$ such that $S(x^*)(x^*)\geq S(x^*)(x)$ and $T(x^*)(x^*)\geq T(x^*)(x)$, for all $x\in X$.

Proof. By Theorem 12, there exists $x^*\in X$ such that $x^*\in\widehat{S}x^*\cap\widehat{T}x^*$. Then by Lemma 7, we have

$$S(x^*)(x^*)\geq S(x^*)(x)\qquad\text{and}\qquad T(x^*)(x^*)\geq T(x^*)(x),$$

for all $x\in X$. □

Corollary 19. *Let (X,d) be a complete metric space and let $S,T:X\to\mathcal{F}(X)$ be fuzzy mappings such that for all $x\in X$, $\widehat{S}(x),\widehat{T}(x)$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta\in\Omega$ and $k\in(0,1)$such that*

$$\Theta\left(H\left(\widehat{S}(x),\widehat{T}(y)\right)\right)\leq\Theta(d(x,y))^k,$$

for all $x,y\in X$ with $H(\widehat{S}(x),\widehat{T}(y))>0$. Then there exists a point $x^*\in X$ such that $S(x^*)(x^*)\geq S(x^*)(x)$ and $T(x^*)(x^*)\geq T(x^*)(x)$ for all $x\in X$.

Proof. Taking $L=0$ in Corollary 18. □

Example 20. Let $X=[0,1]$ and define $d:X\times X\to\mathbb{R}^+$ as follows:

$$d(x,y)=|x-y|.$$

Then (X,d) is a complete metric space. Define a pair of mappings $S,T:X\to\mathcal{F}(X)$, for $\alpha\in[0,1]$ as follows:

For $x\in X$, we have

$$S(x)(t)=\begin{cases}\alpha, & \text{if } 0\leq t\leq\frac{x}{30},\\ \frac{\alpha}{2}, & \text{if } \frac{x}{30}<t\leq\frac{x}{20},\\ \frac{\alpha}{3}, & \text{if } \frac{x}{20}<t\leq\frac{x}{10},\\ \frac{\alpha}{5}, & \text{if } \frac{x}{10}<t\leq 1\end{cases},$$

and

$$T(x)(t) = \begin{cases} \alpha, & \text{if } 0 \leq t \leq \frac{x}{15}, \\ \frac{\alpha}{3}, & \text{if } \frac{x}{15} < t \leq \frac{x}{10}, \\ \frac{\alpha}{4}, & \text{if } \frac{x}{10} < t \leq \frac{x}{5}, \\ \frac{\alpha}{7}, & \text{if } \frac{x}{5} < t \leq 1 \end{cases},$$

such that

$$\begin{aligned} [Tx]_\alpha &= \left[0, \frac{x}{15}\right], \\ [Sx]_\alpha &= \left[0, \frac{x}{30}\right]. \end{aligned}$$

Let $\Theta(t) = e^{\sqrt[k]{t}}$. Then there exists some $k = \frac{1}{\sqrt{15}} \in (0,1)$ and $L = 0$ such that

$$\Theta\left(H\left([Sx]_{\alpha_{S(x)}}, [Ty]_{\alpha_{T(y)}}\right)\right) \leq \Theta(d(x,y))^k + LM(x,y),$$

for all $x, y \in X$ with $H([Sx]_{\alpha_{S(x)}}, [Ty]_{\alpha_{T(y)}}) > 0$, where

$$\begin{aligned} M(x,y) = \min\Big\{ & d\left(x, [Sx]_{\alpha_{S(x)}}\right), d\left(y, [Ty]_{\alpha_{T(y)}}\right), d\left(x, [Ty]_{\alpha_{T(y)}}\right), \\ & d\left(y, [Sx]_{\alpha_{S(x)}}\right)\Big\} \end{aligned}$$

is satisfied to obtain $0 \in [S0]_\alpha \cap [T0]_\alpha$.

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Accepted: 19.12.2018

Characterizations of almost PP-ring for three important classes of rings

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Abstract. A ring is called an almost $pp-$ ring if the annihilator of each element of R is generated by its idempotents. We prove that for a ring R and an Abelian group G, if the group ring RG is an almost $pp-$ ring then so is R, Moreover, if G is a finite Abelian group then $|G|^{-1} \in R$. Then we give a counter example to the converse of this. Also, we prove that RG is an almost $pp-$ring if and only if RH is an almost $pp-$ ring for every subgroup H of G. It is proved that the polynomial ring $R[x]$ is an almost $pp-$ ring if and only if R is an almost $pp-$ ring. Finally, we prove that the power series ring $R[[x]]$is an almost $pp-$ ring if and only if for any two countable subsets S and T of R such that $S \subseteq Ann_R(T)$, there exists an idempotent $e \in Ann_R(T)$ such that $b = be$ for all $b \in S$.
Keywords: almost $pp-$ring, group ring, polynomial ring, power series ring.

1. Introduction

All rings considered in this paper are assumed to be commutative with unity $1 \neq 0$, and all groups are Abelian. Recall that a ring R is called a $pf-$ring if every principal ideal is a flat $R-$module. An ideal I of a ring R is called pure if for every $a \in I$, there exists $b \in I$ such that $ab = a$. It is well known that a ring R is a $pf-$ring if and only if $Ann_R(a) = \{x \in R : xa = 0\}$ is a pure ideal for every $a \in R$, see [1]. There are different characterizations of $pf-$rings, see [7] and [3]. A ring R is called a $pp-$ring if for each $a \in R$, $Ann_R(a)$ is generated by an idempotent element in R. These rings were studied extensively in literatures, see [5], [10], and [3]. As a generalization of $pp-$ring, Al-Ezeh in [4] introduced a new class of rings called almost $pp-$rings. A ring R is called an almost $pp-$ring if for each $a \in R$, the annihilator ideal $Ann_R(a)$ is generated by its idempotents.

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It is known that a ring R is an almost $pp-$ring if and only if for every $a \in R$ and $b \in Ann_R(a)$, there exists an idempotent $e \in Ann_R(a)$ such that $b = eb$, see [4, Theorem 1]. Some properties of almost $pp-$rings were investigated in [4] and [13].

Clearly, every $pp-$ring is an almost $pp-$ring and every almost $pp-$ring is a $pf-$ring. Every $pf-$ring is a reduced ring (a ring has no nonzero nilpotent elements), see [1, Lemma 2]. Al-Ezeh in [4] gave an example of an almost $pp-$ring which is not a $pp-$ring, and another example of a $pf-$ring which is not an almost $pp-$ring.

Our aim in this paper to characterize when group rings RG are almost $pp-$rings. Furthermore, we characterize when polynomial rings and power series rings are almost $pp-$rings.

2. Almost $PP-$Rings

In this section, we establish general results on almost $pp-$rings.

Definition 2.1. *Let R be a ring. Then R is said to be an almost $pp-$ring if for every $a \in R$, the annihilator $Ann_R(a)$ is generated by its idempotents.*

In some research, an almost $pp-$ring is called an "almost weak Baer" ring.

Lemma 2.2. *Let R be a Noetherian ring. Then R is an almost $pp-$ring if and only if R is a $pp-$ring.*

Proof. Clearly, if R is $pp-$ring, then R is an almost $pp-$ring.

Assume that R is an almost $pp-$ring and $a \in R$. Then since R is a Noetherian ring, $Ann_R(a)$ is finitely generated ideal. Also, $Ann_R(a)$ is generated by its idempotents. Hence, $Ann_R(a) = \sum_{i=1}^{n} e_i R = eR$ where $1 - e = \prod_{i=1}^{n}(1 - e_i)$. Thus, R is a $pp-$ring. □

Lemma 2.3. *Let $(R_i)_{i \in I}$ be a family of commutative rings. Then $R = \prod_{i \in I} R_i$ is an almost $pp-$ring if and only if R_i is an almost $pp-$ring for all $i \in I$.*

Proof. Assume that R_i is an almost $pp-$ring for each $i \in I$ and $R = \prod_{i \in I} R_i$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be two elements of R such that $y \in Ann_R(x)$. Then, $xy = (x_i)_{i \in I}(y_i)_{i \in I} = (x_i y_i)_{i \in I} = 0$. Since R_i is an almost $pp-$ring for every $i \in I$, there exists an idempotent $e_i \in Ann_{R_i}(x_i)$ such that $y_i = e_i y_i$.

Hence, $y = (y_i)_{i \in I} = (e_i)_{i \in I}(y_i)_{i \in I}$, $(e_i)_{i \in I} \in Ann_R(x)$ and $\left((e_i)_{i \in I}\right)^2 = (e_i)_{i \in I}$.

Therefore, R is an almost $pp-$ring.

Conversely, assume that $R = \prod_{i \in I} R_i$ is an almost $pp-$ring. Let $i \in I$ and let x_i, y_i be two elements of R_i such that $y_i \in Ann_{R_i}(x_i)$.

Consider $x = (\alpha_j)_{j \in I}$, with $\alpha_j = \begin{cases} x_i, & j = i \\ 0, & j \neq i \end{cases}$ and $y = (\beta_j)_{j \in I}$, with $\beta_j = \begin{cases} y_i, & j = i \\ 0, & j \neq i. \end{cases}$. So, $y \in Ann_R(x)$. Since R is an almost $pp-$ring, then there exists an idempotent $e = (e_j)_{j\in I} \in Ann_R(x)$ such that $y = ey$. That is for all $j \in I$, $\beta_j = e_j\beta_j$, $e_j \in Ann_{R_j}(\alpha_j)$ and $e_j^2 = e_j$.

Hence, $y_i = e_i y_i$, $e_i \in Ann_{R_i}(x_i)$ and $e_i^2 = e_i$.

Thus, R_i is an almost $pp-$ring for all $i \in I$. $\square$

The following Lemma is well known, see for example [12, Proposition 3.2.7 and its corollary]

Lemma 2.4. *Let R be a subring of a ring S both with the same identity. Suppose that S is a free $R-$module with a basis G such that G is multiplicatively closed and $1 \in G$. Let $\varepsilon : S \longrightarrow R$ be a map defined by*

$$\varepsilon\left(\sum_{i=0}^{n} a_i g_i\right) = \sum_{i=0}^{n} a_i.$$

Then ε is a ring epimorphism.

Let R and S be two rings such that $R \subseteq S$. Let e be an idempotent in S. Then, $e^2 = e$, and so, $(\varepsilon(e))^2 = \varepsilon(e^2) = \varepsilon(e)$. Hence, $\varepsilon(e)$ is an idempotent in R.

Suppose $a \in R$ and $e \in Ann_S(a)$. Then, $ea = 0$ in S. So, $\varepsilon(e)a = 0$ in R since $a \in R$, $\varepsilon(a) = a$.Therefore $\varepsilon(e) \in Ann_R(a)$ and $(\varepsilon(e))^2 = \varepsilon(e)$.

Theorem 2.5. *Let R be a subring of a ring S both with the same identity. Suppose that S is a free $R-$module with a basis G such that G is multiplicatively closed and $1 \in G$. If S is almost $pp-$ring, then so is R.*

Proof. Let $a, b \in R$ such that $b \in Ann_R(a) \subseteq Ann_S(a)$. Then, since S is an almost $pp-$ring, there exists an idempotent $e \in Ann_S(a)$ such that $b = be$. Taking ε to both sides, we get

$$b = \varepsilon(b) = \varepsilon(b)\varepsilon(e) = b\varepsilon(e),$$

$\varepsilon(e) \in Ann_R(a)$ and $(\varepsilon(e))^2 = \varepsilon(e)$.

Therefore, R is an almost $pp-$ring. $\square$

3. Group rings

Given a ring R and a group G, we will denote the group ring of G over R by RG. Elements of the ring RG are just formal finite sums of the form $\sum_{g\in G} a_g g$ with all but a finite number of a_g are 0_R. We write C_n for the cyclic group of order n, $\mathbb{Z}$ for the ring of integers, $\mathbb{Z}_n$ for the ring of integers modulo n, and $\mathbb{C}$ is the field of complex numbers. The imaginary unit is denoted by $\mathbf{i}$.

The following facts are consequences of Theorem 2.5.

Corollary 3.1. *Let R be a ring and G be a group. If RG is an almost $pp-$ring, then so is R.*

Proof. $S = RG$ is a free $R-$module with a basis G satisfying the assumptions of Theorem 2.5. $\square$

Corollary 3.2. *If RG is an almost $pp-$ring and H is a subgroup of G, then RH is an almost $pp-$ring too.*

Proof. RH is a subring of RG and RG is a free $RH-$module on the set $\{g_1, g_2, ...\}$, the coset representatives of H in G. $\square$

Recall that a group G is called locally finite, if every finitely generated subgroup of G is finite.

Theorem 3.3. *Let G be a locally finite group. If RH is an almost $pp-$ring for all finite subgroup H of G, then RG is an almost $pp-$ring.*

Proof. Let $u = \sum_{i=1}^{n} a_i g_i \in RG$ and $v = \sum_{i=1}^{n} b_i g_i \in Ann_{RG}(u)$. Let $H = \langle g_1, ..., g_n \rangle$. Then H is finite since G is locally finite. Since $u, v \in RH$, $v \in Ann_{RH}(u) \subseteq Ann_{RG}(u)$. But RH is an almost $pp-$ring by assumption. Thus there exists an idempotent $e \in Ann_{RH}(u) \subseteq Ann_{RG}(u)$ such that $v = ev$. So, RG is an almost $pp-$ring. $\square$

Corollary 3.4. *Let G be a locally finite group. Then RG is an almost $pp-$ring if and only if for every finite subgroup H of G, RH is an almost $pp-$ring.*

Proof. If RG is an almost $pp-$ring, by Corollary 3.2, RH is an almost $pp-$ring for all subgroup H of G. So, in particular, for every finite subgroup H of G, RH is an almost $pp-$ring. By Theorem 3.3, the other direction holds. $\square$

Using the same technique used in Theorem 3.3, we get the following:

Corollary 3.5. *The group ring RG is an almost $pp-$ring if and only if RH is an almost $pp-$ring for each subgroup H of G.*

Theorem 3.6. *If RG is an almost $pp-$ring, then R is an almost $pp-$ring and the order of each finite order element $g \in G$ is a unit in R.*

Proof. By Corollary 3.1, R is almost $pp-$ring. Now let $g \in G$ with $|g| = n < \infty$. Let H be the cyclic subgroup generated by g. Then, by Corollary 3.2, RH is an almost $pp-$ring too. Now $1 + g + g^2 + ... + g^{n-1} \in Ann_{RH}(1 - g)$.

Since RH is an almost $pp-$ring, there exists an idempotent $e = a_0 + a_1 g + ... + a_{n-1} g^{n-1} \in Ann_{RH}(1 - g)$ such that $\left(1 + g + g^2 + ... + g^{n-1}\right) e = 1 + g + g^2 + ... + g^{n-1}$ and $e(1 - g) = 0$.

Thus, $a_0 = a_1 = ... = a_{n-1}$ since $e = eg$.

So, $\left(a_0 + a_0 g + ... + a_0 g^{n-1}\right)\left(1 + g + ... + g^{n-1}\right) = 1 + g + ... + g^{n-1}$. Hence, $a_0 n = 1$.

Therefore, n is a unit in R. $\square$

Corollary 3.7. *If G is a finite group and RG is an almost $pp-$ring, then $|G|^{-1} \in R$.*

Proof. Let G be a finite group and $|G| = n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are distinct primes and $\alpha_i \geq 1$ are positive integers for all $i = 1, ..., k$. Then by Cauchy Theorem, there exists $g_i \in G$ such that $|g_i| = p_i$, for all $i = 1, ..., k$.

Thus, since RG is an almost $pp-$ring and by Theorem 3.6, $p_i^{-1} \in R$ for all $i = 1, ..., k$. But the product of units is a unit.

So, $\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)^{-1} \in R$ and hence $|G|^{-1} \in R$. □

We will see in Example 3.23 that the converse of this Corollary needs not be true.

Example 3.8. If G is a finite group and $\mathbb{Z}_{p^r}G$ is an almost $pp-$ring, p is prime integer and $r \geq 1$, then by the previous Corollary $|G|$ is a unit in $\mathbb{Z}_{p^r}$. So, $p \nmid |G|$ and hence $\gcd(p, |G|) = 1$.

Example 3.9. $\mathbb{Z}_{p^r}G$ is not an almost $pp-$ring for any finite $p-$group G, where p is prime integer and $r \geq 1$. More generally, if $p \mid n$ and G is a $p-$group, then $\mathbb{Z}_nG$ is not an almost $pp-$ring.

The following example shows that if R is an almost $pp-$ring, it is not necessary that RG is an almost $pp-$ring.

Example 3.10. $\mathbb{Z}G$ is not almost $pp-$ring for any nontrivial finite group G.

If R is a ring and G is an Abelian group, then RG is a Noetherian ring if and only if R is a Noetherian ring and G is a finitely generated group, see [9, Theorem 2].

Theorem 3.11. *Let G be a finite Abelian group and n be an integer with $n > 1$. Then the following are equivalent:*

(1) *$\mathbb{Z}_nG$ is an almost $pp-$ring.*

(2) *$\mathbb{Z}_nG$ is a $pp-$ring.*

(3) *$\gcd(n, |G|) = 1$ and n is square free.*

Proof. (1) $\Longleftrightarrow$ (2) Since $\mathbb{Z}_n$ is Noetherian ring and G is finite Abelian group, it follows that $\mathbb{Z}_nG$ is Noetherian ring. Then the result follows by Lemma 2.2.

(2) $\Longleftrightarrow$ (3) See [15, Example 1.8]. □

The following Lemma exists in [12, page 134]

Lemma 3.12. $(R_1 \times R_2 \times ... \times R_n)\, G \cong \prod_{i=1}^{n} R_iG$

Theorem 3.13. *If $R = R_1 \times R_2 \times ... \times R_n$, then RG is an almost $pp-$ring if and only if R_iG is an almost $pp-$ring for all $i = 1, ..., n$.*

Proof. The proof follows from Lemma 2.3 and Lemma 3.12. □

Theorem 3.14. *If $R[x] / \left(x^n + a_1x^{n-1} + ... + a_n\right)$ is an almost $pp-$ring, where $a_1, ..., a_n \in R$ and n is a positive integer, then R is an almost $pp-$ring.*

Proof. $S = R[x] / \left(x^n + a_1x^{n-1} + ... + a_n\right)$ is free $R-$module with a basis $\left\{1, x, ..., x^{n-1}\right\}$ satisfying the assumptions of Theorem 2.5. □

The following proposition was introduced by Yi and Zhou [14].

Proposition 3.15. *Let R be a ring. Then*
(1) *If $2^{-1} \in R$, then $RC_2 \cong R \times R$ and $RC_4 \cong R \times R \times \left(R[x] / \left(x^2 + 1\right)\right)$*
(2) *If $R \subseteq \mathbb{C}$ and $3^{-1} \in R$, then $RC_3 \cong R \times \left(R[x] / \left(x^2 + x + 1\right)\right)$.*

Theorem 3.16. *RC_2 is almost $pp-$ring if and only if R is almost $pp-$ring and $2^{-1} \in R$.*

Proof. The proof follows from Corollary 3.1, Corollary 3.7 and Lemma 2.3. □

Theorem 3.17. *RC_4 is an almost $pp-$ring if and only if $R[x] / \left(x^2 + 1\right)$ is an almost $pp-$ring and $2^{-1} \in R$.*

Proof. By Proposition 3.15, if $2^{-1} \in R$ then $RC_4 \cong R \times R \times (R[x]/(x^2+1).)$. So, using Theorem 3.14, we get RC_4 is an almost $pp-$ring if and only if $R[x]/(x^2+1)$. is an almost $pp-$ring. □

Theorem 3.18. *If $R \subseteq \mathbb{C}$, then RC_3 is an almost $pp-$ring if and only if $R[x] / \left(x^2 + x + 1\right)$ is an almost $pp-$ring and $3^{-1} \in R$.*

Proof. By Proposition 3.15, if $3^{-1} \in R$, then $RC_3 \cong R \times (R[x]/(x^2 + x + 1).)$. So, using Theorem 3.14, we get RC_3 is an almost $pp-$ring if and only if $R[x]/(x^2 + x + 1)$. is an almost $pp-$ring. □

Note that, if $G = H \times K$, then $RG = R(H \times K) \cong (RH) K$.

Theorem 3.19. *If $R \subseteq \mathbb{C}$, then RC_6 is an almost $pp-$ring if and only if $6^{-1} \in R$ and $R[x] / \left(x^2 + x + 1\right)$ is an almost $pp-$ring.*

Proof. Since $C_6 \cong C_3 \times C_2$, then $RC_6 \cong (RC_3) C_2$.

So, RC_6 is an almost $pp-$ring if and only if $2^{-1} \in RC_3$ and RC_3 is an almost $pp-$ring.

Since $R \subseteq \mathbb{C}$, RC_3 is an almost $pp-$ring if and only if $3^{-1} \in R$ and $R[x] / \left(x^2 + x + 1\right)$ is an almost $pp-$ring.

Hence, RC_6 is an almost $pp-$ring if and only if $2^{-1} \in RC_3$, $3^{-1} \in R$ and $R[x] \big/ \big(x^2 + x + 1\big)$ is an almost $pp-$ring.

Now, $2^{-1} \in RC_3$ if and only if $2^{-1} \in R$. To see this, assume that $2^{-1} \in RC_3$. Then $2\left(\sum_{i=0}^{2} a_i g^i\right) = 1$ for some $a_i \in R$, $i = 0, 1, 2$. So, $2a_0 = 1$ and $2a_1 = 2a_2 = 0$. Thus, $2^{-1} \in R$. The converse is clear.

Therefore, we are done. □

Theorem 3.20. *Let R be a von Neumann regular ring and G be a locally finite group. Then the following are equivalent:*

(1) *RG is a $pp-$ring.*

(2) *RG is an almost $pp-$ring.*

(3) *The order of every finite subgroup of G is a unit in R.*

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) Since RG is an almost $pp-$ring, we have RH is an almost $pp-$ring for every subgroup H of G. So, if H is finite subgroup of G, then $|H|^{-1} \in R$.

(3) $\Longrightarrow$ (1) See [15, Proposition 1.9]. □

Lemma 3.21. *Let R_1 and R_2 be two integral domains, and let T be a non-integral domain subring of $R = R_1 \times R_2$ containing the identity element $(1, 1)$. Then T is an almost $pp-$ring if and only if $(0, 1) \in T$.*

Proof. Assume that T is an almost $pp-$ring. Since T is not an integral domain, there are non-zero elements $(a, b), (c, d) \in T$ such that $(a, b) . (c, d) = (0, 0)$. Since $(a, b) \neq (0, 0)$, either $a \neq 0$ or $b \neq 0$, say $a \neq 0$. Thus $c = 0$ and $d \neq 0$. Since $(c, d) \in Ann_T((a, b))$, there exists an idempotent $(x, y) \in Ann_T((a, b))$ such that $(c, d)(x, y) = (c, d)$. So, $dy = d$ and $d \neq 0$ in R_2. Thus $y = 1$. Since $xa = 0$ and $a \neq 0$ in R_1, $x = 0$.So, $(x, y) = (0, 1) \in T$.

Now, assume that $(0, 1) \in T$. Then, $(1, 1) - (0, 1) = (1, 0) \in T$. Consider any $(0, 0) \neq (a, b) \in T$. If $a \neq 0, b \neq 0$, $Ann_T((a, b)) = \{(0, 0)\}$. If $a = 0$, $b \neq 0$, $Ann_T((a, b)) = (1, 0)T$ and if $a \neq 0$, $b = 0$, $Ann_T((a, b)) = (0, 1)T$. Also if $a = b = 0$, then $Ann_T((a, b)) = T$ So, $Ann_T((a, b))$ is generated by its idempotents for all $(a, b) \in T$. Hence T is an almost $pp-$ring. □

Theorem 3.22. *Let R be an integral domain and let $Q(R)$ denotes the quotient field of R. Consider the polynomial $x^2 + a_1x + a_2 \in R[x]$ with α, β are its roots in some field extension, and $\alpha - \beta$ is a unit in R. Then $R[x] \big/ \big(x^2 + a_1x + a_2\big)$ is an almost $pp-$ring if and only if either $\alpha \in R$ or $\alpha \notin Q(R)$.*

Proof. Let $T = R[x] \big/ \big(x^2 + a_1x + a_2\big)$ and $x^2 + a_1x + a_2 = (x - \alpha)(x - \beta)$. By hypothesis, $\alpha \neq \beta$. First suppose $\alpha \notin Q(R)$. Then $x^2 + a_1x + a_2$ is irreducible over $Q(R)$ and hence it is irreducible over R since the polynomial is monic. Thus, T is an integral domain. In particular T is an almost $pp-$ring.

If $\alpha \in Q(R)$, then define $\Phi : R[x] \longrightarrow Q(R) \times Q(R)$ by $\Phi(f(x)) = (f(\alpha), f(\beta)) \in Q(R) \times Q(R)$. Then Φ is a ring homomorphisim with $Ker(\Phi) = \big(x^2 + a_1x + a_2\big)$. Hence, T is a subring of $Q(R) \times Q(R)$.

Assume now that T is an almost $pp-$ring, and so it follows by Lemma 3.21 that $(0,1) \in T$.

Thus there exists $ax+b \in R[x]$ such that $a\alpha + b = 0$ and $a\beta + b = 1$. But since $x^2 + a_1x + a_2 = (x-\alpha)(x-\beta)$, $\alpha + \beta = -a_1$ and $\alpha\beta = a_2$. So,

$$2(b-1)b = 2(-a\beta)(-a\alpha) = 2a^2\alpha\beta = 2a^2a_2.$$

Also,

$$2(b-1)b = (2b-2)b = -(1+a(\alpha+\beta)b = -(1-aa_1)b = (aa_1-1)b.$$

So,

$$2a^2a_2 = (aa_1-1)b.$$

Thus,

$$-b = 2a^2a_2 - aa_1b.$$

Hence

$$\alpha = -\frac{b}{a} = 2aa_2 - a_1b \in R.$$

So, $\alpha \in R$.

Now, assume that $\alpha \in R$,and define $p(x) = \dfrac{x-\alpha}{\beta-\alpha}$. Then $p(x) \in R[x]$, since $\beta - \alpha$ is a unit. But $\Phi(p(x)) = (p(\alpha), p(\beta) = (0,1)$. Thus it follows by Lemma 3.21 that T is an almost $pp-$ring. □

Example 3.23. Let $S = \left\{\frac{n}{3^k} : n,k \in \mathbb{Z},\ k \geq 0\right\}$. Then S is a subring of $\mathbb{Q}$. Set $R = \left\{a + \sqrt{3}b\mathbf{i} : a,b \in S\right\}$. Then R is a subring of $\mathbb{C}$ with $\frac{1}{3} \in R$. Because R is a domain, it is certainly almost $pp-$ring.

Let $f(x) = x^2 + x + 1 \in R[x]$. Then $\alpha = \dfrac{-1+\sqrt{3}\mathbf{i}}{2} \notin R$.

Let $r = 2\sqrt{3}\mathbf{i}$, $s = -\left(3+\sqrt{3}\mathbf{i}\right)$. Then $r,s \in R$ and $\alpha = \dfrac{s}{r} \in Q(R)$.

Since $(\alpha-\beta)^{-1} = \left(\sqrt{3}\mathbf{i}\right)^{-1} = -\dfrac{\sqrt{3}}{3}\mathbf{i} \in R$, $RC_3 \cong R \times \left(R[x]/\left(x^2+x+1\right)\right)$ is not an almost $pp-$ring.

The above example shows that RC_3 is not an almost $pp-$ring although 3 is a unit in R and R is an almost $pp-$ring.

4. Polynomial rings

Let R be a reduced ring and $h(x) = h_0 + h_1x + \dots + h_nx^n \in R[x]$. Then $Ann_{R[x]}(h(x)) = N[x]$, where N is the annihilator of the ideal generated by $h_0, h_1, \dots, h_n$ (i.e. $N = Ann_R(h_0, h_1, \dots, h_n) = \bigcap_{i=0}^{n} Ann_R(h_i)$). Moreover, if $f(x) = a_0 + a_1x + \dots + a_mx^m \in Ann_{R[x]}(h(x))$, then $a_ih_j = 0$, $\forall i = 0,1,\dots,m$, $j = 0,1,\dots,n$, see [6, Theorem 10].

Theorem 4.1. *If $R[x]$ or $R\left[x,x^{-1}\right]$ are almost pp$-$ rings, then so is R.*

Proof. $R[x]$ and $R[x,x^{-1}]$ are free $R-$modules with bases $\{x^i : i = 0,1,...\}$ and $\{x^i : i = 0,\pm1,...\}$, respectively, satisfying the assumptions of Theorem 2.5. □

Theorem 4.2. *The polynomial ring $R[x]$ is an almost pp$-$ring if and only if the ring R is an almost pp$-$ring.*

Proof. By Theorem 4.1, if $R[x]$ is an almost *pp*$-$ring then R is an almost *pp*$-$ring. So, let R be an almost *pp*$-$ring and $f(x) = a_0+a_1x+...+a_nx^n \in R[x]$. Then, since R is a reduced ring, $Ann_{R[x]}(f(x)) = N[x]$ where $N = \bigcap_{i=0}^{n} Ann_R(a_i)$.

Now, let $g(x) = b_0+b_1x+...+b_mx^m \in Ann_{R[x]}(f(x))$. Then, $b_j \in Ann_R(a_i)$ for all $j = 0,...,m$, $i = 0,...,n$. But since R is an almost *pp*$-$ring, there is an idempotent $e_{ji} \in Ann_R(a_i)$ such that $b_je_{ji} = b_j$ for all $j = 0,...,m$, $i = 0,...,n$. Taking $e_j = \prod_{i=0}^{n} e_{ji}$, we have $e_j \in N$ and $b_je_j = b_j$ for all $j = 0,...,m$. Taking $1-e = \prod_{j=0}^{m}(1-e_j)$, we have $e \in N$ is an idempotent and $b_je = b_j$ for all $j = 0,...,m$.

Thus $eg(x) = g(x)$ and $e \in N[x] = Ann_{R[x]}(f(x))$ and hence $R[x]$ is an almost *pp*$-$ring. □

The ring of Laurent polynomials $R\left[x,x^{-1}\right]$ is the localization of the polynomial ring at the multiplicative set consisting of the non negative powers of x. The ring R is an almost *pp*$-$ring if and only if for every $R \subset S \subset Q(R)$, S is an almost *pp*$-$ring where $Q(R)$ is the total quotient ring of R, see [4, Theorem 3].

So, by Theorem 4.2 and Theorem 4.1, $R\left[x,x^{-1}\right]$ is an almost *pp*$-$ring if and only if R is an almost *pp*$-$ring.

In fact, the Laurent polynomial ring $R\left[x,x^{-1}\right]$ is isomorphic to the group ring of the group $\mathbb{Z}$ of integers over R. Thus, $R\mathbb{Z}$ is an almost *pp*$-$ring if and only if R is an almost *pp*$-$ring.

5. Power series rings

Let $R[[x]]$ be the power series ring over the ring R. For any reduced ring R, it was proved in Brewer [6] that $Ann_{R[[x]]}(a_0+a_1x\ldots) = N[[x]]$ where N is the annihilator of the ideal generated by the coefficients $a_0, a_1, ...$ Moreover if $b_0+b_1x+... \in Ann_{R[[x]]}(a_0+a_1x\ldots)$, then $b_ia_j = 0$ for all $i = 0,1,...$; $j = 0,1,\ldots$

Theorem 5.1. *If $R[[x]]$ is an almost pp$-$ring, then so is R.*

Proof. $R[[x]]$ is free $R-$module with basis $\left\{x^i : i = 0,1,...\right\}$ satisfying the assumption of Theorem 2.5. □

Theorem 5.2. *The power series $R[[x]]$ is an almost pp$-$ring if and only if for any two countable sets $S = \{b_0,b_1,...\}$ and $T = \{a_0,a_1,...\}$ such that $S \subseteq$*

$Ann_R(T)$, there exists an idempotent $e \in Ann_R(T)$ such that $b_i = b_i e$ for all $i = 0, 1, ...$

Proof. Assume that $R[[x]]$ is an almost $pp-$ring.

Let $\{b_0, b_1, ...\} \subseteq Ann_R(a_0, a_1, ...)$. Let $g(x) = b_0 + b_1x + ...$ and $f(x) = a_0 + a_1x \ldots$ Then $g(x) \in Ann_{R[[x]]}(f(x))$. Therefore, there exists an idempotent $e \in Ann_{R[[x]]}(f(x)) \cap R$ such that $eg(x) = g(x)$. Thus $e \in Ann_R(a_0, a_1, ...)$ and $eb_i = b_i$, for all $i = 0, 1, ...$

Conversely, the ring R is an almost $pp-$ring because for all $b \in Ann_R(a)$ there exists an idempotent $e \in Ann_R(a)$ such that $be = b$, and so, R is a reduced ring.

Let $g(x) = b_0 + b_1x + ...$, $f(x) = a_0 + a_1x \ldots \in R[[x]]$ such that $g(x) \in Ann_{R[[x]]}(f(x))$. Then $g(x) f(x) = 0$.

Thus $b_i a_j = 0$ for all $i = 0, 1, ...$; $j = 0, 1, ...$

So, $\{b_0, b_1, ...\} \subseteq Ann_R(a_0, a_1, ...)$. By assumption, there exists an idempotent $e \in Ann_R(a_0, a_1, ...)$ such that $eb_i = b_i$ for all $i = 0, 1, ...$

Hence $eg(x) = g(x)$ and $e \in Ann_{R[[x]]}(f(x))$.

Thus $R[[x]]$ is an almost $pp-$ring. □

Example 5.3. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$ be the direct sum. Then R is an almost $pp-$ring since every element in R is an idempotent. But $R[[x]]$ is not an almost $pp-$ring, because $S = \{(0, 1, 0, ..), (0, 0, 1, 0, ...), (0, 0, 0, 1, 0, ...), ...\} \subseteq Ann_R((1, 0, 0, ...))$ and there are no idempotent elements in R that can fix the set S.

Corollary 5.4. *If R has a finite number of idempotents, then $R[[x]]$ is an almost pp−ring if and only if R is an almost pp−ring.*

Proof. If $R[[x]]$ is an almost $pp-$ring, then by Theorem 5.1 R is an almost $pp-$ring.

Conversely, assume that R is an almost $pp-$ring and let $g(x) = b_0 + b_1x + ...$, $f(x) = a_0 + a_1x \ldots \in R[[x]]$ such that $g(x) \in Ann_{R[[x]]}(f(x))$. Since R is reduced ring, then $bi \in Ann_R(a_j)$ for all $i = 0, 1, ...$; $j = 0, 1, \ldots$

So, there exists an idempotent $e_{ji} \in Ann_R(a_j)$ such that $e_{ji} b_i = b_i$ for all $i = 0, 1, ...$; $j = 0, 1, \ldots$ But R has a finite number of idempotent and so, we can find an idempotent $e \in Ann_R(a_j)$ such that $eb_i = b_i$ for all $i = 0, 1, ...$; $j = 0, 1, \ldots$ Hence, $eg(x) = g(x)$ and $e \in Ann_{R[[x]]}(f(x))$.

Thus, $R[[x]]$ is an almost $pp-$ring. □

Kim in [11] proved that if R is a Noetherian ring, then $R[[x]]$ is a $pp-$ring if and only if R is a $pp-$ring, see [11, Theorem 4]. We now give an analogue result for almost $pp-$rings.

Corollary 5.5. *If R is a Noetherian ring, then $R[[x]]$ is an almost pp−ring if and only if R is an almost pp−ring.*

Acknowledgements

This paper is a part of Ph.D. thesis prepared by the first author, under supervision of the other authors, in The University of Jordan.

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Accepted: 15.12.2018

Stability of fixed point sets of generalized multivalued α-ψ contraction of Ciric-Berinde type

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Abstract. The purpose of our paper is to study the existence of fixed point theorems for generalized multivalued α-ψ contraction of Ciric-Berinde type by using Hausdorff distance in metric spaces and obtain stability of fixed point sets for such multivalued contraction. Examples are providing to indicate the usefulness of our main result. Moreover, an application to single value mapping is also given.

Keywords: fixed point, multivalued α-admissible, generalized multivalued α-ψ contraction of Ciric-Berinde type, h-upper semicontinuous, Hausdorff metric, stability.

1. Introduction and preliminaries

Stability of fixed points of set valued contractions was defined by Nadler [14] and Markin [10]. The stability results for multivalued contractions have been useful in the area of generalized differential equation, discrete and continuous dynamical system. Stability of fixed points sets for multivalued mapping has been considered in ([6], [11], [13], [15], [18]).

In 2012, Samet et al. [4] introduced the notion of α-ψ contractive mappings and α-admissible mappings in metric space and gave sufficient condition for the existence of fixed points for the class of mappings. Many authors discussed the fixed point results of α-admissible mappings and gave their generalization, extensions in several works like ([1, 2, 3], [5], [12], [16], [17]). Recently, Chaudury and Bandyopadhyay [7] defined multivalued α-admissible mappings, multivalued α-ψ contractions mappings and obtained some stability results for fixed point sets associated with a sequence of multivalued mappings using Hausdorff distance in metric space.

The purpose of this paper is to introduce the concept of generalized multivalued α-ψ contraction of Ciric-Berinde type and to establish fixed point theorems for such mappings which generalizes the results of ([7], [8]). We also show that the fixed point sets of uniformly convergent sequences for the newly defined generalized multivalued α-ψ contraction of Ciric-Berinde type which are also α-admissible and h-upper semicontinuous are stable under certain condition.

Lastly, we obtain fixed point results of single valued mappings by giving applications of our main results of multivalued mappings.

Let X and Y be non-empty sets. T is said to be a multivalued mapping from X to Y if T is a function from X to the power set of Y. We denote a multivalued mapping by $T : X \to 2^y$. A point $x \in X$ is said to be a fixed point of multivalued mapping T if $x \in Tx$. We denote the set of fixed points of T by $Fix(T)$.

The following are the concepts from set valued analysis which we shall use in this paper. Let (X, d) be a metric space. Then

$$N(X) = \{A : A \text{ is a non-empty subset of } X\},$$
$$CL(X) = \{A : A \text{ is a non-empty closed subset of } X\},$$
$$C(X) = \{A : A \text{ is a non-empty compact subset of } X\} \text{ and}$$
$$CB(X) = \{A : A \text{ is a non-empty closed and bounded subset of } X\}.$$

For $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \to R^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(a, B), a \in A\}, \quad \delta(B, A) = \sup\{d(b, A), b \in B\}$$

and

$$d(a, C) = \inf\{d(a, x), x \in C\}.$$

Note that H is called Hausdorff metric induced by the metric d.

Let $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ be two functions such that ψ is a continuous and non-decreasing function with $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\Phi(t) \to 0$ as $t \to 0$, where ψ^n denotes nth iterate of the function ψ. It is well known that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ for $t = 0$.

Lemma 1.1 ([14])**.** *Let (X, d) be a metric space and $A, B \in C(X)$. Let $q \geq 1$. Then for each $x \in A$, there exists $y \in B$ such that $d(x, y) \leq qH(A, B)$.*

Lemma 1.2 ([8])**.** *Let A and B be two non-empty compact subsets of a metric space (X, d) and $T : A \to C(B)$ be a multivalued mapping. Let $q \geq 1$. Then for $a, b \in A$ and $x \in Ta$, there exists $y \in Tb$ such that $d(x, y) \leq qH(Ta, Tb)$.*

Definition 1.1 ([7],[12])**.** Let X be a non-empty set. A multivalued mapping $T : X \to N(X)$ is said to be multivalued α-admissible with respect to a function $\alpha : X \times X \to [0, \infty)$, if for $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(a, b) \geq 1, \quad \text{for all } a \in Tx \text{ and } b \in Ty. \tag{1.1}$$

If $T : X \to X$, a single-valued mapping then condition (1.1) of α-admissible reduces to $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ for $x, y \in X$.

Example 1.1 ([7]). Let $X = R$, $\alpha : R\times R \to [0,\infty)$. We define $\alpha(x,y) = x^2+y^2$, where $x, y \in R$. Define $T : R \to N(X)$ by $Tx = \{\sqrt{|x|}, -\sqrt{|x|}\}$. Then T is multivalued α-admissible.

Definition 1.2 ([9]). Let (X,d) be a metric space and $T : X \to 2^X$ be a closed valued multifunction. We say that T is an α_*-ψ contractive multifunction whenever

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le \psi(d(x,y)) \text{ for } x,y \in X, \tag{1.2}$$

where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$.

Definition 1.3 ([7]). Let (X,d) be a metric space. A multivalued mapping $T : X \to C(X)$ is called multivalued α-ψ contraction if

$$\alpha(x,y)H(Tx,Ty) \le \psi(d(x,y)) \text{ for } x,y \in X, \tag{1.3}$$

Remark 1.1 ([7]). In (1.3) of Definition 1.3, we have $\alpha(x,y)$ instead of $\alpha_*(Tx,Ty)$ which has been considered in (1.2) of Definition 1.2. $\alpha_*(Tx,Ty)$ is defined as

$$\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\} \text{ for } x,y \in X.$$

From the definition it is clear that $\alpha_*(Tx,Ty)$ is not necessarily equal to $\alpha(x,y)$ and also we cannot compare $\alpha(x,y)$ with $\alpha_*(Tx,Ty)$. Therefore Definition 1.3 is independent of Definition 1.2.

Definition 1.4. Let (X,d) be a metric space. A self mapping $T : X \to X$ is said to be h-upper semicontinuous if and only if, for each $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n,x) = 0$, we have $\lim_{n\to\infty} d(Tx_n,Tx) = 0$.

Definition 1.5 ([12], [17]). Let (X,d) be a metric space. A multivalued mapping $T : X \to C(X)$ is said to be h-upper semicontinuous if and only if, for each $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n,x) = 0$, we have $\lim_{n\to\infty} \delta(Tx_n,Tx) = 0$.

2. Main results

We introduce *generalized multivalued α-ψ contraction of Ciric-Berinde type* which differs from Definition 1.2 and generalization of Definition 1.3.

Definition 2.1. Let (X,d) be a metric space. A multivalued mapping $T : X \to C(X)$ is called a generalized multivalued α-ψ contraction of Ciric-Berinde type if

$$\begin{aligned}\alpha(x,y)H(Tx,Ty) &\le \psi(M(x,y)) \\ &+ L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\end{aligned} \tag{2.1}$$

for all $x,y \in X$ with $\alpha(x,y) \ge 1$, where $L \ge 0$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be a generalized multivalued α-ψ contraction of Ciric-Berinde type. Also suppose that the following conditions are satisfied:*

(i) *T is multivalued α-admissible,*

(ii) *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,*

(iii) *T is h-upper semi continuous.*

Then T has a fixed point.

Proof. By the condition (ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Clearly if $x_0 = x_1$ or $x_1 \in Tx_1$, we find that x_1 is a fixed point of T and so, we can conclude the proof. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$ and hence $d(x_1, Tx_1) > 0$. By Lemma 1.2 for $x_1 \in Tx_0$ there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1).$$

Applying (2.1) and using the monotone property of ψ, we have

$$\begin{aligned}
d(x_1, x_2) &\leq \alpha(x_0, x_1)H(Tx_0, Tx_1) \\
&\leq \psi\left(\max\left\{d(x_0, x_1), \frac{d(x_0, Tx_0)d(x_1, Tx_1)}{d(x_0, x_1)}, \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right) \\
&\quad + L\min\{d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0)\} \\
&\leq \psi\left(\max\left\{d(x_0, x_1), \frac{d(x_0, x_1)d(x_1, x_2)}{d(x_0, x_1)}, \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\}\right) \\
&\quad + L\min\{d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), d(x_1, x_1)\} \\
&\leq \psi\left(\max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\}\right).
\end{aligned}$$

Since

$$\frac{d(x_0, x_2)}{2} \leq \frac{d(x_0, x_1) + d(x_1, x_2)}{2} \leq \max\{d(x_0, x_1), d(x_1, x_2)\},$$

it follow that

$$d(x_1, x_2) \leq \psi(\max\{d(x_0, x_1), d(x_1, x_2)\}),$$

if $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$, then we have

$$0 < d(x_1, x_2) \leq \psi(d(x_1, x_2)) < d(x_1, x_2),$$

which is a contradiction. Thus $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$ and since ψ is strictly increasing, we have

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)). \tag{2.2}$$

Since T is α-admissible, from condition (ii) and $x_2 \in Tx_1$, we have $\alpha(x_1, x_2) \geq 1$. If $x_2 \in Tx_2$ then x_2 is a fixed point. Assume that $x_2 \notin Tx_2$, that is $d(x_2, Tx_2) > 0$. By Lemma 1.2 for $x_2 \in Tx_1$ there exists $x_3 \in Tx_2$ such that

$$\begin{aligned}
d(x_2, x_3) &\leq \alpha(x_1, x_2) H(Tx_1, Tx_2) \\
&\leq \psi\left(\max\left\{d(x_1, x_2), \frac{d(x_1, Tx_1)d(x_2, Tx_2)}{d(x_1, x_2)}, \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right\}\right) \\
&\quad + L\min\{d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, Tx_1)\} \\
&\leq \psi\left(\max\left\{d(x_1, x_2), \frac{d(x_1, x_2)d(x_2, x_3)}{d(x_1, x_2)}, \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\}\right) \\
&\quad + L\min\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3), d(x_2, x_2)\} \\
&\leq \psi\left(\max\left\{d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3)}{2}\right\}\right).
\end{aligned}$$

Since

$$\frac{d(x_1, x_3)}{2} \leq \frac{d(x_1, x_2) + d(x_2, x_3)}{2} \leq \max\{d(x_1, x_2), d(x_2, x_3)\},$$

it follow that

$$d(x_2, x_3) \leq \psi(\max\{d(x_1, x_2), d(x_2, x_3)\}),$$

if $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_2, x_3)$, then we have

$$0 < d(x_2, x_3) \leq \psi(d(x_2, x_3)) < d(x_2, x_3),$$

which is a contradiction.

Thus $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_1, x_2)$ and since ψ is strictly increasing, we have

$$d(x_2, x_3) \leq \psi(d(x_1, x_2)) < \psi^2(d(x_0, x_1)) \tag{2.3}$$

Since $x_2 \in Tx_1$, $x_3 \in Tx_2$ and $\alpha(x_1, x_2) \geq 1$, the α-admissibility of T implies that $\alpha(x_2, x_3) \geq 1$. Continuing this process, we construct a sequence $\{x_n\}$ such that for all $n \geq 0$,

$$\begin{aligned}
&x_n \notin Tx_n, x_{n+1} \in Tx_n, \\
&\alpha(x_n, x_{n+1}) \geq 1
\end{aligned}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})). \tag{2.4}$$

By repeated application (2.4) and monotonic property of ψ, we have

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \psi^2(d(x_{n-1}, x_n)) \leq \cdots \leq \psi^{n+1}(d(x_0, x_1)).$$

Then by property of ψ, we have

$$\sum_n d(x_n, x_{n+1}) \le \sum_n \psi^n(d(x_0, x_1) = \Phi(d(x_0, x_1)) < \infty\,.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence, there exists $z \in X$, such that $x_n \to z$ as $n \to \infty$, that is $\lim_{n\to\infty} d(x_n, z) = 0$.

Consider

$$\begin{aligned} d(z, Tz) &\le d(z, x_{n+1}) + d(x_{n+1}, Tz) \\ &\le d(z, x_{n+1}) + \delta(Tx_n, Tz). \end{aligned} \tag{2.5}$$

Since $\lim_{n\to\infty} d(z, x_{n+1}) = 0$ and by using h-upper semicontinuity of T we have $\lim_{n\to\infty} \delta(Tx_n, Tz) = 0$. By letting $n \to \infty$ in the inequality (2.5), we obtain $d(z, Tz) = 0$. Since Tz is compact and hence Tz is closed, that is, $Tz = \overline{Tz}$, where $\overline{Tz}$ denotes the closure of Tz. Now $d(z, Tz) = 0$ implies that $z \in \overline{Tz} = Tz$, that is, z is a fixed point of T. □

Notice that one can relax the h-upper semicontinuity hypothesis on T, by introducing another regularity condition as shown in next theorem.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be a generalized multivalued α-ψ contraction of Ciric-Berinde type. Also suppose that the following conditions are satisfied:*

(i) T is multivalued α-admissible,

(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$,

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n, where $x_{n+1} \in Tx_n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all n.

Then T has a fixed point.

Proof. By the condition (ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$. Following the proof of Theorem 2.1, we obtain a sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, z) = 0$ for some $z \in X$ such that $x_n \notin Tx_n, x_{n+1} \in Tx_n$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all n. By condition (iii), there exists a sequence $\{x_n\}$ such that $\alpha(x_n, z) \ge 1$ for all n. Now we prove that $z \in Tz$. We have

$$\begin{aligned} M(x_n, z) &= \max\left\{d(x_n, z), \frac{d(x_n, Tx_n)d(z, Tz)}{d(x_n, z)}, \frac{d(x_n, Tz) + d(z, Tx_n)}{2}\right\} \\ &\le \max\left\{d(x_n, z), \frac{d(x_n, x_{n+1})d(z, Tz)}{d(x_n, z)}, \frac{d(x_n, Tz) + d(z, x_{n+1})}{2}\right\} \\ &\le \max\left\{d(x_n, z), \frac{d(x_n, x_{n+1})d(z, Tz)}{d(x_n, z)}, \frac{[d(x_n, z)+d(z, Tz)]+d(z, x_{n+1})}{2}\right\}. \end{aligned}$$

From $x_n \to z$, we deduce that

$$\lim_{n\to\infty} M(x_n, z) = d(z, Tz).$$

Since T is generalized multivalued α-ψ contraction of Ciric-Berinde type, for all n we have

$$\begin{aligned} d(z,Tz) &\le d(z,x_{n+1}) + d(x_{n+1},Tz) \le d(z,x_{n+1}) + H(Tx_n,Tz) \\ &\le d(z,x_{n+1}) + \psi(M(x_n,z)) \\ &\quad + L\min\{d(x_n,Tx_n), d(z,Tz), d(x_n,Tz), d(z,Tx_n)\} \\ &\le d(z,x_{n+1}) + \psi(M(x_n,z)) \\ &\quad + L\min\{d(x_n,x_{n+1}), d(z,Tz), d(x_n,Tz), d(z,x_{n+1})\}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality and $\psi(t) < t$, we have

$$d(z,Tz) \le \psi(d(z,Tz)) < d(z,Tz)$$

which implies $d(z,Tz) = 0$. Since Tz is compact and hence Tz is closed, that is, $Tz = \overline{Tz}$, where $\overline{Tz}$ denotes the closure of Tz. Now $d(z,Tz) = 0$ implies that $z \in \overline{Tz} = Tz$, that is, z is a fixed point of T. □

Remark 2.1. By taking $L = 0$ in Theorem 2.2, replacing $M(x,y)$ by $d(x,y)$, Theorem 2.2 reduces to [7, Theorem 2.1] which is as follows:

Corollary 2.1 ([7])**.** *Let (X,d) be a complete metric space and $T : X \to CL(X)$ be a multivalued α-ψ contraction. Also suppose that the following conditions are satisfied:*

(i) T is multivalued α-admissible,

(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$,

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all n, where $x_{n+1} \in Tx_n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x) \ge 1$ for all n.

Then T has a fixed point.

Example 2.1. Let $X = [0,\infty)$ and $d(x,y) = |x-y|$. Let define the multivalued mapping $T : X \to C(X)$ as

$$Tx = \begin{cases} \left\{1, \frac{1}{2x}\right\}, & \text{if } x > 1, \\ \left\{0, \frac{x}{8}\right\}, & \text{if } 0 \le x \le 1. \end{cases}$$

Now we define the functions $\alpha : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 2, & \text{if } x,y \in (0,1], \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi(t) = \frac{1}{2}t.$$

Obviously, conditions (ii) of Theorem is satisfied with $x_0 = \frac{1}{2}$ and T is multivalued α-admissible. Now we show that T is generalized multivalued α-ψ contraction of Ciric-Berinde type.

Taking for $x, y \in [0,1]$, we have

$$\begin{aligned}
d(0,Ty) &= \inf\left\{0, \frac{y}{8}\right\} = 0,\\
d\left(\frac{x}{8}, Ty\right) &= \inf\left\{\left|0 - \frac{x}{8}\right|, \left|\frac{x}{8} - \frac{y}{8}\right|\right\},\\
d(0,Tx) &= \inf\left\{0, \frac{x}{8}\right\} = 0,\\
d\left(\frac{y}{8}, Tx\right) &= \inf\left\{\left|0 - \frac{x}{8}\right|, \left|\frac{x}{8} - \frac{y}{8}\right|\right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
H(Tx,Ty) &= \max\left\{\sup_{x\in Tx} d(x,Ty), \sup_{y\in Ty} d(y,Tx)\right\}\\
&= \max\left\{\inf\left\{\left|\frac{x}{8}\right|, \left|\frac{x}{8} - \frac{y}{8}\right|\right\}, \inf\left\{\left|\frac{y}{8}\right|, \left|\frac{y}{8} - \frac{x}{8}\right|\right\}\right\}\\
&= \left|\frac{x}{8} - \frac{y}{8}\right|.
\end{aligned}$$

Now,

$$\begin{aligned}
\alpha(x,y)H(Tx,Ty) &= 2 \times \left|\frac{x}{8} - \frac{y}{8}\right|\\
&= \frac{1}{4}|x-y|\\
&\leq \frac{1}{2}|x-y|\\
&= \psi(d(x,y)\\
&\leq \psi(M(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.
\end{aligned}$$

Thus condition (2.1) is satisfied.

Hence all the conditions of Theorems 2.1 and 2.2 are satisfied and T has a fixed point at $x = 0$.

3. Stability of fixed point sets

Stability is a concept associated with the limiting behavior of a system. The study of the relationship between the convergence of a sequence of mappings and their fixed points, known as the stability of fixed points. A sequence of fixed point sets is said to be stable when it converges to the corresponding fixed point sets of the limiting function in the Hausdorff metric. Multivalued maps often have more fixed points than single valued maps. Therefore, the set of fixed points of multivalued mappings becomes larger and hence more interesting for study of stability.

In this section, we consider the stability of fixed point sets of the multivalued contractions mentioned in section 2.

Theorem 3.1. *Let (X,d) be a complete metric space, and $F(T_1)$, $F(T_2)$ are the fixed point sets of T_1 and T_2 respectively, where $T_i : X \to C(X)$, $i = 1,2$. Each T_i is generalized multivalued α-ψ contraction of Ciric-Berinde type as defined in Definition 2.1 with the same α, ψ and L. Also each T_i satisfies the following conditions:*

(i) *each T_i is multivalued α-admissible,*

(ii) *each T_i is h-upper semi continuous,*

(iii) *for any $x \in F(T_1)$, we have $\alpha(x,y) \geq 1$ whenever $y \in T_2x$ and for any $x \in F(T_2)$, we have $\alpha(x,y) \geq 1$ whenever $y \in T_1x$.*

Then $H(F(T_1), F(T_2)) \leq \Phi(k)$, where $k = \sup_{x\in X} H(T_1x, T_2x)$.

Proof. From Theorem 2.1, the set of fixed points of T_i $(i = 1,2)$ are non-empty, that is $F(T_i) \neq \phi$, for $i = 1,2$. Let $x_0 \in F(T_1)$, that is $x_0 \in T_1x_0$. Then by Lemma 1.1, there exists $x_1 \in T_2x_0$ such that

$$d(x_0, x_1) \leq H(T_1x_0, T_2x_0).$$

Since $x_0 \in F(T_1)$ and $x_1 \in T_2x_0$, by condition(iii) of the theorem, we have $\alpha(x_0, x_1) \geq 1$. By Lemma 1.2, for $x_1 \in T_2x_0$ there exists $x_2 \in T_2x_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)H(T_2x_0, T_2x_1).$$

Then, arguing similarly as in the proof of Theorem 2.1, we construct a sequence $\{x_n\}$ such that for all $n \geq 0$,

$$\begin{aligned} &x_{n+1} \in T_2x_n, \\ &\alpha(x_n, x_{n+1}) \geq 1, \\ &d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \end{aligned}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \psi^2(d(x_{n-1}, x_n)) \leq \ldots \leq \psi^{n+1}(d(x_0, x_1)).$$

Then by property of ψ, we have

$$\sum_n d(x_n, x_{n+1}) \leq \sum_n \psi^n(d(x_0, x_1)) = \Phi(d(x_0, x_1)) < \infty\,.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, we prove that $z \in T_2z$. For all $n \geq 0$, $x_{n+1} \in T_2x_n$. Suppose that T_2 is h-upper semicontinuous. We have

$$\begin{aligned} d(z, T_2z) &\leq d(z, x_{n+1}) + d(x_{n+1}, T_2z) \\ &\leq d(z, x_{n+1}) + \delta(T_2x_n, T_2z). \end{aligned}$$

Since $\lim_{n\to\infty} d(z, x_{n+1}) = \lim_{n\to\infty} \delta(T_2x_n, T_2z) = 0$, by letting $n \to \infty$ in the above inequality, we obtain $d(z, T_2z) = 0$. Thus $z \in T_2z$, that is, z is a fixed point of T_2.

Using $d(x_0, x_1) \leq H(T_1x_0, T_2x_0)$ and the definition of k, we have

$$d(x_0, x_1) \leq H(T_1x_0, T_2x_0) \leq k = \sup_{x\in X} H(T_1x, T_2x).$$

Now using triangular inequality,

$$\begin{aligned} d(x_0, z) &\leq \sum_{i=0}^{n} (d(x_i, x_{i+1})) + d(x_{n+1}, z) \\ &\leq \sum_{i=0}^{\infty} (d(x_i, x_{i+1})) \\ &\leq \sum_{i=0}^{\infty} \psi^i(d(x_0, x_1)) \\ &\leq \sum_{i=0}^{\infty} \psi^i(k) = \Phi(k). \end{aligned}$$

Thus, given arbitrary $x_0 \in F(T_1)$, we can find $z \in F(T_2)$ for which $d(x_0, z) \leq \Phi(k)$. Reversing the roles of T_1 and T_2 we also conclude that for each $y_0 \in F(T_2)$ there exists $w \in F(T_1)$ such that $d(y_0, w) \leq \Phi(k)$. Hence $H(F(T_1), F(T_2)) \leq \Phi(k)$. □

Theorem 3.2. *Let (X, d) be a complete metric space, and $F(T_1)$, $F(T_2)$ are the fixed point sets of T_1 and T_2 respectively, where $T_i : X \to C(X)$, $i = 1, 2$. Each T_i is generalized multivalued α-ψ contraction of Ciric-Berinde type as defined in Definition 2.1 with the same α, ψ and L. Also each T_i satisfies the following conditions:*

- *(i) for any $x \in F(T_1)$, we have $\alpha(x, y) \geq 1$ whenever $y \in T_2x$ and for any $x \in F(T_2)$, we have $\alpha(x, y) \geq 1$ whenever $y \in T_1x$.*
- *(ii) each T_i is multivalued α-admissible,*
- *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n, where $x_{n+1} \in T_ix_n$, $i = 1, 2$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then $H(F(T_1), F(T_2)) \leq \Phi(m)$, where $m = \sup_{x\in X} H(T_1x, T_2x)$.

Proof. From Theorem 2.2, the set of fixed points of T_i $(i = 1, 2)$ are non empty, that is $F(T_i) \neq \phi$, for $i = 1, 2$. Let $x_0 \in F(T_1)$, that is $x_0 \in T_1x_0$. Arguing similarly as in the proof of Theorem 3.1, we prove that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exist $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, we prove that $z \in T_2z$. For all $n \geq 0$, $x_{n+1} \in T_2x_n$. Therefore $d(x_{n+1}, T_2z) \leq H(T_2x_n, T_2z)$. By (iii), $\alpha(x_n, z) \geq 1$ for all n. Hence we have for all n,

$$d(x_{n+1}, T_2z) \leq \alpha(x_n, z)H(T_2x_n, T_2z).$$

Letting $n \to \infty$ in the above inequality and $\psi(t) < t$, we have

$$d(z, T_2z) \leq \psi(d(z, T_2z)) \leq d(z, T_2z),$$

which implies that $d(z, T_2z) = 0$. Thus $z \in T_2z$, that is, z is a fixed point of T_2.

Using $d(x_0, x_1) \leq H(T_1x_0, T_2x_0)$ and the definition of k, we have

$$d(x_0, x_1) \leq H(T_1x_0, T_2x_0) \leq m = \sup_{x \in X} H(T_1x, T_2x).$$

Now using triangular inequality,

$$\begin{aligned} d(x_0, z) &\leq \sum_{i=0}^{n}(d(x_i, x_{i+1})) + d(x_{n+1}, z) \\ &\leq \sum_{i=0}^{\infty}(d(x_i, x_{i+1})) \\ &\leq \sum_{i=0}^{\infty} \psi^i(d(x_0, x_1)) \\ &\leq \sum_{i=0}^{\infty} \psi^i(k) = \Phi(m). \end{aligned}$$

Thus, given arbitrary $x_0 \in F(T_1)$, we can find $z \in F(T_2)$ for which $d(x_0, z) \leq \Phi(m)$. Reversing the role of T_1 and T_2 we also conclude that for each $y_0 \in F(T_2)$ there exists $w \in F(T_1)$ such that $d(y_0, w) \leq \Phi(m)$. Hence $H(F(T_1), F(T_2)) \leq \Phi(m)$. □

Lemma 3.1 ([7])**.** *Let (X, d) be a complete metric space. If $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued α-admissible with the same α and which is uniformly convergent to a multivalued mapping $T : X \to C(X)$, then T is multivalued α-admissible if the following condition is satisfied:*

$$\alpha(x_n, y_n) \geq 1, \quad \textit{for every } n \in \mathbb{N} \Rightarrow \alpha(a, b) \geq 1, \tag{3.1}$$

where $x_n \to a$ and $y_n \to b$ as $n \to \infty$.

Proof. Let $\alpha(x,y) \geq 1$, for some $x, y \in X$. Suppose $a \in Tx$ and $b \in Ty$ be arbitrary. Since $T_n \to T$ uniformly, there exist two sequences $\{x_n\}$ in $\{T_n x\}$ and $\{y_n\}$ in $\{T_n y\}$, such that $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Since $\alpha(x,y) \geq 1$ and T_n is multivalued α-admissible for each $n \in \mathbb{N}$, it follows that $\alpha(x_n, y_n) \geq 1$ for each $n \in \mathbb{N}$. Thus by (3.1), it follows that $\alpha(a,b) \geq 1$. Thus for $x, y \in X$, we have

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(a,b) \geq 1, \quad \text{where } a \in Tx \text{ and } b \in Ty.$$

Hence T is multivalued α-admissible. □

Lemma 3.2. *Let (X,d) be a complete metric space. If $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ be a sequence of generalized multivalued α-ψ contraction of Ciric-Berinde type which is uniformly convergent to a multivalued mapping $T : X \to C(X)$, then T is generalized multivalued α-ψ contraction of Ciric-Berinde type.*

Proof. Since each T_n is generalized multivalued α-ψ contraction of Ciric-Berinde type for every $n \in \mathbb{N}$, therefore

$$\begin{aligned}
&\alpha(x,y)H(T_n x, T_n y) \\
&\leq \psi\left(\max\left\{d(x,y), \frac{d(x,T_n x)d(y,T_n y)}{d(x,y)}, \frac{d(x,T_n y) + d(y,T_n x)}{2}\right\}\right) \\
&\quad + L\min\{d(x,T_n x), d(y,T_n y), d(x,T_n y), d(y,T_n x)\}.
\end{aligned}$$

Since the sequence $\{T_n\}$ is uniformly convergent to T and ψ is continuous, taking limit $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
\alpha(x,y)H(Tx,Ty) &\leq \psi\left(\max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \frac{d(x,Ty) + d(y,Tx)}{2}\right\}\right) \\
&\quad + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},
\end{aligned}$$

for all $x, y \in X$. Hence T is generalized multivalued α-ψ contraction of Ciric-Berinde type. □

Lemma 3.3. *Let (X,d) be a complete metric space. If $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ be a sequence of h-upper semicontinuous which is uniformly convergent to a multivalued mapping $T : X \to C(X)$, then T is h-upper semicontinuous mapping.*

Proof. Since each T_n is h-upper semicontinuous for all $n \geq 1$. Then by definition of h-upper semicontinuous mapping for each $x \in X$ and $x_n \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$, we have

$$\lim_{n\to\infty} \delta(T_n(x_n), T_n(x)) = 0.$$

Since $T_n \to T$ uniformly, letting $n \to \infty$, we have for each $x \in X$ and $x_n \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} \delta(Tx_n, Tx) = 0$, which implies that T is h-upper semicontinuous mapping. □

Theorem 3.3. *Let* (X, d) *be a complete metric space. Let* $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ *be a sequence of generalized multivalued* α-ψ *contraction of Ciric-Berinde type which are also multivalued* α*-admissible and* h*-upper semicontinuous with the same* α, ψ *and* L *is uniformly convergent to a multivalued mapping* $T : X \to C(X)$. *Also suppose that the following hold:*

(*i*) *if* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences in* X *with* $x_n \to a$ *and* $y_n \to b$ *as* $n \to \infty$, *then* $\alpha(x_n, y_n) \geq 1$, *for every* $n \in \mathbb{N} \Rightarrow \alpha(a, b) \geq 1$,

(*ii*) *for every* $n \in \mathbb{N}$, *for any* $x \in F\{T_n\}$, *we have* $\alpha(x, y) \geq 1$ *whenever* $y \in Tx$, *and for any* $x \in F(T)$, *we have* $\alpha(x, y) \geq 1$ *whenever* $y \in T_n x$.

Then $H(F(T_n), F(T)) \to 0$ *as* $n \to \infty$, *that is, the fixed point of* T_n *are stable.*

Proof. By Lammas 3.1, 3.2 and 3.3, it follows that T is generalized multivalued α-ψ contraction of Ciric-Berinde type, multivalued α-admissible and h-upper semicontinuous. Let $k_n = \sup_{x \in X} H(T_n x, Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to T on X. Therefore

$$\lim_{n \to \infty} k_n = \lim_{n \to \infty} \sup_{x \in X} H(T_n x, Tx) = 0.$$

Using Theorem 3.1, we get

$$H(F(T_n), F(T)) \leq \Phi(k_n), \quad \text{for every } n \in \mathbb{N}.$$

Since ψ is continuous and $\Phi(t) \to 0$ as $t \to 0$, we have

$$\lim_{n \to \infty} H(F(T_n), F(T)) \leq \lim_{n \to \infty} \Phi(k_n) = 0,$$

that is $\lim_{n \to \infty} H(F(T_n), F(T)) = 0$. Hence the proof is complete. □

Theorem 3.4. *Let* (X, d) *be a complete metric space. Let* $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ *be a sequence of generalized multivalued* α-ψ *contraction of Ciric type which are also multivalued* α*-admissible with the same* α, ψ *and* L *is uniformly convergent to a multivalued mapping* $T : X \to C(X)$. *Also suppose that the following hold:*

(*i*) *if* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences in* X *with* $x_n \to a$ *and* $y_n \to b$ *as* $n \to \infty$, *then* $\alpha(x_n, y_n) \geq 1$, *for every* $n \in \mathbb{N} \Rightarrow \alpha(a, b) \geq 1$,

(*ii*) *for every* $n \in \mathbb{N}$, *for any* $x \in F\{T_n\}$, *we have* $\alpha(x, y) \geq 1$ *whenever* $y \in Tx$ *and for any* $x \in F(T)$, *we have* $\alpha(x, y) \geq 1$ *whenever* $y \in T_n x$.

Then $H(F(T_n), F(T)) \to 0$ *as* $n \to \infty$, *that is, the fixed point of* T_n *are stable.*

Proof. By Lammas 3.1 and 3.2, it follows that T is generalized multivalued α-ψ contraction of Ciric-Berinde type and multivalued α-admissible.
Let $m_n = \sup_{x\in X} H(T_n x, Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to T on X. Therefore

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} \sup_{x\in X} H(T_n x, Tx) = 0.$$

Using Theorem 3.2, we get

$$H(F(T_n), F(T)) \leq \Phi(m_n), \quad \text{for every } n \in \mathbb{N}.$$

Since ψ is continuous and $\Phi(t) \to 0$ as $t \to 0$, we have

$$\lim_{n\to\infty} H(F(T_n), F(T)) \leq \lim_{n\to\infty} \Phi(m_n) = 0,$$

that is $\lim_{n\to\infty} H(F(T_n), F(T)) = 0$. Hence the proof is complete. □

Example 3.1. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Let define the multivalued mappings $T : X \to CL(X)$ as

$$T_n x = \begin{cases} \left\{1 + \frac{1}{n}, \frac{1}{2x} + \frac{1}{n}\right\}, & \text{if } x > 1, \\ \left\{\frac{1}{n}, \frac{1}{n} + \frac{x}{8}\right\}, & \text{if } 0 < x \leq 1, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Now we define the functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ as follows:

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x, y \in (0, 1], \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi(t) = \frac{1}{2}t.$$

Each T_n is multivalued α-admissible. $T_n \to T$ as $n \to \infty$. Then T is define by

$$Tx = \begin{cases} \left\{1, \frac{1}{2x}\right\}, & \text{if } x > 1, \\ \left\{0, \frac{x}{8}\right\}, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Each T_n is generalized multivalued α-ψ contraction of Ciric-Berinde type and T is also. Let $x, y \in (0, 1]$,

$$\begin{aligned} H(T_n x, T_n y) &= \max\left\{ \sup_{x\in Tx} d(x, Ty), \sup_{y\in Ty} d(y, Tx) \right\} \\ &= \max\left\{ \inf\left\{ \left|\frac{x}{8}\right|, \left|\frac{x}{8} - \frac{y}{8}\right| \right\}, \inf\left\{ \left|\frac{y}{8}\right|, \left|\frac{y}{8} - \frac{x}{8}\right| \right\} \right\} \\ &= \left|\frac{x}{8} - \frac{y}{8}\right|. \end{aligned}$$

Therefore,

$$\alpha(x, y) H(T_n x, T_n y) \leq \psi(M(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Thus conditions of Theorem 3.3 and Theorem 3.4 are satisfied. $F(T_1) = \{0, 1\}$ and $F(T_n) = \{0\}$ for $n \geq 2$. Hence $\mathrm{H}(F(T_n), F(T)) \to 0$ as $n \to \infty$.

4. Application to single valued mappings

In this section we obtain some fixed point results for single valued mappings by an application of the corresponding results of section 2.

Theorem 4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a single valued mapping. Suppose that for all $x, y \in X$,*

$$\alpha(x,y)d(Tx,Ty) \le \psi\left(\max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$
$$+ L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{4.1}$$

where the conditions α, ψ and L are same as in Definition 2.1. Also suppose that the following conditions are satisfied:

(i) T is α-admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,

(iii) T is h-upper semi continuous.

Then T has a fixed point.

Proof. We know that for every $x \in X, \{x\}$ is compact in X. Now, we define multivalued mapping $F : X \to C(X)$ as $Fx = \{Tx\}$, for $x \in X$. Let $x_0, y_0 \in X$ such that $\alpha(x_0, y_0) \ge 1$. Then by α-admissible of T, we have $\alpha(Tx_0, Ty_0) \ge 1$, that is, $\alpha(x_1, y_1) \ge 1$, where $x_1 \in Fx_0 = \{Tx_0\}$ and $y_1 \in Fy_0 = \{Ty_0\}$. Therefore, for $x_0, y_0 \in X$,

$$\alpha(x_0,y_0) \ge 1 \Rightarrow \alpha(x_1,y_1) \ge 1, \quad \text{where } x_1 \in Fx_0 \text{ and } y_1 \in Fy_0,$$

that is, F is a multivalued α-admissible mapping.

Let $x, y \in X$. Then by using (4.1), we have

$$\begin{aligned}\alpha(x,y)H(Fx,Fy) &= \alpha(x,y)d(Tx,Ty)\\ &\le \psi\left(\max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)\\ &\quad + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\\ &\le \psi\left(\max\left\{d(x,y), \frac{d(x,Fx)d(y,Fy)}{d(x,y)}, \frac{d(x,Fy)+d(y,Fx)}{2}\right\}\right)\\ &\quad + L\min\{d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx)\},\end{aligned}$$

that is, F satisfy condition (2.1). Therefore, F is a generalized multivalued α-ψ contraction of Ciric-Berinde type of the Theorem 2.1.

Suppose there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Let $x_1 \in Fx_0 = \{Tx_0\}$. Then $\alpha(x_0, Tx_0) \ge 1$ means $\alpha(x_0, x_1) \ge 1$. Therefore, there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0, x_1) \ge 1$.

Then by h-upper semi continuity of T for $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} d(Tx_n, Tx) = 0 = \lim_{n\to\infty} \delta(Fx_n, Fx) = 0$, where $Fx_n = \{Tx_n\}$. Therefore for $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} \delta(Fx_n, Fx) = 0$, that is F is h-upper semi continuous. So, all the conditions of Theorem 2.1 are satisfied and hence F has a fixed point z in X. Then $z \in Fz = \{Tz\}$, that is, $z = Tz$. Hence z is a fixed point of T in X. □

Theorem 4.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a single valued mapping. Suppose that* (4.1) *is satisfied, where the conditions α, ψ and L are same as in Definition 2.1. Also suppose that the following conditions are satisfied:*

(*i*) *T is α-admissible,*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n, where $x_{n+1} \in Tx_n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then T has a fixed point.

Proof. Similarly as in the proof of Theorem 4.1, we define the multivalued mapping $F : X \to C(X)$ as $Fx = \{Tx\}$, for $x \in X$ and we prove that there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0, x_1) \geq 1$, also prove that F is a multivalued α-admissible mapping, which satisfies (2.1). So all the conditions of Theorem 2.2 are satisfied and hence F has a fixed point z in X. Then $z \in Fz = \{Tz\}$, that is, $z = Tz$. Hence z is a fixed point of T in X. □

5. Conclusion

In fixed point theory, most of works have been derived for α-ψ contractions and α-admissible conditions for different mappings defined on various spaces. A multivalued version of α-ψ contractions and α-admissible mapping was introduced in [4]. We introduce generalized multivalued α-ψ contraction of Ciric-Berinde type which is different from other mentioned contractions. This paper deals with fixed point theorems and stability of fixed point sets associated with a sequence of multivalued mappings.

Acknowledgement

The second author sincerely acknowledges the University Grants Commission, Government of India for awarding fellowship to conduct the research work.

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Accepted: 28.12.2018

Modules closed full large extensions of cyclic submodules are summands

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Abstract. This paper introduced a new generalization of extending modules, namely modules in which every closed full large extension of a cyclic submodule is a direct summand, introduced a new generalization of the concept of injective modules. In fact, we give and study the properties of the concept of full-LE-Cy-injective modules. Although full-LE-Cy-injective modules are far from injective modules, they are exactly the same on some kind of rings. Then we make use of relatively full-LE-Cy-injectivity on modules to study direct sums of two (C_1-LE-Cy)-modules. We show that a direct sum of two relatively full-LE-Cy-injective modules is a (C_1-LE-Cy)-module if and only if each one of them is a (C_1-LE-Cy)-module. Examples are provided to illustrate and delimit the theory.

Keywords: (C_1-LE-Cy)-modules, full LE-Cy-modules, full-LE-Cy-injective modules

1. Introduction

all rings are associative with unity, R denotes such a ring, and all modules considered are unitary right R-modules. A module M is said to be an extending module (or module with the condition (C_1)), if every closed submodule C of M is a direct summand of M. The notion of extending modules was generalized recently by many authors see ([1], [7], [3] and [9]). Some of such generaliza-

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tions were named in [4] by principally extending modules, in [6] by generalized extending modules.

In [8], Nicholson and Yousif have introduced and studied the structure of principally injective rings, and have given some characterizations of such rings in terms of the internal properties of these rings. They defined a module M over a ring R to be principally injective if every R-homomorphism from a principal right ideal of R to M can be extended to R. In [4], Kamal and El-mnophy adopt the concept of principally injective rings, in [8], and generalize it to modules. They also introduced the concept of principally extending, (denoted by P-extending). A module M is called a P-extending module if every cyclic submodule is large in a direct summand of M, or equivalently, every EC-closed submodule of M is a direct summand. A submodule N of M is called an EC-submodule of M if there exists m in M such that mR is large in N.

The present paper studies the concept of modules with the condition that every full LE-Cy-submodule is large in a direct summand. This new concept, in turn, generalizes the concept of extending modules. A module M is a large extension of cyclic (denoted by LE-Cy-module) if $mR \leq^L M$ for some $m \in M$. A module M is said to be a full large extension of cyclic module (denoted by full LE-Cy-module) if every submodule of M is a LE-Cy-module. A full LE-Cy-submodule N of M is said to be full LE-Cy-closed in M if N has no proper full LE-Cy-large extensions in M. Consider the following condition on a module M: (C_1-LE-Cy): Every full LE-Cy-submodule of M is large in a direct summand of M. A module M which satisfies the condition (C_1-LE-Cy) is called a (C_1-LE-Cy)-module, equivalently, every full LE-Cy-closed submodule of M is a direct summand of M. Let $M = M_1 \oplus M_2$. It is well known that M_1 is M_2-injective if and only if for every submodule K of M with $K \cap M_1 \neq 0$, there exists a submodule L of M such that $K \leq L$, and $M = M_1 \oplus L$. In analogue, we introduce here the concept of full large extensions of cyclic injectivity (relative full large extensions of cyclic injectivity) which is one of the generalizations of the concept of injectivity (relative injectivity). This generalization is extremely useful in analyzing the structure of direct sums of (C_1-LE-Cy)-modules. We show that if $M = M_1 \oplus M_2$, then M_1 is M_2-full-LE-Cy-injective if and only if for every full LE-Cy-submodule K of M with $K \cap M_1 = 0$, there exists a submodule $M^{\backslash}$ of M such that $K \subseteq M^{\backslash}$, and $M = M_1 \oplus M^{\backslash}$. We also show that if $M = M_1 \oplus M_2$, M_i is M_j-full-LE-Cy-injective ($i \neq j$), then M is a (C_1-LE-Cy)-module if and only if M_i is a (C_1-LE-Cy)-module (i= 1, 2).

In Section 2, Example 2.1., shows that there are LE-Cy-modules, which are not full LE-Cy-modules. We consider connections between relative full large extensions of cyclic injectivity and relative injectivity, Theorem 2.1., gives an equivalent condition of a module M to be full large extensions of cyclic injective relative to a module N in module decompositions. In Section 3, we introduce the concept of modules with the condition that every full LE-Cy-submodule is large in a direct summand. This new concept, in turn, generalizes the concept of extending modules. Example 3.1., shows that a (C_1-LE-Cy)-module not necessary to be an extending module, this example also shows that a direct sum of two (C_1-LE-Cy)-modules not necessary to be a (C_1-LE-Cy)-module. In Proposition 3.3., we show that a direct sum of two relatively full-LE-Cy-injective modules is a (C_1-LE-Cy)-module if and only if each one of them is a (C_1-LE-Cy)-module.

Let R be a ring and M, N be R-modules and ϕ be an R-homomorphism from M into N. If $N \subseteq M$, then $N \leq M$, $N \leq^L M$, $N \leq^c M$, $N \leq^{\oplus} M$, $Z(M)$, $Z_2(M)$ and $\langle\phi\rangle = \{m + \phi(m) : m \in M\}$ denote N is a submodule of M, N is a large submodule of M, N is a closed submodule of M, N is a direct summand submodule of M, the singular submodule of M, the second singular submodule of M, and the graph of a module homomorphism $\phi : M \to N$, respectively.

2. Full LE-Cy-injective modules

In this section, we introduce the concept of full large extensions of cyclic injectivity (relative full large extension of cyclic injectivity) which is one of the generalizations of injectivity (relative injectivity). This generalization is extremely useful in analyzing the structure of modules whose closed full large extension of cyclic are summands.

Definition 2.1. *A module M is a large extension of cyclic (denoted by LE-Cy-module) if $mR \leq^L M$ for some $m \in M$. A module M is said to be a full large extension of cyclic module (denoted by full LE-Cy-module) if every submodule of M is a LE-Cy-module.*

Remark 2.1. 1. Every uniform module is a full LE-Cy-module.

2. Let R be a principal right ideal ring, then R_R is a full LE-Cy-module.

3. There are semisimple modules, which are full LE-Cy-modules, for example $\mathbb{Z}_n$, where $n = p_1p_2...p_n$ (for distinct primes) as a $\mathbb{Z}$-module is full LE-Cy-module. It is clear that $\mathbb{Z}_n$ as a $\mathbb{Z}$-module for each nonzero n in $\mathbb{Z}$ is a full LE-Cy-module.

4. Every non-Noetherian semisimple module is not a full LE-Cy-module.

5. There are LE-Cy-modules, which are not full LE-Cy-modules, for example let S be the set of all functions: $\mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers). S is a commutative ring with +,. defined by, $(f + g)(r) = f(r) + g(r), (f.g)(r) = f(r).g(r))$ for all $f, g \in S$ and $r \in \mathbb{R}$. Hence S_S is an $LE - Cy$-module, and not full LE-Cy; for $I = \{f \in S : f(r) = 0$ for all $\mid r \mid > n$ (for some positive integer $n \in \mathbb{Z}$)$\}$ is not an LE-Cy-module.

Definition 2.2. *Let M and N be modules. We say that M is full large extension of cyclic injective relative to N (for short M is N-full-LE-Cy-injective) if, for each monomorphism $\alpha : K \longrightarrow N$, with K a full LE-Cy-module, and each homomorphism $\beta : K \longrightarrow M$, there exists a homomorphism $\phi : N \longrightarrow M$ such* *that $\phi\alpha = \beta$. M is called a full- LE-Cy-injective module, if M is N-full-LE-Cy-injective for every module N.*

Fully cyclic large extensions injectivity is one of the generalizations of injectivity. We are going to give some properties of such modules.

Lemma 2.1. *Isomorphic copy of a full LE-Cy-module is a full LE-Cy-module.*

Proof. Let M be a full LE-Cy-module, and $\alpha : M \longrightarrow N$ be an R-isomorphism. Let L be a nonzero submodule of N. Since M is a full LE-Cy-module, there exists m in M such that mR is large in $\alpha^{-1}(L)$. It is easy to check that $\alpha(m)R$ is large in L. Therefore, N is a full LE-Cy-module. □

Proposition 2.1. *Let M and N be R-modules. Then the following are equivalent:*

1) *M is N-full-LE-Cy-injective.*

2) *For each full LE-Cy-submodule K of N each homomorphism $\beta : K \longrightarrow M$ can be extended to N.*

Proof. 1) $\Rightarrow$ 2) It is clear.

2) $\Rightarrow$ 1) Let K be a full LE-Cy-module, and $\alpha : K \longrightarrow N$ be an R-monomorphism and $\beta : K \longrightarrow M$ be an R-homomorphism. Since $K \cong \alpha(K)$ and K is an LE-Cy-module. By Lemma 2.1., we have $\alpha(K)$ is an LE-Cy-module. By assumption; there exists an R-homomorphism $\phi : N \longrightarrow M$ such that $\phi\alpha(x) = \beta(x)$ for all $x \in K$. □

Remark 2.2. 1) N-full-LE-Cy-injectivity and N-injectivity are equivalent, whenever N be a full LE-Cy-module.

2) Full-LE-Cy-injectivity and injectivity are the same for modules over principal right ideal rings. In particular, injectivity and full-LE-Cy-injectivity are the same for $\mathbb{Z}$-modules.

3) Let $M = M_1 \oplus M_2$ be an R-module, and $\alpha : M_1 \longrightarrow M_2$ is a homomorphism. Then the following are well known:

i) $\langle\alpha\rangle = \{m_1 + \alpha(m_1) : m_1 \in M_1\}$ is a complementary summand of M_2 in M.

ii) $\langle\alpha\rangle \cong M_1$.

iii) If α is an monomorphism, then $\langle\alpha\rangle \cap M_1 = 0$.

Proposition 2.2.

1. *If M is N-full-LE-Cy-injective, then M is N'-full-LE-Cy-injective; for each submodule N' of M.*

2. *If M is N-full-LE-Cy-injective and $M' \leq^{\oplus} M$, then M' is N-full-LE-Cy-injective.*

Proof. It is clear. □

Theorem 2.1. *Let M_1 and M_2 be an R-modules and let $M = M_1 \oplus M_2$. Then the following are equivalent:*

1) *M_1 is M_2-full-LE-Cy-injective.*

2) *For every full LE-Cy-submodule H of M such that $H \cap M_1 = 0$, there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$, and $H \leq M_3$.*

Proof. 1) $\Rightarrow$ 2) Let H be a full LE-Cy-submodule of M such that $H \cap M_1 = 0$. Let $\pi_i : M \longrightarrow M_i (i = 1, 2)$ be the projections. Observe that $\pi_2|_H : H \longrightarrow M_2$ is an monomorphism. Since M_1 is full M_2-LE-Cy-injective, there exists a R-homomorphism $\alpha : M_2 \longrightarrow M_1$ such that $\alpha \circ \pi_2|_H = \pi_1|_H$. Take $M_3 = \langle \alpha \rangle$, thus, by Remark 2.1., we have $M = M_1 \oplus M_3$. Now, for all $h \in H$, $h = \pi_1(h) + \pi_2(h) = \alpha \circ \pi_2(h) + \pi_2(h) \in M_3$. Therefore, $H \leq M_3$.

2) $\Rightarrow$ 1) Let K be a full LE-Cy submodule of M_2, $g : K \longrightarrow M_1$ be R-homomorphism. By Remark 2.1., we have $\langle g \rangle = \{k - g(k) : k \in K\} \cong K$. Thus, by Lemma 2.1., $\langle g \rangle$ is a full LE-Cy-submodule of M. Since $\langle g \rangle \cap M_1 = 0$, there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $\langle g \rangle \leq M_3$. Let $\pi_1 : M_1 \oplus M_3 \longrightarrow M_1$ be the projection. Then for all $k \in K$, we have $\pi_1(k) = \pi_1(k - g(k) + g(k)) = \pi_1(g(k)) = g(k)$. Therefore, π_1 extends g and hence M_1 is M_2-full-LE-Cy-injective. $\square$

Corollary 2.1. *If $M = M_1 \oplus M_2$ and M_1 is M_2-full-LE-Cy-injective, then $M = M_1 \oplus C$ for every full large extension of cyclic and complement C of M_1 in M.*

Proof. Let C be a full large extension of cyclic and complement of M_1 in M. Since M_1 is M_2-full-LE-Cy-injective, there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $C \leq M_3$. Since $C \oplus M_1$ is large submodule of $M_1 \oplus M_3$, C is a large submodule of M_3. Therefore, $C = M_3$. $\square$

3. Modules with closed full LE-Cy-submodules summands

In this section, we introduce the concept of modules with the condition that every full LE-Cy-submodule is large in a direct summand. This new concept, in turn, generalizes the concept of extending modules.

Definition 3.1. *Consider the following condition on a module M:*

(C_1-LE-Cy): Every full LE-Cy-submodule of M is large in a direct summand of M. A module M, which satisfies the condition (C_1-LE-Cy) is called a (C_1-LE-Cy)-module.

Definition 3.2. *A full LE-Cy-submodule N of M is said to be full LE-Cy-closed in M if N has no proper full LE-Cy-large extension in M.*

Lemma 3.1. *Let M be a module, and K be a full LE-Cy-submodule of M. Then L is a full LE-Cy-submodule, for each Large extension L of K in M.*

Proof. Let K be a full LE-Cy-submodule of M, and L be a large extension of K in M. Let D be a submodule of L. It follows that $D \cap K$ is large in D. Since K is a full LE-Cy-submodule of M, we have that $D \cap K$ is an LE-Cy-module.

Hence D is an LE-Cy-submodule of L. Therefore, L is a full LE-Cy-submodule of M. □

Corollary 3.1. *Every full LE-Cy-closed submodule of a module M is a closed submodule of M.*

Proof. Let K be a full LE-Cy-closed submodule of M and let N be a large extension of K in M. By Lemma 3.1., we have that N is full LE-Cy-submodule of M. Therefore, $K = N$. □

Corollary 3.2. *The following are equivalent for a module M:*

1. *M is a (C_1-LE-Cy)-module.*
2. *Every full LE-Cy-closed submodule of M a direct summand of M.*
3. *For every full LE-Cy-submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $N \leq M_1$ and $N \oplus M_2$ is large in M.*

Proof. 1) $\Rightarrow$ 2) Let H be a full LE-Cy-closed submodule of M, then there exists a direct summand submodule D of M such that H is large in D. By Corollary 3.1., we have that $H = D$. Therefore, H is a direct summand of M.
2) $\Rightarrow$ 3) Let H be a full LE-Cy-submodule of M and let M_1 be a maximal large extension of H in M, then by Lemma 3.1., M_1 is full LE-Cy-closed in M. Therefore, $M = M_1 \oplus M_2$ such that $H \leq M_1$ and $H \oplus M_2$ is large in M.
3) $\Rightarrow$ 1) It is clear. □

Proposition 3.1. *Let M be an indecomposable module and a full LE-Cy-module. Then M is a (C_1-LE-Cy)-module if and only if M is uniform.*

Proof. Let M be a (C_1-LE-Cy)-module and $0 \neq X$ be submodule of M. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $X \leq M_1$ and $X \oplus M_2$ is large in M. It is clear that $M_2 = 0$. Therefore, M is uniform. □

Lemma 3.2 (Theorem 5, [5]). *Let M be a torsion free reduced module over a commutative integral domain R. If M is extending, then M is a finite direct sum of uniform submodules.*

Lemma 3.3 (Theorem 7, [5]). *Let M be a torsion free reduced module over an integral domain R with extension field K. Then the following are equivalent:*
1) M is extending.
2) $M = \oplus_{i=1}^{n} M_i$, with all M_i uniform, and for all $q_1, q_2, ..., q_n \in K$ (not all zero) there exist $\alpha_1, \alpha_2, ..., \alpha_n \in K$ such that $\sum_{k=1}^{n} \alpha_k = 1$ and $\alpha_k q_i M_k \subset q_k M_i$ for all k, i.

Example 3.1. 1) Every extending module is a (C_1-LE-Cy)-module, while there exists (C_1-LE-Cy)-modules, which are not extending, for example the $\mathbb{Z}$-module $M = \bigoplus_{i=1}^{\infty} M_i$, where $M_i = \mathbb{Z}$ for all $i \in \mathbb{N}$. It is clear, from Lemma 3.2., that M

is not extending, and from Lemma 3.3, that each finite subsum of $M = \bigoplus_{i=1}^{\infty} M$ is extending. Since every full *LE-Cy*-submodule of M is contained in a finite subsum of M, we have that M is a (C_1-*LE-Cy*)-module.

2) A direct sum of two (C_1-*LE-Cy*)-modules need not be a (C_1-*LE-Cy*)-module, for example the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ is not a (C_1-*LE-Cy*)-module. In fact the submodule $(2, \bar{1})\mathbb{Z}$ is full-LE-Cy-closed in M, while it is not a direct summand of M.

3) Let $\mathbb{F}$ be a field, then R_R is not a full cyclic large extending module where,

$$R = \begin{pmatrix} \mathbb{F} & \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} & 0 \\ 0 & 0 & \mathbb{F} \end{pmatrix}$$

In fact, R_R contains a simple and closed submodule which is not a direct summand.

Lemma 3.4 ([8], 1.10 (4)). *If K is closed in L and L is closed in M then K is closed in M.*

Lemma 3.5. *Let M be a (C_1-LE-Cy)-module and N be a direct summand submodule of M. Then N is a (C_1-LE-Cy)-module.*

Proof. Let C be full *LE-Cy*-closed in N. Since $N \leq^{\oplus} M$, we have, by Lemma 3.4., that C is full *LE-Cy*-closed in M. As M is a (C_1-*LE-Cy*)-module, we have that $C \leq^{\oplus} M$; and hence $C \leq^{\oplus} N$. Therefore, N is (C_1-*LE-Cy*)-module. □

Proposition 3.2. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then the following are equivalent:*

1) M is a (C_1-LE-Cy)-module.

2) Every full LE-Cy-closed submodule K of M, with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.

Proof. 1) ⇒ 2). It is clear by Lemma 3.5.,.

2) ⇒ 1). Let L be a full *LE-Cy*-closed in M. Let X be a maximal large extension of $L \cap M_2$ in L. Since, by Lemma 3.1., X is full *LE-Cy*-closed in M, with $X \cap M_1 = 0$. By 2), $M = X \oplus Y$ for some submodule Y of M. As $L = X \oplus (Y \cap L)$, by Lemma 3.4., we have that $(Y \cap L)$ is a full *LE-Cy*-closed submodule of M. $(L \cap M_2) \leq X$, then $(Y \cap L) \cap M_2 = 0$. Again by 2), $Y \cap L$ is a direct summand of M, and hence it is a direct summand of Y. Therefore, $M = X \oplus (Y \cap L) \oplus K = L \oplus K$ for some submodule K of M. □

Lemma 3.6. *Let M and N be isomorphic R-modules. If M is a (C_1-LE-Cy)-module, then N is a (C_1-LE-Cy)-module.*

Proof. Let $f : M \longrightarrow N$ be an R-isomorphism and let C be a full *LE-Cy*-closed submodule of N. It is clear, by Lemma 1, that $f^{-1}(C)$ is a full *LE-Cy*-closed submodule of M. By the condition (C_1-*LE-Cy*) for M, $f^{-1}(C)$ is a direct summand of M. Therefore, C is a direct summand of N, i.e. N is a (C_1-*LE-Cy*)-module. □

Proposition 3.3. *Let $M = M_1 \oplus M_2$ be a module, where M_i is M_j-full-LE-Cy-injective $(i \neq j = 1, 2)$. Then the following are equivalent :*
1) M is a (C_1-LE-Cy)-module.
2) M_i is a (C_1-LE-Cy)-module, $(i = 1, 2)$.

Proof. 1) $\Rightarrow$ 2) It is clear from Lemma 3.5.,.
2) $\Rightarrow$ 1) Let K be a full *LE-Cy*-closed submodule of M, with $K \cap M_1 = 0$. By Theorem 2.1., there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $K \leq M_3$. As $M_3 \cong M_2$ and M_2 is a (C_1-*LE-Cy*)-module, we have, by Lemma 3.6., that M_3 is a (C_1-*LE-Cy*)-module. Therefore, K is a direct summand submodule of M_3, and hence K is a direct summand submodule of M. By proposition 3.2., we have that M is a (C_1-*LE-Cy*)-module. □

Corollary 3.3. *Let $M = M_1 \oplus ... \oplus M_n$, where M_i is M_j-full-LE-Cy-injective, for all $i \neq j, (i, j = 1, 2, ..., n)$ for some positive integer n. Then the following are equivalent :*
1) M is a (C_1-LE-Cy)-module.
2) M_i is a (C_1-LE-Cy)-module, $(i = 1, .., n)$.

Proof. 1) $\Rightarrow$ 2) It is clear from Lemma 3.5.,.
2) $\Rightarrow$ 1) By induction on n, it is enough to prove that M is a (C_1-*LE-Cy*)-module by consider in the case, when $n = 2$, which is shown in Proposition 3.3.,. □

Corollary 3.4. *Let $M = Z_2(M) \oplus N$ be a module, where $Z_2(M)$ is the second singular submodule of M. If $Z_2(M)$ and N are both (C_1-LE-Cy)-modules, and $Z_2(M)$ is N-full-LE-Cy-injective, then M is a (C_1-LE-Cy)-module.*

Proof. It is clear that $Hom(Z_2(M), N) = 0$, (due to N is non-singular), and hence N is $Z_2(M)$-injective. By Proposition 3.3., we have that M is a (C_1-*LE-Cy*)-module. □

Proposition 3.4. *Let M be a (C_1-LE-Cy)-module and $Z_2(M)$ be a full LE-Cy-module. Then we have the following :*
1) $M = Z_2(M) \oplus N$, for some submodule N of M, and both $Z_2(M)$, N are (C_1-LE-Cy)-modules.
2) $Z_2(M)$ is N-full-LE-Cy-injective.

Proof. 1) As $Z_2(M)$ is full *LE-Cy*-closed submodule of M, we have that $M = Z_2(M) \oplus N$. By Lemma 3.5., we have $Z_2(M)$ and N are (C_1-*LE-Cy*)-modules.
2) Let L be a full *LE-Cy*-submodule of M with $L \cap Z_2(M) = 0$. Let C be a maximal large extension of L in M. By Lemma 3.1., we have that C is full *LE-Cy*-closed submodule of M. By hypothesis, we have $M = C \oplus C'$ for some submodule C' of M. As $C \cap Z_2(M) = 0$, we have that $Z_2(M) \leq C'$. Thus $M = C' \oplus C = Z_2(M) \oplus (N \cap C') \oplus C$ and $L \leq (N \cap C') \oplus C$. Therefore, by Theorem 2.1., $Z_2(M)$ is N-full-*LE-Cy*-injective. □

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Accepted: 21.12.2018

Some generalized forms of soft compactness and soft Lindelöfness via soft α-open sets

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Abstract. By using a notion of soft α-open sets, we generalize the concepts of soft compact and soft Lindelöf spaces. We define the concepts of soft α-compact, soft α-Lindelöf, almost (approximately, mildly) soft α-compact and almost (approximately, mildly) soft α-Lindelöf spaces. We present two new kinds of the finite intersection property and utilize them to characterize almost soft α-compact and approximately soft α-compact spaces. To elucidate the relationships among the introduced spaces and to illustrate our main results, we supply several interesting examples. Also, we point out that the initiated spaces are preserved under soft α-irresolute mappings and we investigate certain of results which associate an extended soft topology with the introduced soft spaces. In the end, we conclude some findings which associate the introduced spaces with some soft topological notions such as soft α-connectedness, soft α-T_2-spaces, soft α-partition and soft subspaces.

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Keywords: soft α-compactness, almost (approximately, mildly) soft α-compactness, soft α-irresolute mapping and soft subspace.

1. Introduction and preliminaries

In the year 1999, the Russian researcher Molodtsove [24] initiated and studied a new mathematical approach for solving problems associated with uncertainties, namely soft sets. He pointed out that there are no limited conditions to the description of objects, so researchers can choose the form of parameters they need, which simplifies the decision-making process and make the process more efficient in the case of incomplete information. Then Maji et al. [22] in 2003, carried out a systematic study to construct some basic operations between two soft sets. Despite of a lot of shortcomings of Maji et al's study, it formed the first block of the soft set theory. To remove this weakness and to define some soft operators in a way that preserves the crisp properties via the soft set theory, Ali et al. [3] formulated some new operators such as restricted union, restricted intersection and restricted difference of two soft sets and a relative complement of a soft set.

Shabir and Naz [28] in 2011, employed the idea of soft sets to define the soft topological spaces concept. They examined the main properties of soft closed operators and soft separation axioms. Later on, Min [23] showed that Example 9 of [28] does not satisfy a condition of soft T_2-spaces and proved that every soft T_3-space is soft T_2. Zorlutuna et al. [29] showed the connection between fuzzy sets and soft sets. Also, they initiated the first shape of soft point in order to study some properties of soft interior points and soft neighborhood systems. Aygünoğlu and Aygün [14] introduced a concept of soft compact spaces and investigated its main features. They also presented a notion of enriched soft topological spaces and illustrated its role to verify some results associated with constant soft mappings and soft compact spaces. The authors of [16, 25] simultaneously modified the first shape of soft point to be more effective for studying soft limit points and soft metric spaces. Chen [15] started studying generalized soft open sets by defining and investigating soft semi-open sets. Then Arockiarani and Lancy [13] presented a concept of soft pre-open sets and studied its fundamental properties. Akdag and Ozkan introduced a soft α-open sets notion in [1] and then carried out a detailed study on soft α-separation axioms in [2]. Ozturk and Bayramov [26] defined a soft compact-open topology concept and studied its main features. Kandil et al. [20] presented γ-operations and investigated their main properties. The authors of [4, 5, 7, 18] showed some alleged results on soft axioms and corrected them with the help of examples. Al-shami and Kočinac [12] proved the equivalence between the enriched and extended soft topologies and then they derived that $(int(H), K) = int(H, K)$ and $(cl(H), K) = cl(H, K)$, for any soft subset (H, K) of an extended soft topological space. Recently, Al-shami [6, 9] introduced a new class of generalized soft open sets, namely soft somewhere dense sets; and a newly soft mathematical structure, namely soft topological ordered spaces. Al-shami et al. [10, 11] studied new types of soft ordered maps by using soft α-open and soft β-open sets.

This paper is an attempt to open up the theoretical aspects of soft sets by extending the notions of soft compact and soft Lindelöf spaces to the framework of soft sets. We begin this work by presenting certain of concepts of soft set theory and soft topological spaces that we will need to demonstrate our new findings. The goal of this study is to employ soft α-open sets to initiate eight kinds of generalized soft compact spaces, namely soft α-compact, soft α-Lindelöf, almost (approximately, mildly) soft α-compact and almost (approximately, mildly) soft α-Lindelöf spaces. We characterize each one of these concepts and provide various examples to elucidate the relationships among these spaces. Moreover, we offer some soft topological concepts such as soft α-hyperconnectedness and soft α-partition spaces, and we establish some properties which associate them with the introduced generalized soft compact spaces. We demonstrate the relationships between an extended soft topology and the introduced soft spaces. The sufficient conditions for the eight initiated generalized soft compact spaces to be soft hereditary properties are investigated. last but not least, we point out that the soft α-irresolute maps preserve all of the given generalized soft compact spaces.

In order to investigate and discuss our new results, we recollect the following definitions and results which will be needed in the sequels. We shall write these definitions with respect to a fixed set of parameters because we only utilize them on the frame of soft topological spaces which defined on a fixed set of parameters.

Definition 1.1 ([24]). *A pair (G,K) is called a soft set over X provided that G is a mapping of a parameters set K into the family of all subsets of X. It can be expressed as follows: $(G,K)=\{(k,G(k)) : k\in K$ and $G(k)\in 2^X\}$.*

Definition 1.2 ([17, 28]). *Let (G,K) be a soft set over X. We say that:*

(i) *$x\in(G,K)$ if $x\in G(k)$ for each $k\in K$; and $x\notin(G,K)$ if $x\notin G(k)$ for some $k\in K$.*

(ii) *$x\Subset(G,K)$ if $x\in G(k)$ for some $k\in K$; and $x\not\Subset(G,K)$ if $x\notin G(k)$ for each $k\in K$.*

Definition 1.3 ([17]). *A soft subset (G,K) over X is called stable if there is a subset S of X such that $G(k)=S$, for each $k\in K$ and it is denoted by $\widetilde{S}$.*

Definition 1.4 ([3]). *We say that (G,K) is a soft subset of (H,K), denoted by $(G,K)\widetilde{\subseteq}(H,K)$, provided that $G(k)\subseteq H(k)$, for each $k\in K$.*

Definition 1.5 ([3]). *The relative complement of a soft set (G,K), denoted by $(G,K)^c$, is given by $(G,K)^c=(G^c,K)$, where a mapping $G^c : K\to 2^X$ is defined by $G^c(k)=X\setminus G(k)$, for each $k\in K$.*

Definition 1.6 ([3, 27]). *The soft union and intersection of two soft sets (G, K), (F, K) is given by the following rule:*

(i) $(G, K)\widetilde{\bigcup}(F, K) = (H, K)$, *where* $H(k) = G(k) \bigcup F(k)$ *for each* $k \in K$.

(ii) $(G, K)\widetilde{\bigcap}(F, K) = (H, K)$, *where* $H(k) = G(k) \bigcap F(k)$ *for each* $k \in K$.

Definition 1.7 ([22]). *A soft set (G, K) over X is called:*

(i) *An absolute soft set if $G(k) = X$ for each $k \in K$. It is denoted by $\widetilde{X}$;*

(ii) *A null soft set if $G(k) = \emptyset$ for each $k \in K$. It is denoted by $\widetilde{\emptyset}$.*

Definition 1.8 ([28]). *A collection τ of soft sets over X with a fixed set of parameters K is called a soft topology on X if it satisfies the following three axioms:*

(i) *The null soft set $\widetilde{\emptyset}$ and the absolute soft set $\widetilde{X}$ are members of τ;*

(ii) *τ is closed under an arbitrary soft union and*

(iii) *τ is closed under a finite intersection.*

The triple (X, τ, K) is called a soft topological space (For short, STS). Each soft set in τ is called soft open and its relative complement is called soft closed.

Proposition 1.9 ([28]). *Let (X, τ, K) be an STS. Then $\tau_k = \{G(k) : (G, K) \in \tau\}$ defines a topology on X, for each $k \in K$.*

Definition 1.10 ([14]). *A soft topology τ on X is said to be an enriched soft topology if axiom **(i)** of Definition (1.8) is replaced by the following condition: (G, K) is soft open if and only if $G(k) = X$ or $\emptyset$, for each $k \in K$. In this case, the triple (X, τ, K) is called an enriched STS over X.*

Proposition 1.11 ([25]). *Consider (X, τ, K) is an STS and τ_k is a topology on X as in the above proposition. Then $\tau^\star = \{(G, K) : G(k) \in \tau_k$, for each $k \in K\}$ is a soft topology on X finer than τ.*

Remark 1.12. The authors of [12] termed $\tau^\star$ an extended soft topology and demonstrated that the extended and enriched soft topologies are identical.

Theorem 1.13 ([12]). *We have the following two results, for any soft subset (H, K) of an extended soft topological space $(X, \tau^\star, K)$.*

(i) $(int(H), K) = int(H, K)$.

(ii) $(cl(H), K) = cl(H, K)$.

Definition 1.14 ([12, 28]). *Let (F, K) be a soft subset of an STS (X, τ, K). Then:*

(i) *$(cl(F), K)$ is defined as $cl(F)(k) = cl(F(k))$, where $cl(F(k))$ is the closure of $F(k)$ in (X, τ_k), for each $k \in K$.*

(ii) *$(int(F), K)$ is defined as $int(F)(k) = int(F(k))$, where $int(F(k))$ is the interior of $F(k)$ in (X, τ_k), for each $k \in K$.*

Definition 1.15 ([25])**.** *Let (X, τ, K) be an STS and (Y, K) be a non-null soft subset of $\widetilde{X}$. Then $\tau_{(Y,K)} = \{(Y, K)\widetilde{\bigcap}(G, K) : (G, K) \in \tau\}$ is said to be a relative soft topology on (Y, K) and $((Y, K), \tau_{(Y,K)}, K)$ is called a soft subspace of (X, τ, K).*

Definition 1.16 ([16, 25])**.** *A soft subset (P, K) over X is called soft point if there is $k \in K$ and there is $x \in X$ such that $P(k) = \{x\}$ and $P(e) = \emptyset$, for each $e \in K \setminus \{k\}$. A soft point will be shortly denoted by P_k^x.*

Definition 1.17 ([16])**.** *A soft subset (H, K) of $\widetilde{X}$ is called a finite (resp. countable) soft set if $H(k)$ is finite (resp. countable) for each $k \in K$. A soft set is called an infinite (resp. uncountable) soft set if it is not finite (resp. countable).*

Definition 1.18. *A soft subset (A, K) of an STS (X, τ, K) is said to be:*

(i) *Soft α-open [1] if $(A, K)\widetilde{\subseteq}int(cl(int(A, K)))$.*

(ii) *Soft semi-open [15] if $(A, K)\widetilde{\subseteq}cl(int(A, K))$.*

(iii) *Soft pre-open [13] if $(A, K)\widetilde{\subseteq}int(cl(A, K))$.*

Proposition 1.19 ([1])**.** *The union of an arbitrary class of soft α-open sets is soft α-open and the intersection of an arbitrary class of soft α-closed sets is soft α-closed.*

Definition 1.20 ([1])**.** *For a soft subset (G, K) of (X, τ, K), we define the following two operators:*

(i) *$int_\alpha(G, K)$ is the soft union of all soft α-open sets contained in (G, K).*

(ii) *$cl_\alpha(G, K)$ is the soft intersection of all soft α-closed sets containing (G, K).*

Definition 1.21 ([2])**.** *An STS (X, τ, K) is said to be soft α T_2-space if for every $x \neq y$ in X, there are two disjoint soft α-open sets (G, K) and (F, K) such that $x \in (G, K)$ and $y \in (F, K)$.*

Proposition 1.22 ([8])**.** *Consider $((U, K), \tau_{(U,K)}, K)$ is a soft subspace of (X, τ, K) and let cl_U and int_U stand for the soft closure and soft interior operators, respectively, in $((U, K), \tau_{(U,K)}, K)$. Then:*

(i) *$cl_U(A, K) = cl(A, K)\widetilde{\bigcap}(U, K)$, for each $(A, K)\widetilde{\subseteq}(U, K)$.*

(ii) *$int(A, K) = int_U(A, K)\widetilde{\bigcap}int(U, K)$, for each $(A, K)\widetilde{\subseteq}(U, K)$.*

Throughout this work, the two notations (X, τ, K) and (Y, θ, K) stand for soft topological spaces and a notation S stands for a countable set.

2. Soft α-compact spaces

Definition 2.1. **(i)** *A family* $\{(G_i, K) : i \in I\}$ *of soft* α*-open sets is called a soft* α*-open cover of* (X, τ, K) *if* $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(G_i, K)$.

(ii) *An STS* (X, τ, K) *is called soft* α*-compact (resp. soft* α*-Lindelöf) if every soft* α*-open cover of* $\widetilde{X}$ *has a finite (resp. countable) soft sub-cover of* $\widetilde{X}$.

For the purpose of brevity, we shall omit the proofs of the following three propositions.

Proposition 2.2. *Every soft* α*-compact space is soft* α*-Lindelöf.*

Proposition 2.3. *A finite (resp. countable) union of soft* α*-compact (resp. soft* α*-Lindelöf) subsets of* (X, τ, K) *is soft* α*-compact (resp. soft* α*-Lindelöf).*

Proposition 2.4. *Every soft* α*-compact (resp. soft* α*-Lindelöf) space is soft compact (resp. soft Lindelöf).*

The converse of Proposition (2.4) is incorrect as it is evident in the example below.

Example 2.5. Consider a set of parameters K is the set of irrational numbers $\mathcal{Q}^c$ and let a collection $\tau = \{\widetilde{\emptyset}, \widetilde{X}, (G, K)$ such that $G(k) = \{1\}$, for each $k \in K\}$ be a soft topology on $X = \{1, 2\}$. Obviously, (X, τ, K) is soft compact. On the other hand, a collection $\{(G, E)$: There exists $k \in K$ such that $G(k) = X$ and $G(k_j) = \{1\}$, for each $k_j \neq k\}$ forms a soft α-open cover of $\widetilde{X}$. Since this collection has not a countable sub-cover of $\widetilde{X}$, then (X, τ, K) is not soft α-Lindelöf.

Proposition 2.6. *Every soft* α*-closed subset* (D, K) *of a soft* α*-compact (resp. soft* α*-Lindelöf) space* (X, τ, K) *is soft* α*-compact (resp. soft* α*-Lindelöf).*

Proof. We will start with the proof for soft α-Lindelöf spaces, as the proof for soft α-compact spaces is analogous.

Let (D, K) be a soft α-closed subset of $\widetilde{X}$ and let $\{(H_i, K) : i \in I\}$ be a soft α-open cover of (D, K). Then (D^c, K) is soft α-open and $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since $\widetilde{X}$ is soft α-Lindelöf, then $\widetilde{X} = \widetilde{\bigcup}_{i\in S}(H_i, K)\widetilde{\bigcup}(D^c, K)$. This implies that $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i, K)$. Hence (D, K) is soft α-Lindelöf. □

Corollary 2.7. *If* (G, K) *is soft* α*-compact (resp. soft* α*-Lindelöf) and* (D, K) *is soft* α*-closed subsets of* $\widetilde{X}$*, then their soft intersection is soft* α*-compact (resp. soft* α*-Lindelöf).*

To show that the converse of the above proposition is not necessarily correct, we give the following example.

Example 2.8. Take $K = \{k_1, k_2\}$ and assume that τ is the same as in Example (2.5). Then (X, τ, K) is soft α-compact. Take a soft set (G, K) such that $G(k_1) = \{1\}$ and $G(k_2) = \emptyset$. Then (G, K) is a soft α-compact, but it is not soft α-closed.

Theorem 2.9. *An STS (X, τ, K) is soft α-compact (resp. soft α-Lindelöf) if and only if every soft collection of soft α-closed subsets of (X, τ, K), satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X, τ, K) is soft α-Lindelöf, the other case can be made similarly.

Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft α-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As (X, τ, K) is soft α-Lindelöf, then $\widetilde{\bigcup}_{i\in S}(F_i^c, K) = \widetilde{X}$. Therefore $\widetilde{\bigcap}_{i\in S}(F_i, K) = \widetilde{\emptyset}$.

Conversely, let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft α-open cover of $\widetilde{X}$. Suppose that Λ has no a countable soft sub-collection which cover $\widetilde{X}$. Then $\widetilde{\bigcup}_{i\in S}(H_i, K) \neq \widetilde{X}$, for any countable set S. Now, $\widetilde{\bigcap}_{i\in S}(H_i^c, K) \neq \widetilde{\emptyset}$ implies that $\{(H_i^c, K) : i \in I\}$ is a soft collection of soft α-closed subsets of $\widetilde{X}$ which has the countable intersection property. Thus $\widetilde{\bigcap}_{i\in I}(H_i^c, K) \neq \widetilde{\emptyset}$. This implies that $\widetilde{X} \neq \widetilde{\bigcup}_{i\in I}(H_i, K)$. But this contradicts that Λ is a soft α-open cover of $\widetilde{X}$. Hence (X, τ, K) is soft α-Lindelöf. □

Definition 2.10. *A soft mapping $g : (X, \tau, K) \to (Y, \theta, K)$ is called soft α-irresolute if the inverse image of each soft α-open subset of $\widetilde{Y}$ is a soft α-open subset of $\widetilde{X}$.*

We investigate the following theorem which will be useful to prove Theorem (3.11) and Theorem (4.15).

Theorem 2.11. *The following five statements are equivalent for a soft mapping $g : (X, \tau, K) \to (Y, \theta, K)$:*

(i) *g is soft α-irresolute;*

(ii) *The inverse image of each soft α-closed subset of $\widetilde{Y}$ is a soft α-closed subset of $\widetilde{X}$;*

(iii) *$cl_\alpha(g^{-1}(A, K)) \widetilde{\subseteq} g^{-1}(cl_\alpha(A, K))$, for each soft subset (A, K) of $\widetilde{Y}$;*

(iv) *$g(cl_\alpha(E, K)) \widetilde{\subseteq} cl_\alpha(g(E, K))$, for each soft subset (E, K) of $\widetilde{X}$;*

(v) *$g^{-1}(int_\alpha(A, K)) \widetilde{\subseteq} int_\alpha(g^{-1}(A, K))$, for each soft subset (A, K) of $\widetilde{Y}$.*

Proof. **(i) ⇒ (ii)**: Suppose that (F, K) is a soft α-closed subset of $\widetilde{Y}$. Then (F^c, K) is soft α-open. Therefore $g^{-1}(F^c, K)$ is a soft α-open subset of $\widetilde{X}$. It is well known that $g^{-1}(F^c, K) = \widetilde{X} - g^{-1}(F, K)$. Hence $g^{-1}(F, K)$ is a soft α-closed subset of $\widetilde{X}$.

(ii) $\Rightarrow$ (iii): For any soft subset (A, K) of $\widetilde{Y}$, we get that $cl_\alpha(A, K)$ is a soft α-closed subset of $\widetilde{Y}$. Since $g^{-1}(cl_\alpha(A, K))$ is a soft α-closed subset of $\widetilde{X}$, then $cl_\alpha(g^{-1}(A, K))\widetilde{\subseteq}cl_\alpha(g^{-1}(cl_\alpha\ (A, K))) = g^{-1}(cl_\alpha(A, K))$.

(iii) $\Rightarrow$ (iv): For any soft subset (E, K) of $\widetilde{X}$, we know that $cl_\alpha(E, K)\widetilde{\subseteq}cl_\alpha(g^{-1}(g(E, K))$. By **(iii)**, we find that $cl_\alpha(g^{-1}(g(E, K)))\widetilde{\subseteq}g^{-1}(cl_\alpha(g(E, K)))$.

Hence $g(cl_\alpha(E, K))\widetilde{\subseteq}g(g^{-1}\ (cl_\alpha(g(E, K))))\widetilde{\subseteq}cl_\alpha(g(E, K))$.

(iv) $\Rightarrow$ (v): Let (A, K) be any soft subset of $\widetilde{Y}$.

Then $g(cl_\alpha(X - g^{-1}(A, K)))\widetilde{\subseteq}cl_\alpha(g(X - g^{-1}(A, K)))$. Therefore $g(X - int_\alpha(g^{-1}(A, K))) = g(cl_\alpha(X - g^{-1}(A, K)))\widetilde{\subseteq}cl_\alpha(\widetilde{Y} - (A, K)) = \widetilde{Y} - int_\alpha(A, K)$. Thus $\widetilde{X} - int_\alpha(g^{-1}(A, K))\widetilde{\subseteq}g^{-1}(\widetilde{Y} - int_\alpha(A, K)) = g^{-1}(\widetilde{Y}) - g^{-1}(int_\alpha(A, K))$. Hence $g^{-1}(int_\alpha(A, K))\widetilde{\subseteq}int_\alpha(g^{-1}(A, K))$.

(v) $\Rightarrow$ (i): Suppose that (A, K) is any soft α-open subset of $\widetilde{Y}$. Since $g^{-1}(int_\alpha(A, K))\ \widetilde{\subseteq}int_\alpha(g^{-1}(A, K))$, then $g^{-1}(A, K)\widetilde{\subseteq}int_\alpha(g^{-1}(A, K))$. Since $int_\alpha(g^{-1}(A, K))\widetilde{\subseteq}g^{-1}(A, K)$, then $g^{-1}(A, K) = int_\alpha(g^{-1}(A, K))$. Therefore $g^{-1}(A, K)$ is a soft α-open set. Hence g is a soft α-irresolute map. □

Proposition 2.12. *The soft α-irresolute image of a soft α-compact (resp. soft α-Lindelöf) set is soft α-compact (resp. soft α-Lindelöf).*

Proof. For the proof, let $g : X \to Y$ be a soft α-irresolute mapping and let (D, K) be a soft α-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i, K) : i \in I\}$ is a soft α-open cover of $g(D, K)$. Then $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Now, $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}g^{-1}(H_i, K)$ and $g^{-1}(H_i, K)$ is soft α-open, for each $i \in I$. By hypotheses, (D, K) is soft α-Lindelöf, then $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}g^{-1}(H_i, K)$. Therefore $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}g(g^{-1}(H_i, K)) \widetilde{\subseteq} \widetilde{\bigcup}_{i\in S}(H_i, K)$. Thus $g(D, K)$ is soft α-Lindelöf.

A similar proof is given in case of a soft α-compact space. □

Proposition 2.13. *A soft subset (H, K) of (X, τ, K) is soft α-open if and only if there exists a soft open set (G, K) such that $(G, K)\widetilde{\subseteq}(H, K)\widetilde{\subseteq}int(cl((G, K))$.*

Proof. Necessity: Consider (H, K) is a soft α-open set.

Then $int(H, K)\widetilde{\subseteq}(H, K)\widetilde{\subseteq}int\ (cl(int(H, K)))$. Taking $int(H, K) = (G, K)$. Hence $(G, K)\widetilde{\subseteq}(H, K)\widetilde{\subseteq}int(cl((G, K))$.

Sufficiency: Suppose that (H, K) is a soft set such that there exists a soft open set (G, K) satisfies that $(G, K)\widetilde{\subseteq}(H, K)\widetilde{\subseteq}int(cl(G, K))$.

Then $cl(G, K)\widetilde{\subseteq}cl(int(H, K))$. So $int(cl(G, K))\widetilde{\subseteq}int(cl(int(H, K)))$. By our assumption, $(H, K)\widetilde{\subseteq}int(cl((G, K))$. Thus $(H, K)\widetilde{\subseteq}int(cl(int(H, K)))$. Hence the proof is complete. □

Corollary 2.14. *A soft subset (H, K) of (X, τ, K) is soft α-closed if and only if there exists a soft closed set (F, K) such that $cl(int(F, K))\widetilde{\subseteq}(H, K)\widetilde{\subseteq}(F, K)$.*

In what follows, we list some properties of an extended soft topology (Definition (1.11)) and its relationship with soft α-compact and soft α-Lindelöf spaces.

Proposition 2.15. *If H is an α-open subset of (X, τ_k), then there exists a soft α-open subset (F, K) of an extended soft topological space (X, τ, K) such that $F(k) = H$.*

Proof. Suppose that H is an α-open subset of (X, τ_k). Then there exists an open subset $G(k)$ of (X, τ_k) such that $G(k) \subseteq H \subseteq int(cl(G(k)))$. Since $G(k)$ is an open subset of (X, τ_k), then a soft set (F, K), which is defined as $F(k) = G(k)$ and $F(k_i) = \emptyset$, for each $k_i \neq k$, is soft open. Also, we define a soft set (L, K) as $L(k) = H$ and $L(k_i) = \emptyset$, for each $k_i \neq k$. So we infer that $(G, K)\widetilde{\subseteq}(L, K)\widetilde{\subseteq}int(cl(G, K))$. From Proposition (2.13), we obtain that (L, K) is soft α-open. □

Theorem 2.16. *If (X, τ, K) is an extended soft α-compact (resp. extended soft α-Lindelöf) space, then (X, τ_k) is α-compact (resp. α-Lindelöf), for each $k \in K$.*

Proof. We prove the theorem in case of an extended soft α-Lindelöf space and the other case is proven similarly.

Let $\{H_j(k) : j \in J\}$ be an α-open cover of (X, τ_k). We construct a soft α-open cover of (X, τ, K) consisting of the following soft sets:

(i) From the above proposition, we can choose all soft α-open sets (F_j, K) in which $F_j(k) = H_j(k)$, for each $j \in J$.

(ii) Since (X, τ, K) is extended, then we take a soft open set (G, K) which satisfies that $G(k) = \emptyset$ and $G(k_i) = X$, for all $k_i \neq k$.

Obviously, $\{(F_j, K)\widetilde{\bigcup}(G, K) : j \in J\}$ is a soft α-open cover of (X, τ, K). As (X, τ, K) is soft α-Lindelöf, then $\widetilde{X} = \bigcup_{j \in S}(F_j, K)\widetilde{\bigcup}(G, K)$. So $X = \bigcup_{j \in S} F_j(k) = \bigcup_{j \in S} H_j(k)$. Hence (X, τ_k) is an α-Lindelöf space. □

To show that the converse of the above theorem fails, we consider the example below.

Example 2.17. Let a set of parameters be the set of irrational numbers $\mathcal{Q}^c$ and τ be a soft discrete topology on $X = \{1, 2, 3\}$. A collection Λ which consists of all soft points of $\widetilde{X}$ forms a soft open cover of $\widetilde{X}$. Obviously, Λ has not a countable subcover. So $\widetilde{X}$ is not soft α-Lindelöf. But (X, τ_k) is soft α-compact, for each $k \in \mathcal{Q}^c$.

Now, we give a condition which guarantees the converse of the above theorem holds.

Proposition 2.18. *Let (X, τ_k) be extended and K be finite (resp. countable). Then (X, τ, K) is soft α-compact (resp. soft α-Lindelöf) space iff (X, τ_k) is α-compact (resp. α-Lindelöf), for each $k \in K$.*

Proof. Necessity: It is obtained from the theorem above.

Sufficiency: Let $\{(G_j, K) : j \in J\}$ be a soft α-open cover of (X, τ, K) and $| K |= m$. Then $X = \bigcup_{j\in J} G_j(k)$, for each $k \in K$.

It follows, from Proposition (2.13), that there exists a soft open set (H_j, K) such that $(H_j, K)\widetilde{\subseteq}(G_j, K)\widetilde{\subseteq}int(cl((H_j, K))$. By Theorem (1.13), we find that $int(cl((H_j, K)) = (int(cl(H_j)), K)$. So $G_j(k)$ is soft α-open subset of (X, τ_k), for each $j \in J$. As (X, τ_k) is α-compact for each $k \in K$, then $X = \bigcup_{j=1}^{j=n_1} G_j(k_1)$, $X = \bigcup_{j=n_1+1}^{j=n_2} G_j(k_2), \ldots, X = \bigcup_{j=n_{m-1}+1}^{j=n_m} G_j(k_m)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{j=1}^{j=n_m}(G_j, K)$. Thus (X, τ, K) is soft α-compact.

A similar proof can be given for the case between parentheses. □

Proposition 2.19. *If (U, K) is soft open and (H, K) is soft α-open subsets of (X, τ, K), then $(U, K)\widetilde{\bigcap}\ (H, K)$ is a soft α-open subset of $((U, K), \tau_{(U,K)}, K)$.*

Proof. Since (U, K) is soft open and (H, K) is soft α-open subsets of (X, τ, K), then
$(U, K)\widetilde{\bigcap}(H, K)\widetilde{\subseteq}\ (U, K)\widetilde{\bigcap}int(cl(int(H, K)))\widetilde{\subseteq}int_U[(U, K))\widetilde{\bigcap}cl(int(H, K)]$
$\widetilde{\subseteq}int_U(cl(U, K))\ \widetilde{\bigcap}int(H, K)$.

So $(U, K)\widetilde{\bigcap}(H, K)\ \widetilde{\subseteq}int_U[cl(U, K))\widetilde{\bigcap}int(H, K)]\widetilde{\bigcap}(U, K) = int_U[cl(U,\ K))\widetilde{\bigcap}$ $int(H, K)\widetilde{\bigcap}(U, K)] = int_U[cl_U[(U, K))\widetilde{\bigcap}int(H, K)]]$. Since (U, K) is soft open, then $int_U[cl_U[(U, K))\widetilde{\bigcap}int(H, K)]] = int_U[cl_U[int[(U, K))\widetilde{\bigcap}(H, K)]]]\widetilde{\subseteq}$ $int_U[cl_U[int_U[(U, K))\widetilde{\bigcap}\ (H, K)]]]$. Hence the proof is complete. □

Proposition 2.20. *For each soft open set (A, K) and soft set (B, K) in (X, τ, K), we have $(A, K)\widetilde{\bigcap}\ cl_\alpha(B, K)\widetilde{\subseteq}cl_\alpha((A, K)\widetilde{\bigcap}(B, K))$.*

Proof. Let $P^x_k \in (A, K)\widetilde{\bigcap}cl_\alpha(B, K)$. Then $P^x_k \in (A, K)$ and $P^x_k \in cl_\alpha(B, K)$. Therefore for each soft α-open set (U, K) containing P^x_k, we have $(U, K)\widetilde{\bigcap}(B, K)$ $\neq \widetilde{\emptyset}$. Since $(U, K)\widetilde{\bigcap}(A, K)$ is a non-null soft α-open set and $P^x_k \in (U, K)\widetilde{\bigcap}(A, K)$, then $((U, K)\widetilde{\bigcap}(A, K))\ \widetilde{\bigcap}(B, K) \neq \widetilde{\emptyset}$. Now, $(U, K)\widetilde{\bigcap}((A, K)\widetilde{\bigcap}(B, K)) \neq \widetilde{\emptyset}$ implies that $P^x_k \in cl_\alpha((A, K)\widetilde{\bigcap}(B, K))$.

Therefore $(A, K)\widetilde{\bigcap}cl_\alpha(B, K)\widetilde{\subseteq}\ cl_\alpha((A, K)\widetilde{\bigcap}(B, K))$. □

Lemma 2.21. *If (U, K) is a soft open subset of (X, τ, K) and (H, K) is soft α-open subset of $((U, K), \tau_{(U,K)}, K)$, then (H, K) is soft α-open subset of (X, τ, K).*

Proof. Since (H, K) is soft α-open subset of $((U, K), \tau_{(U,K)}, K)$, then
$(H, K)\widetilde{\subseteq}int_U[\ cl_U[int_U(H, K)]]\ \widetilde{\bigcap}(U, K) = int[cl_U[int_U(H, K)]]$
$\widetilde{\subseteq}int[cl[int_U(H, K)]] = int[cl[int_U\ [(H, K)\widetilde{\bigcap}(U, K)]]] = int[cl[int(H, K)]]$. So (H, K) is a soft α-open subset of (X, τ, K). □

Now, we are in a position to verify the following result.

Theorem 2.22. *A soft open subset (A,K) of (X,τ,K) is soft α-compact (resp. soft α-Lindelöf) if and only if a soft open subspace $((A,K),\tau_{(A,K)},K)$ is soft α-compact (resp. soft α-Lindelöf).*

Proof. We prove the theorem in case of soft α-compactness and the proof of the case between parentheses is made similarly.

Necessity: Let $\{(H_i,K) : i \in I\}$ be a soft α-open cover of $((A,K),\tau_{(A,K)},K)$. Since (A,K) is soft open containing (H_i,K), then it follows, by the above lemma, that (H_i,K) is soft α-open subsets of (X,τ,K).

By hypotheses, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i,K)$.

Thus a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft α-compact.

Sufficiency: Let $\{(G_i,K) : i \in I\}$ be a soft α-open cover of (A,K) in (X,τ,K). Now, $(A,K)\widetilde{\bigcap}(G_i,K)$ is a soft α-open subset of (X,τ,K).

By Proposition (2.19), we find that $(A,K)\widetilde{\bigcap}(G_i,K)$ is soft α-open subset of $((A,K),\tau_{(A,K)},K)$. As a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft α-compact, then $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}((A,K)\widetilde{\bigcap}(G_i,K))$. So $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(G_i,K)$. Thus (A,K) is a soft α-compact subset of (X,τ,K). □

Definition 2.23. *An STS (X,τ,K) is said to be soft α T_2'-space if for every two distinct soft points P_k^x and P_k^y, there are two disjoint soft α-open sets (G,K) and (F,K) such that $P_k^x \in (G,K)$ and $P_k^y \in (F,K)$.*

Lemma 2.24. *The soft intersection of a finite family of soft α-open sets is soft α-open.*

Proposition 2.25. *If (A,K) is a soft α-compact subset of a soft α T_2'-space (X,τ,K), then (A,K) is soft α-closed.*

Proof. Let the given conditions be satisfied and let $P_k^x \in (A,K)^c$. Then for each $P_k^y \in (A,K)$, there are two disjoint soft α-open sets (G_i,K) and (W_i,K) such that $P_k^x \in (G_i,K)$ and $P_k^y \in (W_i,K)$. It follows that $\{(W_i,K) : i \in I\}$ forms a soft α-open cover of (A,K). Consequently, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(W_i,K)$. By the above lemma, it follows that $\widetilde{\bigcap}_{i=1}^{i=n}(G_i,K) = (H,K)$ is a soft α-open set and since $(H,K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n}(W_i,K)] = \widetilde{\emptyset}$, then $(H,K)\widetilde{\subseteq}(A,K)^c$. Thus $(A,K)^c$ is a soft α-open set. Hence (A,K) is soft α-closed. □

Corollary 2.26. *If (A,K) is a stable soft α-compact subset of a soft α T_2-space (X,τ,K), then (A,K) is soft α-closed.*

Proof. Since (A,K) is stable, then $P_k^x \in (A,K)$ if and only if $x \in (A,K)$. So by using similar technique of the above proof, the corollary holds. □

3. Almost soft α-compact spaces

Definition 3.1. *An STS (X, τ, K) is called almost soft α-compact (resp. almost soft α-Lindelöf) if every soft α-open cover of $\widetilde{X}$ has a finite (resp. countable) soft sub-collection in which the soft α-closures of whose members cover $\widetilde{X}$.*

Definition 3.2. *A soft set (F, K) is said to be:*

(i) *Soft α-clopen provided that it is soft α-open and soft α-closed.*

(ii) *Soft α-dense set provided that $cl_\alpha(F, K) = \widetilde{X}$.*

For the purpose of brevity, we shall omit the proofs of the following three propositions.

Proposition 3.3. *Every almost soft α-compact space is almost soft α-Lindelöf.*

Proposition 3.4. *A finite (resp. countable) union of almost soft α-compact (resp. almost soft α-Lindelöf) subsets of (X, τ, K) is almost soft α-compact (resp. almost soft α-Lindelöf).*

Proposition 3.5. *Every soft α-compact (resp. soft α-Lindelöf) space is almost soft α-compact (resp. almost soft α-Lindelöf).*

The converse of Proposition (3.5) is incorrect as it is evident in the example below.

Example 3.6. We illustrate that the given soft topological space (X, τ, K) in Example (2.5) is not soft α-Lindelöf. On the other hand, we can note that any soft α-open subset of (X, τ, K) must contain a soft open set (G, K). Since (G, K) is soft α-dense, then any soft α-open set is soft α-dense. So (X, τ, K) is almost soft α-compact.

Proposition 3.7. *Every soft α-clopen subset (D, K) of an almost soft α-compact (resp. almost soft α-Lindelöf) space (X, τ, K) is almost soft α-compact (resp. almost soft α-Lindelöf).*

Proof. Let us prove the proposition in case of (X, τ, K) is almost soft α-compact, the case between parentheses can be achieved similarly.

Let (D, K) be a soft α-clopen subset of $\widetilde{X}$ and let $\{(H_i, K) : i \in I\}$ be a soft α-open cover of (D, K). Then (D^c, K) is soft α-clopen. Therefore $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since $\widetilde{X}$ is almost soft α-compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(H_i, K)\ \widetilde{\bigcup}(D^c, K)$. This implies that $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(H_i, K)$. Hence (D, K) is almost soft α-compact. $\square$

Corollary 3.8. *If (G, K) is an almost soft α-compact (resp. almost soft α-Lindelöf) subset of $\widetilde{X}$ and (D, K) is a soft α-clopen subset of $\widetilde{X}$, then (G, K) $\widetilde{\bigcap}(D, K)$ is almost soft α-compact (resp. almost soft α-Lindelöf).*

In Example (3.6), let (H,K) be a soft subset of (X,τ,K), where $H(k_1) = \{1,4\}$ and $H(k_i) = \{5\}$, for each $i \neq 1$. Then a soft set (H,K) is almost soft α-compact, but it is not soft α-clopen. So the converse of the above proposition is not necessarily correct.

Definition 3.9. *A collection $\Lambda = \{(F_i,K) : i \in I\}$ of soft sets is said to have the first type of finite (resp. countable) α-intersection property if $\widetilde{\bigcap}_{i=1}^{i=n} int_\alpha(F_i,K) \neq \widetilde{\emptyset}$, for any $n \in \mathcal{N}$ (resp. $\widetilde{\bigcap}_{i\in S} int_\alpha(F_i,K) \neq \widetilde{\emptyset}$, for any countable set S).*

It is clear that any collection satisfies the first type of finite (resp. countable) α-intersection property is also satisfies the finite (resp. countable) intersection property.

Theorem 3.10. *An STS (X,τ,K) is almost soft α-compact (resp. almost soft α-Lindelöf) if and only if every soft collection of soft α-closed subsets of (X,τ,K), satisfying the first type of finite (resp. countable) α-intersection property, has, itself, a non-null soft intersection.*

Proof. We will start with the proof for almost soft α-compactness, because the proof for almost soft α-Lindelöfness is analogous.

Let $\Lambda = \{(F_i,K) : i \in I\}$ be a soft α-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i,K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c,K)$. As (X,τ,K) is almost soft α-compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(F_i^c,K)$. Therefore $\widetilde{\emptyset} = (\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(F_i^c,K))^c = \widetilde{\bigcap}_{i=1}^{i=n} int_\alpha(F_i,K)$. Hence the necessary condition holds.

Conversely, let Λ be a soft α-closed subsets of $\widetilde{X}$ which satisfies the first type of finite α-intersection property. Then it also satisfies the finite intersection property. Since Λ has a non-null soft intersection, then (X,τ,K) is a soft α-compact space. It follows, by Proposition (3.5), that (X,τ,K) is almost soft α-compact. $\square$

Theorem 3.11. *The soft α-irresolute image of an almost soft α-compact (resp. almost soft α-Lindelöf) set is almost soft α-compact (resp. almost soft α-Lindelöf).*

Proof. For the proof, let $g : X \to Y$ be a soft α-irresolute mapping and (D,K) be an almost soft α-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i,K) : i \in I\}$ is a soft α-open cover of $g(D,K)$. Then $g(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i,K)$. Now, $(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I} g^{-1}(H_i,K)$ and $g^{-1}(H_i,K)$ is soft α-open, for each $i \in I$. By hypotheses, (D,K) is almost soft α-Lindelöf, then $(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S} cl_\alpha(g^{-1}(H_i,K))$. So $g(D,K)\widetilde{\subseteq}\ \widetilde{\bigcup}_{i\in S} g(cl_\alpha(g^{-1}(H_i,K)))$. From item **(iv)** of Theorem (2.11), we obtain that $g(cl_\alpha(g^{-1}(H_i,K)))\widetilde{\subseteq} cl_\alpha(g(g^{-1}\ (H_i,K)))\widetilde{\subseteq} cl_\alpha(H_i,K)$.

Thus $g(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S} cl_\alpha(H_i,K)$. Hence $g(D,K)$ is almost soft α-Lindelöf.

A similar proof is given in case of an almost soft α-compact space. $\square$

Definition 3.12. *An STS (X, τ, K) is said to be soft α-hyperconnected if it does not contain disjoint soft α-open sets.*

Proposition 3.13. *Every soft α-hyperconnected space is almost soft α-compact.*

Proof. Since any soft α-open set in a soft α-hyperconnected space is soft α-dens, then the space is almost soft α-compact. □

The converse of the above proposition need not be correct in general, as the following example shall illustrates.

Example 3.14. Let $K = \{k_1, k_2\}$ be a set of parameters and consider $\tau = \{\widetilde{\emptyset}, \widetilde{X}, (G,\ K), (H, K), (L, K)$ be a soft topology on $X = \{33, 44\}$ such that:

$(G, K) = (k_1, \{33\}), (k_2, \emptyset)$;
$(H, K) = (k_1, \emptyset), (k_2, \{44\})$ and
$(L, K) = (k_1, \{33\}), (k_2, \{44\})\}$.

Obviously, (X, τ, K) is almost soft α-compact. On the other hand, the soft sets (G, K) and (H, K) are two disjoint soft α-open sets. Then (X, τ, K) is not soft α-hyperconnected.

Theorem 3.15. *If (X, τ, K) is an extended almost soft α-compact (resp. extended almost soft α-Lindelöf) space, then (X, τ_k) is almost α-compact (resp. almost α-Lindelöf), for each $k \in K$.*

Proof. We prove the theorem in case of an almost soft α-compact space and the other proof follows similar lines.

Let $\{H_j(k) : j \in J\}$ be an α-open cover for (X, τ_k). We construct a soft α-open cover for $\widetilde{X}$ like the introduced soft α-open cover in the proof of Theorem (2.16). Now, (X, τ, K) is almost soft α-compact implies that $\widetilde{X} = \bigcup_{j=1}^{j=n} cl_\alpha[(F_j, K)\widetilde{\bigcup}(G, K)] = \bigcup_{j=1}^{j=n}[(cl_\alpha(F_j), K)\ \widetilde{\bigcup}(G, K)]$.

Therefore, $X = \bigcup_{j=1}^{j=n} cl_\alpha(F_j(k)) = \bigcup_{j=1}^{j=n} cl_\alpha(H_j(k))$. Hence (X, τ_k) is an almost α-compact space. □

It can be seen from Example (2.17) that the converse of the above theorem need not be true in general.

Proposition 3.16. *Let (X, τ_k) be extended and K be finite (resp. countable). Then (X, τ, K) is almost soft α-compact (resp. almost soft α-Lindelöf) space iff (X, τ_k) is almost α-compact (resp. almost α-Lindelöf), for each $k \in K$.*

Proof. Necessity: It is obtained from the theorem above.

Sufficiency: Let $\{(G_j, K) : j \in J\}$ be a soft α-open cover of (X, τ, K). By similar discussion of the proof of the sufficient part of of Proposition (2.18), we obtain that $X = \bigcup_{j=1}^{j=n_1} cl_\alpha(G_j(k_1))$, $X = \bigcup_{j=n_1+1}^{j=n_2} cl_\alpha(G_j(k_2))$,..., $X =$

$\bigcup_{j=n_{m-1}+1}^{j=n_m} cl_\alpha(G_j(k_m))$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{j=1}^{j=n_m} cl_\alpha(G_j, K)$. Since τ is extended, then it follows from Theorem (1.13), that $\widetilde{X} = \widetilde{\bigcup}_{j=1}^{j=n_m}(cl_\alpha(G_j), K) = \widetilde{\bigcup}_{j=1}^{j=n_m} cl_\alpha(G_j, K)$ Hence (X, τ, K) is almost soft α-compact.

A similar proof is given for the case between parentheses. □

Remark 3.17. If (X, τ, K) is an extended almost soft α-compact (resp. extended almost soft α-Lindelöf) space, then K is finite (resp. countable).

Proposition 3.18. *Consider $((U, K), \tau_{(U,K)}, K)$ is a soft subspace of (X, τ, K). Let cl_α and int_α stand for the soft α-closure and soft α-interior operators, respectively, in (X, τ, K) and let $cl_{\alpha U}$ and $int_{\alpha U}$ stand for the soft α-closure and soft α-interior operators, respectively, in $((U, K), \tau_{(U,K)}, K)$. Then:*

(i) *$cl_{\alpha U}(A, K) = cl_\alpha(A, K)\widetilde{\bigcap}(U, K)$, for each $(A, K)\widetilde{\subseteq}(U, K)$.*

(ii) *$int_\alpha(A, K) = int_{\alpha U}(A, K))$, for each $(A, K)\widetilde{\subseteq}(U, K)$.*

Theorem 3.19. *A soft open subset (A, K) of (X, τ, K) is almost soft α-compact (resp. almost soft α-Lindelöf) if and only if a soft open subspace $((A, K), \tau_{(A,K)}, K)$ is almost soft α-compact (resp. almost soft α-Lindelöf).*

Proof. Necessity: Let $\{(H_i, K) : i \in I\}$ be a soft α-open cover of $((A, K), \tau_{(A,K)}, K)$. Since (A, K) is soft open containing (H_i, K), then it follows, from Lemma (2.21), that (H_i, K) is soft α-open subsets of (X, τ, K).

By hypotheses, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(H_i, K) = \widetilde{\bigcup}_{i=1}^{i=n}[cl_\alpha(H_i, K)\widetilde{\bigcap}(A, K)$ $= \widetilde{\bigcup}_{i=1}^{i=n} cl_{\alpha U}(H_i, K)$. Thus a soft open subspace $((A, K), \tau_{(A,K)}, K)$ is almost soft α-compact.

Sufficiency: Let $\{(G_i, K) : i \in I\}$ be a soft α-open cover of (A, K) in (X, τ, K). Now, $(A, K)\widetilde{\bigcap}(G_i, K)$ is a soft α-open subset of (X, τ, K). By Proposition (2.19), we find that $(A, K)\widetilde{\bigcap}(G_i, K)$ is a soft α-open subset of $((A, K), \tau_{(A,K)}, K)$. As a soft open subspace $((A, K), \tau_{(A,K)}, K)$ is almost soft α-compact, then $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{\alpha U}[(A, K)\widetilde{\bigcap}(G_i, K)]\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{\alpha U}(G_i, K)$.

So, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(G_i, K)$. Thus (A, K) is an almost soft α-compact subset of (X, τ, K).

A case between parentheses can be proven similarly. □

Proposition 3.20. *If (A, K) is an almost soft α-compact subset of a soft α T_2'-space (X, τ, K), then (A, K) is soft α-closed.*

Proof. Let the given conditions be satisfied and let $P_k^x \in (A, K)^c$. Then for each $P_k^y \in (A, K)$, there are two disjoint soft α-open sets (G_i, K) and (W_i, K) such that $P_k^x \in (G_i, K)$ and $P_k^y \in (W_i, K)$. It follows that $\{(W_i, K) : i \in I\}$ forms a soft α-open cover of (A, K). Consequently, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(W_i, K)$.

By Lemma (2.24), we obtain $\widetilde{\bigcap}_{i=1}^{i=n}(G_i, K) = (H, K)$ is a soft α-open set and since $(H, K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n}(W_i, K)] = \widetilde{\emptyset}$, then $(H, K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n} cl_\alpha(W_i, K)] = \widetilde{\emptyset}$. So $(H, K)\widetilde{\subseteq}(A, K)^c$. Thus $(A, K)^c$ is a soft α-open set. Hence (A, K) is soft α-closed. □

Corollary 3.21. *If (A, K) is an almost soft α-compact stable subset of a soft α T_2-space (X, τ, K), then (A, K) is soft α-closed.*

4. Approximately soft α-compact spaces

Definition 4.1. *An STS (X, τ, K) is called approximately soft α-compact (resp. approximately soft α-Lindelöf) space if every soft α-open cover of $\widetilde{X}$ has a finite (resp. countable) soft sub-collection in which its soft α-closure cover $\widetilde{X}$.*

Proposition 4.2. *Every approximately soft α-compact space is approximately soft α-Lindelöf*

Proof. Straightforward. □

We give an example below in order to show that the converse of the above proposition is not correct in general.

Example 4.3. Consider $(\mathcal{R}, \tau, K)$ is a soft topological space such that $K = \{k_1, k_2\}$ is a set of parameters and $\tau = \{\widetilde{\emptyset}, (G_i, K)\widetilde{\subseteq}\mathcal{R}$ such that for each $k \in K$, $G_i(k) = \{n\}$ or their soft union$\}$. Then any soft set (G, K) is soft α-open if and only if there exists $n \in \mathcal{N}$ such that $n \in (G_i, K)$. We define a soft α-open cover Λ of $\widetilde{X}$ as follows, $\Lambda = \{(G, K) : G(k) = \{1, x\}$, for each $k \in K\}$. This soft α-open cover has not a finite sub-cover in which its soft α-closure cover $\widetilde{X}$, hence $(\mathcal{R}, \tau, K)$ is not approximately soft α-compact. On the other hand, for any soft α-open cover of $\widetilde{\mathcal{R}}$, we can find countable soft α-open subsets of Λ contains a soft open set $\{(G(k_1), \mathcal{N}), (G(k_2), \mathcal{N})\}$. This a soft α-open set is soft α-dense, hence $(\mathcal{R}, \tau, K)$ is approximately soft α-Lindelöf.

Proposition 4.4. *A finite (resp. countable) union of approximately soft α-compact (resp. approximately soft α-Lindelöf) subsets of (X, τ, K) is approximately soft α-compact (resp. approximately soft α-Lindelöf).*

Proof. Let $\{(A_s, K) : s \in S\}$ be approximately soft α-Lindelöf subsets of (X, τ, K) and let $\{(G_i, K) : i \in I\}$ be a soft α-open cover of $\widetilde{\bigcup}_{s\in S}(A_s, K)$. Then there exist countable sets M_s such that $(A_1, K)\widetilde{\subseteq} cl_\alpha(\widetilde{\bigcup}_{i\in M_1}(G_i, K))$, $\ldots$, $(A_n, K)\widetilde{\subseteq} cl_\alpha(\widetilde{\bigcup}_{i\in M_n}(G_i, K))$, $\ldots$.

Therefore, $\widetilde{\bigcup}_{s\in S}(A_s, K)\widetilde{\subseteq} cl_\alpha(\widetilde{\bigcup}_{i\in M_1}(G_i, K))\widetilde{\bigcup} \ldots \widetilde{\bigcup} cl_\alpha(\widetilde{\bigcup}_{i\in M_n}(G_i, K))\widetilde{\bigcup} \ldots$ $\widetilde{\subseteq} cl_\alpha$ $(\widetilde{\bigcup}_{i\in \bigcup_{s\in S} M_s}(G_i, K))$. Since $\bigcup_{s\in S} M_s$ is a countable set, then the desired result is proved.

A similar proof is given in case of an approximately soft α-compact space. □

Proposition 4.5. *Every almost soft α-compact (resp. almost soft α-Lindelöf) space is approximately soft α-compact (resp. approximately soft α-Lindelöf).*

Proof. The proof is obtained directly from the fact that
$\widetilde{\bigcup}_{i\in I} cl_\alpha(G_i, K) \widetilde{\subseteq} cl_\alpha(\widetilde{\bigcup}_{i\in I}(G_i, K))$. □

Corollary 4.6. *Every soft α-hyperconnected space is approximately soft α-Lindelöf.*

We give an example below in order to show that the converse of the above proposition is not correct in general.

Example 4.7. Consider $(\mathcal{R}, \tau, K)$ is a soft topological space such that $K = \{k_1, k_2\}$ and $\tau = \{\widetilde{\emptyset}, \widetilde{\mathcal{R}}, (G_1, K), (G_2, K), (G_3, K)$, where

$$(G_1, K) = \{(k_1, \{1\}), (k_2, \{1\}\};$$
$$(G_2, K) = \{(k_1, \{2\}), (k_2, \{2\}\};$$
$$(G_3, K) = \{(k_1, \{1, 2\}), (k_2, \{1, 2\}\}.$$

Then any soft set (G, K) is soft α-open if and only if $1 \in (G, K)$ or $2 \in (G, K)$. We define a soft α-open cover Λ of $\widetilde{X}$ as follows, $\Lambda = \{$for each $k \in K$, $(G, K) : G(k) = \{1, x\} : x \neq 2$ and $(H, K) : H(k) = \{2\}\}$. This soft α-open cover has not a countable sub-cover in which its soft α-closure of whose members cover $\widetilde{X}$, hence $(\mathcal{R}, \tau, K)$ is not almost soft α-Lindelöf. On the other hand, any soft α-open cover contains a soft α-open set (G_3, K). A soft α-open set (G_3, K) is soft α-dense, hence $(\mathcal{R}, \tau, K)$ is approximately soft α-compact.

Definition 4.8. *A collection $\Lambda = \{(F_i, K) : i \in I\}$ of soft sets is said to have the second type of finite (resp. countable) α-intersection property if $int_\alpha[\widetilde{\bigcap}_{i=1}^{i=n}(F_i, K)] \neq \widetilde{\emptyset}$, for any $n \in \mathcal{N}$ (resp. $int_\alpha[\widetilde{\bigcap}_{i\in S}(F_i, K)] \neq \widetilde{\emptyset}$, for any countable set S).*

It is clear that any collection satisfies the second type of finite (resp. countable) α-intersection property is also satisfies the first type of finite (resp. countable) α-intersection property.

Theorem 4.9. *An STS (X, τ, K) is approximately soft α-compact (resp. approximately soft α-Lindelöf) if and only if every soft collection of soft α-closed subsets of (X, τ, K), satisfying the second type of finite (resp. countable) α-intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X, τ, K) is approximately soft α-compact, the other case can be made similarly.

Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft α-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As (X, τ, K) is approximately soft α-compact, then $\widetilde{X} = cl_\alpha(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K))$. Therefore $\widetilde{\emptyset} = (cl_\alpha(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K)))^c = int_\alpha(\widetilde{\bigcap}_{i=1}^{i=n}(F_i, K))$. Hence the necessary condition holds.

Conversely, Let Λ be a soft α-closed subsets of $\widetilde{X}$ which satisfies the second type of finite α-intersection property. Then it also satisfies the first type of finite α-intersection property. Since Λ has a non-null soft intersection, then (X, τ, K) is an almost soft α-compact space. It follows, by Proposition (4.5), that (X, τ, K) is approximately soft α-compact. □

Definition 4.10. *A topological space (X, τ) is called approximately α-compact (resp. approximately α-Lindelöf) space if every α-open cover of X has a finite (resp. countable) sub-cover in which its α-closure cover X.*

Theorem 4.11. *A soft open subset (A, K) of (X, τ, K) is approximately soft α-compact (resp. approximately soft α-Lindelöf) if and only if a soft subspace $((A, K),\ \tau_{(A,K)}, K)$ is approximately soft α-compact (resp. approximately soft α-Lindelöf).*

Proof. The proof is similar of that Theorem (3.19). □

Definition 4.12. *An STS (X, τ, E) is called soft α-separable provided that it contains a countable α-dense soft set.*

Proposition 4.13. *If there exists a finite (resp. countable) soft α-dense subset of an STS (X, τ, K) such that K is finite (resp. countable), then (X, τ, K) is approximately soft α-compact (resp. approximately soft α-Lindelöf).*

Proof. Let $\{(G_i, K) : i \in I\}$ be a soft α-open cover of (X, τ, K) and let (B, K) be a finite (countable) soft α-dense subset of (X, τ, K). Then for each $P_{k_s}^{x_s} \in (B, K)$, there exists (G_{x_s}, K) containing $P_{k_s}^{x_s}$. This implies that $\widetilde{X} = cl_\alpha[\widetilde{\bigcup}(G_{x_s}, K)]$. Since (B, K) and K are finite (countable), then the collection $\{(G_s, K)\}$ is finite (countable). Hence the proof is complete. □

Corollary 4.14. *Every soft α-separable with a countable set of parameters K is approximately soft α-Lindelöf.*

Theorem 4.15. *The soft α-irresolute image of an approximately soft α-compact (resp. approximately soft α-Lindelöf) set is approximately soft α-compact (resp. approximately soft α-Lindelöf).*

Proof. We prove the theorem by using a similar technique of the proof of Theorem (3.11) and employing item **(iii)** of Theorem (2.11). □

Proposition 4.16. *If (A, K) is an approximately soft α-compact subset of a soft α T_2'-space (X, τ, K), then (A, K) is soft α-closed.*

Proof. The proof is similar of that Proposition (3.20). □

Corollary 4.17. *If (A, K) is an approximately soft α-compact stable subset of a soft α T_2-space (X, τ, K), then (A, K) is soft α-closed.*

5. Mildly soft α-compact spaces

Definition 5.1. *An STS (X, τ, K) is called mildly soft α-compact (resp. mildly soft α-Lindelöf) if every soft α-clopen cover of $\widetilde{X}$ has a finite (resp. countable) soft subcover.*

The proofs of the next two propositions are easy and will be omitted.

Proposition 5.2. *A finite (resp. countable) union of mildly soft α-compact (resp. mildly soft α-Lindelöf) subsets of (X, τ, K) is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Proposition 5.3. *Every mildly soft α-compact space is mildly soft α-Lindelöf.*

It can be seen from Example (4.3) that the converse of above proposition fails.

Proposition 5.4. *Every almost soft α-compact (resp. almost soft α-Lindelöf) space (X, τ, K) is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Proof. We only prove the proposition in case of (X, τ, K) is almost soft α-Lindelöf, the other case can be achieved similarly.

Let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft α-clopen coverof (X, τ, K). Then $\widetilde{X} = \widetilde{\bigcup}_{s \in S} cl_\alpha(H_i, K)$. Now, $cl_\alpha(H_i, K) = (H_i, K)$. Therefore (X, τ, K) is mildly soft α-Lindelöf. □

Corollary 5.5. *Every soft α-compact (resp. soft α-Lindelöf) space is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Corollary 5.6. *If (X, τ, K) is soft α-hyperconnected, then the following six concepts are equivalent:*

(i) *Almost soft α-compact;*

(ii) *Almost soft α-Lindelöf;*

(iii) *Approximately soft α-compact;*

(iv) *Approximately soft α-Lindelöf;*

(v) *Mildly soft α-compact;*

(vi) *Mildly soft α-Lindelöf.*

Proposition 5.7. *Every soft α-connected space (X, τ, E) is mildly soft α-compact.*

Proof. Because (X, τ, K) is soft α-connected, then the only soft α-clopen subsets of (X, τ, K) are $\widetilde{X}$ and $\widetilde{\emptyset}$. Therefore (X, τ, K) is mildly soft α-compact.

□

One can be easily seen from Example (3.14) that the two soft sets (M, K) and (V, K), where $M(k_1) = M(k_2) = \{33\}$ and $V(k_1) = M(k_2) = \{44\}$, are disjoint soft α-open and their soft union is $\widetilde{X}$. So the converse of the above proposition is not always true.

In the next example, we illuminate that an approximately soft α-compact space need not be mildly soft α-Lindelöf.

Example 5.8. Assume that $(\mathcal{R}, \tau, K)$ is the same as in Example (4.7). We illustrated that $(\mathcal{R}, \tau, K)$ is an approximately soft α-Lindelöf space. The given soft collection Λ forms an α-clopen cover of $\mathcal{R}$. Since Λ has not a countable sub-cover, then $(\mathcal{R}, \tau, K)$ is not a mildly soft α-Lindelöf space.

Theorem 5.9. *An STS (X, τ, K) is mildly soft α-compact (resp. mildly soft α-Lindelöf) if and only if every soft collection of soft clopen subsets of (X, τ, K), satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. The proof is similar to that of Theorem (2.9). □

Proposition 5.10. *The soft α-irresolute image of a mildly soft α-compact (resp. mildly soft α-Lindelöf) set is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Proof. By using a similar technique of the proof of Proposition (2.12), the proposition holds. □

For the sake of economy, the proofs of the following two results will be omitted.

Proposition 5.11. *If (D, K) is a soft α-clopen subset of a mildly soft α-compact (resp. mildly soft α-Lindelöf) space (X, τ, K), then (D, K) is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Corollary 5.12. *If (G, K) is a mildly soft α-compact (resp. mildly soft α-Lindelöf) subset of $\widetilde{X}$ and (D, K) is a soft α-clopen subset of $\widetilde{X}$, then $(G, K)\widetilde{\bigcap}(D, K)$ is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Definition 5.13. *An STS (X, τ, K) is said to be soft α-partition provided that a soft set is soft α-open if and only if it is soft α-closed.*

Theorem 5.14. *Let (X, τ, K) be a soft α-partition topological space. Then the following four statements are equivalent.*

(i) *(X, τ, K) is soft α-Lindelöf (resp. soft α-compact);*

(ii) *(X, τ, K) is almost soft α-Lindelöf (resp. almost soft α-compact);*

(iii) *(X, τ, K) is approximately soft α-Lindelöf (resp. approximately soft α-compact);*

(iv) *(X, τ, K) is mildly soft α-Lindelöf (resp. mildly soft α-compact).*

Proof. **(i) $\to$ (ii)**: It follows from Proposition (3.5).

(ii) $\to$ (iii): It follows from Proposition (4.5).

(iii) $\to$ (iv): Let $\{(G_i, K) : i \in I\}$ be a soft α-clopen cover of $\widetilde{X}$. As (X, τ, K) is approximately soft α-Lindelöf, then $\widetilde{X} \widetilde{\subseteq} cl_\alpha(\widetilde{\bigcup}_{s \in S}(G_i, K))$ and as (X, τ, K) is soft α-partition, then $cl_\alpha(\widetilde{\bigcup}_{s \in S}(G_i, K)) = \widetilde{\bigcup}_{s \in S}(G_i, K)$. Therefore (X, τ, K) is mildly soft α-Lindelöf.

(iv) $\to$ (i): Let $\{(G_i, K) : i \in I\}$ be a soft α-open cover of $\widetilde{X}$. As (X, τ, K) is soft α-partition, then $\{(G_i, K) : i \in I\}$ is a soft α-clopen cover of $\widetilde{X}$ and as (X, τ, K) is mildly soft α-Lindelöf, then $\widetilde{X} = \widetilde{\bigcup}_{s \in S}(G_i, K)$.

A similar proof can be given for the case between parentheses. □

Lemma 5.15. *If H is an α-clopen subset of (X, τ_k), then there exists a soft α-clopen subset (F, K) of an extended soft topological space (X, τ, K) such that $F(k) = H$.*

Proof. Suppose that H is an α-clopen subset of (X, τ_k). Then $cl(int(cl(H))) \subseteq H \subseteq int(cl(int(H)))$. Now, we define a soft set (L, K) as $L(k) = H$ and $L(k_i) = \emptyset$, for each $k_i \neq k$.

Since (X, τ, K) is extended, then we can conclude that $(cl(int(cl(L))), K) = cl(int(cl(L, K))) \widetilde{\subseteq} (L, K) \widetilde{\subseteq} (int(cl(int(L))), K) = int(cl(int(L, K)))$.

Hence (L, K) is a soft α-clopen subset of (X, τ, K) □

Theorem 5.16. *If (X, τ, K) is an extended mildly soft α-compact (resp. extended mildly soft α-Lindelöf) space, then (X, τ_k) is mildly α-compact (resp. mildly α-Lindelöf), for each $k \in K$.*

Proof. We prove the theorem in case of an extended mildly soft α-Lindelöf space and the other case is proven similarly.

Let $\{H_j(k) : j \in J\}$ be an α-clopen cover of (X, τ_k). We construct a soft α-open cover of (X, τ, K) consisting of the following soft sets:

(i) From the above lemma, we can choose all soft α-clopen sets (F_j, K) in which $F_j(k) = H_j(k)$, for each $j \in J$.

(ii) Since (X, τ, K) is extended, then we take a soft clopen set (G, K) which satisfies that $G(k) = \emptyset$ and $G(k_i) = X$, for all $k_i \neq k$.

Obviously, $\{(F_j, K) \widetilde{\bigcup} (G, K) : j \in J\}$ is a soft α-clopen cover of (X, τ, K). As (X, τ, K) is mildly soft α-Lindelöf, then $\widetilde{X} = \bigcup_{j \in S}(F_j, K) \widetilde{\bigcup} (G, K)$. So $X = \bigcup_{j \in S} F_j(k) = \bigcup_{j \in S} H_j(k)$. Hence (X, τ_k) is a mildly α-Lindelöf space. □

It can be seen from Example (2.17) that the converse of the above theorem need not be true in general.

Proposition 5.17. *Let (X, τ, K) be extended and K be finite (resp. countable). Then (X, τ, K) is soft mildly α-compact (resp. extended soft mildly α-Lindelöf) space iff (X, τ_k) is mildly α-compact (resp. mildly α-Lindelöf), for each $k \in K$.*

Proof. Necessity: It is obtained from the theorem above.

Sufficiency: Let $\{(G_j, K) : j \in J\}$ be a soft α-clopen cover of (X, τ, K). Then $X = \bigcup_{j\in J} G_j(k)$ for each $k \in K$. It follows, from Lemma (5.15), that there exists a soft clopen set (H_j, K) such that $H_j(k) = G_j(k)$ and $H_j(k_i) = X$, for each $k_i \neq k$. As (X, τ_k) is mildly α-compact, for each $k \in K$, then $X = \bigcup_{j=1}^{j=n_1} G_j(k_1)$, $X = \bigcup_{j=n_1+1}^{j=n_2} G_j(k_2)$,..., $X = \bigcup_{j=n_{m-1}+1}^{j=n_m} G_j(k_m)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{j=1}^{j=n_m}(G_j, K)$. Thus (X, τ, K) is mildly soft α-compact.

A similar proof can be given for the case between parentheses. □

Remark 5.18. If (X, τ, K) is an extended mildly soft α-compact (resp. extended mildly soft α-Lindelöf) space, then K is finite (resp. countable).

Definition 5.19. *A collection β of soft α-open sets is called soft α-base of (X, τ, K) if every soft α-open subset of $\widetilde{X}$ can be written as a soft union of members of β*

Theorem 5.20. *Consider (X, τ, K) has a soft α-base consists of soft α-clopen sets. Then (X, τ, K) is soft α-compact (resp. soft α-Lindelöf) if and only if it is mildly soft α-compact (resp. mildly soft α-Lindelöf).*

Proof. The necessary condition is obvious.

To verify the sufficient condition, assume that Λ is a soft α-open cover of a mildly soft α-compact space (X, τ, K). Since $\widetilde{X}$ is a soft union of members of the soft α-base and $\widetilde{X}$ is mildly soft α-compact, then we can find a finite member (H_s, K) of the soft α-base satisfies that $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(H_s, K)$. So for each member (G_s, K) of Λ, there exists a member (H_s, K) of the soft α-base such that $(H_s, K)\widetilde{\subseteq}(G_s, K)$. Thus $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(G_s, K)$. Hence (X, τ, K) is soft α-compact.

The proof in case of a mildly soft α-Lindelöf space is similar. □

Proposition 5.21. *If (A, K) is a mildly soft α-compact subset of a soft α T_2'-space (X, τ, K), then (A, K) is soft α-closed.*

Proof. The proof is similar to that of Proposition (2.25). □

Corollary 5.22. *If (A, K) is a mildly soft α-compact stable subset of a soft α T_2-space (X, τ, K), then (A, K) is soft α-closed.*

Conclusion

The purpose of the present study is to establish and introduce eight generalized forms of soft compactness and soft Lindelöfness, namely soft α-compactness, soft α-Lindelöfness, almost (approximately, mildly) soft α-compactness and almost (approximately, mildly) soft α-Lindelöfness. With the help of illustrative examples, the relationships among these concepts are shown and the image of these spaces under soft α-irresolute maps is investigated. Some properties of soft α-open sets which enable us to prove certain of our results are studied and verified. The relationships of some of the introduced spaces with soft α T_2-spaces and soft α T_2'-spaces are given. We study the equivalent conditions for all of the initiated spaces and illustrate under what conditions the four types of soft α-compact (the four types of soft α-Lindelöf) spaces are equivalent. The eight introduced concepts are compared in relation with many soft topological notions such as soft α-connectedness, soft subspaces and soft α-partition. The concepts presented in this study are fundamental for further researches and will open a way to improve more applications on soft topology.

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Accepted: 8.01.2019

Trust-based Testbed for P2P digital library

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Abstract. Retrieving relevant information from trust-based Peer-to-Peer (P2P) networks is a challenging research task. The users on P2P networks endeavor to ensure that the peers have to provide guaranteed relevant information (or documents) for their information needs. Recent researches in evaluating the trustworthiness or even the reputation of peers lack a well-formulated testbed. As such, building a robust testbed for evaluating trustworthy P2P networks motivates to propose a ground truth testbed as a contribution in P2P systems. In this paper, a trust-based testbed is developed especially in P2P digital libraries using TREC WT10g collection which has been used for evaluation in many P2P networks. The testbed contains a set of peers of trustworthy values for each peer that is estimated using peer rank approach. The statistical factors such as the distribution of peer trustworthiness, distribution of relevant documents and the location of relevant documents within each setting, and the importance of proposed testbed to reach 100% Recall in information retrieval are analyzed. The results are compared with two well-known testbeds.
Keywords: P2P, trustworthiness, page rank, peer rank, digital library.

1. Introduction

Peer-to-Peer (P2P) overlay network is a distributed computing system emerging as a popular way to share a huge amount of data [14]. The nature of the P2P

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overlay network gives the flexibility to form many application scenarios from open file sharing (content exchange) and resource sharing (CPU, storage and bandwidth) to digital libraries [8]. Digital Library (DL) application is considered one of the common applications of such a network for content storage and retrieval [16]. These libraries have common characteristics such as (i) contain a large number of documents, (ii) their documents are fairly uniformly distributed and (iii) they share the same popularity in the pattern of documents access. In order to build a testbed to evaluate the performance of IR systems related to DL applications, many factors should be taken into account such as the digital library trustworthiness in order to retrieve pure and high-quality materials. Although the traditional search engines that are centralized or distributed over self-organized P2P overlay networks mainly focus on improving the retrieval effectiveness and efficiency, the users of P2P networks concern with relevant and trustworthy documents. These documents are relevant to the issued query and at the same time are not corrupted or contain viruses to satisfy their information needs. Because of the content distribution in P2P information Retrieval (P2PIR) is subject to constant change as results of operations such as insertion, deletion, and/or modification, the quality of the published contents (i.e. their trustworthiness) becomes an important issue in these applications. This means that there are some peers more trustworthy than the others. Thus, trust-aware P2P overlay networks are concerned with the retrieval of not only relevant but also high-quality documents from trusted peers.

Several types of research have been dedicated to essentially improve and focus on the retrieval effectiveness of DL applications. In [10] proposed a hybrid P2PIR approach using a large scale testbed and a set of queries. The retrieval effectiveness is estimated where the efficiency of the system is evaluated based on the number of routed queries. They considered, in their testbed, the combined documents that have similar content (topic) by using the similarity-based soft-clustering algorithm. They only created the testbed depending on the topic of documents ignoring any factors that are related to documents or peers trustworthiness. In [7] proposed a cluster-based P2P architecture that is evaluated using six testbeds developed from WT10g collection as real suite large-scale testbeds [9]. In [9] considered three factors for building their testbeds: (i) document distribution across peer-collection (which follows a power-law distribution), (ii) the location of relevant documents (Recall) and (iii) coverage of the topics (precision). However, they did not mention or even tackled the trust factor associated with DL applications. The same testbeds are used by [1,2,3] for evaluating a set of routing techniques on a cluster-based architecture P2P network to retrieved relevant documents of a given query maintained into clusters (or super-peers). To our knowledge, there is no common ground truth testbed that is realistic to be used for P2P system evaluation depending on the trust factor. Therefore, an approach is proposed to build a testbed by measuring the importance of the peers (i.e. their trustworthiness) by estimating and assigning a trust value, named peer rank. These values facilitate routing the given query to the relevant and trustworthy peers in providing high-quality contents. *The contribution here is suggesting a proposed testbed for P2P overlay architectures that focusing on peer trustworthiness evaluation. In particular, the testbed utilizes the well-known page rank algorithm to estimate the peers rank based on the connected graph between each other.*

The remainder of the paper is organized as follows. Section 2 presents research related to trustworthiness; Section 3 discusses the proposed approach on building the testbed. Section 4 presents the experimental settings and the data set used for constructing the proposed testbed and also analyzes the created testbeds from trustworthiness and effectiveness point of views, followed by Section 5 that concludes the findings of the paper and the future work.

2. Related works

Identifying the quality of documents stored in P2P network nodes has revealed a new concept in P2P information retrieval research area called entity trustworthiness. The trustworthiness of an entity could be defined as the quality of relevant documents or service providers for satisfying the user's information needs [3]. There are many techniques to estimate the trust values of peers, one of them is gathering the number of positive or negative feedback from the other peers to be exploited for estimating the reputation of a specific peer [17]. The trustworthiness of peers in P2PIR must be taken into consideration by researchers due to apparently the widespread applications that depend on P2P systems such as block-chain systems which might be a security mechanism for authentication and authorization [13]. In addition, the users in P2P networks have full permission to publish their content which might lead to overwhelming the network with malicious or selfish peers or even untrustworthy documents or feedback. Therefore, the necessity for robust techniques in order to filter such malicious (or selfish) peers or untrustworthy documents (or feedback) is high.

Many techniques have been proposed to compute the trustworthiness of documents and peers. For instances, EigenTrust [6] is a reputation-based system that depends on filtering out inauthentic files in P2P file-sharing networks. In this technique, a unique global trust value for each peer in the system is estimated based on the local opinions of all the other peers. The local trust value is a summation of user ranks that are gathered directly or indirectly of either positive or negative rate. In [15] proposed a multi-agent trustworthiness model where each agent maintains models of its acquaintances that describe their expertise (the quality of the services they provide) and sociability (the quality of the referrals they provide). These two works used their own testbeds to evaluate their systems and compute the trust factor through the lifetime of the system. In [12] suggested two trustworthiness methods to calculate the trustworthiness of a specific peer: subjective and objective trustworthiness. The subjective trustworthiness is estimated through direct interaction with the acquaintance peer where the objective trustworthiness is estimated by collecting trustworthiness information from other peers. This is similar to the reputation-based trust model that uses experience or the experiences of others as a recommendation. In [18] also proposed to estimate documents trustworthiness depend on document reputation (objective trustworthiness) and peer trustworthiness (subjective and objective trustworthiness). In [4] built an approach that depends on the combination of some metrics to calculate the trust values of peers. The system suggested many factors to calculate the trustworthy peer value, where the increase in factors will force these systems to built by a small number of peers for evaluation. VectorTrust [20] is a trust management system for aggregating distributed trust scores, which is built on a Trust Overlay Network (TON) on the top of decentralized P2P networks. VectorTrust uses a Bellman-Ford based distributed algorithm for fast trust score aggregation.

Although there are many models for P2P overlay networks that cover the trust factors in these type of the systems, these proposed systems use their own testbeds for trust-based evaluation which are small in their nature. Therefore, they do not reflect the real environments and provide a useful testbed for comparison. As such, this motivates us to build a testbed simulate the trust values of peers. The proposed testbed is built based formally on the peers rank values; which is the number of access to a specific peer by the link structure of documents inside the peer as a factor for trustworthiness.

3. Trust-based Testbed for digital libraries

The proposed trust-aware testbed has the characteristics of real DL applications. The documents are distributed evenly over the peers with almost the same number of documents. This paper proposes a method to build a testbed that satisfies such characteristic called Digital Library Peer Rank (DLPR). The UWOR (Uniform WithOut Replication) testbed is used built by [9] to be used for systems with uniformly documents distribution, as initial testbed to build DLPR. The UWOR testbed contains 11,680 domains (or peers) that guarantee the equal number of documents on each peer for the proposed testbed. The process of building DLPR testbed has three phases include estimating page ranking, selecting top domains and documents transfer. These phases are discussed in more detail as follows:

Page ranking: page rank indicates the importance of the page by the number of times a user visits it through browsing the web giving either an implicit or explicit feedback. Implicit feedback is derived from different signals depending on the user behaviors such as the time spent on viewing documents during the search process (or dwell time), page scrolling or browsing actions [5]. On the other hand, the explicit feedback is estimated directly from the user during click-through documents as relevant documents. The feedback gives an indication of the importance related to the visited documents. The documents in the web, however, contains links to other topically related documents which forms the building block link structure for estimating the page rank of the web

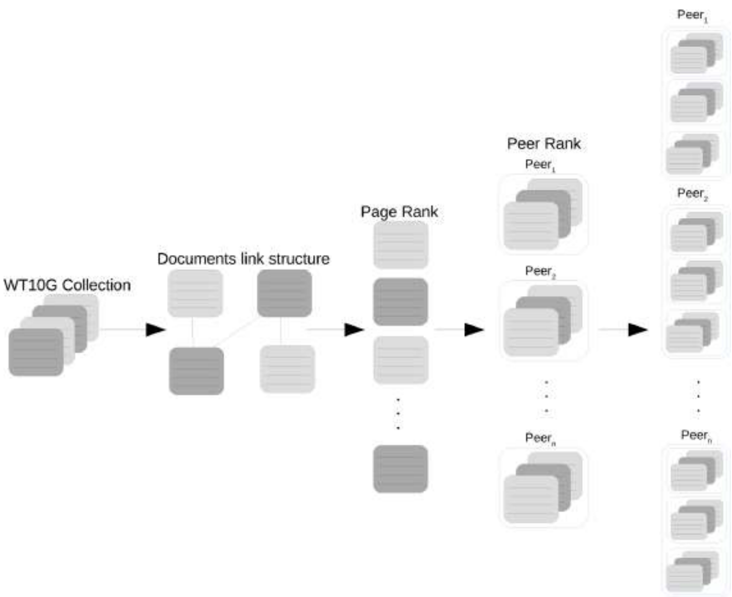

Figure 1: The process of building trust-based Testbed

pages. This rank value has the ability to distinguish the popular (those with incoming links or those that have links from popular pages) and unpopular web pages by using the in-link and out-link files [11,19]. In order to estimate the page rank of web documents, a graph of documents has to be built and traversed for derived the page rank. The graph comprises a set of nodes (or vertices) represent the documents in the corpus (or collection of documents) and a set of links (or edges) refer to the hyperlinks (in-links or out-links) between pages. Such a graph is used to estimate the document rank by walking through the graph and estimate the number of times document is visited as a probability chain the link is being visited by the others. The general formula to calculate the Page Rank for any page u is:

$$PR(u) = \frac{\lambda}{N} + (1-\lambda) \times \sum_{v \in B_u} \frac{PR(v)}{L_v} \tag{1}$$

where N is the number of pages being considered, λ is a constant with typical value 0.15, B_u is the set of pages that point to u, and L_v is the number of outgoing links from page v (not counting duplicate links).

Peer ranking: The process of estimating the ranks of peers relies on the rank of their documents. In particular, the page ranks are estimated for each WT10g document that has in-link and out-link structures. The peer rank based on the rank of its documents is evaluated, then the peers are sorted based on their rank values in descending order. The DLPR is then built by selecting the top 8 peers from the sorted list, this number was selected to guarantee a small number of libraries and on the same time ensure the equality in the number of documents in each peer. These top peers are used as attractors for other peers where the documents of the other peers are distributed over these attractors. This process is continued until there are no peers left in the list. As a result, this method generates 1460 number of peers each of which has 8 domains (or peers) with an average of 1160 documents.

Document transfer: The transfer of documents in the testbed ensures the uniform distribution of documents that represent the characteristics of DL applications. Nevertheless, the domain of trustworthiness needs a rank system for assigning trust values to each peer as discussed. The question is how can exploit the rank values of peers to be used as trust metric value for re-ranking the peers at query time. In summary, building a realistic testbed for DL applications reflecting the trustworthiness manner leads to the necessity for a metric that measures the trust factor of each peer collection. As a result, this paper proposes a peer-rank based metric for building a trust-based testbed.

4. Experimental setting and results

4.1 Experimental settings

Test collection. The TRECs Web 10G Collection (WT10g) is used that consists of 11,680 web domains, 1,692,096 documents and relevance assessment comprises of 100 topic query (Ad-hoc TREC 2000 task). The motivation behind using such collection is because the testbeds that are built by [9] have a set of characteristics and they are large enough to reflect the real situation in P2P architectures.

Evaluation metrics. Two metrics have used that estimate the trustworthiness values of documents and peers used in the proposed testbed that is being compared with the other two testbeds proposed by [7] and [10] as baseline testbeds. These two metrics include document ranking (or page rank) and peer ranking (extracted from the average page rank of the peers documents). Moreover, in order to evaluate the effectiveness and efficiency of the proposed trust-based testbed, a set of metrics were examined such as peer rank, the distribution of documents, fraction of peers to reach 100% Recall values and average Precision values. The peer rank value is obtained from its documents that represents the importance of a peer regarding the number of visits using the peer link structure. The distribution of the documents across peers reveals the equal number of documents over each peer to reflect the specific features characterize the DL applications. The fraction of peers required to achieve 100% Recall (for each topic) clarifies for each query how many peers in average is required to reach Recall value of 100%. The average precision represents the number of relevant documents in the results list to the whole number of documents. This metric is used in three testbeds to analyze the effect of trustworthiness on the distribution of the relevant documents over peers within the testbeds.

4.2 Experimental results and testbed anaylsis

The trustworthiness of Testbeds. The proposed trust-based testbed ranks the peers using the page rank of its documents which is estimated overall the collection documents. Here, the evaluation metrics are examined and the results are compared to the baseline testbeds. Figure 2 demonstrates the peer rank values that are estimated in the proposed testbed and the other two baseline testbeds. The x-axis represents the fraction of peers that are taken into account and the y-axis represents the average peers rank value for the fraction interval. The figure sorts out the testbeds based on the obtained average values of peers ranks across the peers fractions. The findings show that the DLPR substantially follows a power law distribution including peers rank values. This also suggests that there are in the testbed some peers are more trustworthy than the other peers and the distribution of this pattern follows the realistic DL scenario. However, such property cannot be observed for the other two testbeds, i.e. DL WOR and DL LC testbeds. As a result, these two testbeds are not suitable for DL applications depending on the trustworthiness concept to retrieve relevant and/or high quality (or trustworthy) documents from the peer-based IR systems.

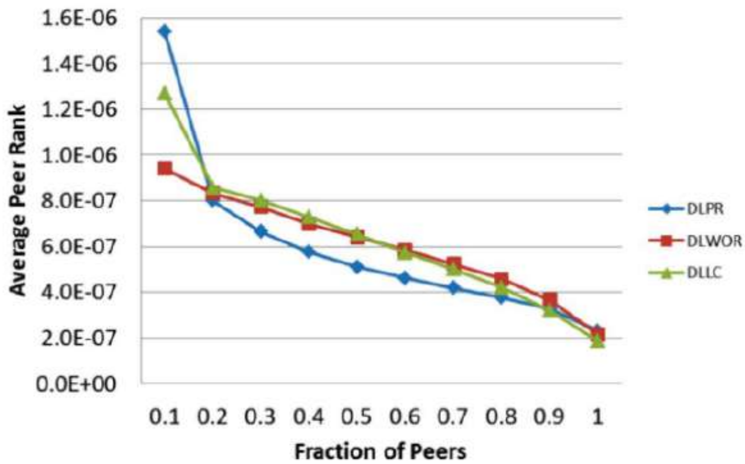

Figure 2: Average peer rank of fraction peers

Documents distribution of Testbeds. One of the P2P DLs scenario characteristics is the even distribution of documents over each digital library. Thereby, it is important to create the testbed which almost has evenly documents distribution. Figure 3 determines the average distribution of documents over fraction number of peers, where the x-axis demonstrates the fraction peers taken into account and the y-axis figures out the average number of documents in each fraction. The proposed testbed (DLPR) is the best one of the two testbeds in an equally large number of documents distribution which is more real as an initial sprinkling of documents over peer collection.

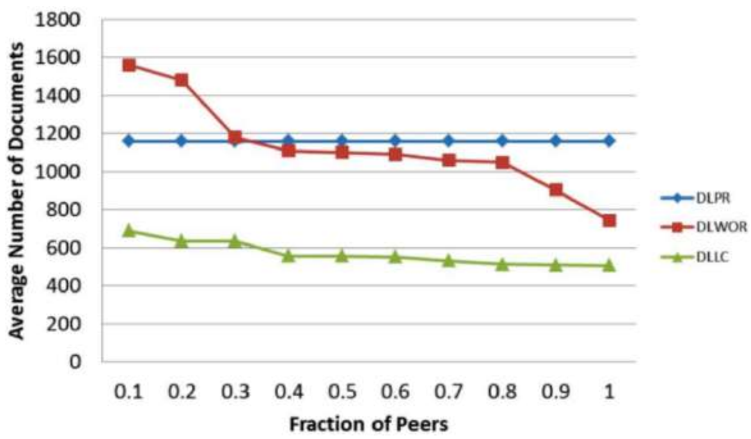

Figure 3: Average number of documents for each peer fraction

The effectiveness of Testbeds. The location of relevant documents has important attention in the P2P system architecture due to increasing the efficiency of these models. So two concepts are briefly being discussed that related to the distribution of relevant documents (topics) over peer-collection to reach 100% Recall and average precision of relevant documents over peer-collections.

Recall: In order to compute the number of peers to reach 100% Recall, the fraction of peer population is computed as shown in Figure 4. As clarified from the figure, a conclusion is that for each topic the DLPR testbeds almost has a high topical distribution over collections (or peer) than the other two testbeds. But the DLPR testbed distributes topics over peers with redundancy in an efficient manner compared to the other two baseline testbeds. As such, the probability for reaching 100% Recall is high in the DLPR over peer collections. In addition to the topics distributions, the DLPR testbed provides other important factors such as trust factors that focus on the trustworthiness of peers in providing trustworthy documents.

Precision: The other metric that is more important is the Precision within the peer-collection. The Precision represents the proportion at which topics (or relevant documents) are presented within the peer collections. The precision is calculated by using one of the [9] precision metric that considered all the peer collections have at least one relevant document and measured their average precision as follows:

$$P_{avg} = \frac{1}{n}\sum_{i=1}^{n} P_i \tag{2}$$

where n is the number of peer-collections that have at least one relevant document and P_i is the precision as measured by the number of relevant documents over the total number of documents shared in the ith collection.

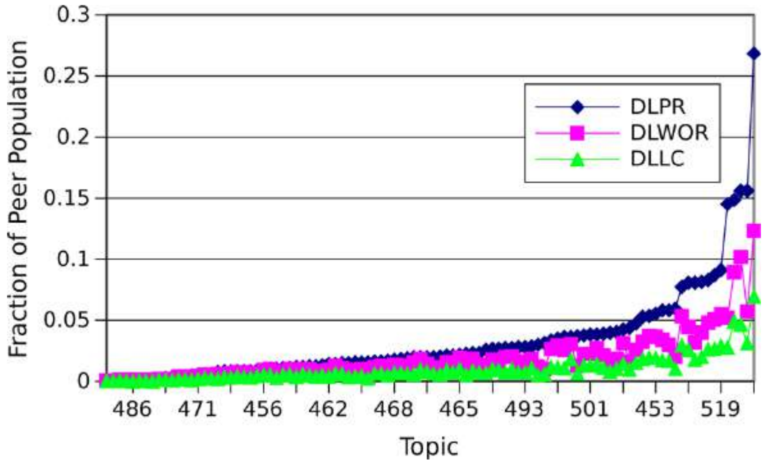

Figure 4: Fraction of peer population to reach 100% recall

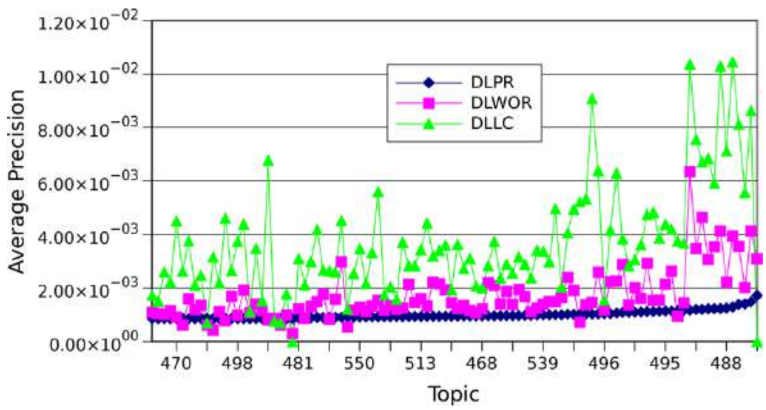

Figure 5: Average precision of topics distribution

In Figure 5, the DLLC (Digital Libraries Lu and Callan) and DLWOR (Digital Library WithOut Replication) have high Precision values than the proposed testbed. The Precision and Recall have an effect to the testbed from an effectiveness perspective, but their results do not reflect the quality (or trustworthy) of documents as the number of trustworthy documents for some peers in the testbed significantly are larger than the other two testbeds. Consequently, the DLPR testbed specifically focuses on the importance of trustworthiness that is related to some peers comparing to the other peers. The reasons behind such results back to that high number of non-relevant documents have the highest trust values, due to the trust value computed through the number of peers access to these documents or more recommend by other peers as to be the trustworthy documents [18].

5. Conclusions and future works

Building a testbed reflecting the trustworthiness factor is one of the demanding tasks. In addition, the trustworthiness concept is subjective in its nature. Therefore, estimating such factor value occurs during the lifetime of the system as deployed in dynamic systems. As a result, this paper proposed and built a testbed that comprises the trustworthiness factor required by many researchers to tackle the problem of retrieving more relevant and trustworthy (or reputable) documents. The DLPR (or proposed) testbed is constructed to reflect digital libraries by taken into considerations many factors related to them such as a large number of peers, uniformly distribution of relevant documents and the most important factor that is related to trustworthiness that depends on the page rank of the documents inside each peer. In the construction phase, a well-known dataset is selected reflecting the Web documents and the power low distribution of documents over the Internet. This dataset is the WT10g Collection that is a real collection and since it used by many other systems as a standard collection to evaluate their models.

The proposed testbed suffers from precision and recall which is expected results, as the nature of systems that depends on the trustworthiness changes content through the lifetime of systems and some peers might join and leaf the system to increase their reputation and existence so the distribution of topics on the type of these systems was not taken into consideration.

Evaluating the trustworthiness in the P2P overlay networks is one of the most demanding and neglected tasks because the trustworthiness is subjective in nature and it supposed to be computed through the lifetime of the system. This paper presented a proposed trust-based testbed that depends on the peer rank as a trust factor used as a criterion for building the testbed. The distribution of peers ranks in peer collections of DLPR, DLWOR and DLLC testbeds is also discussed along with demonstrating the other factors such as the distribution of relevant documents (presented by topics) over peers collection in order to reach 100% recall and the distribution of relevant documents (presented by topics) over peers. The proposed testbed is concentrated on the importance of peer on peer-collection and reflects it as peer trustworthiness, so another factor may be relatively different. In the future work, the proposed testbed is being expanded to support the model with the actual behavior of a P2P system to check how much accurate the testbed in trust-based P2P systems.

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Accepted: 16.01.2019

A new contraction and existence theorems on fuzzy metric space with a graph

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Abstract. In the present paper our aim is to develop coupled fixed point theorems in fuzzy metric space with graph. We introduce the concept of $\mathcal{J}$-γ-contraction mapping using the control function developed by Wardowski [16]. In current paper, we show the existence of coupled coincidence fixed point in fuzzy metric space with respect to graph. We also give the result having particular value of control function such that the $\mathcal{J}$-γ-contraction change to $\mathcal{J}$-fuzzy contraction.
Keywords: coupled fixed point, $\mathcal{J}$-γ-contraction mapping, fuzzy metric space($\mathcal{FM}$-space).

1. Introduction

Banach contraction principle remains as the backbone of fixed point theory to relate different areas like differential equations, integral equations, game theory etc. The concept of graph and fixed point theory were also combined to prove fixed point theorems in R-trees by Espinola and Kirk [5]. Jackymski [7] stepped in the fixed point theory with the language of graph theory and gave result with a directed graph on Banach contractions in a metric space. Metric space with

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graph theory is the developing area in the field of research. Also, the coupled and common fixed point theorem were executed by Bhaskar and Lakshmikantham [1]. In 2014, Chifu and Petrusel [2] worked on coupled fixed point results in metric space endowed with directed graph. In the same year, Shukla [15] also defined G-fuzzy contraction on fuzzy metric space endowed with graph. Shukla's work in fuzzy metric space gave us a new direction to think with graphs. The justified extension of coupled fixed point result to fuzzy metric was done by Zhu et al. [23]. The control functions introduced by Wardowski [16] helped us to use the concept of coupled fixed point theorem in fuzzy metric space endowed with graph theory. Fuzzy metric space was introduced by Kramosil and Michalek [8] and modified by George and Veeramani [6] using the concept of fuzzy sets, introduced by Zadeh [22] on metric spaces. Different work has been done in fuzzy metric space by [10], [12]-[14], [17]-[21]. In present paper, we apply the concept of coupled fixed point theory on graph theory in fuzzy metric space to find common coupled fixed point.

Definition 1.1 ([11])**.** A binary operation $* : [0,1]^2 \to [0,1]$ is called continuous t-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,

(ii) $u * 1 = u$ for all $u \in [0,1]$,

(iii) $u * v \leq w * r$ whenever $u \leq w$ and $v \leq r$ for all $u, v, w, r \in [0,1]$,

(iv) $*$ is continuous.

George and Veeramani [6] introduced the following definition of fuzzy metric space. This definition of fuzzy metric space is utilized in our paper.

Definition 1.2 ([6])**.** The 3-tuple $(\mathcal{K}, M, *)$ is called a fuzzy metric space if $\mathcal{K}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $\mathcal{K}^2 \times (0,\infty)$ satisfying the following conditions for each $u, y, z \in \mathcal{K}$ and $t, s > 0$:

(FM1) $M(u, y, t) > 0$,

(FM2) $M(u, y, t) = 1$ if and only if $u = y$,

(FM3) $M(u, y, t) = M(y, u, t)$,

(FM4) $M(u, y, t) * M(y, z, s) \geq M(u, z, t + s)$, and

(FM5) $M(u, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Definition 1.3 ([6])**.** Let $(\mathcal{K}, M, *)$ be a fuzzy metric space.

(i) A sequence $\{u_n\}$ in $\mathcal{K}$ is said to be convergent to a point $x \in \mathcal{K}$ if $\lim_{n\to\infty} M(u_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{u_n\}$ in $\mathcal{K}$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(u_n, u_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.4 ([16]). Denote by W a family of mappings $\gamma : (0, 1] \to [0, \infty)$ satisfying the following two conditions:

(W1) γ transforms $(0, 1]$ onto $[0, \infty)$;

(W2) for all $s, t \in (0, 1]$, $s < t \Rightarrow \gamma(s) > \gamma(t)$ (i.e. γ is strictly decreasing).

Note that (W1) and (W2) imply $\gamma(1) = 0$ and $\gamma(\alpha_n) \to 0$ whenever $\alpha_n \to 1$ as $n \to \infty$.

Example of γ-function is $\gamma(t) = \frac{1}{t} - 1$, $t \in (0, 1]$.

Lemma 1.1 ([3]). *Let $(\mathcal{K}, M, *)$ be a fuzzy metric space and let $\gamma \in H$. The sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{K}$ are convergent to the points $x \in \mathcal{K}$ and $y \in \mathcal{K}$ if $\lim_{n\to\infty} \gamma(M(x_n, x, t) * M(y_n, y, t)) = 0$ for all $t > 0$.*

Lemma 1.2 ([3]). *The sequence $\{x_n\}$ and $\{y_n\}$ in $\mathcal{K}$ are Cauchy sequences if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $\gamma(M(x_n, x_m, t) * M(y_n, y_m, t)) \leq \epsilon$ for all $n, m \geq n_0$.*

The concepts of graphs are similar to those in [7]. Let $(\mathcal{K}, M, *)$ be a fuzzy metric space. Let a directed graph $\mathcal{J}$ such that the set $V(\mathcal{J})$ of its vertices, consider as elements of $\mathcal{K}$, the set $\mathcal{E}(\mathcal{J})$ of its edges contains all loops, i.e. $\mathcal{E}(\mathcal{J}) \supseteq \Delta$, where Δ denote the diagonals of Cartesian product $\mathcal{K} \times \mathcal{K}$. We identify $\mathcal{J}$ with the pair $(V(\mathcal{J}), \mathcal{E}(\mathcal{J}))$ having no parallel edges. We also treat $\mathcal{J}$ as weighted graph by assigning to each edge the fuzzy distance between its vertices.

Also, $\mathcal{J}^{-1}$ be the graph obtained from $\mathcal{J}$ by reversing the direction of edges, i.e.

$$\mathcal{E}(\mathcal{J}^{-1}) = \{(x, y) \in \mathcal{K} \times \mathcal{K} : (y, x) \in \mathcal{E}(\mathcal{J})\}.$$

Definition 1.5 ([9]). An element $(x, y) \in \mathcal{K} \times \mathcal{K}$ is called a coupled coincidence point of the functions $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ if

$$H(x, y) = gx \quad \text{and} \quad H(y, x) = gy.$$

Let us denote the set of all coupled coincidence points of H and g by $C(Hg)$.

Definition 1.6 ([9]). An element $(x, y) \in \mathcal{K} \times \mathcal{K}$ is called a coupled common fixed point of the functions $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ if

$$H(x, y) = g(x) = x \quad \text{and} \quad H(y, x) = g(y) = y.$$

Definition 1.7 ([9])**.** Let $\mathcal{K}$ be a non-empty set. Then the function $H : \mathcal{K}\times\mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ are said to be commutative if

$$g(H(x, y)) = H(gx, gy) \quad \text{for all } x, y \in \mathcal{K}.$$

Definition 1.8 ([7])**.** A function $\mathcal{J} : \mathcal{K} \to \mathcal{K}$ is $\mathcal{J}$-continuous if

(i) for all $x, x^* \in \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{x_{n_i}\} \to x^*$ and $(x_{n_i}, x_{n_{i+1}}) \in \mathcal{E}(\mathcal{J})$ for $n \in N$ implies $\{g(x_{n_i})\} \to gx^*$.

(ii) for all $y, y^* \in \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{y_{n_i}\} \to y^*$ and $(y_{n_i}, y_{n_{i+1}}) \in \mathcal{E}(\mathcal{J}^{-1})$, for $n \in N$, implies $\{g(y_{n_i})\} \to gy^*$.

Definition 1.9 ([2])**.** A function $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ in $\mathcal{J}$-continuous if for all $(x, y), (x^*, y^*) \in \mathcal{K} \times \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{x_{n_i}\} \to x^*$, $\{y_{n_i}\} \to y^*$ as $i \to \infty$ and $(x_{n_i}, x_{n_{i+1}}) \in \mathcal{E}(\mathcal{J})$, $(y_{n_i}, y_{n_{i+1}}) \in \mathcal{E}(\mathcal{J}^{-1})$ for $n \in N$, implies $\{H(x_{n_i}, y_{n_i})\} \to H(x^*, y^*)$ and $\{H(y_{n_i}, x_{n_i})\} \to H(y^*, x^*)$.

2. Main result

To find our main result, we first define some definitions and lemmas as follows:
First we define property (A) for graph in fuzzy metric space.

Definition 2.1. Let $(\mathcal{K}, M, *)$ be a complete fuzzy metric space endowed with a directed graph $\mathcal{J}$. Then the tuple $(\mathcal{K}, M, *, \mathcal{J})$ has the property (A) if

(i) for any sequence $\{x_n\}_{n\in N}$ in $\mathcal{K}$ such that $\{x_n\} \to x^*$ and $(x_n, x_{n+1}) \in \mathcal{E}(\mathcal{J})$, $(x_n, x^*) \in \mathcal{E}(\mathcal{J})$;

(ii) for any sequence $\{y_n\}_{n\in N}$ in $\mathcal{K}$ such that $\{y_n\} \to y^*$ and $(y_n, y_{n+1}) \in \mathcal{E}(\mathcal{J}^{-1})$, $(y_n, y^*) \in \mathcal{E}(\mathcal{J}^{-1})$.

Next, let us consider $(\mathcal{K}, M, *)$ be a fuzzy metric space endowed with a directed graph $\mathcal{J}$ and $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ be the mappings.
Define the set $(\mathcal{K} \times \mathcal{K})_{Hg}$ as

$$(\mathcal{K}\times\mathcal{K})_{Hg} = \{(x, y) \in \mathcal{K}\times\mathcal{K} : (gx, H(x, y)) \in \mathcal{E}(\mathcal{J}) \text{ and } (gy, H(y, x)) \in \mathcal{E}(\mathcal{J}^{-1})\}.$$

Definition 2.2. The mapping $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is called a $\mathcal{J} - \gamma$-contraction if

(i) g is edge preserving, i.e., $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and $(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1})$
$\Rightarrow (g(gx), g(gu)) \in \mathcal{E}(\mathcal{J})$ and $(g(gy), g(gv)) \in \mathcal{E}(\mathcal{J}^{-1})$;

(ii) H is g-edge preserving, i.e., $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and
$(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1}) \Rightarrow (H(x, y), H(u, v) \in \mathcal{E}(\mathcal{J}))$ and $(H(y, x), H(v, u) \in \mathcal{E}(\mathcal{J}^{-1})$;

(iii) for all $x, y, u, v \in \mathcal{K}$ such that $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and $(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1})$

$$\begin{aligned}&\gamma(M(H(x,y),H(u,v),t)) * M(H(y,x),H(v,u),t))\\ &\quad\leq k\gamma(M(gx,gu,t) * \gamma(M(gy,gv,t))\end{aligned}$$

where $k \in (0,1)$ is called contraction constant of H.

Lemma 2.1 ([4])**.** *Suppose that $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is g-edge preserving and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Also, let $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences in fuzzy metric space $(\mathcal{K}, M, *)$ endowed with a directed graph $\mathcal{J}$. Then the following statements are true:*

(i) $(gx, gu) \in \mathcal{E}(\mathcal{J})$ *and* $(gy, gu) \in \mathcal{E}(\mathcal{J}^{-1})$
$\Rightarrow (H(x_n, y_n), H(u_n, v_n)) \in \mathcal{E}(\mathcal{J})$ *and* $(H(y_n, x_n), H(v_n, u_n)) \in \mathcal{E}(\mathcal{J}^{-1})$ *for all* $n \in N$

(ii) $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg} = (H(x_n, y_n), H(x_{n+1}, y_{n+1})) \in \mathcal{E}(\mathcal{J})$ *and* $(H(y_n, x_n), H(y_{n+1}, x_{n+1})) \in \mathcal{E}(\mathcal{J}^{-1})$ *for all* $n \in N$.

(iii) $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg} \Rightarrow (H(x_n, y_n), H(y_n, x_n)) \in (\mathcal{K} \times \mathcal{K})_{Hg}$ *for all* $n \in N$.

Lemma 2.2. *Let $(\mathcal{K}, M, *)$ be a fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0,1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Also suppose that $\{x_n\}$, $\{y_n\}$ be sequences in $\mathcal{K}$. Then, for $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, there exists $p(x, y, t) \geq 0$ and $k \in (0,1)$ such that*

$$\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \leq k^n \gamma(p(x, y, t)).$$

where

$$p(x, y, t) = (M(gx_0, gx_1, t) * M(gy_0, gy_1, t))$$

Proof. $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$

$$\begin{aligned}&\Rightarrow \quad (gx, H(x,y)) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy, H(y,x)) \in \mathcal{E}(\mathcal{J}^{-1})\\ &\Rightarrow \quad (gx_0, gx_1) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_0, gy_1) \in \mathcal{E}(\mathcal{J}^{-1})\end{aligned}$$

Then, by Lemma 2.1,

$$\begin{aligned}&(H(x_n,y_n),H(x_{n+1},y_{n+1}))\in\mathcal{E}(\mathcal{J})\\ &\quad\text{and} \quad (H(y_n,x_n),H(y_{n+1},x_{n+1}))\in\mathcal{E}(\mathcal{J}^{-1}) \quad \text{for all } n\in N\\ &\quad\Rightarrow (gx_n, gx_{n+1}) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (g(y_n), g(y_{n+1})) \in \mathcal{E}(\mathcal{J}^{-1}) \quad \text{for all } n \in N\end{aligned}$$

But H is a $\mathcal{J} - \gamma$-contraction, so

$$\begin{aligned}&\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t))\\ &\quad= \gamma(M(H(x_{n-1}, y_{n-1}), H(x_n, y_n), t) * M(H(y_{n-1}, x_{n-1}), H(y_n, x_n), t))\\ &\quad\leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)),\end{aligned}$$

that is,

$$\begin{aligned} &\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \\ &\leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)) \end{aligned} \tag{2.1}$$

From (2.1)we can get for all $n \geq 1$, $t > 0$,

$$\begin{aligned} &\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \\ &\leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)) \\ &\leq k^2\gamma(M(gx_{n-2}, gx_{n-1}, t) * M(gy_{n-2}, gy_{n-1}, t)) \\ &\leq k^3\gamma(M(gx_{n-3}, gx_{n-2}, t) * M(gy_{n-3}, gy_{n-2}, t)) \\ &\vdots \\ &\leq k^n\gamma(M(gx_0, gx_1, t) * M(gy_0, gy_1, t)) \end{aligned} \tag{2.2}$$

From the definition of γ-function we have

$$\gamma(M(gx_n, gx_{n+1}) * M(gy_n, gy_{n+1}, t)) \geq k^n\gamma(p(x, y, t)),$$

where

$$p(x, y, t) = (M(gx_0, gx_1, t) * M(gy_0, gy_1, t)) \tag{2.3}$$

Hence the lemma is proved. □

Lemma 2.3. *Let $(\mathcal{K}, M, *)$ be fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0, 1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(\mathcal{K})$.*

If the mapping H satisfies the conditions:

(*i*) *There exists x_0 and y_0 in $\mathcal{K}$ such that*

$$\prod_{i=1}^{l}(M(gx_0, H(x_0, y_0), t_i) * M(gy_0, H(y_0, x_0), t)) \neq 0, \text{ for all } l \in N,$$

(*ii*) *$r * s > 0 \Rightarrow \gamma(r * s) \leq \gamma(r) + \gamma(s)$ for all $r, s \in \{M(gx_0, H(x_0, y_0), t) * M(qy_0, H(y_0, x_0), t)$ for all $x_0, y_0 \in \mathcal{K}$, $t > 0\}$,*

(*iii*) *$\{\gamma(M(gx_0, H(x_0, y_0), t_i) * M(gy_0, H(y_0, x_0), t_i)) : i \in N\}$ is bounded for all x_0 and y_0 in $\mathcal{K}$ and any sequence $\{t_i\}_i \subset (0, \infty)$,*

Also, suppose that $\{x_n\}$, $\{y_n\}$ be sequences in $\mathcal{K}$. Then, for $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, there exist $x^*, y^* \in \mathcal{K}$ such that $\{gx_n\} \to x^*$ and $\{gy_n\} \to y^*$, as $n \to \infty$.

Proof. Let for any $n, m \in N$, $n > m$, $t > 0$ and let $\{a_i\}_{i \in N}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. From (2.3) and using the property of γ, we have

$$M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t) \geq M(gx_0, gx_1, t) * M(gy_0, gy_1, t). \tag{2.4}$$

From (2.4) and condition (i) given in Lemma 2.3 we have

$$\begin{aligned}
&M(gx_m, gx_n, t) * M(gy_m, gy_n, t)\\
&\geq \left(M(gx_m, gx_m, t - \sum_{i=m}^{n-1} a_i t) * M(gx_m, gx_n, \sum_{i=m}^{n-1} a_i t) \right)\\
&\quad * \left(M\left(gy_m, gy_m, \sum_{i=m}^{n-1} a_i t\right) * M\left(gy_m, gy_n, \sum_{i=m}^{n-1} a_i t\right)\right)\\
&= \left(1 * M\left(gx_m, gx_n, \sum_{i=m}^{n-1} a_i t\right)\right) * \left(1 * M\left(gy_m, gy_n, \sum_{i=m}^{n-1} a_i t\right)\right)\\
&\geq \prod_{i=m}^{n-1} (M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t))\\
&\geq \prod_{i=m}^{n-1} (M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t))
\end{aligned} \tag{2.5}$$

By (2.5) and the condition (ii) of Lemma 2.3 , we have

$$\begin{aligned}
&\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t))\\
&\leq \gamma\left(\prod_{i=m}^{n-1} (M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t)) \right)\\
&\leq \sum_{i=m}^{n-1} \gamma\left(M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t)\right).
\end{aligned} \tag{2.6}$$

From (2.2) and (2.6), we have

$$\begin{aligned}
&\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t))\\
&\leq \sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t)).
\end{aligned} \tag{2.7}$$

Here the sequence $\gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t))$ for all $i \in N$, is increasing and by condition (iii) of the Lemma 2.3, we find the convergence of the series $\sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t))$.

For given $\epsilon > 0$ there exists $n_0 \in N$ such that

$$\sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx, a_i t) * M(gy_0, gy_1, a_i t)) < \epsilon \text{ for all } n, m \geq n_0,\ n > m. \tag{2.8}$$

From (2.7) and (2.8) we have

$$\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t)) \le \epsilon.$$

So, by Lemma 2.2, we conclude that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Also $(\mathcal{K}, M, *)$ is complete, therefore there exists $x^*, y^* \in \mathcal{K}$ such that

$$\lim_{n\to\infty} gx_n = x^* \quad \text{and} \quad \lim_{n\to\infty} gy_n = y^* .$$

Theorem 2.1. *Suppose that $(\mathcal{K}, M, *)$ be a complete fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0, 1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(\mathcal{K})$. Let g be $\mathcal{J}$-continuous and commutes with H. Also, we assume, either*

(i) H is $\mathcal{J}$-continuous, or

(ii) the four tuple $(\mathcal{K}, M, *, \mathcal{J})$ has the Property (A). Then $C(Hg) \neq \phi$ iff $(\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi$.

$C(Hg)$ denotes the set of coupled coincidence points.

Proof. Suppose that $C(Hg) \neq \phi$.

Then these exists some $(x^*, y^*) \in C(Hg)$, i.e. $gx^* = H(x^*, y^*)$ and $gy^* = H(y^*, x^*)$. So,

$$\begin{aligned}
&(gx^*, H(x^*, y^*)) = (gx^*, gx^*) \in \Delta \subseteq \mathcal{E}(\mathcal{J})\\
&\text{and } (gy^*, H(y^*, x^*)) = (gy^*, gy^*) \in \Delta \subseteq \mathcal{E}(\mathcal{J}^{-1}).\\
&\Rightarrow (x^*, y^*) \in (\mathcal{K} \times \mathcal{K})_{Hg}.\\
&\Rightarrow (\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi.
\end{aligned}$$

Next, let us assume $(\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi$.

Then there exists $(x_0, y_0) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, i.e., $(gx_0, H(x_0, y_0)) \in \mathcal{E}(\mathcal{J})$ and $(gy_0, H(y_0, x_0)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then by Lemma 2.1, we have a sequence $\{n_i\}_{i\in N}$ of positive integers such that

$$(H(x_{n_i}, y_{n_i}), H(x_{n_i+1}, y_{n_i+1})) \in \mathcal{E}(\mathcal{J}) \text{ and } (H(y_{n_i}, x_{n_i}), H(y_{n_i+1}, x_{n_i+1})) \in \mathcal{E}(\mathcal{J}^{-1}).$$

Also, $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Therefore

$$(gx_{n_i+1}, gx_{n_i+2}) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_{n_i+1}, y_{n_i+2}) \in \mathcal{E}(\mathcal{J}^{-1}). \tag{2.9}$$

Also, from Lemma 2.3

$$\lim_{n\to\infty} gx_{n_i} = x^* \quad \text{and} \quad \lim_{n\to\infty} gy_{n_i} = y^*. \tag{2.10}$$

But g is $\mathcal{J}$-continuous

$$\Rightarrow \quad \lim_{n\to\infty} g(gx_{n_i}) = gx^* \quad \text{and} \quad \lim_{n\to\infty} g(gy_{n_i}) = gy^*.$$

Also, since H and g are commutative

$$g(g(x_{n_i+1})) = g(H(x_{n_i}, y_{n_i})) \text{ and } g(g(y_{n_i+1})) = g(H(y_{n_i}, x_{n_i}))$$

implies

$$g(gx_{n_i+1}) = H(gx_{n_i}, gy_{n_i}) \text{ and } g(gy_{n_i+1}) = H(gy_{n_i}, gx_{n_i}). \tag{2.11}$$

Finally, we show that

$$gx^* = H(x^*, y^*) \text{ and } gy^* = H(y^*, x^*).$$

Let H be $\mathcal{J}$-continuous.

Then, from (2.11), we have

$$\lim_{n\to\infty} g(gx_{n_i+1}) = \lim_{n\to\infty} H(gx_{n_i}, gy_{n_i}) \quad \text{gives} \quad gx^* = H(x^*, y^*)$$
$$\text{and} \quad \lim_{n\to\infty} g(gy_{n_i+1}) = \lim_{n\to\infty} H(gy_{n_i}, gx_{n_i}),$$

implies $gy^* = H(y^*, x^*)$.

Thus, (x^*, y^*) is a coupled coincidence point of the mapping H and g, i.e. $C(Hg) \neq \phi$.

Next, we assume that Property (A) holds.

From (2.9) and (2.10), we have $\{gx_{n_i}\} \to x^*$ as $i \to \infty$ and $(gx_{n_i}, gx_{n_i+1}) \in \mathcal{E}(\mathcal{J})$ and $\{gy_{n_i}\} \to y^*$ as $i \to \infty$ and $(gy_{n_i}, gy_{n_i+1}) \in \mathcal{E}(\mathcal{J}^{-1})$. Therefore, using property (A),

$$(gx_{n_i}, x^*) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_{n_i}, y^*) \in \mathcal{E}(\mathcal{J}^{-1}).$$

Therefore,

$$\begin{aligned}
&M(gx^*, H(x^*, y^*), t) * M(gy^*, H(y^*, x^*), t)\\
&\geq (M(gx^*, g(gx_{n_i+1}), t/2) * M(g(gx_{n_i+1}), H(x^*, y^*), t/2))\\
&* (M(gy^*, g(gy_{n_i+1}), t/2) * M(g(gy_{n_i+1}), H(y^*, x^*), t/2))\\
&= (M(gx^*, g(gx_{n_i+1}), t/2) * M(H(gx_{n_i}, gy_{n_i}), H(x^*, y^*), t/2))\\
&* (M(gy^*, g(gy_{n_i+1}), t/2) * M(H(gy_{n_i}, gx_{n_i}), H(y^*, x^*), t/2))\\
&= (1 * M(H(gx_{n_i}, gy_{n_i}), H(x^*, y^*), t/2))\\
&\cdot (1 * M(H(gy_{n_i}, gx_{n_i}), H(y^*, x^*), t/2)).
\end{aligned}$$

Now, taking the limit $n \to \infty$,

$$M(gx^*, H(x^*, y^*), t) = 1 \quad \text{gives} \quad gx^* = H(x^*, y^*).$$

Also, $M(gy^*, H(y^*, x^*)t) = 1$ implies $gy^* = H(y^*, x^*)$. □

Theorem 2.2. *Suppose that the hypotheses of Theorem 2.1 hold. Beside, let for every* $(x,y),(x^*,y^*) \in \mathcal{K}\times\mathcal{K}$ *there exist* $(u,v)\in\mathcal{K}\times\mathcal{K}$ *such that*
$(H(x,y),H(u,v)) \in \mathcal{E}(\mathcal{J}),(H(y,x),H(v,u)) \in \mathcal{E}(\mathcal{J}^{-1})$ *and*
$(H(x^*,y^*),H(u,v)) \in \mathcal{E}(\mathcal{J}),(H(y^*,x^*),H(v,u)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then H *and* g *have a unique coupled common fixed point.*

Proof. Let (x,y) and (x^*,y^*) be coupled coincidence points, i.e.,

$$gx = H(x,y),\ gy = H(y,x) \quad \text{and} \tag{2.12}$$

$$gx^* = H(x^*,y^*),\ gy^* = H(y^*,x^*). \tag{2.13}$$

By hypothesis, we have

$$(H(x,y),H(u,v)) \in \mathcal{E}(\mathcal{J}) \ \text{ and } \ (H(y,x),H(v,u)) \in \mathcal{E}(\mathcal{J}^{-1}) \tag{2.14}$$

$$(H(x^*,y^*),H(u,v)) \in \mathcal{E}(\mathcal{J}) \ \text{ and } \ (H(y^*,x^*),H(v,u)) \in \mathcal{E}(\mathcal{J}^{-1}) \tag{2.15}$$

Set $H(u_n,v_n) = gu_{n+1}$, $u = u_0$ and $H(v_n,u_n) = gv_{n+1}$, $v = v_0$.

Then, using (2.12), (2.13), (2.14) and (2.15) we get

$$\begin{aligned}
&(gx,gu_1) \in \mathcal{E}(\mathcal{J}), \quad (gy,gv_1) \in \mathcal{E}(\mathcal{J}),\\
&(gx^*,gu_1) \in \mathcal{E}(\mathcal{J}), \quad (gy^*,gv_1) \in \mathcal{E}(\mathcal{J}^{-1}).
\end{aligned}$$

But H is g-edge preserving, so

$$\begin{aligned}
&(H(x,y),H(u_1,v_1)) \in \mathcal{E}(\mathcal{J}),\ (H(y,x),H(v_1,u_1)) \in \mathcal{E}(\mathcal{J}^{-1}) \text{ and}\\
&(H(x^*,y^*),H(u_1,v_1)) \in \mathcal{E}(\mathcal{J}),\ (H(y^*,x^*),H(v_1,u_1)) \in \mathcal{E}(\mathcal{J}^{-1})
\end{aligned}$$

this implies $(gx,gu_2) \in \mathcal{E}(\mathcal{J})$, $(gy,gv_2) \in \mathcal{E}(\mathcal{J}^{-1})$ and $(gx^*,gu_2) \in \mathcal{E}(\mathcal{J})$, $(gy^*,gv_2) \in \mathcal{E}(\mathcal{J}^{-1})$. Using the g-edge preserving property of H repeatedly, for all $n \geq 1$, one can obtain

$$\begin{aligned}
&(gx,gu_n) \in \mathcal{E}(\mathcal{J}), \quad (gy,gv_n) \in \mathcal{E}(\mathcal{J}^{-1}) \quad \text{and}\\
&(gx^*,gu_n) \in \mathcal{E}(\mathcal{J}), \quad (gy^*,gv_n) \in \mathcal{E}(\mathcal{J}^{-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\gamma(M(gx,gu^*,t) * M(gy,gy^*,t))\\
&\leq \gamma((M(gx,gu_{n+1},t/2) * M(gu_{n+1},gx^*,t/2))\\
&* (M(gy,gv_{n+1},t/2) * M(gv_{n+1},gy^*,t/2)))\\
&\leq \gamma((M(gx,gu_{n+1},t/2) * M(gu_{n+1},gx^*,t/2))\\
&+ \gamma(M(gy,gv_{n+1},t/2) * M(gv_{n+1},gy^*,t/2))\\
&\leq k^n\gamma(p(x,y,t)). \qquad \text{(by Lemma (2.2))}
\end{aligned}$$

Letting $n \to \infty$, we have

$$\gamma(M(gx, gx^*t) * M(gy, gy^*, t)) = 0$$

$$\text{implies} \quad M(gx, gx^*, t) = 1 \text{ and } M(gy, gy^*, t) = 1, \text{this gives}$$

$$gx = gx^* \text{ and } gy = gy^*. \tag{2.16}$$

Let $gx = gx^* = r$ and $gy = gy^* = s$.

Then using the commutativity of H and g, (2.16) gives
$g(gx) = g(H(x, y)) = H(gx, gy)$ and $g(gy) = g(H(y, x)) = H(gy, gx)$,
$gr = H(r, s)$ and $gs = H(s, r)$.

Thus, (r, s) is a coupled coincidence point.

Now, the same for (x, y) as (r, s),

$$gx = gr \quad \text{and} \quad gy = gs \quad \text{this gives}$$
$$r = gr \quad \text{and} \quad s = gs.$$

Thus, $r = gr = H(r, s)$ and $s = gs = H(s, r)$. So, (r, s) is a coupled common fixed point of H and g.

Finally, we prove that the coupled fixed point of H and g is unique.

Let us suppose that (a, b) is another coupled common fixed point of H and g. Then

$$a = ga = H(a, b) \quad \text{and} \quad b = gb = H(b, a). \tag{2.17}$$

But, from (2.17) we get

$$ga = gr = r \quad \text{and} \quad gb = gs = s, \tag{2.18}$$

implies $a = r$ and $b = s$.

Hence the coupled common fixed point of H and g is unique.

Corollary 2.1. *Let $(\mathcal{K}, M,^*)$ be a complete fuzzy metric space. Let $P : \mathcal{K} \times \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction and $\gamma : (0, 1] \to [0, \infty]$ satisfies the properties (W1) and (W2). If the mapping P satisfies the conditions:*

(i) *There exists x_0 and y_0 in $\mathcal{K}$ such that*

$$\sum_{i=1}^{m} \{M(x_0, P(x_0, y_0), t_i) * M(y_0, P(y_0, x_0), t_i)\} \neq 0$$

for all $m \in N$;

(ii) *$r^*s > 0$ implies $\gamma(r^*s) \leq \gamma(r) + \gamma(s)$ for all $r, s \in \{M(x_0, P(x_0, y_0), t) * M(y_0, P(y_0, x_0), t)$ for all $x_0, y_0 \in \mathcal{K}, t > 0\}$;*

(iii) *$\{\gamma(M(x_0, P(x_0, y_0), t_i) * M(y_0, P(y_0, x_0), t_i)) : i \in N\}$ is bounded for all x_0 and y_0 in $\mathcal{K}$ and any sequence $\{t_i\}_i \subset (0, \infty)$;*

(iv) $\left(\frac{1}{M(P(x,y),P(u,v),t)*M(P(y,x),P(v,u),t)}\right) \leq k\left(\frac{1}{M(x,u,t)*M(y,v,t)}\right)$
for all $x, y, u, v \in \mathcal{K}$, $t > 0$*;*

(v) $P(\mathcal{K} \times \mathcal{K}) \subseteq \mathcal{K}$;

(vi) P *is* $\mathcal{J}$*-continuous or*
the tuple $(\mathcal{K}, M, *, \mathcal{J})$ *has the Property (A);*

(vii) *For every* $(x, y), (x^*, y^*) \in \mathcal{K} \times \mathcal{K}$*, there exist* $(u, u) \in \mathcal{K} \times \mathcal{K}$ *such that*
$(P(x, y), P(u, v)) \in \mathcal{E}(\mathcal{J})$, $(P(y, x), P(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$ *and*
$(P(x^*, y^*), P(u, u)) \in \mathcal{E}(\mathcal{J})$,$(P(y^*, x^*), P(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then P *has a coupled fixed point in* $\mathcal{K}$*.*
Putting $\gamma(t) = \frac{1}{t} - 1$*, and then proof follows by Theorem2.2.*

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Accepted: 9.01.2019

Reliability bounds of dependent linear consecutive k-out-of-n:G systems

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Abstract. Most of researches in the reliability theory dealt to study the independence between components in a system. But, in many real systems, dependence between the components is one of the intractable realistic assumptions that need to be carefully considered. Then, the main purpose of this paper is to provide sharp upper and lower bounds for the reliability of linear consecutive k-out-of-n:G systems consisting of dependent components with identical or arbitrary distribution functions. Some comparisons are done and many examples are treated to prove the performance of the proposed method.

Keywords: linear consecutive k-out-of-n:G system, upper bound (B_U), lower bound (B_L), reliability, dependent components, Copula.

1. Introduction

Reliability is an important task especially in complex and high technology systems. Problems related to reliability are particularly critical when there are concerns over the consequences of system failures in terms of safety and cost. Studies elaborated for linear consecutive k-out-of-n systems have attracted a great importance on theoretical and practical fields, indifferently. These systems appear primordially in various engineering fields, such as: mechanical, civil, electronic engineering, telecommunication and network domain, etc.

A linear consecutive k-out-of-n:G (F) system (denoted Lin/Con/k/n:G (F)) consists of n linearly arranged components such as the system works (fails) if and only if at least k consecutive components work (fail). Note that there is a duality between the two systems.

Reliability and opened problems which are related to consecutive k-out-of-n:G (F) systems have been widely studied in the literature under various assumptions and have been resolved, either in the binary case (there are only two states: function, failure): [19], [3], [4], [7], [12], or in the multi-state case

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(the system and the components can assume more than two states): [10], [22], [9], [21], [1], [6], [2], ... Where special attention has been paid to i.i.d coherent systems, because computation of reliability characteristics of a system that consists of dependent components is difficult especially when the specific type of dependence is not known.

However, in several real situations, a great number of systems operate with dependence structures. So, it is necessary to study this category. Recently, some authors started to study systems with dependence structures and similarly, evaluating the exact value of reliability for such systems is so difficult. For this reason, conditions oblige us to estimate this reliability via bounds limits approach. Although, it should be noted that obtaining bounds under an assumption of unknown independence for such systems is harder than for independence one. In the literature, there are few papers investigating the system's reliability with dependence assumption. In this domain, [8] obtained the reliability of consecutive k-out-of-n:G system with dependent elements using a matrix formulation. [16] proposed bounds for the reliability and the expectations of lifetimes of coherent systems based on dependent exchangeable components (which means that the joint distribution of the component lifetimes is invariant in law under permutation) using the concept of Samaniego's signature. [15] investigated the properties of coherent systems with dependent components using the concept of hyperminimal and hypermaximal distributions and proved that the lifetime distribution of any coherent system is a generalized mixture of a series (parallel) subsystem lifetime distributions. Then, from well-known properties of mixtures, bounds and moments for the hyperminimal and hypermaximal distributions for coherent systems are obtained. [20] studied the residual lifetime of both linear and circular consecutive k-out-of-n systems with independent but non identical components. They obtained expressions for residual lifetime distributions and mean residual lifetime functions in terms of permanents when $2k \geq n$ for linear systems and $2k + 1 \geq n$ for circular systems. The failure rate functions and their asymptotic behavior of consecutive k-out-n systems, using mixture representations, were investigated by [7]. They studied the two cases: i.i.d component lifetimes, then independent but not identically distributed component lifetimes. They also obtained some results for the case of exchangeable dependent component lifetimes. [5] investigated the bounding systems reliability, especially for k-out-of n:F linear and circular systems. For the k-out-of-n:F system, the upper and lower bounds are illustrated. But for the other structures, they provided upper and lower bounds for only linear consecutive 2-out-of-3:F system and circular 2-out-of-4:F system. [14] derived bounds for the reliability and the expected lifetime of a coherent system with heterogeneous components, (based on $\overline{G_1}$: the average of the reliability functions of the components and on $\overline{G_2}$: the average of the reliability functions of the series systems obtained from the minimal pats sets). They showed by treating some examples that the bounds obtained by $\overline{G_2}$ seem to be better than that from $\overline{G_1}$, but not always (they presented a counter-example). With the particularity that $\overline{G_1}$ does not depend on the system structure, the permutation of the component reliability functions, and the dependence structure, while $\overline{G_2}$ may depend on these three characteristics. In their other paper, [13] extended the bounds obtained in the precedent one to the case where the components are ordered (the usual stochastic order). Then, using these bounds, they studied the optimal allocation of the components at a given system structure in order to improve the system reliability. Moreover a similar procedure was applied to get bounds for mixtures and the generalized proportional hazard rate model when the baseline populations are ordered. Note that, for this last case, the authors didn't know if the proposed bounds remain optimal. Our contribution is devoted to establish reliability bounds for dependent linear consecutive k-out-of-n:G system for any value of n and k satisfying the relationship $2k \geq n$. From well-known properties of coherent systems and using [6] formula, upper and lower bounds are established. The performance of the provided bounds is quite satisfactory and their calculation is very easy.

The paper is organized as follows: Section 2 is devoted to some notations, assumption and defintions which will be used in the whole paper. In Section 3, we provide sharp upper and lower bounds for the reliability of linear consecutive k-out-of-n:G system with component lifetimes of the systems are dependent and both of an arbitrary joint distribution and identically distributed, by using formulas provided by [6],[20] and [18]. In section 4, we present briefly the context of copula applied in reliability computation. We compare reliability bounds for Lin/con/2/3:G system obtained by our proposed method and the reliability evaluated by using the copula context elaborated by [11]. We also compare our obtained results with those provided by [14] and [13].

In the last section, we study the influence of the number of components on the reliability bounds. In each section, we treat some examples to illustrate the proposed results.

2. Notations and Definitions

Notations 1. n: number of components in the system.

k : the minimum number of consecutive components required for the system to be good.

T_j : lifetime of component j ,$j = 1, \ldots, n$.

$F_j(t) = P(T_j \leq t)$: distribution function of T_j.

$R(t) = P(T_{k/n:G} > t)$: reliability of the system.

$T_{k/n:G}$: lifetime of the system.

Assumptions 1. We assume that $T = (T_1, ..., T_n)$ an n-dimensional random vector is positively lower and upper orthant dependent. (In reliability, the component lifetimes are usually positively dependent).

Definition 1. *Let $T = (T_1, ..., T_n)$ an n-dimensional random vector, in [18], the positively lower orthant dependent and positively upper orthant dependent are defined as:*

1. T is positively lower orthant dependent (PLOD) if for all $(t_1, t_2, \ldots, t_n)$ in R^n

$$P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n) \geq \prod_{j=1}^{n} P(T_j \leq t_j).$$

2. T is positively upper orthant dependent (PUOD) if for all $(t_1, t_2, \ldots, t_n)$ in R^n

$$P(T_1 > t_1, T_2 > t_2, \cdots, T_n > t_n) \geq \prod_{j=1}^{n} P(T_j > t_j).$$

Definition 2. *A consecutive k-out-of-n:G system consists of n linearly arranged components, this system works if and only if at least k consecutive components work.*

In the following section, we propose bounds of reliability for the dependent linear consecutive k-out-of-n:G system (Lin/con/k/n:G system).

3. Bounds of linear consecutive k-out-of-n:G system

Proposition 1. *Let the Lin/con/k/n:G system with arbitrarily distributed dependent components. For $2k \geq n$ the system's reliability is bounded as follows*

$$B_L \leq R(t) \leq B_U,$$

where

$$B_U = 1 + \sum_{m=1}^{m=n-k} F_{k+m}(t) \left[1 - \max_{m \leq j \leq k+m-1} F_j(t)\right] - \max_{n-k+1 \leq j \leq n} F_j(t). \tag{1}$$

and

$$B_L = (k-n) + \sum_{m=1}^{m=n-k} \max_{m \leq j \leq k+m} F_j(t) + \sum_{m=1}^{m=n-k+1} \prod_{j=m}^{j=k+m-1} [1 - F_j(t)]. \tag{2}$$

Corollary 1. *If distribution functions are identical ($F_j(t) = F(t)\ \forall j$), the above formula can be written in this simple form*

$$\begin{aligned} (k-n) + (n-k)F(t) + (n-k+1)(1-F(t))^k &\leq R(t) \\ &\leq 1 + (n-k-1)F(t) - (n-k)F(t)^2. \end{aligned} \tag{3}$$

Remark 1. To ensure the reliability value of a system in the tolerable interval $[0,1]$, it's necessary to establish the following relation

$$
\begin{aligned}
&\max\left\{(k-n)+\sum_{m=1}^{m=n-k}\max_{m\le j\le k+m}F_j(t)+\sum_{m=1}^{m=n-k+1}\prod_{j=m}^{j=k+m-1}[1-F_j(t)],0\right\}\\
(4)\quad &\le\ R(t)\\
&\le\min\left\{1+\sum_{m=1}^{m=n-k}F_{k+m}(t)\left[1-\max_{m\le j\le k+m-1}F_j(t)\right]-\max_{n-k+1\le j\le n}F_j(t),1\right\}.
\end{aligned}
$$

When components are identically distributed, formula (4) becomes

$$
\begin{aligned}
&\max\{(k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k,0\}\\
(5)\quad &\le\ R(t)\\
&\le\min\{1+(n-k-1)F(t)-(n-k)F(t)^2,1\}.
\end{aligned}
$$

The bellow property can be used to quantify bounds of the reliability of the studied system

Properties 1.

$$\max_j\{P(T_j\le t)\}\le P(\bigcup_j T_j\le t)\le\min\{\sum_j P(T_j\le t),1\}.$$

Proof of Proposition 1. The reliability of dependent linear consecutive k-out-of-n:G system is defined by

$$R(t)=P(T_{k/n:G}>t).$$

The lifetime of this system can be represented as follows

$$T_{k/n:G}=\max\{T_{[1:k]},T_{[2:k+1]},\cdots,T_{[n-k+1:n]}\},$$

such as $T_{[1:n]}=\min\{T_1,\cdots,T_n\}$.

[6] formula leads to

$$R(t)=P(T_{k/n:G}>t)=\sum_{i=k}^{n}\left[P(T_{[i-k+1:i]}>t)-P(T_{[i-k+1:i+1]}>t)\right]\tag{6}$$

with $P(T_{[n-k+1:n+1]>t)}=0$.

Certainly, formula (6) gives the expression of reliability for this system, but it is quite difficult to calculate it and it is not straightforward to use it especially for large systems. For this reason, we have resorted to the calculation of the following bounds.

Equation (6) can be written as

$$R(t) = 1 + \sum_{m=1}^{m=n-k} P(\bigcup_{j=m}^{j=k+m} T_j \leq t) - \sum_{m=1}^{m=n-k+1} P(\bigcup_{j=m}^{j=k+m-1} T_j \leq t) \tag{7}$$

Mathematical substitutions have been done in equation (7) that leads to

$$R(t) = 1 + \sum_{m=1}^{m=n-k} \left[F_{k+m}(t) - P(T_{k+m} \leq t \bigcap (\bigcup_{j=m}^{j=k+m-1} T_j \leq t)) \right]$$
$$- P \left(\bigcup_{j=n-k+1}^{j=n} T_j \leq t \right). \tag{8}$$

We begin by establishing the upper bound.

We have

$$-P(\bigcup_{j=n-k+1}^{j=n} (T_j \leq t) \leq - \max_{n-k+1 \leq j \leq n} F_j(t)$$

and

$$- P(T_{k+m} \leq t \bigcap (\bigcup_{j=m}^{j=k+m-1} T_j \leq t)) \leq -P(T_{k+m} \leq t) P(\bigcup_{j=m}^{j=k+m-1} T_j \leq t)$$
$$\leq -F_{k+m}(t) \max_{m \leq j \leq k+m-1} F_j(t).$$

Thus, the integration of obove inequalities in equation (8), the upper bound of the reliability is given as

$$R(t) \leq 1 + \sum_{m=1}^{m=n-k} F_{k+m}(t) \left[1 - \max_{m \leq j \leq k+m-1} F_j(t) \right] - \max_{n-k+1 \leq j \leq n} F_j(t). \tag{9}$$

The lower bound expression can be provided as follows:

Using equation (7), we have

$$P(\bigcup_{j=m}^{j=k+m} T_j \leq t) \geq \max_{m \leq j \leq k+m} F_j(t)$$

and

$$-P(\bigcup_{j=m}^{j=k+m-1} T_j \leq t) = -1 + P(\bigcap_{j=m}^{j=k+m-1} T_j > t) \geq -1 + \prod_{j=m}^{j=k+m-1} [1 - F_j(t)].$$

Then, we obtain

$$R(t) \geq (k-n) + \sum_{m=1}^{m=n-k} \max_{m \leq j \leq k+m} F_j(t) + \sum_{m=1}^{m=n-k+1} \prod_{j=m}^{j=k+m-1} [1 - F_j(t)]. \tag{10}$$

$\square$

3.1 Numerical examples

We suppose that all components are exponentially distributed with parameter λ ($T_j \sim \xi(\lambda)$)

Case 1: The linear consecutive 2-out-of-3:G system. The reliability bounds using equation (5) are plotted in figure 1

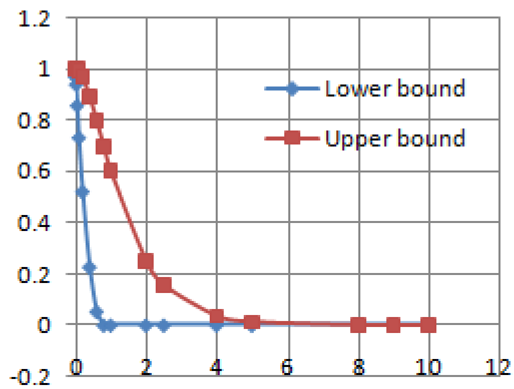

Figure 1: Reliability bounds for linear consecutive 2-out-of-3:G system

Case 2: The linear consecutive 2-out-of-4:G system

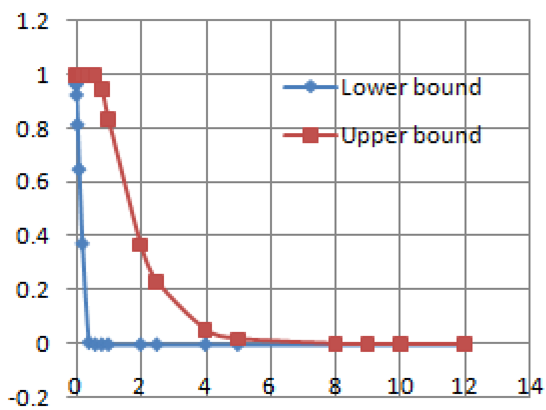

Figure 2: Reliability bounds for linear consecutive 2-out-of-4:G system

Case 3: The linear consecutive 3-out-of-4:G system

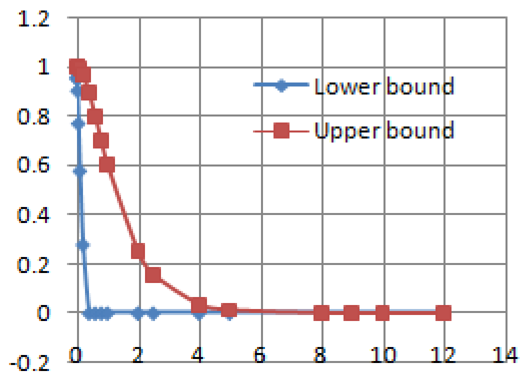

Figure 3: Reliability bounds for linear consecutive 3-out-of-4:G system

Figures 1, 2 and 3 show that for $\lambda t > 5$, the system is less reliable. Then, it's necessary to operate machines until $\lambda t = 5$. We can remark also that when the difference $(n-k)$ decreases, the reliability bounds of the system are getting closer.

4. Comparison with recent results

4.1 Comparison with reliability values using copula

One of the most commonly used methods for modeling dependence between component lifetimes is based on copulas because they contain the information about the dependence structure and can capture the nonlinear dependence. Each copula has its own dependence properties and the detailed review of copulas can be found in [17]. In reliability, the component lifetimes are usually positively dependent, this should be considered while choosing a suitable copula. With the concept of copula, several families of distributions have been constructed such as Gaussian, Clayton, Gumbel, Frank,...etc, with the particularity that Gumbel copula and Clayton copula have simple closed form. Also Clayton copula may characterize the joint distribution of the component lifetimes in the context of stress strength interference, for this reason, [11] computed the reliability of dependent consecutive k-out-of-n:G system. The dependency be either linear or non linear. As an example, in his paper, the dependent linear consecutive 2-out-of-3:G was studied using Clayton copula

$$C_l^{cl}(u_1, u_2) = (u_1^{-1} + u_2^{-1} - 1)^{-1}. \tag{11}$$

The component lifetime distribution functions are

$$F_1(t) = 1 - e^{-t}, F_2(t) = 1 - e^{-2t}, F_3(t) = 1 - e^{-3t}.$$

The system's reliability is obtained as

$$\begin{aligned} R(t) &= e^{-2t} - \{(1-e^{-t})^{-1} + (1-e^{-3t})^{-1} - 1\}^{-1} \\ &\quad + \{(1-e^{-t})^{-1} + (1-e^{-2t})^{-1} + (1-e^{-3t})^{-1} - 2\}^{-1}. \end{aligned} \tag{12}$$

Now, the same example is traited using our proposed approach and applying formula (4), we obtain

$$\begin{aligned} &\max\{-1 + F_3(t) + [1 - F_1(t)][1 - F_2(t)] + [1 - F_2(t)][1 - F_3(t)], 0\} \\ &\leq \ R(t) \ \leq \min\{1 - F_3(t)F_2(t), 1\}. \end{aligned} \tag{13}$$

The figure 4 shows curves of upper and lower bounds elaborated by the proposed approach and the exact value of reliability using the Clayton copula method.

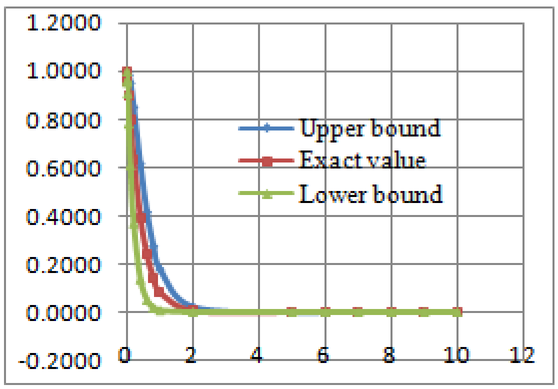

Figure 4: Lower, upper bounds and exact value of Lin/con/2/3:G system

We can remark that there is a well concordance between the results obtained by the two methods. The bounds come very close to exact value of the system's reliability. And when n , k increase, our approach is more flexible, because it didn't need many computations and the time computation is less than the time used in the copula approach.

4.2 Comparaison with Miziula and Navarro bounds

Let us compare the procedure described here and the procedure derived by [14] and [13]. We consider the following examples: linear consecutive 2-out-of-3:G system and linear consecutive 2-out-of-4:G system, with

$$F_1(t) = 1 - e^{-t}, F_2(t) = 1 - e^{-2t}, F_3(t) = 1 - e^{-3t}, F_4(t) = 1 - e^{-4t}$$

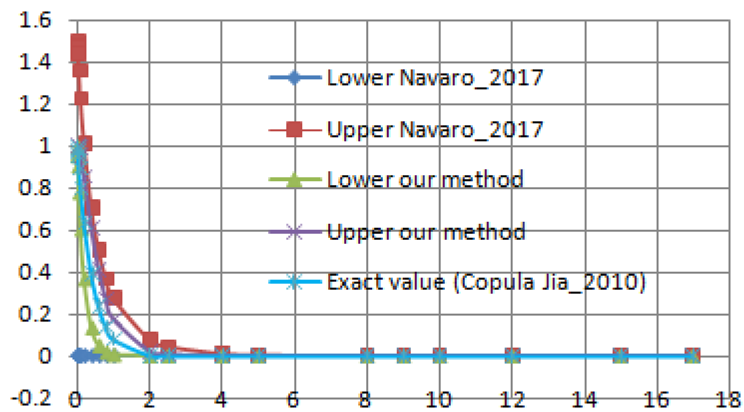

Figure 5: Reliability bounds for linear consecutive 2-out-of-3:G

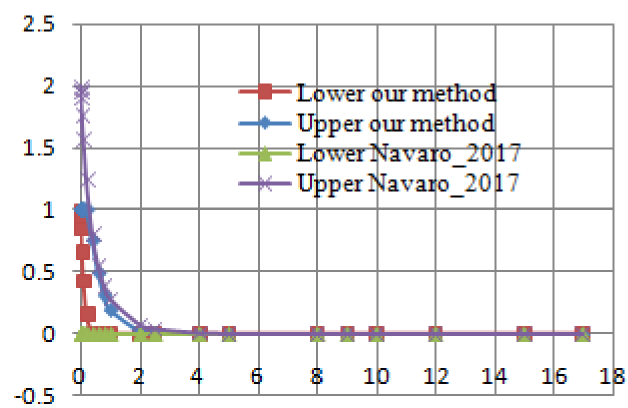

Figure 6: Reliability bounds for linear consecutive 2-out-of-4:G

Figures 5 and 6 show that the bounds obtained in our paper are better than those derived by [13].

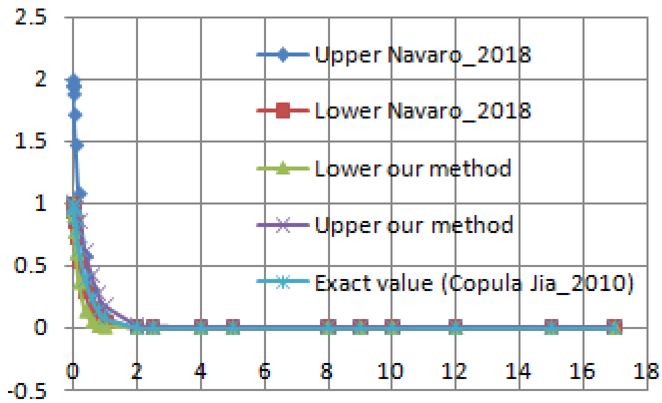

Figure 7: Reliability bounds for linear consecutive 2-out-of-3:G

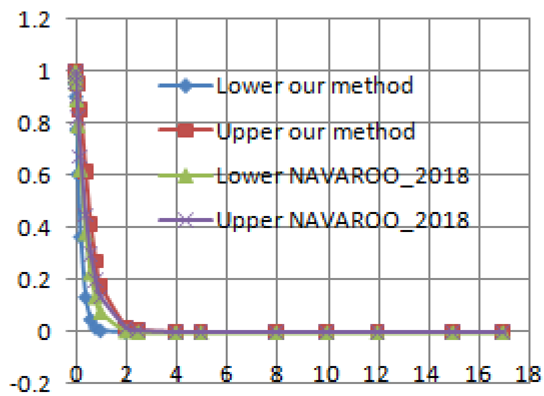

Figure 8: Reliability bounds for linear consecutive 2-out-of-3:G (using (Equ11) in Navarro (2018))

While figure 7 (plotted using $\overline{G_2}$) and figure 8 (plotted using (Equ11) in Navarro 2018) show that the bounds provided by [14] are little better than our bounds, but they stay very close.

5. Influence of system components on reliability bounds

In this section, we will evaluate the difference between the reliability bounds for the case $F_j(t) = F(t)\ \forall j$. In order to determine the influence of variations of n and k on the difference of bounds.

(14) $I = B_U - B_L = (n-k+1) - F(t) - (n-k)F(t)^2 - (n-k+1)(1-F(t))^k.$

Where

$$\begin{aligned}
B_L &= \max\{(k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k, 0\}\\
&= (k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k,\\
B_U &= \min\{1+(n-k-1)F(t)-(n-k)F(t)^2, 1\}\\
&= 1+(n-k-1)F(t)-(n-k)F(t)^2.
\end{aligned}$$

We assume that the component lifetimes are exponentially distributed with parameter $\lambda = 1$, and for some different values of n and k we obtain the following figure.

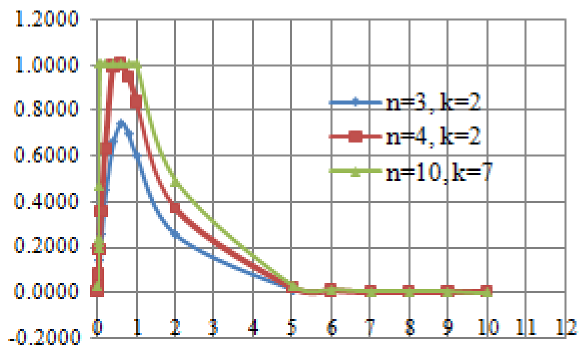

Figure 9: Difference interval for various systems

Figure 9 evaluates the maximum difference between B_U and B_L for different values of n and k.

6. Conclusion

The following results inspired from this work can be drawn as follows

- The development of the reliability bounds of a linear consecutive k-out-of-n:G system in which k consecutive components are positively dependent and arbitrarily distributed.

- The previous results are always valid for any values of k and n satisfying the relationship $2k \geq n$.

- Reliability system using copula approach belongs to the interval of reliability bounds using the developed approach.

- Comparisons between our obtained results and those of [14] and [13] were done. Our approach was applied corresponding to their hypotheses, the corresponding results show a pertinence and a robustness.

Acknowledgments

We would like to thank the editors and the anonymous reviewers for several helpful suggestions that have served to improve the earlier version of this paper.

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Accepted: 15.01.2019

Predator-prey model of Holling-type II with harvesting and predator in disease

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Abstract. A modified predator-prey model is introduced with Holling -type II. Including a constant rate of harvesting in both infected predators by prey, and predators who are prone to disease. An existence of positive biological equilibrium and uniformly boundedness of the present system are obtained as well. Furthermore, the local stability conditions are defined based on Routh-Hurwitz criteria. Finally, an effective Lyapunov function was performed to check the global asymptotic stability of the interior equilibrium point.

Keywords: predator-prey, Holling-type II, harvesting, boundness, stability, Lyapunov function.

1. Introduction

The active research area on classical applications of mathematics to biology, is the study of interactions between populations of various species, by using autonomous differential equations modeling a predator-prey systems [1, 2, 11, 12]. The Lotka-Volterra model was the first to study interactions between predators and prey in 1927. Together with further developed and extended researches such as [3], who divided the prey populations into susceptible and infected. This dynamic relationship between predators and their prey will take into account some aspects that are considered essential to explain the dynamics. Lesile-Gower model has investigated several researchers such as [4]. They studied the boundedness of positive equilibrium points and global stability. Sufficient conditions for the existence and global stability of the model's positive periodic solutions were discussed in [13].

*. corresponding author

The goal of this paper is to give a study of three-dimensional system incorporating a modified version of Lesile-Gower Holling-type II, $p(x) = \frac{x}{m+x}$, where m is the environment that provides production to prey. In our system there are preys and an infected predators which are harvested continuously. However, there are a susceptible predators living on prey and are not harvested. Harvesting infected mathematical dynamics because otherwise, it can lead population density to a dangerously low level of extinction. There are many researches on harvesting such as [5, 9, 10]. Important to note is that our system proposed predators are exposed to the risk of disease. In both mathematical and ecological terms, the effect of disease on the ecological system is an important issue. As a result, many researches [6, 7, 8] proposed and studied different predator-prey models in presence of disease.

This paper is organized as what follows. In section 2, we describe our system (1) then we reduce the number of parameters to get system (2). Next, in section 3, we studied the uniformly boundedness of all positive solutions. Existence of positive equilibrium points are discussed in section 4. After that in section 5, behavior of solutions at each equilibrium points are discussed. Finally, global stability of interior equilibrium point is studied in section 6.

2. The model

We will consider our model under the framework of the following nonlinear differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - \frac{ayx}{m+x} - azx - h_1x,\\
\frac{dy}{dt} &= bxy + \alpha yz + \frac{\gamma yx}{m+x} - h_2y,\\
\frac{dz}{dt} &= bzx - \alpha yz - dz.
\end{aligned} \tag{1}$$

Here, x, y and z are the prey, infected predator and susceptible predator, respectively and $r, k, a, b, \gamma, \alpha, h_1, h_2, d$ are assumed to be positive constants. From the biological point of view, we are only interested in the dynamics of system (1) in the closed octant $\mathbb{R}^3_+$. Thus, we consider the initial conditions are $x(0) = x_0 \geq 0, y(0) = y_0 \geq 0, z(0) = z_0 \geq 0$.

To reduce the number of parameters, we non-dimensionalize system (1) with the following scaling:

$$X = \frac{x}{k},\ Y = \frac{ay}{rm},\ Z = \frac{az}{rm},\ T = rt.$$

Table 1: **Definition of parameters in the model**

Parameter	**Definition**
r	The logistic growth rate of the prey in the absence of predators.
k	The enviromental carrying capacity.
a, b	The capture rates with $(a > b)$.
α	The interaction between y, z.
h_1, h_2	The rates of harvesting where $(h_1 > h_2)$.
γ	The interaction rate of infected predator species.
d	The natural death rate in the absence of prey.

Then system (1) takes the form (after some simplification):

$$\begin{aligned}\frac{dX}{dT} &= X(1-X) - \frac{YX}{1+\beta X} - mZX - \delta_1 X,\\ \frac{dY}{dT} &= cXY + eYZ + \frac{nYX}{1+\beta X} - \delta_2 Y,\\ \frac{dZ}{dT} &= cXZ - eYZ - wZ,\end{aligned} \tag{2}$$

where

$$\frac{k}{m} = \beta,\ \frac{h_1}{r} = \delta_1,\ \frac{bk}{r} = c,\ \frac{d}{r} = w,\ \frac{h_2}{r} = \delta_2,\ \frac{\gamma k}{rm} = n,\ \frac{\alpha m}{a} = e.$$

3. Boundedness of all positive solutions

Theorem 1. *All solutions of system* (2) *that start in* $\mathbb{R}^3_+$ *are uniformly bounded and remain positive, in order to be meaningful from a biological viewpoint.*

Proof. Assuming that $(X(T), Y(T), Z(T))$ be any positive solution of system (2). Let $Q(T) = nX + Y + Z$. Hence

$$\frac{dQ}{dT} + \mu Q \leq -n\Big(X - \frac{D}{2}\Big)^2 + n\Big(\frac{D}{2}\Big)^2,$$

where $D = (1-\delta_1) + \mu$, $(\delta_1 < 1)$, therefore

$$\frac{dQ}{dT} + \mu Q \leq n\Big(\frac{D}{2}\Big)^2.$$

Solving the differential inequality, we obtain

$$Q(T) \leq \frac{nD^2}{4\mu} + ce^{-\mu T},$$

for $T \to \infty$, all solutions of system (2) enter into the region

$$B = \{(X, Y, Z) : 0 \leq Q \leq \frac{nD^2}{4\mu}\}.$$

□

4. Equilibrium points

System (2) has the following points of equilibrium:

(i) The trivial equilibrium $E_0(0,0,0)$.

(ii) The predators free equilibrium $E_1(1-\delta_1,0,0)$.

(iii) The infected predator free equilibrium $E_2(X_2,0,Z_2)$, where $X_2=\frac{w}{c}$ and $Z_2=\frac{c(1-\delta_1)-w}{m}$ if $\frac{c(1-\delta_1)}{w}>1$.

(iv) The susceptible predator free equilibrium $E_3(X_3,Y_3,0)$, from system (2) we get $c\beta X^2+((c+n)-\delta_2\beta)X-\delta_2=0$, we have one positive real root given by $X_3=\frac{-((c+n)-\delta_2\beta)+\sqrt{((c+n)-\delta_2\beta)^2+4c\beta\delta_2}}{2c\beta}$, therefore $Y_3=((1-\delta_1)-X_3)(1+\beta X_3)$ if $(1-\delta_1)>X_3$ hold.

(v) The interior equilibrium $E^*(X^*,Y^*,Z^*)$, given by:

$$\begin{aligned} 1-X^*-\frac{Y^*}{1+\beta X^*}-mZ^*&=\delta_1,\\ cX^*+eZ^*+\frac{nX^*}{1+\beta X^*}&=\delta_2,\\ cX^*-eY^*&=w, \end{aligned} \tag{3}$$

from (3) we get $X^*=\frac{w+eY^*}{c}$, $Z^*=\frac{(1+\beta X^*)(\delta_2-cX^*)-nX^*}{e(1+\beta X^*)}$, if $(X^*<\frac{\delta_2}{c})$. Therefore

$$D_1Y^2+D_2Y-D_3=0, \tag{4}$$

where

$$\begin{aligned} D_1&=\beta e^2(m-\frac{e}{c}),\\ D_2&=me\big(c+n+\beta(2w-\delta_2)-\frac{c}{m}\big)-e^2(1-\beta(1-\frac{2w}{c}-\delta_1)),\\ D_3&=m(c\delta_2+w(\beta(\delta_2-w)-(c+n)))\\ &\quad+ew(\beta((1-\delta_1)-\frac{w}{c})-1)-ec(1-\delta_1). \end{aligned} \tag{5}$$

We have one positive root for equation (4) given by:

$$Y^*=\frac{-D_2+\sqrt{D_2^2+4D_1D_3}}{2D_1},$$

if the following conditions hold, $D_1>0 \iff mc>e$, and $D_3>0 \iff \beta\delta_2>1, \beta c(1-\delta_1)>1$, and $ec(1-\delta_1)<1$.

5. Behaviour of solutions

First, we need to compute the Jacobian matrix of system (2) for general (X, Y, Z).

$$J(X,Y,Z) = \begin{bmatrix} 1-2X-\frac{Y}{(1+\beta X)^2}-mZ-\delta_1 & -\frac{X}{1+\beta X} & -mX \\ cY+\frac{nY}{(1+\beta X)^2} & cX+eZ+\frac{nX}{1+\beta X}-\delta_2 & eY \\ cZ & -eZ & cX-eY-w \end{bmatrix}.$$

We will evaluated $J(X,Y,Z)$ at each equilibrium points. Let J_0 denoted the Jacobian matrix at E_0,

$$J_0 = \begin{bmatrix} 1-\delta_1 & 0 & 0 \\ 0 & -\delta_2 & 0 \\ 0 & 0 & -w \end{bmatrix},$$

this immediately shows that E_0 is saddle point (unstable).

Let J_1 denoted the Jacobian matrix at E_1,

$$J_1 = \begin{bmatrix} -(1-\delta_1) & \frac{-(1-\delta_1)}{1+\beta(1-\delta_1)} & -m(1-\delta_1) \\ 0 & \frac{(1-\delta_1)[c(1+\beta(1-\delta_1))+n]}{1+\beta(1-\delta_1)}-\delta_2 & 0 \\ 0 & 0 & c(1-\delta_1)-w \end{bmatrix},$$

so the eigenvalues are:

$$\begin{aligned} \lambda_1 &= -(1-\delta_1) < 0, \\ \lambda_2 &= \frac{(1-\delta_1)[c(1+\beta(1-\delta_1))+n]}{1+\beta(1-\delta_1)} - \delta_2, \\ \lambda_3 &= c(1-\delta_1) - w > 0. \end{aligned} \tag{6}$$

thus, from (6)E_1 is saddle point (unstable).

Let J_2 denoted the Jacobian matrix at E_2,

$$J_2 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & 0 \end{bmatrix},$$

where

$$\begin{aligned} &A_{11} = 1-2X_2-mZ_2-\delta_1,\ A_{12} = \frac{-X_2}{1+\beta X_2,}, \\ &A_{13} = -mX_2,\ A_{22} = cX_2+eZ_2+\frac{nX_2}{1+\beta X_2}-\delta_2,\ A_{31} = cZ_2,\ A_{32} = -eZ_2. \end{aligned}$$

The characteristic equation of the Jacobian matrix J_2 is given by:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$
\begin{aligned}
a_1 &= -(A_{11} + A_{22}),\\
a_2 &= A_{11}A_{22} - A_{13}A_{31},\\
a_3 &= A_{31}A_{13}A_{22}.
\end{aligned} \tag{7}
$$

Hence,

$$
a_1a_2 - a_3 = -A_{11}A_{22}(A_{11} + A_{22}) + A_{11}A_{13}A_{31}.
$$

If $A_{11} < 0,\ A_{22} < 0$ then $a_1 > 0,\ a_3 > 0$, and $a_1a_2 - a_3 > 0$, then by using Routh-Hurwitz criterion E_2 is asymptotically stable.

Let J_3 denoted the Jacobian matrix at E_3,

$$
J_3 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix},
$$

where

$$
\begin{aligned}
&B_{11} = 1 - 2X_3 - \frac{Y_3}{(1+\beta X_3)^2} - \delta_1,\ B_{12} = \frac{-X_3}{1+\beta X_3},\ B_{13} = -mX_3,\\
&B_{21} = cY_3 + \frac{nY_3}{(1+\beta X_3)^2},\ B_{22} = cX_3 + \frac{nX_3}{1+\beta X_3} - \delta_2,\\
&B_{23} = eY_3,\ B_{33} = cX_3 - eY_3 - w.
\end{aligned}
$$

The characteristic equation of the Jacobian matrix J_3 is given by:

$$
\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,
$$

where

$$
\begin{aligned}
b_1 &= -(B_{11} + B_{22} + B_{33}),\\
b_2 &= B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} - B_{12}B_{21},\\
b_3 &= B_{12}B_{21}B_{33} - B_{11}B_{22}B_{33}.
\end{aligned} \tag{8}
$$

Hence,

$$
\begin{aligned}
b_1b_2 - b_3 = [&-(B_{11} + B_{22} + B_{33})(B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33}) + B_{12}B_{21}B_{11}\\
&+ B_{12}B_{21}B_{22}] + B_{11}B_{22}B_{33}.
\end{aligned} \tag{9}
$$

Let $M_1 = B_{11}B_{22}B_{33}$. If $B_{11} < 0,\ B_{22} < 0$, and $B_{33} < 0$, then $b_1 > 0,\ b_3 > 0, M_1 < 0$, and the first bracket in (9) is positive.

Thus, if $M_1 < [-(B_{11}+B_{22}+B_{33})(B_{11}B_{22}+B_{11}B_{33}+B_{22}B_{33})+B_{12}B_{21}B_{11}+B_{12}B_{21}B_{22}]$, then by using Routh-Hurwitz criterion E_3 is asymptotically stable.

Finally, let J^* denoted the Jacobian matrix at E^*,

$$
J^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},
$$

where

$$C_{11} = 1 - 2X^* - \frac{Y^*}{(1+\beta X^*)^2} - mZ^* - \delta_1,\ C_{12} = \frac{-X^*}{1+\beta X^*},\ C_{13} = -mX^*,$$
$$C_{21} = cY^* + \frac{nY^*}{(1+\beta X^*)^2},$$
$$C_{22} = cX^* + eZ^* + \frac{nX^*}{1+\beta X^*} - \delta_2,\ C_{23} = eY^*,\ C_{31} = cZ^*,\ C_{32} = -eZ^*,$$
$$C_{33} = cX^* - eY^* - w.$$

The characteristic equation of the Jacobian matrix J^* is given by:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where

$$\begin{aligned} c_1 &= -(C_{11} + C_{22} + C_{33}),\\ c_2 &= (C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}) - (C_{12}C_{21} + C_{13}C_{31} + C_{23}C_{32}),\\ c_3 &= (C_{12}C_{21}C_{33} + C_{13}C_{31}C_{22} + C_{11}C_{23}C_{32})\\ &\quad - (C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}). \end{aligned} \tag{10}$$

Hence,

$$\begin{aligned} c_1c_2 - c_3 = {} & [-(C_{11} + C_{22} + C_{33})(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33})\\ & + C_{11}C_{12}C_{21} + C_{11}C_{13}C_{31} + C_{22}C_{12}C_{21} + C_{22}C_{23}C_{32} + C_{33}C_{13}C_{31}\\ & + C_{33}C_{23}C_{32}] + (C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}). \end{aligned} \tag{11}$$

If $C_{11} < 0,\ C_{22} < 0$, and $C_{33} < 0$, then $c_1 > 0$, also the first parenthesis of c_3 in (10) and the square bracket in (11) are positive.

Let $M_2 = [-(C_{11} + C_{22} + C_{33})(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}) + C_{11}C_{12}C_{21} + C_{11}C_{13}C_{31} + C_{22}C_{12}C_{21} + C_{22}C_{23}C_{32} + C_{33}C_{13}C_{31} + C_{33}C_{23}C_{32}]$, and $M_3 = C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}$,then M_2 maybe positive or negative or zero. We have the following theorem using the Routh-Hurwitz criterion.

Theorem 2. *Suppose that $E^*(X^*, Y^*, Z^*)$ exist. Let $C_{11} < 0,\ C_{22} < 0$, and $C_{33} < 0$. Then E^* is asymptotically stable in one of the following cases:*

(*i*) *M_3=0, or*

(*ii*) $0 < M_3 < C_{12}C_{21}C_{33} + C_{13}C_{31}C_{22} + C_{11}C_{23}C_{32}$, *or*

(*iii*) $-M_2 < M_3 < 0$

6. Global stability of interior equilibrium point

Theorem 3. *The interior equilibrium E^* is globally asymptotically stable if the following conditions hold.*

$$1+\beta X^* > \frac{n(4c\mu\epsilon + D^2(n+1))}{4\mu\epsilon(n(1+m)-c)}, \tag{12}$$

$$1+\beta X^* < \max\Big\{\frac{4ne\mu\epsilon - nD^2}{4\mu\epsilon(c+e)}, \frac{n(e-c)}{e-nm}\Big\}, \tag{13}$$

$$c < n + nm < e + n. \tag{14}$$

Proof. We will construct a Lyapunov function V which is continuous and defined on $\mathbb{R}^3_+$. The function V should be zero at E^* and positive for all other values X, Y and Z. Let us define the function V as follow:

$$V(X,Y,Z) = L_1(X - X^* - X^*\ln\frac{X}{X^*}) + L_2(Y - Y^* - Y^*\ln\frac{Y}{Y^*}) + (Z - Z^* - Z^*\ln\frac{Z}{Z^*}),$$

where $L_1 = 1+\beta X^*$ and $L_2 = \frac{1+\beta X^*}{n}$.

The time derivative of V along the solution of (2) is:

$$\begin{aligned}\frac{dV}{dt} =& L_1(X-X^*)\Big(1 - X - \frac{Y}{1+\beta X} - mZ - \delta_1\Big)\\ &+ L_2(Y-Y^*)\Big(cX + eZ + \frac{nX}{1+\beta X} - \delta_2\Big)\\ &+ (Z-Z^*)(cX - eY - w),\end{aligned} \tag{15}$$

and using system (2), we get:

$$\begin{aligned}\frac{dV}{dt} =& L_1(X-X^*)\Big[-(X-X^*) - m(Z-Z^*)\\ &+ \frac{Y^*(1+\beta X) - Y(1+\beta X^*)}{(1+\beta X)(1+\beta X^*)}\Big]\\ &+ L_2(Y-Y^*)\Big[c(X-X^*) + e(Z-Z^*)\\ &+ \frac{nX(1+\beta X^*) - nX^*(1+\beta X)}{(1+\beta X)(1+\beta X^*)}\Big]\\ &+ (Z-Z^*)\Big[c(X-X^*) - e(Y-Y^*)\Big].\end{aligned} \tag{16}$$

Using some manipulation with the result of theorem 1 and assuming $\epsilon = \frac{1}{\beta}$, equation (16) takes the form

$$\begin{aligned}\frac{dV}{dt} \leq& \Big[(1+\beta X^*)\Big(-1 + \frac{c}{n} - m\Big) + c + \frac{D^2(n+1)}{4\mu\epsilon}\Big](X-X^*)^2\\ &+ \Big[(1+\beta X^*)\Big(\frac{c+e}{n}\Big) - \Big(e - \frac{D^2}{4\mu\epsilon}\Big)\Big](Y-Y^*)^2\\ &+ \Big[(1+\beta X^*)\Big(\frac{e}{n} - m\Big) + c - e\Big](Z-Z^*)^2.\end{aligned} \tag{17}$$

From (12)-(14), $\frac{dV}{dt}$ is negative definite. Finally, E^* is globally asymptotically stable. $\square$

7. Conclusion

In this paper, we deal with a modified Holling-type II predator-prey model of one prey and two predators whom are exposed to the risk of disease. As well as constants of harvesting ($h_1 > h_2$) in both prey and infected predator species. Positive biological equilibrium points(E_0, E_1) are direct show unstability, where the local stability of (E_2, E_3, E^*) are discussed by using the Routh-Hurwitz criterion. In addition, global stability of interior equilibrium point (E^*) has been investigated by using a suitable Lyapunov function. In our study it is important to realize that illegal harvesting on prey occures a risk even in the absence of predator.

Acknowledgements

This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

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Accepted: 7.01.2019

Bipolar complex fuzzy sets and their properties

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Abstract. The primary motivation behind this paper is to present a brief overview of the bipolar complex fuzzy sets (in short BCFS) which is an extension of bipolar fuzzy set theory. New operations defined over the bipolar complex fuzzy sets some properties of these operations are discussed.

1. Introduction

Fuzzy sets are a sort of useful mathematical structure representing a vague collection of objects. There are various types of Fuzzy sets in the Fuzzy set theory, such as intuitional Fuzzy sets, valued Fuzzy sets, vague sets, etc.

Zhang [11] introduced bipolar fuzzy sets in 1998. Positive information in a bipolar fuzzy set is what is guaranteed to be possible, while negative information is what is impossible or forbidden or certainly false. Bipolar valued fuzzy set by Lee [3] introduced a further generalization of fuzzy sets in which the degree of membership between $[0, 1]$ and $[-1, 1]$ increased. In bipolar fuzzy sets, membership degree 0 means that elements are irrelevant to corresponding property, membership degree belongs to $(0, 1]$ indicate that somewhat elements are sat-

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isfying the corresponding property and membership degree belongs to $[-1, 0)$ indicate that somewhat elements are satisfying implicit counter property.

Ramot et al. [5] introduced a new innovative concept in 2002 and called it a complex fuzzy set (CFS). This approach is absolutely different from other researchers, where Ramot et al. extended the range of membership function to unit disc in the complex plane, unlike the others who limited to $[0, 1]$. Hence, Ramot et al. [6] added an additional term called the phase term to solve the enigma in translating some complex-valued functions on physical terms to human language and vice versa.

We employ techniques similar to these used earlier by Abdallah Al-Husban and Abdul Razak Salleh ([13], [14]) and Abdallah Al-Husban et al. ([15], [**?**]).

In this work, we introduce some workable concepts about our (BCFS) concept, which are intersection, union and complement. Also some properties of the set theoretic operations of (BCFS).

2. Preliminaries

In this section, we remember the definitions and related results that this work requires.

Definition 2.1 ([9])**.** A fuzzy set (FS) A in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.

Definition 2.2 ([11])**.** Let X be a non-empty set. A bipolar fuzzy set (BFS) A in X is an objective having the form $A = \{(x, r_A^+(x), r_A^-(x)) : x \in X\}$ where $r^+ : X \to [0, 1]$ and $r^- : X \to [-1, 0]$ are mappings.

Definition 2.3 ([11])**.** Let X be every two bipolar fuzzy set.

$A = \{(x, r_A^+(x), r_A^-(x)) : x \in X\}$ and $B = \{(x, r_B^+(x), r_B^-(x)) : x \in X\}$ we define

i) $A \cap B = \{\min(r_A^+(x), r_B^+(x)), \max(r_A^-(x), r_B^-(x)) : x \in X\}$,

ii) $A \cup B = \{\max(r_A^+(x), r_B^+(x)), \min(r_A^-(x), r_B^-(x)) : x \in X\}$,

iii) $A^c = \{(x, 1 - r_A^+(x), -1 - r_A^-(x)) : x \in X\}$.

Definition 2.4 ([5])**.** A complex fuzzy set (CFS) A, defined on a universe of discourse U, is characterized by a membership function $\mu_A(x)$, that assigns to any element $x \in U$ a complex-valued grade of membership in A. By definition, the values of $\mu_A(x)$, may receive all lying within the unit circle in the complex plane, and are thus of the form $\mu_A(x) = r_A(x)e^{i\varpi_A(x)}$, where $i = \sqrt{-1}$, each of $r_A(x)$ and $\varpi_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. The CFS A may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in U\} = \left\{\left(x, r_A(x)e^{i\varpi_A(x)}\right) : x \in U\right\}.$$

Definition 2.5 ([5])**.** A complex fuzzy complement of A may be represented as follows:

$$A^c = \{(x, \mu_{A^c}(x)) : x \in U\} = \left\{\left(x, r_{A^c}(x)e^{i\varpi_{A^c}(x)}\right) : x \in U\right\},$$

where $r_{A^c}(x) = 1 - r_A(x)$ and $\varpi_{A^c}(x) = 2\pi - \varpi_A(x)$.

Definition 2.6 ([5])**.** Let A and B be two complex fuzzy sets on U where

$$A = \left\{\left\langle x, \mu_A(x) = r_A(x)e^{i\arg_{r_A}(x)}\right\rangle : x \in U\right\},$$
$$B = \left\{\left\langle x, \mu_B(x) = r_B(x)e^{i\arg_{r_B}(x)}\right\rangle : x \in U\right\}.$$

The complex fuzzy intersection of A and B denoted by $A \oplus B$, is specified by $A \oplus B = \{\langle x, \mu_{A\oplus B}(x)\rangle : x \in U\}$, where $\mu_{A\oplus B}(x) = r_{A\oplus B}(x)e^{i\arg_{r_{A\oplus B}}(x)} = \max(r_A(x), r_B(x))\, e^{i\max\left(\arg_{r_A}(x),\arg_{r_B}(x)\right)}$.

The complex fuzzy union of A and B denoted by $A \otimes B$, is specified by

$$A \otimes B = \{\langle x, \mu_{A\otimes B}(x)\rangle : x \in U\},$$

where

$$\mu_{A\otimes B}(x) = r_{A\otimes B}(x)e^{i\arg_{r_{A\otimes B}}(x)} = \min(r_A(x), r_B(x))\, e^{i\min\left(\arg_{r_A}(x),\arg_{r_B}(x)\right)}.$$

Definition 2.7 ([5])**.** Let A and B be two complex fuzzy sets on X, $\mu_A(x) = r_A(x)e^{i\arg_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\arg_B(x)}$ their membership functions, respectively. We say that A is greater than B, denoted by $A \supseteq B$ or $B \supseteq A$, if for any $x \in X$, $r_A(x) \geq r_B(x)$ and $\arg_A(x) \geq \arg_B(x)$.

3. Bipolar complex fuzzy sets

In this work, we start with the introduction of a (BCFS) definition and some related properties are discussed.

Definition 3.1. Let X is a non-empty set. A bipolar complex fuzzy set (BCFS) A in X is an objective having the form $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$. where $r_A^+ : X \to [0,1]$ and $r_A^- : X \to [-1,0]$ are mappings. $r_A^+ e^{i\theta_A^+}$ the positive complex membership degree and $r_A^- e^{i\theta_A^-}$ the negative complex membership degree. Also the phase term of bipolar complex positive membership function and bipolar complex negative membership function belongs to $(0, 2\pi]$ and $r_A^+ \in [0,1]$, $r_A^- \in [-1,0]$.

Example 3.1. Let $A = \left\{ \begin{array}{c} \left(a, 0.2e^{2\pi i}, -0.4e^{-1.2\pi i}\right), \left(b, 0.8e^{1.3\pi i}, -0.2e^{0\pi i}\right), \\ \left(c, 0.3e^{\pi i}, -0.4e^{-1.5\pi i}\right) \end{array} \right\}$ is a (BCFS) of $X = \{a, b, c\}$.

Definition 3.2. The complement of a (BCFS) $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$ is denoted by A^c and defined by $A^c = \{(x, 1-r_A^+ e^{i(2\pi-\theta_A^+)}, -1-r_A^- e^{i(2\pi-\theta_A^-)}) : x \in X\}$.

Example 3.2. Let $X = \{a, b, c\}$ be a universe of discourse

$$A = \left\{ \begin{array}{c} \left(a, 0.2e^{2\pi i}, -0.4e^{-1.2\pi i}\right), \left(b, 0.8e^{1.3\pi i}, -0.2e^{0\pi i}\right), \\ \left(c, 0.3e^{\pi i}, -0.4e^{-1.5\pi i}\right) \end{array} \right\} \text{ is a (BCFS).}$$

Then

$$A^c = \left\{ \begin{array}{c} (a, (1-0.2)e^{i(2\pi-2\pi)}, (-1+0.4)e^{i(2\pi+1.2\pi)}), \\ (b, (1-0.8)e^{i(2\pi-1.3\pi)}, (-1+0.2)e^{i(2\pi-0\pi)}), \\ (c, (1-0.3)e^{i(2\pi-\pi)}, (-1+0.4)e^{i(2\pi+1.5\pi i}) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(a, 0.8e^{0i}, -0.6e^{3.2\pi i}\right), \left(b, 0.2e^{0.7\pi i}, -0.8e^{2\pi i}\right), \\ \left(c, 0.7e^{i\pi}, -0.6e^{3.5\pi i}\right) \end{array} \right\}.$$

Definition 3.3. The union of two (BCFS) as follows:

Let A and B be two (BCFS) in X, where $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$ and $B = \{(x, r_B^+ e^{i\theta_B^+}, r_B^- e^{i\theta_B^-}) : x \in X\}$. Then the union of A and B is denoted as $A \cup B$ and is given as:

$$A \cup B = \{(\max(r_A^+, r_B^+)e^{i\max(\theta_A^+, \theta_B^+)}, \min(r_A^-, r_B^-)e^{i\min(\theta_A^-, \theta_B^-)}) : x \in X\}.$$

Example 3.3. Let $X = \{a, b, c\}$ be a universe of discourse. Let A and B be two (BCFS). Let

$$A = \left\{ \begin{array}{c} \left(a, 0.2e^{2\pi i}, -0.4e^{-1.2\pi i}\right), \left(b, 0.8e^{1.3\pi i}, -0.2e^{0\pi i}\right), \\ \left(c, 0.3e^{\pi i}, -0.4e^{-1.5\pi i}\right) \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{c} \left(a, 0.2e^{\pi i}, -0.3e^{-1.4\pi i}\right), \left(b, 0.7e^{1.5\pi i}, -0.1e^{1.5\pi i}\right), \\ \left(c, 0.1e^{\pi i}, -0.3e^{-1.5\pi i}\right) \end{array} \right\},$$

then

$$(A \cup B) = \left\{ \begin{array}{c} \left(a, 0.2e^{2\pi i}, -0.3e^{-1.2\pi i}\right), \left(b, 0.8e^{1.5\pi i}, -0.1e^{0\pi i}\right), \\ \left(c, 0.3e^{\pi i}, -0.3e^{-1.5\pi i}\right) \end{array} \right\}.$$

Definition 3.4. The intersection of two (BCFS) as follows: Let A and B be two (BCFS) in X, where $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$ and $B = \{(x, r_B^+ e^{i\theta_B^+}, r_B^- e^{i\theta_B^-}) : x \in X\}$. Then the intersection of A and B is denoted as $A \cap B$ and is given as:
$(A \cap B)(x) = \{(\min(r_A^+, r_B^+)e^{i\min(\theta_A^+, \theta_B^+)}, \max(r_A^-, r_B^-)e^{i\max(\theta_A^-, \theta_B^-)})\}$.

Example 3.4. Let $X = \{a, b, c\}$ be a universe of discourse. Let A and B be two (BCFS). Let
$A = \{(a, 0.2e^{2\pi i}, -0.4e^{-1.2\pi i}), (b, 0.8e^{1.3\pi i}, -0.2e^{0\pi i}), (c, 0.3e^{\pi i}, -0.4e^{-1.5\pi i})\}$ and
$B = \{(a, 0.2e^{\pi i}, -0.3e^{-1.4\pi i}), (b, 0.7e^{1.5\pi i}, -0.1e^{1.5\pi i}), (c, 0.1e^{\pi i}, -0.3e^{-1.5\pi i})\}$,
then

$$(A \cap B) = \{(a, 0.2e^{\pi i}, -0.3e^{-1.2\pi i}), (b, 0.7e^{1.3\pi i}, -0.2e^{0\pi i}), (c, 0.1e^{\pi i}, -0.3e^{-1.5\pi i})\}.$$

Definition 3.5. If A and B are (BCFSs) in a universe of discourse X , where $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$ and $B = \{(x, r_B^+ e^{i\theta_B^+}, r_B^- e^{i\theta_B^-}) : x \in X\}$, then

1) $A \subset B$ if and only if $r_A^+ < r_B^+$ and $r_A^- > r_B^-$ for amplitude terms and the phase terms (arguments) $\theta_A^+ < \theta_B^+$ and $\theta_A^- > \theta_B^-$ for all $x \in X$.

2) $A = B$ if and only if $r_A^+ = r_B^+$ and $r_A^- = r_B^-$ for amplitude terms and the phase terms (arguments) $\theta_A^+ = \theta_B^+$ and $\theta_A^- = \theta_B^-$ for all $x \in X$.

Proposition 3.6. *Let A, B and R be any three (BCFS) over U. Then the following holds:*

i. $A \cup A = A$,
ii. $A \cap A = A$,
iii. $A \cup B = B \cup A$,
iv. $A \cap B = B \cap A$,
v. $A \cup (B \cap R) = (A \cup B) \cap (A \cup R)$,
vi. $A \cap (B \cup R) = (A \cap B) \cup (A \cap R)$,
vii. $A \cup (B \cup R) = (A \cup B) \cup R$,
viii. $A \cap (B \cap R) = (A \cap B) \cap R$.

Proof. Let A, B and R are three (BCFS) given as: $A = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\}$, $B = \{(x, r_B^+ e^{i\theta_B^+}, r_B^- e^{i\theta_B^-}) : x \in X\}$ and $R = \{(x, r_R^+ e^{i\theta_R^+}, r_R^- e^{i\theta_R^-}) : x \in X\}$.

So, to prove (i) we need to recall 3.3, then we have:

$A \cup A = \{(\max(r_A^+, r_A^+) e^{i\max(\theta_A^+, \theta_A^+)}, \min(r_A^-, r_A^-) e^{i\min(\theta_A^-, \theta_A^-)}) : x \in X\} = \{(x, r_A^+ e^{i\theta_A^+}, r_A^- e^{i\theta_A^-}) : x \in X\} = A$.

Analogously to (i), we can prove (ii) with recalling 3.4.

To prove (iii) we need to recall 3.3, and then we have

$A \cup B = \{(\max(r_A^+, r_B^+) e^{i\max(\theta_A^+, \theta_B^+)}, \min(r_A^-, r_B^-) e^{i\min(\theta_A^-, \theta_B^-)}) : x \in X\} = \{(\max(r_B^+, r_A^+) e^{i\max(\theta_B^+, \theta_A^+)}, \min(r_B^-, r_A^-) e^{i\min(\theta_B^-, \theta_A^-)}) : x \in X\} = B \cup A$.

Analogously to (iii), we can prove (iv) with recalling 3.4.

To prove (v) we need to recall both 3.3 and 3.4, and then we have

$$A \cup (B \cap R) = \left\{ \left\langle \begin{array}{c} x, \max\left(r_A^+(x), r_{B\cap R}^+(x)\right) e^{i\max\left(\theta_A^+(x), \theta_{B\cap R}^+(x)\right)}, \\ \min\left(r_A^-(x), r_{B\cap R}^-(x)\right) e^{i\min\left(\theta_A^-(x), \theta_{B\cap R}^-(x)\right)} \end{array} \right\rangle : x \in X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \max[r_A^+(x), \\ \min(r_B^+(x), r_R^+(x)) e^{i\min(\theta_A^+(x), \min(\theta_B^+(x), \theta_R^+(x))}], \\ \min[r_A^-(x), \max(r_B^-(x), r_R^-(x)) e^{i\max(\theta_A^-, (x)\max(\theta_B^-(x), \theta_R^-(x))}] \end{array} \right\rangle : x \in X \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \begin{array}{c} x, \min[\max(r_A^+(x), r_B^+(x)), \\ \max(r_A^+(x), r_R^+(x))] e^{i\min[\max(\theta_A^+(x), \theta_B^+(x)), \max(\theta_A^+(x), \theta_B^+(x))]}, \\ \max[\min(r_A^-(x), r_B^-(x)), \\ \min(r_A^-(x), r_R^-(x))] e^{i\max[\min(\theta_A^-(x), \theta_B^-(x)), \min(\theta_A^-(x), \theta_B^-(x))]} \end{array} \right\rangle \\ : x \in X \end{array} \right\}$$

$= (A \cup B) \cap (A \cup R)$.

Analogously to prove (v), we can prove (vi).

To prove (vii) we need to recall 3.3, and then we have

$$A \cup (B \cup R) = \left\{ \left\langle \begin{array}{c} x, \max(r_A^+(x), r_{B\cup R}^+(x))e^{i\max(\theta_A^+(x),\theta_{B\cup R}^+(x))}, \\ \min(r_A^-(x), r_{B\cap R}^-(x))e^{i\min(\theta_A^-(x),\theta_{B\cup R}^-(x))} \end{array} \right\rangle : x \in X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \max[r_A^+(x), \max(r_B^+(x), r_R^+(x))e^{i\max(\theta_A^+(x),\min(\theta_B^+(x),\theta_R^+(x))}], \\ \min[r_A^-(x), \min(r_B^-(x), r_R^-(x))e^{i\min(\theta_A^-,(x)\max(\theta_B^-(x),\theta_R^-(x))}] \end{array} \right\rangle : x{\in}X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \max[(r_A^+(x), r_B^+(x), r_R^+(x))]e^{i\max(\theta_A^+(x),\theta_B^+(x),\theta_R^+(x))}, \\ \min[r_A^-(x), r_B^-(x), r_R^-(x))]e^{i\min(\theta_A^-(x),\theta_B^-(x),\theta_R^-(x))} \end{array} \right\rangle : x{\in}X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \max[\max(r_A^+(x), r_B^+(x)), r_R^+(x)]e^{i\max(\max(\theta_A^+(x),\theta_B^+(x)),\theta_R^+(x))}, \\ \min[\min(r_A^-(x), r_B^-(x)), r_R^-(x))]e^{i\min(\min(\theta_A^-(x),\theta_B^-(x)),\theta_R^-(x))} \end{array} \right\rangle : x \in X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \max[r_{A\cup B}^+(x), r_R^+(x)]e^{i\max(\theta_{A\cup B}^+(x),\theta_R^+(x))}, \\ \min[r_{A\cup B}^-(x), r_R^-(x))]e^{i\min(\theta_{A\cup B}^-(x),\theta_R^-(x))} \end{array} \right\rangle : x \in X \right\} = (A \cup B) \cup R$$

Analogously to prove (vii), we can prove (viii). □

Proposition 3.7. *Let A and B be BCFSs over universe of discourse U. Then*

i. $(A^c)^c = A$,

ii. $(A \cup B)^c = A^c \cap B^c$,

iii. $(A \cap B)^c = A^c \cup B^c$.

Proof. Let A and B are two (BCFS) given as:

$A = \{(x, r_A^+(x)e^{i\theta_A^+(x)}, r_A^-(x)e^{i\theta_A^-(x)}) : x \in X\}$ and
$B = \{(x, r_B^+(x)e^{i\theta_B^+(x)}, r_B^-(x)e^{i\theta_B^-(x)}) : x \in X\}$.

So, to prove (i) we need to apply 3.2 twice. Then, we have

$A^c = \{(x, (1 - r_A^+(x))e^{i(2\pi-\theta_A^+(x))}, (-1 - r_A^-(x))e^{i(2\pi-\theta_A^-(x))}) : x \in X\}$, thus

$$(A^c)^c = \left\{ \left(\begin{array}{c} x, \left(1 - \left(1 - r_A^+(x)\right)\right) e^{i\left(2\pi-(2\pi-\theta_A^+(x))\right)}, \\ \left(-1 - \left(-1 - r_A^-(x)\right)\right) e^{i\left(2\pi-(2\pi-i\theta_A^-(x))\right)} \end{array} \right) : x \in X \right\}$$

$$= \left\{ \left(x, r_A^+(x)e^{i\theta_A^+(x)}, r_A^-(x)e^{i\theta_A^-(x)}\right) : x \in X \right\} = A$$

To prove (ii) we need to apply 3.2 and 3.3. Then, we have

$(A \cup B)^c =$complement$(A \cup B)$

$$=\text{complement}\left\{ \left\langle \begin{array}{c} x, \max\left(r_A^+(x), r_B^+(x)\right) e^{i\max\left(\theta_A^+(x),\theta_B^+(x)\right)}, \\ \min\left(r_A^-(x), r_B^-(x)\right) e^{i\min\left(\theta_A^-(x),\theta_B^-(x)\right)} \end{array} \right\rangle : x \in X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \min\left(1 - r_A^+(x), 1 - r_B^+(x)\right) e^{i\min\left(\theta_A^+(x),\theta_B^+(x)\right)-2\pi}, \\ \max\left(-1 - r_A^-(x), -1 - r_B^-(x)\right) e^{i\max\left(\theta_A^-(x),\theta_B^-(x)\right)-2\pi} \end{array} \right\rangle : x \in X \right\}$$

$= A^c \cap B^c$.

Analogously to prove (ii), we can prove (iii). □

4. Conclusion

In this work, introduced (BCFS), also a new operations defined over the (BCFS) some properties of this operations are discussed. These properties illustrate the relationship between the basic set theoretical operations such as: commutative laws, associative laws, distributive laws, and De Morgan's laws.

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Accepted: 23.01.2019

The impact of organizational and human factors on the successful implementation of ERP system in water authority of Jordan

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Abstract. The implementation process of the Enterprise resource planning (ERP) systems is complex, costly and influenced by several critical factors. Organizational and human factors are the most critical success or failure factor for Enterprise Resource Planning (ERP) systems implementation. This paper aims to study the impact of these factors on the successful implementation of ERP system in Water Authority of Jordan; which is a governmental institute responsible of distributing water to participants in Jordan. The methodology of this paper is based on deductive and quantitative method; a questionnaire is designed with (43) questions. The population consists of (77) employees working on the ERP system in the Water Authority of Jordan. Intentional sample of (58) employees working at the main center in Amman was taken. Fifty-seven questionnaires were distributed and (51) were returned. Statistical Package for the Social Sciences (SPSS) program version (12) is used to analyze every item in the questionnaire. While the statistical analysis consists of Cronbach's Alpha, its value is (0. 897), Mean and Standard Deviation, Pearson Correlation, and Linear Regression. The results show that there is statistically significant impact of Organizational and Human factors on the successful implementation of the ERP system in the Water Authority of Jordan.

1. Introduction

ERP system is a major and sensitive tools that helps organizations to enhance their functional abilities, improve their performance, develop their decision-

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making processes and gain competitive advantages [4]. An ERP system is an integrated software package composed by a set of standard functional modules such as: Finance, Human resources, Production Planning, Material Management, Marketing and Sales [3]. These modules integrate and interact to achieve the organizations' goals. ERP system typically attempts to cover all basic functions of business, regardless of the organizations business or charter. Business, Non- profit Organization, Non- Governmental Organization, Governments and other large entities utilize ERP system [5]. ERP have come a long way evolving from the early 1960s to the present time, as illustrated in Figure (1).

Figure 1: The Evolution of ERP System

In 1960's, Most of the software packages were designed to handle inventory based on traditional inventory concepts. Then in 1970's the focus shifted to MRP (Material Requirement Planning) systems which is simply about ensuring the materials are available to manufacture a specific part in a specific volume. After that, in 1980's, the concept of MRP-II systems evolved, and it takes care of all other aspects of a job including ordering, tracking inventory and ensuring capacity. In 1990's the ERP is defined as an integrated information system that serves all departments within an enterprise like engineering, finance, human resources, project management [19].

While in 2000's the ERPII software provides an integral coordination of activates, which are carried out at every department of the enterprise. And in 2010's Alternate ERP Solutions begin implemented like Open Source and On-Demand ERP Applications [12]. And recently in 2015, the cloud ERP starts to be implemented [10]. The Implementation of Enterprise Resource Planning System (ERP) is an advanced, complex and requires a lot of preparation and readiness in addition to its costly to organizations. As it's known, Jordan is a small country and its resources are few and greatly influenced by the circumstances of the global economic. Also, Water Authority of Jordan (WAJ) is a government establishment that regulates administers and oversees activities in the water sector of Jordan for participants. Through its main subsidiaries; it provides waste water treatment and desalination services; distribution of water. WAJ has implemented ERP system for Finance and Human Resources, this paper aims to determine the impact of the Critical Success Factors of Organizational and Human on the successful implementation of ERP system.

2. Related work

ERP systems support the organizations' functionality, sharing data, performance and enhance the process of decision-making in addition to improve the organizations' management of the processes [4,8]. Many factors play a critical role in implementation success or failure for the ERP systems including Organizational, technological and human factors [1,2]. Afaneh et al. [1] studied the influence of the technological and the organizational factors during the ERP system implementation at the Greater Amman Municipality (GAM). The authors found that there is a correlation between the availability of the organizational factors and the success on ERP System Implementation. Also, Afaneh et al. [1] stated that the availability of the advanced technological infrastructure is a major to guarantee the success of ERP implementation. AlHadid et al. [2] investigated the relationship between the human factors and the ERP system implantation, the authors stated that the success of ERP system implementation depends on delivering an effective systems training, in addition to the top management and staff awareness about the importance of the ERP system. AlHadid et al. [2] argued that the human factors are critical and might leads to system failure. Hasan et al. [11] studied the factors that influence the ERP systems implementation by investigating the literature studies between the 2011 and 2016. The research provides a deep understanding related to the success factors that influence ERP implantation which can be adopted to develop a strategic plan to ensure the successful implementation of ERP system. Desalegn and Pettersson [8] investigated the critical success factors of an ERP implementation from a user perspective. Researchers argue that more users should be involved in the ERP implementation in addition to provide users' with the required education and training. Fadelelmoula [9] discussed the effects of the key critical success factors for ERP implementation in the higher education sector. Researcher claimed that organizations should pay more attention to the top management support, tanning, project management, technical resources, business process reengineering and consultant support factors. Researchers stated that the mentioned factors have a positive relationship with adopting ERP in the organization. Gupta et al. [10] and Baskaran [6] inspected the organizational and technological factors that affect the successful implementation of cloud ERP implementation. Researchers also study other factors that influence the cloud ERP implementation such as security, compliance and network. Reitsma et al. [15] stated that the top management should prioritize the following critical success factors when implementing an ERP system. The factors include the users training and education, business process reengineering and communication across organizational levels and teams. Yildirim and Kuşakci [20] studied the critical success factors of ERP selection and implementation in logistics sector. Researchers found that the factors that affect the success of ERP implementation in an organization are the continuous support of top management and the ERP system compatibility with the fundamental business processes in addition to the reengineering of the business processes in the organization. from the previous studies, we find the factors that influence the success or failure of the ERP system implementation are organizational and human factors including the lack or inadequate training for end users, incompatible business process reengineering or lack of management commitment and support in addition to the availability of the technological factors [1,2,7,9,13,14].

3. The paper model

The researchers depended in building the paper model on previous studies that are related to the subject of the research, and then chose the common factors among these studies, which explained the critical factors affecting the application of ERP system. After that the researchers relied on the above mentioned according to what was consistent with Jordan Environment and the objectives of the paper to build a model with the proposed independent factors affecting ERP system implementation, which is described in Figure (2).

3.1 Hypothesis

(H1): There is no statistically significant Impact of Organizational Factors on the successful implementation of ERP System in Water Authority of Jordan. Organizational Factors in the following studies were adopted as factors for the successful implementation of the ERP system [6,9,10,14,15].

(H2): There is no statistically significant Impact of Human Factors on the successful implementation of ERP System in Water Authority of Jordan.

(H2:a) - There is no statistically significant Impact of Top Management Support on the successful implementation of ERP System in Water Authority of Jordan.

(H2:b) - There is no statistically significant Impact of Staff Training on the successful implementation of ERP System in Water Authority of Jordan.

These studies used the Top Management Support and the Staff Training as a Critical Success Factor on the implementation of ERP [6,9,11,13,14,15].

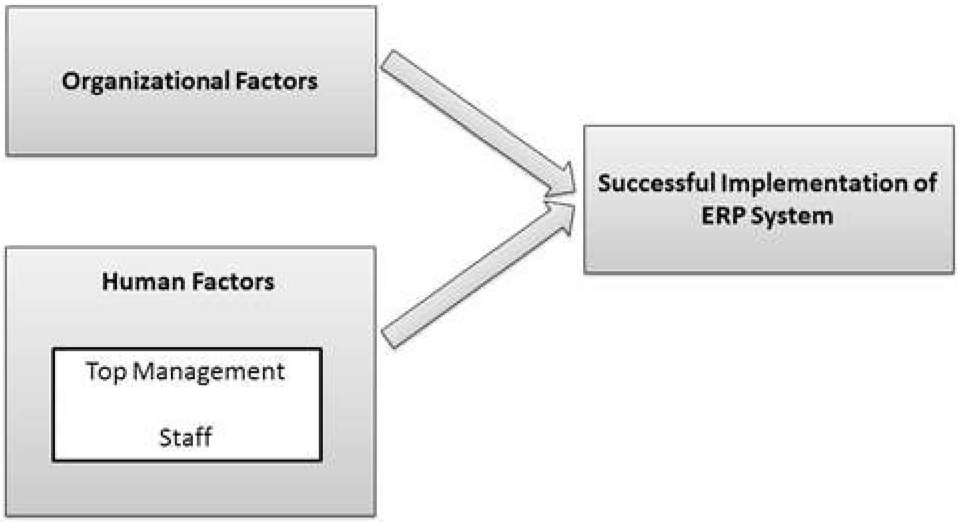

Figure 2: The Model

3.2 The methodology

The methodology of this paper is based on deductive and quantitative method, thus, a questionnaire is designed to measure the impact of the independent and dependent Organizational and Human Factors on WAJ. The questionnaire consists of (43) questions, (23) questions for the Organizational Factors, and (20) questions for the Human Factors.

3.3 The population and sample

The population consists of (77) employees working on the ERP system in the Water Authority of Jordan. Intentional sample of (58) employees working at the main center in Amman was taken. Fifty-seven questionnaires were distributed and (51) were returned. After examination all the questionnaires were valid, and the sample size is acceptable for analysis according to Sekaran [16].

3.4 The statistical analysis

The statistical analysis that is used the following statistical ways by using Statistical Package for the Social Sciences (SPSS) program version (12) to analyze every item in the questionnaire:

- Means and Standard deviation: to examine the hypothesis, and to identify the relative importance.
- Cronbach's Alpha: to verify the reliability of the used questionnaire [17] in this paper, Cronbach's Alpha value is (0. 897).

- Correlation: is a term that refers to the strength of a relationship between two variables. A strong, or high, correlation means that two or more variables have a strong relationship with each other while a weak, or low, correlation means that the variables are hardly related. Correlation coefficients can range from -1.00 to +1.00. The value of -1.00 represents a perfect negative correlation while a value of +1.00 represents a perfect positive correlation. A value of (0.00) means that there is no relationship between the variables being tested. In this paper the researchers used Pearson correlation test to find a correlation between the dependent and independent variables.

- - Linear regression: Regression analysis is used to determine the impact of the independent factor on the dependent factor. In other words, is there an impact for the independent factor on the dependent variable or not? If the sig value in the test is less than 5% It means no effect, If it is greater than 5%, then there is an effect of the independent variable on the child [18].

Each item in the questionnaire was divided into 5-points according to Likert-type scale, and was determined in five levels as the follows: strongly agree given (5) degrees, agree given (4) degrees, neutral given (3) degrees, disagree given (2) degrees, and strongly disagree given (1) one degree. In addition, to use judge scale for the responses, in this paper the level of significance of the measures was distributed according to Table (1).

Table 1: Measures Significance Levels

Likert Scale Levels	**Range**
Strongly Disagree	1-1.80
Disagree	1.81-2.61
Neutral	2.62-3.42
Agree	3.43-4.23
Strongly Agree	4.24-5

The level of the significance was used to analyze and study the arithmetic Mean for the sample, whether it is agree or disagree for each question. So if the Mean from (1) to (2.61) degree, that means disagree, or the range between (2.62) to (3.42) degree it means neutral, and if Mean between (3.43) to (5) degree, that means agree.

4. Results and analysis

The researcher calculated Cronbach's Alpha for all the areas of the questionnaire (with 43 questions) to test the reliability of each section; Cronbach's Alpha value is (0.897) which is accepted, because the value is above or more than (0.6) which is the minimum acceptable value. If the same test is returned to another sample of the same population, the results will be close to (89%).

4.1 Demographic characteristics of the research sample

Figure (3) shows that the number of females in the sample is (18) by (35.3%), and that the number of males is (33) by (64.7%).

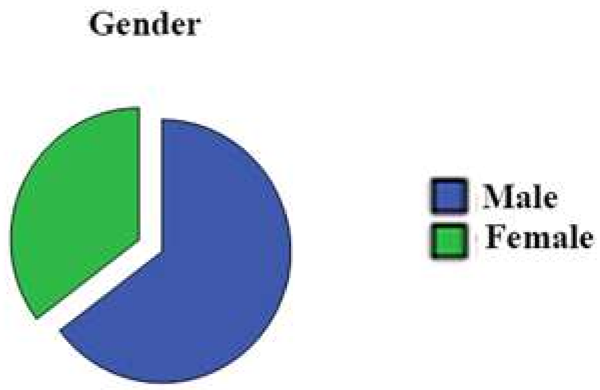

Figure 3: Gender characteristics of the research sample

Figure (4) illustrates the distribution of the sample of Positions, where it is clear that the number of Employees is (36) by (70.5%), Heads of Departments is (12) by (23.5%) and the number of Managers is (3) by (6%).

Figure 4: Position characteristics of the research sample

Figure (5) shows the sample distribution on the number of years of experience in WAJ, where more than (54%) of the sample has more than (10) years' experience.

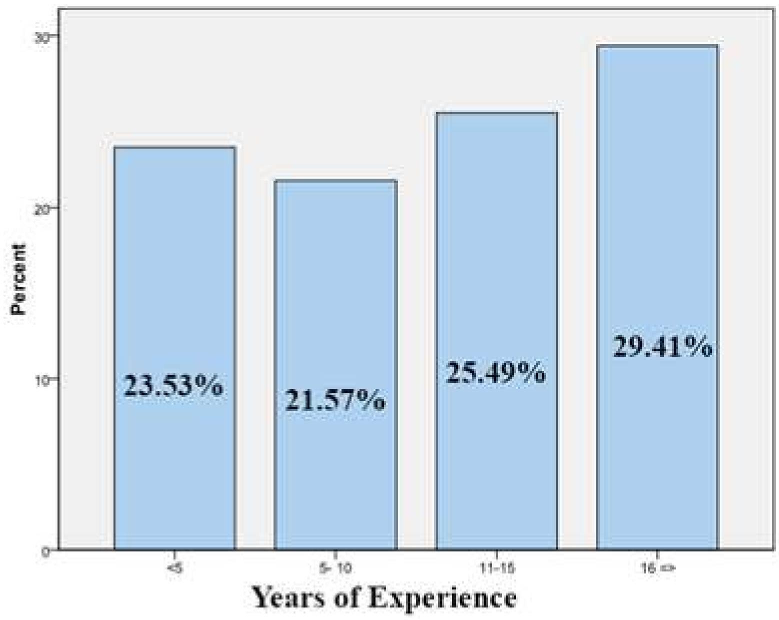

Figure 5: years of experience in WAJ

4.2 Hypotheses testing

(H1): There is **no** statistically significant impact of Organizational factors on the successful implementation of the ERP system in Water Authority of Jordan.

To examining this Hypothesis the researchers first: calculates Mean and Standard deviation for each item for the Organizational Factors in WAJ in Tables (2) and (3). Second calculates Pearson Correlation to examine whether there is a relationship between the Organizational Factors and the successful implementation of the ERP system in Table (4), and then the Linear Regression test is calculated to see if there is an impact of these factors on the implementation of the ERP system in Table (5).

Table (2) shows the descriptive analysis of the questions related to the Organizational factors, from which it can be observed that all the Mean values fall within the acceptance area according to the significance of the measures of the five-dimensional Likert scale, as well as all the standard deviations are less than 1, which means the acceptance of the values of Mean. So, the sample agrees with the availability of Organizational factors in WAJ.

Table 2: Descriptive Statistics of the Organizational Factor

Section	Max.	Min.	Mean	Std.
Group1				
1. The objectives in our organization are obvious for managers.	5	1	3.82	.910
2. The objectives in our organization are obvious for employees.	5	1	3.71	0.923
3. The procedures are clear for managers.	5	1	3.90	0.806
4. The procedures are clear for employees.	5	1	3.88	.765
5. The daily operations and plans are understood by the employees.	5	1	3.94	0.835
6. The control standards in the organization are obvious and specific.	5	1	3.63	0.937
7. The managerial level in the organization is task and duty specified.	5	1	3.82	0.842
8. All the Systems in the organization are integrated & facilitate the communication with the outside.	5	1	3.67	0.792
9. All the Systems in the organization are integrated & facilitate the communication between the organization's departments.	5	1	3.59	0.920
16. The employees in the IT department are experienced with efficient.	5	3	3.98	0.648
17. There is a maintenance staff in the IT department.	5	3	4.02	0.616
18. The role for communication in the organization is clear.	5	1	3.80	0.849
19. The LAN in the organization increases the efficiency of exchanging information between the employees.	5	2	3.78	0.783
23. ERP system provides the suitable Security mechanism.	5	1	3.88	0.864
Total			3.82	0.82

Table (3) shows the descriptive analysis of the questions that are related to the implementation of the ERP system in WAJ. The table shows that all questions have Mean values in the acceptance area according to the statistical significance levels of the five-dimensional Likert scale and that all standard deviations are less than 1. Except the question (13); ERP System contributes in enhancing the exchange of ideas and experiences between the employees in the organization. The Mean is (3.55) but the standard deviation is (1.006), i.e., more than 1. This indicates that there is a different view of the sample members on the extent to which the ERP system can helps to exchange ideas among workers within the authority.

Table 3: Descriptive Statistics of the ERP System

Section	**Max.**	**Min.**	**Mean**	**Std.**
Group2				
10. ERP System supports and achieves our objectives.	5	1	3.82	0.817
11. ERP System improves the operations and services.	5	1	3.80	0.825
12. ERP System supports the decision making processes.	5	1	3.84	0.857
13. ERP System contributes in enhancing the exchange of ideas and experiences between the employees in the organization.	5	1	3.55	1.006
14. The new (ERP) system will minimize the time of the operation.	5	1	3.98	0.905
15. ERP System contributes in facilitating exchange of ideas and experiences with the rest of the governmental institutions' work.	5	1	3.67	0.909
20. The ERP system helps in facilitating the exchange of information via communication channels.	5	1	3.69	0.905
21. ERP system helps to optimize the utilization of Hardware (HW) in WAJ.	5	1	3.67	0.841
22. ERP system helps to optimize the use of available Software (SW) in WAJ.	5	1	3.78	0.832
Total			3.76	0.877

The other side for testing this hypothesis is to use Pearson Correlation test between group1 (q1,q2,q3,q4,q5,q6,q7,q8,q9,q16,q17,q18,q19,q23) and group2 (q10,q11,q12,q13,q14,q15,q20,q21,q22). The table (4) explains there is a positive correlation between group (1) and group (2). The value for Pearson Correlation is (0.799) this Correlation is significant at the 0.01 level (1-tailed). The results in table (4) indicate that there is a significant correlation of Organizational Factors on ERP System implementation, with an error rate of not more than 1%. So there is availability of the Organizational Factors for Success ERP System implementation in WAJ.

Table 4: Correlation between Organizational Factor and ERP System

		Group1	Group2
Group1	Pearson Correlation	1	.799 (**)
	Sig. (1-tailed)	.	0.000
	N	51	51
Group2	Pearson Correlation	.799 (**)	1
	Sig. (1-tailed)	0.000	.
	N	51	51

*** Correlation is significant at the 0.01 level (1-tailed).

Table (5) shows the results of the Linear Regression test between the Group1 related to Organizational Factors and the Group2 related to the ERP system. It is noted that the value of sig = .002 is less than 5%, i.e., it is located in the rejection zone. This means rejecting the null hypothesis and accepting the alternative hypothesis that **there is a statistically significant impact of Organizational Factors on the successful implementation of the ERP system in Water Authority of Jordan**.

Table 5: ANOVA test of Group1 and Group2

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	14.132	1	14.132	46.845	0.002
	Residual	14.782	49	0.302		
	Total	28.914	50			

4.3 Second hypothesis

(H2): There is no statistically significant impact of Human factors on the successful implementation of the ERP system in Water Authority of Jordan.

For testing this hypothesis the researchers do the same steps as in testing the first hypothesis, the results are shown in Tables from (6) to (12). The questions which are related to the Human Factors are divided into 3 groups; Group3 (q24,q25,q26,q27,q28,q29,q30) for Top Management Support, Group4 (q31,q32,q33,q34,q35,q36,q37) for Training, Group5 (q38,q39, q40, q41, q42, q43) for Employees Satisfaction.

(H2-a): There is no statistically significant impact of Top Management Support on the successful implementation of the ERP system in Water Authority of Jordan.

(H2-b): There is no statistically significant impact of Training on the successful implementation of the ERP system in Water Authority of Jordan.

Table (6) shows descriptive analysis of the degree of Top Management Support for the successful implementation of the ERP system. However, the values of Mean are at the beginning of the acceptance range with close to the neutral period, and the standard deviations are less than 1, which means there is a need to increase the support provided by the Top Management to ensure the success of the application of the system. As an example, the statistical analysis of Question (27), shows that the value of Mean is (3.37), which is in the neutral area, not the acceptance, and the highest average value is (3.57) which is at the beginning of the acceptance period.

Table 6: Descriptive Statistics of Top Management

Section	**Max.**	**Min.**	**Mean**	**Std.**
Group3				
24. The Top Management provides specialized managers to supervise the new ERP System in the organization.	5	1	3.51	0.967
25. The Top Management supports the solutions provided by the system.	5	1	3.57	0..946
26. The Top Management provides sufficient support for the work team responsible for the ERP System.	5	1	3.51	0.966
27. Top Management presents sufficient money required for upgrading ERP System from time to time.	5	1	3.37	0.958
28. Top management presents full support to the maintenance operations needed to the system in our organization.	5	1	3.55	0.923
29. The Top Management support the ERP system in order to support the administrative decision making process.	5	1	3.47	0.946
30. Top Management presents support in order to employees' participation in applying ERP system.	5	1	3.45	0.986
Total			3.49	0.956

Table (7) shows the Descriptive Analysis of the questions that are related to the Training of the employees of WAJ on the ERP system. When studying the Mean values, they are all located in the rejection area according to the statistical significance of the five-dimensional Likert scale. That is, the employees are not satisfied with the training courses and they need to develop their skills to be able to deal with the system.

Table 7: Descriptive Statistics of the Training

Section	Max.	Min.	Mean	Std.
Group4				
31. A time table was set to train all the users of the new ERP system.	4	1	2.25	0.917
32. The training program of ERP System provides the users with all the needs to understand and use the system.	4	1	2.29	0.986
33. We have a comprehensive training plan to train the employees on using the new system.	4	1	2.41	0.904
34. The ERP System in the organization is simple and assesses the users in self-training through following the instructions.	4	1	2.35	0.836
35. The organization provides the qualified employees & the requirements needed to train the users on using the system.	4	1	2.24	0.950
36. Training the employees on the ERP System contributes in minimizing mistakes during using the system.	4	1	2.53	0.840
37. WAJ is providing continuous training for the employees.	4	1	2.18	0.990
Total			2.32	0.92

Table (8) shows the Descriptive Analysis of the questions regarding the Opinion of Employees on the ERP system after use and their satisfaction with the system in terms of facilitating and speeding up the business performance. The analysis of the questions shows a high degree of satisfaction with the system by the staff, All Mean values are in the acceptance area and all standard deviations are less than 1.

Table (9) shows a significant correlation with error rate not exceeding 1% between the third group (Top Management Support) and the fifth group (related to the ERP system). This means that the provision of Top Management Support leads to the successful implementation of the system.

Table (10) shows a significant correlation with error rate not exceeding 1% between the fourth group (Staff Training) and the fifth group (related to the ERP system). This means that good Training leads to successful application of the system. Table (11) shows the results of the linear regression test between the Group3 which is related to Top Management Support and the Group5 which

Table 8: Descriptive Statistics of the Employees Satisfaction

Section	**Max.**	**Min.**	**Mean**	**Std.**
Group5				
38. Our employees do not find difficulties in dealing with the ERP System.	5	1	3.45	0.879
39. ERP System in the organization is easily understood due to its capabilities and usage.	5	1	3.63	0.824
40. ERP System in the organization contributes in facilitating the work procedures.	5	1	3.84	0.758
41. The ERP System in the organization is easy to be reached and dealt with.	5	1	3.89	0.883
42. The ERP System in the organization helps in facilitating the tracking of the operations in any department at any time.	5	1	3.97	0.841
43. ERP System in the organization is easy to learn.	5	1	3.63	0.848
Total			3.74	0.839

Table 9: Correlation between Top Management and ERP System

		Group3	**Group5**
Group3	Pearson Correlation	1	0.721 (**)
	Sig. (1-tailed)	.	0.000
	N	51	51
Group5	Pearson Correlation	0.721 (**)	1
	Sig. (1-tailed)	0.000	.
	N	51	51

*** Correlation is significant at the 0.01 level (1-tailed).

is related to the ERP system. It is noted that the value of sig = .003 is less than 5%, i.e. it is located in the rejection zone. This means rejecting the null hypothesis and accepting the alternative hypothesis that there is a statistically significant impact of Top Management Support on the successful implementation of the ERP system in Water Authority of Jordan. Table (12) shows the results of the Linear Regression test between the Group4 which is related to the Staff Training and Group5 which is related to the ERP system. Note that the value of (.001), that is, less than 5%, that is located in the rejection area and this means rejecting the null hypothesis and accept the alternative hypothesis that there is a significant statistical impact to Staff Training on the successful implementation of the ERP system in Water Authority of Jordan.

From the previous tests it can be said that the second main null hypothesis is rejected There is **no** statistically significant impact of Human factors on the successful implementation of the ERP system in Water Authority of Jordan.

Table 10: Correlation between Training and ERP System

		Group4	**Group5**
Group4	Pearson Correlation	1	.758 (**)
	Sig. (1-tailed)	.	0.000
	N	51	51
Group5	Pearson Correlation	. 758 (**)	1
	Sig. (1-tailed)	0.000	.
	N	51	51

*** Correlation is significant at the 0.01 level (1-tailed).

Table 11: ANOVA^b

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	13.361	1	13.361	53.141	0.003
	Residual	12.320	49	0.251		
	Total	25.681	50			

5. Conclusions

As a conclusion thus, the aim of this paper is to study the organizational and human factors that impact the influence of the ERP system implementation success or failure. As a result, there is statistically significant impact of these factors in WAJ. And the Organizational Factors in WAJ are available. While using the ERP system facilitates the administrative Processes. There is a need to increase the support provided by the top management to ensure the success of the ERP system implementation. The employees are not satisfied with the provided training courses and they need to develop their skills to use the ERP system appropriately. However, there is a high degree of satisfaction with the ERP system by the staff.

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Table 12: ANOVA

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	14.740	1	14.740	66.014	0.001
	Residual	10.941	49	0.223	.223	
	Total	25.681	50			

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Accepted: 27.01.2019

Second type nabla Hukuhara differentiability for fuzzy functions on time scales

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Abstract. In this paper, we introduce a new class of derivative called second type nabla Hukuhara derivative for fuzzy functions on time scales under Hukuhara difference. We prove existence and uniqueness of this derivative and obtain its fundamental properties.
Keywords: fuzzy functions, time scales, Hukuhara difference, nabla Hukuhara derivative.

1. Introduction

In modelling a real world phenomenon, some vagueness or impreciseness occurs due to incomplete information about the parameters which we cannot exactly describes the behaviour of the problem. In order to deal with these impreciseness or vagueness Zadeh [29] introduced the theory called fuzzy sets. The fuzzy set theory is an excellent approach which helps us to deal with fuzzy dynamic models. Fuzzy set theory is the preliminary source to study fuzzy differential equations (Fde's) or interval differential equations. Fde's play a vital role in applications of biology, economics and many other engineering problems where uncertainty arises. Hukuhara [12] initiated the difference between two

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sets called Hukuhara difference and developed the theory of derivatives and integrals for set valued mappings. Later, Puri and Ralescu [21] studied Hukuhara derivative for fuzzy functions using Hukuhara difference and it is the primary approach for studying uncertainty of the dynamical systems. Further, Kaleva [14], studied Fde's under Hukuhara differentiability and also studied existence and uniqueness of the solutions to Fde's using Hukuhara derivative which has a disadvantage that the solutions exist only when the functions have an increasing length of support. To overcome this circumstance, Bede and Gal [6] studied generalizations of the differentiability of fuzzy number valued functions Later, Stefanini and Bede [22] studied generalized Hukuhara differentiability of interval value functions and interval differential equations. Further, Malinowski [19, 20] studied the concept of second type Hukuhara derivative for interval differential equations and interval cauchy problem with second type Hukuhara derivative. Furthermore, Zhang and Sun [28] studied stability of Fde's under second type Hukuhara derivative.

Time scales was initiated by the german mathematician Stefan Hilger [10]. For fundamental theory and applications on time scales calculus and dynamic equations on time scales are found in [1, 7]. For alternative solutions of linear dynamic equations on time scales and boundary value problems for dynamic equations on time scales were studied in [3] and [8]. The important features of time scales are extension, unification and generalization. The theory of time scale calculus is applicable to any field in which dynamic process described by continuous or discrete time models. If we take time scales as real numbers, then the derivative of a function is equal to standard differentiation while, if we take time scales as integers then it turns to backward difference operator or forward difference operator. In some recent studies and applications in economics [5], production, inventory models [4], adaptive control [13], neural networks [17] cellular neural networks [9] suggested nabla derivative is more preferable than delta derivative on time scales.

The multivalued functions on time scales under Hukuhara derivative was introduced in [11]. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales was studied in [18]. Recently Vasavi et. al. [23, 24, 25, 26] introduced Hukuhara delta derivative, second type Hukuhara delta derivative and generalized Hukuhara delta derivatives using Hukuhara difference and studied fuzzy dynamic equations on time scales. With the importance and advantages of nabla derivative, we proposed to develop the theory of fuzzy nabla dynamic equations on time scales. In this context, we introduce second type nabla Hukuhara derivative for fuzzy functions on time scales and study their properties. The rest of this paper is organized as follows. In section 2, we present some definitions, properties, basic results relating to fuzzy sets, calculus of fuzzy functions and time scales calculus. Section 3 introduces second type nabla Hukuhara derivative of fuzzy functions on time scales and establish uniqueness, existence of the derivative and also obtain some properties.

2. Preliminaries

It is important to recall some basic results and definitions related to fuzzy calculus. Let $\Re_k(\Re^n)$ be the family of all convex compact nonempty subsets of $\Re^n$. Denote the set addition and scalar multiplication in $\Re_k(\Re^n)$ as usual. Then $\Re_k(\Re^n)$ satisfies the properties of commutative semigroup [14] under addition with cancellation laws. Further, if $\alpha, \beta \in \Re$ and $S, T \in \Re_k(\Re^n)$, then

$$\alpha(S+T) = \alpha S + \alpha T, \ \ \alpha(\beta S) = (\alpha\beta)S, \ \ 1.S = S,$$

and if $\beta, \alpha \geq 0$ then $(\beta+\alpha)S = \beta S + \alpha S$. Let S and T be two bounded nonempty subsets of $\Re^n$. By using the Pampeiu-Hausdorff metric we defined the distance between S and T as follows

$$d_H(S,T) = \max\{\sup_{s\in S}\inf_{t\in T}\|s-t\|, \sup_{t\in T}\inf_{s\in S}\|s-t\|\}$$

here $||.||$ is the Euclidean norm in $\Re^n$. Then $(\Re_k(\Re^n), d_H)$ becomes a seperable and complete metric space [14].

Define

$$\mathbb{E}_n = \{\mu : \Re^n \to [0,1]/\mu \ \text{ satisfies(a)-(d) below}\}, \text{ where}$$

(a) If $\exists$ a $t \in \Re^n$ such that $\mu(t) = 1$ then μ is said to be normal,

(b) μ is fuzzy convex,

(c) μ is upper semicontinuous,

(d) the closure of $\{t \in \Re^n/\mu(t) > 0\} = [\mu]^0$ is compact.

For $0 < \lambda \leq 1$, denote $[\mu]^\lambda = \{t \in \Re^n : \mu(t) \geq \lambda\}$, then from the above conditions we have that the λ-level set $[\mu]^\lambda \in \Re_k(\Re^n)$.

According to Zadeh's extension principle we define $g : \mathbb{E}_n \times \mathbb{E}_n \to \mathbb{E}_n$ by

$$g(p,q)(Z) = \sup_{Z=g(p,q)} \min\{p(x), q(y)\}.$$

We know that $[g(p,q)]^\lambda = g([p]^\lambda, [q]^\lambda)$, for all $p, q \in \mathbb{E}_n$ and g is a continuous function. The scalar multiplication $\odot$ and addition $\oplus$ of $p, q \in \mathbb{E}_n$ is defined as

$$[p \oplus q]^\lambda = [p]^\lambda + [q]^\lambda, [k \odot p]^\lambda = k[p]^\lambda, \text{ where } p, q \in \mathbb{E}_n, \ k \in \Re, \ 0 \leq \lambda \leq 1.$$

Theorem 2.1. *[14] If $\mu \in \mathbb{E}_n$, then*

(a) $[\mu]^\lambda \in \Re_k(\Re^n)$ *for all* $0 \leq \lambda \leq 1$,

(b) $[\mu]^{\lambda_2} \subset [\mu]^{\lambda_1}$ *for all* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$,

(c) *If* $\lambda_k \in [0,1]$ *and* $\{\lambda_k\}$ *is a nondecreasing sequence converging to* $\lambda > 0$, *then*

$$[\mu]^\lambda = \bigcap_{k\geq 1}[\mu]^{\lambda_k}.$$

Conversely, if $\{X^\lambda/0 \leq \lambda \leq 1\}$ *is a subsets of family of* $\Re^n$ *satisfying the above conditions from (a)-(c), then* $\exists a\, x \in \mathbb{E}_n \ni$

$$[\mu]^\lambda = X^\lambda, \ \ forall \quad \lambda \in (0,1] \ and$$

$$[\mu]^0 = cl\left\{\bigcup_{0<\lambda\leq 1} X^\lambda\right\} \subset X^0, \ here\ cl\ is\ the\ closure\ of\ the\ set.$$

Theorem 2.2 ([14])**.** *If sequence* $\{X_n\}$ *converges to* X *in* $\Re_k(\Re^n)$ *and* $d(X_n, X) \to 0$ *as* $n \to \infty$ *then*

$$X = \bigcap_{n\geq 1} cl\left\{\bigcup_{m\geq n} X_m\right\}.$$

Define $D_H : \mathbb{E}_n \times \mathbb{E}_n \to [0,\infty)$ by

$$D_H(s,t) = \sup_{0\leq\lambda\leq 1} d_H([s]^\lambda, [t]^\lambda),$$

here d_H is the Pampeiu Hausdorff metric defined in $\Re_k(\Re^n)$. Then $(\mathbb{E}_n, D_H)$ is a complete metric space [14].

The following theorem extend the properties of addition and scalar multiplication of fuzzy number valued functions ($\Re_F = \mathbb{E}_1$) to $\mathbb{E}_n$.

Theorem 2.3 ([2])**.** (a) *If* $\tilde{0}$ *is the zero element in* $\Re_F$, *then* $\hat{0} = (\tilde{0}, \tilde{0}, \ldots, \tilde{0})$ *is the zero element in* $\mathbb{E}_n$. *i.e.* $s \oplus \hat{0} = \hat{0} \oplus s = s\ \forall s \in \mathbb{E}_n$;

(b) *For any* $s \in \mathbb{E}_n$ *has no inverse with respect to* '$\oplus$';

(c) *For any* $\beta, \gamma \in \Re$ *with* $\beta, \gamma \geq 0$ *or* $\beta, \gamma \leq 0$ *and* $s \in \mathbb{E}_n$, *then* $(\beta+\gamma)\odot s = (\beta \odot s) \oplus (\gamma \odot s)$;

(d) *For any* $\beta \in \Re$ *and* $s, t \in \mathbb{E}_n$, *we have* $\beta \odot (s \oplus t) = (\beta \odot s) \oplus (\beta \odot t)$;

(e) *For any* $\beta, \gamma \in \Re$ *and* $s \in \mathbb{E}_n$, *we have* $\beta \odot (\gamma \odot s) = (\beta\gamma) \odot s$.

Let $S, T \in \mathbb{E}_n$. If $\exists$ a $R \in \mathbb{E}_n$ such that $S = T \oplus R$ then we say that R is the H-difference (Hukuhara difference) of S and T and is denoted by $S \ominus_h T$. For any $S, T, R, U \in \mathbb{E}_n$ and $\alpha \in \Re$, the following holds

(a) $D_H(S,T) = 0 \Leftrightarrow S = T$;

(b) $D_H(\alpha S, \alpha T) = |\alpha| D_H(S,T)$;

(c) $D_H(S \oplus R, T \oplus R) = D_H(S, T)$;

(d) $D_H(S \ominus_h R, T \ominus_h R) = D_H(S, T)$;

(e) $D_H(S \oplus T, R \oplus U) \leq D_H(S, R) + D_H(T, U)$;

(f) $D_H(S \ominus_h T, R \ominus_h U) \leq D_H(S, R) + D_H(T, U)$.

provided the H-difference exists.

Now, we present some fundamental definitions and properties of Hukuhara derivative of fuzzy functions on the compact interval $I = [a, b]$, $a, b \in \Re$.

Definition 2.1 ([6]). *Let $I = [a, b] \subset \Re$ be a compact interval. A mapping $G : I \to \mathbb{E}_n$ is said to be Hukuhara form-I differentiable at $\theta \in I$ if $\exists$ a $G^{'}(\theta) \in \mathbb{E}_n, \ni G(\theta + \hbar) \ominus_h G(\theta)$, $G(\theta) \ominus_h G(\theta - \hbar)$ exists for all $\hbar > 0$ sufficiently small and the limit*

$$\lim_{\hbar \to 0^+} \frac{G(\theta + \hbar) \ominus_h G(\theta)}{\hbar}, \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}.$$

exists in the topology of $\mathbb{E}_n$ and equal to $G^{'}(\theta)$. The element $G^{'}(\theta)$ is called the Hukuhara derivative of G at θ in the metric space $(\mathbb{E}_n, D_H)$. Consider only the one-sided derivatives at the end points of I.

Definition 2.2 ([6]). *A mapping $G : I \to \mathbb{E}_n$ is said to be Hukuhara form-II differentialble at $\theta \in I$, if $\exists$ a $G^{'}(\theta) \in \mathbb{E}_n \ni G(\theta) \ominus_h G(\theta + \hbar), G(\theta - \hbar) \ominus_h G(\theta)$ exists, $\forall\, \hbar > 0$ sufficiently small $\ni$ the limit exists*

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar}, \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar}.$$

and are equal to $G^{'}(\theta)$. Here $G^{'}(\theta)$ is called the Hukuhara form-II derivative at θ.

Remark 2.1 ([14]). A function $G : I \to \mathbb{E}_n$ is said to be differentiable if the multivalued mapping $G_\lambda : I \to \Re_k(\Re^n)$ is Hukuhara differentiable for all $\lambda \in [0, 1]$ and

$$[G_\lambda(\theta)]^{'} = [G^{'}(\theta)]^\lambda,$$

where $[G_\lambda]^{'}$ is the H-derivative of G_λ.

Definition 2.3 ([14]). *A mapping $G : I \to \mathbb{E}_n$ is said to be strongly measurable if for each $\lambda \in [0, 1]$, the fuzzy function $G_\lambda : I \to \Re_k(\Re^n)$ defined by $G_\lambda(\theta) = [G(\theta)]^\lambda$ is measurable.*

Remark 2.2 ([14]). If $\{\lambda_k\}$ is a nonincreasing sequence converges to 0 for all $x \in \mathbb{E}_n$, then

$$\lim_{k \to \infty} d_H([x]^0, [x]^{\lambda_k}) = 0.$$

Now, we present some fundamental definitions and results of time scales.

Definition 2.4 ([7])**.** (*a*) *Any nonempty closed subset of* $\Re$ *is defined as a time scale which is denoted by* $\mathbb{T}$.

(*b*) $\rho : \mathbb{T} \to \mathbb{T}$ *is the backward jump operator and* $\nu : \mathbb{T} \to \mathbb{R}^+$*, the graininess operator are defined by*

$$\rho(\theta) = \sup\{\theta_0 \in \mathbb{T} : \theta_0 < \theta\},\ \nu(\theta) = \theta - \rho(\theta)\ for\, \theta \in \mathbb{T}.$$

(*c*) *The operator* ρ *is called left dense if* $\rho(\theta) = \theta$*, otherwise left scattered.*

(*d*) $\mathbb{T}_k = \mathbb{T} - \{m\}$*, if* $\mathbb{T}$ *has a right scattered minimum* m*. Otherwise* $\mathbb{T}_k = \mathbb{T}$.

(*e*) *A mapping* $g^\rho : \mathbb{T} \to \Re$ *defined by*

$$g^\rho(\theta) = g(\rho(\theta))\ for\ each\ \theta \in \mathbb{T},$$

where $g : \mathbb{T} \to \Re$ *is a function.*

(*f*) *The interval in time scale* $\mathbb{T}$ *is defined by*

$$\mathbb{T}^{[a,b]} = \{\theta \in \mathbb{T} : a \leq \theta \leq b\} = [a, b] \cap \mathbb{T}$$

and

$$\mathbb{T}_k^{[a,b]} = \begin{cases} \mathbb{T}^{[a,b]}, & if\ a\ is\ right\ dense; \\ \mathbb{T}^{[\sigma(a),b]}, & if\ a\ is\ right\ scattered. \end{cases}$$

Definition 2.5 ([7])**.** *Let* $g : \mathbb{T} \to \Re$ *be a function and* $\theta \in \mathbb{T}_k$*. Then* $g^\nabla(\theta)$ *exists as a number provided for any given* $\epsilon > 0$*,* $\exists$ *a neighbourhood* N_δ *of* θ *(i.e.,* $N_\delta = (\theta - \delta, \theta + \delta) \cap \mathbb{T}$ *for some* $\delta > 0$*) such that*

$$|[g(\rho(\theta)) - g(\theta_0)] - g^\nabla(\theta)[\rho(\theta) - \theta_0]| \leq \epsilon|\rho(\theta) - \theta_0|,\ \ for\ all\ \ \theta_0 \in N_\delta,$$

Here, $g^\nabla(\theta)$ *is called the nabla derivative of* g *at* θ*. Moreover,* g *is said to be nabla (or Hilger) differentiable on* $\mathbb{T}_k$*, if* $g^\nabla(\theta)$ *exists* $\forall\ \theta \in \mathbb{T}_k$*. The function* $g^\nabla : \mathbb{T}_k \to \Re$ *is then called the nabla derivative of* g *on* $\mathbb{T}_k$*.*

Definition 2.6 ([7])**.** *A mapping* $g : \mathbb{T} \to \Re$ *is said to be regulated if its left sided limits exists and are finite at all ld-point (left dense points) in* $\mathbb{T}$ *and its right sided limits exists and are finite at all rd-points (right dense points) in* $\mathbb{T}$

Definition 2.7 ([7])**.** *Let* $g : \mathbb{T} \to \Re$ *be a function.* g *is said to be ld-continuous, if it is continuous at each ld-point in* $\mathbb{T}$ *and* $\lim\limits_{\theta_0 \to \theta^+} g(\theta)$ *exists as a finite number for all rd-points in* $\mathbb{T}$.

Lemma 2.1 ([7])**.** *Let* $G : \mathbb{T} \to \Re$.

(*a*) *If* g *is* ∇*-differentiable at* $\mathbb{T}$*, then* g *is continuous at* θ.

(*b*) *If* g *is continuous at* θ *and* θ *is left scattered, then* g *is* ∇*-differentiable and* $g^\nabla(\theta) = \frac{g(\theta) - g(\rho(\theta))}{\nu(\theta)}$.

(*c*) *If* g *is* ∇*-differentiable at* θ*, then* $g(\rho(\theta)) = g(\theta) + (-1)\nu(\theta)g^\nabla(\theta)$.

3. Nabla Hukuhara differentiability

In this section, first we introduce second type nabla Hukuhara derivative of fuzzy functions on time scales. Later, we establish uniqueness and existence of this derivative and obtain some properties on second type nabla Hukuhara derivative. For further discussion, we use the following notation: for some $\delta > 0$, we define the neighbourhood of $\theta \in \mathbb{T}^{[a,b]}$ by $N_{\mathbb{T}^{[a,b]}}(\theta, \delta) = (\theta - \delta, \theta + \delta) \cap \mathbb{T}^{[a,b]} = N_{\mathbb{T}^{[a,b]}}$

Definition 3.1. *For any given $\epsilon > 0$ $\exists$ a $\delta > 0$, such that the fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ has a unique $\mathbb{T}$-limit $P \in \mathbb{E}_n$ at $\theta \in \mathbb{T}^{[a,b]}$ if $D_H(G(\theta) \ominus_h P, \hat{0}) \leq \epsilon$, for all $\theta \in N_{\mathbb{T}^{[a,b]}}(\theta, \delta)$ and it is denoted by $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta)$.*

Here $\mathbb{T}$-limit denotes the limit on time scale in the metric space $(\mathbb{E}_n, D_H)$.

Remark 3.1. From the above definition, we have

$$\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta) = P \in \mathbb{E}_n \Longleftrightarrow \mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h P) = \hat{0},$$

where $\hat{0}$ is the zero element in $\mathbb{E}_n$.

Definition 3.2. *A fuzzy mapping $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous at $\theta_0 \in \mathbb{T}^{[a,b]}$, if $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta) \in \mathbb{E}_n$ exists and $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta) = G(\theta_0)$, i.e.*

$$\mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h G(\theta_0)) = \hat{0}.$$

Remark 3.2. If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous at $\theta \in \mathbb{T}^{[a,b]}$, then for every $\epsilon > 0$, $\exists$ a $\delta > 0$, such that

$$D_H(G(\theta) \ominus_h G(\theta_0), \hat{0}) \leq \epsilon, \text{ for all } \theta \in N_{\mathbb{T}^{[a,b]}}.$$

Remark 3.3. Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ and $\theta_0 \in \mathbb{T}^{[a,b]}$.

(a) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0)$, then G is said to be right continuous at θ_0.

(b) If $\mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta) = G(\theta_0)$, then G is said to be left continuous at θ_0.

(c) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0) = \mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta)$, then G is continuous at θ_0.

Definition 3.3 ([16])**.** *Suppose $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be a fuzzy function and $\theta \in \mathbb{T}_k^{[a,b]}$. Let $G^{\nabla_h}(\theta)$ be an element of $\mathbb{E}_n$ exists provided for any given $\epsilon > 0$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and for some $\delta > 0$ such that*

$$\begin{aligned} &D_H[(G(\theta + \hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon |\hbar + \nu(\theta)|, \\ &D_H[(G(\rho(\theta)) \ominus_h G(\theta - \hbar), (\hbar - \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon |\hbar - \nu(\theta)|, \end{aligned} \tag{1}$$

for all $(\theta - \hbar, \theta + \hbar) \in N_{\mathbb{T}^{[a,b]}}$ *with* $0 < h < \delta$ *where* $\nu(\theta) = \theta - \rho(\theta)$*. Then* G *is called nabla Hukuhara form-I (nabla-h) differentiable at* θ *and is denoted by* $G^{\nabla_h}(\theta)$*.*

or

A fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *is said to be nabla-h differentiable at* $\theta \in \mathbb{T}_k^{[a,b]}$ *if* $\exists$ *a* $G^{\nabla_h}(\theta) \in \mathbb{E}_n$ *such that the limits*

$$\mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\theta + \hbar) \ominus_h G(\rho(\theta))}{\hbar + \nu(\theta)} \;\&\; \mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\rho(\theta)) \ominus_h G(\theta - \hbar)}{\hbar - \nu(\theta)}$$

exists and are equal to $G^{\nabla_h}(\theta)$*.*

Moreover, if nabla-h derivative exists for each $\theta \in \mathbb{T}_k^{[a,b]}$*, then* G *is nabla-h differentiable on* $\mathbb{T}_k^{[a,b]}$*. We consider only right limit at left scattered points and one-sided limit at the end points of* $\mathbb{T}_k^{[a,b]}$*.*

The above definition does not exists if the fuzzy function has decreasing diameter. So, in order to overcome this circumstance we introduce and study the second type nabla Hukuhara derivative for fuzzy functions on time scales where the results exist for the functions which are having decreasing diameter.

Definition 3.4. *Let* $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *is a fuzzy function and* $\theta \in \mathbb{T}_k^{[a,b]}$*. Let* $G^{\nabla^{sh}}(\theta)$ *be an element of* $\mathbb{E}_n$ *exists provided for any given* $\epsilon > 0$*,* $\exists$ *a neighbourhood* $N_{\mathbb{T}^{[a,b]}}$ *of* θ *and for some* $\delta > 0$ *such that*

$$D_H[(G(\rho(\theta)) \ominus_h G(\theta + \hbar)), -(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon |-(\hbar + \nu(\theta))|, \tag{2}$$

$$D_H[(G(\theta - \hbar) \ominus_h G(\rho(\theta)), -(\hbar - \nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon |-(\hbar - \nu(\theta))|, \tag{3}$$

for all $(\theta - \hbar, \theta + \hbar) \in N_{\mathbb{T}^{[a,b]}}$ *with* $0 < h < \delta$ *where* $\nu(\theta) = \theta - \rho(\theta)$*. Then* G *is called second type nabla Hukuhara form-II differentiable (*∇^{sh}*-differentiable) at* θ *and is denoted by* $G^{\nabla^{sh}}(\theta)$*.*

or

A fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *is* ∇^{sh}*-differentiable at* $\theta \in \mathbb{T}_k^{[a,b]}$ *if* $\exists$ *a* $G^{\nabla^{sh}}(\theta) \in \mathbb{E}_n$ *such that the limits*

$$\mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\rho(\theta)) \ominus_h G(\theta + \hbar)}{-(\hbar + \nu(\theta))} \;\&\; \mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\rho(\theta))}{-(\hbar - \nu(\theta))}$$

exists and are equal to $G^{\nabla^{sh}}(\theta)$*. Moreover, if* ∇^{sh}*-derivative exists for each* $\theta \in \mathbb{T}_k^{[a,b]}$*, then* G *is* ∇^{sh}*-differentiable on* $\mathbb{T}_k^{[a,b]}$*. We consider only right limit at left scattered points and one-sided limit at the end points of* $\mathbb{T}_k^{[a,b]}$*.*

Note. If both $\mathbb{T}$-limits exists at left scattered point, then the ∇^{sh}-derivative is in $\Re^n$ (crisp). It will restrict the ∇^{sh}-differentiability of fuzzy functions on time scales having left scattered points. To avoid this, we consider only right limit at left scattered points.

Example 3.1. Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_1$ be a function defined by $G(\theta) = \frac{1}{\theta^2} \odot n$, where $n = (1, 3, 5)$ is a fuzzy number.

If $\mathbb{T} = \Re$, then from Definition 3.3 $G : \Re \to \mathbb{E}_n$ is not ∇_h-differentiable at $\theta \in \Re$. Since the H-difference $G(\theta) \ominus_h G(\theta - \hbar), G(\theta + \hbar) \ominus_h G(\theta)$ does not exists. Since the H-difference $G(\theta) \ominus_h G(\theta + \hbar), G(\theta - \hbar) \ominus_h G(\theta)$ exists. Therefore, from Definition 3.4, G is ∇^{sh}-differentiable and we have

$$\begin{aligned}
G^{\nabla^{sh}}(\theta) &= \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar))}{-\hbar} \\
&= \lim_{\hbar \to 0} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar} \\
&= \frac{-2}{\theta^3} \odot (1, 3, 5) = \frac{1}{\theta^3} \odot (-10, -6, -2).
\end{aligned}$$

Lemma 3.1. *If G is ∇^{sh}-differentiable at θ, then ∇^{sh}- derivative exists and it is unique.*

Proof. Suppose that $G^{\nabla^{sh}_1}(\theta)$ and $G^{\nabla^{sh}_2}(\theta)$ are ∇^{sh}-derivatives of G at θ. Then

$$D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), G(\rho(\theta)) \ominus_h G(\theta + \hbar)] \le \frac{\epsilon}{2}| - (\hbar + \nu(\theta))|,$$

$$D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta), G(\rho(\theta) \ominus_h G(\theta + \hbar))] \le \frac{\epsilon}{2}| - (\hbar + \nu(\theta))|.$$

Consider

$$\begin{aligned}
&D_H[G^{\nabla^{sh}_1}(\theta), G^{\nabla^{sh}_2}(\theta)] \\
&= \frac{1}{| - (\hbar + \nu(\theta))|} \Big(D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), -(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta)] \Big) \\
&\le \frac{1}{| - (\hbar + \nu(\theta))|} \Big(D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), G(\rho(\theta) \ominus_h G(\theta + \hbar)] \\
&+ D_H[G(\rho(\theta)) \ominus_h G(\theta + \hbar), -(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta)] \Big) \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall \, | - (\hbar + \nu(\theta))| \ne 0.
\end{aligned}$$

Since $\epsilon > 0$, then $D_H[G^{\nabla^{sh}_1}(\theta), G^{\nabla^{sh}_2}(\theta)] = 0$. Therefore, $G^{\nabla^{sh}_1}(\theta) = G^{\nabla^{sh}_2}(\theta)$. Hence ∇^{sh}-derivative exists and is unique. □

Theorem 3.1. *If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ, then G is continuous when θ is left dense and right continuous when θ is left scattered.*

Proof. Assume that $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$. Let $\epsilon^1 \in (0,1)$. Define

$$\epsilon^1 = \epsilon[1 + \|G^{\nabla^{sh}}(\theta)\|]^{-1}.$$

Since G is ∇^{sh}-differentiable $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ such that

$$D_H[(G(\rho(\theta)) \ominus_h G(\theta+\hbar), -(\hbar+\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \leq \epsilon|-(\hbar+\nu(\theta))|,$$
$$D_H[(G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \leq \epsilon|-(\hbar-\nu(\theta))|,$$

for all $\hbar \geq 0$ with $(\theta-\hbar, \theta+\hbar) \in \mathbb{N}_T^{[a,b]}$. Therefore, for all $(\theta-\hbar, \theta+\hbar) \in N_T^{[a,b]} \cap (\theta-\epsilon, \theta+\epsilon)$ with $0 \leq \hbar < \epsilon$.

$$\begin{aligned} D_H[G(\theta-\hbar), G(\theta)] &= D_H[G(\theta-\hbar) \ominus_h G(\theta), \hat{0}] \\ &= D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)) + G(\rho(\theta)) \ominus_h G(\theta), \\ &\quad -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta) - \nu(\theta) \odot G^{\nabla^{sh}} + \hbar G^{\nabla^{sh}}(\theta)] \\ &\leq D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \\ &\quad + D_H[G(\rho(\theta)) \ominus_h G(\theta), (-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \\ &\quad + D_H[\hbar G^{\nabla^{sh}}(\theta), \hat{0}] \\ &\leq \epsilon^1|-(\hbar-\nu(\theta)| + \epsilon^1|(-\nu(\theta)))| + \hbar\|G^{\nabla^{sh}}(\theta)\| \\ &< \epsilon^1(1 + \|G^{\nabla^{sh}}(\theta)\|) \\ &= \epsilon. \end{aligned}$$

Therefore, for θ being left dense or left scattered

$$\mathbb{T} - \lim_{\hbar \to 0^+} G(\theta-\hbar) = G(\theta).$$

For left dense point θ, it is easy to prove that

$$\mathbb{T} - \lim_{\hbar \to 0^+} G(\theta+\hbar) = G(\theta).$$

Hence G is continuous at left dense points and right continuous at left scattered points in $\mathbb{T}_k^{[a,b]}$. □

Theorem 3.2. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be right continuous at θ, θ is left-scattered then G is ∇^{sh}-differentiable at θ and*

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\nu(\theta)} \odot (G(\rho(\theta)) \ominus_h G(\theta)).$$

Proof. Let θ be left-scattered and since G is right continuous, then by Theorem 3.1, we have

$$G^{\nabla^{sh}}(\theta) = \mathbb{T} - \lim_{h \to 0^+} \frac{G(\rho(\theta)) \ominus_h G(\theta+\hbar)}{-(\hbar+\nu(\theta))} = \frac{-1}{\nu(\theta)} \odot (G(\rho(\theta)) \ominus_h G(\theta)).$$

□

Theorem 3.3. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be fuzzy function and let $\theta \in \mathbb{T}_k^{[a,b]}$. If θ is left dense, then G is ∇^{sh}-differentiable at $\mathbb{T}_k^{[a,b]}$ if and only if the limits exists and are equal to $G^{\nabla^{sh}}(\theta)$ i.e.,*

$$\lim_{\hbar\to 0^+} \frac{-1}{h} \odot (G(\theta) \ominus_h G(\theta+\hbar)) = \lim_{\hbar\to 0^+} \frac{-1}{h} \odot (G(\theta-\hbar) \ominus_h G(\theta)) = G^{\nabla^{sh}}(\theta).$$

Proof. Suppose that G is ∇^{sh}-differentiable at θ and θ is ld-point. Since G is ∇^{sh}-differentiable at θ, for any given $\epsilon > 0$, $\exists\ N_{\mathbb{T}^{[a,b]}}$ a neighbourhood of θ $\ni$

$$D_H[(G(\rho(\theta)) \ominus_h G(\theta+\hbar), -(\hbar+\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon| -(\hbar+\nu(\theta))|,$$

$$D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon| -(\hbar-\nu(\theta))|,$$

for all, $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. Since $\rho(\theta) = \theta$, i.e. $\nu(\theta) = 0$,

$$D_H[(G(\theta) \ominus_h G(\theta+\hbar), -\hbar \odot G^{\nabla^{sh}}(\theta)] \le \epsilon\hbar,$$

$$D_H[(G(\theta-\hbar) \ominus_h G(\theta), -\hbar \odot G^{\nabla^{sh}}(\theta)] \le \epsilon\hbar,$$

for all, $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. This implies that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$

$$D_H\left[\frac{G(\theta-\hbar) \ominus_h G(\theta)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$

for all $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. Since ϵ is arbitrary, we have

$$\lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot (G(\theta) \ominus_h G(\theta+\hbar)) = \lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot (G(\theta-\hbar) \ominus_h G(\theta)) = G^{\nabla^{sh}}(\theta)$$

Conversely, suppose that for all $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}}$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and θ is left dense such that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$
$$D_H\left[\frac{G(\theta-\hbar) \ominus_h G(\theta)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon.$$

From the above inequalities, G is ∇^{sh}-differentiable at θ and since θ is ld-point, we have $G^{\nabla^{sh}}(\theta) = G'(\theta)$. □

Theorem 3.4. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be $\nabla^{sh}-$differentiable and $\theta \in \mathbb{T}_k^{[a,b]}$. Then*

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

or

$$G(\theta) = G(\rho(\theta)) \ominus_h (-1)(\nu(\theta)G^{\nabla^{sh}}(\theta).$$

Proof. (a) If θ is left dense then $\rho(\theta) = \theta$ and $\nu(\theta) = 0$. Hence

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

or

$$G(\theta) = G(\rho(\theta)) = G(\rho(\theta)) \ominus_h (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

(b) If θ is left-scattered then $\rho(\theta) < \theta$. From Theorem 3.2 we have

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\nu(\theta)} \odot [G(\rho(\theta)) \ominus_h G(\theta)] \Rightarrow (-1)\nu(\theta) \odot G^{\nabla^{sh}}(\theta) = G(\rho(\theta)) \ominus_h G(\theta).$$

Thus,

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta)$$

or

$$G(\theta) = G(\rho(\theta)) \ominus_h (-1)(\nu(\theta)G^{\nabla^{sh}}(\theta).$$

□

Example 3.2. Let us consider $\mathbb{T} = \Re$ or $\mathbb{T} = t\mathbb{Z} = \{tk : k \in \mathbb{Z}\}$.

(a) If $\mathbb{T} = \Re$, then from Theorem 3.3 $G : \Re \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in \Re$ iff

$$G^{\nabla^{sh}}(\theta) = \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{\hbar} = \lim_{\hbar \to 0} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{\hbar} = G'(\theta).$$

(b) If $\mathbb{T} = t\mathbb{Z}$, then every point $\theta \in \mathbb{T}$ is isolated and

$$\rho(\theta) = \sup\{\theta - nt : n \in \mathbb{N}\} = \theta - t,$$

$$\nu(\theta) = \theta - \rho(\theta) = \theta - (\theta - t) = t.$$

From Theorem 3.2 $G : t\mathbb{Z} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in t\mathbb{Z}$ and

$$G^{\nabla^{sh}}(\theta) = \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} = \frac{G(\theta - t) \ominus_h G(\theta)}{-t} = \frac{-1}{t} \odot \Delta G(\theta),$$

where Δ is the forward Hukuhara difference operator.

Theorem 3.5. *Denote* $[G(\theta)]^\lambda = G_\lambda(\theta)$ *for each* $\lambda \in [0, 1]$, *where* $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *be the fuzzy function and if* G *is* ∇^{sh}*-differentiable, then* G_λ *is also* ∇^{sh}*-differentiable and*

$$[G^{\nabla^{sh}}(\theta)]^\lambda = G_\lambda^{\nabla^{sh}}(\theta).$$

Proof. If θ is left scaterred and G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$, then from Theorem 3.2, we get

$$[G^{\nabla^{sh}}(\theta)]^\lambda = \frac{[G(\rho(\theta))]^\lambda \ominus_h [G(\theta)]^\lambda}{-\nu(\theta)} = \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)} = G_\lambda^{\nabla^{sh}}(\theta),$$

for each $\lambda \in [0,1]$. If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left dense, then for $\lambda \in [0,1]$, we get

$$[G(\theta - \hbar) \ominus_h G(\theta)]^\lambda = [G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\theta)]$$

and multiplying by $\frac{-1}{\hbar} < 0$ and taking the limit $\hbar \to 0^+$, we have

$$\lim_{\hbar \to 0^+} \frac{-1}{\hbar} \odot [G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\theta)] = G_\lambda^{\nabla^{sh}}(\theta).$$

Similarly, we can prove

$$\lim_{\hbar \to 0^+} \frac{-1}{\hbar} \odot \big[G_\lambda(\theta) \ominus_h G_\lambda(\theta + \hbar)\big] = G_\lambda^{\nabla^{sh}}(\theta).$$

Therefore, from Theorem 3.3, we get $[G^{\nabla^{sh}}(\theta)]^\lambda = G_\lambda^{\nabla^{sh}}(\theta)$. □

Remark 3.4. The above Theorem 3.5, states that if G is ∇^{sh} differentiable then the multivalued mapping G_λ is ∇^{sh}-differentiable for all $\lambda \in [0,1]$, but the converse of the theorem need not be true. That is the existence of H-differences of λ-level sets $[p]^\lambda \ominus_h [q]^\lambda$ does not imply the existence of H-difference of $p \ominus_h q$.

However, for the converse of the theorem we have the following:

Theorem 3.6. *Suppose that $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ satisfy the following conditions:*

(1) *For each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left dense*

 (*a*) *$\exists$ a $\beta > 0$, $\ni$ the Hukuhara differences $G(\theta - \hbar) \ominus_h G(\theta)$ and $G(\theta) \ominus_h G(\theta + \hbar)$ exists for all $0 < \hbar < \beta$ and for all $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$;*

 (*b*) *the fuzzy mappings G_λ, $\lambda \in [0,1]$, are uniformly ∇^{sh}-differentiable with derivative $G_\lambda^{\nabla^{sh}}$, i.e., to each $\theta \in \mathbb{T}^{[a,b]}$ and $\epsilon > 0$ $\exists$ a $\delta > 0$ such that*

$$D_H\left\{\frac{G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\rho(\theta))}{-(\hbar - \nu(\theta))}, G_\lambda^{\nabla^{sh}}(\theta)\right\} < \epsilon,$$
$$D_H\left\{\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta + \hbar)}{-(\hbar + \nu(\theta))}, G_\lambda^{\nabla^{sh}}(\theta)\right\} < \epsilon,$$

 for all $0 < \hbar < \delta$, $\theta - \hbar, \theta + \hbar \in N_T^{[a,b]}$, $\lambda \in [0,1]$.

(2) *for each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left scattered*

 (*a*) *the Hukuhara differences $G(\rho(\theta)) \ominus_h G(\theta)$ exists and;*

(*b*) *the fuzzy mappings* G_λ, $\lambda \in [0,1]$, *are uniformly nabla*^{sh}*-differentiable with derivative* $G_\lambda^{\nabla^{sh}}$, *i.e., to each* $\theta \in \mathbb{T}^{[a,b]}$ *and* $\epsilon > 0 \ \exists$ *a* $\delta > 0$ *such that*

$$D_H \left\{ \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_\lambda^{\nabla^{sh}}(\theta) \right\} < \epsilon. \tag{4}$$

Then G *is* ∇^{sh}*-differentiable and its derivative is given by* $G_\lambda^{\nabla^{sh}}(\theta) = [G^{\nabla^{sh}}(\theta)]^\lambda$.

Proof. Case (1): For θ being left dense points in $\mathbb{T}^{[a,b]}$, then the proof is similar to the proof of Theorem 5.1 [14].

Case(2): For θ being left scattered points in $\mathbb{T}^{[a,b]}$, consider $\{G_\lambda^{\nabla^{sh}}(\theta), \lambda \in [0,1]\}$, where $G_\lambda^{\nabla^{sh}}(\theta)$ is convex, compact and nonempty subset of $\Re^n$. If $\lambda_1 \leq \lambda_2$ then by our supposition (a), we have

$$G_{\lambda_1}(\rho(\theta)) \ominus_h G_{\lambda_1}(\theta) \supset G_{\lambda_2}(\rho(\theta)) \ominus_h G_{\lambda_2}(\theta) \tag{5}$$

For $0 < \hbar < \beta$, we have $G_{\lambda_1}^{\nabla^{sh}}(\theta) \supset G_{\lambda_2}^{\nabla^{sh}}(\theta)$. Let $\{\lambda_n\}$ be a nondecreasing sequence coverges to $\lambda > 0$. For $\epsilon > 0$ choose $\hbar > 0 \ni$ the equation (4) holds. Now, let us consider

$$\begin{aligned} D_H(G_\lambda^{\nabla^{sh}}(\theta), G_{\lambda_n}^{\nabla^{sh}}(\theta)) \leq D_H &\left(G_\lambda^{\nabla^{sh}}(\theta), \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)} \right) \\ &+ D_H \left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_{\lambda_n}^{\nabla^{sh}}(\theta) \right) \\ < \epsilon &+ \frac{1}{\nu(\theta)} D_H[G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta), G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta)] \\ &+ \frac{1}{\nu(\theta)} D_H[G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta), -\nu(\theta) G_{\lambda_n}^{\nabla^{sh}}(\theta)] \\ < 2\epsilon &+ \frac{1}{\nu(\theta)} D_H[G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta), G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta)]. \end{aligned}$$

By our supposition $2(a)$, the rightmost term converges to zero as $n \to \infty$ and hence

$$\lim_{n\to\infty} D_H(G_\lambda^{\nabla^{sh}}(\theta), G_{\lambda_n}^{\nabla^{sh}}(\theta)) = 0.$$

From Theorem 2.2 and (5) we have

$$G_\lambda^{\nabla^{sh}}(\theta) = \bigcap_{n\geq 1} cl \left\{ \bigcup_{m \geq n} G_{\lambda_m}^{\nabla^{sh}}(\theta) \right\}.$$

If $\lambda = 0$, we can write it as

$$\lim_{n\to\infty} D_H(G_0^{\nabla^{sh}}(\theta), \nabla^{sh} G_{\lambda_n}^{\nabla^{sh}}(\theta)) = 0,$$

where the nondecreasing sequence $\{\lambda_n\}$ tends to zero , and as a result of this

$$G_0^{\nabla^{sh}}(\theta) = cl\left(\bigcup_{n\geq 1} G_{\lambda_n}^{\nabla^{sh}}(\theta)\right).$$

Then from Theorem 2.1, $\exists$ an element $\tilde{u} \in \mathbb{E}_n$ such that

$$[\tilde{u}]^\lambda = G_\lambda^{\nabla^{sh}}(\theta), \quad \lambda \in [0,1].$$

Let $\theta \in \mathbb{T}^{[a,b]}$, $\epsilon > 0$, $\delta > 0$ and $(\theta - \hbar, \theta + \hbar) \in N_T^{[a,b]}$ be as in supposition (b) then, we have

$$D_H\left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, \tilde{u}^\lambda\right) = D_H\left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_\lambda^{\nabla^{sh}}(\theta)\right) < \epsilon$$

Thus, G is ∇^{sh}-differentiable. □

Theorem 3.7. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ defined by $G(\theta) = g(\theta) \odot u$ for all $\theta \in \mathbb{T}^{[a,b]}$, where $u \in \mathbb{E}_n$ and $g : \mathbb{T}^{[a,b]} \to \mathbb{T}_+$ is nabla differentiable at $\theta_0 \in \mathbb{T}^{[a,b]}$. If $g^\nabla(\theta_0) < 0$, then G is ∇^{sh}-differentiable at θ_0 with $G^{\nabla^{sh}}(\theta_0) = g^\nabla(\theta_0) \odot u$.*

Proof. Since g is nabla differentiable at θ_0, then from Lemma 2.1, g is continuous at θ_0. **Case(i):** If θ_0 is left scattered then, we have

$$g^\nabla(\theta_0) = \frac{g(\theta_0) - g(\rho(\theta_0))}{\nu(\theta_0)}.$$

Since $g^\nabla(\theta_0) < 0$, then

$$g(\rho(\theta_0)) - g(\theta_0) = g^\nabla(\theta_0)(-\nu(\theta_0)) > 0$$

It implies that

$$g(\rho(\theta_0)) = g(\theta_0) + g^\nabla(\theta_0)(-\nu(\theta_0)).$$

Now, multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, then we get

$$g(\rho(\theta_0)) \odot u = [g(\theta_0) \odot u] \oplus \left[g^\nabla(\theta_0)(-\nu(\theta_0)) \odot u\right].$$

It implies that $[g(\rho(\theta_0)) \odot u] \ominus_h [g(\theta_0) \odot u] = [g^\nabla(\theta_0)(-\nu(\theta_0))] \odot u$ and then

$$G(\rho(\theta_0)) \ominus_h G(\theta_0) = [g^\nabla(\theta_0)(-\nu(\theta_0))] \odot u.$$

dividing by $(-\nu(\theta_0))$, we have

$$\frac{G(\rho(\theta_0)) \ominus_h G(\theta_0)}{(-\nu(\theta_0))} = [g^\nabla(\theta_0)] \odot u$$

and hence

$$G^{\nabla^{sh}}(\theta_0) = g^\nabla(\theta) \odot u.$$

Case(ii): If θ_0 is left dense, then $g^{\nabla}(\theta_0) = g^{'}(\theta_0) < 0$ and

$$g^{'}(\theta_0) = \lim_{\hbar \to 0^+} \frac{g(\theta_0) - g(\theta_0 - \hbar)}{\hbar}.$$

It follows that for $\hbar > 0$ sufficiently small, we have $g(\theta_0) - g(\theta_0 - \hbar) < o$ and

$$g(\theta_0 - \hbar) = g(\theta_0) + \phi(\theta_0, \hbar)$$

Now, multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, we get

$$g(\theta_0 - \hbar)) \odot u = [g(\theta_0) \odot u] \oplus [\phi(\theta_0, \hbar) \odot u] .$$

It implies

$$G(\theta_0 - \hbar) = G(\theta_0) \oplus [(\phi(\theta_0, \hbar) \odot u].$$

Therefore, $G(\theta_0 - \hbar) \ominus_h G(\theta_0)$ exists and hence G is left ∇^{sh}-differentiable at θ_0. Similarly, we can prove G is right ∇^{sh}-differentiable at θ_0.

It follows that, G is ∇^{sh}-differentiable at θ_0 with $G^{\nabla^{sh}}(\theta_0) = g^{\nabla_0}(\theta) \odot u$. $\square$

Example 3.3. Let us define $G(\theta) = \frac{1}{\theta} \odot u \,, \forall\, \theta \in \mathbb{T}^{[1,10]}$, $G : \mathbb{T}^{[1,10]} \to \mathbb{E}_1$ is a fuzzy function and $u = (2, 3, 4)$ is the triangular fuzzy number. Here, $g(\theta) = \frac{1}{\theta}$ and $g^{\nabla}(\theta) = \frac{-1}{\theta(\rho(\theta))} < 0 \,, \forall\, \theta \in \mathbb{T}^{[1,10]}$, from Theorem 3.7, we have $G(\theta)$ is ∇^{sh}-differentiable and $G^{\nabla}(\theta) = \frac{-1}{\theta(\rho(\theta))} \odot u \,, \forall\, \theta \in \mathbb{T}^{[1,10]}$.

Theorem 3.8. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}^1$ defined as $[G(\theta)]^{\lambda} = [g_{\lambda}(\theta), h_{\lambda}(\theta)], \lambda \in [0, 1]$ and $G(\theta)$ is ∇^{sh}-differentiable on $\mathbb{T}^{[a,b]}$. Then g_{λ} and h_{λ} are nabla-differentiable on $\mathbb{T}^{[a,b]}$ and*

$$[G^{\nabla^{sh}}(\theta)]^{\lambda} = [h_{\lambda}^{\nabla}(\theta), g_{\lambda}^{\nabla}(\theta)].$$

Proof. If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left scattered, then for any $\lambda \in [0, 1]$,

$$[G(\rho(\theta)) \ominus_h G(\theta)]^{\lambda} = [g_{\lambda}(\rho(\theta)) - g_{\lambda}(\theta), h_{\lambda}(\rho(\theta)) - h_{\lambda}(\theta)].$$

and multiplying with $\frac{-1}{\nu(\theta)}$, we get

$$\begin{aligned}
[G^{\nabla^{sh}}(\theta)]^{\lambda} =& \frac{-1}{\nu(\theta)} \odot [G(\rho(\theta)) \ominus_h G(\theta)]^{\lambda} \\
=& \left[\frac{1}{-\nu(\theta)} \odot [g_{\lambda}(\rho(\theta)) - g_{\lambda}(\theta), h_{\lambda}(\rho(\theta)) - h_{\lambda}(\theta)]\right] \\
=& \left[\frac{h_{\lambda}(\rho(\theta)) - h_{\lambda}(\theta)}{-\nu(\theta)}, \frac{g_{\lambda}(\rho(\theta)) - g_{\lambda}(\theta)}{-\nu(\theta)}\right] \\
=& \left[\frac{h_{\lambda}(\theta) - h_{\lambda}(\rho(\theta))}{\nu(\theta)}, \frac{g_{\lambda}(\theta) - g_{\lambda}(\rho(\theta))}{\nu(\theta)}\right] \\
=& \left[h_{\lambda}^{\nabla}(\theta), g_{\lambda}^{\nabla}(\theta)\right] .
\end{aligned}$$

If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is ld-point, then for any $\lambda \in [0,1]$,

$$[G(\theta-\hbar)\ominus_h G(\theta)]^\lambda = [g_\lambda(\theta-\hbar)-g_\lambda(\theta), h_\lambda(\theta-\hbar)-h_\lambda(\theta)]$$

and multiplying with $\frac{-1}{\hbar} < 0$ and taking limits as $\hbar \to 0^+$, we get

$$\begin{aligned}
&\lim_{\hbar\to 0^+}\frac{-1}{\hbar}\odot[G(\theta-\hbar)\ominus_h G(\theta)]^\lambda\\
&\quad=\lim_{\hbar\to 0^+}\frac{-1}{\hbar}\odot[g_\lambda(\theta-\hbar)-g_\lambda(\theta), h_\lambda(\theta-\hbar)-h_\lambda(\theta)]\\
&\quad=\left[\lim_{\hbar\to 0^+}\frac{h_\lambda(\theta-\hbar)-h_\lambda(\theta)}{-\hbar}, \lim_{\hbar\to 0^+}\frac{g_\lambda(\theta-\hbar)-g_\lambda(\theta)}{-\hbar}\right]\\
&\quad=\left[\lim_{\hbar\to 0^+}\frac{h_\lambda(\theta)-h_\lambda(\theta-\hbar)}{\hbar}, \lim_{\hbar\to 0^+}\frac{g_\lambda(\theta)-g_\lambda(\theta-\hbar)}{\hbar}\right]\\
&\quad=[h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)].
\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+}\frac{-1}{\hbar}[G(\theta)\ominus_h G(\theta+\hbar)]^\lambda = [h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)].$$

Thus, g_λ and h_λ are nabla differentiable on $\mathbb{T}^{[a,b]}$ and $[G^{\nabla^{sh}}(\theta)]^\lambda = [h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)]$. □

Example 3.4. Consider the fuzzy function $G(\theta)$ as in Example 3.3. Then $u^\lambda = [2+\lambda,\ 4-\lambda]$ is λ-level set of u and

$$\begin{aligned}
[G(\theta)]^\lambda &= [g_\lambda(\theta), h_\lambda(\theta)]\\
&= \frac{1}{\theta}\odot[2+\lambda,\ 4-\lambda]\\
&= [\frac{1}{\theta}(2+\lambda),\ \frac{1}{\theta}(4-\lambda)].
\end{aligned}$$

From Example 3.3, $G(\theta)$ is ∇^{sh}-differentiable and $G^{\nabla^{sh}}(\theta) = \frac{-1}{\theta(\rho(\theta))}\odot u$. Clearly, $g_\lambda^\nabla(\theta), h_\lambda^\nabla(\theta)$ are nabla differentiable and $g_\lambda^\nabla(\theta) = \frac{-1}{\theta(\rho(\theta))}(2+\lambda)$, $h_\lambda^\nabla(\theta) = \frac{-1}{\theta(\rho(\theta))}(4-\lambda)$. From Example 3.3 and Theorem 3.5, we have

$$\begin{aligned}
[G^{\nabla^{sh}}(\theta)]^\lambda &= \frac{-1}{\theta(\rho(\theta))}\odot u^\lambda\\
&= \frac{-1}{\theta(\rho(\theta))}\odot[2+\lambda,\ 4-\lambda]\\
&= \left[\frac{-1}{\theta(\rho(\theta))}(4-\lambda), \frac{-1}{\theta(\rho(\theta))}(2+\lambda)\right] = [h^\nabla(\theta), g^\nabla(\theta)].
\end{aligned}$$

Hence Theorem 3.8 is verified. And also if $\mathbb{T}=\mathbb{R}$, then $\rho(\theta)=\theta$ and

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\theta(\rho(\theta))}\odot u^\lambda = \frac{-1}{\theta^2}\odot u^\lambda.$$

If $\mathbb{T}=q^{\mathbb{N}}$, then $\rho(\theta)=\frac{\theta}{q}$ and $G^{\nabla^{sh}(\theta)} = \frac{-1}{\theta(\rho(\theta))}\odot u^\lambda = \frac{-q}{\theta^2}\odot u^\lambda$.

Now, we obtain the ∇^{sh}-derivatives of addition, scalar multiplication and product of second type nabla Hukuhara differentiable for fuzzy functions on time scales.

Theorem 3.9. *Let $G, H : \mathbb{T}^{[a,b]} \to \mathbb{E}^n$ are ∇^{sh}-differentiable at $\theta \in T_k^{[a,b]}$. Then,*

(a) *the sum $G \oplus H : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$(G \oplus H)^{\nabla^{sh}}(\theta) = G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta);$$

(b) *for any constant λ, $\lambda G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$(\lambda \odot G)^{\nabla^{sh}}(\theta) = \lambda \odot G^{\nabla^{sh}}(\theta);$$

(c) *the product $GH : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$\begin{aligned}(GH)^{\nabla^{sh}}(\theta) &= G(\rho(\theta))H^{\nabla^{sh}}(\theta) + H(\theta)G^{\nabla^{sh}}(\theta)\\ &= G(\theta)H^{\nabla^{sh}}(\theta) + H(\rho(\theta))G^{\nabla^{sh}}(\theta).\end{aligned}$$

Proof. Since G and H be ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$. Then from Theorem 3.1, G and H are continuous when θ is left dense and right continuous when θ is left scattered. If θ is left scattered, then from Theorem 3.2, we have

$$\frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} = G^{\nabla^{sh}}(\theta) \tag{6}$$

and

$$\frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)} = H^{\nabla^{sh}}(\theta). \tag{7}$$

If θ is ld-point, then from Theorem 3.3, we have

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar} = \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar} = G^{\nabla^{sh}}(\theta) \tag{8}$$

and

$$\lim_{\hbar \to 0^+} \frac{H(\theta) \ominus_h H(\theta + \hbar)}{-\hbar} = \lim_{\hbar \to 0^+} \frac{H(\theta - \hbar) \ominus_h H(\theta)}{-\hbar} = H^{\nabla^{sh}}(\theta). \tag{9}$$

(a) If θ is left scattered and G, H are ∇^{sh}-differentiable at θ, then from Theorem 3.1, $(G \oplus H)$ is right continuous at θ. From Theorem 3.2 and (6), (7), we have

$$
\begin{aligned}
(G \oplus H)^{\nabla^{sh}}(\theta) =& \frac{(G \oplus H)(\rho(\theta)) \ominus_h (G \oplus H)(\theta)}{-\nu(\theta)} \\
&= \frac{[G(\rho(\theta)) \oplus H(\rho(\theta))] \ominus_h [G(\theta) \oplus H(\theta)]}{-\nu(\theta)} \\
&= \frac{[G(\rho(\theta)) \ominus_h G(\theta)] \oplus [H(\rho(\theta)) \ominus_h H(\theta)]}{-\nu(\theta)} \\
&= \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} \oplus \frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)} \\
&= G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta).
\end{aligned}
$$

If θ is left dense and G, H are ∇^{sh}-differentiable at θ, then from (8) & (9), we have

$$
\begin{aligned}
&\lim_{\hbar \to 0^+} \frac{(G \oplus H)(\theta) \ominus_h (G \oplus H)(\theta + \hbar)}{-\hbar} \\
&\quad = \lim_{\hbar \to 0^+} \frac{(G(\theta) \ominus_h G(\theta + \hbar)) \oplus (H(\theta) \ominus_h H(\theta + \hbar))}{-\hbar} \\
&\quad = \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar} \oplus \lim_{\hbar \to 0^+} \frac{H(\theta) \ominus_h H(\theta + \hbar)}{-\hbar} \\
&\quad = G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta).
\end{aligned}
$$

Similarly, we can prove

$$
\lim_{\hbar \to 0^+} \frac{(G \oplus H)(\theta - \hbar) \ominus_h (G \oplus H)(\theta)}{-\hbar} = G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta).
$$

Therefore, $G \oplus H$ is ∇^{sh}-differentiable at θ and

$$
(G \oplus H)^{\nabla^{sh}}(\theta) = G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta).
$$

(b) For $\gamma = 0$, the result is obvious. Now, let us assume that $\gamma > 0$.

If θ is left scattered, then from Theorem 3.1, $\gamma \odot G$ is right continuous at θ. From Theorem 3.2 and (6), we have

$$
\begin{aligned}
(\gamma \odot G)^{\nabla^{sh}}(\theta) &= \frac{\gamma \odot G(\rho(\theta)) \ominus_h \gamma \odot G(\theta)}{-\nu(\theta)} \\
&= \gamma \odot \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} = \gamma \odot G^{\nabla^{sh}}(\theta).
\end{aligned}
$$

Since G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left dense, then from (8),

$$
\lim_{\hbar \to 0^+} \frac{\gamma \odot G(\theta) \ominus_h \gamma \odot G(\theta + \hbar)}{-\hbar} = \gamma \odot \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar} = \gamma \odot G^{\nabla^{sh}}(\theta).
$$

Similarly, we can prove

$$
\lim_{\hbar \to 0^+} \frac{\gamma \odot G(\theta - \hbar) \ominus_h \gamma \odot G(\theta)}{-\hbar} = \gamma \odot G^{\nabla^{sh}}(\theta).
$$

(c) If θ is left scattered, then from Theorem 3.1, it is clear that GH is right continuous at θ. From Theorem 3.2, (6) & (7), we have

$$G^{\nabla^{sh}}(\theta) = \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} \;\&\; H^{\nabla^{sh}}(\theta) = \frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)}.$$

$$\begin{aligned}(GH)^{\nabla^{sh}}(\theta) &= \frac{GH(\rho(\theta)) \ominus_h GH(\theta)}{-\nu(\theta)} \\ &= \frac{G(\theta)[H(\rho(\theta)) \ominus_h H(\theta)] \oplus [G(\rho(\theta)) \ominus_h G(\theta)]H(\rho(\theta))}{-\nu(\theta)} \\ &= G(\theta)\left[\frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)}\right] \oplus H(\rho(\theta))\left[\frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)}\right] \\ &= G(\theta)H^{\nabla^{sh}}(\theta) \oplus H(\rho(\theta))G^{\nabla^{sh}}(\theta).\end{aligned}$$

Since G, H are ∇^{sh}-differentiable and if θ is left dense, then from (8) & (9), we have

$$\begin{aligned}&\lim_{\hbar\to 0^+} \frac{GH(\theta) \ominus_h GH(\theta+\hbar)}{-\hbar} \\ &= \lim_{\hbar\to 0^+} \frac{G(\theta)[(H(\theta) \ominus_h H(\theta+\hbar)] \oplus [G(\theta) \ominus_h G(\theta+\hbar)]H(\theta)}{-\hbar} \\ &= G(\theta) \lim_{\hbar\to 0^+} \frac{H(\theta) \ominus_h H(\theta+\hbar)}{-\hbar} \oplus \lim_{\hbar\to 0^+} \frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar} \lim_{\hbar\to 0^+} H(\theta+\hbar) \\ &= G(\theta) \lim_{\hbar\to 0^+} \frac{H(\theta) \ominus_h H(\theta+\hbar)}{-\hbar} \oplus \lim_{\hbar\to 0^+} \frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar} H(\theta) \\ &= G(\theta)H^{\nabla^{sh}}(\theta) \oplus G^{\nabla^{sh}}(\theta)H(\theta).\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+} \frac{(GH)(\theta-\hbar) \ominus_h (GH)(\theta)}{-\hbar} = G(\theta)H^{\nabla^{sh}}(\theta) \oplus G^{\nabla^{sh}}(\theta)H(\theta).$$

Thus, $(GH)^{\nabla^{sh}}(\theta) = G(\theta)H^{\nabla^{sh}}(\theta) \oplus H(\rho(\theta))G^{\nabla^{sh}}(\theta)$ holds at θ. We get the another product rule in (c) by interchanging G and H and which follows from the last equation. □

4. Conclusions

The fuzzy nabla Hukuhara derivative of form-I (Definition 3.3) does not exists for a fuzzy function of decreasing diameter on time scales. To overcome this shortcoming, in this paper we introduce and study the fundamental properties of second type nabla Hukuhara derivative for fuzzy functions on time scales. In our future work, we propose to study fuzzy nabla integrals on time scales. Further, these concepts can be applied to study the fuzzy nabla dynamic equations on time scales.

Acknowledgement

The authors would like to thank the anonymous reviewer for his/her careful reading and suggestions.

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Accepted: 8.02.2019

Characterization of generalized projective and injective soft modules

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Abstract. We first introduce the concepts of projective soft LA-modules, free soft LA-modules, split sequence in soft LA-modules and establish various results on projective soft LA-modules. Then, we consider the injective soft LA-modules and give some relevant results by using free soft LA-modules and split sequences in soft LA-modules.
Keywords: soft LA-rings, soft LA-modules, soft LA-ring homomorphism, soft LA-module homomorphism, exact sequence.

1. Introduction

In our daily life, the real world is multifaceted. Thus, there are many problems in different disciplines such as engineering, social sciences, medical sciences etc in the real world and we construct "models" of reality that are simplifications of aspects of the real world. Unluckily, these mathematical models are quite intricate and we are unable to find the precise solutions. Since there are many

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uncertainties mixed up with the data. The traditional tools to deal with these uncertainties are applicable only under certain environment. These may be due to the uncertainties of natural environmental phenomena of human awareness about the real world or to the confines of the means used to measure objects. For example, elusiveness or uncertainty in the boundary between states or between urban and rural areas or the exact growth rate of population in a country's rural area or making decision in a machine based environment using database information. Thus, the classical set theory, which is based on crisp and exact case, may not be fully suitable for conducting such problems of uncertainty.

In order to deal with uncertainties, there are many theories, for example, theory of fuzzy sets [28], theory of intuitionistic fuzzy sets [4], theory of vague sets, the theory of interval mathematics [5], [8] and theory of rough sets [11] have been developed, yet difficulties are seem to be still there. It is noted that the theory of soft sets was first proposed by D. Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainties. In order to model vagueness and uncertainties, D. Molodtsov first introduced the concept of soft sets and it has received much attention since its inception. In his well known paper [13], D. Molodtsov presented some fundamental results of the new theory and successfully applied them into several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, theory of probability etc. A soft set is a collection of approximate description of an object. He also showed how soft set theory is free from parametrization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. In fact, the soft systems provides a very general framework with the involvement of parameters. Nowadays, the research work on soft set theory and its applications in various fields are progressing rapidly.

We notice that P. K. Maji [11], [12] in 2002 and 2003 first presented the application of soft sets in decision making based on the reduction of parameters to keep the optimal choice objects. In addition, D. Chen [6] presented a new definition of soft set parametrization reduction and a comparison of it with attributes reduction in rough set theory. A. Sezgin et al. [17], introduced the union soft subnear-rings and union soft ideals of a near-ring. The application of soft sets in algebraic structures was introduced by H. Aktaş and N. Çağman [1]. They discussed the notion of soft groups and derived some basic properties. They also showed that soft groups extends the concept of fuzzy groups. Recently, X. Liu et al.[9], established some useful fuzzy isomorphism theorems of soft rings. They also discussed the fuzzy ideals of soft rings. In [10], X. Liu et al., have considered the isomorphism theorems for soft rings. In [25], Q. M. Sun et al., have discussed the concept of soft modules and investigated some of their basic properties.

The Left Almost Ring (LA-ring) is actually an off shoot of LA-semigroup and LA-group. In fact, an LA-rings is a non-commutative and non-associative algebraic structure and gradually due to its peculiar characteristics it has been emerging as a useful non-associative class which intuitively would be a quite convinent tool to enhance non-associative ring theory. By an LA-ring, we mean a non-empty set R with at least two elements such that $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R,+,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for all $a,b\in R$, $a\oplus b=b-a$ and $a\cdot b$ is same as in the ring.

Furthermore, T. Shah and I. Rehman [22], have discussed left almost ring (LA-ring) of finitely nonzero functions which is in fact a generalization of a commutative semigroup ring. Recently T. Shah and I. Rehman [23], discussed some properties of LA-rings through their ideals and intuitively ideal theory would be a gate way for investigating the application of fuzzy sets, intuitionistics fuzzy sets and soft sets in LA-rings. For example, T. Shah et al., [20], have applied the concept of intuitionistic fuzzy sets and established some useful results. In [16], some computational work through Mace4, has been done and some interesting characteristics of LA-rings have been explored. Recently, in [18], T. Shah et al., have adopted a new approach to apply the Molodtsov's soft set theory to a class of non-associative rings. And in [19], T. Shah and Asima Razzaque have discussed some basic properties regarding soft M-system, soft P-system and soft I-System in a non-associative left almost rings. T. Shah et al. [21], have promoted the concept of LA-modules and establish some results of isomorphisms theorems and direct sum of LA-modules. Also T. Shah and I. Rehman in [22] utilized the both LA-semigroup and LA-ring and generalizes the notion of a commutative semigroup ring. Furthermore, they defined the notion of LA-modules over an LA-ring, which is a non abelian non-associative structure but closer to abelian group. Hence the study of this algebraic structure is completely parallel to modules which are basically the abelian groups. In this aspect, A. Alghamdi and F. Sahraoui [2], have defined and constructed a tensor product of LA-modules, they extended some simple results from the ordinary tensor to the new setting. Also Asima Razzaque et al., [3] have given the concept of exact sequence in LA-modules. For some further study of LA-rings, the readers are referred to ([15], [21], [24]).

In this paper, we initiate the concepts of projective and injective soft LA-modules. Also we discuss the free soft LA-modules, split sequence in soft LA-modules and prove some of their related results.

2. Preliminaries

In this section, we recall some basic definitions and results which are relevant to soft sets and LA-modules.

Definition 1 ([13]). *Let U be an initial universe and E be a set of parameters. Then we use $P(U)$ to denote the power set of U and A be a non-empty subset of E. A pair (F,A) is called a soft set over U, where F is a mapping given by $F:A\to P(U)$.*

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

For the soft sets, we give the following definitions.

Definition 2 ([14]). *For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if it satisfies the following conditions:*

(i) $A \subseteq B$ and

(ii) for all $e \in A$, $F(e) \subseteq G(e)$.

We write $(F, A)\widetilde{\subset}(G, B)$. Also we call (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote this soft superset by $(F, A)\widetilde{\supset}(G, B)$.

Definition 3 ([12]). *Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Definition 4 ([12]). *A soft set (F, A) over U is said to be a $NULL$ soft set denoted by Φ if for all $\varepsilon \in A$, $F(\varepsilon) = \emptyset$ (null set).*

Definition 5 ([7]). *Let (F, A) be a soft set .Then, the set supp $(F, A) = \{x \in A \mid F(x) \neq \phi\}$ is called the support of the soft set (F, A). A soft set is said to be non null if its support is not equal to the empty set.*

Definition 6 ([22]). *Let $(R, +, .)$ be an LA-ring with left identity e. An LA-group $(M, +)$ is said to be LA-module over R if $R \times M \to M$ defined as $(a, m) \mapsto am \in M$, where $a \in R$, $m \in M$ satisfies the following conditions:*

(i) $(a + b)\, m = am + bm$,

(ii) $a(m + n) = am + an$,

(iii) $a(bm) = b(am)$,

(iv) $1.m = m$,

for all $a, b \in R$, $m, n \in M$.

Left R LA-module is denoted by ${}_RM$ or simply M. Right R LA-module can be defined in a similar manner and is denoted by M_R.

In the following, we give a non-trivial example of LA-module over R constructed by T. Shah and I. Rehman in [22]. We observe that the LA-module constructed in this example is not a module.

Example 1 ([22]). Let $(R, +, .)$ be an LA-ring with a left identity and S is a commutative semigroup. Then $R[S] = \{\sum_{j=1}^{n} a_j s_j : a_j \in R, s_j \in S\}$ and the map $R \times R[S] \mapsto R[S]$ defined by $(a, \sum_{j=1}^{n} a_j s_j) \mapsto \sum_{j=1}^{n} (aa_j) s_j$ is an LA-module over R.

Definition 7 ([21]). *Let M be a left R LA-module. Then, we call an LA-subgroup N of M over an LA-ring R is called left R LA-submodule of M, if $RN \subseteq N$, i.e., $rn \in N$ for all $r \in R$ and $n \in N$. This is denoted by $N \leq M$.*

By the above definition, we immediately have the following theorem.

Theorem 1 ([21]). *If A and B are two LA-submodules of an LA-module M over an LA-ring R, then $A \cap B$ is also an LA-submodule of M.*

Corollary 1 ([21]). *The intersection of any number of LA-submodules of an LA-module is a LA-submodule.*

Following is the very useful definition in the study of LA-modules.

Definition 8 ([21]). *Let M, N be LA-modules over an LA-ring R. A map $\varphi : M \longrightarrow N$ is called an LA-module homomorphism(or simply R-homomorphism) if, for all r in R and m, n in M*
(i) $\varphi(m+n) = \varphi(m) + \varphi(n)$
(ii) $\varphi(rm) = r\varphi(m)$

We now describe the LA-modules

Theorem 2 ([21]). *Let $\varphi : M \longrightarrow N$ be an LA-module homomorphism from an LA-module M to an LA-module N, then*
(1) *If A is an LA-submodule of M, then $\varphi(A)$ is an LA-submodule of N.*
(2) *If B is an LA-submodule of N, then $\varphi^{-1}(B)$ is an LA-submodule of M.*

Definition 9 ([21]). *Let M be an LA-module and $A \subset M$ is an LA-submodule. We define quotient module or factor module M/A by $M/A = \{A+m : m \in M\}$. That is, M/A is the set of equivalence classes of elements of M. An equivalence class is denoted by $A+m$ or by $[m]$. Each element in the class $A+m$ is called a representative of the class.*

Lemma 1 ([21]). *With the canonical operations, by choosing representatives, $(A+m)+(A+n) = A+(m+n)$, the set M/A is an LA-group. A, the equivalence class of $0 \in M$ is the left identity of M/A. The map $\pi : M \longrightarrow M/A$, $\pi(m) = A+m$ is surjective LA-group homomorphism.*

Definition 10 ([21]). *Let M be an LA-module over an LA-ring R. Let A and B be LA-submodules of M. Then M is said to be the internal direct sum of A and B, if every element $m \in M$ can be written in one and only one way as $m = a+b$, where $a \in A$ and $b \in B$. Symbolically, the direct sum is represented by the notation $M = A \oplus B$.*

Definition 11 ([25]). *Let $\{M_i \mid i \in I\}$ be a nonempty family of R-modules, $P = \prod_{i\in I} M_i = \{(x_i) \mid x_i \in M_i\}$ is a direct product set, if the operations on the product are given by $(x_i)+(y_i) = (x_i+y_i)$ and $r(x_i) = (rx_i)$, then P induce a left R module structure called direct product of $\{M_i \mid i \in I\}$, which is denoted by $\prod_{i\in I} M_i$.*

Proposition 1 ([25]). *Let $\{M_i \mid i \in I\}$ be a nonempty family of submodules of M. Then $\cap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are all submodules of M.*

Definition 12 ([25]). *All the elements (x_i) in the direct product $\prod_{i \in I} M_i$, where x_i is zero for almost all $i \in I$ except finite one, establish a submodule of $\prod_{i \in I} M_i$ which is called direct sum of $\{M_i \mid i \in I\}$, will be denoted by $\coprod_{i \in I} M_i$ or $\oplus_{i \in I} M_i$.*

For the LA-modules over an LA-ring, we state the following theorem.

Theorem 3 ([21]). *Let M be an LA-module over an LA-ring R. If A and B are LA-submodules of M, then M is the internal direct sum of A and B if and only if*

(1) $M = A + B$.

(2) $A \cap B = \{0\}$.

Definition 13 ([25]). *For a sequence of R-homomorphisms and R-modules $\cdots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$ is called an exact sequence if $Imf_{n-1} = kerf_n$ for all $n \in \mathbb{N}$. An exact sequence of the form $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is called a short exact sequence.*

By applying the above definition, we immediately obtain the following Proposition concerning the morphisms of modules.

Proposition 2 ([27]). *Let $f : M \longrightarrow N$ be R-homomorphism for R-modules M and N. Then*

(1) $0 \longrightarrow M \xrightarrow{f} N$ *is exact if and onlt if f is monomorphism.*

(2) $M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is epimorphism.*

(3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an isomorphism.*

Theorem 4 ([26]). *Every left R module is a homomorphic image of free left R module.*

Proposition 3 ([26]). *For a short exact sequence $O \to A \xrightarrow{f} B \xrightarrow{g} C \to O$ of R-modules and homomorphisms, the following statements are equivalent:*

(1) *there exists a homomorphism $\alpha : B \to A$ such that $\alpha f = \perp_A$*

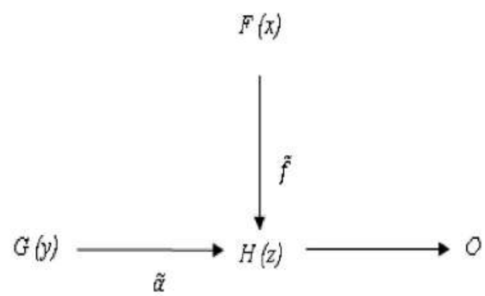
(2) *there exists a homomorphism $\beta : C \to B$ such that $g\beta = \perp_C$*

(3) *Imf is a direct summand of B.*

3. Projective soft LA-modules

We initiate in this section with the definition of projective soft LA-modules.

Definition 14. *Let M be a left LA-module over an LA-ring R. Then (F, A) is called a projective soft LA-module over M, if the given diagram of soft LA-modules and soft LA-homomorphisms with row exact, there exists a soft LA-homomorphism $\widetilde{g} : F(x) \to G(y)$ which makes the completed diagram commutative, that is $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$.*



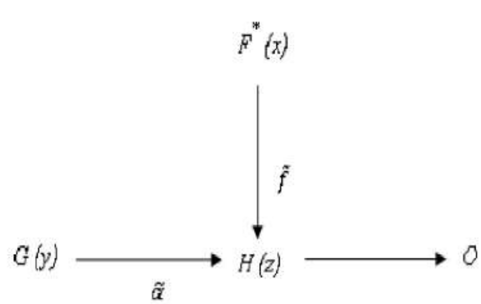
Definition 15. *A soft LA-module* (F^*, A) *over* M *is called a free soft LA-module on a basis* $(\bar{X}, A) \neq \phi$, *if there is a map* $\widetilde{\alpha} : \bar{X}(x) \to F^*(x)$ *such that given any map* $\widetilde{f} : \bar{X}(x) \to G^*(y)$, *where* (G^*, B) *is any soft LA-module, there exists a unique soft LA-homomorphism* $\widetilde{g} : F^*(x) \to G^*(y)$ *such that* $\widetilde{f} = \widetilde{g}\widetilde{\alpha}$.

Throughout this paper homomorphism is always considered as a soft LA-homomorphism.

The following result is a crucial result.

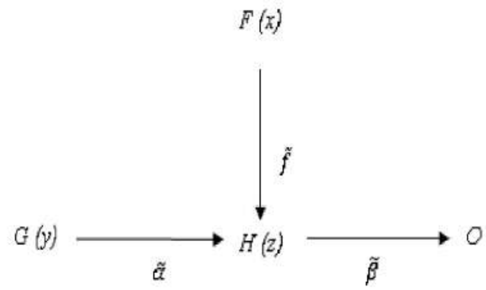
Theorem 5. *Every free soft LA-module is a projective soft LA-module.*

Proof. Let (F^*, A) be a free soft LA-module with basis $(\bar{X}, A)$. Let be a diagram of soft LA-modules and soft LA-homomorphism in which row is exact. Let $x \in \bar{X}(x)$ for all $x \in A$. Then $\widetilde{f}(x) \in H(z)$ for all $z \in C, x \in A$ and as $\widetilde{\alpha}$ is onto, so there exists $b \in G(y)$ for all $y \in B$ such that $\widetilde{\alpha}(b) = \widetilde{f}(x)$. Define $\widetilde{g} : \bar{X}(x) \to G(y)$, for all $x \in A$, for all $y \in B$ by $\widetilde{g}(x) = b$ and extend this function $\widetilde{g} : F^*(x) \to G(y)$ where $x \in A$, $y \in B$. It can be observed that $\widetilde{\alpha}\widetilde{g}(x) = \widetilde{\alpha}(\widetilde{g}(x)) = \widetilde{\alpha}(b) = \widetilde{f}(x)$. Hence it follows that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Therefore it is proved that every free soft LA-module is projective soft LA-module.

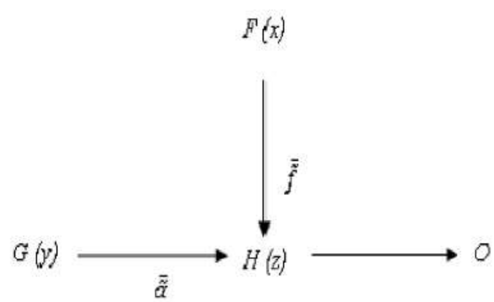


□

Proposition 4. *Let (F, A) be a projective soft LA-module. If the diagram of soft LA-modules and soft LA-homomorphisms the row is exact and $\widetilde{\beta}\widetilde{f} = 0$, then there exist a homomorphism $\widetilde{g} : F(x) \to G(y)$ for all $x \in A$ and $y \in B$ such that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$.*



Proof. Let $\bar{H} = Im\widetilde{\alpha} = ker\widetilde{\beta}$ and $\bar{\widetilde{\alpha}} : G(y) \to \bar{h}(z)$ be the homomorphism induced by $\widetilde{\alpha}$ where $y \in B$ and $z \in C$. So that $\widetilde{\beta}\widetilde{f} = 0$ this implies that $Im\widetilde{f}$ is contained in $ker\widetilde{\beta} = Im\widetilde{\alpha} = \bar{H}$. Therefore $\widetilde{f}$ induces a homomorphism $\bar{\widetilde{f}} : F(x) \to \bar{H}(z)$ where $z \in C$ and $x \in A$, such that if $\widetilde{i} : \bar{H}(z) \to H(z)$ is the inclusion map then $\widetilde{\alpha} = i\bar{\widetilde{\alpha}}$ and $\widetilde{f} = i\bar{\widetilde{f}}$. We have then a diagram in which row is exact. The soft LA-module $F(x)$ for all $x \in A$ is projective soft LA-module, so there exists a homomorphism $\widetilde{g} : F(x) \to G(y)$ such that $\bar{\widetilde{\alpha}}\widetilde{g} = \bar{\widetilde{f}}$ for $x \in A$ and $y \in B$. But $\widetilde{\alpha}\widetilde{g} = i\bar{\widetilde{\alpha}}\widetilde{g} = i\bar{\widetilde{f}} = \widetilde{f}$ .This implies that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Hence the theorem is proved.

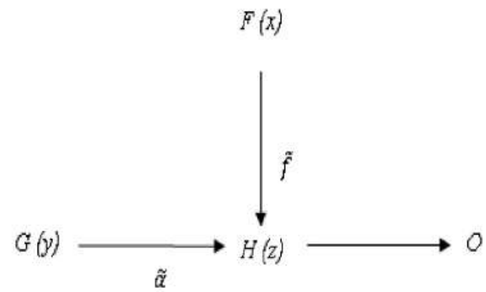


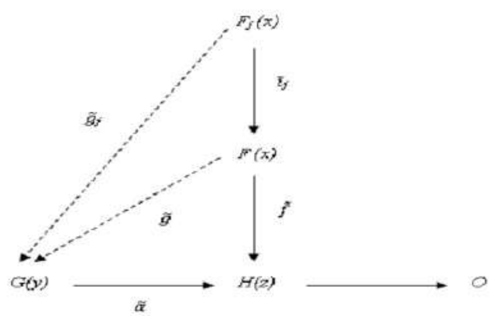
□

In the following Proposition, we give a characterization for the projective soft LA-modules.

Proposition 5. *The soft LA-module* (F, A) *is projective soft LA-module if and only if* (F_j, A) *is a projective soft LA-module for every* $j \in J$.

Proof. Suppose that every (F_j, A) is a projective soft LA-module for every $j \in J$. Consider, a diagram with row exact. We have a soft LA-homomorphism $\widetilde{f}\,\widetilde{i}_j : F_j(x) \to H(z)$ for all $x \in A, z \in C$ and $j \in J$. $F_j(x)$ being projective soft LA-module, so there exists a homomorphism $\widetilde{g}_j : F_j(x) \to G(y)$ such that $\widetilde{\alpha}\widetilde{g}_j = \widetilde{f}\,\widetilde{i}_j$for all $x \in A, y \in B$ and $j \in J$. Now define $\widetilde{g} : F(x) \to G(y)$ for all $x \in A, y \in B$ by $\widetilde{g}(\varepsilon) = \sum_j \widetilde{g}_j \widetilde{\pi}_j(\varepsilon)$, for $\varepsilon \in F(x)$, $x \in A$.





It can be observed that the sum on the right hand side is finite. Then $\widetilde{g}$ is a soft homomorphism. Now to show projective soft LA-module, so for $\varepsilon \in F(x)$ where $x \in A$, $\widetilde{\alpha}\widetilde{g}(\varepsilon) = \widetilde{\alpha}(\sum_j \widetilde{g}_j \widetilde{\pi}_j(\varepsilon)) = \sum_j \widetilde{\alpha}\widetilde{g}_j \widetilde{\pi}_j(\varepsilon) = \sum_j \widetilde{f}\,\widetilde{i}_j \widetilde{\pi}_j(\varepsilon) = \widetilde{f}(\sum_j \widetilde{i}_j \widetilde{\pi}_j(\varepsilon)) = \widetilde{f}((\sum_j \widetilde{i}_j \widetilde{\pi}_j)(\varepsilon)) = \widetilde{f}(\varepsilon)$. Therefore it shows that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Hence, we have proved that (F, A) is projective soft LA-module. Conversely, suppose that $F(x)$ for all $x \in A$ is projective soft LA-module. For any $j \in J$, consider a diagram with row exact. Then for all $x \in A$ and $z \in C$, $\widetilde{f}\widetilde{\pi}_j : F(x) \to H(z)$ is a homomorphism and $F(x)$ being projective soft LA-module, there exists a homomorphism $\widetilde{g} : F(x) \to G(y)$ such that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}\widetilde{\pi}_j$where $x \in A$ and $y \in B$. Now let take $\widetilde{g}_j = \widetilde{g}\widetilde{i}_j$ which is a homomorphism from $F_j(x) \to G(y)$, then $\widetilde{\alpha}\widetilde{g}_j = \widetilde{\alpha}\widetilde{g}\widetilde{i}_j = \widetilde{f}\widetilde{\pi}_j\widetilde{i}_j = \widetilde{f}$. Hence it is proved that $F_j(x)$ is projective soft LA-modules.

$$\begin{tikzcd} & F_j(x) \arrow[d, "\widetilde{f}"] & \\ G(y) \arrow[r, "\widetilde{\alpha}"'] & H(z) \arrow[r] & O \end{tikzcd}$$

□

Definition 16. *A short exact sequence of the form $O \to F(x) \xrightarrow{\widetilde{i}} G(y) \xrightarrow{\widetilde{j}} H(z) \to O$ of soft LA-modules and soft homomorphism is said to splits or split sequence of soft LA-modules, if any of the following these condition holds*

(i) there exists a homomorphism $\widetilde{\gamma} : G(y) \to H(z)$ such that $\widetilde{\gamma}\widetilde{i} = \perp_{F(x)}$

(ii) there exists a homomorphism $\widetilde{\theta} : H(z) \to G(y)$ such that $\widetilde{j}\widetilde{\theta} = \perp_{H(z)}$

(iii) $Im\widetilde{i}$ is a direct summand of $G(y)$
where for all $x \in A, y \in B$ and $z \in C$.

We now state the following theorem concerning the soft LA-modules.

Theorem 6. *Every soft LA-module is a homomorphic image of a free soft LA-module.*

Proof. Let (F, A) be a soft LA-module over M. Let $(\bar{X}, B) \neq \phi$ be soft set of the elements of which are in one to one correspondence with the elements of (F, A). Let the elements of $(\bar{X}, B)$ corresponding to the element $\varepsilon \in (F, A)$ be denoted $\bar{x_\varepsilon}$. Let (F^*, B) be the free soft LA-module over N with basis $(\bar{X}, B)$. For all $x \in A$ and $y \in B$, let $\widetilde{f}$: $\bar{X}(y) \to F(x)$ be the map given by $\widetilde{f}(\bar{x_\varepsilon}) = \varepsilon$. Then by definition 15, for all $x \in A$ and $y \in B$, $\widetilde{g}$: $F^*(y) \to F(x)$ is the unique homomorphism which satisfies $\widetilde{g}(\bar{x_\varepsilon}) = \varepsilon = \widetilde{f}(\bar{x_\varepsilon})$. Thus $\widetilde{g}$ is an epimorphism and hence it is proved that (F, A) is a homomorphic image of (F^*, B). □

Remark 1. *Since every soft LA-module is homomorphic image of a free soft LA-module and every free soft LA-module is a projective soft LA-module, then we have the following lemma.*

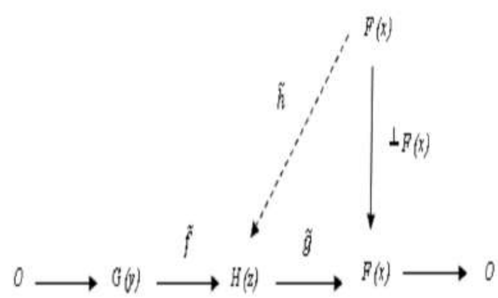
Lemma 2. *Every soft LA-module is a homomorphic image of a projective soft LA-module.*

Proof. The proof follows straightforwardly by theorem 6 and theorem 5. □

We now give below a characterization theorem of projective soft LA-modules.

Theorem 7. *A soft LA-module (F, A) is projective soft LA-module if and only if every exact sequence $O \to G(y) \to H(z) \to F(x) \to O$ splits for all $x \in A, y \in B$ and $z \in C$.*

Proof. Suppose that for all $x \in A$, $F(x)$ is a projective soft LA-module. Consider the following exact sequence, then by definition of projective soft LA-module, there exists a homomorphism $\widetilde{h} : F(x) \to H(z)$ where $x \in A$ and $z \in C$, such that $\widetilde{g}\widetilde{h} = \perp_{F(x)}$which shows that the sequence splits. Now conversely, suppose that the sequence of the form $O \to G(y) \to H(z) \to F(x) \to O$ splits. Since every soft LA-module being homomorphic image of a free soft LA-module. Let (F^*, D) being free soft LA-module and $\widetilde{\alpha} : F^*(t) \to F(x)$ be an epimorphism for all $t \in D$ and $x \in A$. If $G(y)$ denotes the kernel $\widetilde{\alpha}$, we get an exact sequence $O \to G(y) \overset{\widetilde{i}}{\to} F^*(t) \overset{\widetilde{\alpha}}{\to} F(x) \to O$ which by hypothesis splits. Thus $F^*(t) \cong F(x) \oplus G(y)$. The soft LA-module (F^*, D) being free is projective soft LA-module and hence this implies that $F(x)$ and $G(y)$ are projective. Hence, the theorem is proved.



□

4. Injective soft LA-modules

In this section, we define the injective soft LA-modules and establish some relevant results.

Definition 17. *Let (I, A) be a soft LA-module over M, (I, A) is called injective soft LA-module, if given diagram of soft LA-modules and soft LA-homomorphisms with row exact, then there exists a homomorphism $\widetilde{g} : G(z) \to I(x)$ for all $z \in C$ and $x \in A$, which makes the completed diagram commutative that is $\widetilde{g}\widetilde{\alpha} = \widetilde{f}$.*

$$\begin{tikzcd} O \arrow[r] & F(y) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{f}"'] & G(z) \\ & I(x) & \end{tikzcd}$$

For the injective soft LA-modules, we have the following Propositions.

Proposition 6. *If (I, A) is an injective soft LA-module, then given diagramwith row exact and $\widetilde{f}\widetilde{\alpha} = 0$, there exists a homomorphism $\widetilde{g} : H(t) \to I(x)$ such that the completed diagram is commutative $\widetilde{g}\widetilde{\beta} = \widetilde{f}$ for all $x \in A$ and $t \in D$.*

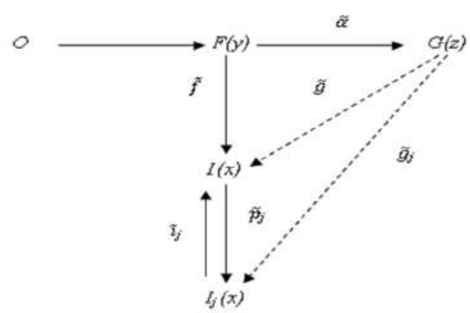
$$\begin{tikzcd} F(y) \arrow[r, "\tilde{\alpha}"] & G(z) \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{f}"'] & H(t) \\ & I(x) & \end{tikzcd}$$

Proof. Let $X(z) \subseteq G(z) = ker\widetilde{\beta} = Im\widetilde{\alpha}$ where $z \in C$. Then $\widetilde{\beta}$ induces a monomorphism $\overline{\widetilde{\beta}} : G(z)/X(z) \to H(t)$ for all $t \in D$ and $z \in C$, given by $\overline{\widetilde{\beta}}(\varepsilon + X(z)) = \widetilde{\beta}(\varepsilon)$, where $\varepsilon \in G(z)$. Let $\overline{\widetilde{\beta}}(\varepsilon_1 + X(z)) = \overline{\widetilde{\beta}}(\varepsilon_2 + X(z)) \Rightarrow \widetilde{\beta}(\varepsilon_1) = \widetilde{\beta}(\varepsilon_2) \Rightarrow \widetilde{\beta}(\varepsilon_1) - \widetilde{\beta}(\varepsilon_2) = 0 \Rightarrow \widetilde{\beta}(\varepsilon_1 - \varepsilon_2) = 0 \Rightarrow \varepsilon_1 - \varepsilon_2 \in ker\widetilde{\beta} = X(z) \Rightarrow \varepsilon_1 - \varepsilon_2 \in X(z) \Rightarrow \varepsilon_1 - \varepsilon_2 + X(z) = X(z) \Rightarrow \varepsilon_1 + X(z) = \varepsilon_2 + X(z)$, hence it shows that $\overline{\widetilde{\beta}}$ is a monomorphism. Also $\widetilde{f}\widetilde{\alpha} = 0 \Rightarrow \widetilde{f}(\widetilde{\alpha}(\varepsilon')) = 0(\varepsilon') = 0 \Rightarrow \widetilde{f}(\widetilde{\alpha}(\varepsilon')) = 0 \Rightarrow \widetilde{\alpha}(\varepsilon') \in ker\widetilde{f}$ but $\widetilde{\alpha}(\varepsilon') \in Im\widetilde{\alpha} \Rightarrow Im\widetilde{\alpha} \subseteq ker\widetilde{f} \Rightarrow X = Im\widetilde{\alpha} \subseteq ker\widetilde{f}$ and, therefore $\widetilde{f}$ induces a homomorphism $\overline{\widetilde{f}} : G(z)/X(z) \to I(x)$ for all $x \in A$ and $z \in C$, by $\overline{\widetilde{f}}(\varepsilon + X(z)) = \widetilde{f}(\varepsilon)$, where $\varepsilon \in G(z)$. Let $\widetilde{\pi} : G(z) \to G(z)/X(z)$ denotes the natural projection. Then $\overline{\widetilde{f}}\widetilde{\pi}(\varepsilon) = \overline{\widetilde{f}}(\widetilde{\pi}(\varepsilon)) = \overline{\widetilde{f}}(\varepsilon + X(z)) = \widetilde{f}(\varepsilon)$

for all $\varepsilon \in G(z)$, hence $\widetilde{\overline{f}}\widetilde{\pi} = \widetilde{f}$ and $\widetilde{\overline{\beta}}\widetilde{\pi}(\varepsilon) = \widetilde{\overline{\beta}}(\widetilde{\pi}(\varepsilon)) = \widetilde{\overline{\beta}}(\varepsilon + X(z)) = \widetilde{\beta}(\varepsilon)$ for all $\varepsilon \in G(z)$, therefore $\widetilde{\overline{\beta}}\widetilde{\pi} = \widetilde{\beta}$. Since $I(x)$ is an injective soft LA-module, so there exists a homomorphism $\widetilde{g} : H(t) \to I(x)$ for all $t \in D$ and $x \in A$ such that $\widetilde{\overline{f}} = \widetilde{g}\widetilde{\overline{\beta}}$ then $\widetilde{g}\widetilde{\beta} = \widetilde{g}\widetilde{\overline{\beta}}\widetilde{\pi} = \widetilde{\overline{f}}\widetilde{\pi} = \widetilde{f}$. Hence proved. □

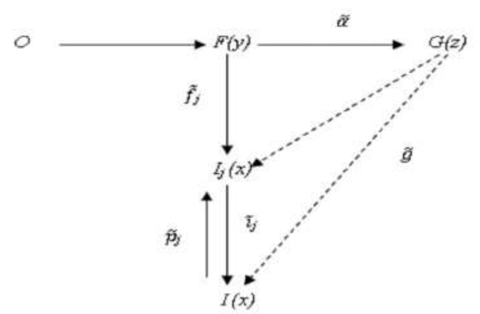
Proposition 7. *Let $(I_j, A)_{j \in J}$ be a family of soft LA-modules and $I(x) = \prod_{j \in J} I_j(x)$ (=direct product of I_j) for all $x \in A$. Then $I(x)$ is an injective soft LA-module if and only if $I_j(x)$ is an injective soft LA-module.*

Proof. Since for all $x \in A$, $I(x) = \prod_{j \in J} I_j(x)$, there exists a homomorphism $\widetilde{i}_j : I_j(x) \to I(x)$ and $\widetilde{p}_j : I(x) \to I_j(x)$ such that $\widetilde{p}_j \widetilde{i}_j = \perp_{I_j}$and $\widetilde{p}_k \widetilde{i}_j = 0$, the zero map if $j \neq k$. Let $\widetilde{\alpha} : F(y) \to G(z)$ for all $y \in B$ and $z \in C$ be a monomorphism of soft LA-modules. Suppose that every $I_j(x)$ for all $x \in A$, is an injective soft LA-module, by considering the following diagram.



Let $\widetilde{f} : F(y) \to I(x)$ be a homomorphism for $y \in B$ and $x \in A$, then $\widetilde{p}_j \widetilde{f} : F(y) \to I_j(x)$ is a homomorphism and as $I_j(x)$ being injective soft LA-module, then there exists a homomorphism $\widetilde{g}_j : G(z) \to I_j(x)$ such that $\widetilde{g}_j \widetilde{\alpha} = \widetilde{p}_j \widetilde{f}$, where $z \in C$ and $x \in A$. Now define $\widetilde{g} : G(z) \to I(x)$ by $\widetilde{g}(\varepsilon) = (\widetilde{g}_j(\varepsilon)), \varepsilon \in G(z)$. Then $\widetilde{g}$ is a homomorphism and for $\varepsilon' \in F(y)$ where $y \in B$, $\widetilde{g}(\widetilde{\alpha}(\varepsilon')) = (\widetilde{g}_j \widetilde{\alpha}(\varepsilon')) = \widetilde{p}_j \widetilde{f}(\varepsilon') = \widetilde{f}(\varepsilon')$, which shows that $\widetilde{g}\widetilde{\alpha} = \widetilde{f}$. Hence proved that $I(x)$ is an injective soft LA-module. Conversely, suppose that $I(x)$ is an injective soft LA-module. For any $j \in J$, let $\widetilde{f}_j : F(y) \to I_j(x)$ be a homomorphism where $y \in B$ and $x \in A$.

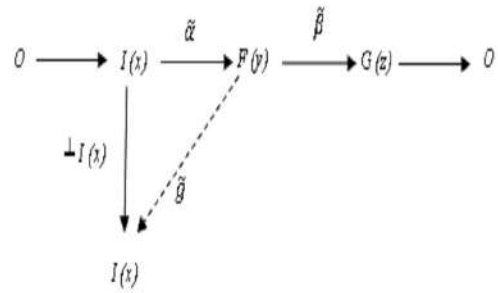
The soft LA-module $I(x)$ being injective soft LA-module, there exists a homomorphism $\widetilde{g} : G(z) \to I(x)$ where $z \in C$ and $x \in A$, such that $\widetilde{g}\widetilde{\alpha} = \widetilde{i}_j \widetilde{f}_j$. Then $\widetilde{p}_j \widetilde{g} : G(z) \to I_j(x)$ is a homomorphism such that $\widetilde{p}_j \widetilde{g}\widetilde{\alpha} = \widetilde{p}_j \widetilde{i}_j \widetilde{f}_j = \widetilde{f}_j$. Hence we have proved that $I_j(x)$ is an injective soft LA-module. □



Proposition 8. *If* (I, A) *is an injective soft LA-module, then every exact sequence of the form* $O \to I(x) \overset{\widetilde{\alpha}}{\to} F(y) \overset{\widetilde{\beta}}{\to} G(z) \to O$ *splits for every* $x \in A, y \in B$ *and* $z \in C$.

Proof. Consider the following diagram,

since $I(x)$ is an injective soft LA-module, so there exists a homomorphism $\widetilde{g} : F(y) \to I(x)$ such that $\widetilde{g}\widetilde{\alpha} = \perp_{I(x)}$. Hence we have proved that the exact sequence of the form $O \to I(x) \overset{\widetilde{\alpha}}{\to} F(y) \overset{\widetilde{\beta}}{\to} G(z) \to O$ splits.



□

5. Conclusion

Our paper can be regarded as a systematic study of soft LA-modules. We study soft LA-modules by giving the concepts of split sequence in soft LA-modules, free soft LA-modules, projective and injective soft LA-modules and their related properties. One can further develop the theory of homological algebra of soft LA-modules by defining functors, pullback and pushouts etc

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Accepted: 28.02.2019

Fixed point theorem for contraction mappings in probabilistic normed spaces

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Abstract. In this paper, the concept of contractive mappings and $\phi-$ contraction mappings on Menger's probabilistic normed spaces are defined with suitable examples. The unique fixed point theorem for contractive mappings and $\phi-$ contraction mappings are established in Menger's probabilistic normed spaces.

Keywords: Menger's Probabilistic normed spaces, sequential continuity, fixed point, probabilistic bounded.

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1. Introduction and preliminaries

Probabilistic functional analysis has risen as one of the vital mathematical disciplines in view of the needs in dealing with probabilistic models in applied problems. Probabilistic Normed spaces were introduced by Serstnev and its new definition was proposed by C.Alsina, B.Schwerier and A.Sklar ([1], [2]). The general theory of Probabilistic Metric spaces and Probabilistic Normed spaces can be read in ([8], [13], [14]). The theory of probabilistic normed spaces (briefly. PN spaces) is important as a generalization of deterministic results of linear normed spaces and in the study of random operator equations. The PN spaces may also provide us with the suitable tools to contemplate the geometry of nuclear physics and have important applications in quantum particle physics especially in string theory and in ε_∞ theory. Some of the recent developments in different types of probabilistic normed spaces, translation invariant topologies induced by probabilistic norms and linear 2-normed spaces are discussed in ([3], [6], [7], [8]).

A triangle function [4] is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, non-decreasing in each place and has ε_0 as identity, this is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$,

(TF2) $\tau(F, G) = \tau(G, F)$,

(TF3) $F \le G \Longrightarrow \tau(F, H) \le \tau(G, H)$,

(TF4) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions [4] are

$$\tau_T(F, G)(x) = \sup_{s+t=x} \{T(F(s), G(t))\}$$

and

$$\tau_{T^*}(F, G) = \inf_{s+t=x} \{T^*(F(s), G(t)\}.$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0, 1]$ that is commutative, associative, non-decreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0, 1]$ which is related to the continuous t-norm T through $T^*(x, y) = 1-T(1-x, 1-y)$. Let us recall among the triangular function one has the function defined via $T(x, y) = \min(x, y) = M(x, y)$ and $T^*(x, y) = \max(x, y)$ or $T(x, y) = \Pi(x, y) = xy$ and $T^*(x, y) = \Pi^*(x, y) = x + y - xy$.

Some more examples of t-norms [4] are W and Z, defined respectively by

$$W(x,y) := \max\{x+y-1, 0\}.$$

$$Z(x,y) := \begin{cases} 0, & \text{if } (x,y) \in [0,1] \times [0,1], \\ x, & \text{if } x \in [0,1], y = 1, \\ y, & \text{if } x = 1, y \in [0,1] \end{cases}$$

then we have

$$Z < W < \Pi < M$$

and every t-norm T,

$$Z \leq T \leq M.$$

For every t-norm Π, W, Z and M, it is defined that

$$\Pi_\Pi(F,G)(x) := \Pi(F(x), G(x)),$$
$$\Pi_W(F,G)(x) := W(F(x), G(x)),$$
$$\Pi_Z(F,G)(x) := Z(F(x), G(x)),$$
$$\Pi_M(F,G)(x) := M(F(x), G(x)).$$

A few more interesting examples of t-norms and t-conorms can be found in the recent paper [10].

We recall the definition of probabilistic normed space (briefly, PN space) as given in [2], together with the notation that will be needed [13]. We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left-continuous and non-decreasing functions from $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ into $[0,1]$ such that $F(0) = 0$ and $F(+\infty) = 1$, will be denoted by Δ^+, while the subset $\mathcal{D}^+ \subset \Delta^+$ will denote the set of all proper distance d.f.'s, namely those for which $\ell^- F(+\infty) = 1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point $x \in \mathbb{R}$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbb{R}$. For any $a \geq 0$, ε_a is the d.f. given by

$$\varepsilon_a = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

The space Δ^+ can be metrized in several ways [13], but we shall here adopt the Sibley metric d_S. If F, G are d.f.'s and h is in $]0,1[$, let $(F, G; h)$ denote the condition:

$$G(x) \leq F(x+h) + h \text{ for every} x \in \left]0, \frac{1}{h}\right[.$$

Then the Sibley metric is defined by

$$d_S(F,G) := \inf\{h \in]0,1[: \text{ both } (F,G;h) \text{ and } (G,F;h) \text{ hold }\}. \tag{1.1}$$

Since every d.f. F is bounded, $0 \leq F(t) \leq 1$, for all $t \in \overline{\mathbb{R}}$, one has $d_S \leq 1$.

Definition 1.1 ([1], [4])**.** A Menger's Probabilistic Normed Space (briefly, a Menger's PN space), is a quadruple (X, ν, τ, T), where X is a real vector space, τ is a triangle function, T is a t-norm and the mapping $\nu : X \to \triangle^+$ satisfies the conditions:

1. $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in X);
2. $\nu_{\alpha p}(t) \geq \nu_p \left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}\backslash\{0\}$ and $t \in \mathbb{R}$;
3. $\nu_{p+q}(s+t) \geq \tau_T\left(\nu_p(s), \nu_q(t)\right)$ for all $p, q \in X$ and $s, t \in \mathbb{R}$.

The function ν is called the Menger's probabilistic norm.

Example 1.1 ([1])**.** Let $(V, \|.\|)$ be a normed space and define $\nu_p := \varepsilon_{\|p\|}$. Let τ be a triangle function such that

$$\tau\left(\varepsilon_a, \varepsilon_b\right) = \varepsilon_{a+b},$$

for all $a, b \geq 0$ and let τ^* be a triangle function with $\tau \leq \tau^*$. For instance, it suffices to take $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where T is a continuous t-norm and T^* is its t-conorm. Then (V, ν, τ, τ^*) is a Menger's PN space.

Definition 1.2 ([8])**.** Let (X, ν, τ, T) be a Menger's PN space, and (x_n) be a sequence of X then the sequence (x_n) is said to be convergent to x if for all $t > 0$ and $\lambda \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\nu_{x_n - x}(t) > 1 - \lambda$ for every $n > n_0$.

Definition 1.3 ([8])**.** Let (X, ν, τ, T) be a Menger's PN space, and (x_n) be a sequence of X then the sequence (x_n) is said to be Cauchy sequence if for all $t > 0$ and $\lambda \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\nu_{x_n - x_m}(t) > 1 - \lambda$ for all $n, m > n_0$.

Definition 1.4 ([8])**.** A Menger's PN space (X, ν, τ, T) is said to be complete if every Cauchy sequence in X is convergent to some point in X.

A compltete Menger's PN space is called Menger probabilistic Banach space.

Definition 1.5 ([8])**.** Let (X, ν, τ, T) be a Menger's PN space, E be a subset of X, then the closure of E is defined as $\overline{E} = \{x \in X : \text{ there exists } (x_n) \subset E \text{ such that } x_n \to x\}$, that is, for $\alpha \in (0, 1)$ and $r > 0, x \in \overline{E}$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ one has $\nu_{x_n - x}(r) \geq \alpha$.

We say, E is sequentially closed if $E = \overline{E}$,

Definition 1.6 ([8])**.** The probabilistic radius R_A of a nonempty set A in a Menger's PN space (X, ν, τ, T) is defined by

$$R_A := \begin{cases} \ell^- \varphi_A(t), & t \in [0, +\infty[\\ 1, & t = +\infty \end{cases}$$

where $\varphi_A(t) := \inf\{\nu_p(t) : p \in A\}$.

Definition 1.7 ([8]). A nonempty set A in a Menger's PN space (X, ν, τ, T) is said to be:

(a) certainly bounded, if $R_A(t_0) = 1$ for some $t_0 \in]0, +\infty[$;

(b) perhaps bounded, if one has $R_A(t) < 1$ for every $t \in]0, +\infty[$, but

$$\lim_{t \to +\infty} R_A(t) = 1;$$

(c) perhaps unbounded, if $R_A(t_0) > 0$ for some $t_0 \in]0, +\infty[$ and

$$\lim_{t \to +\infty} R_A(t) \in]0, 1[;$$

(d) certainly unbounded, if $\lim_{t \to +\infty} R_A(t) = 0$, i.e., if $R_A = \varepsilon_\infty$. Moreover, the set A will be said to be distributionally bounded (henceforth $\mathcal{D}-$bounded) if either (a) or (b) holds, i.e.,if $R_A \in \mathcal{D}^+$; otherwise, i.e., if R_A belongs to $\Delta^+ \setminus \mathcal{D}^+$, A will be said to be $\mathcal{D}-$unbounded.

Definition 1.8 ([8]). Let (X, ν, τ, T) and (Y, μ, τ, T) be two Menger's PN spaces then a mapping $T : X \to Y$ is said to be sequentially continuous if a sequence (x_n) in X converges to $x \in X$ implies (Tx_n) converges to Tx in Y.

Lemma 1.9 ([8]). Let (X, ν, τ, T) be a Menger's PN space. If $|\alpha| \leq |\beta|$, then $\nu_{\alpha x} \geq \nu_{\beta x}$ for every $x \in X$.

Definition 1.10 ([8]). Let (X, ν, τ, T) be a Menger's PN space and A be a nonempty subset of X then A is said to be probabilistically bounded if for each $r \in (0, 1)$ there exists $t > 0$ such that $\nu_x(t) > 1 - r$ for all $x \in A$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. In connection with the function φ we consider the following properties [5]:

1. φ is monotonic increasing;
2. $\varphi(t) < t$ for all $t > 0$;
3. $\varphi(0) = 0$;
4. φ is continuous;
5. $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$;
6. $\sum_{n=0}^{\infty} \{\varphi^n(t)\}$ converges for all$t > 0$;
7. $t - \varphi(t) \to \infty$ as $t \to \infty$;
8. φ is subadditive.

Lemma 1.11 ([5]). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function then:

1. φ is monotonic increasing and $\varphi(t) < t$ for all $t > 0$ implies $\varphi(0) = 0$.
2. $\varphi(t) < t$ for all $t > 0$ and φ is continuous implies $\varphi(0) = 0$.
3. φ is monotonic increasing and $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$ then $\varphi(t) < t$ for all $t > 0$.

Definition 1.12 ([5], [11]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if φ is monotonic increasing and $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$.

Lemma 1.13 ([5], [11]).

1. Any comparison function φ satisfies $\varphi(0) = 0$;
2. Any subadditive comparison function φ is continuous;
3. If φ is a comparison function then, for any $k \in \mathbb{N}, \varphi^k$ is a comparison function;
4. If φ is a comparison function then, the function $s : \mathbb{R}_+ \to \mathbb{R}_+$ such that $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$ for $t \in \mathbb{R}_+$ satisfies that s is monotonic increasing and $s(0) = 0$.

Example 1.14 ([5][11]). Some examples for the function φ satisfying the properties mentioned in definition (1.11):

1. Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = kt$ for $k \in [0, 1]$ then φ satifies all the conditions in definition (1.11).
2. Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = \frac{t}{t+1}$ then φ is monotonic increasing,$\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$ and $t - \varphi(t) \to \infty$ as $t \to \infty$.

2. Contraction and $\varphi-$contraction mappings in Menger's PN spaces

Definition 2.1 ([12]). Let (X, ν, τ, T) be a Menger's PN space, a subset L of X of the form $\{x + ty; t \in \mathbb{R}_+\}$ where $x, y \in X$ and $y \neq 0$ is called a line.

Definition 2.2. Let (X, ν, τ, T) be a Menger's PN space,$f : X \to X$ is said to be a contraction on X if and only if there is a $k \in (0, 1)$ such that $\nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ for every $x, y \in X$ and $t > 0$.

Example 2.3. Let (X, ν, τ, T) be a Menger's PN space, and S be a subset of $L = \{x + \alpha y; \alpha \in \mathbb{R}_+\}$. Define $f : S \to X$ by $f(x + \alpha y) = (\frac{\alpha}{1+\alpha})y$ for $x, y \in X$, then f is a contraction on X.

Lemma 2.4. Let (X, ν, τ, T) be a Menger's PN space then, every contraction $f : X \to X$ is sequentially continuous.

Proof. Since f is a contraction on X, we have for $k \in (0, 1)$ $\nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$, for all $x, y \in X$ and $t > 0$.

Let (x_n) be a sequence in X such that $x_n \to x$ i.e., for all $t > 0$ and $\lambda \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have $\nu_{x_n-x}(t) > 1 - \lambda$.

As $x_n \in X$ for all n, we have $\nu_{fx_n - fx}(kt) \geq \nu_{x_n - x}(t) > 1 - \lambda$ for every $n > n_0$.

Hence $fx_n \to fx$. So f is sequentially continuous. □

Now we prove a unique fixed point theorem for contractive mappings in Menger's PN space.

Theorem 2.5. Let (X, ν, τ, T) be a Menger's probabilistic Banach space and E be a nonempty closed and probabilistic bounded subset of X. Let $f : E \to E$ be a contraction then f has a unique fixed point on X.

Proof. Since f is a contraction on X, for $k \in (0, 1), \nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ for all $x, y \in X$ and $t > 0$.

Let $x_0 \in X$. Construct a sequence $\{x_n\}$ depending on x_0. Let $x_1 = fx_0, x_2 = fx_1, x_3 = fx_2, ..., x_n = fx_{n-1}$ then $fx_0 = x_1, f^2x_0 = f(fx_0) = fx_1 = x_2$, $..., f^n x_0 = x_n$.

We have, $\nu_{f^2(x)-f^2(y)}(kt) = \nu_{f(fx)-f(fy)}(kt) \geq \nu_{fx-fy}(t) \geq \nu_{x-y}\left(\frac{t}{k}\right)$.

Similarly, $\nu_{f^3x-f^3y}(kt) \geq \nu_{x-y}\left(\frac{t}{k^2}\right)$.

Continuing like this we get

$$\nu_{f^n x - f^n y}(kt) \geq \nu_{x-y}\left(\frac{t}{k^{n-1}}\right).$$

We have E is probabilistically bounded then for each $r \in (0, 1)$ there exists $t > 0$ such that $\nu_x(t) > 1 - r$ for all $x \in E$.

We prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n > 0$ with $m > n$. Take $m = n + p$ then

$$\begin{aligned}
&\nu_{x_n - x_{n+p}}(kt) = \nu_{(x_n - x_{n+1})+(x_{n+1}-x_{n+p})}(kt) \\
&\geq \tau_T\{\nu_{x_n - x_{n+1}}\left(\frac{kt}{2}\right), \nu_{x_{n+1}-x_{n+p}}\left(\frac{kt}{2}\right)\} \\
&= \tau_T\{\nu_{f^n x_0 - f^n x_1}\left(\frac{kt}{2}\right), \nu_{(x_{n+1}-x_{n+2})+(x_{n+2}-x_{n+p})}\left(\frac{kt}{2}\right)\} \\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \tau_T\{\nu_{f^n x_1 - f^n x_2}\left(\frac{kt}{4}\right), \nu_{(x_{n+2}-x_{n+p})}\left(\frac{kt}{4}\right)\}\} \\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \tau_T\{\nu_{x_1 - x_2}\left(\frac{t}{4k^{n-1}}\right), \nu_{(x_{n+2}-x_{n+p})}\left(\frac{kt}{4}\right)\}\} \\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \nu_{x_1 - x_2}\left(\frac{t}{4k^{n-1}}\right), \tau_T\{\nu_{(x_{n+2}-x_{n+3})+(x_{n+3}-x_{n+p})}\left(\frac{kt}{4}\right)\}\} \\
&> \tau_T\{1 - r, 1 - r, ..., 1 - r\} = 1 - r.
\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in X. Therefore, $\{x_n\}$ converges to some point $x \in X$.

Since f is sequentially continuous, we have $\lim_{n\to\infty} fx_n = fx$ and $fx = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = x$. Hence f has a fixed point in X.

It remains to prove that such a fixed point is unique.

Let $y \in X$ with $y \neq x$ such that $fy = y$ then, $\nu_{x-y}(kt) = \nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ implies $\nu_{\frac{1}{k}(x-y)}(t) \geq \nu_{x-y}(t)$ implies $|\frac{1}{k}| \leq 1$ implies $k \geq 1$, a contradiction to $k \in (0,1)$. Hence the fixed point of f is unique. $\square$

Definition 2.6. Let (X, ν, τ, T) be a Menger's PN space then a mapping $f : X \to X$ is said to be a $\varphi-$contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\nu_{fx-fy}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c))$ for $0 < c < 1$, for all $x, y \in X$ and $t > 0$.

Example 2.7. Let (X, ν, τ, T) be a Menger's PN space. Define $f : X \to X$ by $f(t) = t$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = \frac{t}{c}$, for $0 < c < 1$ then φ is a comparison function and hence f is a $\varphi-$contraction on X.

Lemma 2.8. Let (X, ν, τ, T) be a Menger's PN space then, every $\varphi-$contraction $f : X \to X$ is sequentially continuous.

Proof. Let (x_n) be a sequence in X such that $x_n \to x$ in X i.e., for all $t > 0$ and $\lambda \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, we have $\nu_{x_n - x} > 1 - \lambda$.

Since f is a $\varphi-$contraction, we have, for $0 < c < 1$, for all $x, y \in X$ and $t > 0$

$$\nu_{fx_n - fx}(\varphi(t)) \geq \nu_{x_n - x}(\varphi(t/c)) > 1 - \lambda, \quad \text{for every} \quad n > n_0.$$

Hence f is sequentially continuous. $\square$

Theorem 2.9. Let (X, ν, τ, T) be a Menger's probabilistic Banach space and E be a nonempty closed and $\mathcal{D}-$ bounded subset of X. Let $f : E \to E$ be a $\varphi-$contraction then f has a unique fixed point on X.

Proof. Since f is a $\varphi-$contraction on X, then there exists a comparison function φ for $0 < c < 1$, for all $x, y \in X$ and $t > 0$

$$\nu_{fx-fy}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c)).$$

We have

$$\begin{aligned}
\nu_{f^2(x)-f^2(y)}(\varphi(t)) &= \nu_{f(fx)-f(fy)}(\varphi(t)) \\
&\geq \nu_{fx-fy}(\varphi(t/c)) \\
&\geq \nu_{x-y}\left(\varphi(t/c^2)\right).
\end{aligned}$$

Similarly, $\nu_{f^3x-f^3y}(\varphi(t)) \geq \nu_{x-y}\left(\varphi(t/c^3)\right)$. Continuing like this we get,

$$\nu_{f^nx-f^ny}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c^n)).$$

Let $x_0 \in X$. Construct a sequence $\{x_n\}$ depending on x_0. Let $x_1 = fx_0, x_2 = fx_1, x_3 = fx_2, ..., x_n = fx_{n-1}$ then $fx_0 = x_1, f^2x_0 = f(fx_0) = fx_1 = x_2$, $..., T^nx_0 = x_n$.

We prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n > 0$ with $m > n$. Take $m = n + p$ then

$$\begin{aligned}
\nu_{x_n - x_{n+p}}(\varphi(t)) &= \nu_{(x_n - x_{n+1}) + (x_{n+1} - x_{n+p})}(\varphi(t)) \\
&\geq \tau_T\{\nu_{x_n - x_{n+1}}(\varphi(t)), \nu_{x_{n+1} - x_{n+p}}(\varphi(t))\} \\
&= \tau_T\{\nu_{f^n x_0 - f^n x_1}(\varphi(t)), \nu_{(x_{n+1} - x_{n+2}) + (x_{n+2} - x_{n+p})}(\varphi(t))\} \\
&\geq \tau_T\{\nu_{x_n - x_1}(\varphi(t)), \tau_T\{\nu_{(x_{n+1} - x_{n+2})}(\varphi(t)), \nu_{(x_{n+2} - x_{n+p})}(\varphi(t))\}\} \\
&> \tau_T\{1 - r, 1 - r\} = 1 - r.
\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in X. Therefore, $\{x_n\}$ converges to some point $x \in X$.

Since f is sequentially continuous, we have $\lim_{n\to\infty} fx_n = fx$ and $fx = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = x$. Hence f has a fixed point in X.

Now, we have to prove that such a fixed point is unique.

Let $y \in X$ with $y \neq x$ such that $fy = y$ then, for $t > 0$,

$$\begin{aligned}
\nu_{x-y}(\varphi(t)) &= \nu_{fx-fy}(\varphi(t)) \\
&\geq \nu_{x-y}(\varphi(t/c)) \\
&\geq \nu_{fx-fy}\left(\varphi(t/c^2)\right) \\
&\cdots \\
&\geq \nu_{x-y}(\varphi(t/c^n)).
\end{aligned}$$

We have $\lim_{n\to\infty} \varphi(t/c^n) = \infty$ and as a consequence $\nu_{x-y}(\infty) = 1$ if and only if ν_x is in D^+. i.e., $\lim_{t\to\infty} \nu_x(t) = 1$. Then, $\nu_{x-y}(t) = 1$ for all $t > 0$ if and only if $\nu_{x-y} = \varepsilon_0$ if and only if $x = y$. This completes the proof. $\square$

Acknowledgement

The first author acknowledges MIT, Manipal Academy of Higher Education (Deemed to be University), India; the second author acknowledges Universidad de Almeria, Spain, the third author acknowledges Gyeongsang National University, South Korea, and the fourth author acknowledges Gurudev Arts and Science College, Kannur University, India, for their kind encouragement.

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Accepted: 23.02.2019

Criteria and geometric properties for bounded univalent functions in the unit disk

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Abstract. In this article, we establish new univalence criteria for normalized analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. Indeed, we prove for any $n \geq 2$ that the condition $|(f(z)/z)^{(n)}| \leq (n!/(n+1))(1 - \sum_{k=2}^{n} k|a_k|)$ is sufficient and sharp for f to be univalent in $\mathbb{U}$. The equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$, where $\sum_{k=2}^{n} k|a_k| = 1$. We investigate interesting geometric properties for such classes of functions. Namely, subordinations, inclusions, distortion and growth theorems, area estimate, starlikeness and convexity.

Keywords: univalent functions, univalence criteria, subordination, starlike functions, convex functions, area theorem, distortion theorem.

1. Introduction and preliminaries

It is well known that the condition $\mathrm{Re} f'(z) > 0$ is sufficient for analytic function f to be univalent (one-to-one) in any convex domain. In 1962, MacGregor [1] investigated such functions in the unit disk $\mathbb{U} = \{z : \ |z| < 1\}$, whenever f is normalized by $f(0) = f'(0) - 1 = 0$. In fact, the class of normalized analytic functions, denoted by $\mathcal{A}$, is analytically characterized by functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Afterwards, in [2, 3], the authors studied the subclass

$$F = \{f \in \mathcal{A} : \ |f'(z) - 1| < 1, \, z \in \mathbb{U}\}$$

of the class $\mathcal{R} = \{f \in \mathcal{A} : \mathrm{Re}\, f'(z) > 0, \, z \in \mathbb{U}\}$, for various geometric properties. A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

Also, $f \in \mathcal{A}$ is called convex in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

The classes of starlike functions and convex functions are denoted, respectively, by $\mathcal{S}^*$ and $\mathcal{K}$. A function f is said to be subordinate to g, written $f \prec g$, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$, for $z \in \mathbb{U}$. The problems of finding criteria and investigating geometric properties for univalent functions has been extensively studied by many authors, see for example [4-11].

In this article, for every $n \geq 2$, we introduce the classes

$$F_n = \left\{ f \in \mathcal{A} : \left| \left(\frac{f(z)}{z} \right)^{(n)} \right| \leq \frac{n!}{n+1} \left(1 - \sum_{k=2}^{n} k|a_k| \right), \quad \frac{f(z)}{z} \neq 0, \;\; z \in \mathbb{U} \right\}$$

of univalent functions in $\mathbb{U}$. Indeed, we prove that F_n is included in the class F, for every $n \geq 2$. As a motivation of our univalence criteria is that, for $n = 2$ and $f(z)/z \neq 0$ in $\mathbb{U}$, the condition

$$\sum_{k=3}^{\infty} (k-1)(k-2)|a_k| \leq \frac{2}{3}(1 - 2|a_2|) \tag{1.1}$$

is sufficient for $f \in \mathcal{A}$ to be in F_2 and hence it is univalent in $\mathbb{U}$. Also, the condition

$$\left| \left(\frac{f(z)}{z} \right)'' \right| \leq \frac{1}{5} \left(-7|a_2| + \sqrt{10 - |a_2|^2} \right), \quad (z \in \mathbb{U})$$

is sufficient for $f \in \mathcal{A}$ to be starlike in $\mathbb{U}$, where equality attained for $f(z) = z + a_2 z^2$, $(|a_2| = 1/\sqrt{5})$. Moreover, the condition

$$\left| \left(\frac{f(z)}{z} \right)'' \right| \leq \frac{2}{11}(1 - 4|a_2|), \quad (z \in \mathbb{U})$$

is sufficient for $f \in \mathcal{A}$ to be convex in $\mathbb{U}$, where equality attained for $f(z) = z + a_2 z^2$, $(|a_2| = 1/4)$. Further, we investigate several geometric properties for the classes F_n. Mainly, inclusions, subordinations, distortion and growth theorems, area estimate, starlikeness and convexity. As consequences from the subordination results, we obtain univalence criteria in terms of bounded derivatives of f.

The following lemmas are needed in the sequel.

Lemma 1.1 ([12]). *If $\omega(z)$ is analytic in $\mathbb{U}$ and $|\omega(z)| \leq 1$ in $\mathbb{U}$, then for each $m \geq 1$, the function $\Phi_m(z)$ defined by*

$$\Phi_m(z) = \int_0^z m u^{m-1} \omega(u) du = z^m \int_0^1 m t^{m-1} \omega(tz) dt = z^m \Psi_m(z)$$

is clearly analytic in $\mathbb{U}$ and moreover, $\Psi_m(z)$ is analytic in $\mathbb{U}$ such that $|\Psi_m(z)| \leq 1$ in $\mathbb{U}$.

Lemma 1.2 ([13]). *Let g be a convex function in $\mathbb{U}$, and let $h(z) = g(z) + k\alpha z g'(z)$ for $z \in \mathbb{U}$, where $\alpha > 0$ and k is a positive integer. If $p(z) = g(0) + p_k z^k + p_{k+1} z^{k+1} + \cdots$, $z \in \mathbb{U}$, is holomorphic in $\mathbb{U}$ and $p(z) + \alpha z p'(z) \prec h(z)$, $z \in \mathbb{U}$, then $p(z) \prec g(z)$, $z \in \mathbb{U}$, and this result is sharp.*

Lemma 1.3 ([14, p.7]). *For analytic functions $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in $\mathbb{U}$ with $|g(z)| \leq 1$, we have*

$$\sum_{k=0}^{\infty} |b_k|^2 \leq 1.$$

Lemma 1.4. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then, for every $n \geq 2$, we have*

$$f^{(n)}(z) = z\left(\frac{f(z)}{z}\right)^{(n)} + n\left(\frac{f(z)}{z}\right)^{(n-1)}. \tag{1.2}$$

Proof. From the following expansions

$$f^{(n)}(z) = n!\, a_n + \sum_{k=n+1}^{\infty} \frac{k!}{(k-n)!} a_k\, z^{k-n},$$

$$\left(\frac{f(z)}{z}\right)^{(n)} = \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n-1)!} a_k\, z^{k-n-1},$$

and

$$\left(\frac{f(z)}{z}\right)^{(n-1)} = (n-1)!\, a_n + \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n)!} a_k\, z^{k-n}.$$

It can be easily verified that (1.2) holds. □

2. Univalence and inclusions

Let us start by proving that $F_n \subseteq F$, for every $n \geq 2$.

Theorem 2.1 (Univalence Criteria). *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in $\mathbb{U}$ and let*

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \leq \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right), \quad (z \in \mathbb{U}). \tag{2.1}$$

for some $n \geq 2$. Then $f \in F$ and hence it is univalent in $\mathbb{U}$. The inequality (2.1) is sharp, where equality attained for functions of the form

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad \textit{where} \ \sum_{k=2}^{n} k|a_k| = 1 \tag{2.2}$$

and for $f(z) = z \pm (1/(n+1)) z^{n+1}$.

Proof. Let

$$\beta_n = \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right).$$

Then, for $n = 2$, condition (2.1) is equivalent to

$$\left|\left(\frac{f(z)}{z}\right)''\right| \leq \beta_2$$

and

$$\left(\frac{f(z)}{z}\right)'' = \beta_2\, \phi_1(z), \tag{2.3}$$

where ϕ_1 is analytic in $\mathbb{U}$ and $|\phi_1(z)| \leq 1$ in $\mathbb{U}$. Now, by the virtue of

$$\left(\frac{f(z)}{z}\right)^{(k)}|_{z=0} = k!\, a_{k+1}$$

and using Lemma 1.1, then integrating (2.3) from 0 to z yields

$$\left(\frac{f(z)}{z}\right)' = a_2 + \beta_2 z \int_0^1 \phi_1(tz)\, dt := a_2 + \beta_2 z \phi_2(z). \tag{2.4}$$

The relation (2.4), by integration and multiplying by z, gives

$$f(z) - z = a_2 z^2 + \beta_2 z \int_0^z u\, \phi_2(u)\, du. \tag{2.5}$$

By differentiating both sides of (2.5), we have

$$\begin{aligned} f'(z) - 1 &= 2a_2 z + \beta_2 \left(z^2 \phi_2(z) + \int_0^z u\, \phi_2(u)\, du\right) \\ &= 2a_2 z + \beta_2 \left(z^2 \phi_2(z) + \frac{1}{2} z^2 \int_0^1 2t\, \phi_2(tz)\, dt\right). \end{aligned}$$

Therefore,

$$\left|f'(z) - 1\right| < 2|a_2| + \frac{3}{2}\beta_2 = 1,$$

and hence $f \in F$. For $n = 3$, we have

$$\left(\frac{f(z)}{z}\right)''' = \beta_3 \psi_1(z), \tag{2.6}$$

where ψ_1 is analytic in $\mathbb{U}$ and $|\psi_1(z)| \leq 1$ in $\mathbb{U}$. By integration (2.6) from 0 to z, we have

$$\left(\frac{f(z)}{z}\right)'' = 2a_3 + \beta_3 z \int_0^1 \psi_1(tz)\, dt := 2a_3 + \beta_3 z\, \psi_2(z). \tag{2.7}$$

Again by integration (2.7), we get

$$\left(\frac{f(z)}{z}\right)' = a_2 + 2a_3 z + \beta_3 \frac{z^2}{2}\int_0^1 2t\psi_2(tz)\, dt := a_2 + 2a_3 z + \frac{\beta_3}{2} z^2 \psi_3(z). \tag{2.8}$$

Integration (2.8) and then multiplying by z, gives

$$f(z) - z = a_2 z^2 + a_3 z^3 + \frac{\beta_3}{2} z \int_0^z u^2 \psi_3(u)\, du. \tag{2.9}$$

By differentiating both sides of (2.9) and using Lemma 1.1, we have

$$\begin{aligned} f'(z) - 1 &= 2a_2 z + 3a_3 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \int_0^z u^2\psi_3(u)\, du\right) \\ &= 2a_2 z + 3a_3 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \frac{1}{3} z^3 \int_0^1 3t^2 \psi_3(tz)\, dt\right). \end{aligned}$$

Therefore,

$$\left|f'(z) - 1\right| < 2|a_2| + 3|a_3| + \frac{2}{3}\beta_3 = 1,$$

and hence $f \in F$. In general, if

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \le \beta_n, \quad (z \in \mathbb{U}),$$

then

$$\begin{aligned} f'(z) - 1 = &\sum_{k=2}^{n} k a_k z^{k-1} \\ &+ \frac{1}{(n-1)!}\beta_n \left(z^n \varphi_n(z) + \frac{z^n}{n}\int_0^1 n t^{n-1} \varphi_n(tz)\, dt\right), \end{aligned} \tag{2.10}$$

where φ_n is analytic in $\mathbb{U}$ and $|\varphi_n(z)| \le 1$ in $\mathbb{U}$ by Lemma 1.1. Therefore,

$$\left|f'(z) - 1\right| < \sum_{k=2}^{n} k|a_k| + \frac{1}{(n-1)!}\left(1 + \frac{1}{n}\right)\beta_n = 1,$$

and hence $f \in F$. To show that the result is sharp for $n \ge 2$, we consider

$$f_\epsilon(z) = z + \frac{1+\epsilon}{n+1} z^{n+1}, \quad (\epsilon \ge 0). \tag{2.11}$$

Clearly, $a_k = 0$ for $2 \le k \le n$. A computation shows that

$$\left(\frac{f_\epsilon(z)}{z}\right)^{(k)} = \frac{(1+\epsilon)n(n-1)\cdots(n-k+1)}{n+1} z^{n-k}, \ (2 \le k \le n).$$

Therefore,

$$\left(\frac{f_\epsilon(z)}{z}\right)^{(n)} = \frac{n!}{n+1}(1+\epsilon). \tag{2.12}$$

Letting $\epsilon = 0$ in (2.11) and (2.12) implies that $f_0(z)$ satisfies the equality in (2.1). However, for every $\epsilon > 0$ and $n \geq 2$, we have

$$f'_\epsilon\left(\left(\frac{-1}{1+\epsilon}\right)^{\frac{1}{n}}\right) = 0.$$

Hence f_ϵ is not univalent in $\mathbb{U}$, for $\epsilon > 0$ and the result is sharp. It can be easily check that functions of the form (2.2) are also satisfying the equality in (2.1), where both sides will be zero. This completes the proof of Theorem 2.1. $\square$

Setting $n = 2$ in Theorem 2.1 implies, for $f(z)/z \neq 0$ in $\mathbb{U}$, that

$$\left|\left(\frac{f(z)}{z}\right)''\right| \leq \frac{2}{3}(1 - 2|a_2|), \quad (z \in \mathbb{U})$$

is sufficient condition for f to be in F_2. Now, expanding $(f(z)/z)^{(n)}$ as a Taylor series gives the following corollary.

Corollary 2.2. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in $\mathbb{U}$ and let*

$$\sum_{k=2}^{n} k|a_k| + \frac{n+1}{n!} \sum_{k=n+1}^{\infty} (k-n)_n\, |a_k| \leq 1, \quad (z \in \mathbb{U}), \tag{2.13}$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)...(x+n-1).$$

Then $f \in F_n$.

Proof. Since

$$\left(\frac{f(z)}{z}\right)^{(n)} = \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n-1)!} a_k z^{k-n-1}.$$

Then, we easily conclude that (2.13) is sufficient condition for f to be in F_n. $\square$

As a consequent result from Corollary 2.2, we observe that condition (1.1) is sufficient for f to be in F_2. Note that, $f(z) = z \pm (1/n)z^n \in F_{n-1} \cap F_n$. Indeed, we have the following result.

Theorem 2.3 (Inclusions)**.** *For $n \geq 3$, $F_n \subseteq F_{n-1}$ in the disk $|z| \leq (n+1)/n^2$. Moreover, if $f \in F_n$ with $\sum_{k=2}^{n} k\,|a_k(f)| = 1$, then $f \in F_{n-1}$ in the unit disk $\mathbb{U}$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \in F_n$. Then

$$\left(\frac{f(z)}{z}\right)^{(n)} = \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)\mu_1(z),$$

where μ_1 is analytic in $\mathbb{U}$ and $|\mu_1(z)| \le 1$ in $\mathbb{U}$. Therefore, by integration from 0 to z, we observe that

$$\left(\frac{f(z)}{z}\right)^{(n-1)} = (n-1)!\, a_n + \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right) z\,\mu_2(z),$$

where μ_2 is analytic in $\mathbb{U}$ and $|\mu_2(z)| \le 1$ in $\mathbb{U}$. Now, if $|z| \le (n+1)/n^2$, then a simple computation gives

$$\begin{aligned}\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right| &\le (n-1)!|a_n| + \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)|z| \\ &\le \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n-1} k\,|a_k|\right).\end{aligned}$$

This completes the proof of the first part. To prove the second part, let $f \in F_n$ with $\sum_{k=2}^{n} k\,|a_k| = 1$. Then $(f(z)/z)^{(n)} = 0$ and

$$\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right| = (n-1)!|a_n| = \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n-1} k\,|a_k|\right).$$

Therefore $f \in F_{n-1}$. □

We note that F_n is not included in F_{n-1}, consider the following example

Example 2.4. The function

$$f(z) = z + \frac{1}{4}z^4$$

belongs to F_3. However, it does not belong to F_2.

3. Subordination and an area theorem

Theorem 3.1 (Subordination)**.** *Let $f \in \mathcal{A}$ be satisfying*

$$f^{(n)}(z) \prec n!\, a_n + \beta(n+1)z, \quad (z \in \mathbb{U}). \tag{3.1}$$

Then,

$$\left(\frac{f(z)}{z}\right)^{(n-1)} \prec (n-1)!\, a_n + \beta z, \quad (z \in \mathbb{U}), \tag{3.2}$$

for every $n \ge 3$ and $\beta \in \mathbb{C}$. The result is sharp.

Proof. In view of Lemma 1.4, the subordination (3.1) can be written as

$$\frac{z}{n}\left(\frac{f(z)}{z}\right)^{(n)} + \left(\frac{f(z)}{z}\right)^{(n-1)} \prec (n-1)!\,a_n + \frac{n+1}{n}\beta z, \quad (z \in \mathbb{U}).$$

Applying Lemma 1.2 for $p(z) = (f(z)/z)^{(n-1)}$, $g(z) = (n-1)!\,a_n + \beta z$, $k = 1$ and $\alpha = 1/n$ to the above subordination yields the desired. □

Corollary 3.2. *If $f \in \mathcal{A}$ satisfies*

$$|f^{(n)}(z) - n!\,a_n| < \frac{(n+1)!}{n^2}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right), \quad (z \in \mathbb{U}), \tag{3.3}$$

for some $n \geq 3$, then $f \in F_{n-1}$.

Proof. Set

$$\beta = \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right).$$

Then, in view of (3.3), $f^{(n)}(z) - n!\,a_n \prec \beta(n+1)z$ and hence (3.1) holds. Therefore, from (3.2), we obtain

$$\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right| < \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n-1} k\,|a_k|\right).$$

Hence, $f \in F_{n-1}$. □

For the case $n = 2$, we have the following

Theorem 3.3. *If $f \in \mathcal{A}$ satisfies*

$$f''(z) - 2\,a_2 \prec \frac{3}{2}\,(1 - 2|a_2|)\,z, \tag{3.4}$$

or equivalently, $|f''(z) - 2\,a_2| < \frac{3}{2}\,(1 - 2|a_2|)$, $(z \in \mathbb{U})$, then $f \in F$. In particular, if $a_2 = 0$ and $|f''(z)| < 3/2$ then $f \in F$ and hence it is univalent in $\mathbb{U}$.

Proof. In view of (3.4), we have $f''(z) = 2\,a_2 + \frac{3}{2}\,(1 - 2|a_2|)\,w(z)$, where w is analytic function in $\mathbb{U}$ and $|w(z)| \leq |z|$ in $\mathbb{U}$ by Schwarz lemma. Therefore,

$$\begin{aligned}
|f'(z) - 1| &= \left|\int_0^z f''(s)\,ds\right| \\
&\leq |z|\int_0^1 |f''(tz)|\,dt \\
&\leq |z|\int_0^1 \left(2|a_2| + \frac{3}{2}(1 - 2|a_2|)|z|t\right)\,dt \\
&\leq 2|a_2||z| + \frac{3}{4}(1 - 2|a_2|)|z|^2 \\
&< \frac{3}{4} + \frac{1}{2}|a_2| \leq 1.
\end{aligned}$$

Thus $f \in F$. □

Next, we study the area covered by functions in F_n. Applying Lemma 1.3 to $(f(z)/z)^{(n)}$, where $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \in F_n$, implies that

$$\sum_{k=n+1}^{\infty} \left(\frac{(k-n)_n\,(n+1)}{n!\left(1-\sum_{j=2}^{n} j\,|a_j|\right)} \right)^2 |a_k|^2 \leq 1.$$

Using this inequality, we may derive the following theorem.

Theorem 3.4 (An Area Theorem)**.** *The area of the image of* $\mathbb{U}$ *under each function in* F_n *satisfies*

$$A \leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)^2 \right\}.$$

Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$*, where* $\sum_{k=2}^{n} k|a_k| = 1$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$ be in the class F_n. It is well known, (see [2]), that the area of the image of $\mathbb{U}$ under f is given by

$$A = \pi \left\{ 1 + \sum_{k=2}^{\infty} k|a_k|^2 \right\}.$$

By making use of Lemma 1.3 and Corollary 2.2, we obtain

$$\begin{aligned}
A &= \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \sum_{k=n+1}^{\infty} k|a_k|^2 \right\} \\
&\leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{\left(1-\sum_{k=2}^{n} k\,|a_k|\right)^2}{n+1} \sum_{k=n+1}^{\infty} \left(\frac{(k-n)_n\,(n+1)}{n!\left(1-\sum_{j=2}^{n} j\,|a_j|\right)} \right)^2 |a_k|^2 \right\} \\
&\leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)^2 \right\}.
\end{aligned}$$

This completes the proof. □

We may observe, for $f \in F_2$, that

$$A \leq \frac{4\pi}{3}\left(1 - |a_2| + \frac{10}{4}|a_2|^2\right)$$

and the maximum value of A is $(3/2)\pi$ which attained at $|a_2| = 1/2$ for the function $f(z) = z + a_2 z^2$, $(z \in \mathbb{U},\ |a_2| = 1/2)$. In general, the maximum value of A for functions in F_n is $\pi(1 + (1/n))$ which attained for the function $f(z) = z + a_n z^n$, $(|a_n| = 1/n)$.

4. Distortion, starlikeness and convexity

The following theorem introduces bounds for functions in F_n and for their derivatives. It will be useful for investigating starlike and convex functions in F_n.

Theorem 4.1 (Distortion and Growth). *Let $f(z) = z+\sum_{k=2}^{\infty} a_k z^k \in F_n$. Then, for $|z| = r < 1$ and $g(r) = \sum_{k=2}^{n} |a_k| r^k + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right) r^{n+1}$, we have*

$$|f(z) - z| \le g(r); \tag{4.1}$$

$$\left|f'(z) - 1\right| \le g'(r); \tag{4.2}$$

$$r - g(r) \le |f(z)| \le r + g(r); \tag{4.3}$$

$$1 - g'(r) \le |f'(z)| \le 1 + g'(r). \tag{4.4}$$

Equalities in (4.1), (4.2), (4.3) and (4.4) are attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$, where $\sum_{k=2}^{n} k|a_k| = 1$.

Proof. In view of relation (2.10), we have

$$f'(z) - 1 := \sum_{k=2}^{n} k a_k z^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right) z^n\, W(z),$$

where

$$W(z) = \frac{n}{n+1}\left(\varphi_n(z) + \frac{1}{n}\int_0^1 n t^{n-1} \varphi_n(tz)\, dt\right)$$

is analytic in $\mathbb{U}$ and $|W(z)| \le 1$ in $\mathbb{U}$. Therefore,

$$|f'(z) - 1| \le \sum_{k=2}^{n} k|a_k||z|^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right) |z|^n$$

and

$$\begin{aligned}
|f(z) - z| &= \left|\int_0^z (f'(s) - 1)\,\mathrm{d}s\right| \\
&\le |z| \int_0^1 |f'(tz) - 1|\,\mathrm{d}t \\
&\le |z| \int_0^1 \left\{\sum_{k=2}^{n} k|a_k||z|^{k-1} t^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right) |z|^n t^n\right\} \mathrm{d}t \\
&= \sum_{k=2}^{n} |a_k||z|^k + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right) |z|^{n+1}.
\end{aligned}$$

The estimates (4.3) and (4.4) are immediate consequences from (4.1) and (4.2), respectively. □

For functions $f \in F_n$, one can write

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \le \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k\,|a_k|\right)\alpha,$$

for some $0 \le \alpha \le 1$. In the next results, we find values for α such that $f \in F_n$ is starlike or convex in the unit disk $\mathbb{U}$.

Theorem 4.2 (Starlikeness)**.** *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in $\mathbb{U}$ be satisfying*

$$\begin{aligned}\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| &\le \frac{n!}{2(n^2+2n+2)}\left\{-(2n+3)\sum_{k=2}^{n} k\,|a_k|\right.\\ &+ \left.\sqrt{4(n^2+2n+2)-(n-1)^2\left(\sum_{k=2}^{n} k\,|a_k|\right)^2}\right\}\end{aligned}$$

for some $n \ge 2$ and for every $z \in \mathbb{U}$. Then $f \in \mathcal{S}^*$. Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$ such that $\sum_{k=2}^{n} k\,|a_k| = \sqrt{4/5}$.

Proof. Let

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \le \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k\,|a_k|\right)\alpha, \tag{4.5}$$

for some $\alpha \ge 0$. In view of relation (2.10) and from (4.2) and (4.5), we observe that

$$\begin{aligned}|f'(z)-1| &\le \sum_{k=2}^{n} k|a_k||z|^{k-1} + \alpha\left(1-\sum_{k=2}^{n} k|a_k|\right)|z|^n\\ &< \sum_{k=2}^{n} k|a_k| + \alpha\left(1-\sum_{k=2}^{n} k|a_k|\right).\end{aligned}$$

Also, from (2.10), (4.1) and (4.5), we obtain

$$\begin{aligned}\left|\frac{f(z)}{z}-1\right| &\le \sum_{k=2}^{n} |a_k||z|^{k-1} + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right)|z|^n\\ &< \frac{1}{2}\sum_{k=2}^{n} k|a_k| + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right).\end{aligned}$$

Now,

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \left|\arg f'(z)\right| + \left|\arg\frac{f(z)}{z}\right|\\ &< \arcsin\left\{\sum_{k=2}^{n}k|a_k| + \alpha\left(1-\sum_{k=2}^{n}k|a_k|\right)\right\}\\ &+ \arcsin\left\{\frac{1}{2}\sum_{k=2}^{n}k|a_k| + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n}k|a_k|\right)\right\}.\end{aligned}$$

The last bound is equal to $\pi/2$ if

$$\left\{\sum_{k=2}^{n}k|a_k|+\alpha\left(1-\sum_{k=2}^{n}k|a_k|\right)\right\}^2+\left\{\frac{1}{2}\sum_{k=2}^{n}k|a_k|+\frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n}k|a_k|\right)\right\}^2=1.$$

A computation shows that the root α of the above equation is given by

$$\begin{aligned}\alpha &= \frac{n+1}{2(n^2+2n+2)\left(1-\sum_{k=2}^{n}k\,|a_k|\right)}\left\{-(2n+3)\sum_{k=2}^{n}k\,|a_k|\right.\\ &+ \left.\sqrt{4(n^2+2n+2)-(n-1)^2\left(\sum_{k=2}^{n}k\,|a_k|\right)^2}\right\}\end{aligned}$$

and $\alpha \geq 0$ if

$$\sum_{k=2}^{n}k\,|a_k| \leq \sqrt{\frac{4}{5}}.$$

Therefore, for the defined root α, we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}$$

and hence $f \in \mathcal{S}^*$. Finally, (4.5) is equivalent to the assumption condition for the defined root α and this completes the proof of Theorem 4.2. $\square$

For $n = 2$, the above result is reduced to

Corollary 4.3. *Any function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with $f(z)/z \neq 0$ in $\mathbb{U}$, satisfies*

$$\left|\left(\frac{f(z)}{z}\right)''\right| \leq \frac{1}{5}\left(-7|a_2| + \sqrt{10-|a_2|^2}\right), \quad (z \in \mathbb{U})$$

is starlike in $\mathbb{U}$. Equality attained for the function $f(z) = z + a_2\, z^2$, $(|a_2| = 1/\sqrt{5})$.

Theorem 4.4 (Convexity)**.** *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with $f(z)/z \neq 0$ in $\mathbb{U}$, be satisfying*

$$\left| \left(\frac{f(z)}{z} \right)^{(n)} \right| \leq \frac{n!}{2n^2+n+1} \left(1 - \sum_{k=2}^{n} k^2 |a_k| \right), \quad (z \in \mathbb{U}). \tag{4.6}$$

Then $f \in \mathcal{K}$. Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$ such that $\sum_{k=2}^{n} k^2 |a_k| = 1$.

Proof. Let

$$\left| \left(\frac{f(z)}{z} \right)^{(n)} \right| \leq \alpha \beta_n,$$

for some $\alpha \geq 0$ and

$$\beta_n = \frac{n!}{n+1} \left(1 - \sum_{k=2}^{n} k |a_k| \right).$$

In view of relation (2.10), we have

$$f'(z) = 1 + \sum_{k=2}^{n} k a_k z^{k-1} + \frac{\alpha \beta_n}{(n-1)!} \left(z^n \varphi_n(z) + \int_0^z u^{n-1} \varphi_n(u)\, \mathrm{d}u \right).$$

Differentiating the previous relation gives

$$f''(z) = \sum_{k=2}^{n} k(k-1) a_k z^{k-2} + \frac{z^{n-1}}{(n-1)!} \alpha \beta_n \left[(n-1)\varphi_{n-1}(z) + (n+1)\varphi_n(z) \right]$$

where $z^n \varphi_n'(z) = (n-1) z^{n-1} (\varphi_{n-1}(z) - \varphi_n(z))$ by Lemma 1.1. Hence, from the above relations and estimate (4.4), we obtain

$$\left| \frac{z f''(z)}{f'(z)} \right| < \frac{\sum_{k=2}^{n} k(k-1)|a_k| + \frac{2n}{(n-1)!} \alpha \beta_n}{1 - \sum_{k=2}^{n} k a_k - \frac{n+1}{n!} \alpha \beta_n}.$$

A mild computation shows that the last bound is less than or equal to 1 if

$$\alpha \leq \frac{(n+1)\left(1 - \sum_{k=2}^{n} k^2 |a_k|\right)}{(2n^2+n+1)\left(1 - \sum_{k=2}^{n} k |a_k|\right)}.$$

This yields

$$\alpha \beta_n \leq \frac{n!}{2n^2+n+1} \left(1 - \sum_{k=2}^{n} k^2 |a_k| \right).$$

Thus, condition (4.6) implies that f is convex in $\mathbb{U}$. □

Theorem 4.4, when $n = 2$, is reduced to

Corollary 4.5. *Any function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with $f(z)/z \neq 0$ in $\mathbb{U}$, satisfies*

$$\left| \left(\frac{f(z)}{z} \right)'' \right| \leq \frac{2}{11} (1 - 4|a_2|), \quad (z \in \mathbb{U})$$

is convex in $\mathbb{U}$. Equality attained for $f(z) = z + a_2 z^2$, ($|a_2| = 1/4$).

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Accepted: 19.02.2019

Studying the solutions of the delay Sturm Liouville problems

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Abstract. In this paper, the delay Sturm Liouville problems are introduced with the sufficient and necessary conditions, where the solutions of these problems are studied. Besides, definitions, remarks, examples, theorems, and corollaries are submitted to illustrate the delay Sturm Liouville problems properties. In addition, the inverse of the DSLP are shown in section two. Furthermore, the applications of the results in the second section are given in the third section.

Keywords: eventually positive, eventually negative, oscillatory

Introduction

Delay differential equations appear in a large number of fields in science, for instance in biology, mechanics, economic, mathematics, etc. Moreover, the solutions of the delay differential equations are studied by many authors like B. Mehmet, B. Azad, S. Erdogan [1] and K. Gopalsamy [2]. Besides, G. Ladas and I. P. Stavroulakis [3]. Note that, the concepts of eventually positive and eventually negative solutions are introduced by Peiguang W. [4], while the definition of oscillatory solution is given by P. Gimenes Luciene [5].

As well as, the Sturm Liouville Problems are also studied by numerous authors, for instance, they are studied by S.A. Buterin and two others in [6], besides B. Erdal, O. Ramazan [7]. Where M. Seyfollah [8], P. Milenko, V. Vladimir, M. Olivera [9], and C. Elmir, P. Milenko [10] are discussed the inverse SturmLiouville problems.

Studying the behavior of solutions of the delay Sturm Liouville Problems is interested in this work, where the delay Sturm Liouville Problems in this paper are denoted by DSLP. Add to that, the DSLP formulas are introduced with the sufficient and necessary conditions. Furthermore, definitions, remarks, examples, theorems, and corollaries are submitted to illustrate the delay Sturm Liouville problems properties. While the inverse of the DSLP are shown in section two. Also, the results in the second section are discussed on a form of applications in the third section, where they are shown as in examples form.

The Delay Sturm Liouville Problems (DSLP)

In this section, the DSLP as well as the sufficient and necessary conditions are introduced. Where definitions, remarks, examples, theorems, and corollaries are submitted to illustrate DSLP properties. Besides, the inverse of the DSLP is given.

Definition 1. *The DSLP are the following equation and the two inequalities with the boundary conditions, in addition to the sufficient and necessary conditions below*

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) = 0, \tag{1}$$

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) \leqslant 0, \tag{2}$$

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) \geqslant 0. \tag{3}$$

Where $r'(t) > 0, \alpha(t) \geqslant 0, \rho(t) > 0$ *are continuous functions on some interval* $a \leqslant t \leqslant b$, λ *is a positive parameter, and* τ *is a positive constant. The boundary conditions are*

$$a_1y(a) - a_2y'(a) = 0, \quad b_1y(b) - b_2y'(b) = 0 \tag{4}$$

at the end points of the interval, and require that at least on coefficient in each equation be nonzero. Besides, the necessary and sufficient conditions are listed in theorem 1 below under which the following are satisfied:

- *Equation 1 has oscillatory solutions only;*
- *Inequality 2 has eventually negative solutions only;*
- *Inequality 3 has eventually positive solutions only.*

Remark 1. Notice that, the previous results are because of a lateness argument, which they do not exist when $\tau = 0$, as the following example shows

Example 1. Let the DSLP

$$y''(t) - (1+\lambda^2)y(t) \geqslant 0.$$

Then it has a negative solution $y(t) = -e^{-nt}, n = 1, 2, \ldots$ such that $n < \lambda$

Theorem 1. *Suppose that the DSLP 2 exists with* $r'(s) = 1$, *moreover*

$$\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > -\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \alpha(s)ds \tag{5}$$

and

$$\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > 0. \tag{6}$$

Then 2 has eventually negative solutions only.

Proof. Let $y(t)$ be a solution to 2. To show that $y(t)$ is eventually positive, which leads to a contradiction. So,

$$y(t) > 0, \quad t > t_0 \quad \Rightarrow \quad y(t-\tau) > 0, \qquad\qquad t > t_0 + \tau.$$

Also from 2,

$$y''(t) < 0, \quad t > t_0 + \tau \quad \Rightarrow \quad y(t) < y(t-\tau), \quad t > t_0 + 2\tau.$$

Now, dividing both sides of 2 by $y(t)$ and get the following

$$\frac{[r(t)y'(t)]'}{y(t)} + \alpha(t) + \lambda\rho(t)\frac{y(t-\tau)}{y(t)} \leqslant 0, t > t_0 + 2\tau,$$

$$(7) \quad \int_{t-\tau}^{t} \frac{[r(s)y'(s)]'}{y(s)} ds + \int_{t-\tau}^{t} \alpha(s) ds + \frac{y(t-\tau)}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0, t > t_0 + 3\tau.$$

But $r'(s) = 1$ by assumption, besides to solve the first integral from the left of 7 consider

$$y(s) = \tan\theta, \quad y'(s) = \sec^2\theta, \quad y''(s) = 2\sec\theta\sec\theta\tan\theta.$$

The above compensation is using as follows

$$\int_{t-\tau}^{t} \frac{[r(s)y'(s)]'}{y(s)} ds = \int_{t-\tau}^{t} \frac{2\sec\theta\sec\theta\tan\theta}{\tan\theta} d\theta = 2y(t) - 2y(t-\tau).$$

Now, substitute it in 7

$$2y(t) - 2y(t-\tau) + \int_{t-\tau}^{t} \alpha(s) ds + \frac{y(t-\tau)}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0, \qquad t > t_0 + 3\tau.$$

Dividing the previous inequality by $y(t-\tau)$, then $\forall t > t_0 + 3\tau$

$$2\frac{y(t)}{y(t-\tau)} - 2 + \frac{1}{y(t-\tau)} \int_{t-\tau}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0.$$

So,

$$\frac{1}{y(t-\tau)} \int_{t-\tau)}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 2 - 2\frac{y(t)}{y(t-\tau)}.$$

Furthermore, since $\frac{y(t)}{y(t-\tau)} < 1$, then

$$\frac{1}{y(t-\tau)} \int_{t-\tau)}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds < 0.$$

Hence,

$$\int_{t-\tau}^{t} \lambda\rho(s) ds < -\frac{y(t)}{y(t-\tau)} \int_{t-\tau}^{t} \alpha(s) ds,$$

and again since $\frac{y(t)}{y(t-\tau)} < 1$, then

$$\int_{t-\tau}^{t} \lambda\rho(s)ds < -\int_{t-\tau}^{t} \alpha(s)ds.$$

Here, take the limit inferiors on both sides of the preceding inequalities, that is caused to get the following result

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds < -\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \alpha(s)ds,$$

which contradicts the hypothesis 5. Presently you can contradict 6 as follows

Take the integrals in 7 from $t-\frac{\tau}{2}$ to t for all $t > t_0 + \frac{\tau}{2}$ and follow the same way of steps in the current proof to get

$$\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \lambda\rho(s)ds < -\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \alpha(s)ds.$$

But

$$-\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \alpha(s)ds < 0.$$

That leads to

$$\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \lambda\rho(s)ds < 0.$$

As you see, it contradicts 6. □

Theorem 2. *Suppose that the DSLP 3 exists and*

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > 0.$$

Then, 3 has eventually positive solutions only.

Proof. Firstly assume that $y(t)$ is a solution to 3. To prove that $-y(t)$ is eventually negative, which leads to a contradiction. Then

$$-y(t) < 0, \qquad t > t_0.$$

By multiplying both sides by -1, so the results will be

$$y(t) > 0, \qquad t > t_0 \quad \Rightarrow \quad y(t-\tau) > 0, \qquad t > t_0 + \tau$$

3 gives

$$y''(t) > 0, \quad t > t_0 + \tau \quad \Rightarrow \quad y(t) < y(t-\tau), \quad t > t_0 + 2\tau.$$

And continue the proof by the similar steps as in a proof of theorem 1 to get a contradiction. □

Corollary 1. *Consider that the DSLP 1 exists and*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \lambda\rho(s)ds > 0.$$

Then 1 has oscillatory solutions only.

Proof. Remember that a solution is oscillatory, if it is neither eventually nor negative. Hence the proof is similar to the steps of a proof of the theorem 1 and theorem 2 by assuming the converse to get a contradiction. □

Definition 2. *Let $r'(t) > 0, \alpha(t) \geqslant 0, \rho(t) > 0$ be constants on some interval $a \leqslant t \leqslant b$, λ is a positive parameter, and τ is a positive constant.Then the DSLP becomes*

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) = 0, \tag{8}$$

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) \leqslant 0, \tag{9}$$

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) \geqslant 0. \tag{10}$$

The boundary conditions are

$$a_1 y(a) - a_2 y'(a) = 0, \quad b_1 y(b) - b_2 y'(b) = 0.$$

Where the conditions 5 and 6 are reduced to

$$\lambda\rho\tau > \alpha, \qquad \alpha \geqslant 0. \tag{11}$$

Corollary 2. *The necessary and sufficient condition is $\lambda\rho\tau > \alpha, \alpha \geqslant 0$, for which the statements below hold:*

- *Equation 8 has oscillatory solutions only;*
- *Inequality 9 has eventually negative solutions only;*
- *Inequality 10 has eventually positive solutions only.*

Theorem 3. *Suppose that the inverse of the DSLP as follows*

$$[r(t)y'(t)]' - \alpha(t)y(t) - \lambda^4\rho(t)y(t-2n\tau) = 0,$$

where $r(t) = 1, \alpha(t) \geqslant 0, \rho(t) = 1, n = 1, 2, \ldots$ are continuous functions on some interval $a \leqslant t \leqslant b$. Where λ is a positive parameter, and τ is a positive constant with the boundary condition 4. Then each bounded solutions of the inverse of the DSLP are oscillatory.

Proof. Suppose the converse, that there exists a bounded solution $y(t)$ such that

$$y(t) > 0, \qquad t > t_0.$$

Hence

$$y(t - n\tau) > 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \ldots$$

$$y''(t) > 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \ldots$$

Because of $y(t)$ is bounded, it follows that

$$y'(t) < 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \ldots$$

Put the following equation for sufficiently large t is negative, where $n = 1, 2, \ldots$

$$x(t) = r(t)y'(t) - \lambda^2 \rho(t)y(t - n\tau). \tag{12}$$

At the present time, derivation of both sides of 12, with considering that $r(t) = 1, \rho(t) = 1$ by assumption, so 12 becomes as follows

$$x(t) = y'(t) - \lambda^2 y(t - n\tau),$$

$$x'(t) = y''(t) - \lambda^2 y'(t - n\tau),$$

$$x'(t) + \lambda^2 x(t - n\tau) = y''(t) - \lambda^2 y'(t - n\tau) + \lambda^2 y'(t - n\tau) - \lambda^4 y(t - 2n\tau)$$

and substitute instead of $y''(t)$, then it follows

$$x'(t) + \lambda^2 x(t - n\tau) = \alpha(t)y(t) \geqslant 0.$$

Hence

$$x'(t) + \lambda^2 x(t - n\tau) \geqslant 0. \tag{13}$$

Now, notice that $\lambda^2 \rho\tau > \alpha, \alpha \geqslant 0$ is satisfied. So 13 has eventually positive solutions only. This results leads to a contradiction with the assumption 12. □

Applications

This section consists of some applications on the results in the second section, which is represented by the examples that are listed in the current section. Let the DSLP mentioned as below.

Example 2. Let the DSLP mentioned as below

$$y''(t) + 2y(t) + \lambda y(t - \pi) = 0,$$

where $y(0) = 0, y(\pi) = 0, \lambda = 1$. Then DSLP has the following oscillatory solutions, when $\lambda = 1$.

1. y(t)= sint

2. y(t)= -sint
3. y(t)= cost
4. y(t)= -cost

Also it satisfies condition 11, because $r(t) = 1, \alpha(t) = 2, \rho(t) = 1, \lambda\rho\tau = \pi > \alpha = 2$.

Example 3. Put the DSLP

$$y''(t) + \lambda y(t - \pi) \leqslant 0,$$

where $y(0) = 0, y(\pi) = 0, \lambda = k^2, k = 1, 2, \ldots$. Then DSLP when $\lambda = 1$ has an eventually negative solution $y(t) = -cost$, where $t = (2n + 1)\pi, \quad n = 0, \mp 1, \mp 2, \mp 3, \ldots$ Since, when we substitute $y(t) = -cost$ and $y''(t) = cost$ in $y''(t) + \lambda y(t - \pi)$, then we get it equal to $2cost$. And it satisfies condition 11.

Example 4. Suppose that the DSLP

$$y''(t) + \lambda y(t - \pi) \geqslant 0,$$

where

$$y(0) = 0, y(\pi) = 0, \lambda = k^2, k = 1, 2, \ldots$$

Then the DSLP when $\lambda = 1$ has an eventually positive solution

$$y(t) = cost, \quad when \quad t = (2n + 1)\pi, \quad n = 0, \mp 1, \mp 2, \mp 3, \ldots$$

Since, when we substitute $y(t) = cost$ and $y''(t) = -cost$ in $y''(t) + \lambda y(t - \pi)$, then we get it equal to $-2cost$. furthermore, it satisfies condition 11.

Example 5. Consider

$$y''(t) + e^{-t} y(t) + \lambda y(t - 1) \geqslant 0, \qquad \lambda = e^{-2t}.$$

Then the DSLP does not have eventually positive solution, since the condition 11 does not hold That is, $\lambda\rho\tau = e^{-2t}$ is not greater than $\alpha = e^{-t}$

Conclusion

The aim of this paper is studying the behaviour of solutions of the DSLP, and how they would be oscillatory, eventually positive, and eventually negative. Where Theorems, corollaries, examples, and remarks are given to explain each case. Moreover, the sufficient and necessary conditions are introduced with the DSLP forms. The inverse of the DSLP are also introduced in section two. Besides, the applications of the results in the second section are given, which they are represented by examples in the third section.

Acknowledgements

I would like to express my very great appreciation to the staff of the Mathematics Department- college of science- Mustansiriyah University. I would also like to extend my thanks to my husband, brothers, and sister for their patient guidance, enthusiastic encouragement and useful critiques of this research work. Finally I am particularly grateful for the assistance given by my sincerely girlfriend Dr. Muna Jasim.

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Accepted: 15.03.2019

Common fixed point of faintly compatible in fuzzy metric space

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Abstract. Aim of this paper is to establish a common fixed point theorem for faintly compatible and subsequentially continuous self maps of fuzzy metric space and generalizing the result of Jain et al. [6].

Keywords: common fixed point, fuzzy metric space, weakly compatible mappings, occasionally weakly compatible, conditionally compatible, faintly compatible mappings, sub sequentially continuous.

1. Introduction

The concept of fuzzy set was initially investigated by Zadeh [18] as a new way to represent vagueness in everyday life. The special feature of fuzzy set is that it assign partial membership for elements in its domain, while in ordinary set theory particular element has either full membership or no membership, intermediate situation is not considered. A large number of renowned Mathematicians worked with fuzzy sets in different branches of Mathematics. One such is the Fuzzy Metric Space. In this paper, we are considering the fuzzy metric space

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defined by Kramosil and Michalek [9] and modified by George and Veeramani [4] with the help of continuous t-norm. In 1999,Vasuki [13] introduced the concept of R-weak commutative of mappings in fuzzy metric space.

In 1982, Sessa [14] obtained the first weaker version of commutativity by introducing the notion of weak commutativity. This concept was further generalized by Jungck [8] when he defined the concept of compatible mapping. The concept of compatibility in fuzzy metric space was proposed by Mishra et. al. [11]. In 1996, Jungck [7] again generalized the notion of compatible mapping by introducing weak mapping. Cho et. al. [3] introduced the concept of semi-compatible maps in d-topological space. Singh and Jain [15] defined the concept of semi-compatible maps in fuzzy metric space. In 2008, Al-Thagafi and Shahzad [1] generalized the notion of weak compatibility by new notion of occasionally weakly compatible (owc) mappings. Pant et. al. [12] introduced the concept of conditional compatible maps. The use of occasional weak compatibility is a redundancy for fixed point theorems under contractive conditions. To remove this redundancy we use faintly compatible mapping in our paper which is weaker than weak compatibility or semi compatibility. Faintly compatible maps introduced by Bisht and Shahzad [2] as an improvement of conditionally compatible maps.

In 2007, Singh et. al. [17] proved common fixed point theorem using the concept of compatible and weak compatible in fuzzy metric space. Subsequently, in 2014, Jain et. al. [6] established fixed point theorem for six self maps by using concept of occasionally weak compatible maps and generalized the result of Singh et. al. [17]. Jain et. al. [5] introduced the notion of subsequential continuous mappings in fuzzy metric space which is more general than continuous mappings as well as reciprocal continuous mappings and also introduced the concept of occasionally weakly compatible mappings which is more general than weakly compatible mappings in fixed point theory in 2014.

In this paper, we generalize the result of Jain et. al. [6] by replacing the occasionally weakly compatible maps to faintly subsequential continuous maps.

2. Preliminaries

Definition 2.1 ([18])**.** Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 2.2 ([11])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Examples of t-norms are

$a * b = \min\{a, b\}$ (minimum t-norm),
$a * b = ab$ (product t-norm).

Definition 2.3 ([11]). The 3-tuple (X,M,*) is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:
(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous, ; for all $x, y, z \in X$ and $t, s > 0$.
Let (X, d) be a metric space and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.
Let $M(x, y, t) = \frac{t}{t+d(x,y)}$; for all $x, y \in X$ and $t > 0$.
Then (X,M,*) is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [11].

Definition 2.4 ([11]). A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is said to be convergent to a point $x \in X$,if $M(x_n, x, t) = 1$ for all $t > 0$.
Further, the sequence $\{x_n\}$ is said to be Cauchy if $M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and $p > 0$.
The space (X,M,*) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.5 ([10]). Let (X,M,*) be a fuzzy metric space. Then M is non-decreasing for all $x, y \in X$.

Lemma 2.6 ([11]). Let (X,M,*) be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.

Throughout this paper (X,M,*) will denote the fuzzy metric space with the following condition:
(FM-6) $\lim_{n\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Definition 2.7 ([16]). Let $f$and g be self mappings on a fuzzy metric space (X,M,*).
The pair (f, g) is said to compatible if

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \text{ for some } z \in X.$$

Definition 2.8 ([17])**.** Let f and g be self mappings on a fuzzy metric space (X,M,*). Then the mappings are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies $fgx = gfx$.

It is known that a pair of (f, g) compatible maps is weakly compatible but converse is not true in general.

Definition 2.9 ([15])**.** A pair (A, B) of self maps of a fuzzy metric space (X,M,*) is said to be semi-compatible if $\lim_{n\to\infty} ABx_n = Bx$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x.$$

Definition 2.10 ([6])**.** Self maps A and S of a Fuzzy metric space (X,M,*) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of A and S at which A and S commute.

It follows that if (A, B) is semi-compatible and $Ax = Bx$ then $ABx = BAx$ that means every semi-compatible pair of self maps is weak compatible but the converse is not true in general.

Definition 2.11 ([5])**.** Two self maps A and S on a fuzzy metric space are called reciprocal continuous if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ for some t in X whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t.$$

Definition 2.12 ([5])**.** Two self maps A and S on a fuzzy metric space are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \text{ for some in } X \text{ and satisfy}$$

$$\lim_{n\to\infty} ASx_n = At \text{ and } \lim_{n\to\infty} SAx_n = St.$$

Clearly, if A and S are continuous then they are obviously sub-sequentially continuous. The next example shows that there exist sub-sequential continuous pairs of mappings which are neither continuous nor reciprocally continuous.

Example 2.13. Let $X = R$, endowed with metric d and $M_d(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$, all $t > 0$ define the self mappings A, S as follow

$$A(x) = \left\{ \begin{array}{ll} 2, & x < 3 \\ x, & x \geq 3 \end{array} \right\} \text{and } S(x) = \left\{ \begin{array}{rl} 2x-4, & x \leq 3 \\ 3, & x > 3 \end{array} \right\}.$$

Consider a sequence $x_n = 3 + \frac{1}{n}$; then,

$$A(x_n) = \left(3+\frac{1}{n}\right) \to 3, S(x_n) = 3,\ SA(x_n) = S\left(3+\frac{1}{n}\right) = 3 \neq S(3) = 2, \text{as } n \to \infty.$$

Thus A and S are not reciprocally continuous but, if we consider a sequence $x_n = 3 - \frac{1}{n}$, then,

$$A(x_n) = 2,\ S(x_n) = 2\Big(3 - \frac{1}{n}\Big) - 4 = \Big(2 - \frac{2}{n}\Big) = 2 \text{ as } n \to \infty$$
$$AS(x_n) = A\Big(2 - \frac{2}{n}\Big) = 2 = A(2),\ SA(x_n) = S(2) = 0 = S(2) \text{ as } n \to \infty.$$

Therefore, A and S are sub sequentially continuous.

Remark 2.14 ([5])**.** If A and S are continuous or reciprocally continuous then they are obviously sub sequentially continuous, but converse is not true.

Definition 2.15 ([2])**.** Two self maps A and S on a fuzzy metric space are said to be conditionally compatible if and only if whenever the set of sequences y_n satisfying $\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} S(y_n)$ is nonempty, there exists a sequence z_n such that $\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = u$ and

$$\lim_{n\to\infty} M\big(A(S(z_n)), S(A(z_n)), t\big) = 1.$$

Definition 2.16 ([2])**.** Two self-mappings A and S of a metric space (X, d) will be called faintly compatible iff A and S are conditionally compatible and A and S commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

If A and S are compatible, then they are obviously faintly compatible, but the converse is not true in general.

Example 2.17. Let $X = [3, 6]$ and d be the usual metric on X. Define self-mappings A and S on X as follows:

$$A(x) = 3 \text{ if } x = 3 \text{ or } x > 5,\ A(x) = x + 1 \text{ if } 3 < x \leq 5.$$
$$S(3) = 3,\ S(x) = \frac{(x+7)}{3} \text{ if } 3 < x \leq 5,\ S(x) = \frac{(x+1)}{2} \text{ if } x > 5.$$

In this example A and S are faintly compatible but not compatible.

To see this, if we consider the constant sequence $\{y_n = 3\}$, then A and S are faintly compatible.

On the other hand, if we choose the sequence $\Big\{x_n = 5 + \frac{1}{n}\Big\}$, then

$$\lim_{n\to\infty} A(x_n) = 3 = \lim_{n\to\infty} S(x_n) \ \text{ and } \ \lim_{n\to\infty} M\Big(A(S(x_n)), S(A(x_n)), t\Big) \neq 0.$$

Thus A and S are faintly compatible, but they are not compatible.

In 2014, Jain et. al. [6] proved the following result:

Theorem 2.18. Let A, B, S and T be self mappings of a complete Fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

(2.15.1) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$;
(2.15.2) The pairs (A, S) and (B, T) are occasionally weakly compatible,
(2.15.3) There exists $k \in (0, 1)$ such that $\forall\ x, y \in X$ and $t > 0$,

$$M(Ax, By, kt) \geq Min\big\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\big\}.$$

Then A, B, S and T have a unique common fixed point in X.

3. Main result

Theorem 3.1. Let A, B, P, Q, S and T be self-mappings of fuzzy metric space (X,M,*). Suppose that they satisfy the following condition:

[3.1.1] $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$,
[3.1.2] (A, PS) and (B, QT) are faintly compatible and subsequently continuous,
[3.1.3] $AS = SA, BT = TB, QT = TQ$ and $PS = SP$.
[3.1.4] Their exist $k \in (0, 1)$ such that $\forall\, x, y \in X$ and $t > 0$

$$M(Ax, By, kt) \geq \min\{M(By, QTy, t), M(PSx, QTy, t), M(Ax, PSx, t)\}.$$

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point.

From condition [3.1.1], $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$

There exist x_1 and $x_2 \in X$ such that

$$A(x_0) = QT(x_1) \text{ and } B(x_1) = PS(x_2)$$

We can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$\begin{aligned} y_{2n} &= A(x_{2n}) = QT(x_{2n+1}) \\ y_{2n+1} &= B(x_{2n+1}) = PS(x_{2n+2}) \text{ for } n = 0, 1, 2, 3, ... \end{aligned}$$

We show that $\{y_n\}$ is a Cauchy sequence in X.

Using equation (3.1.4) with $x = x_{2n}, y = x_{2n+1}$,

$$\begin{aligned} M(Ax_{2n}, Bx_{2n+1}, kt) &= M(y_{2n}, y_{2n+1}, kt) \\ &\geq \min\{M(Bx_{2n+1}, QTx_{2n+1}, t), M(PSx_{2n}, QTx_{2n+1}, t), \\ &\quad M(Ax_{2n}, PSx_{2n}, t)\} \end{aligned}$$

$$\begin{aligned} M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\} \\ &\geq \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\} \end{aligned}$$

Thus, we have

$$\begin{aligned}
M(y_{2n}, y_{2n+1}, t) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t/k)\} \\
M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t/k), \\
&\qquad M(y_{2n}, y_{2n-1}, t)\} \\
M(y_{2n}, y_{2n+1}, kt) &\geq \min\{M(y_{2n+1}, y_{2n}, t/k), M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^2), M(y_{2n}, y_{2n-1}, t/k^2), \\
&\qquad M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^2), M(y_{2n}, y_{2n-1}, t)\} \\
&\geq \min\{M(y_{2n+1}, y_{2n}, t/k^m), M(y_{2n}, y_{2n-1}, t)\}.
\end{aligned}$$

Taking limit as $m \to \infty$

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n-1}, t);\ \forall\ t > 0.$$

Similarly

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t);\ \forall\ t > 0.$$

Thus, for all n and $t > 0$

$$M(y_n, y_{n+1}, kt) \geq M(y_n, y_{n-1}, t).$$

Therefore,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k) \geq M(y_{n-2}, y_{n-1}, t/k^2) \geq \ldots \geq M(y_0, y_1, t/k^n).$$

Hence, $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1;\ \forall\ t > 0$.

Now, for any integer p, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1 = 1 \\
&\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.
\end{aligned}$$

This show that $\{y_n\}$ is Cauchy sequence in X which is complete therefore $\{y_n\}$ converges to $u \in X$.

Then, subsequences $\{A(x_{2n})\}, \{B(x_{2n+1})\}, \{QT(x_{2n+1})\}$ and $\{PS(x_{2n+2})\}$ are also converges to $u \in X$.

$$\lim_{n\to\infty} A(x_{2n}) = \lim_{n\to\infty} PS(x_{2n}) = \lim_{n\to\infty} B(x_{2n+1}) = \lim_{n\to\infty} QT(x_{2n+1}) = u.$$

Case (1): (A, PS) is faintly compatible and sub sequentially continuous.

$\lim_{n\to\infty} A(x_{2n}) = \lim_{n\to\infty} PS(x_{2n}) = u$ and (A, PS) is faintly compatible then, there exist sequence $\{z_n\}$ in X, where, $\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} PS(z_n) = v$ for some $v \in X$ such that

$$\lim_{n\to\infty} M(PSA(z_n), APS(z_n), t) = 1.$$

As (A, PS) sub sequentially continuous, we have

$$\lim_{n\to\infty} Az_n = v \Rightarrow \lim_{n\to\infty} PS(Az_n) = PSv,$$

and

$$\lim_{n\to\infty} PSz_n = v \Rightarrow \lim_{n\to\infty} A(PSz_n) = Av,$$

Since,

$$\lim_{n\to\infty} M(PSAz_n, APSz_n, t) = 1,$$

$$PSv = Av. \tag{1}$$

Case (2): (B, QT) is faintly compatible and subsequently continuous. We know that $\lim_{n\to\infty} B(x_{2n+1}) = \lim_{n\to\infty} QT(x_{2n+1}) = u$ and (B, QT) is faintly compatible then there exist sequence $\{z'_n\}$ in X
where, $\lim_{n\to\infty} B(z'_n) = \lim_{n\to\infty} QT(z'_n) = v'$ for some $v \in X$ such that

$$\lim_{n\to\infty} M\Big(B(QTz'_n), QT(Bz'_n), t\Big) = 1.$$

As (B, QT) subsequently continuous we have

$$\lim_{n\to\infty} Bz'_n = v' \Rightarrow \lim_{n\to\infty} QT(Bz'_n) = QTv'$$
$$\lim_{n\to\infty} QTz'_n = v' \Rightarrow \lim_{n\to\infty} B(QTz'_n) = Bv'$$

Since,

$$\lim_{n\to\infty} M(BQTz'_n, QBTz'_n, t) = 1$$

$$Bv' = QTv'. \tag{2}$$

Since pairs (A, PS) and (B, QT) are faintly compatible, we have

$$Av = PSv$$

$$\Rightarrow AAv = APSv = PSAv = PS(PSv) \tag{3}$$

and $Bv' = QTv'$

$$\Rightarrow BBv' = BQTv' = QTBv' = QT(QTv\prime). \tag{4}$$

Now, we show that $Av = Bv'$

Using inequality [3.1.4] with $x = v$ and $y = v'$,

$$\begin{aligned} M(Ax, By, kt) &\geq \min\{M(By, QTy, t), M(PSx, QTy, t), M(Ax, PSx, t)\} \\ M(Av, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PSv, QTv', t), M(Av, PSv, t)\} \\ &\geq \min\{M(Bv', Bv, t), M(Av, Bv', t), M(Av, Av, t)\} \\ &\geq \min\{1, M(Av, Bv', t), 1\} \quad \text{by using (1) and (2)} \\ M(Av, Bv', kt) &\geq M(Av, Bv', t). \end{aligned}$$

$$Av = Bv'. \tag{5}$$

Now we show that $A(Av) = Av$.

Using equation [3.1.4] with $x = Av$ and $y = v'$,

$$\begin{aligned} M(AAv, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PSAv, QTv', t), M(AAv, PSAv, t)\} \\ M(AAv, Av, kt) &\geq \min\{M(Bv', Bv', t), M(AAv, Bv', t), M(AAv, AAv, t)\} \\ M(AAv, Av, kt) &\geq \min\{1, M(AAv, Av, t), 1\} \\ M(AAv, Av, kt) &\geq M(AAv, Av, t) \end{aligned}$$

$$AAv = Av. \tag{6}$$

Therefore, Av is fixed point of mapping A.

Again, we show that

$$B(Av) = (Av) \quad \text{or} \quad BBv' = Av.$$

Putting $x = v$ and $y = Bv'$ in [3.1.4]

$$M(Av, BBv', kt) \geq \min\{M(BBv', QTBv', t), M(PSv, QTBv', t), M(Av, PSv, t)\}.$$

Using (1) & (4),

$$\begin{aligned} M(Av, BBv', kt) &\geq \min\{M(BBv', BBv', t), M(Av, BBv', t), M(Av, Av, t)\} \\ M(Av, BBv', kt) &\geq \min\{1, M(Av, BBv', t), 1\} \\ M(Av, BBv', kt) &\geq M(Av, BBv', t) \\ M(Av, BBv', kt) &= 1 \\ Av &= BBv' \end{aligned}$$

$$\text{or} \qquad B(Av) = Av. \tag{7}$$

Therefore, Av is fixed point of mapping B.

Using equations (3), (4), (5) and (6)

$$A(Av) = B(Av) = PS(Av) = QT(Av) = Av. \tag{8}$$

Putting $x = Sv$ and $y = v'$ in inequality [3.1.4]

$$\begin{aligned}
M(ASv, Bv', kt) &\geq \min\{M(Bv', QTv', t), M(PS(Sv), QTv', t),\\
&\qquad M(ASv, PSSv, t)\}\\
M(SAv, Bv', kt) &\geq \min\{M(Bv', Bv', t), M(SPSv, Bv', t), M(ASv, S(PSv), t)\}\\
M(SAv, Bv, kt) &\geq \min\{1, M(SAv, Av, t), M(ASv, SAv, t)\}\\
&\geq \min\{1, M(SAv, Av, t), 1\}\\
M(SAv, Bv', kt) &\geq M(SAv, Av, t)
\end{aligned}$$

$$SAv = Av \tag{9}$$

$PS(Av) = Av$ by using equation (8)

$$P(Av) = Av \tag{10}$$

Av is also fixed point of mappings P and S.

Therefore,

$$A(Av) = B(Av) = P(Av) = S(Av) = Av.$$

Now, using equation [3.1.4] with $x = v$ and $y = Tv'$,

$$\begin{aligned}
M(Av, BTv', kt) &\geq \min\{M(BTv', QT(Tv'), t), M(PSv, QT(Tv'), t),\\
&\qquad M(Av, PSv, t)\}\\
M(Av, TBv', kt) &\geq \min\{M(BTv', TQ(Tv'), t), M(PSv, TQ(Tv'), t),\\
&\qquad M(Av, Av, t)\}\\
M(Av, TAv, kt) &\geq \min\{M(BTv', TBv', t), M(Av, TBv', t),\\
&\qquad M(Av, Av, t)\},\ \text{since (2)}\\
&\geq \min\{M(BTv', BTv', t), M(Av, TAv, t), 1\}\\
&\geq \min\{1, M(Av, TAv, t), 1\}\\
M(Av, TAv, kt) &\geq M(Av, TAv, t)\\
M(Av, TAv, kt) &= 1
\end{aligned}$$

$$Av = TAv. \tag{11}$$

Therefore, Av is fixed point of mapping T.

Using equation (8) and (11)

$$\begin{aligned}
QT(Av) &= Av\\
Q(Av) &= Av. \qquad (12)
\end{aligned}$$

Av is also fixed point of mapping Q

We get that there is a point Av in set X such that

$$A(Av) = B(Av) = S(Av) = P(Av) = T(Av) = Q(Av) = Av.$$

Av is a common fixed point of mappings A, B, S, P, T and Q in X.

Uniqueness. Let v and w are two common fixed points of mappings A, B, S, P, T and Q. Then,

$$Av = Bv = Sv = Pv = Tv = Qv = v \tag{13}$$

$$\text{and} \qquad Aw = Bw = Sw = Pw = Tw = Qw = w. \tag{14}$$

Now we have to show that $v = w$.

Putting $x = v$ and $y = w$ in inequality [3.1.4],

$$\begin{aligned} M(Av, Bw, kt) &\geq \min\{M(Bw, QTw, t), M(PSv, QTw, t), M(Av, PSv, t)\} \\ M(v, w, kt) &\geq \min\{M(w, Qw, t), M(Pv, Qw, t), M(v, PSv, t)\} \\ M(v, w, kt) &\geq \min\{1, M(v, w, t), 1\} \\ M(v, w, kt) &\geq M(v, w, t). \end{aligned}$$

$v = w$. If we take $S = T = I$ the identity mappings on X in theorem 3.1.

Corollary 3.2. Let A, B, P and Q be self mappings of complete fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

[3.1.1] $A(X) \subseteq Q(X)$ and $B(X) \subseteq P(X)$,

[3.1.2] (A, P) and (B, Q) are faintly compatible and subsequently continuous,

[3.1.3] Then exist $k \in (0, 1)$ such that $\forall\ x, y \in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \min\{M(By, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}.$$

then A, B, P and Q have a unique common fixed point in X.

Proof. The proof is similar to the proof of theorem (3.1).

Corollary 3.3. Let A, P and Q be self mappings of complete fuzzy metric space (X,M,*) satisfy the following conditions:

[3.1.1] $A(X) \subseteq P(X) \cap Q(X)$.

[3.1.2] (A, P) and (A, Q) are faintly compatible and subsequently continuous,

[3.1.3] $M(Ax, Ay, kt) \geq \min\{M(Ay, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}$, for all $x, y \in X,\ t > 0$ and $k \in (0, 1)$.

Then A, P and Q have a unique common fixed point in X.

If X is not complete and (3.2.1) may or may not be satisfy for these four self mappings.

Corollary 3.4. Let A, B, P and Q be self mappings of fuzzy metric space (X,M,*). Suppose that they satisfy the following conditions:

[3.4.1] $M(Ax, By, kt) \geq \min\{M(By, Qy, t), M(Px, Qy, t), M(Ax, Px, t)\}$;
for all $x, y \in X,\ t > 0$ and $k \in (0, 1)$

[3.4.2] If pairs (A, P) and (B, Q) are non-compatible faintly compatible and subsequently continuous.

Then, A, B, P and Q mappings have a unique common fixed point in X.

Proof. (A, P) and (B, Q) are non-compatible then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $A(x_n) = P(x_n) = u,$ for some $u \in X$. But $M(APx_n, PAx_n, t) \neq 1$ and $\lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} Q(y_n) = u'$ for some $u' \in X$,

But $M(By_n, Qy_n, t) \neq 1$.

(A, P) and (B, Q) are non-compatible faintly compatible, so it implies that

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} P(x_n) = u \text{ and } (A, P)$$

is faintly compatible subsequentially continuous

$$\text{and } \lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} Q(y_n) = u' \text{ and } (B, Q)$$

is faintly compatible subsequentially continuous

Rest proof is similar to case (I) and case (II).

4. Conclusion

Our result is a generalization of the result of Jain et.al. [6] in the sense that we have replaced occasionally weakly compatible (owc) to faintly compatible and prove a theorem on common fixed point theorems for six self mappings in complete fuzzy metric space. Corollary 3.4 is also another generalization of Jain et.al. [6] where completeness is not necessary.

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Accepted: 29.03.2019

Operator inequalities involving improved Young inequality

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Abstract. The main purpose of this paper is to present some operator inequalities and matrices inequalities for the Hilbert-Schmidt norm. We first give the refined Young inequality for scalars. After that, based on this inequality, we establish operator inequalities and matrices inequalities for the Hilbert-Schmidt norm.
Keywords: operator inequality, Young inequality, Hilbert-Schmidt norm.

1. Introduction

Throughout this paper, let M_n be the set of $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$. $\mathcal{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We reserve m, m', M, M' for scalars and I for the identity operator.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators and $0 \le v \le 1$. The v-weighted arithmetic mean $A\nabla_v B$ and the v-weighted geometric mean $A\sharp_v B$ are defined by

$$A\nabla_v B = (1-v)\,A + vB, A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v} A^{\frac{1}{2}},$$

when $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively.

It is well-known that

$$A\sharp B \le H_v(A,B) \le A\nabla B,$$

where $H_v(A,B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2}$ is called Heinz mean.

The famous v-weighted arithmetic-geometric mean inequality says that if $a, b \ge 0$ and $0 \le v \le 1$, then

$$a^v b^{1-v} \le va + (1-v)b \tag{1.1}$$

with equality if and only if $a = b$. (1.1) is often called Young inequality. When $v = \frac{1}{2}$, (1.1) is just the arithmetic-geometric mean inequality

$$\sqrt{ab} \le \frac{a+b}{2}. \tag{1.2}$$

F. Kittaneh and Y. Manasrah ([1] [2]) obtained the improvement of (1.1) :

$$a^{1-v}b^v + s(\sqrt{a}-\sqrt{b})^2 \geq (1-v)a + vb \geq a^{1-v}b^v + r(\sqrt{a}-\sqrt{b})^2, \tag{1.3}$$

where $a, b \geq 0$, $r = \min\{v, 1-v\}$ and $s = \max\{v, 1-v\}$.

Zou et al.[3] refined Young inequality as follows:

$$(1-v)a + vb \geq K(h)^r a^{1-v}b^v, \tag{1.4}$$

where $r = \min\{v, 1-v\}$, $h = \frac{a}{b}$ and $K(h) = \frac{(h+1)^2}{4h}$, so that $K(h)$ is called Kantorovich constant.

The Kantorovich constant satisfies the following properties:

(i) $K(1) = 1$,

(ii) $K(h) = K(\frac{1}{h})$ for $h > 0$,

(iii) $K(h)$ is monotone increasing on $[1, +\infty)$ and monotone decreasing on $(0, 1]$.

For more results on Young inequality, the reader is referred to [4, 5, 6], and references therein.

In this paper, we obtain Lemma 1 which is the refinement of inequality (1.1). Meanwhile, we establish the operator and matrix version of inequality (2.1) for the Hilbert-Schmidt norm.

2. Main results

We start this section with two lemmas which are important to prove the main results.

Lemma 1. *Let* $0 < m \leq a \leq m' < M' \leq b \leq M$ *and* $v \in [0, 1]$*, then*

$$(1-v)a + vb \geq a^{1-v}b^v + 2r(\sqrt{K(h')} - 1)\sqrt{ab}, \tag{2.1}$$

where $r = \min\{v, 1-v\}$ *and* $h' = \frac{M'}{m'}$.

Proof. Let $h = \frac{b}{a}$, then $h \geq h' > 1$.

By the second inequality in (1.3) and inequality (1.4), we have

$$\begin{aligned} (1-v)a + vb &\geq a^{1-v}b^v + r(2\sqrt{K(h)}\sqrt{ab} - 2\sqrt{ab}) \\ &= a^{1-v}b^v + 2r(\sqrt{K(h)} - 1)\sqrt{ab} \\ &\geq a^{1-v}b^v + 2r(\sqrt{K(h')} - 1)\sqrt{ab}. \end{aligned}$$

This completes the proof. □

Lemma 2 ([7])**.** *Let* $X \in \mathcal{B}(\mathcal{H})$ *be self-adjoint and let* f *and* g *be continuous real functions such that* $f(t) \geq g(t)$ *for all* $t \in Sp(X)$ *(the spectrum of* X*), then*

$$f(X) \geq g(X).$$

Theorem 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive invertible operators and $0 < mI \le A \le m'I < M'I \le B \le MI$, then*

$$A\nabla_v B \ge A\sharp_v B + 2r(\sqrt{K(h') - 1})A\sharp B,$$

where $v \in [0,1]$, $r = \min\{v, 1-v\}$ and $h' = \frac{M'}{m'}$.

Proof. By (2.1), we have

$$(1-v) + vb \ge b^v + 2r(\sqrt{K(h')} - 1)\sqrt{b}, \tag{2.2}$$

for any $b > 0$.

Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and thus $Sp(X) \subseteq (0, +\infty)$, (2.2) holds for any $b \in Sp(X)$. By Lemma 2 and (2.2), we get

$$(1-v) + vX \ge X^v + 2r(\sqrt{K(h')} - 1)X^{\frac{1}{2}}. \tag{2.3}$$

Multiplying both sides of (2.3) by $A^{\frac{1}{2}}$, we obtain

$$A\nabla_v B \ge A\sharp_v B + 2r(\sqrt{K(h') - 1})A\sharp B.$$

This completes the proof. $\square$

Corollary 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive invertible operators and $0 < mI \le A \le m'I < M'I \le B \le MI$, then*

$$A\nabla B \ge H_v(A,B) + 2r(\sqrt{K(h') - 1})H_{\frac{1}{2}}(A,B),$$

where $v \in [0,1]$, $r = \min\{v, 1-v\}$ and $h' = \frac{M'}{m'}$.

Proof. By Theorem 1, we get

$$(1-v)A + vB \ge A\sharp_v B + 2r(\sqrt{K(h')} - 1)A\sharp B$$

and

$$vA + (1-v)B \ge A\sharp_{1-v} B + 2r(\sqrt{K(h')} - 1)A\sharp B.$$

Summing up the above two inequalities, we obtain

$$A\nabla B \ge H_v(A,B) + 2r(\sqrt{K(h') - 1})H_{\frac{1}{2}}(A,B).$$

This completes the proof. $\square$

Theorem 2. *Let $A, B, X \in M_n$ such that A, B are two positive definite matrices and $0 < mI \le A \le m'I < M'I \le B \le MI$, then*

$$||(1-v)AX + vXB||_2^2 \ge ||A^{1-v}XB^v||_2^2 + 4r^2(\sqrt{K(h')} - 1)^2||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2,$$

where $v \in [0,1]$, $r = \min\{v, 1-v\}$ *and* $h' = \frac{M'}{m'}$.

Proof. Since every positive definite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n$ such that $A = UDU^*$ and $B = VEV^*$, where$D = diag\,(\lambda_1, \cdots, \lambda_n)\,, E = diag\,(\mu_1, \cdots, \mu_n)\,$, and $\lambda_i, \mu_i > 0, i = 1, \cdots, n$. Let $Y = U^*XV = (y_{ij})$, then

$$(1-v)AX + vXB = U[((1-v)\lambda_i + v\mu_j) \circ Y]V^*,$$

$$A^{1-v}XB^v = U[(\lambda_i^{1-v}\mu_j^v) \circ Y]V^*.$$

Utilizing the unitarily invariant property of $||\cdot||_2$ and (2.1), we have

$$\begin{aligned}
||(1-v)AX + vXB||_2^2 &= ||((1-v)\lambda_i + v\mu_j) \circ Y||_2^2 \\
&= \sum_{i,j=1}^{n} ((1-v)\lambda_i + v\mu_j)^2 |y_{ij}|^2 \\
&\geq \sum_{i,j=1}^{n} (\lambda_i^{1-v}\mu_j^v + 2r(\sqrt{K(h')} - 1)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})^2 |y_{ij}|^2 \\
&\geq \sum_{i,j=1}^{n} [(\lambda_i^{1-v}\mu_j^v)^2 + 4r^2(\sqrt{K(h')} - 1)^2(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})^2] |y_{ij}|^2 \\
&= ||A^{1-v}XB^v||_2^2 + 4r^2(\sqrt{K(h')} - 1)^2 ||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2.
\end{aligned}$$

This completes the proof. □

Acknowledgments

This research was supported by Scientific Research Fund of Yunnan Provincial Education Department (Grant No. 2019J0350).

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Accepted: 23.05.2019

On minimal λ_{rc}-open sets

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Abstract. We introduce and discuss the notions of minimal λ_{rc}-open sets in topological spaces. We investigate some its fundamental properties. We show that the notions of minimal open sets and minimal λ_{rc}-open sets are independent and finally we obtain some applications of a minimal λ_{rc}-open sets.
Keywords: minimal λ_{rc}-Open Sets, λ_{rc}-locally finite space.

1. Introduction

The study of semi open sets in topological spaces was initiated by Levine [10]. The complement of a subset A of X is denoted by $X \setminus A$. In 1937, M. Stone [22], defined regular closed set, a subset A is said to be regular-closed if $A = Cl(Int(A))$. The family of all regular-closed sets of (X, τ) is denoted by $RC(X)$. The concept of operation γ was initiated by Kasahara [4]. He also introduced γ-closed graph of a function. Using this operation, Ogata [21] introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ. He further investigated general operator approaches of closed graph of mappings. Further Ahmad and Hussain [1] continued studying the properties of γ -open(γ-closed) sets. In 2009, Hussain and Ahmad [3], introduced the concept of minimal γ-open sets. In 2011 [5] (resp. in 2013 [6]) Khalaf and Namiq, defined an operation γ called s-operation. They work in operation in topology in [14], [8], [9], [15], [16], [17],[18], [19]. They defined $\lambda_{\beta c}$-open set[13] by using s-operation and β-closed set and also investigated several properties of $\lambda_{\beta c}$-derived, $\lambda_{\beta c}$-interior and $\lambda_{\beta c}$-closure points in

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topological spaces. In 2017, Carpintero et al. [2], investigated the notions of minimal open sets in a generalized topological spaces and investigated its fundamental properties.
In this paper, we introduce and discuss minimal λ_{rc}-open sets in topological spaces and investigate some of their fundamental properties. We show that the notions of minimal λ_{rc}-open sets and minimal open sets are independent. Finally we obtain some applications of minimal λ_{rc}-open sets. First, we recall some definitions and results used in this paper.

2. Preliminaries

Throughout, X denotes a topological space. Let A be a subset of X, then the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X,τ) is said to be semi open [10] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is called semi closed [10]. The family of all semi open (resp. semi closed) sets in a topological space (X,τ) is denoted by $SO(X,\tau)$ or $SO(X)$ (resp. $SC(X,\tau)$ or $SC(X)$). We consider λ as a function defined on $SO(X)$ into $P(X)$ and $\lambda : SO(X) \to P(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each non-empty semi open set V. It is assumed that $\lambda(\emptyset) = \emptyset$ and $\lambda(X) = X$ for any s-operation λ. Let X be a topological space and $\lambda : SO(X) \to P(X)$ be an s-operation, then a subset A of X is called a λ^*-open set [12] which is equivalent to λ-open set [5] and λ_s-open set [6] if for each $x \in A$ there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ^*-open set is called λ^*-closed. The family of all λ^*-open (resp. λ^*-closed) subsets of a topological space (X,τ) is denoted by $SO_\lambda(X,\tau)$ or $SO_\lambda(X)$ (resp. $SC_\lambda(X,\tau)$ or $SC_\lambda(X)$).

Proposition 2.1 ([13]). *For a topological space X, $SO_\lambda(X) \subseteq SO(X)$.*

The following example shows that the contention of the above proposition may be strict..

Example 2.2 ([13]). Let $X = \{a,b,c\}$, and $\tau = \{\emptyset, \{a\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $b \in A$ and $\lambda(A) = X$ otherwise. Here, we have $\{a,c\}$ is semi open but it is not λ^*-open.

Definition 2.3 ([13]). *An s-operation λ on X is said to be s-regular which is equivalent to λ-regular [7] if for every semi open sets U and V of X containing the point $x \in X$, there exists a semi open set W containing x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.*

The proof of the following two propositions are in [7].

Proposition 2.4. *Let $\{A_\alpha\}_{\alpha\in I}$ be any collection of λ^*-open sets in a topological space (X,τ), then $\bigcup_{\alpha\in I} A_\alpha$ is a λ^*-open set.*

Proposition 2.5. *Let λ be semi-regular operation. If A and B are λ^*-open sets in X, then $A \cap B$ is also a λ^*-open set.*

Definition 2.6. *A λ^*-open [12] (λ-open [5], λ_s-open [6]) subset A of a topological space X is called λ_{rc}-open [20] if for each $x \in A$ there exists a regular closed set F such that $x \in F \subseteq A$. The complement of a λ_{rc}-open set is called λ_{rc}-closed. The family of all λ_{rc}-open (resp. λ_{rc}-closed) subsets of a topological space (X, τ) is denoted by $SO_{\lambda_{rc}}(X, \tau)$ or $SO_{\lambda_{rc}}(X)$(resp. $SC_{\lambda_{rc}}(X, \tau)$ or $SC_{\lambda_{rc}}(X)$).*

Definition 2.7. *Let X be a topological space and $\lambda : SO(X) \to P(X)$ be an s-operation, then a subset A of X is called a λ_{rc}-open neighbourhood of a point $x \in X$ if A is a λ_{rc}-open set and $x \in A$.*

Proposition 2.8 ([20])**.** *For a topological space X, $SO_{\lambda_{rc}}(X) \subseteq SO_{\lambda}(X) \subseteq SO(X)$.*

The following example shows that the contention of the above proposition may be strict.

Example 2.9. In Example 2.2, we have $\{a, c\}$ is semi open but it is not λ^*-open. And also $\{a, b\}$ is λ^*-open set but it is not λ_{rc} -open.

Definition 2.10 ([20])**.** *Let A be a subset of X. Then:*

1. *The λ_{rc}-closure of A ($\lambda_{rc}Cl(A)$) is the intersection of all λ_{rc}-closed sets containing A.*
2. *The λ_{rc}-interior of A ($\lambda_{rc}Int(A)$) is the union of all λ_{rc}-open sets of X contained in A.*

Proposition 2.11 ([20])**.** *For each point $x \in X$, $x \in \lambda_{rc}Cl(A)$ if and only if $V \cap A \neq \emptyset$ for every $V \in SO_{\lambda_{rc}}(X)$ such that $x \in V$.*

Proposition 2.12 ([20])**.** *Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of λ_{rc}-open sets in a topological space (X, τ), then $\bigcup_{\alpha \in I} A_\alpha$ is a λ_{rc}-open set.*

Proposition 2.13 ([20])**.** *Let λ be an s-regular operation. If A and B are λ_{rc}-open sets in X, then $A \cap B$ is also a λ_{rc}-open set.*

Definition 2.14 ([11])**.** *Let X be a space and $A \subseteq X$ be an open set. Then A is called a minimal open set if $\emptyset$ and A are the only open subsets of A.*

3. Minimal λ_{rc}-open sets

Definition 3.1. *Let X be a space and $A \subseteq X$ be a λ_{rc}-open set. Then A is called a minimal λ_{rc}-open set if $\emptyset$ and A are the only λ_{rc}-open subsets of A.*

The following example show that minimal open set and minimal λ_{rc}-open set are independent.

Example 3.2. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\}, \{b\}$ and $\lambda(A) = \{a, b\}$ if $A = \{a\}$ or $\{b\}$. The λ_{rc}-open sets are $\emptyset, \{a, c\}, \{b, c\}$ and X. We have $\{a, c\}$ is minimal λ_{rc}-open set, but it is not minimal open set. And also $\{a\}$ is minimal open set, but it is not minimal λ_{rc}-open set.

Proposition 3.3. *Let A be a nonempty λ_{rc}-open subset of a space X. If $A \subseteq \lambda_{rc}Cl(C)$, then $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(C)$, for any nonempty subset C of A.*

Proof. For any nonempty subset C of A,we have $\lambda_{rc}Cl(C) \subseteq \lambda_{rc}Cl(A)$. On the other hand, by supposition, we see $\lambda_{rc}Cl(A) \subseteq \lambda_{rc}Cl(\lambda_{rc}Cl(C)) = \lambda_{rc}Cl(C)$ implies $\lambda_{rc}Cl(A) \subseteq \lambda_{rc}Cl(C)$. Therefore we have $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(C)$ for any nonempty subset C of A. □

Proposition 3.4. *Let A be a nonempty λ_{rc}-open subset of a space X. If $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(C)$, for any nonempty subset C of A, then A is a minimal λ_{rc}-open set.*

Proof. Suppose that A is not a minimal λ_{rc}-open set. Then there exists a nonempty λ_{rc}-open set B such that $B \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin B$. Then we have $\lambda_{rc}Cl(\{x\}) \subseteq X \setminus B$ implies that $\lambda_{rc}Cl(\{x\})$ is a proper subset of $\lambda_{rc}Cl(A)$. And the result follows. □

Remark 3.5. For simplify, we assume that λ is an s-regular operation in the remainder of this section three, such as in Proposition 3.7, 3.8, 3.9, 3.10, Corollary 3.11, 3.12 and Theorem 3.13. Observe that if the condition of λ is not an s-regular operation, then the intersection of two λ_{rc}-open sets not necessarily is a λ_{rc}-open set, as we can see in the following Example.

Example 3.6. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\}, \{b\}$ and $\lambda(A) = \{a, b\}$ if $A = \{a\}$ or $\{b\}$.
$SO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
$SO_\lambda(X) = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$.
$SO_{\lambda_{rc}}(X) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$.
Clearly λ is not a s-regular operation and the intersection of the λ_{rc}-open sets $\{a, c\}$ and $\{b, c\}$ is not a λ_{rc}-open.

Proposition 3.7. *The following statements are true:*

1. *If A is a minimal λ_{rc}-open set and B a λ_{rc}-open set, By proposition 2.13, $A \cap B$ is a λ_{rc}-open set. Then $A \cap B = \emptyset$ or $A \subseteq B$.*

2. *If B and C are minimal λ_{rc}-open sets. Then $B \cap C = \emptyset$ or $B = C$.*

Proof. (1) Let B be a λ_{rc}-open set such that $A \cap B \neq \emptyset$. Since A is a minimal λ_{rc}-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore $A \subseteq B$.
(2) If $A \cap B \neq \emptyset$, then by (1), we have $B \subseteq C$ and $C \subseteq B$. Therefore, $B = C$. □

Proposition 3.8. *Let A be a minimal λ_{rc}-open set. If $x \in A$, then $A \subseteq B$ for any λ_{rc}-open neighborhood B of x.*

Proof. Let B be a λ_{rc}-open neighborhood of x such that A is not contained in B. Since λ is a s-regular operation, then $\emptyset \neq A \cap B$ is a λ_{rc}-open set. This contradicts our assumption that A is a minimal λ_{rc}-open set. □

Proposition 3.9. *Let A be a minimal λ_{rc}-open set. Then for any $x \in A$, $A = \cap\{B : B \text{ is } \lambda_{rc} - \text{open neighborhood of } x\}$.*

Proof. By Proposition 3.4 and the fact that A is λ_{rc}-open neighborhood of x, we have $A = \cap\{B : B \text{ is } \lambda_{rc} - \text{open neighborhood of } x\} \subseteq A$. Therefore, the result follows. □

Proposition 3.10. *If A is a minimal λ_{rc}-open set in X not containing $x \in X$. Then for any λ_{rc}-open neighborhood C of x, either $C \cap A = \emptyset$ or $A \subseteq C$.*

Proof. Since C is a λ_{rc}-open set, we have the result by Proposition 3.3. □

Corollary 3.11. *If A is a minimal λ_{rc}-open set in X not containing a point $x \in X$. If $A_x = \cap\{B : B \text{ is } \lambda_{rc} - \text{open neighborhood of } x\}$. Then either $A_x \cap A = \emptyset$ or $A \subseteq A_x$.*

Proof. If $A \subseteq B$ for any λ_{rc}-open neighborhood B of x, then $A \subseteq \cap\{B : B \text{ is } \lambda_{rc} - \text{open neighborhood of } x\}$. Therefore $A \subseteq A_x$. Otherwise there exists a λ_{rc}-open neighborhood B of x such that $B \cap A = \emptyset$. Then we have $A_x \cap A = \emptyset$. □

Corollary 3.12. *If A is a nonempty minimal λ_{rc}-open set of X, then for a nonempty subset C of A, $A \subseteq \lambda_{rc}Cl(C)$.*

Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any λ_{rc}-open neighborhood of y. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus we have $B \cap C \neq \emptyset$ and hence $y \in \lambda_{rc}Cl(C)$. This implies that $A \subseteq \lambda_{rc}Cl(C)$. □

Combining Corollary 3.12 and Propositions 3.3 and 3.4, we have:

Theorem 3.13. *Let A be a nonempty λ_{rc}-open subset of space X. Then the following are equivalent:*

1. *A is minimal λ_{rc}-open set, where λ is s-regular.*
2. *For any nonempty subset C of A, $A \subseteq \lambda_{rc}Cl(C)$.*
3. *For any nonempty subset C of A, $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(C)$.*

4. Finite λ_{rc}-open sets

In this section, we study some properties of minimal λ_{rc}-open sets in finite λ_{rc}-open sets and λ_{rc}-locally finite spaces.

Proposition 4.1. *Let (X, τ) be a topological space and $\emptyset \neq B$ a finite λ_{rc}-open set in X. Then there exists at least one (finite) minimal λ_{rc}-open set A such that $A \subseteq B$.*

Proof. Suppose that B is a finite λ_{rc}-open set in X. Then we have the following two possibilities:

1. B is a minimal λ_{rc}-open set.
2. B is not a minimal λ_{rc}-open set.

In case (1), if we choose $A = B$, then the proposition is proved. If the case (2) is true, then there exists a nonempty (finite) λ_{rc}-open set B_1 which is properly contained in B. If B_1 is minimal λ_{rc}-open, we take $A = B_1$. If B_1 is not a minimal λ_{rc}-open set, then there exists a nonempty (finite)λ_{rc}-open set B_2 such that $B_2 \subseteq B_1 \subseteq B$. We continue this process and have a sequence of λ_{rc}-open sets $\subseteq B_m \subseteq ... \subseteq B_2 \subseteq B_1 \subseteq B$. Since B is a finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal λ_{rc}-open set B_k such that $A = B_k$. This completes the proof. □

Definition 4.2. *A space X is said to be a λ_{rc}-locally finite space, if for each $x \in X$ there exists a finite λ_{rc}-open set A in X such that $x \in A$.*

Definition 4.3. *Let $X = \mathbb{R}$ and $\tau = P(\mathbb{R})$. We define an s-operation $\lambda : SO(\mathbb{R}) \to P(\mathbb{R})$ as $\lambda(A) = A$ for every subset A of $\mathbb{R}$. Then $(\mathbb{R}, \tau)$ is a λ_{rc}-locally finite space*

Corollary 4.4. *Let X be a λ_{rc}-locally finite space and B a nonempty λ_{rc}-open set. Then there exists at least one (finite) minimal λ_{rc}-open set A such that $A \subseteq B$, where λ is s-regular.*

Proof. Since B is a nonempty set, there exists an element x of B. Since X is a λ_{rc}-locally finite space, we have a finite λ_{rc}-open set B_x such that $x \in B_x$. Since $B \cap B_x$ is a finite λ_{rc}-open set, we get by Proposition 4.1, a minimal λ_{rc}-open set A such that $A \subseteq B \cap B_x \subseteq B$. □

Proposition 4.5. *Let X be a space and for any $\alpha \in I$, B_α a λ_{rc}-open set and $\emptyset \neq A$ a finite λ_{rc}-open set. Then $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a finite λ_{rc}-open set, where λ is s-regular.*

Proof. We see that there exists an integer n such that $A \cap (\bigcap_{\alpha \in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_{\alpha_i}))$ and hence we have the result. □

Using Proposition 4.5, we can prove the following:

Theorem 4.6. *Let X be a space and for any $\alpha \in I$, B_α is a λ_{rc}-open set and for any $\beta \in I$, B_β is a nonempty finite λ_{rc}-open set. Then $(\bigcup_{\beta\in I} B_\beta) \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a λ_{rc}-open set, where λ is s-regular.*

It is important to know that the notions of λ_{rc}-locally finite space and locally finite space are independent, because the family of λ_{rc}-open sets and the family of open set in a topological (X, τ) with an s-operation $\lambda : SO(X) \to P(X)$ are independent.

5. More properties

Let A be a nonempty finite λ_{rc}-open set. It is clear, by Proposition 3.3 and 4.1, that if λ is s-regular, then there exists a natural number m such that$\{A_1, A_2, \dots, A_m\}$ is the class of all minimal λ_{rc}-open sets in A satisfying the following two conditions:

1. For any i, n with $1 \le i, n \le m$ and $i \ne n$, $A_i \cap A_n = \emptyset$.
2. If C is a minimal λ_{rc}-open set in A, then there exists i with $1 \le i \le m$ such that $C = A_i$.

Theorem 5.1. *Let X be a space and $A \ne \emptyset$ be a finite λ_{rc}-open set which is not a minimal λ_{rc}-open set. Let $\{A_1, A_2, \dots, A_m\}$ be a class of all minimal λ_{rc}-open sets in A and $y \in A \setminus (\bigcup_{i=1}^m A_i)$. Define $A_y = \cap\{B_y\}$, where B_y is a λ_{rc}-open neighborhood of y. Then there exists a natural number $k \in \{1, 2, 3, \dots, m\}$ such that A_k is contained in A_y, where λ is s-regular.*

Proof. Suppose on the contrary that for any natural number $k \in \{1, 2, 3, \dots, m\}$, A_k is not contained in A_y. By Proposition 3.7, for any minimal λ_{rc}-open set A_k in A, $A_k \cap A_y = \emptyset$. By Proposition 4.5, $\emptyset \ne A_y$ is a finite λ_{rc}-open set. Therefore by Proposition 4.1, there exists a minimal λ_{rc}-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have C is a minimal λ_{rc}-open set in A. By supposition, for any minimal λ_{rc}-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \emptyset$. Therefore, for any natural number $k \in \{1, 2, 3, \dots, m\}$, $C \ne A_k$. This contradicts our assumption. Hence the proof. □

Proposition 5.2. *Let X be a space and $A \ne \emptyset$ be a finite λ_{rc}-open set which is not a minimal λ_{rc}-open set. Let $\{A_1, A_2, \dots, A_m\}$ be a class of all minimal λ_{rc}-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, 3, \dots, m\}$, such that for any λ_{rc}-open neighborhood B_y of y, A_k is contained in B_y, where λ is s-regular.*

Proof. This follows from Theorem 5.1. □

Theorem 5.3. *Let X be a space and $A \ne \emptyset$ be a finite λ_{rc}-open set which is not a minimal λ_{rc}-open set. Let $\{A_1, A_2, \dots, A_m\}$ be a class of all minimal λ_{rc}-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, 3, \dots, m\}$, such that $y \in \lambda_{rc}Cl(A_k)$. where λ is s-regular.*

Proof. It follows from Proposition 5.2, that there exists a natural number $k \in \{1, 2, 3, \ldots, m\}$ such that $A_k \subseteq B$ for any λ_{rc}-open neighborhood B of y. Therefore $\emptyset \neq A_k \subseteq A_k \cap B$ and then, $y \in \lambda_{rc}Cl(A_k)$. □

Proposition 5.4. *Let $A \neq \emptyset$ be a finite λ_{rc}-open set in a space X and for each $k \in \{1, 2, 3, \ldots, m\}$, A_k is a minimal λ_{rc}-open set in A. If the class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal λ_{rc}-open sets in A, then for any $\emptyset \neq B_k \subseteq A_k$, $A \subseteq \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, where λ is s-regular.*

Proof. If A is a minimal λ_{rc}-open set, then this is the result of Theorem 3.13(2). Otherwise, when A is not a minimal λ_{rc}-open set. If x is any element of $A \setminus (A_1 \cup A_2 \cup A_3 \cup ... \cup A_m)$, then by Theorem 5.3, $x \in \lambda_{rc}Cl(A_1) \cup \lambda_{rc}Cl(A_2) \cup ....\lambda_{rc}Cl(A_m)$. Therefore, by Theorem 3.13 (3), we obtain that $A \subseteq \lambda_{rc}Cl(A_1) \cup \lambda_{rc}Cl(A_2) \cup ....\lambda_{rc}Cl(A_m) = \lambda_{rc}Cl(B_1) \cup \lambda_{rc}Cl(B_2) \cup ....\lambda_{rc}Cl(B_m) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$. □

Proposition 5.5. *Let $A \neq \emptyset$ be a finite λ_{rc}-open set and for each $k \in \{1, 2, 3, \ldots, m\}$, A_k is a minimal λ_{rc}-open set in A. If for any $\emptyset \neq B_k \subseteq A_k$, $A \subseteq \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, then $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$.*

Proof. For any $\emptyset \neq B_k \subseteq A_k$, with $k \in \{1, 2, 3, \ldots, m\}$, we have $\lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) \subseteq \lambda_{rc}Cl(A)$. Also, we have $\lambda_{rc}Cl(A) \subseteq \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m) = \lambda_{rc}Cl(B_1) \cup \lambda_{rc}Cl(B_2) \cup \lambda_{rc}Cl(B_3) \cup ... \cup \lambda_{rc}Cl(B_m)$. Therefore, $\lambda_{rc}Cl(A) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, for any nonempty subset B_k of A_k with $k \in \{1, 2, 3, \ldots, m\}$. □

Proposition 5.6. *Let $A \neq \emptyset$ be a finite λ_{rc}-open set and for each $k \in \{1, 2, 3, \ldots, m\}$, A_k is a minimal λ_{rc}-open set in A. If for any $\emptyset \neq B_k \subseteq A_k$, $\lambda_{rc}Cl(A_k) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, then the class $\{A_1 \cup A_2 \cup A_3 \cup ... \cup A_m\}$ contains all minimal λ_{rc}-open sets in A.*

Proof. Suppose that C is a minimal λ_{rc}-open set in A and $C \neq A_k$ for $k \in \{1, 2, 3, \ldots, m\}$. Then we have $C \cap \lambda_{rc}Cl(A_k) = \emptyset$ for each $k \in \{1, 2, 3, \ldots, m\}$. It follows that any element of C is not contained in $\lambda_{rc}Cl(A_1 \cup A_2 \cup A_3 \cup ... \cup A_m)$. This is a contradiction to the fact that $C \subseteq A \subseteq \lambda_{rc}Cl(A) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$. This completes the proof. □

Combining Propositions 5.4, 5.5 and 5.6, we have the following theorem:

Theorem 5.7. *Let A be a nonempty finite λ_{rc}-open set and A_k a minimal λ_{rc}-open set in A for each $k \in \{1, 2, 3, \ldots, m\}$. Then the following three conditions are equivalent:*

1. *The class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal λ_{rc}-open sets in A.*
2. *For any $\emptyset \neq B_k \subseteq A_k$, $A_k \subseteq \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$.*
3. *For any $\emptyset \neq B_k \subseteq A_k$, $\lambda_{rc}Cl(A_k) = \lambda_{rc}Cl(B_1 \cup B_2 \cup B_3 \cup ... \cup B_m)$, where λ is s-regular.*

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Accepted: 20.03.2019

Pseudo analytic approach to estimate drug transport and release in the annular section of human limbs

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Abstract. A mathematical model has been developed to estimate the concentration of transdermal drug transport in an annular section of the human forearm. The formulation of the model is based on the radial and angular diffusion equation together with appropriate boundary conditions. An analytic method has been employed to determine the steady-state concentration of the drug in the annular region of dermal system and the unsteady-state concentration of drug release and transport has been computed using finite difference explicit method. The proposed model may be useful for drug transport in human subjects especially for the application of drug through transdermal drug delivery system. The model has applications in biomedical sciences especially while dealing with the patients having oral and intravenous drug issues.
Keywords: transdermal drug diffusion, Dirichlet's problem, separation of variables method, finite difference method.

1. Introduction

The suitable administration route for strong and low molecular weight drugs is considered to be transdermal drug delivery. Its main advantage is that it is a substitute to tablets and injections. This delivery system is mainly concerned with the delivery device and anatomy of dermal region, which consists of uppermost stratum corneum and underlying layers of stratum germinativum, dermis and subcutaneous tissue[2]. The transdermal drug delivery system(TDDs) is a suitable format in which the drug is applied externally either through a reservoir in contact with the outermost layer or through periodic application. The drug and the delivery system are designed in such a way that the drug reaches the targeted area with prescribed concentration.

Over the last 50 years, mathematical modeling on the diffusional and release processes has been used to design a number of simple and complex drug delivery systems and devices to predict the overall release behaviour and diffusion of the

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drug. R.A. Gatenby and E.T. Gawlinski developed a reaction-diffusion model of cancer invasion[3]. They have developed a model encompassing the key components of their hypothesis predicting an acidic pH gradient extending into the peritumoral tissue, which they confirmed by reanalysis of extant experimental data. A two-layer reaction-diffusion-convection model for drug delivery in biological tissues was developed by S. McGinty and G. Pontrelli[11]. They have presented a general model of drug release from a drug delivery device(DDD) and the subsequent drug transport in biological tissue. Feizabadi *et al* have developed a two compartment interacting with the dynamic drugs[7]. They have combined the total cell evolution curve and a two compartment model interacting with dynamic anti-cancer agents. They have analytically obtained the evolution of subpopulations. N.A. Peppas and B.Narasimhan have developed mathematical models in drug delivery to predict how the new drug delivery systems can be designed[12].

Further Khanday and Rafiq have studied the absorption rate of drug at various compartments through TDD system[8]. Khanday *et al* have also developed some mathematical models for drug diffusion through the compartments of blood and tissue medium[6]. They have established mathematical models to understand the distribution of drug administration in human body through oral and intravenous routes. They formulated three models based on diffusion process using Fick's principle and law of mass action. Distribution of drug in a sample of five layers of human skin was also studied by A. Sharma and V.P. Saxena[13]. They have constructed a mathematical model to study the drug concentration in the different layers of skin through transdermal drug delivery system. They have used the finite element method with linear shape functions to obtain the solution of governing one dimensional partial differential equation for unsteady state case. Further, mathematical and computational models of drug transport in tumors were also developed by C.M. Groh *et al*[5]. They considered three different modelling approaches, each of which represented drug delivery from a central blood vessel to a surrounding tumor cord. Their models were based on the assumption of axial uniformity- the dependence of drug concentration on the distance from the central vessel does not vary along the vessel, to reduce the complexity of their models.

The mathematical models based on radial and angular diffusion were studied by various researchers. Heat and mass diffusions were extensively studied by Khanday and his co-workers[6, 8, 9, 10] however, the drug diffusion in an annular region of human forearm has not been studied so far. The transport of drug has been studied in the dermal and the muscular regions of the forearm. In the steady-state, we have taken the three regions of the dermal section viz: epidermis, dermis and hypodermis, while as in the unsteady-state case, epidermis, dermis and hypodermis have been collectively taken as the skin(dermal) region. Consider the boundary of the region with one end at the boundary of the bone and the other end at the skin surface. The formulation is based on the mass diffusion equation with appropriate boundary and initial conditions, and the solutions have been established analytically and numerically respectively for the steady-state and unsteady-state cases. The proposed work can provide the details of drug transport in the annular region defined above.

2. Mathematical model

Consider the annular section of the human forearm with radii respectively as r_1 *cm* and r_2 *cm* such that $0 < r_1 < r_2$ as shown in Fig. 1 and Fig. 2.

Figure 1: Cylindrical sample section of human forearm.

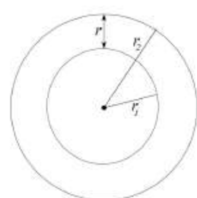

Figure 2: Annular Cross-section of the limb.

The drug is applied at the outer skin surface $r = r_2$ and the estimation of the drug transport in the ring shaped(annular) region(skin and muscle) can be established using suitable partial differential equation. The diffusion equation in plane polar coordinates for the transport and diffusion of a drug in the cross-section of the annular region of human forearm is given by Crank[1]

$$\frac{1}{D}\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{1}{r^2}\frac{\partial^2 C}{\partial \theta^2} + R(r,\theta) \tag{1}$$

where $C(r,\theta,t)$ is the drug concentration in the annular dermal region $r_1 \leq r \leq r_2$, t denotes the time, θ determines the angular direction of drug, D is the diffusion coefficient and R is the rate of metabolic drug consumption.

It is imperative to see that the transport of the drug from the skin surface towards the deep core has a non-linear behaviour and at any radial distance, the oscillation of the drug flow has been assumed by using the following initial and boundary conditions :

$$C(r_2, \theta, t) = c_0 \; at \; t = 0 \; and \; \forall \; 0 \leq \theta \leq \; 2\pi \tag{2}$$

$$C(r_2, \theta, t) = \sin(\theta - \frac{\pi}{2}t) \; \forall \; 0 \leq \theta \leq \; 2\pi \; and \; \forall \; t > 0 \tag{3}$$

$$C(r_1, \theta, t) = \cos(\theta - \frac{\pi}{2}t) \; \forall \; 0 \leq \theta \leq \; 2\pi \; and \; \forall \; t > 0 \tag{4}$$

2.1 Steady-state case

In this case, Eq. (1) reduces to

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{1}{r^2}\frac{\partial^2 C}{\partial \theta^2} = -R(r, \theta) \tag{5}$$

which is a non homogeneous partial differential equation.

The solution of Eq.(5) is sum of two terms, namely $C_1(r, \theta)$ which represents the complementary function corresponding to the solution of the homogeneous part of Eq.(5) and $C_2(r, \theta)$, which represents the particular integral corresponding to the non-homogeneous part of Eq.(5). Also, the boundary conditions in Eqs.(2) ,(3),(4) for the homogeneous part of Eq.(5), become respectively as :

$$C_1(r_2, \theta) = \sin(\theta) \; \forall \; 0 \leq \theta \leq \; 2\pi \tag{6}$$

$$C_1(r_1, \theta) = \cos(\theta) \; \forall \; 0 \leq \theta \leq \; 2\pi \tag{7}$$

Assume the periodicity condition as:

$$C_1(r, \theta + 2\pi) = C(r, \theta), \;\; r_1 \leq r \leq r_2 \tag{8}$$

The formulation part of the model is given by the boundary-value problem defined by Eqs.(5) - (8). The solution of the model can help us to estimate the steady-state drug concentration at various sections of the annular region given in Fig. (2).

2.1.1 Method

The model Eq. (5) is a non homogeneous equation in nature, its complete solution will be of the form

$$C(r, \theta) = C_1(r, \theta) + C_2(r, \theta) \tag{9}$$

where $C_1(r, \theta)$ and $C_2(r, \theta)$ respectively represent the solution of homogeneous part and the particular solution of Eq.(5).

Now considering the homogeneous part of Eq.(5) along with the conditions given by Eqs.(6) -(8), it becomes Dirichlet's boundary value problem for the annulus. The analytical solution of the problem is given as

$$\begin{aligned} C_1(r, \theta) = a_0 + \alpha_0 log(r) + \sum_{n=1}^{\infty} \big[&(a_n r^n + \alpha_n r^{-n}) \cos(n\theta) + (b_n r^n \\ &+ \beta_n r^{-n}) \sin(n\theta) \big] \end{aligned} \tag{10}$$

where the coefficients a_0,α_0,a_n,α_n,b_n and β_n are determined and are given in Appendix.

Also the particular integral of Eq.(5) is

$$C_2(r,\theta) = \frac{1}{D_1^2 + D'^2} F(r,\theta) \tag{11}$$

where $D_1 = \frac{\partial}{\partial z}$, $z = log\ r$, $D' = \frac{\partial}{\partial \theta}$ and $F(r,\theta) = -r^2 R(r,\theta) = \frac{c_0}{r}\cos(\theta)$ (because the concentration decreases radially).

Now,

$$C_2(r,\theta) = \frac{1}{D_1^2 + D'^2}\frac{c_0}{r}\cos(\theta) = c_0 \frac{1}{D_1^2 + D'^2} e^{-z}\cos(\theta) = -\frac{c_0}{2r}\cos(\theta)$$

Therefore the complete solution of Eq.(5) is

$$C(r,\theta) = \left[(a_1 r + \alpha_1 r^{-1})\cos(\theta) + (b_1 r + \beta_1 r^{-1})\sin(\theta)\right] - \frac{c_0}{2r}\cos(\theta) \tag{12}$$

where the coefficients are defined in Appendix.

After finding the solution completely, we now assign different values to the physiological parameters used in the model depending on the sample of the annular section of the forearm under study. Some of the values of the parameters were taken from M.A. Khanday *et al*[10] in order to determine the drug concentrations in the mentioned regions using Eq. (12). Eq. (12) can help us in finding out the drug absorption and release in steady-state case by assigning values of the parameters.

2.1.2 Results

The drug concentration profiles in the annular region were computed taking the initial concentration $c_0 = 2.5 mg/cm^3$. The numerical values of the physiological parameters used and the results obtained are shown in Table 1. The computed values are given as follows: $C(r = 0.25) = C(0.25) = 0.42 mg/cm^3$, $C(r = 0.29) = C(0.29) = 0.21 mg/cm^3$, $C(r = 0.16) = C(0.16) = 0.43 mg/cm^3$, $C(r = 1.28) = C(1.28) = 0.02 mg/cm^3$.

For fixed $\theta = \pi/4$ and applying Lagranges's interpolation formula [4], the interpolation polynomial is given as

$$C(r) = 39.6r^3 - 67.3r^2 + 22.4r - 1.5 \tag{13}$$

The graph in Fig. 3 has been plotted between drug release/absorption versus radial distance of the annular region. The size of the ring was taken as $(1.02 \leq r \leq 3.0) cm$, where it was assumed that bone radius is 1.02 cm and the size of the limb is 3.0 cm.

Further the graphs given in Figs.[4] and [5] are plotted according to the equation(12).

Table 1: Physiological and Numerical values of parameters and Concentration in the different sections of the annular region[10].

Parameter	Epidermis	Dermis	Hypodermis	Muscle
r_1	$2.75\,cm$	$2.46\,cm$	$2.30\,cm$	$1.02\,cm$
r_2	$3.00\,cm$	$2.75\,cm$	$2.46\,cm$	$2.30\,cm$
r	$0.25\,cm$	$0.29\,cm$	$0.16\,cm$	$1.28\,cm$
Q	1.09	1.11	1.06	2.25
Q^{-1}	0.91	0.89	0.94	0.44
$Q - Q^{-1}$	0.18	0.22	0.12	1.81
a_1	$-1.85\ cm^{-1}$	$-1.65\ cm^{-1}$	$-3.38\ cm^{-1}$	$-0.24\ cm^{-1}$
α_1	$16.66\ cm$	$12.5\ cm$	$20.5\ cm$	$1.27\ cm$
b_1	$2.02\ cm^{-1}$	$1.84\ cm^{-1}$	$3.62\ cm^{-1}$	$0.54\ cm^{-1}$
β_1	$-15.27\ cm$	$-11.18\ cm$	$-19.16\ cm$	$-0.54\ cm$
C	$0.42\ mg/cm^3$	$0.21\ mg/cm^3$	$0.43\ mg/cm^3$	$0.02\ mg/cm^3$

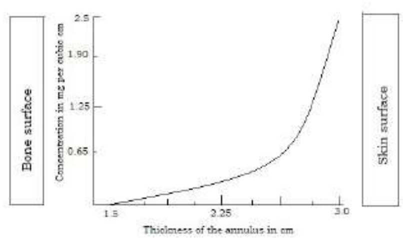

Figure 3: Drug Concentration in the annular section from the skin towards the core with $c_0 = 2.5mg/cm^3$

2.2 Un-steady state case

The unsteady-state diffusion equation in polar coordinates for the transport and diffusion of a drug is given in Eq.(1) along with the associated conditions given in Eqs.(2)-(4). In order to determine the time dependent drug distribution in the annular region of the human limb, the two subsections denoted by 1 and 2 in Fig. $6(a)$ were studied and the uniform treatment can be followed for other regions. The discretization of these regions is shown in Fig. $6(b)$, where nine nodal points are shown for bone, muscle and skin regions. The value of each of the nine nodal points has to be calculated for $k = 0, 1, 2$ time levels, so that we actually have twenty-seven nodal points. The subscripts d and m represent the parameters related to these regions respectively. The physiological and parametric behaviour of these regions is given below:

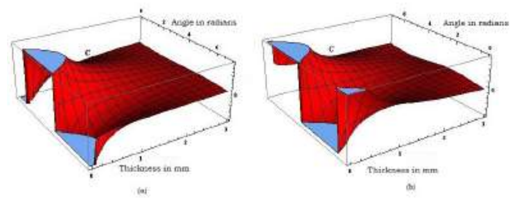

Figure 4: Drug flow in (a) epidermis and (b) dermis.

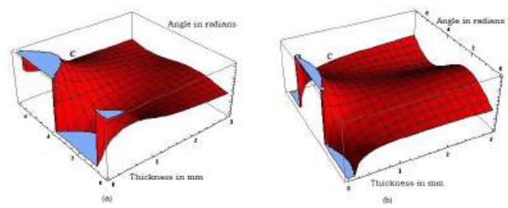

Figure 5: Drug flow in (a) hypodermis and (b) muscle.

Skin: In the uppermost layer of this region(epidermis), there are no blood vessels and hence almost negligible metabolic activity [9], thus we take $D_d = D_1 = constant = 0.002cm^2/min$ and $R_d = R_0 = 0$.

Muscular region: For this case, we take $D_m = D_2 = constant = 0.00204cm^2/min$ and $R_m = R_i = \frac{c_0}{(ih)^3}\cos(\pi/4)$ for $i = 1, 2$.

2.2.1 Method

On employing the explicit finite difference method to solve the boundary value problem given in Eqs.(1)-(4), we have

$$\begin{aligned} a_iC_{i,j,k+1} = \{a_i - d_i - 2(b_i + 1)\}C_{i,j,k} + (b_i + d_i)C_{i+1,j,k} + b_iC_{i-1,j,k} \\ + C_{i,j+1,k} + C_{i,j-1,k} + e_i \end{aligned} \tag{14}$$

$$C_{0,j,0} = c_0 = 2.5\ mg/cm^3\ \forall\ j \tag{15}$$

$$C_{0,j,k} = \sin(jk' - \frac{\pi}{2}kl)\ \forall\ k \neq 0\ and\ \forall\ j \tag{16}$$

$$C_{2,j,k} = \cos(jk' - \frac{\pi}{2}kl)\ \forall\ k \neq 0\ and\ \forall\ j \tag{17}$$

where the values of unknown parameters a_i, b_i, d_i and e_i are given in Appendix.

For $j = 0, 1, 2$ in Eq.(15), we obtain the values of $C_{0,0,0}, C_{0,1,0}$ and $C_{0,2,0}$, and are all found to be equal to $c_0 = 2.5\ mg/cm^3$. Also we obtained that $R_1 = 5.1mg/cm^6$, $R_2 = 0.6mg/cm^6$.

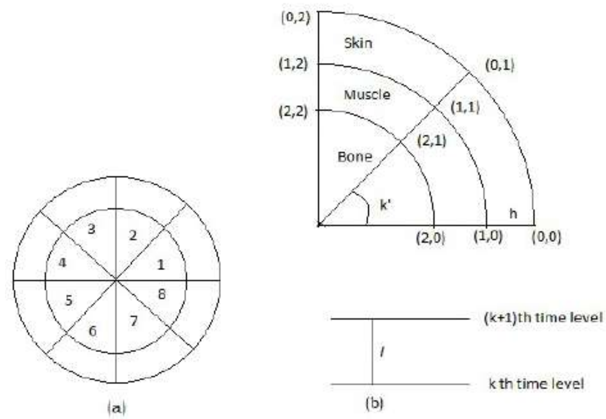

Figure 6: (a) Division of the annular cross-section into 8 sub-sections and (b) Layer-wise discretization of skin and muscle,$C_{i,j,k}$(i-space step parameter in cm, j-angular step parameter in radians and k-time step parameter in min based on finite difference method) is the nodal concentration at $(i, j, k)th$ place of the interface.

Now, using $j = 0, 1, 2$ and then $k = 1, 2$ in Eqs. (16) and (17), we obtain the nodal concentrations $C_{0,0,1}, C_{0,0,2}, , C_{0,1,1}, C_{0,1,2}, C_{0,2,1}, C_{0,2,2}, C_{2,0,1}, C_{2,0,2}, C_{2,1,1}$, $C_{2,1,2}, C_{2,2,1}$ and $C_{2,2,2}$.

Continuing in this way, we shall be able to compute the values of all nodal points of the sub-domains given by the symbols $C_{i,j,k}$, representing the nodal values of the drug concentration of the discretized region shown in Fig. 5 using Eq.(14).

2.2.2 Results

Assigning different values to the physiological parameters used in the model depending on the properties and the sample of the forearm(annular section) of skin and muscle under study, we have calculated the drug concentrations at various nodal points. The values have been calculated for $h = 0.70\ cm$ for skin, $h = 1.28\ cm$ for muscle,$l = 5\ min$ and $k' = \pi/4$, as given below in the following Tables $2 - 5$. Since,when the time has not started or when the drug has not been applied, there will be no diffusion of the drug inside the skin or muscle, so we choose $C_{1,0,0} = C_{1,1,0} = C_{1,2,0} = C_{2,0,0} = C_{2,1,0} = C_{2,2,0} = 0$.

3. Discussion

The mathematical model has been formulated to study the drug transport in an annular region of human forearm. The model has been solved for steady

Table 2: Numerical values of the coefficients appearing in Eq.(14).

Parameter	Value	Parameter	Value
a_0	0	d_0	0
a_1	30.1	d_1	0.6
a_2	120.7	d_2	1.2
b_0	0	e_0	0
b_1	0.6	e_1	$-1.5\ mg/cm^3$
b_2	2.4	e_2	$-0.6\ mg/cm^3$

Table 3: Concentration at nodal points corresponding to $i = 0$(i.e; at exposed skin surface).

Concentration-nodal points	Values in mg/cm^3
$C_{0,0,0}$	2.5
$C_{0,1,0}$	2.5
$C_{0,2,0}$	2.5
$C_{0,0,1}$	1
$C_{0,0,2}$	0
$C_{0,1,1}$	0.7
$C_{0,1,2}$	0.7
$C_{0,2,1}$	0
$C_{0,2,2}$	1

and unsteady-state cases using analytical and numerical methods respectively. For the steady-state case, the solution has been established using Dirichlet's boundary value problem and Lagrange's interpolation scheme, and the boundary conditions were constructed on the basis of drug/cream/ointment pasted on the skin surface. For the unsteady-state case, the solution was established using the explicit finite difference method where the domain is taken as the skin(dermal regions) and the muscular region. The transport of drug takes place through the dermal regions. The presence of pores on the skin surface and other biophysical and physiological parameters support the flow of drug towards the inner core. Since the formulation of the model has been carried out on a ring shaped annular region, both radial and angular variables play a key role to understand the mechanism of drug transport in this region. We assumed that the flux of drug at the skin surface satisfies sinusoidal pattern due the fact that drug transports in all directions and the rare presence of holes at the skin surface. The same argument is applicable at the bone surface which is infact the interior of the annulus. The partial differential equation (1) together with the appropriate boundary conditions were used to estimate the drug transport in the region $(0 <)r_1 < r < r_2$. The model equations were solved by using method of separation

Table 4: Concentration at nodal points corresponding to $i = 1$(i.e; at the interface between the skin and the muscle).

Concentration-nodal points	Values in mg/cm^3
$C_{1,0,1}$	0.08
$C_{1,0,2}$	0.04
$C_{1,1,1}$	0.00066
$C_{1,1,2}$	0.0046
$C_{1,2,1}$	0.00066
$C_{1,2,2}$	0.0093

Table 5: Concentration at nodal points corresponding to $i = 2$(i.e; at the interface between the bone and the muscle).

Concentration-nodal points	Values in mg/cm^3
$C_{2,0,1}$	0
$C_{2,0,2}$	1
$C_{2,1,1}$	0.7
$C_{2,1,2}$	0.7
$C_{2,2,1}$	1
$C_{2,2,2}$	0

of variables and Fourier expansions for the steady-state case and by the explicit finite difference method for the unsteady-state case. Our proposed model has an advantage that at any time we can find the value of the drug concentration inside the given region if the initial concentration is known.

4. Conclusion

It has been observed that the drug release and absorption in dermal and muscular regions has a continuous pattern from the outer surface towards the inner core through TDD route. The pattern of drug transport is shown in Fig. 3 with initial drug concentration of $c_0 = 2.5\ mg/cm^3$ while keeping one variable (θ) fixed. Also graphs given in Figs.[4] and [5] are plotted according to the Eqn.(12) which show the drug flow in the regions epidermis, dermis, hypodermis and muscle respectively, which describe that the concentration of the drug in each region is maximum near the boundaries of the regions and remains nearly uniform in the centre of each region in the steady state case.

The maximum drug absorption takes place in the papillary and reticular regions of human dermal system due to dense network of blood vessels. In the muscular region, the flow is comparatively steep as shown by the curve of the graph in Fig. 3. The solution for the unsteady-state case as given by Eqn.(14) depicts that the drug concentration can be found at any time in any region of the domain. This study can help the medical scientists and allied researchers to understand the residual drug concentration and absorption at various tissues of the human forearm. Further, since the model does not possess an exact solution for the unsteady state case, this is a severe limitation of this model, hence this work can further be improved if an exact solution can be found.

Acknowledgements

The authors are highly thankful to National Board for Higher Mathematics, Department of Atomic Energy, Govt. of India for providing financial support under award no. 2/48(15)/2016NBHM(RP)-RND&II/16941 to carry out this work.

Appendix

$a_0 = 0;\ \alpha_0 = 0;\ a_1 = -\frac{r_2^{-1}}{Q-Q^{-1}};$

$\alpha_1 = \frac{r_2}{Q-Q^{-1}};\ b_1 = \frac{r_1^{-1}}{Q-Q^{-1}};\ \beta_1 = -\frac{r_1}{Q-Q^{-1}};\ a_n = \alpha_n = b_n = \beta_n = 0\ \forall\ n \neq 1\ ;$

$Q = \frac{r_2}{r_1}$

$a_i = \frac{i^2 h^2 k'^2}{Dl};\ b_i = i^2 k'^2;\ d_i = ik'^2;\ e_i = -Ri^2 h^2 k'^2\ ;$

$i = 0,\ 1,\ 2.$

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Accepted: 25.03.2019

Some cryptographic properties of near bent functions over finite fields

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Abstract. We present a method for construction of near bent function with the help of Gold power functions. We have also investigated the cryptographical properties, i.e., non-linearity, correlation immunity, algebraic immunity and algebraic degree of these functions.

Keywords: boolean functions, near bent functions, trace, walsh coefficient, cryptography.

1. Introduction

In the modern days the communication between two individuals, or among groups of people, or social establishments requires high security of the message

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or information. Boolean functions have been studied due to their cryptographic properties for the last two decades. These functions play a significant role in constructing components of symmetric ciphers. Boolean functions used in cryptographic applications provide security of a cipher against different kinds of attacks.

Nonlinearity of a Boolean function is an essential property and functions with high non-linearity have applications in cryptography. Shannon in [1] identified that confusion and diffusion are the essential technique for performing security. Confusion could be achieved by the nonlinearity of a Boolean function. Thus, certainly one can efficiently use Boolean functions with high nonlinearity in coding theory and cryptography. For security reasons, the nonlinearity of Boolean functions must be high since the existence of affine approximations of the Boolean functions involved in a cryptosystem allows to build attacks on this system. In the case of stream ciphers, high nonlinearity is important to prevent fast correlation attacks and Linear Cryptanalysis for Block Ciphers [2,3,4].

Siegenthaler proposed the concept of correlation immunity in 1984. Correlation immunity is an interesting cryptographic property, which is to measure the level of resistance against correlation attacks. It is a safety measure for the correlation attack of nonlinear combiners. When used in a stream cipher as a combining function for linear feedback shift registers, a Boolean function with low-order correlation-immunity is more susceptible to a correlation attack than a function with correlation immunity of high order [4,5].

Algebraic immunity is a cryptographic property to measure the resistance against algebraic attack for stream ciphers. The concept of algebraic immunity of Boolean functions comes from the algebraic attack on stream ciphers proposed by Courtois and Meier in 2003 in [6], which has proven to be a very effective attack for both stream ciphers and block ciphers [4,7].

Zheng and Zhang in 1999 introduced plateaued Boolean functions for designing cryptographic functions as they have various cryptographic characteristics [8]. If squared Walsh transform of a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ takes only one nonzero value then the function is known as plateaued [9]. Moreover, if the values of its Walsh transform belong to the set $\{0, \pm 2^{\frac{n+r}{2}}\}$ for some fixed r, $0 \leq r \leq n$ then the n-variable Boolean function is said to be r-plateaued. The cases of $r = 0, 1$ and 2 have attracted much attention due to their cryptographic algebraic and combinatorial properties [5].

Bent functions are 0-plateaued functions introduced by Rothaus in the year 1976 [10]. Bent functions are perfect nonlinear functions and have interesting implications to design block ciphers as well as stream ciphers. But these functions may not be compatible with other cryptographic design criteria as these functions cannot be implemented in conjunction with balance or highest nonlinear order [11]. Near bent functions are 1- plateaued functions on $\mathbb{F}_{2^n}$ exist only when n is odd and Semi bent functions are 2- plateaued functions on $\mathbb{F}_{2^n}$ exist only when n is even, introduced by Chee et al. in the year 1995 [12]. Similar to bent functions, semi bent functions and near bent functions are also widely studied in sequences and cryptography. Unlike bent functions, semi bent functions and near bent functions are nearly perfect nonlinear so that they can be balanced and resilient. These functions are desirable for cryptographic applications as these functions have the low autocorrelation, a maximal nonlinearity among balanced plateaued functions, the high algebraic degree and satisfy the propagation criteria. They are also used for constructing the cryptographically robust S-blocks and widely used in code division multiple access (CDMA) communication systems for sequence design [5,13,14]. These semi bent and near bent functions are one of the most intensively studied topics related to bent functions.

Khoo et al. in 2002 gave the construction of n- variable quadratic semi bent functions in polynomial forms for both odd and even n [15]. Before his work, most of the researchers constructed semi bent function from power polynomials, that is, for suitably chosen d $f(x) = Tr(x^d)$. Dillon and McGuire in 2008 presented a general criterion for near bent functions to be bent on a hyperplane, and they showed that the Kasami-Welch function $Tr(x^d)$ is a bent function when restricted to the hyperplane of trace 0 elements in $\mathbb{F}_{2^n}$ [16]. Dong et al. in 2013 presented a new method for constructing semi bent function in polynomial form for both odd and even n with the help of few trace terms [17]. S. K. Pandey et al. presented an exhaustive construction of bent and balanced symmetric generalized functions (in form of ANF) on smaller domains [18].

From the above all observations, most of the researchers have focused on the construction of monomial semi bent and near bent functions. And few researchers have constructed these functions via composition and constructed in the polynomial form using few trace terms. To the best of our knowledge no work has been carried out so far to construct near bent function of the form $f(x) = (x^2+x)^d$ where d is Gold exponent (2^i+1) and we have also investigated some of the cryptographical properties as mentioned above. The improvement of cryptographic properties can be possibly expected with suitable modifications on the homomorphism functions similar to the functions used in [19].

2. Preliminaries

Definition of the near bent function is given by using Walsh-Hadamard coefficients.

Definition 2.1 ([9]). *The Walsh-Hadamard transform of a function f in n variables is the integer-valued function on $\mathbb{F}_2^n$, whose value at $a \epsilon \mathbb{F}_2^n$ is defined as*

$$W_f(a) = \sum_{x \epsilon \mathbb{F}_2^n} (-1)^{f(x) + <a \cdot x>}.$$

As an example, one can use the usual inner product over $\mathbb{F}_2^n$, which is $a \cdot x = \sum_{i=1}^n a_i x_i \pmod 2$. If the vector space $\mathbb{F}_2^n$ is viewed as the structure of the finite field $\mathbb{F}_{2^n}$, the usual inner product is nothing but $a \cdot x = tr_n(ax)$, where $tr_n(x) = \sum_{i=0}^{n-1} x^{2^i}$ is the trace function.

Definition 2.2 ([5]). *A Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is called a near bent if its Walsh transform satisfies:*

$$W_f(a) \,\epsilon\, \{0, \pm 2^{\frac{n+1}{2}}\}, \text{ for all } a \,\epsilon\, \mathbb{F}_{2^n}$$

Near bent functions on $\mathbb{F}_{2^n}$ exist only when n is odd.

Definition 2.3 ([20]). *The non-linearity of a Boolean function f is denoted by $nl(f)$ and is defined as*

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{a \epsilon \mathbb{F}_2^n} W_f(a).$$

Definition 2.4 ([14]). *A sub function of order k of Boolean function f in variables $x_1, \ldots, x_n$ is a function $f_{a_1,\ldots,a_k}^{i_1,\ldots,i_k}$ where each variable x_{i_j} is fixed by the value a_{i_j}, $j = 1, \ldots, k$, $0 < k \leq n$.*

Definition 2.5 ([14]). *A Boolean function f in n variables is called correlation immune of order k if the weight of any of its sub function of order k equals $wt(f)/2^k$.*

Definition 2.6 ([14]). *The minimum algebraic degree of a Boolean function g, $g \neq 0$, such that $f \cdot g = 0$ or $(f \oplus 1) \cdot g = 0$ is called the algebraic immunity of f, and is denoted by AI(f).*

Definition 2.7 ([21]). *If c is an element of $K = GF(q^n)$, its trace relative to the subfield $F = GF(q)$ is defined as follows:*

$$Tr_F^K(c) = c + c^q + c^{q^2} + \ldots + c^{q^{n-1}}.$$

Theorem 2.8 ([21]). *For all α, $\beta \,\epsilon\, K$ we have*

$$Tr(\alpha + \beta) = Tr(\alpha) + Tr(\beta).$$

3. Algebraic construction of near bent function

Theorem 3.1. *The function of the form*

$$f(x) = Tr(x^2 + x)^{(2^i+1)},$$

is a near bent function with $\gcd(i, n) = 1$.

Proof. $(x^2 + x)^{2^i+1} = x^{2*(2^i+1)} + x^{2*2^i} * x + x^2 * x^{2^i} + x^{2^i+1}$.

Note that in the finite field $\mathbb{F}_{2^n}$ holds $x^{2*(2^i)} = x^{2^i}$. Therefore, $(x^2+x)^{2^i+1} = x^{2^i+1}+x^{2^i}*x+x^2*x^{2^i}+x^{2^i+1} = x^{2^i+1}+x^{2^i+2}$. Walsh transform for the function $f(x) = Tr((x^2+x)^{2^i+1})$ is

$$W_f(a) = \sum_{x \epsilon \mathbb{F}_2^n} (-1)^{Tr(x^{2^i+1}+x^{2^i+2}) \oplus <a \cdot x>} \tag{1}$$

Walsh transform for the function $f(x) = Tr((x)^{2^i+1})$ is

$$W_f(a) = \sum_{x \epsilon \mathbb{F}_2^n} (-1)^{Tr(x)^{2^i+1} \oplus <a \cdot x>} \tag{2}$$

Equation (1) has same terms as that of equation (2), except that the one term $Tr(x^{2^i+2})$ is extra. So we consider the following two cases regarding extra term.

Case 1. When $Tr(x^{2^i+2}) = 0$.

The term of the Boolean function will be $Tr((x)^{2^i+1})+0$, which is $Tr((x)^{2^i+1})$. Hence it is near bent function by [14].

Case 2. When $Tr(x^{2^i+2}) = 1$.

The terms of the Boolean function will be $Tr((x)^{2^i+1})+1$, whose values will be exactly opposite to that of equation (2). As $f(x) = Tr((x)^{2^i+1})$ is near bent function, the function $f(x) = Tr((x^2+x)^{2^i+1})$ is also near bent over $\mathbb{F}_2^n$.

Thus the function defined on $\mathbb{F}_2^n$ by

$$f(x) = Tr((x^2+x)^{2^i+1})$$

with $\gcd(i,n) = 1$ is a near bent over $\mathbb{F}_2^n$. □

If $\gcd(i,n) \neq 1$, then the functions $Tr((x^2+x)^{2^i+1})$ cannot be near bent function, which can be observed over some fields from the following examples.

Example 3.2. Over $\mathbb{F}_2^5$, the function $f(x) = Tr((x^2+x)^{2^5+1})$ is not a near bent function.

In fact, the values of Walsh transform $W_f(a)$ are 0 or -32, for all $a \epsilon \mathbb{F}_2^5$.

Hence $f(x) = Tr((x^2+x)^{33})$ is not a near bent function.

Example 3.3. Over $\mathbb{F}_2^9$, the function $f(x) = Tr((x^2+x)^{2^3+1})$ is not a near bent function.

In fact, the values of Walsh transform $W_f(a)$ are 0 or ± 64, for all $a \epsilon \mathbb{F}_2^9$.

Hence $f(x) = Tr((x^2+x)^9)$ is not a near bent function.

4. Cryptographic properties of above constructed near bent functions

There are many different kinds of attacks on the stream ciphers, and hence the Boolean functions used in the stream ciphers should have essential properties. Boolean functions play an important role in both error correcting coding activities and cryptography. Indeed, cryptographic transformations can be developed by the appropriate composition of nonlinear Boolean functions. Moreover, every code of length 2^n, for some nonnegative integer n, can be interpreted as a set of Boolean functions. In both frameworks, n is rarely large, in practice. The error correcting codes derived from n-variable Boolean functions have length 2^n; so, taking $n = 11$ already gives codes of length 2048. In the case of stream ciphers, n was in general at most equal to 11 until recently [4,5,14].

Moreover, Some of the important and very common cryptographic properties of near bent functions are briefly described in next sections.

4.1 Nonlinearity

The nonlinearity of the functions which are constructed using the theorems above is tabled for some values of i and n as follows.

Table 1: Nonlinearity of function $f(x) = Tr((x^2+x)^{2^i+1})$, with (i) $\gcd(i, n) = 1$ (ii) $\gcd(i, n) \neq 1$

(i)

i \ n	3	5	7	9	11
1	2	12	56	240	992
2	2	12	56	240	992
3		12	56		992
4	2	12	56	240	992
5	2		56	240	992

(ii)

i \ n	3	5	7	9	11
1					
2					
3	0			224	
4					
5		0			

The above Tables represent the nonlinearity values of the constructed near bent functions and Boolean functions for different values of i and n. For instance, for $i = 3$ and $n = 11$, the Table (1(i)) shows that nonlinearity of the function is 992. The same comparison is true for other values of i and n.

From the above Tables (1) and (1(ii)), it is clear that the nonlinearity of a newly constructed near bent functions in the Tables (1(i)) is more than that of Boolean functions in the Tables (1(ii)).

4.2 Correlation immunity

The below Tables (2(i), 2(ii)), represent the correlation immunity of the newly constructed near bent function and Boolean function. The correlation immunity for the constructed near bent functions in Tables (2(i)) found to have the low number when compared to the Boolean function of the form in the Tables (2(ii)).

Table 2: Correlation Immunity of function $f(x) = Tr((x^2+x)^{2^i+1})$, with (i) $\gcd(i,n)=1$ (ii) $\gcd(i,n)\neq 1$.

(i)

i \ n	3	5	7	9	11
1	0	0	0	0	0
2	0	0	0	0	0
3		0	0		0
4	0	0	0	0	0
5	0		0	0	0

(ii)

i \ n	3	5	7	9	11
1					
2					
3	2			0	
4					
5		4			

4.3 Algebraic immunity

The below Tables (3(i),3(ii), represent the algebraic immunity values of the near bent functions and Boolean functions. The constructed near bent function exhibit more algebraic immunity, which can be observed by in Tables (3(i)) and (3(ii)). Further, it is worth noticing that the use of near bent functions will enhance the security of cryptosystems.

Table 3: Algebraic immunity of function $f(x) = Tr((x^2+x)^{2^i+1})$, with (i) $\gcd(i,n)=1$ (ii) $\gcd(i,n)\neq 1$.

(i)

i \ n	3	5	7	9	11
1	1	2	2	2	2
2	1	2	2	2	2
3		2	2		2
4	1	2	2	2	2
5	1		2	2	2

(ii)

i \ n	3	5	7	9	11
1					
2					
3	0			2	
4					
5		0			

5. Conclusion

The cryptographic properties nonlinearity, correlation immunity, and algebraic immunity are exhibited remarkably by near bent functions which are constructed using Gold power functions. The similar properties are expected to in case of large values of n too.

6. Acknowledgements

The corresponding author and the second author acknowledges Manipal Institute of Technology (MIT), Manipal Academy of Higher Education, India for their kind encouragement. The first author is grateful to Manipal Academy of Higher Education for their support through the Dr. T. M. A. Pai Ph. D. scholarship program.

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Accepted: 1.03.2019

On (m, n)-fully stable Banach algebra modules

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Abstract. In this paper the concept of fully-(m, n) stable Banach Algebra-module $(F-(m,n)-S-B-A$-module), we study some properties of $F-(m,n)-S-B-A$-module and another characterization have been given.

Keywords: fully stable Banach A-module, fully (m, n)-stable Banach A - module, multiplication $(m, n) - A$-module.

1. Introduction

A non-empty set A is an algebra if, $(A, +, .)$ is a vector space over a field F, $(A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in F$, for every $a, b \in A$" [1]. In [2]" a ring R is an algebra $\langle R, +, \cdot, -, 0\rangle$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $\langle R, +, -, 0\rangle$ is an abelian group, $\langle R, .\rangle$ is a semigroup and $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y + z)$". "Let A be an algebra, recall that a Banach space E is a Banach left A-module ($B - A$ - module) if E is a left A-module, and $\|a.x\| \leq \|a\| \|x\|(a \in A,\ x \in E)$" [1]. Following [3] "a map from a left $B - A$-module X into a left Banach A-module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a.x) = a.Tx$ for all $a \in A$, $x \in X$ ". In [4], "a submodule N of an R-module M is said to be stable, if $f(N) \subseteq N$ for each R-homomorphism $f : N \longrightarrow M$. M is called a fully stable module, each submodule of M is stable". "A Banach algebra module M is called $F - S - B - A$-module if for every submodule N of M and for each multiplier $\theta : N \longrightarrow M$ such that $\theta(N) \subseteq N$" [5]. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of

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all formal $m \times n$ matrices whose entries are elements of N. Let M be a right Banach Algebra-module and N be a left R-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in M^{n\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well defined element in $M^{l\times m}$ (resp. $N^{n\times k}$). "If $X \subseteq M^{l\times m}$, $S \subseteq R^{m\times n}$ and $Y \subseteq N^{n\times k}$ define

$$\begin{aligned}
\ell_{M^{l\times m}}(S) &= \left\{u \in M^{l\times m} \mid us = 0; \forall s \in S\right\}, \\
r_{N^{n\times k}}(S) &= \left\{v \in N^{n\times k} \mid sv = 0; \forall s \in S\right\}, \\
\ell_{R^{m\times n}}(Y) &= \left\{s \in R^{m\times n} \mid sy = 0; \forall y \in Y\right\}, \\
r_{R^{m\times n}}(X) &= \left\{s \in R^{m\times n} \mid xs = 0; \forall x \in X\right\}.
\end{aligned}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$ [6]. In this paper for two fixed positive integers n, m the concept of fully (m, n)-stable Banach algebra modules has been introduced.

2. Fully (m,n)-stable Banach algebra modules

"A left $B - A$-module X is n-generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1}^{n} a_k.x_k$ for some $a_1, \dots, a_n \in A$. A module which is 1-generated is called a cyclic module" [7] .

Definition 2.1. Let K be $B - A$-module, K is called (m, n)-fully stable B-A-module, if for every n-generated submodule L of K^m and for each multiplier $\theta : L \to K^m$ satisfy $\theta(L) \subseteq L$, for two fixed positive integers n, m.

In [5] "for a nonempty subset M in a left $B - A$ - module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A | a.x = 0 \, \forall x \in M\}$".

Notation. Let X be a $B - A$-module

1. $L_{x_1,x_2,\dots,x_n} = \{\oplus l_{x_i} | n \in N, x_i \in X, i = 1, 2, \dots, n\}$,
   $K_{y_1,y_2,\dots,y_n} = \{\oplus k_{y_i} | k \in K, y_i \in X, i = 1, 2, \dots, n\}$,
2. $\ell_{A^{m\times n}} L_{x_1,x_2,\dots,x_n} = \{a \in A^{m\times n}, a.(\oplus l_{x_i}) = 0, \forall \oplus l_{x_i} \in L_{x_1,x_2,\dots,x_n}\}$,
   $\ell_{A^{m\times n}} K_{y_1,y_2,\dots,y_n} = \{a \in A^{m\times n}, a.(\oplus k_{y_i}) = 0, \forall k_{y_i} \in K_{y_1,y_2,\dots,y_n}\}$.

Proposition 2.2. *A $B - A$-module M is fully-(m, n) stable, if and only if any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \dots, L_{x_1,x_2,\dots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \dots, K_{y_1,y_2,\dots,y_m}\}$ of M_n, if $\beta_j \notin \sum_{i=1}^{n} A\alpha_i$, for each $j = 1, \dots, m$ implies $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \dots, L_{x_1,x_2,\dots,x_m}\}) \nsubseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \dots, K_{y_1,y_2,\dots,y_m}\})$.*

Proof. Assume that K is $F-(m, n)-S-B-A$-module and there exist two m - element subsets $\{L_{x_1}, L_{x_1,x_2}, \dots, L_{x_1,x_2,\dots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \dots, K_{y_1,y_2,\dots,y_m}\}$ of M_n such that if $K_{y_j} \notin \sum_{i=1}^{n} A\alpha_i$, for each $j = 1, \dots, m$ and

$$\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \dots, L_{x_1,x_2,\dots,x_m}\}) \subseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \dots, K_{y_1,y_2,\dots,y_m}\}).$$

Define $f:\sum_{i=1}^n \alpha_i A \longrightarrow M^m$ by $f(\sum_{i=1}^n a_i L_{x_i}) = \sum_{i=1}^n a_i K_{y_i}$.

Let $L_{x_i} = (k_{1i}, k_{2i}, \ldots, k_{ni})$. If $\sum_{i=1}^n a_i L_{x_i} = 0$, then $\sum_{i=1}^n a_i k_{ij} = 0, j = 1, 2, \ldots, m$, implies that $rL_{x_j} = 0$ where $r = (r_1, \ldots, r_n)$ and hence $r \in \ell_{A^n}\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$. By assumption $rK_{y_j} = 0, j = 1, \ldots, m$ so $\sum_{i=1}^n r_i K_{y_i} = 0$. This shows that f is well defined. It is an easy matter to see that f is multiplier. Fully-(m,n) stability of M implies that there exists $t = (t_1, \ldots, t_n) \in A^n$ such that $f(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n t_k(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n \sum_{i=1}^n (t_k r_i) L_{x_i}$ for each $\sum_{i=1}^n r_i L_{x_i} \in \sum_{i=1}^n AL_{x_i}$.

Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = f(L_{x_i}) = \sum_{k=1}^n t_k L_{x_i} \in \sum_{i=1}^n AL_{x_i}$, which is contradiction. Conversely assume that there exists n-generated $B-A$-submodule of M^m and multiplier $\mu : \sum_{i=1} AL_{x_i} \to M^m$ such that $\mu(\sum_{i=1}^n AL_{x_i}) \nsubseteq \sum_{i=1}^n AL_{x_i}$. Then there exists an element $\beta(= \sum_{i=1}^n r_i L_{x_i}) \in \sum_{i=1}^n AL_{x_i}$ such that $\mu(K_y) \notin \sum_{i=1}^n AL_{x_i}$. Take $K_{y_i} = K_y$, $j = 1, \ldots, m$, then we have m-element subset $\{\mu(K_y), \ldots, \mu(K_y)\}$, such that $\mu(K_y) \notin \sum_{i=1}^n AL_{x_i}$, $j = 1, \ldots, m$. Let $\eta = (t_1, \ldots, t_n) \in \ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\})$, then $\eta\alpha_j = 0$, i.e $\sum_{i=1}^n t_i a_{ij} = 0$, for each $j = 1, \ldots, m$, $L_{x_j} = (a_{1j}, a_{2j}, \ldots, a_{nj})$ and $\{\mu(K_y), \ldots, \mu(K_y)\}\eta = \sum_{k=1}^n t_k \mu(K_y) = \sum_{k=1}^n t_k \mu(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n \mu(\sum_{i=1}^n t_k r_i L_{x_i}) = 0$ hence $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{\mu(K_y), \ldots, \mu(K_y)\})$, thus $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{\mu(K_{y_1}), \ldots, \mu(K_{y_1,y_2,\ldots,y_m})\}$ which is a contradiction. Thus M is $F-(m,n)-S-B-A$-module. $\square$

Corollary 2.3. *Let M be an $F-(m,n)-S-B-A$-module, then for any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n, $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\})$ implies that $AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_m} = AK_{y_1} + AK_{y_1,y_2} + AK_{y_1,y_2,\ldots,y_m}$.*

In [9], "$AB-A$ - module X is said to satisfy Baer criterion if each submodule of X satisfies Baer criterion, that is for every submodule N of X and A-multiplier $\theta : N \to X$, there exists an element a in A such that $\theta(n) = an$ for all $n \in N$".

Definition 2.4. A $B-A$ - module X is said to satisfy Baer (m,n)-criterion if each submodule of X satisfies Baer (m,n)-criterion, that is for every n-generated submodule L of X and A- multiplier $\theta : L \to X^m$, there exists an element a in A such that $\theta(l) = al$ for all $l \in L$".

Proposition 2.5. *If X satisfies Baer $(m,1)$-criterion and $\ell_A(L \cap M) = \ell_A(L) + \ell_A(M)$ for each n-generated submodules of X^m, then X satisfies Baer (m,n)-criterion.*

Proof. Let $P = Ax_1 + Ax_2 + \ldots + Ax_n$ be an n-generated submodule of X^m and $f : P \to X^m$ a multiplier. We use induction on n. It is clear that M satisfies Bear (m,n) - criterion, if $n = 1$. Suppose that M satisfies Bear (m,n)-criterion for all k-generated submodule of X^m, for $k \le n-1$. Write $L = Ax_1$, $M = Ax_2 + \ldots + Ax_n$, then for each $w_1 \in L$ and $w_2 \in M$, $f|_L(w_1) = y_1 w_1$,

$f|_M(w_2) = y_2w_2$ for some $y_1, y_2 \in A$. It is clear $y_1 - y_2 \in \ell_A(L \cap M) = \ell_A(L) + l_A(M)$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in \ell_A(L), z_2 \in \ell_A(M)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $w = w_1 + w_2 \in P$ with $w_1 \in L$ and $w_2 \in M$, $f(w) = f(w_1) + f(w_2) = w_1y_1 + w_2y_2 = w_1(y - 1 - z_1) + w_2(y_2 + z_2) = w_1y + w_2y = (w_1 + w_2)y = wy$. □

Proposition 2.6. *Let X be a $B - A$ - module. Then X satisfies Baer (m, n) criterion if and only if* $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ *for n-element subset* $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ *of X_n.*

Proof. Suppose that Baer (m, n)-criterion holds for n-generated submodule of X^m let $L_{x_i} = (k_{i1}; k_{i2}, \ldots, k_{im})$, for each $i = 1, \ldots, n$ and $K_y = \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_n}\} \in r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $K_{y_i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$.

Define $\mu : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \to X_m$ by $\mu(\sum_{i=1}^n a_iL_{x_i}) = \sum_{i=1}^n a_iK_{y_i}$. If $\sum_{i=1}^n a_iL_{x_i}$, then $\sum_{i=1}^n a_ik_{ij} = 0$. $j = 1, \ldots, m$, this implies that $rL_{x_i} = 0$ where $r = (r_1, \ldots, r_n)$ and hence $r \in \ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$. By assumption $rL_{x_i} = 0$, $i = 1, \ldots, n$ so $\sum_{i=1}^n a_iK_{y_i} = 0$. This show that f is well defined. It is an easy matter to see that μ is an multiplier. By assumption there exists $t \in A$ such that $\mu(\sum_{i=1}^n a_iL_{x_i}) = t(\sum_{i=1}^n a_iK_{y_i}) = \sum_{i=1}^n (ta_i)K_{y_i}$ for each $\sum_{i=1}^n a_iL_{x_i} \in \sum_{i=1}^n AL_{x_i}$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = \mu(\sum_{i=1}^n L_{x_i}) = \sum_{i=1}^n tL_{x_i} \in \sum_{i=1}^n AL_{x_i}$ which is contradiction. This implies that $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) \subseteq AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, the other inclusion is trivial.

Conversely, assume that $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for each $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ in X_n.

Then for each multiplier $f : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \longrightarrow X_m$ and $s = (s_1, \ldots, s_n) \in \ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $\sum_{k=1}^n s_k(\sum_{i=1}^n t_iL_{x_i}) = 0$, for each $\sum_{i=1}^n t_iL_{x_i} \in \sum_{i=1}^n AL_{x_i}$, hence

$$\sum_{k=1}^n s_kf(\sum_{i=1}^n t_iL_{x_i}) = \sum_{k=1}^n f(\sum_{i=1}^n s_kt_iL_{x_i}) = 0,$$

thus $f(\sum_{i=1}^n t_iL_{x_i}) \in r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for some $t \in A$. Then X satisfies Baer (m, n)-criterion. □

Corollary 2.7. *Let X be a $B - A$ - module. Then X is $F - (m, n) - S - B - A$ - module if and only if* $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ *for n-element subset* $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ *of X_n*

Following [8], let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi α-injective if, $\varphi : N \to X$ is A-module homomorphisms such that

$\|\varphi\| \leq 1$, there exists A-module homomorphism $\theta : X \to X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α".

The concepts quasi $(m,n) - \alpha$ - injective for some α and multiplication $(m,n) - B - A$ - module has been introduced.

Definition 2.8. Let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi $(m,n) - \alpha$ - injective if,$\varphi : N \longrightarrow X$ is A - module homomorphisms such that $\|\varphi\| \leq 1$, there exists A - module homomorphism $\theta : X \longrightarrow X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$ where i is an isometry from n - generated submodule N of X. We shall say that X is quasi (m,n) - injective if it is quasi $(m,n) - \alpha$ - injective for some α.

Definition 2.9. $B - A$-module X is called multiplication $(m,n) - A$ - module if each n - generated submodule of X is of the form KX_n for some ideal K of $A^{m \times n}$.

Proposition 2.10. *Let X be multiplication $(m,n) - B - A$ - module. If X is quasi $(m,n) - B - A$ -module then X is $F - (m,n) - S - B - A$-module.*

Proof. Let N be n-generated submodule of X, let $\alpha > 1$ and f be any A-module homomorphism from N to X^m such that $\|f\| \leq 1$. Since X is multiplication $(m,n) - B - A$-module, then $N = KX_n$, and since X is quasi $(m,n) - B - A$ - module, then there exist A-module homomorphism $g : X^m \longrightarrow X^m$ such that $f(N) = g(N) = g(KX_n) = Kg(X_n) \subseteq KX_n = N$. $\square$

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Accepted: 15.04.2019

Vertex (n, k)-choosability of graphs

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Abstract. Let $G = (V, E)$, connected, simple graph of order n and size m and let $V(G) = \{1, 2, ..., n\}$. A graph $G = (V, E)$ is said to be vertex (n, k)-choosable, if there exists a collection of subsets of the vertex set, $\{S_k(v) : v \in V\}$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$. This paper initiates a study on vertex (n, k)-choosable graphs and finds the different integer values of k, for which the given graph is vertex (n, k)-choosable.
Keywords: choosability, vertex (n, k)-choosability.

1. Introduction

Throughout this article, unless otherwise mentioned, by a graph we mean a connected, simple graph and any terms which are not mentioned here, the reader may refer to [8]. Let $G = (V, E)$, be a graph of order n and size m, where $V(G) = \{1, 2, ..., n\}$. Given a graph G, a *list assignment* L (or a *list coloring)* of G is a mapping that assigns to every vertex v of G, a finite list $L(v)$ of colors[12]. Also, G is said to be $\mathcal{L}$-*list colorable* if the vertices of G can be properly colored so that each vertex v is colored with a color from $\mathcal{L}(v)$.

Invoking the concept of list-assignments of graphs, the concept of $(a : b)$-choosability was defined and studied in [4].

Definition 1.1. A graph $G = (V, E)$ is $(a : b)$-choosable, if for every family of sets $\{S(v) : v \in V\}$ of cardinality a, there exist subsets $C(v) \subset S(v)$, where $|C(v)| = b$ for every $v \in V$, and $C(u) \cap C(v) = \emptyset$, whenever $u, v \in V$ are adjacent.

The k^{th} choice number of G, denoted by $ch_k(G)$, is the minimum integer n so that G is $(n : k)$-choosable. A graph $G = (V, E)$ is k-choosable if it is $(k : 1)$-choosable. The choice number of G, denoted by $ch(G)$, is equal to $ch_1(G)$. Following this, some interesting studies on choosability of graphs have been done (see [1, 5, 6]).

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Motivated by the studies on $(a:b)$-choosability of graphs, we initiate a study on the vertex (n,k)-choosable graphs, where n is the cardinality of the vertex set of G, and discuss the various parameter for the integer values of k.

2. Vertex (n,k)-choosability of graphs

Definition 2.1. A graph $G=(V,E)$ is said to be *vertex (n,k)-choosable*, if there exists a collection of subsets $\{S_k(v) : v \in V\}$ of $V(G)$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$.

Definition 2.2. The maximum value of k for which the given graph G is vertex (n,k)-choosable is called *vertex choice number of* G, and is denoted by $\mathcal{V}_{ch}(G)$.

Not all graphs admit vertex (n,k)-choosability for all values of k. A trivial bound for k is, $k \leq n-1$. One may verify that when $k=n-1$, the only vertex (n,k)-choosable graph is the trivial graph K_2. And, for $k=n-2$, the vertex (n,k)-choosable graph is isomorphic to P_3. However, every graph G of order n is vertex $(n,1)$-choosable. That is, the minimum value of k for which the given graph G is vertex (n,k)-choosable is $k=1$. Hence, finding the positive integer values of k, and also the maximum value of k, where $1 \leq k \leq n$, for which the graph G is vertex (n,k)-choosable is an interesting problem.

First, let us look at the vertex choice number of certain classes of graphs. The following observations are immediate.

Observation 2.3. The vertex choice number of a path P_n is $\lfloor \frac{n}{2} \rfloor$. That is, P_n is vertex (n,k)-choosable for all k, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Consider two disjoint k-element subsets of $V(P_n)$. Since, the path P_n is a bipartite graph, one k-element set can be assigned to all vertices in the first partition and other k-element set can be assigned to all vertices in the second partition. That is, by atleast two disjoint k-element sets, all vertices of P_n can be covered. Hence, the maximum value of k will be, $\lfloor \frac{n}{2} \rfloor$. Let $k > \lfloor \frac{n}{2} \rfloor$, say, $k = \lfloor \frac{n}{2} \rfloor + 1$. Take any subset V_1 of $V(P_n) = \{u_1, u_2, \ldots, u_n\}$, of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. Let $u_1 \in P_n$ be assigned by this set of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. . Then, for the second vertex u_2, we cannot find a subset V_2 of $V(P_n)$, of order $\lfloor \frac{n}{2} \rfloor + 1$, disjoint from V_1. That is, P_n is not vertex (n,k) choosable for $k = \lfloor \frac{n}{2} \rfloor + 1$. Hence, in general P_n is not vertex (n,k) choosable for any $k > \lfloor \frac{n}{2} \rfloor$.

Observation 2.4. The vertex choice number of the star graph S_n is $\lfloor \frac{n}{2} \rfloor$.

That is, for the star graph S_n, the vertex (n,k)-choosability is possible if there exists two disjoint k-element subsets of $V(S_n)$. Then, one k-element set should necessarily be assigned to the central node and the other k-element set should be assigned to all other nodes that are at a distance one from the central node. Therefore, S_n is vertex (n,k)-choosable for all $k \leq \lfloor \frac{n}{2} \rfloor$.

Proposition 2.5. *The complete graph K_n is vertex (n,k)-choosable if and only if $k=1$.*

Proof. Let $V(G) = \{1, 2, 3, \ldots, n\}$. Clearly there are n number of disjoint one element subsets of $V(G)$, and hence these one element subsets may be assigned to every vertex of K_n in a one-to-one manner. And hence, K_n is vertex $(n, 1)$-choosable. If $k \geq 2$, then the number of k-element subsets are less than n. Hence K_n is not vertex (n, k)-choosable, for $k \geq 2$. Also the vertex choice number of the complete graph K_n is 1, for all n. □

Theorem 2.6. *An even cycle C_n is vertex (n, k)-choosable if and only if $k \leq \frac{n}{2}$.*

Proof. Let $V(C_n) = \{1, 2, \ldots, n\}$, and n be even.

Consider, $f : V(C_n) \to \mathcal{P}(V(C_n)) - \emptyset$ defined by,

$$f(i) = \begin{cases} \{1, 2, \ldots, k\}, \text{if i is odd,} \\ \{k+1, k+2, \ldots, k+k\}, \text{if i is even.} \end{cases}$$

Then, for C_n to be vertex (n, k)-choosable, k should necessarily be such that, $1 \leq k \leq \frac{n}{2}$, if n is even.

Conversely, when n is even and $k > \frac{n}{2} + 1$, we reach a contradiction that whenever $ij \in E(C_n), f(i) \cap f(j) \neq \emptyset$.

□

Remark. An odd cycle C_n is vertex (n, k)-choosable if and only if $\leq \lfloor \frac{n}{2} \rfloor$.

Theorem 2.7. *A complete bipartite graph $K_{m,n}$ is vertex $(m+n, k)$-choosable, for $1 \leq k \leq \frac{m+n}{2}$, if and only if both m and n are simultaneously even or simultaneously odd.*

Proof. Without loss of generality, assume that both m and n are even. Let the vertex set of $K_{m,n}$ be V, where $V = A \cup B$ so that $|A| = m$ and $|B| = n$. Here, $|V| = m + n$. Now we have to find the values of k, for which $K_{m,n}$ is $(m + n, k)$-choosable.

Trivially there exists vertex $(m + n, 1)$ choosability, since there are $m + n$ disjoint one element subsets of V. Hence, $k \geq 1$.

Now, $V = A \cup B$, and every vertices in A is adjacent to all vertices in B. Also, there is no adjacency among the vertices in A and similarly, no two vertices in B are adjacent to each other. Let $A = \{1, 2, \ldots, m\}$ and $B = \{m + 1, m + 2, \ldots, m + n\}$.

Choose a k-element subset of V for a vertex in A. For example, let $f(1) = \{1, 2, \ldots, k\}$. Since i and j are not adjacent for all $i, j \in \{1, 2, \ldots, m\}$, it is possible to choose the same set for each vertex in A. That is, $f(i) = \{1, 2, \ldots, k\}$ for all i such that $1 \leq i \leq m$. Since, every vertices in A is adjacent to all other vertices in B, we cannot give the same set to any element in B. Hence, we need other k-element set. For this, let $f(m + i) = \{k + 1, k + 2, \ldots, k + k\}$ for all i such that $1 \leq i \leq n$. This is possible since, no two vertices in B are adjacent to each other. Hence, if there are two disjoint k element sets then vertex (n, k)-choosability is possible for $K_{m,n}$. Which gives $k \leq \frac{m+n}{2}$.

Now, suppose that $K_{m,n}$ is $(m+n, k)$-choosable for $1 \leq k \leq \frac{m+n}{2}$, and let m is odd and n is even. That is add $fracm + n2$ is not an integer. Hence, the complete bipartite graph $K_{m,n}$ is vertex $(m+n, k)$-choosable for $1 \leq k \leq \frac{m+n}{2}$, if and only if, both m and n are simultaneously even or simultaneously odd. □

Theorem 2.8. *A tree of order n is vertex (n, k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let T be the given tree of order n and let $V(T) = \{v_1, v_2, \ldots, v_n\}$. Apply BFS algorithm to the given tree T, by choosing a vertex v_i with maximum degree as root. If there are more than one vertices $v_j \in V(T)$ of maximum degree, choose one such vertex arbitrarily. Without loss of generality, denote the chosen root vertex as v_1. Then, by the choice of v_1, there will be $|deg(v_1)|$ number of vertices in the first level. Define a function $f : V(T) \to \mathcal{P}(V(T)) - \emptyset$ by $f(v_1) = \{1, 2, \ldots, i\}$, $1 \leq i \leq \frac{n}{2}$. Let v_k^j denote any vertex in j^{th} level which is adjacent to vertices in the $(j-1)^{th}$ level. Hence, the vertex v_1 can be denoted by v_1^0.

Define

$$f(v_k^j) = \begin{cases} \{1, 2, \ldots, k\}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, & \text{if j is even,} \\ \{k+1, k+2, \ldots, k+k\}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, & \text{if j is odd.} \end{cases}$$

With this labeling the tree T admits vertex (n, k)-choosability $\forall k$ such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Conversely, it is sufficient to prove that if $k > \lfloor \frac{n}{2} \rfloor$, then the tree T is not vertex (n, k)-choosable.

If possible, let $k = \lfloor \frac{n}{2} \rfloor + 1$ Let $f(v_1) = \left\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + 1\right\}$.

Let v_m^1 be any vertex in the first level adjacent to the root vertex v_1. Then, we should necessarily have,

$$f(v_m^1) = \left\{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, 2(\lfloor \frac{n}{2} \rfloor + 1)\right\}.$$

Clearly, $|f(v_m^1)| < \lfloor \frac{n}{2} \rfloor + 1$, a contradiction. Hence, $k \leq \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.9. *The complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ is vertex $(m_1 + m_2 + \cdots + m_r, k)$-choosable for, $1 \leq k \leq \lfloor \frac{m_1+m_2+\cdots+m_r}{r} \rfloor$.*

Proof. Denote the given complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ by G. Here, $|S(V)| = m_1 + m_2 + \cdots + m_r$. Now, let $V(G) = A_1 \cup A_2 \cup \cdots \cup A_r$. Since, every vertex in A_i is adjacent to all other vertices in A_j, for all $i \neq j$. Hence, atleast r k-element sets are needed. First choose a k-element set for the first set A_1. Since there is no adjacency between any pair of vertices in A_1, the same set can be choosen for all vertices in A_i. similarlly for each A_i, this method can be followed. That is, only r k-element sets are needed to cover all the vertices in G. Hence, $k \leq \lfloor \frac{m_1+m_2+\cdots+m_r}{r} \rfloor$. □

Theorem 2.10. *Any unicyclic graph G of order n with the unique cycle C_p is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let G be a unicyclic graph of order n with the unique cycle C_p. Suppose that p is even. By theorem 2.4, an even cycle C_n is vertex (n,k)-choosable if and only if, $k \leq \frac{n}{2}$. Hence, the cycle C_p alone is vertex (n,k)-choosable in G, for $k \leq \lfloor \frac{n}{2} \rfloor$. We note that $G - C_p$ is a forest. Consider the components of $G - C_p$.

For the vertex (n,k)-choosabiliy of trees, we need two distinct k-element subsets. We can choose the same sets that are assigned for the vertices in the cycle, for the vertices in the tree also. For this, let $\{1,2,\ldots,p\}$ be the vertex set of C_p and $\left\{j_1, j_2, \ldots, j_{p_j}\right\}$ be the vertex set of the tree with the root vertex j in the cycle. Also, we have $p+p_1+p_2+...+p_p = n$. Then, if there are two distinct k-element sets, then the vertex (n,k)-choosabiliy of the cycle C_p is given by,

$$f(i) = \begin{cases} \{1,2,\ldots,k\}, \text{if i is odd,} \\ \{k+1, k+2, \ldots, k+k\}, \text{if i is even,} \end{cases}$$

where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Then, in the tree if j is odd, then for j_1, we can choose the set assigned for even vertices in the cycle C_p. By applying BFS algorithm, we can seen that, by two distinct k element sets, we can cover all the vertices in the tree. Hence, the unicyclic graph G of order n is vertex (n,k)-choosable for $k \leq \lfloor \frac{n}{2} \rfloor$, if the unique cycle C_p is even. Now, let p be odd. That is C_p is an odd cycle.

We have an odd C_n is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$. First label the vertices of C_p by $\lfloor \frac{n}{2} \rfloor$ element subsets of $V(G)$. Next, consider the remaining vertices in the tree. If the root vertex of the tree is an even(odd) vertex in the cycle, then for the next vertex in the tree, we can choose the set assigned for the neighbouring odd (even) vertices in the cycle. Using these two sets all vertices in the tree can be labelled. In a similar manner all the trees attached with the vertices of C_p can be labelled.

Hence, the unicyclic graph G of order n with the unique cycle C_p is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

Theorem 2.11. *A graph G is vertex (n,k)-choosable, if it does not contain a complete subgraph K_m of order $m \geq \lfloor \frac{n}{k} \rfloor + 1$.*

Proof. Let $G = (V,E)$ be a vertex (n,k)-choosable graph. Suppose that G contains a complete subgraph of order $m = \lfloor \frac{n}{k} \rfloor + 1$. Since, G is vertex (n,k)-choosable, every vertex of G can choose a set of k elements. Let $V(K_m) = \{1,2,\ldots,m\}$. Now, define the function $f: V(G) \to \mathcal{P}(V(G)) - \emptyset$. Consider the vertex 1 in the complete graph, and let $f(1) = \{1,2,\ldots,k\}$. Since, 1 is adjacent to all the remaining vertices i, where $i = 2,3,\ldots,m$ in K_m, they cannot chose the same set $f(1)$. That is, for all vertices in the complete graph K_m we need disjoint k element sets. Hence, atleast m disjoint k-element sets are needed to cover all the vertices in K_m. Since G is a connected graph and K_m is a complete subgraph of G, atleast one vertex in K_m will be adjacent to a vertex not in K_m. Hence we have, $mk < n$. This implies $m < \frac{n}{k}$. That is, $m = \lfloor \frac{n}{k} \rfloor + 1 < \frac{n}{k}$, which is a contradiction. Hence, a graph G is vertex (n, k)-choosable, if it does not contain a complete subgraph of order $m \geq \lfloor \frac{n}{k} \rfloor + 1$. □

3. Conclusion

In this paper, we introduced a new concept namely, vertex (n, k)-choosability of graph. We also discussed the vertex (n, k)-choosability of certain fundamental graph classes. There is a wide scope for further investigation on the vertex (n, k)-choosability of many other graph classes, graph operations and graph products. The edge (m, k)-choosability is another interesting area for further investigation.

Acknowledgement

The second author would like to acknowledge her gratitude to Council of Scientific and Industrial Research (CSIR), India for the financial support under CSIR Junior Research Fellowship scheme vide order no: 09/1108(0016)/2017-EMR-I.

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Accepted: 28.04.2019

Rough approximate operators based on fuzzy soft relation

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Abstract. Fuzzy soft set is a mapping from a parameter set to the collection of fuzzy subset of universal set. In this paper fuzzy soft relation is presented based on the cartesian product of fuzzy soft sets and the notion of fuzzy soft equivalence relation is introduced. We prove that every fuzzy soft equivalence relation on an arbitrary fuzzy soft set partition the given fuzzy soft set into equivalence classes and thus induces a new relation on the parameter set. Basic properties of the induced relation are studied. A pair of rough approximate operators are investigated and their related properties are given. Relationship between a fuzzy soft topological space and rough approximate operators based on fuzzy soft relation is further established.
Keywords: fuzzy soft set, fuzzy soft relation, fuzzy soft topology.

1. Introduction

Theory of fuzzy sets and fuzzy relation first developed by Zadeh [1] has been applied to many branches of mathematics. Fuzzy equivalence relation introduced by Zadeh as a generalization of the concept of an equivalence relation has been widely studied in [2], [3], [4], [5], [6], [7] as a way to measure the degree of distinguishability or similarity between the objects of a given universe of discourse. And it have been shown to be useful in different context such as fuzzy control [8], approximate reasoning [9], fuzzy cluster analysis [10]. Depending on the authors and the context in which they appeared, it have received other names such as similarity relations, indistinguishability operators [11], many valued equivalence relations[12], etc. Later V. Murali [13] studied the cuts of fuzzy equivalence relation and lattice theoretic properties of fuzzy equivalence relation. In 1999 Molodtsov [14] proposed the novel concept of soft theory which provides a completely new approach for modelling vagueness and uncertainty. Theory of soft

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set has gained popularity among the researchers working in diverse areas. It is getting richer with new developments. Application of soft set theory can be seen in [15], [16], [17], [18], [19], [20], [21], [22], [23].

Relations in soft set have been studied in [24], [25], [26]. Structures of soft set have been studied by many authors [27], [28]. Recently Ali et al. [29] have shown that a collection of soft set with reference to so called new operations give rise to many algebraic structures and form certain complete modular lattice structures. The theory of fuzzy soft set [30], fuzzification of the notion of soft set has the ability of hybridization. In this reagard fuzzy soft set and their applications has been investigated by many authors [31], [32], [33], [34], [35], [36], [37]. Fuzzy soft set is the parametrized collection of fuzzy sets. Fuzzy soft sets can be used to crunch the volume of data. Collection of fuzzy soft set form a complete modular lattice structures with respect to certain binary operations defined on them [38]. Topological structure of fuzzy soft sets can be seen in [39].

Based on these concepts fuzzy soft relation is introduced and it provides both a general and flexible method for the designing of fuzzy logic controller and more generally for the modelling of any decision making process. Fuzzy soft relation stores data in terms of relation between parameters which we define by membership function.

Theory of rough sets proposed by Pawalk [40]is considered as an alternative tool for imperfect data analysis. The rough set approach has fundamental importance in the area of knowledge acquisition, machine learning, decision analysis and many other fields [41], [42]. Approximation space is the basic structure of Rough set theory. Lower and upper approximation induced from an approximation space can be used to reveal and express the knowledge hidden in information systems in the form of decision rules. Various fuzzy generalizations of rough approximations have been proposed in [43], [44]. The most common fuzzy rough set, obtained by replacing the crisp relations with fuzzy relations on the universe and crisp subsets with fuzzy sets, have been used to solve practical problems such as data mining [45], approximate reasoning [46], and medical time series. An interesting topic in rough set theory is to study the relationship between rough sets and topologies. Many authors studied topological properties of rough sets [47], [48]. Using the concept of fuzzy soft relation, R rough set is introduced. Fuzzy soft equivalence relation is the key notion used in R rough set model. The equivalence classes generated by the fuzzy soft equivalence relation are the building blocks for the construction of these approximations.

This paper is organized in the following manner. In Section 2 basic definitions related to fuzzy soft sets are given. These basic concepts are required in later sections. In Section 3 fuzzy soft relation is defined and its properties are studied. Also this Section is devoted to the study of composition of fuzzy soft relation and fuzzy soft equivalence relation. We define a new relation on parameter set induced by the fuzzy soft relation with example and its theoretical aspects are studied. Also we prove that the new relation induced by a fuzzy soft eqivalence relation partition the given fuzzy soft set. In Section 4 a pair of rough approximate operators has been defined and fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive relation have been characterized by these rough approximate operators. In Section 5 relation between a fuzzy soft topological space and rough approximate operators based on fuzzy soft relation is further established. The last section concludes the paper and points out further research work.

2. Preliminaries

Through out this paper X refers to an initial universe, $\wp$ is a set of parameters in relation to objects in X. Parameters are often attributes, characteristics or properties of objects. I^{X} denote the set of all fuzzy subsets of X and P,Q $\subset \wp$.

Definition 2.1 ([30]). The pair (f,P) is called a fuzzy soft set over X if f is a mapping given by f:P$\to$$I^X$. Each element in the fuzzy soft set (f,P) corresponding to the parameter p$\in$ P can be denoted by f_p, where f_p is a function from X to [0,1].

The fuzzy soft set(f,P) is said to be a null fuzzy soft set, denoted by $\tilde{0}$, if $f_p(x)=0, \forall$ p $\in$ P and $\forall$ x $\in$ X

Definition 2.2 ([30]). Let (f,P) and (g,Q) be two fuzzy soft set over X. Then (f,P) is called fuzzy soft subset of (g,Q) denoted by (f,P) $\subseteq$(g,Q) if P $\subseteq$ Q and $f_p(x) \leq g_p(x)$, $\forall$ p $\in$ P.

Collection of all fuzzy soft subsets of(f,P) be denoted as S_f(X,P)

Definition 2.3 ([30]). Let (f,P) and (g,Q) be two fuzzy soft sets over X. Then (f,P)-(g,Q) is the fuzzy soft set (h,C) where C= P-Q and $h_c(x)=f_c(x), \forall$ c $\in$ C

Definition 2.4 ([30]). Union of two fuzzy soft sets (f,P) and (g,Q) over X is defined as the fuzzy soft set (h,C)=(f,P) $\cup$ (g,Q) where C=P$\cup$Q and for all c$\in$C

$$h_c(x) = \begin{cases} f_c(x), & \text{if c} \in P - Q \\ g_c(x), & \text{if c} \in Q - P \\ f_c(x) \vee g_c(x)\ , & \text{if c} \in P \cap Q. \end{cases}$$

Definition 2.5 ([30]). Intersection of two fuzzy soft sets (f,P) and (g,Q) over X is defined as the fuzzy soft set (h,C)=(f,P) $\cap$ (g,Q) where C=P$\cap$Q and for all c$\in$ C, $h_c(x)=f_c(x) \wedge g_c(x)$

Definition 2.6 ([30]). Let (f,P) and (g,Q) be two fuzzy soft sets over a universe X. Then cartesian product of (f,P) and (g,Q) is defined as (f,P) x (g,Q) = (h, P x Q) where h: P x Q$\to$$I^X$ and $h_{(p,q)}(x) = \min(f_p(x), g_q(x))$, $\forall$ (p,q) $\in$ P x Q.

Example 2.7. Consider the various investment avenues as x_1-bank deposit, x_2-Insurance, x_3-postal savings, x_4- shares and stocks, x_5-mutual funds, x_6-gold, x_7-real estate as the universal state X, and factors influencing investment decision such as e_1-safety of funds, e_2-liquidity of funds, e_3-high returns, e_4-maximum profit in minimum time period, e_5-stable returns, e_6-easy accessibility, e_7-tax concession, e_8-minimum risk of parameters.

Decision maker P is good at the parameters e_1 and e_5. Decision maker Q is good at the parameters e_3 and e_4. This information can be expressed by two fuzzy soft sets (f,P) and (g,Q) respectively.

$$(f,P) = \begin{Bmatrix} e_1 = \{\frac{x_1}{1}, \frac{x_2}{0.9}, \frac{x_3}{1}, \frac{x_4}{0.2}, \frac{x_5}{0.3}, \frac{x_6}{0.8}, \frac{x_7}{0.4}\} \\ e_5 = \{\frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{0.1}, \frac{x_5}{0.1}, \frac{x_6}{0.3}, \frac{x_7}{0.7}\} \end{Bmatrix}$$

and

$$(g,Q) = \begin{Bmatrix} e_3 = \{\frac{x_1}{0.5}, \frac{x_2}{0.5}, \frac{x_3}{0.5}, \frac{x_4}{0.7}, \frac{x_5}{0.6}, \frac{x_6}{0.8}, \frac{x_7}{0.9}\} \\ e_4 = \{\frac{x_1}{0.4}, \frac{x_2}{0.2}, \frac{x_3}{0.4}, \frac{x_4}{0.8}, \frac{x_5}{0.6}, \frac{x_6}{0.8}, \frac{x_7}{0.9}\} \end{Bmatrix}$$

A typical element of (h,P x Q) will look like

$$h(e_1, e_3) = \left\{\frac{x_1}{0.5}, \frac{x_2}{0.5}, \frac{x_3}{0.5}, \frac{x_4}{0.2}, \frac{x_5}{0.3}, \frac{x_6}{0.8}, \frac{x_7}{0.4}\right\}$$

3. Fuzzy soft relations and partition on fuzzy soft set

Fuzzy soft Relation is a suitable tool for describing correspondence between the parameters in a fuzzy soft set, which makes the theory of fuzzy soft set a hot subject for research. It plays an important role in modeling and decision making of systems. In this section we discuss a variety of different properties of a fuzzy soft relation may possess.

Definition 3.1. Fuzzy Soft Relation R from (f,P) to (g,Q) is a fuzzy soft subset of (f,P) x (g,Q). If R is a fuzzy soft subset of (f,P) x (f,P) then it is called a Fuzzy Soft Relation on (f,P).

If R is a Fuzzy Soft Relation on (f,P) then $R^{-1}_{pq} = R_{qp}$, $\forall$ (p,q) $\in$ PxQ. If R is a fuzzy soft relation from (f,P) to (g,Q) then R^{-1} is a fuzzy soft relation from (g,Q) to (f,P).

Definition 3.2. Let R_1 and R_2 be two Fuzzy Soft Relations from (f,P) to (g,Q) and (g,Q) to (h,S) respectively. Composition of R_1 and R_2 denoted by $R_1 \circ R_2$ is a Fuzzy Soft Relation from (f,P) to (h,S) defined as $(R_1 \circ R_2)_{ps} = \bigvee_{q\in Q}((R_1)_{pq} \wedge (R_2)_{qs})$ where (p,q) $\in$ PxQ and (q,s) $\in$ QxS.

Theorem 3.3. *Let Q, R, S be fuzzy soft relation on (f, P) then:*

1) $(R^{-1})^{-1} = R$;
2) $R \subseteq S \implies R^{-1} \subseteq S^{-1}$;
3) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;
4)$R \subseteq S \implies R \circ Q \subseteq S \circ Q$;

5)(Q∘R)∘S = Q∘(R∘S).

Definition 3.4. Let (f,P) be a fuzzy soft set over the universal set X and R be a fuzzy soft relation on (f,P) then R is said to be:

1) Fuzzy soft reflexive if $\forall$ p,q $\in$ P with p $\neq$ q and $\forall$ x $\in$ X, $R_{pq}(x) \leq R_{pp}(x)$ and $R_{qp}(x) \leq R_{pp}(x)$;

2) Fuzzy soft symmetric relation if R = R^{-1};

3) Fuzzy soft transitive relation if R $\circ$ R $\subseteq$ R;

4) Fuzzy soft equivalence relation if it is fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive.

Definition 3.5. The relation R on fuzzy soft set (f,P) induces a new relation R^λ on (f,P) as follows:

Let $\lambda \in [0,1]$. Define the relation R^λ on (f,P) such that f_p R^λ f_q if and only if $R_{pq}(x) \geq \lambda$, $\forall$ x $\in$ X.

Theorem 3.6. *If R is a fuzzy soft equivalence relation on (f,P) and* $\alpha = \bigwedge_{x\in X} \bigwedge_{p\in P} R_{pp}(x)$ *then for each* $\lambda \in [0,\alpha]$, R^λ *is an equivalence relation on (f,P).*

Proof. For each p $\in$ P and $\forall$ x $\in$ X, $R_{pp}(x) \geq \alpha \geq \lambda \implies f_p R^\lambda f_p \implies R^\lambda$ is reflexive.

Let $f_p R^\lambda f_q$ i.e. $R_{pq}(x) \geq \lambda$.

Since R is a fuzzy soft symmetric relation, $R^{-1}_{pq}(x) = R_{pq}(x) \implies R_{qp}(x) = R_{pq}(x) \geq \lambda \implies f_q R^\lambda f_p$. This implies R^λ is a symmetric relation.

Finally let $f_p R^\lambda f_r$ and $f_r R^\lambda f_q \implies R_{pr}(x) \geq \lambda$ and $R_{rq}(x) \geq \lambda$, $\forall$ x $\in$ X. Since R is a fuzzy soft transitive relation, $R_{pq}(x) \geq (R \circ R)_{pq}(x) = \bigvee_{r\in P}(R_{pr}(x) \wedge R_{rq}(x)) \geq (R_{pr}(x) \wedge R_{rq}(x)) \geq \lambda$. Hence R^λ is a transitive relation.

Definition 3.7. Equivalence class of f_p denoted by $[f_p]$ is defined as $[f_p] = \{f_q: f_p R^\lambda f_q\}$.

Example 3.8. Let the fuzzy soft set (f,P) over universal set x=$\{x_1, x_2, x_3, x_4\}$ and parameter set P =$\{$p, q, r$\}$ be given by.

$$(f,P) = \left\{ \begin{array}{c} f_p = \{\frac{x_1}{.76}, \frac{x_2}{0.5}, \frac{x_3}{.82}, \frac{x_4}{0.64}\} \\ \\ f_q = \{\frac{x_1}{.58}, \frac{x_2}{.075}, \frac{x_3}{.6}, \frac{x_4}{0.56}\} \\ \\ f_r = \{\frac{x_1}{.66}, \frac{x_2}{.5}, \frac{x_3}{.7}, \frac{x_4}{0.75}\} \end{array} \right\}$$

Consider the fuzzy soft equivalence relations R on (f,P) as follows

R	x_1	x_2	x_3	x_4
R_{pp}	0.76	0.5	0.82	0.64
R_{pq}	0.58	0.075	0.6	0.56
R_{qp}	0.58	0.075	0.6	0.56
R_{qq}	0.6	0.7	0.82	0.9

Since $R_{rr}(x) = 0$, $\forall x \in X$ we have $\lambda = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x) = 0$. Then $R^0 = \{(f_p,f_p), (f_q,f_q), (f_p,f_q), (f_q,f_p), (f_r,f_r)\}$ is an equivalence relation on (f,P).

If $\lambda = 0.55$, $R^{0.55} = \{(f_q,f_q), (f_p,f_q), (f_q,f_p)\}$ is not an equivalence relation on fuzzy soft set (f,P).

Lemma 3.9. *Let R be a fuzzy soft equivalence relation on (f,P) and* $\alpha = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$*. For* $p,q \in P$ *and* $\lambda \in [0,\alpha]$*,* $[f_p] = [f_q]$ *if and only if* $f_p R^\lambda f_q$ *if and only if* $[f_p] \cap [f_q] \neq \phi$*.*

Proof. Suppose $[f_p] = [f_q]$.

Since R is a fuzzy soft equivalence relation on (f,P), R^λ is an equivalence relation $\implies$ R^λ is a reflexive relation $\implies$ $f_p R^\lambda f_p$.

Hence $f_p \in [f_p] = [f_q]$ i.e. $f_q R^\lambda f_p$. By Theorem 3.6 we have R^λ is a symmetric relation. Hence $f_p R^\lambda f_q$. Conversely suppose that $f_p R^\lambda f_q$.

Let $f_r \in [f_p]$, then $f_p R^\lambda f_r$. Using symmetric and transitive property of R^λ we have $f_q R^\lambda f_r \implies f_r \in [f_q]$. Hence $[f_p] \subseteq [f_q]$.

Using a similar argument we can show that $[f_p] \subseteq [f_q]$. Hence $[f_p] = [f_q]$.

Now, let $[f_p] \cap [f_q] \neq \phi \iff f_r \in [f_p] \cap [f_q] \iff f_p R^\lambda f_r$ and $f_q R^\lambda f_r \iff f_p R^\lambda f_q$.

Definition 3.10. A collection of fuzzy soft subset $\{(f_i, P_i)$: $i \in I\}$ of a fuzzy soft set (f,P) is called a partition of (f,P) if:

1) $(f,P) = \bigcup_i (f_i,P_i)$;

2) $P_i \cap P_j = \phi$ when ever $i \neq j$.

Theorem 3.11. *Corresponding to every fuzzy soft equivalence relation defined on the fuzzy soft set (f,P) there exist a partition of (f,P) and this partition precisely consist of the equivalence class of* R^λ *where* $\lambda = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$*.*

Proof. Let $[f_p]$ be an equivalence class corresponding to the relation R^λ on (f,P) and $P_p = \{ q \in P$: $f_p R^\lambda f_q\}$. Also denote $[f_p]$ as (f,P_p).

We have to show that $\{(f,P_p)$: $p \in P\}$ of such distinct set forms a partition of (f,P), i.e. we have to prove that:

1) $(f,P) = \bigcup_{p\in P}(f, P_p)$;

2) if P_p, P_q are not identical then $P_p \cap P_q = \phi$.

Since R^λ is a reflexive relation $f_p R^\lambda f_p$ $\forall$ $p \in P$, so that 1) can be easily proved.

Let $r \in P_p \cap P_q$. Then, $f_r \in (f,P_p)$ and $f_r \in (f,P_q) \implies f_r R^\lambda f_p$ and $f_r R^\lambda f_q$. Using transitive property of R^λ we have $f_p R^\lambda f_q$ hence by Lemma 3.9 $[f_p]=[f_q]$ $\implies P_p = P_q$.

Definition 3.12. Let R^λ be the relation induced by the fuzzy soft relation R on (f,P). $\forall$ $(g,P_g) \in S_f(X,P)$,Define $P_g(f,R)= \{f_p \in (f,P):[f_p] \subseteq (g,P_g)\}$ and $P^g(f,R) = \{f_p \in (f,P) : [f_p] \cap (g,P_g) \neq \tilde{0}\}$.

Example 3.13. In Example 3.8 let $(g,P_g) \in S_f(X,P)$ be given by

$$(g,P_g) = \left\{ g_p = \{\frac{x_1}{.76}, \frac{x_2}{0.5}, \frac{x_3}{.82}, \frac{x_4}{0.64}\} \right\}.$$

Consider the relation $R^{0.5}$ induced by the fuzzy soft equivalence relation R. $[f_p]$ = $[f_q]$=$\{f_p,f_q\}$ not a subset of (g,P_g). Hence $P_g(f,R)$ is an empty set with repect to the equivalence classes $[f_p]$ and $[f_q]$.

Since P_g contains only one parameter p, we have $[f_r]=\tilde{0} \nsubseteq (g,P_g) \implies P_g(f,R)= \phi$ with repect to the equivalence classes $[f_r]$. Next, we can compute $P^g(f,R)$.

$[f_p] \cap (g,P_g) \neq \tilde{0} \implies P^g(f,R)=\{f_p\}$ with respect to the equivalence classes $[f_p]$. Using a similar argument we can prove that $P^g(f,R)=\{f_q\}$ w.r.t to the equivalence classes $[f_q]$. $[f_r] \cap (g,P_g)=\tilde{0} \implies P^g(f,R)=\phi$ w.r.t to the equivalence class $[f_r]$.

Lemma 3.14. *Let R be a fuzzy soft relation defined on (f,P) then* $\forall$ $(g,P_g),(h,P_h) \in S_f(X,P)$:

1. $P_f(f,R)=(f,P)$;
2. *If R^λ is a reflexive relation on (f,P) then $P_g(f,R) \subseteq (g,P_g) \subseteq P^g(f,R)$;*
3. a) $(g,P_g) \subseteq (h,P_h) \implies P_g(f,R) \subseteq P_h(f,R)$;
b) $(g,P_g) \subseteq (h,P_h) \implies P^g(f,R) \subseteq P^h(f,R)$;
4. a) $P^l(f,R)= P^g(f,R) \cup P^h(f,R)$ where $(l,P_l)=(g,P_g) \cup (h,P_h)$;
b)$P_l(f,R)= P_g(f,R) \cap P_h(f,R)$ where $(l,P_l)=(g,P_g) \cap (h,P_h)$.

Proof. 1. This is obvious.

2. Let $f_p \in P_g(f,R)$.

Since R^λ is a reflexive relation on (f,P), $f_p \in [f_p] \subseteq (g,P_g) \implies [f_p] \cap (g,P_g) \neq 0 \implies f_p \in P^g(f,R)$. Hence the proof.

3. a) $f_p \in P_g(f,R) \implies [f_p] \subseteq (g,P_g) \implies [f_p] \subseteq (h,P_h) \implies f_p \in P_h(f,R)$.

Hence $P_g(f,R) \subseteq P_h(f,R)$.

b) Proof is similar to a).

4. a) Let $f_p \in P^l(f,R) \implies [f_p] \cap (l,P_l) \neq 0$.Since $(l,P_l)=(g,P_g) \cap (h,P_h)$,either $[f_p] \cap (g,P_g) \neq 0$ or $[f_p] \cap (h,P_h) \neq 0 \implies f_p \in P^g(f,R)$ or $f_p \in P^h(f,R) \implies f_p \in P^g(f,R) \cup P^h(f,R)$. Conversely this is obvious.

b) Proof is similar to a).

4. Rough approximate operators of fuzzy soft relation

In this section we propose two rough approximate operators for a given fuzzy soft relation.

Definition 4.1. Define the following operations- $\underline{apr}$, $\overline{apr}$: S_f(X,P)$\rightarrow S_f$(X,P) by $\underline{apr}$(g,P_g)= $\cup$ P_g(f,R) and $\overline{apr}$(g,P_g)= $\cup$ P^g(f,R).

$\underline{apr}$ and $\overline{apr}$ are called the R lower approximation operator and R upper approximation operator respectively. The soft subset (g,P) is called R definable if $\underline{apr}$(g,P_g) = $\overline{apr}$(g,P_g). (g,P_g) is called R rough set if $\underline{apr}$(g,P_g) $\neq$ $\overline{apr}$(g,P_g).

Theorem 4.2. *Let R be a fuzzy soft relation defined on (f,P). Then $\forall$ (g,P_g),(h,P_h) $\in S_f$(X,P).*

1. *If (g,P_g)$\subseteq$ (h,P_h), then:*
   a)$\underline{apr}$(g,P_g) $\subseteq$ $\underline{apr}$(h,P_h);
   b) $\overline{apr}$(g,P_g)$\subseteq$ $\overline{apr}$(h,P_h).
2. a)$\underline{apr}$((g,P_g)$\cap$(h,P_h))=$\underline{apr}$(g,P_g)$\cap\underline{apr}$(h,P_h);
   b) $\overline{apr}$((g,P_g) $\cup$ (h,P_h))=$\overline{apr}$(g,P_g) $\cup$ $\overline{apr}$(h,P_h).

Proof. 1. This obviously hold by Lemma 3.14.

2. a) Let (g,P_g)$\cap$(h,P_h)=(l,P_l) then by Lemma 3.10,P_l(f,R)= P_g(f,R) $\cap$ P_h(f,R).

Let $f_p \in \underline{apr}$(l,P_l) $\implies$ $f_p \in \cup$ P_l(f,R) $\implies$ $f_p \in P_l$(f,R) $\implies f_p \in P_g$(f,R) and $f_p \in P_h$(f,R) $\implies$ $f_p \in \underline{apr}$(g,P_g) $\cap$ $\underline{apr}$(h,P_h).

Hence $\underline{apr}$(l,P_l)$\subseteq$ $\underline{apr}$(g,P_g) $\cap$ $\underline{apr}$(h,P_h).

Similarly we can prove the Converse part $\underline{apr}$(g,P_g)$\cap\underline{apr}$(h,P_h) $\subseteq$ $\underline{apr}$(l,P_l).

b) This can be proved similarly as above.

Theorem 4.3. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) then R^λ, $\lambda \in$[0,1] is reflexive iff $\forall$ (g,P_g) $\in S_f$(X,P),$\underline{apr}$(g,P_g)$\subseteq$(g,P_g) $\subseteq$ $\overline{apr}$(g,P_g).*

Proof. First part of above statement follows from Lemma 3.14.

Now, suppose that $\forall$ (g,P_g) $\in S_f$(X,P), $\underline{apr}$(g,P_g)$\subseteq$(g,P_g) $\subseteq$ $\overline{apr}$(g,P_g).

Let $f_p \in$ (f,P) and (g,P_g)= f_p. By our assumption $f_p \in \overline{apr}$(g,P_g) $\implies$ $f_p \in$ P^g(f,R) $\implies$ [f_p] $\cap$ (g,P_g) $\neq \tilde{0}$ $\implies$ $f_p \in$ [f_p], $\forall$ $f_p \in$ (f,P) $\implies$ R^λ is reflexive.

Theorem 4.4. *Let R be any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P). If R^λ,$\lambda \in$ [0,1] is reflexive then:*

1) $\underline{apr}$(f,P)= $\overline{apr}$(f,P)=(f,P);

2) $\underline{apr}\tilde{0}$= $\overline{apr}\tilde{0}$=$\tilde{0}$.

Proof. 1) By Theorem 4.3 we have $\underline{apr}$(f,P)$\subseteq$(f,P) $\subseteq$ $\overline{apr}$(f,P). Conversely since P^f(f,R) $\subseteq$(f,P) $\implies$ $\overline{apr}$(f,P) $\subseteq$ (f,P) $\implies$ $\overline{apr}$(f,P) = (f,P). By Lemma 3.14 P_f(f,R)=(f,R) $\implies$ $\underline{apr}$(f,P)=(f,P). Hence the result.

2) This is obvious.

Theorem 4.5. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) and α= $\bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$ then R^λ,$\lambda \in$[0,α] is symmetric iff $\forall$ (g,P_g) $\in S_f$(X,P), $\overline{apr}$($\underline{apr}$(g,P_g))$\subseteq$(g,P_g) $\subseteq$ $\underline{apr}$($\overline{apr}$(g,P_g)).*

Proof. Let (g,P_g) $\in$ S_f(X,P) and denote (k,P_k)=$\underline{apr}$(g,P_g),(h,P_h)=$\overline{apr}$(g,P_g), (w,P_w)=$\overline{apr}$(k,P_k), (l,P_l)=$\underline{apr}$(h,P_h).

Suppose R^λ is symmetric and (w,P_w)-(g,P_g)$\neq \phi$. Pick $f_p \in$ (w,P_w)-(g,P_g) $\Longrightarrow$ [f_p]$\cap$ (k,P_k)$\neq\tilde{0}$.

Let $f_k\in$ [f_p]$\cap$ (k,P_k) $\Longrightarrow$ $f_k\in$(k,P_k)=$\cup P_g$(f,R) $\Longrightarrow$ $f_k\in P_g$(f,R) $\Longrightarrow$ [f_p]$\subseteq$(g,P_g). Since R^λ is symmetric,$f_k\in$ [f_p] $\Longrightarrow$ $f_p\in$ [f_k]. Hence $f_p\in$ [f_k]$\subseteq$(g,P_g), a contradiction. Hence $\overline{apr}$($\underline{apr}$(g,P_g))$\subseteq$(g,P_g). Next suppose that (g,P_g)-(l,P_l)$\neq \phi$. Pick f_p $\in$(g,P_g)-(l,P_l) $\Longrightarrow$ f_p $\in$(g,P_g) and f_p $\notin$(l,P_l).

f_p $\notin$(l,P_l) $\Longrightarrow$ f_p $\notin\underline{apr}$(h,P_h) $\Longrightarrow$ f_p $\notin P_h$(f,R) $\Longrightarrow$ [f_p] $\nsubseteq$(h,P_h). Pick $f_q\in$[f_p] such that $f_q\notin$ (h,P_h) $\Longrightarrow$ $f_q\notin P^g$(f,R) $\Longrightarrow$ [f_q]$\cap$(g,P_g)=$\tilde{0}$.

Since R^λ is symmetric $f_p\in$[f_q], i.e $f_p\notin$(g,P_g), a contradiction. Conversely suppose $\forall$ (g,P_g) $\in$ S_f(X,P),$\overline{apr}$($\underline{apr}$(g,P_g))$\subseteq$(g,P_g) $\subseteq$ $\underline{apr}$($\overline{apr}$(g,P_g)).

Let $f_q\in$[f_p] and (g,P_g)=f_p, then $f_p\in\underline{apr}$($\overline{apr}$(g,P_g)) $\Longrightarrow$ $f_p\in P_h$(f,R) $\Longrightarrow$ [f_p]$\subseteq$(h,P_h)=$\overline{apr}$(g,P_g) $\Longrightarrow$ $f_q\in\overline{apr}$(g,P_g) $\Longrightarrow$ $f_q\in P^g$(f,R) $\Longrightarrow$ [f_q]$\cap$(g,P_g)$\neq\tilde{0}$ $\Longrightarrow$ $f_p\in$[f_q] $\Longrightarrow$ R^λ is symmetric.

Lemma 4.6. *Let R be a fuzzy soft relation defined on fuzzy soft set(f,P) and R^λ,λ $\in$[0,1] is symmetric then for each $f_p\in$(f,P), $\overline{apr}$(f_p) =[f_p].*

Proof. Let (g,P_g)= f_p and (h,P_h)= $\overline{apr}$(g,P_g) f_k $\in$ (h,P_h) $\Longleftrightarrow$ f_k $\in P^g$(f,R) $\Longleftrightarrow$ [f_k] $\cap$ (g,P_g)$\neq\tilde{0}$ $\Longleftrightarrow$ $f_p\in$[f_k] $\Longleftrightarrow$ $f_k\in$[f_p]. Hence the proof.

Theorem 4.7. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) and R^λ,λ $\in$[0,1] is reflexive and symmetric then following statements are equivalent:*

1) *R^λ is transitive;*

2) $\forall$ (g,P_g)$\in$ S_f(X,P),$\underline{apr}$(g,P_g) $\subseteq$ $\underline{apr}$($\underline{apr}$(g,P_g))$\subseteq$(g,P_g) $\subseteq$ $\overline{apr}$($\overline{apr}$(g,P_g)) $\subseteq\overline{apr}$(g,P_g).

Proof. 1 $\Longrightarrow$ 2. Let (h,P_h)=$\underline{apr}$(g,P_g),(k,P_k)=$\overline{apr}$(g,P),and (l,P_l)=$\overline{apr}$(k,P_k). First, we prove that if (h,P_h)=$\underline{apr}$(g,P_g) then P_h(f,R)=P_g(f,R).

By Lemma 3.14 (h,P_h)$\subseteq$ $\underline{(g,P_g)}$ $\Longrightarrow$ P_h(f,R) $\subseteq$ P_g(f,R) Now let P_g(f,R)-P_h(f,R)$\neq$ ϕ and $f_p\in$ P_g(f,R)-P_h(f,R)$f_p\in$ P_g(f,R) $\Longrightarrow$ [f_p]$\subseteq$ (g,P_g) and $f_p\notin$ P_h(f,R) $\Longrightarrow$ [f_p]$\nsubseteq$ (h,P_h) $\Longrightarrow$ there exist $f_k\in$ [f_p] and $f_k\notin$ (h,P_h).

Since R^λ is transitive [f_k] $\subseteq$[f_p]. Hence [f_k] $\subseteq$ (g,P_g) $\Longrightarrow$ f_k $\in$ P_g(f,R) $\Longrightarrow$ f_k $\in$ $\underline{apr}$(g,P_g) $\Longrightarrow$ f_k $\in$ (h,P_h), a contradiction.

This implies P_g(f,R)$\subseteq P_h$(f,R) and hence P_h(f,R)=P_g(f,R). Let $f_p\in$ (h,P_h) $\Longrightarrow$ $f_p\in P_g$(f,R) $\Longrightarrow$ $f_p\in P_h$(f,R) $\Longrightarrow$ $f_p\in$ $\underline{apr}$(h,P_h). $\Longrightarrow$ $\underline{apr}$(g,P_g) $\subseteq$ $\underline{apr}$($\underline{apr}$(g,P_g)).

By Theorem 4.3 we have $\underline{apr}$($\underline{apr}$(g,P_g))$\subseteq$(g,P_g) $\subseteq$ $\overline{apr}$($\overline{apr}$(g,P_g)) To prove (l,P_l)$\subseteq$ (k,P_k). Suppose(l,P_l)- $\overline{(k,P_k)}$$\neq\phi$ and $f_p\in$(l,P_l)- (k,P_k) $f_p\notin$ (k,P_k) $\Longrightarrow$ $f_p\notin$ P^g(f,R) $f_p\in$(l,P_l) $\Longrightarrow$ $f_p\in$ P^k(f,R) $\Longrightarrow$ [f_p] $\cap$ (k,P_k)$\neq\tilde{0}$. Let f_k $\in$[f_p] and f_k $\in$ (k,P_k) f_k $\in$ (k,P_k) $\Longrightarrow$ [f_p] $\cap$ (g,P_g)$\neq\tilde{0}$. Since R^λ is transitive, f_k $\in$[f_p] $\Longrightarrow$ [f_k] $\subseteq$[f_p] $\Longrightarrow$ [f_p] $\cap$ (g,P_g)$\neq\tilde{0}$ $\Longrightarrow$ $f_p\in P^g$(f,R), a contradiction.

Hence $\overline{apr}(\overline{apr}(g,P_g))\subseteq\overline{apr}(g,P_g)$.

$2 \Longrightarrow 1$. Let $f_p,f_q,f_r\in R^\lambda$ such that $(f_p,f_q),(f_q,f_r)\in R^\lambda \Longrightarrow f_q\in[f_p]$and $f_q\in[f_r]$ $\Longrightarrow$ $[f_p] \cap [f_r]\neq \phi$. Let $(h,P_h)=\overline{apr}(f_p)$. By previous Lemma $[f_r] \cap(h,P_h)\neq \tilde{0}$ $\Longrightarrow f_r\in P^h(f,R) \Longrightarrow f_r\in\overline{apr}(h,P_h)$. By our assumption $\overline{apr}((h,P_h))\subseteq(h,P_h) \Longrightarrow$ $f_r\in(h,P_h)=[f_p] \Longrightarrow (f_r,f_p)\in R^\lambda \Longrightarrow R^\lambda$ is transitive.

5. Fuzzy soft topology induced by the fuzzy soft relation

Definition 5.1 ([39])**.** Let $\{P_i:i\in I\}\subset\wp$.

The union of non empty family $\{(f_i,P_i):i\in I\}$ of fuzzy soft sets over the common universe X, denoted by $\bigcup_{i\in I}(f_i,P_i)$ is defined as the fuzzy soft set(h,C) such that $C=\bigcup_{i\in I}P_i$ and for each $c\in C$, $h_c(x)=\bigvee_{i\in I_C}(f_i)_c(x),\forall$ $x\in X$, where $I_c=\{i\in I$: $c\in P_i\}$

Definition 5.2 ([39])**.** Let $\{P_i:i\in I\}\subset\wp$. The intersection of non empty family $\{(f_i,P_i):i\in I\}$ of fuzzy soft sets over the common universe X, denoted by $\bigcap_{i\in I}(f_i,P_i)$ is defined as the fuzzy soft set(h,C) such that $C=\bigcap_{i\in I}P_i$ and for each $c\in C,h_c(x)=\bigwedge_{i\in I_C}(f_i)_c(x),\forall$ $x\in X$.

Definition 5.3 ([39])**.** Let (f,P) be a fuzzy soft set defined over the universal set X and the parameter set $P \subset \wp$. τ be the sub family of $S_f(X,P)$. Then τ is called the fuzzy soft topology on (f,P) if the following conditions are satisfied:

1) $\phi,(f,P)\in\tau$;

2) $(f_1,P_1),(f_2,P_2)\in \tau \Longrightarrow (f_1,P_1)\cap (f_2,P_2)\in \tau$;

3) $\{(f_i,P_i):i\in I\}\subset\tau \Longrightarrow \bigcup_{i\in I}(f_i,P_i)\in \tau$.

The pair $(X_f,\tau)$is called the fuzzy soft topological space. Every member of τ is called open fuzzy soft set.

Theorem 5.4. *Let R be any arbitrary surjective fuzzy soft relation defined on fuzzy soft set (f,P) and if for $\lambda \in[0,1],R^\lambda$ is reflexive then $\tau=\{(g,P_g)\in S_f(X,P)$; $\underline{apr}(g,P_g)=(g,P_g)\}$is a fuzzy soft topology defined on (f,P).*

Proof. By Theorem 4.4 we have ϕ, $(f,P)\in \tau$.

Let $(g,P_g),(h,P_h)\in \tau \Longrightarrow \underline{apr}(g,P_g)= (g,P_g)$ and $\underline{apr}(h,P_h)= (h,P_h)$ (g,P_g) $\cap(h, P_h) = \underline{apr}(g,P_g)\cap \underline{apr}(h,P_h)=\underline{apr}((g,P_g)\cap (h,P_h)) \Longrightarrow (g,P_g)\cap (h,P_h)\in$ τ.

Now let $(g_i,P_i)\in \tau$, $\forall i\in I$. By Theorem 4.3, $\underline{apr}(\bigcup(g_i,P_i))\subseteq\bigcup(g_i,P_i)$. Conversely, since $\underline{apr}(g_i, P_i) = (g_i, P_i)$ we have $\bigcup(g_i,P_i)= \bigcup\underline{apr}(g_i, P_i)\subseteq\underline{apr}(\bigcup(g_i, P_i))$.

Hence $\tau=\{(g,P_g)\in S_f(X,P)$; $\underline{apr}(g,P_g)=(g,P_g)\}$is a fuzzy soft topology defined on (f,P) and τ is called the fuzzy soft topology induced by the fuzzy soft relation R on (f,P).

6. Conclusion

In this paper we have proposed fuzzy soft relation on fuzzy soft sets. By means of relations on parameter set induced by a fuzzy soft relation, a pair of rough approximate operators are defined. Also we have investigated fuzzy soft topology generated by the rough approximate operators. As a future work with the motivation of ideas presented in this paper one can think of axiomatization of proposed rough approximate operators based on fuzzy soft relation and consider some applications of proposed notions.

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Accepted: 30.04.2019

New classes of uniformly convex functions of fractional power on Banach space

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Abstract. The aim of this paper is to define new certain subclasses of analytic functions of fractional parameters in the well-known unit disk $\mathbb{U}$. Then introduce and study a new integral operator type fractional in the sense of Noor integral on Banach space. In addition, some of its applications are discussed by utilizing a Owa-Hadamard product.

Keywords: analytic functions, Owa-Hadamard product, uniformly convex functions, Banach space.

1. Introduction

In general, one of the most significant problems facing many analytical applications of geometric functions is how to introduce and study operators type fractional of analytic and univalent functions on complex Banach spaces for example (see [1, 2, 3, 4, 5, 6]). Specifically, the theory of analytic functions includes the following format:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1.1}$$

which are analytic and univalent in $\mathcal{S} \in \mathbb{U} := \{z : |z| < 1\}$, normalized by $f'(0) = 1$ and $f(0) = 0$. For instance the functions of class $\mathcal{S}$ are convex if

$$f(z) = \frac{z}{(1-z)} = z + z^2 + z^3 + \dots,$$

and are starlike if

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots.$$

The aim of this paper is to define a new analytic family type of fractional power $\mathcal{A}_\upsilon$ by

$$\mathbb{F}(z) = z + \sum_{m=2}^{\infty} a_m z^{\upsilon m}, \qquad (|z| < 1), \tag{1.2}$$

where $\upsilon := \frac{m+k-1}{k}$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Specifically, when $m = k = 1$ in (1.2) we get the *Koebe* function in $\mathbb{U}$. Then, introduce subclasses of analytic and univalent functions defined by applying fractional integral operators involving the well-known integral operator such as the Noor integral operator. For two functions $\mathbb{F}$ given by (1.2) and $\mathbb{G}(z) = z + \sum_{m=2}^{\infty} h_m z^{\upsilon m}$, the product functions $\mathbb{F} * \mathbb{G}$ is known as the convolution (or Hadamard product) and defined by [7]

$$(\mathbb{F} * \mathbb{G})(z) = z + \sum_{m=2}^{\infty} a_m h_m z^{\upsilon m} \tag{1.3}$$

and

$$(\mathbb{F} * \mathbb{G})'(z) = \mathbb{F} * \mathbb{G}'(z), \quad (|z| < 1).$$

By utilising (1.3) we introduce a new integral operator type of fractional power denote by $\mathcal{I}_{\rho,\upsilon} : \mathcal{A}_\upsilon \to \mathcal{A}_\upsilon$. In this effort, let defined the analytical fractional function $\mathbb{F}_\rho$ by

$$\mathbb{F}_\rho = \frac{z^\upsilon}{(1 - z^\upsilon)^{\rho+1}}, \quad (z \in \mathbb{U}, \upsilon \geq 1, \rho \geq 1)$$

such that

$$\mathbb{F}_\rho(z) * \mathbb{F}_\rho^{-1}(z) = \frac{z^\upsilon}{1 - z^\upsilon}. \tag{1.4}$$

Consequently, we receive the integral operator $\mathcal{I}_{\rho,\upsilon}$ defined by

$$\begin{aligned} \mathcal{I}_{\rho,\upsilon}\mathbb{F}(z) &= \left(\frac{z^\upsilon}{(1 - z^\upsilon)^{\rho+1}}\right)^{-1} * F(z) \\ &= z + \sum_{m=2}^{\infty} \frac{(m-1)!}{(\rho+1)_{m-1}} a_m z^{\upsilon m}. \end{aligned} \tag{1.5}$$

For $\rho = 1$ and $\upsilon \geq 1$, then the integral operator $\mathcal{I}_{1,\upsilon}$ is closed to the Noor integral (see [8]) of the m-th order of function $\mathbb{F} \in \mathcal{A}_\upsilon$. Corresponding to (1.5), we have the following conclusion:

$$z(\mathcal{I}_{\rho,\upsilon} F(z))' = z + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(\rho+1)}{\Gamma(m+\rho)} \upsilon a_m z^{\upsilon m}. \tag{1.6}$$

In the following section, we study some properties of the integral operator $\mathcal{I}_{\rho,\upsilon}$ given by (1.5) in the class of uniformly convex functions type of fractional power on Banach spaces.

2. Class of uniformly convex functions

Let $\mathbb{X}$ be a Banach space and $\mathbb{X}^\dagger$ its dual. For any $A \in \mathbb{X}^\dagger$, we interest the set $\mathcal{W}(A) := \{w \in \mathbb{X} : A(w) \neq 0\}$ and let the set $\gamma(A) := \{w \in \mathbb{X} : \mathbb{X} \setminus \mathcal{W}(A)\}$.

If $A \neq 0$ then $\mathcal{W}(A)$ is dense in $\mathbb{X}$ and $\mathcal{W}(A) \cap \hat{\mathcal{B}}$ is dense in $\hat{\mathcal{B}}$, where $\hat{\mathcal{B}} := \{w \in \mathbb{X} : ||w|| = 1\}$. Let define $\mathcal{B}$ be a complex Banach space and $\mathcal{H}(\mathcal{B}, \mathbb{C})$ be a family of all functions $f : \mathcal{B} \to \mathbb{C}$, such that $f(w)|_{w=0} = 0$, this means that these functions are holomorphic in $\mathcal{B}$ and have the Fréchet derivative $f'(w)$ for all points $w \in \mathcal{B}$.

Recall that : Let Υ and Ξ be two Banach spaces, such that $\Omega \subset \Upsilon$ an open subset in V. A function $\phi : \Omega \to \Xi$ is known as Fréchet differentiable at $y \in \Omega$ if there exists a bounded linear operator $\Lambda : \Upsilon \to \Xi$ such that [9]

$$\lim_{h \to 0} \left[\frac{\|\phi(y+h) - \phi(y) - \Lambda h\|_{\Xi}}{\|h\|_{\Upsilon}} \right] = 0.$$

If $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$, then

$$f(w) = \sum_{m=1}^{\infty} \mathcal{P}_m(w). \tag{2.1}$$

Remark 1. We note that, the series $\mathcal{P}_m : \mathbb{X} \to \mathbb{C}$ are

1- Uniformly convergent on some neighborhood V of the origin.

2- Continuous and homogeneous polynomials of degree m.

In unit disk $\mathbb{U}$, let denote the family CV of functions by

$$\mathbb{F}(z) = z + \sum_{m=2}^{\infty} a_m z^{\upsilon m}, \tag{2.2}$$

are convex in $\mathbb{U}$. In geometrically sense Goodman [10] considered the class $\mathbb{U}CV \subset CV$ of uniformly convex functions in $\mathbb{U}$ and stated that, if f normalized and every (positive oriented) circular arc γ with center ζ in $\mathbb{U}$ such that the image arc $f(\gamma)$ is a convex arc, then $f \in CV$. Moreover, proved the function f given by (1.1) analytic in $\mathbb{U}CV$ if and only if satisfied

$$R\{(z-\zeta)\frac{f''(z)}{f'(z)} + 1\} \geq 0, \quad (z, \zeta) \in \mathbb{U} \times \mathbb{U}. \tag{2.3}$$

Lemma 1 ([10])**.** *If $f \in \mathbb{U}CV$, then*

$$|a_n| \leq \frac{1}{n}, \quad n \geq 2.$$

Rønning [11] declared that, the function f analytic in $\mathbb{U}CV$ if and only if

$$R\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad |z| < 1. \tag{2.4}$$

Now, let $A \in \mathbb{X}^{\dagger}$, $A \neq 0$. For any $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form

$$\mathbb{F}(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_m(w), \quad w \in \mathcal{B} \tag{2.5}$$

and for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ we put

$$\mathbb{F}_a(z) = \frac{\mathbb{F}(za)}{A(a)}, \quad \upsilon \geq 1, z \in \mathbb{U}. \tag{2.6}$$

It is clear that, for all $|z| < 1$

$$\mathbb{F}_a(z) = z + \sum_{m=2}^{\infty} \frac{P_m(a)}{A(a)} z^{\upsilon m} \tag{2.7}$$

and

$$\mathbb{F}_a^{(m)}(z) = \frac{F_a^{(m)}(za)(a, \cdots, a)}{A(a)}, \quad m \in \mathbb{N}. \tag{2.8}$$

Let $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ denote the family of all functions $\mathbb{F} \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form (2.5) such that, for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ the function $\mathbb{F}_a$ belongs to the class $\mathbb{U}CV$. From the following results, we investigate some properties of the functions $\mathbb{F}$ in the class $\mathbb{U}CV$.

Theorem 1 (Bounded coefficient)**.** *If the function $\mathbb{F}$ is belong in $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ and $a \in \hat{\mathcal{B}}$. Then*

$$\big|\mathcal{P}_m(a)\big| \leq \frac{1}{m}\big|A(a)\big|, \quad m \geq 2.$$

Proof. Assume that, the function $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, if $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, then $\mathbb{F}_a \in \mathbb{U}CV$. In another side, if $a \in \gamma(A) \cap \hat{\mathcal{B}}$, clearly that $a = \lim_{n\to\infty} a_n$, where $a_n \in \mathcal{W}(A), n \in \mathbb{N}$. There exists $r_n \in \mathbb{R}^+$ such that $\frac{a_n}{r_n} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}, n \in \mathbb{N}$, it is clear that $(r_n, n > 0)$ is bounded for the origin is an interior point of $\mathcal{B}$. For $\frac{a_n}{r_n} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, $n \in \mathbb{N}$, we obtain

$$\big|\mathcal{P}_m(\frac{a_n}{r_n})\big| \leq \frac{1}{m}\big|A(\frac{a_n}{r_n})\big|, \quad m \geq 2$$

consequence

$$\big|\mathcal{P}_m(a_n)\big| \leq \frac{r_n^{m-1}}{m}\big|A(a_n)\big|, \quad m \geq 2$$

by letting $n \to \infty$, we get $\mathcal{P}_m(a) = 0$. □

Corollary 1. All the functions $\mathbb{F}$ in $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ are vanish on $\gamma(A) \cap \mathcal{B}$.

Corollary 2. If $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, then

$$||\mathcal{P}_m|| \leq \frac{1}{m}||A||, \quad m \geq 2.$$

Theorem 2 (Sufficient condition)**.** *If* $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$ *and* $\mathbb{F}'(w) \neq 0$*, for all* $w \in \mathcal{B}$*, then*

$$R\left\{1 + \frac{\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right\} \geq \left|\frac{\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B}. \tag{2.9}$$

Proof. Let $w \in \mathcal{W}(A) \cap \mathcal{B}$, $w \neq 0$, then $a = \frac{w}{||w||} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ and thus the $\mathbb{F}_a \in \mathbb{U}CV$. By using (2.4), we get

$$R\left\{1 + \frac{z\mathbb{F}_a''(z)}{\mathbb{F}_a'(z)}\right\} \geq \left|\frac{z\mathbb{F}_a''(z)}{\mathbb{F}_a'(z)}\right|, \quad z \in \mathbb{U}.$$

By recall the equality

$$\frac{z\mathbb{F}_a''(z)}{\mathbb{F}_a'(z)} = \frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}$$

then, we have

$$\left|\frac{z\mathbb{F}_a''(z)}{\mathbb{F}_a'(z)}\right| = \left|\frac{z\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}\right| \leq \left|1 + \frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}\right|,$$

by putting $za = ||w||$, we obtain (2.9). □

Corollary 3. For $\mathbb{F} \in \mathcal{H}(\mathcal{B},\mathcal{C})$, $\mathbb{F}'(w)|_{w=0} = A$ and $\mathbb{F}'(w) \neq 0$, for all $w \in \mathcal{B}$. If

$$R\left\{1 + \frac{z\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right\} \geq \left|\frac{z\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B} \tag{2.10}$$

then $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$

Proof. Let $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$. Then $\mathbb{F}_a'(z) = \mathbb{F}'(za)(a) \neq 0$, $|z| < 1$ and

$$\frac{z\mathbb{F}_a''(z)}{\mathbb{F}_a'(z)} = \frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)}, \quad |z| < 1.$$

□

From (2.10), we get $\mathbb{F}_a \in \mathbb{U}CV$, for all $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ hence $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$.

3. Owa-Hadamard product

In this section, we set up some certain results which dealing with the Owa-Hadamard product functions $\mathbb{F}(w)$ of form (2.5). First, let define

$$\mathbb{F}_j(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_{m,j}(w) z^{\upsilon m}, \quad j = \{1, 2, \cdots l\}, \tag{3.1}$$

and

$$\mathbb{F}_{a_j}(z) = z + \sum_{m=2}^{\infty} \frac{\mathcal{P}_m(a_j)}{A(a_j)} z^{\upsilon m}, \quad j = \{1, 2, \cdots l\}, z \in \mathbb{U}. \tag{3.2}$$

Let define the Owa-Hadamard product of two functions $\mathbb{F}(w)$ and $\mathbb{G}(w)$ in class $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ by

$$(\mathbb{F} * \mathbb{G})(w) := A(w) + \sum_{m=2}^{\infty} \mathcal{P}(w)\Phi(w) z^{\upsilon m},$$

where $G(w) := A(w) + \sum_{m=2}^{\infty} \Phi(w) z^{\upsilon m} \quad w \in \mathcal{B}$.

Theorem 3. Let $\mathbb{F}_j$ given by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_\upsilon}$ for every $j = 1, 2, \cdots l$; and let the function $\mathbb{G}_i$ defined by

$$\mathbb{G}_i(w) = A(w) + \sum_{m=2}^{\infty} \Phi_{m,i}(w) z^{\upsilon m}, i = 1, 2, \cdots, s$$

then the Owa-Hadamard product of more two functions $\mathbb{F}_1 * \mathbb{F}_2 * \cdots * \mathbb{F}_l * \mathbb{G}_1 * \mathbb{G}_2, \cdots * \mathbb{G}_s(z)$ belongs to the class $\mathbb{U}CV_{\mathcal{A}_\upsilon}^{l+s}$.

Proof. Let

$$H(w) = A(w) + \sum_{m=2}^{\infty} \left\{ \prod_{j=1}^{l} \mathcal{P}_{m,j}(w) \prod_{i=1}^{s} \Phi_{m,i}(w) \right\} z^{\upsilon m}.$$

We aim to show that

$$\sum_{m=2}^{\infty} m^{l+s} \left\{ \prod_{j=1}^{l} \mathcal{P}_m(a_j) \prod_{i=1}^{s} \Phi_m(a_i) \right\} \le \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i).$$

Since $\mathbb{F}_j \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, then from Theorem 1, we obtain (3.3) and (3.4)

$$\sum_{m=2}^{\infty} m\mathcal{P}_m(a_j) \le A(a_j),$$

for every $j = 1, 2, \cdots, l$. then we have

$$\mathcal{P}_m(a_j) \le \frac{A(a_j)}{m} \tag{3.3}$$

for every $j = 1, , 2, \cdots, s$. In a similar way, for $G_i \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$ we get

$$\sum_{m=2}^{\infty} m\Phi_m(a_i) \le A(a_i).$$

Therefore

$$\Phi_m(a_i) \le \frac{A(a_i)}{m}, \tag{3.4}$$

for every $i = 1, 2, \cdots, s$. By (3.3) and (3.4), for $j = 1, 2, \cdots, l$ and $i = 1, 2, \cdots, s$, we attain

$$\begin{aligned}\sum_{m=2}^{\infty} &\left[m^{l+s} \left\{ \prod_{j=1}^{l} \mathcal{P}_m(a_j) \prod_{i=1}^{s} \Phi_m(a_i) \right\} \right] \\ &\le \left[m^{l+s} \left\{ m^{-s} m^{-l} \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i) \right\} \right] \le \left\{ \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i) \right\}.\end{aligned}$$

Hence $H(w) \in \mathbb{U}CV_{A_v}^{l+s}$. □

Corollary 4. Let the function $\mathbb{F}_j(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_{m,j}(w) z^{\upsilon m}$ given by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_v}$ for every $j = 1, 2, \cdots l$. Then the Hadamard product $\mathbb{F}_1 * \mathbb{F}_2, \cdots * \mathbb{F}_l(z)$ belongs to the class $\mathbb{U}CV_{\mathcal{A}_v}^{l}$.

Corollary 5. Let the function $\mathbb{G}_i(w) = A(w) + \sum_{m=2}^{\infty} \Phi_{m,i}(w) z^{\upsilon m}$, defined by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_v}$ for every $i = 1, 2, \cdots s$. Then the Hadamard product $\mathbb{G}_1 * \mathbb{G}_2, \cdots * \mathbb{G}_s(z)$ belongs to the class $\mathbb{U}CV_{\mathcal{A}_v}^{s}$.

4. Conclusion

We generalized a class of analytic functions (Koebe type), by utilizing the concept of fractional calculus. Moreover, by utilising the above class, we defined fractional operator type of integral in the sense of Noor integral operator. Some geometrical properties are illustrated in Banach space. The generalized product (Owa-Hadamard product) is discussed in some subclasses.

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Accepted: 18.04.2019

Asymptotic stability analysis of nonlinear systems with impulsive effects and disturbance input

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Abstract. In this paper, a sufficient condition for asymptotic stability of nonlinear systems with impulse time window is derived, which avoids solving linear matrix inequalities. For the system with disturbance input and bounded gain error due to limit of equipment and technology in practical applications, another sufficient condition is also obtained. Numerical examples are carried out to validate effectiveness of the proposed results.

Keywords: impulse time window, disturbance input, bounded gain error, impulsive control, asymptotic stability, Chua's oscillator.

1. Introduction

Customarily, R^n is an $n-$ dimensional real Euclidean space with norm $\|\cdot\|$. $R^{m\times n}$ denotes the set of all $m \times n-$ dimensional real matrices. A^T, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the transpose, the maximal and the minimal eigenvalue of a real matrix A, respectively. $A > 0$ means the matrix A is symmetric

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and positive definite. I is the identity matrix of proper dimension. Define $f\left(x\left(t_0^-\right)\right) = \lim_{t \to t_0^-} f\left(x\left(t\right)\right)$.

Impulsive control theory has received considerable attention and many scholars have been researching on this topic because it can be applied in many fields. For instance, HIV prevention model [3], chaotic systems [6], neural networks [17], etc. A lot of results of impulsive control and its applications have been reported, see [9, 10, 14, 16] and reference therein.

In the previous literature of impulsive control [2, 13], the assumption of impulses occur at fixed times. Recently, Feng, Li and Huang [4] discussed the following nonlinear impulsive control systems with impulse time window:

$$\begin{cases} \dot{x}(t) = Ax\left(t\right) + f\left(x\left(t\right)\right), & mT \leq t < mT + \tau_m, \\ x(t) = x\left(t^-\right) + Jx\left(t^-\right), & t = mT + \tau_m, \\ \dot{x}(t) = Ax\left(t\right) + f\left(x\left(t\right)\right), & mT + \tau_m < t < (m+1)T, \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ is the state vector, $f : R^n \to R^n$ is said to be a continuous nonlinear function if $f(0) = 0$, there exists a constant $l \geq 0$ such that $\|f\left(x\right)\| \leq l\left\|x\right\|$, $A, J \in R^{n\times n}$ are constant matrices, $T > 0$ is the control period, τ_m is unknown within impulse time window $\left(mT, \left(m+1\right)T\right)$. Since the impulsive effects can be stochastically occurred in an impulse time window in system (1.1), which is more general and more applicable than ones impulses occurred at fixed times. Some results related to impulse time window can be found in [8, 15].

Zou et al. [18] considered system (1.1) with bounded gain error and parameter uncertainty. The corresponding system was described as

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x\left(t\right) + f\left(x\left(t\right)\right), & mT \leq t < mT + \tau_m, \\ x(t) = (J_m + \Delta J_m)x\left(t^-\right), & t = mT + \tau_m, \\ \dot{x}(t) = Ax\left(t\right) + f\left(x\left(t\right)\right), & mT + \tau_m < t < (m+1)T, \end{cases} \tag{1.2}$$

where $\Delta A = DG\left(t\right)E$ is the parameter uncertainty, $\Delta J_m = mF(t)J_m$ is gain error which is often time-varying and bounded.

In many practical applications, we can not guarantee the input and impulses without any error due to the limit of equipment and technology. In what follows, we will consider system (1.1) with disturbance input and bounded gain error. System (1.1) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = Ax\left(t\right) + Bw\left(t\right) + f\left(x\left(t\right)\right), & mT \leq t < mT + \tau_m, \\ x(t) = x\left(t^-\right) + Jx\left(t^-\right) + \phi\left(x\left(t^-\right)\right), & t = mT + \tau_m, \\ \dot{x}(t) = Ax\left(t\right) + Bw\left(t\right) + f\left(x\left(t\right)\right), & mT + \tau_m < t < (m+1)T, \end{cases} \tag{1.3}$$

where $w\left(t\right) \in R^r$ is the disturbance input, $\phi\left(x\left(t\right)\right)$ is gain error, $\phi : R^n \to R^n$ is a continuous nonlinear function satisfying $\phi(0) = 0$, $B \in R^{n\times r}$ is a constant matrix. Without loss of generality, we assume that

$$\left\|w\left(t\right)\right\| \leq l_1\left\|x\left(t\right)\right\|, \left\|\phi\left(x\left(t\right)\right)\right\| \leq l_2\left\|x\left(t\right)\right\|,$$

where $l_1, l_2 \geq 0$ are two constants. In fact, for the case of parameter uncertainties, they are commonly assumed that $w(t) = \Delta A x(t) = DG(t) Ex(t)$ and $\phi(x(t)) = \Delta J_m x(t) = mF(t) Jx(t)$, where $\|G(t)\| \leq 1$ and $\|F(t)\| \leq 1$, while D and E are appropriate known matrices, $m > 0$ is a known constant. At the same time, we have

$$\begin{aligned} \|w(t)\|^2 &= (DG(t) Ex(t))^T (DG(t) Ex(t)) \\ &= x^T(t) E^T G^T(t) D^T DG(t) Ex(t) \\ &\leq \|D\|^2 \|E\|^2 \|x(t)\|^2 \end{aligned}$$

and

$$\begin{aligned} \|\phi(x(t))\|^2 &= (mF(t) Jx(t))^T (mF(t) Jx(t)) \\ &= x^T(t) J^T F^T(t) mmF(t) Jx(t) \\ &\leq m^2 \|J\|^2 \|x(t)\|^2 . \end{aligned}$$

We can choose $l_1 = \|D\| \|E\|$ and $l_2 = m \|J\|$. Clearly, (1.3) is more general than (1.1) and (1.2).

The aim of this paper is to present a new sufficient condition for the asymptotic stability of system (1.1). Compared with the results shown in [4, 5], our result is simpler. At the same time, we investigate system (1.3) and establish a new sufficient condition for the asymptotic stability of system (1.3). Furthermore, numerical examples are given to demonstrate effectiveness of our theoretical results.

2. Main results

We begin this section with the following lemmas.

Lemma 2.1 ([7]). *Let $x, y \in R^n$, then*

$$\left|x^T y\right| \leq \|x\| \|y\| .$$

Lemma 2.2 ([7]). *Let A be a real symmetrical matrix. Then for any $x \in R^n$,*

$$\lambda_{\min}(A) x^T x \leq x^T A x \leq \lambda_{\max}(A) x^T x.$$

Theorem 2.1. *Let $0 < P \in R^{n\times n}$ such that*

$$gT + \ln \beta < 0,$$

where $\beta = \lambda_{\max}(P^{-1}(I+J)^T P(I+J))$, $\beta_1 = \lambda_{\max}(P^{-1}(PA + A^T P))$, $\beta_2 = \lambda_{\max}(P)$, $\beta_3 = \lambda_{\min}(P)$, $g = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, then system (1.1) is asymptotically stable at origin.

Proof. Consider the following Lyapunov function:

$$V(x(t)) = x^T(t) Px(t) .$$

If $t \in [mT, mT + \tau_m)$, then by Lemmas 2.1 and 2.2, we have

$$\begin{array}{rcl} D^+ (V (x (t))) & = & 2x^T (t) P (Ax (t) + f (x (t))) \\ & = & 2x^T (t) PAx (t) + 2x^T (t) Pf (x (t)) \\ & = & x^T (t) \left(PA + A^T P\right) x (t) + 2x^T (t) P^{\frac{1}{2}} P^{\frac{1}{2}} f (x (t)) \\ & \leq & \beta_1 x^T (t) Px (t) + 2\sqrt{x^T (t) Px (t) f^T (x (t)) Pf (x (t))} \\ & \leq & \beta_1 x^T (t) Px (t) + 2\sqrt{x^T (t) Px (t) \beta_2 f^T (x (t)) f (x (t))} \\ & \leq & \beta_1 x^T (t) Px (t) + 2\sqrt{x^T (t) Px (t) \beta_2 l^2 x^T (t) x (t)} \\ & \leq & \beta_1 x^T (t) Px (t) + 2l\sqrt{x^T (t) Px (t) \dfrac{\beta_2}{\beta_3} x^T (t) Px (t)} \\ & = & gV (x (t)) , \end{array}$$

which implies that

$$V (x (t)) \leq V (x (mT)) e^{g(t-mT)}. \tag{2.1}$$

Similarly, if $t \in (mT + \tau_m, (m + 1) T)$, we also have

$$D^+ (V (x (t))) \leq gV (x (t)) ,$$

which leads to

$$V (x (t)) \leq V (x (mT + \tau_m)) e^{g(t-mT-\tau_m)}. \tag{2.2}$$

If $t = mT + \tau_m$, then we have

$$\begin{array}{rcl} V (x (t)) & = & \left(x (t^-) + Jx (t^-)\right)^T P (x (t^-) + Jx (t^-)) \\ & = & x^T (t^-) (I + J)^T P (I + J) x (t^-) \\ & = & x^T (t^-) P^{\frac{1}{2}} P^{-\frac{1}{2}} (I + J)^T P (I + J) P^{-\frac{1}{2}} P^{\frac{1}{2}} x (t^-) \\ & \leq & \beta V (x (t^-)) . \end{array} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$V (x (t)) \leq \beta V \left(x \left((mT + \tau_m)^-\right)\right) e^{g(t-mT-\tau_m)}, \tag{2.4}$$

where $t \in [mT + \tau_m, (m + 1) T)$.

When $m = 0$, if $t \in [0, \tau_0)$, then by (2.1) we have

$$V (x (t)) \leq V (x (0)) e^{gt}$$

and so

$$V \left(x \left(\tau_0^-\right)\right) \leq V (x (0)) e^{g\tau_0}. \tag{2.5}$$

If $t \in [\tau_0, T)$, then by (2.4) and (2.5) we have

$$V (x (t)) \leq \beta V (x (0)) e^{gt}$$

and so

$$V(x(T)) \le \beta V(x(0))\, e^{gT}. \tag{2.6}$$

When $m = 1$, if $t \in [T, T+\tau_1)$, then by (2.1) and (2.6) we have

$$\begin{aligned} V(x(t)) &\le V(x(T))\, e^{g(t-T)} \\ &\le \beta V(x(0))\, e^{gt}. \end{aligned} \tag{2.7}$$

If $t \in [T+\tau_1, 2T)$, then by (2.4) and (2.7) we have

$$\begin{aligned} V(x(t)) &\le \beta V\left(x\left((T+\tau_1)^-\right)\right) e^{g(t-T-\tau_1)} \\ &\le \beta^2 V(x(0))\, e^{gt}. \end{aligned}$$

By induction, when $m = k,\ k = 0, 1, \cdots$, if $t \in [kT, kT+\tau_k)$, then we have

$$\begin{aligned} V(x(t)) &\le \beta^k V(x(0))\, e^{gt} \\ &\le \begin{cases} V(x(0))\, e^{gT+k(gT+\ln\beta)}, & g > 0, \\ V(x(0))\, e^{k(gT+\ln\beta)}, & g \le 0. \end{cases} \end{aligned} \tag{2.8}$$

If $t \in [kT+\tau_k, (k+1)T)$, we obtain

$$\begin{aligned} V(x(t)) &\le \beta^{k+1} V(x(0))\, e^{gt} \\ &\le \begin{cases} V(x(0))\, e^{(k+1)(gT+\ln\beta)}, & g > 0, \\ \beta V(x(0))\, e^{k(gT+\ln\beta)}, & g \le 0. \end{cases} \end{aligned} \tag{2.9}$$

It follows from (2.8), (2.9), and $gT + \ln\beta < 0$ that

$$\lim_{t\to\infty} V(x(t)) = 0.$$

This completes the proof. □

Remark 2.1. The computation amount of solving linear matrix inequalities is not small [1]. Compared with the results shown in [4], Theorem 2.1 avoids solving linear matrix inequalities.

Theorem 2.2. *Let $0 < P \in R^{n\times n}$ such that*

$$hT + \ln\gamma < 0,$$

where

$$\beta = \lambda_{\max}(P^{-1}(I+J)^T P(I+J)), \beta_1 = \lambda_{\max}(P^{-1}(PA + A^T P)), \beta_2 = \lambda_{\max}(P),$$
$$\beta_3 = \lambda_{\min}(P), \beta_4 = \lambda_{\max}(B^T P B), \beta_5 = \lambda_{\max}(B^T B), \beta_6 = \lambda_{\max}((I+J)^T(I+J)),$$
$$h = \beta_1 + 2\sqrt{\frac{\beta_4 l_1^2 + \beta_2 l^2 + 2 l l_1 \beta_2 \sqrt{\beta_5}}{\beta_3}},$$
$$\gamma = \beta + \frac{l_2 \beta_2 (2\sqrt{\beta_6} + l_2)}{\beta_3},$$

then system (1.3) is asymptotically stable at origin.

Proof. Consider the following Lyapunov function:

$$V(x(t)) = x^T(t) P x(t).$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}(Bw(t))^T Pf(x(t)) &\le \sqrt{(Bw(t))^T Bw(t)(Pf(x(t)))^T Pf(x(t))} \\ &\le \sqrt{\beta_5\beta_2^2 w^T(t) w(t) f^T(x(t)) f(x(t))} \\ &\le \sqrt{\beta_5\beta_2^2 l_1^2 l^2 x^T(t) x(t)} \\ &\le \tfrac{ll_1\beta_2}{\beta_3}\sqrt{\beta_5} x^T(t) P x(t).\end{aligned} \tag{2.10}$$

Similarly, we have

$$w^T(t) B^T PBw(t) \le \frac{\beta_4 l_1^2}{\beta_3} x^T(t) P x(t) \tag{2.11}$$

and

$$f^T(x(t)) Pf(x(t)) \le \frac{\beta_2 l^2}{\beta_3} x^T(t) P x(t). \tag{2.12}$$

By Lemmas 2.1, 2.2, (2.10), (2.11) and (2.12), we have

$$\begin{aligned}&x^T(t) P(Bw(t) + f(x(t))) = x^T(t) P^{\frac{1}{2}} P^{\frac{1}{2}}(Bw(t) + f(x(t))) \\ &\le \sqrt{x^T(t) Px(t)(Bw(t) + f(x(t)))^T P(Bw(t) + f(x(t)))}\end{aligned} \tag{2.13}$$

$$\begin{aligned}&= \sqrt{x^T(t)Px(t)(w^T(t)B^TPBw(t) + 2w^T(t)B^TPf(x(t)) + f^T(x(t))Pf(x(t)))} \\ &\le \sqrt{\frac{\beta_4 l_1^2 + \beta_2 l^2 + 2ll_1\beta_2\sqrt{\beta_5}}{\beta_3}} x^T(t) P x(t).\end{aligned}$$

If $t \neq mT + \tau_m$, by (2.13) we have

$$\begin{aligned}D^+(V(x(t))) &= 2x^T(t) P(Ax(t) + Bw(t) + f(x(t))) \\ &= 2x^T(t) PAx(t) + 2x^T(t) P(Bw(t) + f(x(t))) \\ &\le (\beta_1 + 2\sqrt{\frac{\beta_4 l_1^2 + \beta_2 l^2 + 2ll_1\beta_2\sqrt{\beta_5}}{\beta_3}}) x^T(t) P x(t) \\ &= hV(x(t)),\end{aligned}$$

which implies that

$$V(x(t)) \le V(x(mT)) e^{h(t-mT)}.$$

If $t = mT + \tau_m$, then we have

$$
\begin{aligned}
V(x(t)) &= ((I+J)x(t^-) + \phi(x(t^-)))^T P((I+J)x(t^-) + \phi(x(t^-))) \\
&= x^T(t^-)(I+J)^T P(I+J)x(t^-) + 2x^T(t^-)(I+J)^T P\phi(x(t^-)) \\
&+ \phi^T(x(t^-))P\phi(x(t^-)) \qquad (2.14)\\
&\leq \beta x^T(t^-)Px(t^-) + \frac{2l_2\beta_2\sqrt{\beta_6}}{\beta_3} x^T(t^-)Px(t^-) + \frac{\beta_2 l_2^2}{\beta_3} x^T(t^-)Px(t^-) \\
&= (\beta + \frac{l_2\beta_2(2\sqrt{\beta_6}+l_2)}{\beta_3})V(x(t^-)).
\end{aligned}
$$

The rest of proof is same as that of Theorem 2.1, so we omit it here for simplicity. This completes the proof. □

Remark 2.2. If we choose $B = 0$ and $\phi(x(t)) = 0$, Theorem 2.2 is reduced into Theorem 2.1.

3. Numerical examples

In this section, we illustrate the effectiveness of our theoretical results employing the Chua's oscillator.

Example 3.1. The Chua's oscillator [12] is described by

$$
\begin{cases}
\dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\
\dot{x}_2 = x_1 - x_2 + x_3, \\
\dot{x}_3 = -\eta x_2,
\end{cases} \qquad (3.1)
$$

where α and η are parameters and $g(x_1)$ is the piecewise linear characteristics of the Chua's diode, which is defined by $g(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$, where $a < b < 0$ are two constants.

By decomposing the linear and nonlinear parts of the system in (3.1), we can rewrite it into the following form, $\dot{x}(t) = Ax + f(x)$, where

$$
A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}, f(x) = \begin{bmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.
$$

By simple calculation, we can choose $l^2 = \alpha^2(a-b)^2$.

In the initial condition $x(0) = (0.5, 0.3, -0.5)^T$, Chua's circuit (3.1) has chaotic phenomenon when $\alpha = 9.2156$, $\eta = 15.9946$, $a = -1.24905$, $b = -0.75735$, as shown in Figure 1. Meanwhile, for the sake of convenience, we can choose $P = I$, $J = diag(-0.5, -0.5, -0.5)$. A small calculations show that $\beta = 0.25$, $\beta_1 = 16.5498$, $l = 4.5313$ and $g = 25.6124$. By the condition of Theorem 2.1, we have $T < 0.0541$. Thus by Theorem 2.1 we know that the

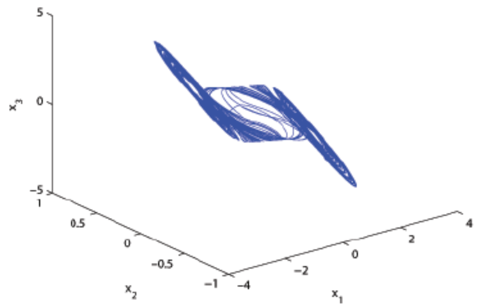

Figure 1: The chaotic phenomenon of Chua's oscillator with the initial condition $x(0) = (0.5, 0.3, -0.5)^T$.

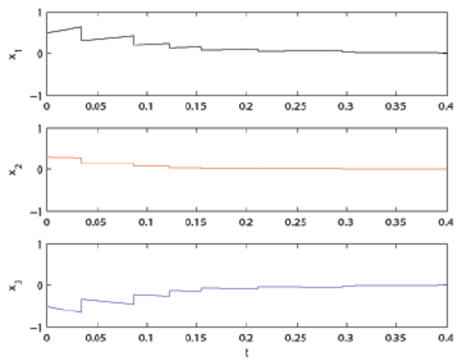

Figure 2: Time response curves of controlled Chua's oscillator with the initial condition $x(0) = (0.5, 0.3, -0.5)^T$.

origin of the system (1.1) is asymptotically stable. The simulation results with $T = 0.0500$ are shown in Figure 2.

Example 3.2. In this example, the coefficient matrix A and the impulsive control gain matrix J are same as Example 3.1. Suppose that $w(t) = 0.05x(t)cost$, $\phi(t) = 0.05x(t)sintJ$, $B = I$. Simple calculations show that $l_1 = 0.05$, $l_2 = 0.025\sqrt{3}$, $\beta = 0.25$, $\beta_1 = 16.5498$, $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 1$, $\beta_6 = 0.25$, $h = 25.7124$ and $\gamma = 0.2952$. By the condition of Theorem 2.2, we have $T < 0.0475$. Thus by Theorem 2.2 we know that the origin of the system (1.3) is asymptotically stable. The simulation results with $T = 0.0400$ are shown in Figure 3.

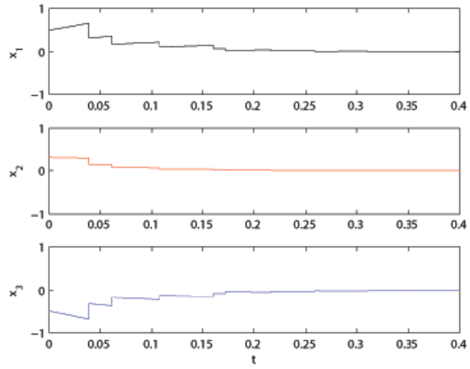

Figure 3: Time response curves of controlled Chua's oscillator with the initial condition $x(0) = (0.5, 0.3, -0.5)^T$.

4. Conclusions

In this paper, we discuss asymptotic stability of nonlinear impulsive control systems with impulse time window. The stability conditions avoid solving linear matrix inequalities. At the same time, we consider the nonlinear impulsive control systems with impulse time window, disturbance input and bounded gain error. Obviously, system (1.3) is more general and more applicable than [4, 5, 11, 18]. Finally, numerical examples demonstrate the effectiveness of the theoretical results.

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Accepted: 15.05.2019

Numerical simulation of nonlinear fractional integrodifferential equations of Volterra type via power series expansion

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Abstract. In this article, an effective recent analytical treatment is presented to solve a certain class of nonlinear fractional integrodifferential equations of Volterra type based on the residual error functions. The solution methodology of the fractional power series (FPS) approach is to replace the n-term truncated solution by generalized fractional power series to minimize the residual error function through the derivation of those functions under the Caputo concept. Anyhow, the approximate solution is obtained

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directly in a rapidly convergent fractional power series without needed to linearization, perturbation, or discretization. Numerical examples are performed to show the validity and reliability of the FPS method. Numerical analysis of the results indicates that the RPS approach is simple, efficient and systematic tool in solving fractional nonlinear issues arising in applied mathematics, physics and engineering.
Keywords: fractional derivative, residual power series method, nonlinear fractional models, integro-differential equations.

1. Introduction

The fractional differentiation and integration theory is indeed a generalization of ordinary calculus theory that deals with differentiation and integration to an arbitrary order, which is utilized to describe various real-world phenomena arising in natural sciences, applied mathematics, and engineering fields with great applications for these tools, for instance, fractional fluid-dynamic traffic, economics, solid mechanics, viscoelasticity, the nonlinear oscillation of earthquakes, control theory [1-8]. The major cause behind this is that- modeling of a specific phenomenon doesn't depend only at the time instant but also the historical state, so the fractional differential and integral operators superb tool to describe the hereditary and memory properties for different engineering and physical phenomena. However, several mathematical forms of above-mentioned issues contain nonlinear fractional integro-differential equations (FIDEs) [9-11]. Since most fractional differential and integro-differential equations cannot be solved analytically, thus it is necessary to find an accurate numerical and analytical methods to deal with the complexity of fractional operators involving such equations. Anyhow, in recent times, many experts have devoted their interest in finding solutions of the fractional integro-differential equations and other nonlinear differential equations utilizing different analytic-numeric methods [12-17]. The Adomian decomposition method, variational iteration method, homotopy perturbation method, Taylor expansion method, multistep approach, and reproducing kernel method are powerful and reliable numerical tools for handling many real-world problems [18-23].

The basic goal of the present work is to introduce a recent analytic-numeric method based on the use of residual power series technique for obtaining the approximate solution for a class of nonlinear fractional Volterra integro-differential equations in the form

$$\mathcal{D}^{\beta}_{a^+}\varphi(t) + \int_0^t h(t,s)(\varphi(s))^r ds = f(t), \quad 0 < \beta \leq 1, r \geq 2, \tag{1}$$

with the initial condition

$$\varphi(0) = \varphi_0, \tag{2}$$

where $\mathcal{D}^{\beta}_{a^+}$ denotes the Caputo fractional derivative, $f(t)$ and $h(t,s)$ are smooth functions. Here, $\varphi(t)$ is unknown analytic function to be determined.

The fractional power series (FPS) method is a recent analytic-numeric treatment method based on power series expansion, which is easy and applicable to find the series solutions for several types of the non-linear differential equation and integro-differential equations without being linearized, discretized, or exposed to perturbation [24-27]. The RPS method has been successfully applied to solve linear and non-linear ordinary, partial and fuzzy differential equations for more details, see [28-30].

The rest of the current paper is as follow: In next section, we introduce some essential preliminaries related to fractional calculus and fractional power series representations. In Section 3, we illustrate the solution methodology by using the RPS technique. In Section 4, illustrative problems are provided to demonstrate the simplicity, accuracy, and performance of the present method. Finally, we give a concluding remark in the final section.

2. Preliminaries

In this section, we recall some definitions and basic results concerning fractional calculus and fractional power series representations.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order β, over the interval $[a,b]$ for a function $\varphi \in L_1[a,b]$ is defined by*

$$\mathcal{J}_{a^+}^{\beta}\varphi(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\beta}}, & 0 < \tau < t, \beta > 0 \\ \varphi(t), & \beta = 0 \end{cases}$$

For $\beta_1, \beta_2 \geq 0$, and $q \geq -1$, the operator $\mathcal{J}_{a^+}^{\beta}$ has the following basic properties:

1) $\mathcal{J}_{a^+}^{\beta}(t-a)^q = \frac{\Gamma(q+1)}{\Gamma(q+1+\beta)}(t-a)^{q-\beta}$.

2) $\mathcal{J}_{a^+}^{\beta_1}\mathcal{J}_{a^+}^{\beta_2}\varphi(t) = \mathcal{J}_{a^+}^{\beta_1+\beta_2}\varphi(t)$.

3) $\mathcal{J}_{a^+}^{\beta_1}\mathcal{J}_{a^+}^{\beta_2}\varphi(t) = \mathcal{J}_{a^+}^{\beta_2}\mathcal{J}_{a^+}^{\beta_1}\varphi(t)$.

Definition 2.2. *For $\beta > 0$, $a, t, \beta \in \mathbb{R}$ The following fractional differential operator of order β*

$$\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau,$$

$n-1 < \beta < n$ for $n \in \mathbb{N}$, is called the Caputo fractional derivative of order β. In case $\beta = n$, then $\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{d^n}{dt^n}\varphi(t)$.

The following are some interesting properties of the operator $\mathcal{D}_{a^+}^{\beta}\varphi(t)$:

1) For any constant $c \in \mathbb{R}$, then $\mathcal{D}_{a^+}^{\beta} c = 0$.

2) $\mathcal{J}^{\beta}_{a^+}\mathcal{D}^{\beta}_{a^+}\varphi(t) = \varphi(t)$.

3) $\mathcal{J}^{\beta}_{a^+}\mathcal{D}^{\beta}_{a^+}\varphi(t) = \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a^+)}{k!}(t-a)^k$.

Definition 2.3. *The representation of the fractional power series (FPS) about $t = t_0$ is given by $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta} = c_0 + c_1(t-t_0)^{\beta} + c_1(t-t_0)^{2\beta} + \ldots$, where $0 \leq n-1 < \beta \leq n$ and $t \geq t_0$ and c_m's are the coefficients of the series.*

Remark 2.1. Let R be the radius of convergence for the FPS $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$, then the following are only the possibilities for the FPS

1) For all $t = t_0$, the series $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$ converges with R.

2) The series converges for all $t \geqslant t_0$ whenever the radius of convergence is equal to ∞.

3) The series converges for $t \in [t_0, t_0 + \mathrm{R})$, for some positive R and diverges for $t > t_0 + \mathrm{R}$.

Theorem 2.1. *Assuming that the FPS expansion $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$ $0 \leq n-1 < \alpha \leq n$, has radius of convergence $\mathrm{R} > 0$. If function $\varphi(t)$ is defined by $f(t) = \sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$, then for $t_0 < t < t_0 + \mathrm{R}$, we have the following:*

1) $\mathcal{D}^{\beta}_{t_0} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m-1)\beta+1)}(t-t_0)^{(m-1)\beta}$;

2) $\mathcal{J}^{\beta}_{t_0} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m+1)\beta+1)}(t-t_0)^{(m+1)\beta}$

Proof. For the first part, by using the definition of the Caputo fractional derivative and certain properties of the operator $\mathcal{D}^{\beta}_{t_0}$, we conclude that

$$\begin{aligned}
\mathcal{D}^{\beta}_{t_0}\varphi(t) &= \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t} \frac{\varphi^{(n)}(\varepsilon)}{(t-\varepsilon)^{\beta-n+1}} d\varepsilon \\
&= \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t} (t-\varepsilon)^{n-\beta-1}\left(\frac{d^n}{dt^n}\sum_{m=0}^{\infty} c_m(\varepsilon-t_0)^{m\beta}\right) d\varepsilon \\
&= \sum_{m=1}^{\infty} c_m \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t} (t-\varepsilon)^{n-\beta-1}\left(\frac{d^n}{dt^n}(\varepsilon-t_0)^{m\beta}\right) d\varepsilon \\
&= \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m-1)\beta+1)}(t-t_0)^{(m-1)\beta}.
\end{aligned} \tag{3}$$

For the second part, apply the Riemann-Liouville fractional integral operator $\mathcal{J}^{\beta}_{t_0}$ and by using the property $\mathcal{J}^{\beta}_{a^+}(t-a)^q = \frac{\Gamma(q+1)}{\Gamma(q+1+\beta)}(t-a)^{q-\beta}$, we conclude that

$$\mathcal{J}^{\beta}_{t_0}\varphi(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t} \frac{\varphi(\varepsilon)}{(t-\varepsilon)^{1-\beta}} d\varepsilon = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t} (t-\varepsilon)^{\beta-1}\left(\sum_{m=0}^{\infty} c_m(\varepsilon-t_0)^{m\beta}\right) d\varepsilon$$

$$
\begin{aligned}
(4) \qquad &= \sum_{m=0}^{\infty} c_m \left(\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\varepsilon)^{\beta-1} (\varepsilon - t_0)^{m\beta} \right) d\varepsilon = \sum_{m=1}^{\infty} c_m \mathcal{J}_{t_0}^{\beta} (t-t_0)^{m\beta} \\
&= \mathcal{J}_{t_0}^{\beta} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m+1)\beta+1)} (t-t_0)^{(m+1)\beta}.
\end{aligned}
$$

□

Theorem 2.2. *Assuming that* $\mathcal{D}_{t_0}^{m\beta} = \mathcal{D}_{t_0}^{\beta}.\mathcal{D}_{t_0}^{\beta}...\mathcal{D}_{t_0}^{\beta}$ *(m-times)and* $\varphi(t)$ *has the following representation of the FPS at* $t = t_0$

$$\varphi(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\beta}$$

where $0 \le n-1 < \beta \le n, t_0 < t < t_0 + \mathrm{R}, \varphi(t) \in \mathcal{C}[t_0, t_0+\mathrm{R})$ *and* $\mathcal{D}_{t_0}^{m\beta}\varphi(t) \in \mathcal{C}[t_0, t_0+\mathrm{R})$, *for* $m = 0, 1, 2, ...$ *then the coefficients* c_m *is given as follows* $c_m = \frac{\mathcal{D}_{t_0}^{m\beta}\varphi(t_0)}{\Gamma(m\beta+1)}$.

3. Construction solution by RPS algorithm

The purpose of this section is to construct FPS solution for non-linear fractional Volterra integro-differential equations (1) and (2) by substitute its FPS expansion among its truncated residual function.

The RPS algorithm proposed the solution of Eqs. (1) and (2) about $a = 0$ has the following FPS expansion:

$$\varphi(t) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}. \tag{5}$$

For obtaining the approximate values of (5), consider the following k-th FPS approximate solution

$$\varphi_k(t) = \sum_{m=0}^{k} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}. \tag{6}$$

Clearly, if $m = 0$, $\varphi(0) = \varphi_0$. So, the expansion (6) can be written as

$$\varphi_k(t) = \varphi_0 + \sum_{m=1}^{k} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}. \tag{7}$$

Define the so-called the residual function for equations (1) and (2) as follows:

$$Res(t) = \mathcal{D}_{0^+}^{\beta}\varphi(t) + \int_0^t h(t,s)(\varphi(s))^r ds - f(t), \tag{8}$$

and the following k-th residual function

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi_k(t) + \int_0^t h(t,s)(\varphi_k(s))^r ds - f(t). \tag{9}$$

As in [31-36], some useful properties of residual function

1) $\lim_{k\to\infty} Res_k(t) = Res(t) = 0$, for each $t \in (0,1)$.

2) $\mathcal{D}^{m\beta}_{0^+} Res(0) = \mathcal{D}^{m\beta}_{0^+} Res_k(0)$, for each $m = 0, 1, 2, ...k$.

For obtaining the coefficients $c_m, m = 0, 1, 2, , k$, solve the solution of the following relation:

$$\mathcal{D}^{(k-1)\beta}_{0^+} Res_k(0) = 0, \quad k = 1, 2, 3, ... \tag{10}$$

Lemma 3.1 Assuming that $\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R}), \mathrm{R} > 0, \mathcal{D}^{j\beta}_{t^0}\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R}), 0 < \beta \leq 1$. Then for any $j \in \mathbb{N}$, we have

$$\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t) = \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}.$$

Proof. From the properties of the fractional integral operator, it follows that

$$\begin{aligned}
&\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t)\\
&= \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{\beta}_{t_0}\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)\\
&= \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{\beta}_{t_0}(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0})\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)\\
&= \mathcal{J}^{j\beta}_{t_0}\left[\left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right)\right] - \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\left(\mathcal{D}^{\beta}_{t_0}\right)\varphi(t),
\end{aligned} \tag{11}$$

Hence, for $\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\left(\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)$, one can obtain

$$\begin{aligned}
&\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t)\\
&= \mathcal{J}^{j\beta}_{t_0}\left[\left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right) - \left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right) + \mathcal{D}^{j\beta}_{t_0}\varphi(t)\right]\\
&= \mathcal{J}^{j\beta}_{t_0}\left[\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right] = \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}
\end{aligned} \tag{12}$$

with $c = \mathcal{D}^{j\beta}_{t_0}\varphi(t_0)$ □

Theorem 3.1. *If $\varphi(t)$ has the FPS of (8) with $\mathrm{R} > 0$, such that $\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R} > 0), \mathcal{D}^{j\beta}_{t_0}\varphi(t) \in \mathcal{C}(t_0, t_0 + \mathrm{R})$ for $j = 0, 1, 2, ..., N+1$. Then,*

$$\varphi(t) = \varphi_N(t) + R_N(\zeta), \tag{13}$$

where $\varphi_N(t) = \sum_{j=0}^{N} \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}$ and $R_N(\zeta) = \sum_{j=0}^{N} \frac{\mathcal{D}^{(N+1)\beta}_{t_0}\varphi(t_0)}{\Gamma((N+1)\beta+1)}(t-t_0)^{j\beta}$, for some $\zeta \in (t_0, t)$

Proof. From the properties of fractional operators, we have

$$\varphi(t)=\sum_{j=0}^{N}\left[\left(\mathcal{J}_{t_0}^{j\beta}\mathcal{D}_{t_0}^{j\beta}\right)\varphi(t)-\left(\mathcal{J}_{t_0}^{(j+1)\beta}\mathcal{D}_{t_0}^{(j+1)\beta}\right)\varphi(t)\right].$$

By using Lemma 3.1, it follows

$$\varphi(t)-\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)=\sum_{j=0}^{N}\frac{\mathcal{D}_{t_0}^{j\beta}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}. \tag{14}$$

So, $\varphi(t)=\sum_{j=0}^{N}\frac{\mathcal{D}_{t_0}^{j\beta}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}+\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)$. But

$$\begin{aligned}\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)&=\mathcal{J}_{t_0}^{(N+1)\beta}\left(\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)\\&=\frac{1}{\Gamma((N+1)\beta)}\int_0^t\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(\tau)(t-\tau)^{(N+1)\beta-1}d\tau\\&=\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)}\int_0^t(t-\tau)^{(N+1)\beta-1}d\tau,\end{aligned} \tag{15}$$

by the MVT of integrals

$$\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)}\frac{(t-t_0)_{t_0}^{(N+1)\beta}}{((N+1)\beta)}=\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)+1}(t-t_0)^{(N+1)\beta}. \tag{16}$$

□

Remark 3.1. The representation of $\varphi_N(t)$ in (13) gives an approximation of $\varphi(t)$, and $R_N(\zeta)$ is remainder term. Furthermore, if $\mid\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)\mid< M$ on$[t_0,t_0+\mathrm{R})$, then the upper bound of the error can be computed by

$$|R_N(\zeta)|=\sup_{t\in[t_0,t_0+\mathrm{R}]}\frac{M(t-t_0)^{(N+1)\beta}}{\Gamma(N+1)\beta+1}.$$

Remark 3.2. To solve the fractional IVP in (1) and (2) by the FPS method, let

$$\varphi_N(t)=\sum_{n=0}^{N}c_n\frac{t^{n\beta}}{\Gamma(n\beta+1)}, \tag{17}$$

with radius of convergence $\mathrm{R}_0>0$. If $\varphi(t)\in\mathcal{C}[0,\mathrm{R}_0),\mathcal{D}_{t_0}^{j\beta}\varphi(t)\in\mathcal{C}(0,\mathrm{R}_0)$ then $\varphi(t)=\varphi_N(t)+R_N(\zeta)$.

Algorithm 3.1. To find the coefficients $c_m, m=1,2,3,k$, in (7), do the following steps:

Step 1: Substitute the expansion (6) function $\varphi_k(t)$ into the k-th residual residual function (7) such that

$$Res_k(t) = \mathcal{D}^{\beta}_{a^+}\left(\varphi_0 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right) + \int_0^t h(t,s)\left(\varphi_0 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^r ds - f(t). \tag{18}$$

Step 2: Find the relation of fractional formula $\mathcal{D}^{(k-1)\beta}_{t_0}$ of $Res_k(t)$ at $t = t_0$.
Step 3: Do the following:

For $k = 1$, obtain $Res_1(t)\Big|_{t=0} = 0$.

For $k = 2$, obtain $\mathcal{D}^{\beta}_{0^+} Res_2(t)\Big|_{t=0} = 0$.

$\vdots$

For $k = m$, obtain $\mathcal{D}^{m\beta}_{0^+} Res_m(t)\Big|_{t=0} = 0$.

Step 4: Solve the obtained system $\mathcal{D}^{(k-1)\beta}_{0^+} Res_k(0)$, $k = 1, 2, 3, \ldots$.
Step 5: Substitute the values of c_m back into Eq. (4) and then STOP.

4. Numerical examples

In this section, we demonstrate the efficiency, accuracy of the RPS approach by applying to two nonlinear fractional VIDEs. All numerical calculations are performed using Mathematica 10.

Example 4.1. Consider the following nonlinear fractional VIDE

$$\mathcal{D}^{\beta}_{0^+}\varphi(t) = e^t + \frac{t}{3}(1 - e^{3t}) + \int_0^t e^{t-s}(\varphi(s))^3 ds, \quad 0 < \beta \le 1, \tag{19}$$

with the initial condition

$$\varphi(0) = 1. \tag{20}$$

Here, the exact solution at $\beta = 1$ is given by $\varphi(t) = e^t$.

Using the FPS algorithm, The k-th residual function $Res_k(t)$ is given by

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi(t) - \int_0^t e^{t-s}(\varphi_k(s))^3 ds - \left(e^t + \frac{t}{3}(1 - e^{3t})\right), \tag{21}$$

where $\varphi_k(t)$ has the form

$$\varphi_k(t) = 1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)},$$

Consequently,

$$\begin{aligned} Res_k(t) = \mathcal{D}_{0^+}^{\beta} & \left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right) \\ & - \int_0^t e^{t-s}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^3 ds - \left(e^t + \frac{t}{3}(1-e^{3t})\right), \end{aligned} \tag{22}$$

The absolute errors are listed in Table 1. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at $\beta = 1, n = 6$ and step size 0.2. While Table 2 show approximate solutions at different values of β such that $\beta \in 1, 0.9, 0.8, 0.7$ with step size 0.16. From the table, one can be found that the RPS method provides us with an accurate approximate solution, which is in good agreement with the exact solutions for all values of t in $[0, 1]$.

Table 1. The numerical results of Absolute error for Example 4.1 at $\beta = 1$.

t	Exact solution	Numerical solution	Absolute error
0.2	1.221402758160169	1.2214027555555556	2.60461×10^{-9}
0.4	1.491824697641270	1.4918243555555555	3.42085×10^{-7}
0.6	1.822118800390509	1.8221128000000000	6.00039×10^{-6}
0.8	2.225540928492468	2.2254947555555558	4.61729×10^{-5}

Table 2. Numerical results for Example 4.1 for different values of β.

t	6^{th} FPS solution			
	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.16	1.1735108704	1.2236588706	1.2896293585	1.3781327965
0.32	1.3771276933	1.4620068483	1.5701199217	1.7112052001
0.48	1.6160731635	1.7354009578	1.8854925107	2.0791905574
0.64	1.8964714019	2.0527277406	2.2480109650	2.4982975959
0.80	2.2254947555	2.4227191207	2.6681456160	2.9806767189
0.96	2.6115273760	2.8549680620	3.1566853797	3.5379276118

Example 4.2. Consider the following nonlinear fractional VIDE

$$\mathcal{D}_{0^+}^{\beta}\varphi(t) = \int_0^t cost - s(\varphi(s))^2 ds - \frac{2}{3}sin(t)(2 + cos(t)), \quad 0 < \beta \leq 1, \tag{23}$$

with the initial condition

$$\varphi(0) = 1. \tag{24}$$

Here, the exact solution is $\varphi(t) = cos(t)$ for $\beta = 1$.

Using the FPS algorithm, The k-th residual function $Res_k(t)$ is given by

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi_k(t) - \int_0^t cos(t-s)(\varphi_k(s))^2 ds + \frac{2}{3}sin(t)(2+cos(t)), \tag{25}$$

where $\varphi_k(t)$ has the form

$$\varphi_k(t) = 1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)},$$

Consequently,

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)$$
$$- \int_0^t \cos(t-s)\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^2 ds + \frac{2}{3}\sin(t)(2+\cos(t)),$$

The absolute errors are given in Table 3. The results obtained by the FPS method show that the exact solutions are in good agreement with approximate solutions at $\beta = 1, n = 8$ and step size 0.25. While Table 4 show approximate solutions at different values of β such that $\beta = 1, 0.95, 0.85$, and $\beta = 0.75$ with step size 0.2. In Figure 1, the behavior of the 8th FPS-approximation is plotted for different values of β in $[0, 1]$, where $\beta = 1, 0.95, 0.85$, and $\beta = 0.75$. From these results, it can be observed that the behavior of the approximate solutions for different values of β is in good agreement with each other that depends on the fractional order β.

Table 3. The numerical results of Absolute error for Example 4.2 at $\beta = 1$.

t	Exact solution	Numerical solution	Absolute error
0.25	0.9689124217106447	0.9689124217109074	2.62679×10^{-13}
0.50	0.8775825618903728	0.8775825621589781	2.68605×10^{-10}
0.75	0.7316888688738209	0.7316888843263899	1.54526×10^{-8}
1.0	0.5403023058681397	0.5403025793650793	2.73497×10^{-7}

Table 4. Numerical results for Example 4.1 for different values of β.

t	6^{th} FPS solution			
	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
0.2	0.98006657784	0.97441161411	0.95844541415	0.93403621799
0.4	0.921060994032	0.90575062873	0.86795945799	0.82005645947
0.6	0.82533561657	0.80058551593	0.74520770139	0.68453141148
0.8	0.69670673879	0.66520694561	0.60100540330	0.54042995454

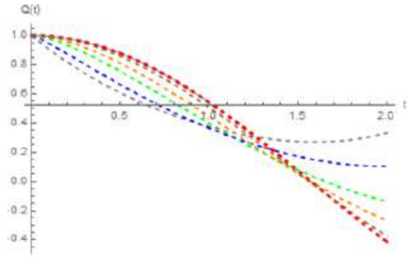

Figure 1: FPS-solution plots of Example 4.2 for different values of β.

5. Concluding remarks

The present paper aims to solve a class of nonlinear fractional Volterra integro-differential equations of order β, $0 < \beta \leq 1$, based on the use of RPS algorithm. The solution methodology depends on the constructing of the residual function and applying the generalized Taylor formula under the Caputo fractional derivative. The proposed algorithm provides the solutions in the form of rapidly convergent series with no need linearization, limitation on the problems nature, sort of classification or perturbation. Graphical and numerical results are performed by Mathematica 10. The results demonstrate the accuracy, efficiency and the capability of the present method. Therefore, the RPS algorithm is reliable, effective, simple, straightforward tool for handling a wide range of nonlinear fractional integro-differential equations.

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Accepted: 22.05.2019

Qualitative behavior of a SIRS epidemic model with vaccination on heterogeneous networks

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Abstract. This paper studies the dynamics of a SIRS epidemic model with varying population size and vaccination in a complex network. Using an analytical method, we mainly investigate the stability of the model according to the threshold $\mathcal{R}_0$. That is, if $\mathcal{R}_0$ is less than one, then the disease will die out. Alternatively, the system admits a unique endemic equilibrium which is globally asymptotically stable if $\mathcal{R}_0 > 1$. Moreover, we investigate the case when $\mathcal{R}_0 = 1$. Finally, some numerical simulations are provided to illustrate the effectiveness of the theoretical results.

Keywords: SIRS model, vaccination, stability, complex network.

1. Introduction

In order to prevent and understand the spreading of diseases, mathematical epidemic models have been developed. Based on the pioneering work by Kermark and Mckendrick [9], Many researches have studied the spread of infectious diseases in a population by compartmental models such as SIS, SIR, SIRS, SEIR or SVIS, see for instance [5, 10, 19, 11, 12]. Arino et al [2], incorporated vacci-

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nation of both newborns and susceptible individuals into an SIRS model. They shown that a backward bifurcation leading to bistability can occur depending to the efficacy of the vaccine. In [4], Onofrio studied the use of a pulse vaccination strategy to eradicate infectious diseases. However, the early models were presented on homogeneous networks which implies that all individuals are equally likely to contact each other. Obviously, this assumption is unrealistic in some sense because physical contacts between individuals vary with each individual. To deal with the effect of contact heterogeneity, another approach came to analyze the spreading of diseases using the complex network theory. In a complex network, each node represents an individual in its corresponding epidemiological state, and each edge between two nodes stands for an interaction that may allow disease transmission. Several forms of computer-generated networks have been studied in the context of disease transmission. Each of these idealized networks can be defined in terms of how individuals are distributed in space and how connections are formed. One of the most studied network is scale-free network, see for instance [3, 7, 15, 20]. Scale-free network provides a means of achieving extreme levels of heterogeneity. In such networks nodes degree followed the power-law distribution. Namely, $P(k) \sim k^{-\gamma}$, the parameter γ must be larger than zero to ensure a finite average connectivity $< k >$. One special case of scale-free networks is the Barabasi Albert (BA) model [3]. In this model $P(k) \sim k^{-3}$. It incorporates two important general concepts: growth and preferential attachment. Growth means that the number of nodes in the network increases over time. Preferential attachment means that the more connected a node is, the more likely it is to receive new links. Scale-free networks can be constructed dynamically by adding new individuals to a network one by one with a connection mechanism that imitates the natural formation of social contacts. In the preferential attachment model of Barabasi Albert [3], the existence of individuals of arbitrarily large degree means that there is no level of random vaccination that is sufficient to prevent an epidemic [1, 17, 20]. On the other hand, when there is some upper limit imposed on the degree of individuals [21], or when a scale-free network is generated by nearest neighbor attachment within a lattice [22], it becomes possible to control infection through random vaccination [8]. In addition, Li et al [13] proposed a SIRS network-based model in constant population size and studied the global dynamics through theoretical analysis and numerical simulation.

In this paper, based on the previous works, we will study a SIRS epidemic model on the scale-free networks with vaccination in a non-constant population including births and deaths, where a fraction q of the newly born individuals are vaccinated at birth . Due to the complexity of network structure, the nodes in network are divided into n classes with respect to their degrees, where n denotes the maximum degree of the network. That is to say that the nodes i and j belong to the k-th class if they both have degree k , where $k \in \{1, 2, ..., n\}$. So, the dynamical behaviour of our model can be described as

$$
(1) \qquad \begin{cases} \frac{dS_k}{dt} = & (1-q)\Lambda - (\mu_1+\nu)S_k - \beta k \Theta S_k + \gamma R_k, \\ \frac{dI_k}{dt} = & -(\mu_2+\lambda)I_k + \beta k \Theta S_k, \\ \frac{dR_k}{dt} = & q\Lambda - (\mu_3+\gamma)R_k + \nu S_k + \lambda I_k, \end{cases}
$$

where the initial states satisfy

$$
(2) \quad S_k(0), I_k(0), R_k(0) > 0 \quad \text{and} \quad S_k(0)+I_k(0)+R_k(0) \leq \frac{\Lambda}{\mu},\ k = 1,2,...,n.
$$

Denote the meaningful domain for system (1) by

$$
\Delta = \{(S_1, I_1, R_1, ..., S_n, I_n, R_n) \in \mathrm{R}_+^{3n},\ S_k + I_k + R_k \leq \frac{\Lambda}{\mu},\ k = 1,2,...,n\}.
$$

The meaning of the variables and parameters in system (1) is as follows. $S_k(t)$, $I_k(t)$ and $R_k(t)$ represent the relative densities of the susceptible, infected and recovered nodes with degree k, Λ is the birth rate (and $q \in [0,1]$ is a percentage of new born vaccinated children). μ_1, μ_2, and μ_3 represents the death rates of susceptible, infected and recovered individuals, respectively. β is the infection coefficient, λ is the rate at which the infective individuals become recovered, ν is the proportional coefficient of vaccinated for susceptible, and γ is the average loss of immunity rate. Also, it is assumed that the connectivity of nodes on the network is uncorrelated, thus, the probability that an edge points to an infected node with degree k is proportional to $kP(k)I_k(t)$ such that $\Theta(t) = \sum_{k=1}^{n} \frac{kP(k)I_k(t)}{<k>}$, where $P(k)$ is the connectivity distribution and $<k>= \sum_{k=1}^{n} kP(k)$ is the average degree of the network.
The rest of this paper is organized as follows. In Section 2, we discuss the positivity and boundedness of the solutions. Then, we establish the basic reproduction number and the existence of equilibrium points. Section 3 is devoted to explore the convergence of solution of system (1) to the disease-free equilibrium and the global stability of the endemic equilibrium. Finally, conclusions and simulations are drawn in Section 4.

2. Positivity of solutions and the epidemic threshold

In this section, we will provide some basic properties of system (1). First we establish that the domain Δ is positively invariant with respect to system (1).

Lemma 2.1. *Let* $(S_1, I_1, R_1, ..., S_n, I_n, R_n)$ *be the solution of system* (1) *with initial conditions* (2) *and* $\Theta(0) > 0$. *Then, the set* Δ *is positively invariant for model* (1) *and* $\Theta(t) > 0$ *for all* $t > 0$.

Proof. First, we will show $\Theta(t) > 0$. In fact, from the second equation of system (1) we have

$$
(3) \qquad \frac{d\Theta(t)}{dt} = \left(-(\mu_2+\lambda) + \beta \sum_{k=1}^{n} \frac{k^2 P(k) S_k(t)}{<k>} \right) \Theta(t).
$$

Then

$$\Theta(t)=\Theta(0)\exp\Bigg(-(\mu_2+\lambda)t+\frac{\beta}{<k>}\int_0^t\sum_{k=1}^n k^2P(k)S_k(s)ds\Bigg)>0.$$

On the other hand, we have $S_k(0)>0$ for $k=1,...,n$. So, by continuity there exists δ_1 such that $S_k(t)>0$ for $t\in(0,\delta_1)$ and $k=1,...,n$.
Let $\delta_k=\sup\{\tau>0: S_k(t)>0, \forall t\in(0,\tau)\}$. Now, we will show $S_k(t)>0$ for all $t>0$ and $k=1,...,n$. To this end, we have to proof that $\delta_k=\infty$ for $k=1,...,n$. Suppose not, so there exists $m\in\{1,...,n\}$ such that $\delta_m<\infty$. Then, $S_m(\delta_m)=0$ and $S_m(t)>0$ for all $t\in(0,\delta_m)$. From the second equation of (1), we get $I'_m(t)+(\lambda+\mu_2)I_m(t)>0$ for $t\in(0,\delta_m)$. Then, $I_m(t)>I_m(0)e^{-(\lambda+\mu_2)t}\geq 0$ for $t\in(0,\delta_m)$. Since $I_m(t)>0$ and $S_m(t)>0$ for all $t\in(0,\delta_m)$. It follows $R'_m(t)+(\mu_3+\gamma)R_m(t)>0$ for $t\in(0,\delta_m)$, using the similar arguments to those given for $I_m(t)$, we get $R_m(t)>0$ for $t\in(0,\delta_m)$. By continuity of $R_m(t)$ we have $R_m(\delta_m)\geq 0$. Thus, $S'_m(\delta_m)=(1-q)\Lambda+\gamma R_m(\delta_m)>0$. So, there exists some $t\in(0,\delta_m)$ such that $S_m(t)<0$. This is apparently a contradiction. Consequently $\delta_k=\infty$ for $k=1,...,n$, which means $S_k(t)>0$ for all $t>0$ and $k=1,...,n$. Finally, by the second and the third equation of (1), we conclude that $I_k(t)>0$ and $R_k(t)>0$ for all $t>0$ and $k=1,...,n$.

Now, let denote $N_k(t)=S_k(t)+I_k(t)+R_k(t)$ for all $t\geq 0$ and $k=1,...,n$. By summing the three equations of (1), we get

$$\begin{array}{rcl}\dfrac{dN_k(t)}{dt} &=& \Lambda-\mu_1S_k-\mu_2I_k-\mu_3R_k\\ &\leq& \Lambda-\mu(S_k+I_k+R_k)\\ &\leq& \Lambda-\mu N_k(t),\end{array}\tag{4}$$

where $\mu=\min(\mu_1,\mu_2,\mu_3)$. Using the comparison principle of ODEs we deduce

$$N_k(t)\leq\frac{\Lambda}{\mu}+(N_k(0)-\frac{\Lambda}{\mu})\exp(-\mu t).$$

Hence, $S_k(t),I_k(t),R_k(t)\leq\frac{\Lambda}{\mu}$ for all $t>0$ and $k=1,...,n$, which implies that Δ is positively invariant. □

Obviously, system (1) admits the disease-free equilibrium $E^0=(S^0,0,R^0,...,S^0, 0,R^0)\in\mathbb{R}^{3n}$, where

$$S^0=\frac{((1-q)\mu_3+\gamma)\Lambda}{\mu_1(\mu_3+\gamma)+\nu\mu_3}\quad \textit{and}\quad R^0=\frac{(q\mu_1+\nu)\Lambda}{\mu_1(\mu_3+\gamma)+\nu\mu_3}.$$

Now, we will investigate the existence of a positive equilibrium state in terms of the number

$$\mathcal{R}_0=\frac{<k^2>}{<k>}\frac{\beta S^0}{\mu_2+\lambda}.$$

Lemma 2.2. *The system* (1) *admits a unique endemic equilibrium* $E^* = (S_1^*, I_1^*, R_1^*, ..., S_n^*, I_n^*, R_n^*)$ *if and only if* $\mathcal{R}_0 > 1$.

Proof. By letting the right side of system (1) equal to zero, we get the following equations

$$S_k^* = \frac{(\mu_2+\lambda)I_k^*}{\beta k \Theta^*} \tag{5}$$

$$R_k^* = \frac{q}{\mu_3+\gamma}\Lambda + \left(\frac{\nu(\mu_2+\lambda)}{(\mu_3+\gamma)\beta k\Theta^*} + \frac{\lambda}{\mu_3+\gamma}\right) I_k^* \tag{6}$$

$$I_k^* = \frac{((1-q)\mu_3+\gamma)\beta k\Theta^*\Lambda}{(\mu_2(\mu_3+\gamma)+\mu_3\lambda)\beta k\Theta^* + (\mu_2+\lambda)(\mu_1(\mu_3+\gamma)+\mu_3\nu)}, \tag{7}$$

which determine the endemic equilibrium E^* of system (1). We know that $\Theta^* = \frac{\sum kP(k)I_k^*}{<k>}$. So, from (7) we get $f(\Theta^*) = 1$, where

$$f(x) = \frac{1}{<k>}\sum_k^n \frac{((1-q)\mu_3+\gamma)\beta k^2 P(k)\Lambda}{(\mu_2(\mu_3+\gamma)+\mu_3\lambda)\beta k x + (\mu_2+\lambda)(\mu_1(\mu_3+\gamma)+\mu_3\nu)}. \tag{8}$$

Since, $f(0) = \mathcal{R}_0$ and f is a decreasing function, the equation $f(x) = 1$ has unique root if and only if $\mathcal{R}_0 > 1$. □

3. Disease-free equilibrium dynamics

In this section, we will first prove that the solution of system (1) converges in the mean to the disease-free equilibrium E^0 when $\mathcal{R}_0 < 1$, and next we show that under the same condition, the solution $(S_k(t), I_k(t), R_k(t))$ of system (1) converges to $(S^0, 0, R^0)$ for all $k \in \{1, ..., n\}$. Finally, we explore the crucial case when $\mathcal{R}_0 = 1$.

3.1 Convergence in the mean

Theorem 3.1. *If* $\mathcal{R}_0 < 1$, *then for all* $k \in \{1, ..., n\}$ *we have*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S_k(s)ds = S^0, \lim_{t\to\infty}\frac{1}{t}\int_0^t I_k(s)ds = 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t R_k(s)ds = R^0. \tag{9}$$

Proof. From the second equation of (1) we get

$$\dot{\Theta} = -(\mu_2+\lambda)\Theta + \beta\Theta\frac{\sum k^2 P_k S_k}{<k>}.$$

Hence

$$\begin{aligned} d\log\Theta &= -(\mu_2+\lambda) + \beta\frac{\sum k^2 P_k S_k}{<k>} \\ &= -(\mu_2+\lambda) + \beta\frac{\sum k^2 P_k S^0}{<k>} + \beta\frac{\sum k^2 P_k (S_k - S^0)}{<k>} \\ &= -(\mu_2+\lambda)(1-\mathcal{R}_0) + \beta\frac{\sum k^2 P_k (S_k - S^0)}{<k>}. \end{aligned}$$

Using the following identities

$$\begin{aligned}(1-q)\Lambda &= (\mu_1+\nu)S^0+\gamma R^0,\\ \Lambda &= (\mu_3+\gamma)R^0+\nu S^0,\end{aligned}$$

we have

$$\begin{aligned}\dot{S}_k &= -(\mu_1+\nu)(S_k-S^0)-\beta k\Theta S_k+\gamma(R_k-R^0), && (10)\\ \dot{R}_k &= -(\mu_3+\gamma)(R_k-R^0)+\nu(S_k-S^0)+\lambda I_k. && (11)\end{aligned}$$

So

$$\begin{aligned}\frac{\mu_3+\gamma}{\gamma}\dot{S}_k+\dot{R}_k &= \frac{-(\mu_3+\gamma)}{\gamma}(\mu_1+\nu)(S_k-S^0)-\beta\frac{\mu_3+\gamma}{\gamma}k\Theta S_k\\ &\quad +\nu(S_k-S^0)+\lambda I_k.\end{aligned}$$

Then

$$\frac{\mu_3(\mu_1+\nu)+\gamma\mu_1}{\gamma}(S_k-S^0) = -\frac{\mu_3+\gamma}{\gamma}\dot{S}_k-\dot{R}_k-\beta\frac{\mu_3+\gamma}{\gamma}k\Theta S_k+\lambda I_k. \quad (12)$$

By integrating the above equality both sides from 0 to t we get

$$\begin{aligned}\int_0^t(S_k(s)-S^0)ds &= \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\Bigg(-\frac{\mu_3+\gamma}{\gamma}S_k(t)+\frac{\mu_3+\gamma}{\gamma}S_k(0)\\ &\quad -R_k(t)+R_k(0)-\beta\frac{\mu_3+\gamma}{\gamma}k\int_0^t\Theta(s)S_k(s)ds\\ &\quad +\lambda\int_0^t I_k(s)ds)\Bigg),\end{aligned}$$

which together with (2) implies

$$\begin{aligned}\int_0^t(S_k(s)-S^0)ds &\le \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\frac{\Lambda}{\mu}\left(1+\frac{\mu_3+\gamma}{\gamma}\right)+\lambda\int_0^t I_k(s)ds\right)\\ &\triangleq a_0+a_1\int_0^t I_k(s)ds, && (13)\end{aligned}$$

where

$$\begin{cases}a_0=\dfrac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\dfrac{\Lambda}{\mu}(1+\dfrac{\mu_3+\gamma}{\gamma})\right)\\ a_1=\dfrac{\lambda\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}.\end{cases}$$

From (10) and (13) we have

$$\log\Theta(t)-\log\Theta(0) \le -(\mu_2+\lambda)(1-\mathcal{R}_0)t+\beta n a_0+\beta a_1 n\int_0^t\Theta(s)ds.$$

Hence

$$\Theta(t)\exp\left(-\beta a_1 n\int_0^t\Theta(s)ds\right) \quad\leq\quad \Theta(0)\exp(\beta n a_0)\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big).$$

Therefore

$$\frac{d\frac{-1}{\beta a_1 n}\exp\left(-\beta a_1 n\int_0^t\Theta(s)ds\right)}{dt} \quad\leq\quad \Theta(0)\exp(\beta n a_0)\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big).$$

By integrating the above inequality both sides from 0 to t we obtain

$$\exp\left(-\beta a_1 n\int_0^t\Theta(s)ds\right)\geq 1 + \frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\left(\exp\Big(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big)-1\right).$$

Which implies

$$-\beta a_1 n\int_0^t\Theta(s)ds \geq \log\left(1+\frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\Big(\exp(-(\mu_2+\lambda)(1-\mathcal{R}_0)t)-1\Big)\right).$$

Then,

$$\frac{1}{t}\int_0^t\Theta(s)ds \leq \frac{-1}{\beta a_1 n t}\log\left(1+\frac{\Theta(0)\beta a_1 n\exp(\beta n a_0)}{(\mu_2+\lambda)(1-\mathcal{R}_0)}\Big(\exp(-(\mu_2+\lambda)(1-\mathcal{R}_0)t\Big)\right).$$

Consequently, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Theta(s)ds=0 \quad\text{and then}\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t I_k(s)ds=0,\quad k\in\{1,...,n\}. \tag{14}$$

By (3.1) we get

$$\begin{aligned}\frac{1}{t}\int_0^t(S_k(s)-S^0)ds \quad=\quad & \frac{\gamma}{\mu_3(\mu_1+\nu)+\gamma\mu_1}\left(\frac{-(\mu_3+\gamma)}{\gamma}\frac{(S_k(t)-S_k(0))}{t}\right.\\ & -\frac{R_k(t)-R_k(0)}{t}-\beta\frac{(\mu_3+\gamma)}{\gamma}k\frac{1}{t}\int_0^t\Theta(s)S_k(s)ds\\ & \left.+\lambda\frac{1}{t}\int_0^t I_k(s)ds\right).\end{aligned}$$

Combining $(S_k(t), I_k(t), R_k(t)) \in \Delta$ with (2) and (14) yields to $\lim_{t\to\infty} \Upsilon(t) = 0$, where

$$\Upsilon(t) = \frac{S_k(t) - S_k(0)}{t} + \frac{R_k(t) - R_k(0)}{t} + \frac{1}{t}\int_0^t \Theta(s) S_k(s) ds.$$

Which, together with (14), implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (S_k(s) - S^0) ds = 0, \quad k \in \{1, ..., n\}.$$

Finally, from (11) one can easily deduce

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (R_k(s) - R^0) ds = 0, \quad k \in \{1, ..., n\}.$$

□

3.2 Pointwise convergence

Theorem 3.2. *If $\mathcal{R}_0 < 1$, then for all $k \in \{1, ..., n\}$ we have*

$$\lim_{t\to\infty} S_k(t) = S^0, \quad \lim_{t\to\infty} I_k(t) = 0, \quad \textit{and} \quad \lim_{t\to\infty} R_k(t) = R^0.$$

Proof. Combining (9) and (10) leads to

$$\lim_{t\to\infty} \sup \frac{1}{t} \log \Theta(t) \le -(\mu_2 + \lambda)(1 - \mathcal{R}_0).$$

Hence, $\lim_{t\to\infty} \Theta(t) = 0$ which leads to $\lim_{t\to\infty} I_k(t) = 0$. Using (10), we obtain

$$(S_k(t) - S^0)' + (\mu_1 + \nu)(S_k(t) - S^0) = -\beta k \Theta(t) S_k(t) + \gamma(R_k(t) - R^0)$$
$$\Big((S_k(t) - S^0)e^{(\mu_1+\nu)t}\Big)' = \left[-\beta k \Theta(t) S_k(t) + \gamma(R_k(t) - R^0)\right] e^{(\mu_1+\nu)t}.$$

Integrating the above equality both sides from 0 to t yields to

$$\begin{aligned} S_k(t) - S^0 &= (S_k(0) - S^0)e^{-(\mu_1+\nu)t} \\ &\quad + \left(\int_0^t \Big(- \beta k \Theta(s) S_k(s) + \gamma(R_k(s) - R^0)e^{(\mu_1+\nu)s}\Big) ds\right) e^{-(\mu_1+\nu)t}. \end{aligned}$$

So

$$\lim_{t\to\infty} \sup(S_k(t) - S^0) \le \Big(\frac{\gamma}{\mu_1 + \nu}\Big) \lim_{t\to\infty} \sup(R_k(t) - R^0).$$

Similarly, we also get

$$\lim_{t\to\infty} \sup(R_k(t) - R^0) \le \Big(\frac{\nu}{\mu_3 + \gamma}\Big) \lim_{t\to\infty} \sup(S_k(t) - S^0).$$

Consequently

$$\lim_{t\to\infty} R_k(t) = R^0 \quad \text{and} \quad \lim_{t\to\infty} S_k(t) = S^0, \quad k \in \{1, ..., n\}$$

□

Theorem 3.3. *If $\nu = 0$ and $\mathcal{R}_0 = 1$, then the solution $(S_k(t), I_k(t), R_k(t))$ of system* (1) *converges to $(S^0, 0, R^0)$ for all $k \in \{1, ..., n\}$.*

Proof. Let $\varepsilon > 0$ such that $\varepsilon < \Theta(0)$. We define

$$\begin{array}{ll} \tau_1 = \inf\{t > 0, \quad \Theta(t) \leq \varepsilon\}, & \tau' = \inf\{t \geq \tau_1, \quad \Theta(t) \geq \varepsilon\}, \\ \tau_m' = \inf\{t \geq \tau_m, \quad \Theta(t) \geq \varepsilon^m\}, & \tau_{m+1} = \inf\{\tau_m \leq t \leq \tau_m', \quad \Theta(t) \leq \varepsilon^{m+1}\}. \end{array}$$

Let's show that $\tau_m < \infty$, $\forall m \geq 1$. We will proceed by contradiction. Suppose that $\tau_1 = \infty$, so

$$\Theta(t) \geq \varepsilon, \forall t > 0. \tag{15}$$

Which implies that

$$\Gamma_n(I(t)) \triangleq \sum \frac{k^2 P_k}{< k >} I_k(t) > \Theta(t) \geq \varepsilon, \qquad \forall t > 0. \tag{16}$$

From the differential system (1), we have

$$\frac{d \log \Theta(t)}{dt} = -(\mu_2 + \lambda)(1 - \mathcal{R}_0) + \beta \sum \frac{k^2 P_k}{< k >}(S_k(t) - S^0) \triangleq \beta\ \Gamma_n(S(t) - S^0). \tag{17}$$

The 3^{rd} equation of (1) implies that

$$R_k(t) - R^0 = (R_k(0) - R^0)e^{-(\mu_3+\gamma)t} + \lambda \int_0^t I_k(s) e^{-(\mu_3+\gamma)(t-s)} ds. \tag{18}$$

So

$$\Gamma_n(R(t) - R^0) = \Gamma_n(R(0) - R^0)e^{-(\mu_3+\gamma)t} + \lambda \int_0^t \Gamma_n(I(s)) e^{-(\mu_3+\gamma)(t-s)} ds.$$

It follows from (16) that

$$\Gamma_n(R(t) - R^0) \geq \Gamma_n(R(0) - R^0)e^{-(\mu_3+\gamma)t} + \frac{\lambda\varepsilon}{\mu_3 + \gamma}\Big(1 - e^{-(\mu_3+\gamma)t}\Big). \tag{19}$$

On the other hand

$$d(N_k - N^0) \quad = \quad -\mu_1(N_k - N^0) - (\mu_2 - \mu_1)I_k - (\mu_3 - \mu_1)(R_k - R^0).$$

Which leads to

$$\begin{aligned} N_k(t) - N^0 \quad &= \quad (N_k(0) - N^0)e^{-\mu_1 t} - (\mu_2 - \mu_1)\int_0^t I_k(s) e^{-\mu_1(t-s)} ds \\ &\quad -(\mu_3 - \mu_1)\int_0^t (R_k(s) - R^0) e^{-\mu_1(t-s)} ds. \end{aligned} \tag{20}$$

Then we get

$$\begin{aligned}\Gamma_n(N(t)-N^0) &= \Gamma_n(N(0)-N^0)e^{-\mu_1 t}-(\mu_2-\mu_1)\int_0^t \Gamma_n(I(s))e^{-\mu_1(t-s)}ds\\ &\quad -(\mu_3-\mu_1)\int_0^t \Gamma_n(R(s)-R^0)e^{-\mu_1(t-s)}ds.\end{aligned}$$

According to (16) and (19), we have

$$\begin{aligned}\Gamma_n(N(t)-N^0) &\leq \Gamma_n(N(0)-N^0)e^{-\mu_1 t}-\frac{\mu_2-\mu_1}{\mu_1}\varepsilon\Big(1-e^{-\mu_1 t}\Big)\\ &\quad -(\mu_3-\mu_1)\Gamma_n(R(0)-R^0)\frac{e^{-\mu_1 t}\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\\ &\quad -\frac{\mu_3-\mu_1}{\mu_3+\gamma}\lambda\varepsilon\left[\frac{1}{\mu_1}\Big(1-e^{-\mu_1 t}\Big)-\frac{e^{-\mu_1 t}\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\right].\end{aligned} \tag{21}$$

In views of (17), we get

$$d\log\Theta=\beta\Big(\Gamma_n(N(t)-N^0)-\Gamma_n(I(t))-\Gamma_n(R(t)-R^0)\Big). \tag{22}$$

Substituting (19) and (21) into (22), we obtain

$$d\log\Theta(t)\leq -\mathcal{H}\varepsilon+\ F(\varepsilon,t). \tag{23}$$

Where

$$\mathcal{H}\varepsilon=\beta\frac{(\mu_2-\mu_1)}{\mu_1}\varepsilon+\beta\frac{(\mu_3-\mu_1)\lambda\varepsilon}{(\mu_3+\gamma)\mu_1}+\beta\varepsilon,$$

and

$$\begin{aligned}F(\varepsilon,t) &= \beta\ \Gamma(N(0)-N^0)e^{-\mu_1 t}+\frac{\beta(\mu_2-\mu_1)\varepsilon e^{-\mu_1 t}}{\mu_1}\\ &\quad -\beta(\mu_3-\mu_1)\Gamma_n(R(0)-R^0)e^{-\mu_1 t}\frac{\Big(1-e^{-(\mu_3+\gamma-\mu_1)t}\Big)}{\mu_3+\gamma-\mu_1}\\ &\quad +\left[\frac{\beta(\mu_3-\mu_1)\lambda\varepsilon}{\mu_3+\gamma}\left(\frac{1}{\mu_1}+\frac{1-e^{-(\mu_3+\gamma\mu_1)t}}{\mu_3+\gamma-\mu_1}\right)\right]e^{-\mu_1 t}\\ &\quad -\beta e^{-(\mu_3+\gamma)t}\Big(\Gamma_n(R(0)-R^0)+\frac{\lambda\varepsilon}{\mu_3+\gamma}\Big).\end{aligned} \tag{24}$$

Since, there exists t_0 such that $t>t_0$ and $F(\varepsilon,t)\leq\frac{\mathcal{H}\varepsilon}{2}$. Then, for $t\geq t_0$ we get

$$\int_{t_0}^t d\log\Theta(s)ds\leq -\frac{\mathcal{H}\varepsilon}{2}(t-t_0).$$

Which implies that $\Theta(t) \leq \Theta(0)\ e^{-\frac{\mathcal{H}\varepsilon}{2}(t-t_0)}$ and then $\lim\limits_{t\to\infty}\Theta(t) = 0$. This contradicts the assumption that in (15). Let's suppose that $\tau_m < \infty$ and $\tau_{m+1} = \infty$. We have $\tau_m^{'} = \infty$, which gives $\Theta(t) \geq \varepsilon^{m+1}$ for all $t > \tau_m$. Then, by using similar arguments to those given in the case when $\tau_1 = \infty$, we get

$$\begin{aligned}\Gamma_n\big(N(t) - N^0\big) \quad \leq \quad & \Gamma_n\big(N(\tau_m) - N^0\big)e^{-\mu_1 t} - \tfrac{\mu_2-\mu_1}{\mu_1}\varepsilon^{m+1}\Big(1 - e^{-\mu_1(t-\tau_m)}\Big)\\ & \frac{(-(\mu_3-\mu_2)\Gamma_n(R(\tau_m)-R^0)e^{-\mu_1 t})\Big(1-e^{-(\mu_3+\gamma-\mu_1)(t-\tau_m)}\Big)}{\mu_3+\gamma-\mu_1}\\ & -\tfrac{\mu_3-\mu_1}{\mu_3+\gamma}\lambda\varepsilon^{m+1}\Bigg[\tfrac{1}{\mu_1}\Big(1 - e^{-\mu_1(t-\tau_m)}\Big)\\ & -e^{-\mu_1 t}\frac{\Big(1-e^{-(\mu_3+\gamma-\mu_1)(t-\tau_m)}\Big)}{\mu_3+\gamma-\mu_1}\Bigg].\end{aligned}$$

By (23), we have

$$d\log\Theta(t) \leq -\mathcal{H}\varepsilon^{m+1} + F(\varepsilon^{m+1}, t-\tau_m),$$

there exists $t_0^{'}$ such that $t > t_0^{'} \vee \tau_m$ and $F(\varepsilon^{m+1}, t-\tau_m) \leq \frac{\mathcal{H}\varepsilon^{m+1}}{2}$. Which yields to

$$\int_{t_0^{'}\vee\tau_m}^{t} d\log(\Theta(s))ds \leq -\frac{\mathcal{H}\varepsilon^{m+1}}{2} \times (t - (t_0^{'} \vee \tau_m)), \tag{25}$$

thus

$$\Theta(t) \leq \Theta(0)\ e^{-\frac{\mathcal{H}\varepsilon^{m+1}}{2}\times(t-(t_0^{'}\vee\tau_m))}. \tag{26}$$

Hence, $\Theta(t) \longrightarrow 0$ as $t \longrightarrow \infty$, which contradicts the assumption that in (15). Beside, $\tau_m < \infty$ for all m in $\mathbb{N}$. By construction, the sequence $(\tau_m)_{m\in\mathbb{N}}$ is increasing. Hence, τ_m converges to τ_∞ . Also, We have $\tau_\infty = \infty$ (otherwise $\Theta(\tau_m) = \varepsilon^m$ which leads to $\Theta(\tau_\infty) = 0$, contradiction with $\Theta(t) > 0$ for all $t > 0$).

Finally, let $\eta > 0$ and $m_0 = \left[\frac{\log\eta}{\log\varepsilon}\right] + 1$, where $[.]$ denotes the integer part. For all $t \geq \tau_{m_0}$, there exists $m \geq m_0$ such that $\tau_m \leq t \leq \tau_{m+1}$ and $\Theta(t) \leq \varepsilon^m \leq \varepsilon^{m_0} \leq \varepsilon^{\log\eta/\log\varepsilon} = \eta$. So, $\Theta(t)$ converges to 0 and automatically $I_k(t)$ converges to 0 for all $k \in \{1, .., n\}$. It follows that

$$\lim_{t\to\infty}\int_0^t I_k(s)e^{-(\mu_3+\gamma)(t-s)}ds = 0.$$

Which implies by (18) that $R_k(t)$ converges to R^0 for all $k \in \{1, .., n\}$. Similarly we obtain

$$\lim_{t\to\infty}\int_0^t (R_k(s) - R^0)e^{-\mu_1(t-s)}ds = 0.$$

Then, From (20), one can deduce that $N_k(t)$ converges to N^0 and it immediately yields $S_k(t)$ converges to S^0. Finally, we have shown that $(S_k(t), I_k(t), R_k(t))$ converge towards to $(S^0, 0, R^0)$ for all $k \in \{1, .., n\}$. $\square$

4. Stability of the endemic equilibrium

In this section, We show the global asymptotical stability of the equilibrium E^* of the system (1), by means of a suitable Lyapunov function.

Theorem 4.1. *If $\mathcal{R}_0 > 1$ and the following assumptions hold*

$$\lambda > \nu\frac{\mu_2+\mu_3}{\mu_1+\mu_3} \quad \textit{and} \quad \gamma < \frac{4(\mu_1+\nu)\big(\mu_1+\mu_3-\frac{\nu}{\lambda}(\mu_2+\mu_3)\big)\mu_2}{(\mu_1+\mu_2)^2}.$$

Then E^* is globally asymptotically stable.

Proof. We consider the following Lyapunov function $\mathcal{W} = \sum_{i=1}^{4}\mathcal{W}_i$, where

$$\begin{aligned}
\mathcal{W}_1 &= \frac{a_1}{2}\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2,\ \mathcal{W}_2 = a_1\left(\Theta-\Theta^*-\Theta^*\log\frac{\Theta}{\Theta^*}\right),\\
\mathcal{W}_3 &= \frac{a_3}{2}\sum_k \frac{kP_k}{<k>S_k^*}(R_k-R^*)^2, \mathcal{W}_4 = \frac{a_4}{2}\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*+I_k-I_k^*\\
&\qquad +R_k-R_k^*)^2,
\end{aligned}$$

and a_1, a_3, a_4 are positive constants to be determined suitably. We now give the derivative of each of the previous functions.

$$\begin{aligned}
\mathcal{W}_1'+\mathcal{W}_2' &= a_1\textstyle\sum_k \frac{kP_k}{<k>S_k^*}(S_k-S_k^*)\Big[-\mu_1(S_k-S_k^*)-\beta k\Theta(S_k-S_k^*)\\
&\quad -\beta kS_k^*(\Theta-\Theta^*)+\gamma(R_k-R_k^*)\Big]\\
&\quad +a_1(\Theta-\Theta^*)\beta\textstyle\sum_k\frac{k^2P_k}{<k>}(S_k-S_k^*)\\
&= -a_1(\mu_1+\nu)\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2-a_1\beta\Theta\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2\\
&\quad +a_1\gamma\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(S_k-S^*)(R_k-R_k^*).\\
\mathcal{W}_3' &= a_3\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)\Big[-(\mu_3+\gamma)(R_k-R_k^*)+\nu(S_k-S_k^*)\\
&\quad +\lambda(I_k-I_k^*)\Big]\\
&= -a_3(\mu_3+\gamma)\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2\\
&\quad +a_3\nu\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(S_k-S_k^*)\\
&\quad +a_3\lambda\textstyle\sum_k\frac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\\
\mathcal{W}_4' &= a_4\textstyle\sum_k\frac{kP_k}{<k>S_k^*}\Big(S_k-S_k^*+I_k-I_k^*+R_k-R_k^*\Big)\times\\
&\quad \Big(-\mu_1(S_k-S_k^*)-\mu_2(I_k-I_k^*)-\mu_3(R_k-R_k^*)\Big).
\end{aligned}$$

So

$$\begin{aligned}\mathcal{W}' \;=\;& \sum_k \tfrac{kP_k}{<k>S_k^*}\Big[-a_1(\mu_1+\nu)(S_k-S_k^*)^2 - a_4\mu_2(I_k-I_k^*)^2 \\ & -a_4(\mu_1+\mu_2)(S_k-S_k^*)(I_k-I_k^*)\Big] - a_1\beta\Theta\sum_k \tfrac{kP_k}{<k>S_k^*}(S_k-S_k^*)^2 \\ & -\Big((\mu_3+\gamma)+a_4\mu_3\Big)\sum_k \tfrac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2 \\ & +\Big[a_1\gamma+a_3\nu-a_4(\mu_1+\mu_3)\Big]\sum_k \tfrac{kP_k}{<k>S_k^*}(S_k-S_k^*)(R_k-R_k^*) \\ & +\Big[a_3\lambda-a_4(\mu_2+\mu_3)\Big]\sum_k \tfrac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{W}' \;\le\;& -a_1(\mu_1+\nu)\sum \tfrac{kP_k}{<k>S_k^*}\Big(S_k-S_k^*+\tfrac{a_4(\mu_1+\mu_2)}{2a_1(\mu_1+\nu)}(I_k-I_k^*)\Big)^2 \\ & -\Big[\tfrac{4a_1a_4(\mu_1+\nu)\mu_2-\big(a_4(\mu_1+\mu_2)\big)^2}{4a_1(\mu_1+\nu)}\Big]\sum_k \tfrac{kP_k}{<k>S_k^*}(I_k-I_k^*)^2 \\ & -\Big((\mu_3+\gamma)+a_4\mu_3\Big)\sum_k \tfrac{kP_k}{<k>S_k^*}(R_k-R_k^*)^2 \\ & +\Big[a_1\gamma+a_3\nu-a_4(\mu_1+\mu_3)\Big]\sum_k \tfrac{kP_k}{<k>S_k^*}(S_k-S_k^*)(R_k-R_k^*) \\ & +\Big[a_3\lambda-a_4(\mu_2+\mu_3)\Big]\sum_k \tfrac{kP_k}{<k>S_k^*}(R_k-R_k^*)(I_k-I_k^*).\end{aligned} \tag{27}$$

Consequently, in order to get $\mathcal{W}' \le 0$, the parameters a_1, a_3, and a_4 should satisfy

$$a_1\gamma + a_3\nu - a_4(\mu_1+\mu_3) = 0, \tag{28}$$

$$a_3\lambda - a_4(\mu_2+\mu_3) = 0, \tag{29}$$

$$\frac{4a_1a_4(\mu_1+\nu)\mu_2 - \big(a_4(\mu_1+\mu_2)\big)^2}{4a_1(\mu_1+\nu)} < 0. \tag{30}$$

Therefore, from (28) and (29) we can choose $a_3 = a_4\frac{\mu_2+\mu_3}{\lambda}$ and $a_1 = \frac{a_4}{\gamma}\big(\mu_1 + \mu_3 - \frac{\nu}{\lambda}(\mu_2+\mu_3)\big)$. Thus (30) holds when

$$\gamma < \frac{4(\mu_1+\nu)\big(\mu_1+\mu_3-\frac{\nu}{\lambda}(\mu_2+\mu_3)\big)\mu_2}{(\mu_1+\mu_2)^2}. \tag{31}$$

So, it follows from (27) that $\mathcal{W}' \le 0$. Also, we have $\mathcal{W}' = 0$ if and only if $S_k = S_k^*$, $I_k = I_k^*$ and $R_k = R_k^*$ for $k = 1, 2, ..., n$. According to the LaSalle invariant principle [6], the unique endemic equilibrium state E^* is globally asymptotically stable. This completes the proof. □

5. Simulation and discussion

In this section, several numerical examples are designed to illustrate the dynamics of system (1). Using a preferential attachment algorithm, a BA network can be generated following the methods in [3]. The schema of the scale-free network with different sizes is illustrated in Figure 1.

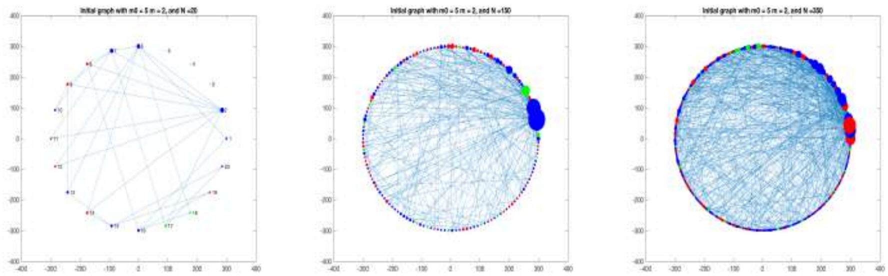

Figure 1: A Barabasi-Albert scale-free network of respectively 20, 150 and 350 nodes, it starts with $m_0 = 5$ fully connected nodes, and then each time a new node is added to the network with $m = 2$ links until the network size is reached.

Example 1. Consider a scale-free network with 20 nodes, and the parameters values $\Lambda = 0.03$, $\lambda = 0.2$, $\beta = 0.01$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $\mu_3 = 0.05$,, $\gamma = 0.6$ and $\nu = 0.015$. In this situation $\mathcal{R}_0 = 0.72 < 1$. Hence, according to Theorem (3.2) the solution of system (1) converges to E^0, (see Figure 2).

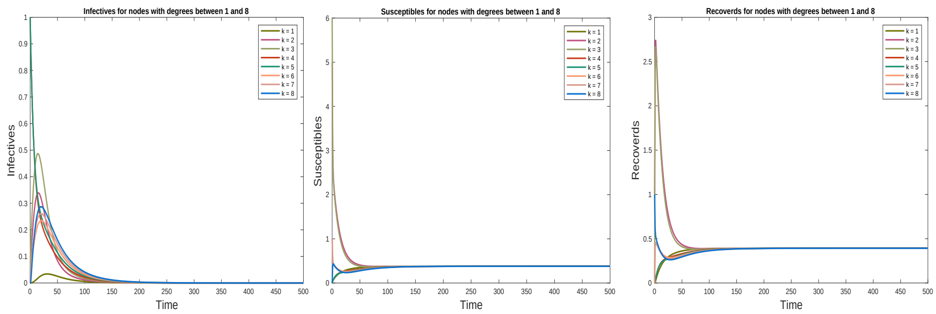

Figure 2: The time evolution of the densities of each state

Example 2. Figure (3) shows the evolution of infectives with several different values of ν, respectively $0.1, 0.2, 0.25, 0.3, 0.35$ and 0.4. We observe that the values of $I(t)$ eventually converge to corresponding equilibrium points at higher speeds as the parameter ν grows, which reveal the important role of vaccination in the stability of system (1).

Example 3. Figure (4) manifest the influence of network size in the time evolution of $I(t)$ of system (1). It is observed that the values of $I(t)$ eventually converge to corresponding equilibrium points at faster rates as the network size increases.

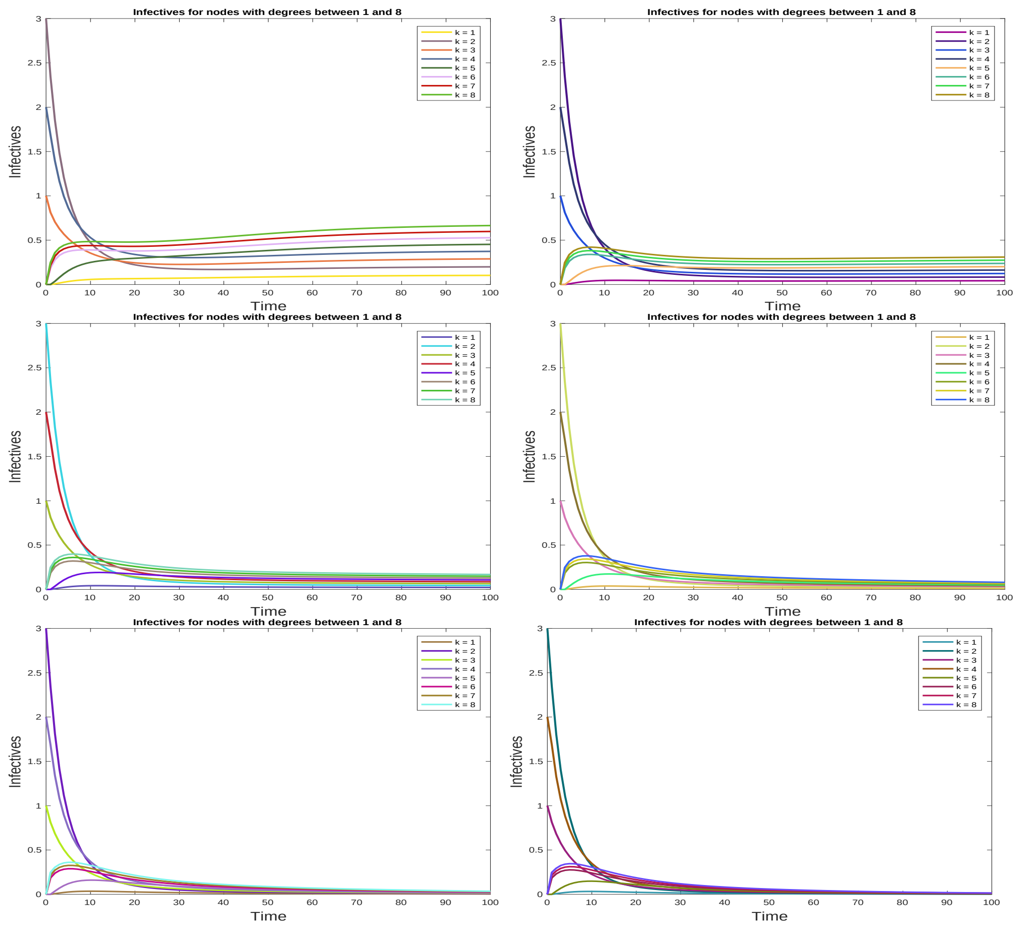

Figure 3: Time evolution of infectives with different values of ν.

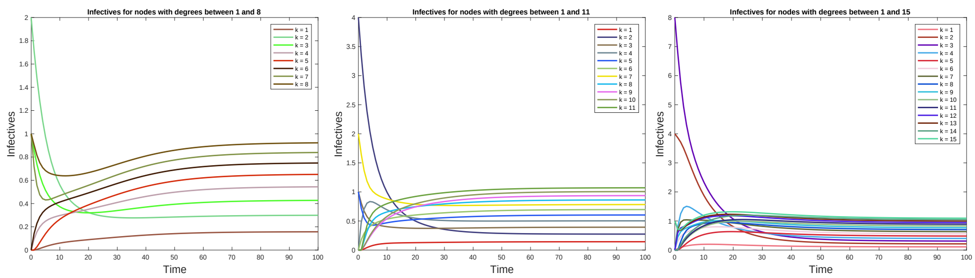

Figure 4: Time evolution of infectives with different network sizes (20, 30, and 50 nodes).

6. Conclusion

In this paper, we have studied an SIRS epidemic model with vaccination in complex heterogeneous networks and where contacts between human are treated as a scale-free social network. We obtain a specific expression of the threshold $\mathcal{R}_0$ through the existence of the endemic equilibrium. It is concluded that the solution of the system (1) converges to the disease free equilibrium E^0 if $\mathcal{R}_0 < 1$, which means from the biological point of view, the disease always dies out eventually. Otherwise the system admits a unique endemic equilibrium, which is globally asymptotically stable if $\mathcal{R}_0 > 1$. We have also treated the crucial case when $\mathcal{R}_0 = 1$ and we have shown that $(S_1, I_1, R_1, ..., S_n, I_n, R_n)$ converges consecutively to the disease free equilibrium $E^0 = (S^0, 0, R^0, ..., S^0, 0, R^0)$. To confirm the accuracy of the theoretical analysis, several numerical simulations are performed. Namely, We have found that the percentages of infectives will increase in the early time and then decrease until achieve a steady state as the parameter of vaccination ν increases. Also, we have shown the impact of network size in the convergence of infectives to the steady states.

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Accepted: 20.06.2019

The non-zero divisor graph of a ring

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Abstract. Let R be a ring, we associate a simple graph $\Phi(R)$ to R, with vertices $V(R) = R\backslash\{0, 1, -1\}$, where distinct vertices $x, y \in V(R)$ are adjacent if and only if either $xy \neq 0$ *or* $yx \neq 0$. In this paper, we prove that if $\Phi(R)$ is connected such that $R \ncong Z_2 \times Z_4$ then the diameter of $\Phi(R)$ is almost 2. Also, we will pay specific attention to investigate the connectivity of certain rings such that, the ring of integers modulo n, Z_n is connected, reduced ring and matrix ring.
Keywords: ring, zero-divisor, connected graph, diameter.

Introduction

It is believed that studying the action of a ring or group on a graph is one of the best comprehensible ways of analysing the structure of the rings or groups. There are many researches appointing a graph on group or ring to study the algebraic properties of that group or ring, for example, see [1, 2, 4, 5 and 10]. Suppose that R is a ring, the non-zero divisor graph, denoted by $\Phi(R)$ has a vertex set $V(R) = R\backslash\{0, 1, -1\}$, along with vertices $x, y \in V(R)$ being connected together on the condition of $x \neq y$ and either $xy \neq 0$ *or* $yx \neq 0$. Therefore, R is domain if and only if $\Phi(R)$ is complete graph. Thus if R is finite commutative ring with one such that $\Phi(R)$ is complete then R is a filed.

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This sort of graphs is inspired by the zero divisor graph which defined as the directed graph $\Gamma(R)$ such that its vertices are all non-zero zero-divisors of R in which any two distinct vertices x and y, $x \longrightarrow y$ is an edge if and only if $xy = 0$. Several fundamental results concerning with the zero divisor graph can be seen in [2].The main object of this paper is to study how the graph theoretical properties of $\Phi(R)$ effect on the ring theoretical properties of R. We assume that all graphs are simple graphs, which means they are undirected graphs with no multiple edges or loops. By abuse of notation, we denote by $V(R)$, the vertices of a graph $\Phi(R)$. We also should mentioned that a graph $\Phi(R)$ is connected if there is a path between any two distinct vertices in $V(R)$. For vertices x and y of $V(R)$, let $d(x, y)$ be the length of a shortest path from x to y. If no such path exist we my let $d(x, y) = \infty$, also $d(x, x) = 0$. Finally, the diameter of $\Phi(R)$ is define as $Diam(\Phi(R)) = sup\{d(x, y)|x \text{ and } y \text{ are vertices of } V(R)\}$ for more details see [5]. This paper is organized as follows: In section 1, we study the non-zero divisor graph $\Phi(R)$ such that R is ring of integers modulo n, Zn. In section 2, we prove that the diameter of the graph is almost 2. Moreover, we investigate the non-zero divisor graph of reduced ring and thus of Boolean ring. In section 3, we scrutinize the connectivity of the non-zero divisor graph for the matrix ring.

1. The non-zero divisor graph of the ring of integers modulo n

This section dedicated to investigate the non-zero divisor graphs for the ring of integers modulo n, Z_n.

Lemma 1.1. $V(R)$ has an invariable element a then $\Phi(R)$ is connected.

Proof. As a is invariable then there is $b \in G$ such that $ba = ab = 1$. Let x in $V(R)$ if $ax = 0$ or $xa = 0$, then we have $x = 0$, which is a contradiction. Thus $|Deg(a)| = |V(R)|$. □

Now, for $n \notin 1, 2, 3, 6$. Let $(Z_n)^{\times}$ be the multiplicative group of integers modulo n. Then its order is given by Euler's phi function [9]

$\phi(n) = \prod_{i=1}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$ Such that n written uniquely as $\prod_{i=1}^{s} p_i^{\delta_i}$, $\delta_i \geqslant 1$ are integers and $pi < p_{i+1}$ are prime numbers. We should note that $\phi(n)$ represent the number of invertible element in the ring of integers modulo n.

Lemma 1.2. For $n > 6$, we have $\phi(n) > 2$.

Proof. We may write n as above, $n = \prod_{i=1}^{s} p_i^{\delta_i}$. Thus $\phi(n) = \prod_{i=1}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$. And one can see immediately that $\phi(n) = 2$ if and only if there is j such that $p_j^{\delta_j} - p_j^{\delta_{j-1}} = 2$ and $\prod_{i=1_{i\neq j}}^{s}(p_i^{\delta_i} - p_i^{\delta_{i-1}})$=1. Which is impossible as $n > 6$. □

Theorem 1.3. *The ring of integers modulo* n, Z_n *is connected if and only if* $n \notin \{1, 3, 2, 6\}$

Proof. By using Lemma 1.2 we get $\phi(n) > 2$. Therefore, there is $x \in (Z_n)^{\times} \cap V(R)$. And our result follow immediately by Lemma 1.1. $\square$

Now, by using MATLAB program [3] we create an algorithm aim to draw $\Phi(Zn)$ for all $n > 3$, and calculate the diameter of the graph. This algorithm summarized as follows :

Algorithm 1

Let zz: seat of the system
Let n: the order of the system
Let $k = 0$
Let no, Initial Conditions
For $i \longleftarrow n - 3$ *to* n_f **Do**
For $j \longleftarrow n - 3$ *to* n_f **Do**
Read all Seats Value $(n_1, n_2, \dot{n}_f)$
Set $ik \longleftarrow \sum_{i \longleftarrow 1}^{n} ik + 1$
Set $K(ik) \longleftarrow zz(i) * zz(j)$
If $mod(K(ik), n) \longleftarrow \; = 0$
Set $w \longleftarrow [zz(i)zz(j)]$;
End if
End for
End for

Magma [7] is a computational algebra system support most research deal with algebra. The following algorithm aim to calculate the diameter of the non-zero divisor graph for a residue class ring $Z/nZ \cong Z_n$, $n > 3$ and the Cartesian product of $Z_n \times Z_m$. Implantations of the procedures associated with this algorithm will be via magma packages. This algorithm employed for the Cartesian product $Z_n \times Z_m$, and for Z_n, $n > 3$, we may take $Zn \cong Z_n \times Z_1$. Moreover, in [8] one can see that $Z_{nm} \cong Z_n \times Z_m$ if and only if $g.c.d(m, n) = 1$. The algorithm is as follows:

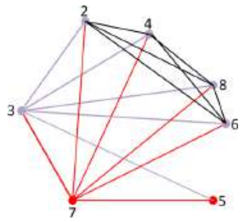
Algorithm 2

Let $Z_n \longleftarrow ResidueClassRing(n)$
Let $Z_m \longleftarrow ResidueClassRing(m)$
Let $R \longleftarrow CartesianProduct(Z_n \times Z_m)$
Let $V(R) \longleftarrow Set(R)\ diff\ \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle n-1, m-1\rangle\}$
Let $Gr \longleftarrow Graph\langle V(R)|\{\}\rangle$
for x, y in S **Do**
if $x \neq y$ **Then**
if $x * y \neq$ **or** $y * x \neq 0$ **Then**
Let $Gr \longleftarrow Gr + \{\{Vertices(Gr)!x, Vertices(Gr)!y\}\}$
End if
End if
End for

.

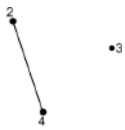
To be familiar with non-zero divisor graph, we utilize the above algorithms in the following examples:

Example 1.4. The non zero divisor graph of Z_{10} is describe as follows:

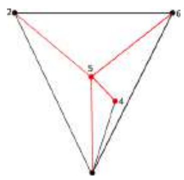


from above graph one can see that $Diam(\Phi(Z_{10}) = 2$.

Example 1.5. $\Phi(Z_6)$ is disconnect:



Example 1.6. Another example of connected non-zero divisor graph is $\Phi(Z_8)$

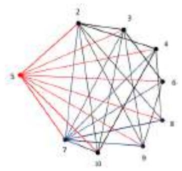


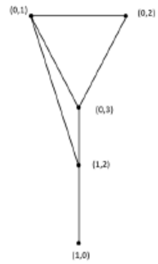
Also, we should note that $Diam(\Phi(Z_8) = 2$.

Example 1.7. The non-zero divisor of $\Phi(Z_{12})$ is describe as follows: Furthermore, $Diam(\Phi(Z_{12}) = 2$.

On the other hand, easy calculating by using Algorithm 2 one can obtain the next example which shows that $Diam(\Phi(Z_2 \times Z_4)) = 3$. Indeed. in the coming section we going to prove that the only connected non-zero divisor graph of diameter 3 is $\Phi(Z_2 \times Z_4)$.

Example 1.8. The non-zero divisor graph of $Z_2 \times Z_4$. This graph with diameter 3 as we see below:





2. Connectivity of the non-zero divisor graph

The results of this section related to connectivity of the graph. The first result shows that diameter of the connected non-zero divisor graph is almost 2.

Theorem 2.1. *Let R be a ring. If $\Phi(R)$ is a connected non-zero divisor graph and $R \ncong Z_2 \times Z_4$. Then $Diam(\Phi(R)) \leqslant 2$.*

Proof. Let $x, y \in V(R)$, such that $x \neq y$. If $xy \neq 0$, then we done. So we may assume that $xy = 0 = yx$. As $\Phi(R)$ is connected then there is $a, b \in V(R)$, such that $d(x, a) = d(y, b) = 1$. Which means either ($ax \neq 0$ or $xa \neq$) and ($by \neq 0$ or $yb \neq 0$). Thus we have the following subcases:

1. If $d(y, a) = 1$ or $d(x, b) = 1$, so in both cases we have $Diam(\Phi(R)) \leqslant 2$.

2. If $bx = xb = ay = ya = 0$, let $w = a \mp b$. Then it's obvious that $d(x, w) = 1 = d(y, w)$. Thus we may assume that both $a - b$ and $a + b$ not in $V(R)$. Hence we need to consider the following subcases:

I.If $a - b = 0$ or $a + b = 0$. Then $ax = bx = 0$. Which is a contradiction.

II. If $a + b = 1$ and $a - b = 1$. Note that with this conditions we have $ax = x$, $2b = 0$, $2a = 2$ and $by = y$. Also we consider the following:

i. If $x^2 = 0$. Then we consider the following cases:

- if $y^2 = 0$, then we take $q = y + 1$, so that $d(q, x) = d(q, y) = 1$. Furthermore, this is not true if $y = \mp 2$ multiply by a we get $0 = ay = 2a = 2$, but this means $y = 0$ which is a contradiction.
- if $y^2 \neq 0$, then $x+1$, $x-1$, $y+a$ and $y-a$ connect x with y and this is not true if $R \ncong Z_2 \times Z_4$. Indeed, in this we have $Diam(\Phi(R)) = 3$ as we showed in Example 1.8. Else, we may take $z \in V(R)$ such that $z \notin \{x, y, a, -a, b\}$. if $d(z, x) = d(y, z) = 1$ we done, if $d(x, z) = 1, d(y, z) = 0$ take $z + b$, and

take $z + a$ if $d(x, z) = 0, d(y, z) = 1$. Finally take $z + 1$ to connect x with y in other cases.

ii. If $x^2 \neq 0$. Then we consider the following cases:

- if $y^2 = 0$, then $d(y + 1, x) = d(y + 1, y) = 1$, also this not true if $y = \mp 2$, multiply by a we get $0 = ay = 2a = 2$, but this mean $y = 0$ which is a contradiction.
- if $y^2 \neq 0$, then $x + y \in V(R)$ connect x with y.

III. If $a + b = -1$ and $a - b = -1$, note that with this conditions we get $ax = -x$, $2b = 0$, $2a = -2$, $by = -y = y$ and $2y = 0$. Then we may apply similar argument as in case II to show that $Diam(\Phi(R))2$.

IV. If $a+b = -1$ and $a-b = 1$, with these conditions we obtain $ax = x$, $2b = b$, $2a = 0$, $by = -y$ and $2x = 0$. Also we consider the following cases:

i. If $x^2 = 0$. Then we consider the following cases:

- if $y^2 = 0$, then we take $w = x + 1$ clearly that $d(w, x) = d(w, y) = 1$, also this not true if $x = \mp 2$, multiply by a we get $0 = bx = 2b = 2$, but this mean $x = 0$ which is a contradiction.
- if $y^2 \neq 0$, then $x+1$ and $x-1$ connect x with y, and this mean $ax = 2a = 0$, again we get a contradiction.

ii. If $x^2 \neq 0$, then we consider the following:

- if $y^2 = 0$ then $y + 1, y - 1, x + b$ and $x - b$ connect x with y. Moreover, this is not true if $R \ncong Z_2 \times Z_4$, which has diameter 3. Otherwise, we may take $z \in V(R)$ such that $z \notin \{x, y, a, -a, b\}$. Now if $d(z, x) = d(y, z) = 1$ we done, if $d(x, z) = 1, d(y, z) = 0$ take $z + b$ in this case, and take $z + a$ if $d(x, z) = 0, d(y, z) = 1$. Finally take $z + 1$ to connect x and y in other cases.
- if $y^2 \neq 0$, then $x + y \in V(R)$ connect x and y.

V. If $a + b = 1$ and $a - b = -1$, with this conditions we obtain $ax = x, 2b = b, 2a = 0, by = y$ and $2x = 0$. Then we may apply similar argument as in case IV to show that $Diam(\Phi(R)) \leqslant 2$. □

The next theorem shows the connectivity of the reduced rings:

Theorem 2.2. *Let R be a reduced ring such that $|V(R)| > 3$. Then $\Phi(R)$ is connected.*

Proof. Let $x, y \in V(R)$, then if $d(x, y) = 1$, then we done. So we may let $d(x, y) \neq 1$, thus $xy = yx = 0$. In this, we need to consider the following subcases:

I. If R without one then $x+y$ connected x with y (if $x+y=0$ then $x^2=0$ which is a contradiction as R is reduced)

II. If R with one, then if $x+y=1$. Since $|V(R)|>3$, then there is $z \neq w \in V(R) \setminus \{x,y\}$. Again we have to consider the following subcases:

1. if $zx=0=zy$ then $z-1$ connect x with y, and this not true if $z=2$, in this case $z+1$ will connect x with y also this is not true if $z=-2$. If both cases not true, then we have 4=0, and this means $z^2=0$, which a contradiction.
2. if $zx \neq 0 \neq zy$, then z connected x with y.
3. if $zx \neq 0$, $zy=0$, then $z+y$ connect x with y only if $z+y \neq -1$. If $z+y=-1$, then we get $2y=0, x^2=x, z^2=-z$ and $x=-z$. Now if w satisfy 1 or 2 then we done, else we have the following subcases:
   i. if $wy \neq 0$, $wx=0$, then $w+x$ connect x with y, and it is not true if $w+x=-1$, and this lead to $2x=0, y^2=y, w^2=-w$ and $y=-w$, this means $x=z$, a contradiction.
   ii. if $wy=0$, $wx \neq 0$, then we get $w+y$ connect x with y, and it is not true if $w+y=-1$, which yields that $w+y=z+y$, thus $w=z$, a contradiction.
4. if $zx=0$, $zy \neq 0$, then similar as case II, 3, one can find a path between x and y.

III. If R with one such that $x+y=-1$. Then the proof is the same as in case II.

□

The above theorem not always correct. Especially when $|V(R)|=3$. A very obvious example can be seen in Example 1.5.

Corollary 2.3. The non-zero divisor graph of a Boolean rings is connected.

3. The non-zero divisor graph of a matrix ring

Let R be a ring. The next theorem illustrate the condition on R in order to get the connectivity of matrix ring $M_n(R)$, such that $n \geq 2$.

Theorem 3.1. *Let $A,B \in V(M)$ such that $d(A,B) \neq 1$. We aim to find $W \in V(M)$ connect A with B. First since $A,B \in V(M)$, thus there are i,j,d and k such that $a_{ij} \neq 0$ and $b_{dk} \neq 0$.To do that we consider the following subcases:*

i. if $a_{ij}=b_{dk}$, then $W \in V(M)$, such that $w_{ji}=1$ and zero otherwise, connect A with B.

ii. *if $n \neq 2$, and no there exist $0 \neq a_{ij} = b_{ij} \neq 0$. Since $A \neq 0$ and $B \neq 0$. Then $W \in V(M)$, connected A with B such that $w_{ji} = 1$, $w_{kd} = 1$ and zero otherwise.*

iii. *if $n = 2$, and no there exist $0 \neq aij = bij \neq 0$, similar as case ii, but this not work if $ij = 11$, and $dk = 22$ or conversely, then $W \in V(M)$, connected A with B such that $w_{11} = 1$, $w_{12} = 1$ and zero otherwise.*

The next theorem shows under what circumstances $\Phi(R)$ become connected when R ring without one.

Theorem 3.2. *Let R be a ring without one, such that $deg(x) > 0$ for all $x \in V(R)$. Then the graph of $M_n(R)$ is connected.*

Proof. Let $A, B \in V(M)$. If $d(A,B) = 1 wedone$. so we may let $d(A,B) \neq 1$. Then since A, B are not equal to zero then there are i, j, k and $d \in n$ such that $a_{ij} \neq 0$ and $b_{dk} \neq 0$. Because of $deg(x) > 0$ for all $x \in V(R)$, then there are $w_1, w_2 V(R)$, such that either $a_{ij}w_1 \neq 0$ and $b_{kh}w_2 \neq 0$ or the other way around. We aim to find $W \in V(R)$ connect A with B. To do that we consider the following cases:

i. if $a_{ij}w_1 \neq 0$ and $b_{ij}w_2 \neq 0$ then let $W = \{w_{js} = w_1, w_{ht} = w_2\}$, such that $t \neq s$ and zero otherwise.

ii. if $w_1 a_{ij} \neq 0$ and $w_2 b_{ij} \neq 0$ then $W = \{w_{is} = w_1, w_{kt} = w_2\}$, such that $t \neq s$ and zero otherwise.

iii. if $w_1 a_{ij} \neq 0$ and $b_{ij}w_2 \neq 0$ then $W = \{w_{is} = w_1, w_{ht} = w_2\}$, such that $t \neq s$ and zero otherwise.

iv. if $a_{ij}w_1 \neq 0$ and $w_2 b_{ij} \neq 0$ then $W = \{w_{js} = w_1, w_{kt} = w_2\}$, such that $t \neq s$ and zero otherwise.

In all cases we can find a path between A and B.

□

It can be shown that Theorem 3.2 not always true. To illustrate that we give the following Exampe 4.1: Let $R = \{0, 2, 4, 6\} \subseteq Z8$, then the non-zero divisor graph of $M = M_n(R)$, such that $n \geqslant 2$, is disconnected. Indeed, if we take $A \in V(M)$ such that $a_{11} = 4$ and $a_{ij} = 0$ for all $i = 2, 3 \ldots, n$ and $j = 2, 3 \ldots, n$. Then $AX = XA = 0$ for all $X \in V(M)$.

Corollary 3.3. *Let R be a ring such that $\Phi(R)$ is connected then $\Phi(M)$ is connected.*

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Accepted: 31.08.2019

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IJPAM – Italian Journal of Pure and Applied Mathematics
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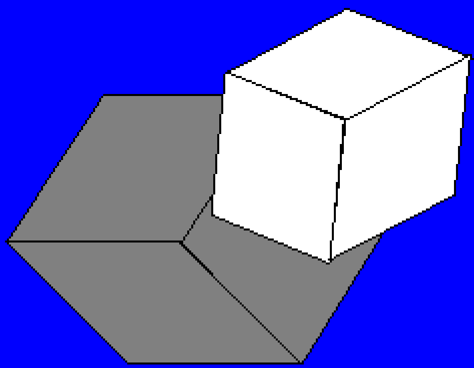

Publisher
Forum Editrice Universitaria Udinese Srl
Via Larga 38 - 33100 Udine
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Rivista semestrale: Autorizzazione Tribunale di Udine n. 8/98 del 19.3.98 - Direttore responsabile: Piergiulio Corsini

ISSN 2239-0227