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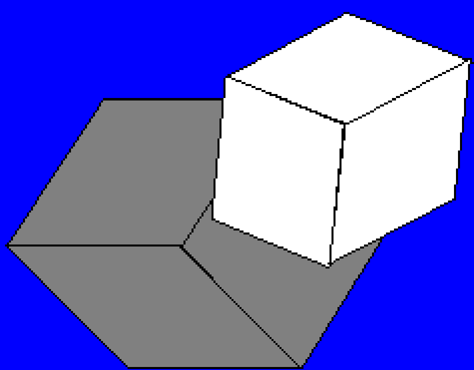

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ON THE GENERALIZED DRAZIN INVERSE IN A BANACH ALGEBRA

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Abstract. We give the representations of $(a+b)^{\mathsf{d}}, (a+ab)^{\mathsf{d}}$ and $(a+ba)^{\mathsf{d}}$ in terms of a, b, a^{d} and b^{d} being elements of a Banach algebra with $a^3b = ba$ and $b^3a = ab$. We also give the representations of $(a+b)^{\mathsf{d}}$ under the assumptions $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1-aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$.
Keywords: generalized Drazin inverse, Banach algebra.

1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra with unit 1. The symbols $\mathscr{A}^{-1}, \mathscr{A}^{\mathsf{nil}}, \mathscr{A}^{\mathsf{qnil}}$ and $\mathscr{A}^{\bullet}$ stand for the sets of all invertible, nilpotent, quasinilpotent and idempotent elements in the Banach algebra $\mathscr{A}$, respectively.

For $a \in \mathscr{A}$, if there exists a unique $x \in \mathscr{A}$ such that

$$xax = x, \qquad ax = xa, \qquad a^{k+1}x = a^k \tag{1.1}$$

x is the Drazin inverse of a (denoted by a^{D}). The least nonnegative integer k for which satisfies the above equations is the Drazin index ind(a) of a. If ind$(a)\leq 1$,

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then a^{D} reduces to the group inverse of a in this case it is customary to denote $a^{\mathsf{D}} = a^{\#}$. Note that $a \in \mathscr{A}$ is invertible if and only if $\operatorname{ind}(a) = 0$.

In [12] Koliha generalized the Drazin invertibility by changing (1.1). Let $a \in \mathscr{A}$. An element $x \in \mathscr{A}$ is called a generalized Drazin inverse of a if

$$xax = x, \qquad ax = xa, \qquad a - a^2x \in \mathscr{A}^{\mathsf{qnil}}. \tag{1.2}$$

If there exists $x \in \mathscr{A}$ satisfying (1.2), the element a is said to be generalized Drazin invertible. Koliha proved in [12] that the set of generalized Drazin inverses of an element of a Banach algebra is or empty or a singleton. In case that a has a generalized Drazin inverse, we shall denote by a^{d} its unique generalized inverse. The subset of $\mathscr{A}$ consisting of elements that have a generalized Drazin inverse will be denoted by $\mathscr{A}^{\mathsf{d}}$. In fact, Koliha proved the following lemma which will be useful. (stated in a unital ring)

Lemma 1.1 ([12, Lemma 2.4])**.** *In a Banach algebra $\mathscr{A}$ with unit, an element $a \in \mathscr{A}$ is generalized Drazin invertible if and only if there is $p \in \mathscr{A}^{\bullet}$ such that*

$$pa = ap, \qquad ap \in \mathscr{A}^{\mathsf{qnil}}, \qquad a + p \in \mathscr{A}^{-1}.$$

In this case, the set consisting of generalized Drazin inverses of a is a singleton and its unique element a^{d} is given by

$$a^{\mathsf{d}} = (a+p)^{-1}(1-p).$$

Let $a \in \mathscr{A}$ be generalized Drazin invertible. If $a - a^2a^{\mathsf{d}} \notin \mathscr{A}^{\mathsf{nil}}$, it is customary to say that $\operatorname{ind}(a) = \infty$. It is easily seen from the proof of [12, Lemma 2.4] that the idempotent p given in Lemma 1.1 is unique which explicit expression is $p = 1 - aa^{\mathsf{d}}$. We shall denote this idempotent by a^{π}.

Every $p \in \mathscr{A}^{\bullet}$ induces a matrix representation of any element $a \in \mathscr{A}$ given by (see [17, Chapter 5])

$$a = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}_p .$$

If $\mathscr{B}$ is a subalgebra of the unital algebra $\mathscr{A}$, for an element $b \in \mathscr{B}^{-1}$, we shall denote by $[b^{-1}]_{\mathscr{B}}$ the inverse of b in $\mathscr{B}$. Let us observe that in general $\mathscr{B}^{-1} \not\subset \mathscr{A}^{-1}$ (if we take $p \in \mathscr{A}^{\bullet}$, $p \neq 1$, and $\mathscr{B} = p\mathscr{A}p$, then $p \in \mathscr{B}^{-1}$ and $p \notin \mathscr{A}^{-1}$). But if the subalgebra $\mathscr{B}$ has unity, then $\mathscr{B}^{-1} \subset \mathscr{A}^{\mathsf{d}}$ and if $b \in \mathscr{B}^{-1}$, then $b^{\mathsf{d}} = [b^{-1}]_{\mathscr{B}}$: let e be the unity of $\mathscr{B}$, since $b[b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}}b = e$, it is easy to see $b[b^{-1}]_{\mathscr{B}}b = b$, $[b^{-1}]_{\mathscr{B}}b[b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}}$, and $[b^{-1}]_{\mathscr{B}}b = b[b^{-1}]_{\mathscr{B}}$.

It is known [3] that any $a \in \mathscr{A}^{\mathsf{d}}$ has the following matrix representation

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p,$$

$$p = 1 - a^{\pi} = aa^{\mathsf{d}}, \quad a_1 \in [p\mathscr{A}p]^{-1}, \quad a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}. \tag{1.3}$$

Then we have

$$a^{\mathsf{d}} = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p . \tag{1.4}$$

If a is Drazin invertible (instead of being generalized Drazin invertible), the only difference with the representation (1.3) is that a_2 is nilpotent instead of being quasinilpotent.

The following result [3] will be useful for our purposes.

Lemma 1.2 ([3, Theorem 2.3]). *Let $\mathscr{A}$ be a Banach algebra, $x, y \in \mathscr{A}$, and $p \in \mathscr{A}^\bullet$. Assume that*

$$x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p , \qquad y = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}_{1-p} .$$

(i) *If $a \in (p\mathscr{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$, then $x, y \in \mathscr{A}^{\mathsf{d}}$ and*

$$x^{\mathsf{d}} = \begin{pmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{pmatrix}_p , \qquad y^{\mathsf{d}} = \begin{pmatrix} b^{\mathsf{d}} & 0 \\ u & a^{\mathsf{d}} \end{pmatrix}_{1-p} , \tag{1.5}$$

where $u = \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} c b^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c (b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}} c b^{\mathsf{d}}$.

(ii) *If $x \in \mathscr{A}^{\mathsf{d}}$ and $a \in (p\mathscr{A}p)^{\mathsf{d}}$, then $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ and x^{d} is given by (1.5).*

Moreover, when an element $x \in \mathscr{A}^{\mathsf{d}}$ commutes with an idempotent $p \in \mathscr{A}$, the generalized Drazin inverse of x has a simple form in terms of the matrix representation relative to p as the following simple (but useful) result shows:

Lemma 1.3. *Let $\mathscr{A}$ be a unital Banach algebra and let $x \in \mathscr{A}, p \in \mathscr{A}^\bullet$. If $x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}_p$, then $x \in \mathscr{A}^{\mathsf{d}}$ if and only if $x_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and $x_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. In this situation, one has $x^{\mathsf{d}} = \begin{pmatrix} x_1^{\mathsf{d}} & 0 \\ 0 & x_2^{\mathsf{d}} \end{pmatrix}_p$.*

The Drazin inverse is used in applications of many areas such that differential and difference equations, Markov chains and control theory [1, 2].

In recent years, the representations of the Drazin inverse of $a+b$ have been considered by many authors (see [4, 5, 6, 7, 8, 9, 11, 13, 14, 15]) (being a, b matrices, operators or elements in a infinite dimensional Banach algebra). In [10] the Drazin inverse in semigroups and associative rings was firstly introduced. The Drazin inverse in a Banach algebra was introduced in [9]. A formula is given for the Drazin inverse of a sum of two matrices in [11]. In [9], Djordjević and Wei considered additive results for the generalized Drazin inverse in a Banach space.

C.Y. Deng in [7] explored the Drazin inverse of bounded operators with commutativity up to a factor in a Banach space, being these extended by Cvetković-Ilić in [4]. In [18], the authors considered the Drazin inverse of a sum of two matrices and derived additive formulas under conditions weaker than those used in some recent papers on the subject. As an application they gave some new representations for the Drazin inverse of a block matrix.

In the rest of this section, we will give some key lemmas. In Section 2 we will discus the representations of the generalized Drazin inverse of $a+b$, $a+ab$ and $a+ba$ in terms of a, b a^d and b^d being elements of a unital Banach algebras under the conditions $a^3b = ba$ and $b^3a = ab$. We also will consider related results under the assumptions $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1-aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$.

We will give some lemmas in the following:

Lemma 1.4. *Let $\mathscr{A}$ be a ring and let $a, b \in \mathscr{A}$ satisfy $a^3b = ba$. If $n \in \mathbb{N}$, then*

(i) $a^{3n}b = ba^n$.

(ii) $(ab)^n = a^{(3^n-1)/2}b^n$.

(iii) $a^{3^n}b^n = b^n a$.

(iv) *If $b^3a = ab$, then $ab = a^{26n}(ab)b^{2n}$.*

Proof. The proofs of (i), (ii), and (iii) can be easily done by induction. To prove (iv), observe that by applying (iii) we have $ab = b^3a = a^{3^3}b^3 = a^{26}(ab)b^2$, and now, the equality $a^{26(n+1)}(ab)b^{2(n+1)} = a^{26}\left[a^{26n}(ab)b^{2n}\right]b^2$ permits finish the proof of (iv) by induction. □

Lemma 1.5. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}$ satisfy $a^3b = ba$. If $a \in \mathscr{A}^{\mathsf{qnil}}$ or $b \in \mathscr{A}^{\mathsf{qnil}}$, then $ab \in A^{\mathsf{qnil}}$ and $aba \in \mathscr{A}^{\mathsf{qnil}}$.*

Proof. Assume that a or b are quasinilpotent. By Lemma 1.4 we have

$$\|(ab)^n\|^{1/n} \leq \|a^{(3^n-1)/2}\|^{1/n}\|b^n\|^{1/n}.$$

Hence $ab \in \mathscr{A}^{\mathsf{qnil}}$. To prove $aba \in \mathscr{A}^{\mathsf{qnil}}$, observe that $aba = aa^3b$ and $(a^4)^3b = a^{3\cdot 4}b = ba^4$. By the first part of this lemma, (notice that $a \in \mathscr{A}^{\mathsf{qnil}} \Rightarrow a^4 \in \mathscr{A}^{\mathsf{qnil}}$) we get that aba is quasinilpotent. □

The proof of the following lemma is inspired by the proof of [4, Theorem 2.2].

Lemma 1.6. *Let $\mathscr{A}$ be a unital Banach algebra. Let $a, b \in \mathscr{A}$ be such that $a^3b = ba$. If $a \in \mathscr{A}^{\mathsf{d}}$, then $a^\pi b = ba^\pi$. If $b \in \mathscr{A}^{\mathsf{d}}$, then $b^\pi a = b^\pi ab^\pi$.*

Proof. Assume $a \in \mathscr{A}^{\mathsf{d}}$. Let $p = aa^{\mathsf{d}}$. To prove $bp = pb$ pick any $n \in \mathbb{N}$ and use Lemma 1.4 (iii). We have

$$\begin{aligned} pb - pbp &= pb(1-p) = p^{3n}b(1-p) = (a^{\mathsf{d}})^{3n}a^{3n}b(1-p) = (a^{\mathsf{d}})^{3n}ba^n(1-p)^n \\ &= (a^{\mathsf{d}})^{3n}ba^n(1-aa^{\mathsf{d}})^n = (a^{\mathsf{d}})^{3n}b[a(1-aa^{\mathsf{d}})]^n. \end{aligned}$$

Since $a(1-aa^{\mathsf{d}}) \in \mathscr{A}^{\mathsf{qnil}}$ we have, by making $n \to \infty$, that $pb = pbp$. Similarly we prove $bp = pbp$. Hence $bp = pbp$.

If $b \in \mathscr{A}^{\mathsf{d}}$, then by setting $q = bb^{\mathsf{d}}$ and by mimicking the above reasoning we have $qa = qaq$. □

2. Main results

Theorem 2.1. *Let $\mathscr{A}$ be a unital Banach algebra and let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $a^3b = ba$. Then*

(i) *$ab \in \mathscr{A}^{\mathsf{d}}$ and $bb^{\mathsf{d}}(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$.*

(ii) *$aba \in \mathscr{A}^{\mathsf{d}}$ and $b^{\mathsf{d}}(aba)^{\mathsf{d}} = (b^{\mathsf{d}}a^{\mathsf{d}})^2$.*

(iii) *$bb^{\mathsf{d}}ab^{\mathsf{d}} = b^{\mathsf{d}}a^3$.*

Proof. (i) and (ii): Since $a \in \mathscr{A}^{\mathsf{d}}$, the elements a and a^{d} can be represented as in (1.3) and (1.4), respectively. Let us represent $b = \begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix}_p$. From Lemma 1.6, we have $pb = bp$, hence $b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p$. Therefore, $ab = \begin{pmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{pmatrix}_p$, $a^3b = \begin{pmatrix} a_1^3b_1 & 0 \\ 0 & a_2^3b_2 \end{pmatrix}_p$, and $ba = \begin{pmatrix} b_1a_1 & 0 \\ 0 & b_2a_2 \end{pmatrix}_p$. From $a^3b = ba$ we get $a_1^3b_1 = b_1a_1$ and $a_2^3b_2 = b_2a_2$. Since a_2 is quasinilpotent and by Lemma 1.5 a_2b_2 is quasinilpotent. i.e. $(a_2b_2)^{\mathsf{d}} = 0$. Let us observe that by Lemma 1.3, $b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$, $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, and $b^{\mathsf{d}} = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$.

Now we consider b_1:

(1) If b_1 is quasinilpotent, then $b_1^{\mathsf{d}} = 0$ and from Lemma 1.5 we get $(a_1b_1)^{\mathsf{d}} = 0$. By using $(a_2b_2)^{\mathsf{d}} = 0$ and Lemma 1.3 we get $ab \in \mathscr{A}^{\mathsf{d}}$ and $(ab)^{\mathsf{d}} = 0$. Since $b \in \mathscr{A}^{\mathsf{d}}$, from Lemma 1.3 we have $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ and $b^{\mathsf{d}} = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$. By that a^{d} is represented as in (1.4) we have $b^{\mathsf{d}}a^{\mathsf{d}} = 0$. Thus (i) holds.

By Lemma 1.5 we have $a_1b_1a_1$ and $a_2b_2a_2$ are quasinilpotent (in their respective subalgebras). By Lemma 1.3 we have $aba \in \mathscr{A}^{\mathsf{d}}$ and $(aba)^{\mathsf{d}} = 0$. Since

$$b^{\mathsf{d}}a^{\mathsf{d}} = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p = 0,$$

(ii) holds.

(2) Assume now that b_1 is invertible (in the subalgebra $p\mathscr{A}p$). Since a_1 is also invertible (in the subalgebra $p\mathscr{A}p$), $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and $(a_1b_1)^{\mathsf{d}} = [a_1b_1{}^{-1}]_{p\mathscr{A}p} =$

$[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p}$. Recall that we have $(a_2b_2)^{\mathsf{d}} = 0$. By Lemma 1.3 we get

$$(ab)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p .$$

Since b_1 is invertible, $b^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$. Thus, (i) holds.

Since $b_1 \in [p\mathscr{A}p]^{-1}$ (recall that a_1 always belongs to $[p\mathscr{A}p]^{-1}$), we have $a_1b_1a_1 \in [p\mathscr{A}p]^{-1}$. Since a_2 is quasinilpotent, by Lemma 1.5, the element $a_2b_2a_2$ is quasinilpotent. From Lemma 1.3 we get $aba \in \mathscr{A}^{\mathsf{d}}$ and $(aba)^{\mathsf{d}} = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p$. Thus

$$b^{\mathsf{d}}(aba)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p$$

and

$$b^{\mathsf{d}}a^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p .$$

Evidently, we have proved (ii).

(3) Assume that b_1 is neither invertible nor quasinilpotent. Setting $q = b_1b_1^{\mathsf{d}}$ we have the representation $b_1 = \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q$, where $b_1' \in [q\mathscr{A}q]^{-1}$ and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$ (recall that $q \in p\mathscr{A}p$ and p is the unity of the algebra $q \in p\mathscr{A}p$). Let us represent a_1 as follows:

$$a_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_q .$$

By Lemma 1.4 (iii)

$$\begin{aligned} qa_1 - qa_1q &= qa_1(p-q) = q^na_1(p-q)^n = (b_1^{\mathsf{d}})^nb_1^na_1(p-q)^n \\ &= (b_1^{\mathsf{d}})^na_1^{3^n}b_1^n(p-q)^n \\ &= (b_1^{\mathsf{d}})^na_1^{3^n}[b_1(p-b_1b_1^{\mathsf{d}})]^n. \end{aligned}$$

Hence $\|qa_1 - qa_1q\|^{1/n} \to 0$, i.e., $qa_1 = qa_1q$, thus

$$a_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q . \tag{2.1}$$

Therefore,

$$a_1b_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q = \begin{pmatrix} a_{11}b_1' & 0 \\ a_{21}b_1' & a_{22}b_1'' \end{pmatrix}_q .$$

To prove $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ we shall apply Lemma 1.2 (i). To this end, we must prove that $a_{11}b_1' \in [q\mathscr{A}q]^{\mathsf{d}}$ and $a_{22}b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$. From $a_1 \in [p\mathscr{A}p]^{-1}$ and (2.1) we get $a_{11} \in [q\mathscr{A}q]^{-1}$ and recall that $b_1' \in [q\mathscr{A}q]^{-1}$ we have $a_{11}b_1' \in [q\mathscr{A}q]^{-1} \subset [q\mathscr{A}q]^{\mathsf{d}}$. Now, from $a_1^3b_1 = b_1a_1$ we get $a_{22}^3b_1'' = b_1''a_{22}$. By using that $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$ and Lemma 1.5 we get $a_{22}b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}} \subset [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$. Moreover, $(a_{22}b_1'')^{\mathsf{d}} = 0$. Therefore, from Lemma 1.2 we get $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$(a_1b_1)^{\mathsf{d}} = \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ x & 0 \end{pmatrix}_q,$$

where x is some element of $\mathscr{A}$ given by Lemma 1.2 (i). To prove $bb^{\mathsf{d}}(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$ it is enough to prove $b_1b_1^{\mathsf{d}}(a_1b_1)^{\mathsf{d}} = b_1^{\mathsf{d}}[a_1{}^{-1}]_{p\mathscr{A}p}$. But we have

$$\begin{aligned} b_1b_1^{\mathsf{d}}(a_1b_1)^{\mathsf{d}} &= \begin{pmatrix} b_1' & 0 \\ 0 & b_2 \end{pmatrix}_q \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ x & 0 \end{pmatrix}_q \\ &= \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \end{aligned}$$

and

$$\begin{aligned} b_1^{\mathsf{d}}[a_1{}^{-1}]_{p\mathscr{A}p} &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \\ &\cdot \begin{pmatrix} [a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ -[a_{22}{}^{-1}]_{(p-q)\mathscr{A}(p-q)}a_{21}[a_{11}{}^{-1}]_{q\mathscr{A}q} & [a_{22}{}^{-1}]_{q\mathscr{A}q} \end{pmatrix}_q \\ &= \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q. \end{aligned} \tag{2.2}$$

We have

$$a_1b_1a_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q = \begin{pmatrix} a_{11}b_1'a_{11} & 0 \\ y & a_{22}b_1''a_{22} \end{pmatrix}_q,$$

where y is some element in $\mathscr{A}$. The invertibility of $a_{11}b_1'a_{11}$ in $q\mathscr{A}q$ follows from the invertibility of a_{11} and b_1' $q\mathscr{A}q$. The quasinilpotency of $a_{11}b_1'a_{11}$ follows from $a_2^3b_1'' = b_1''a_2$, the quasinilpotency of b_1'', and by Lemma 1.5. By Lemma 1.2 (i) we have that $a_1b_1a_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$(a_1b_1a_1)^{\mathsf{d}} = \begin{pmatrix} [a_{11}{}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ z & 0 \end{pmatrix}_q,$$

where z is an element of $\mathscr{A}$ given by Lemma 1.2. Having in mind the quasinilpotency of $a_2b_2a_2$ and that $aba = \begin{pmatrix} a_1b_1a_1 & 0 \\ 0 & a_2b_2a_2 \end{pmatrix}_p$ we get from Lemma 1.3 that $aba \in \mathscr{A}^{\mathsf{d}}$ and

$$(aba)^{\mathsf{d}} = \begin{pmatrix} (a_1b_1a_1)^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_p.$$

Thus, to prove (ii), it is enough to prove $b_1^{\mathsf{d}}(a_1b_1a_1)^{\mathsf{d}} = (b_1^{\mathsf{d}}[a_1{}^{-1}]_{p\mathscr{A}p})^2$. But, this expression follows easily from the above formula and the following computation:

$$\begin{aligned} b_1^{\mathsf{d}}(a_1b_1a_1)^{\mathsf{d}} &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} [a_{11}{}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ z & 0 \end{pmatrix}_q \\ &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q}[a_{11}{}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q. \end{aligned}$$

(iii) As in the beginning of the proof but by changing the roles of a and b, we get that $a, b \in \mathscr{A}^{\mathsf{d}}$ can be represented by

$$a = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}_\pi, \qquad b = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}_\pi,$$

where $f_1, g_1 \in \pi\mathscr{A}\pi$, $f_2, g_2 \in (1-\pi)\mathscr{A}(1-\pi)$, g_1 is invertible and g_2 is quasinilpotent (in their respectives subalgebras) and π is the idempotent bb^{d}. From $a^3b = ba$ we get $f_i^3g_i = g_if_i$ for $i = 1, 2$. To prove (iii) we need to prove $g_ig_i^{\mathsf{d}}f_ig_i^{\mathsf{d}} = g_i^{\mathsf{d}}f_i^3$ for $i = 1, 2$. Since g_1 is invertible, then $g_1g_1^{\mathsf{d}}f_1g_1^{\mathsf{d}} = g_1^{\mathsf{d}}f_1^3$ follows from $f_1[g_1{}^{-1}]_{\pi\mathscr{A}\pi} = [g_1{}^{-1}]_{\pi\mathscr{A}\pi}f_1^3$. Since g_2 is quasinilpotent, then $g_2^{\mathsf{d}} = 0$, and thus, $g_2g_2^{\mathsf{d}}f_2g_2^{\mathsf{d}} = g_2^{\mathsf{d}}f_2^3$ trivially holds. □

Theorem 2.2. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $b^3a = ab$ and $a^3b = ba$. Then*

(i) $ab^{\mathsf{d}} = (b^3)^{\mathsf{d}}a = b^{\mathsf{d}}a^3$,

(ii) $b^{\mathsf{d}}a = b^2ab^{\mathsf{d}}$,

(iii) $ab \in \mathscr{A}^{\mathsf{d}}$ *and* $(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$.

Proof. We represent

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q, \qquad b^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q,$$

where $q = bb^{\mathsf{d}}$, $b_1 \in [q\mathscr{A}q]^{-1}$ and $b_2 \in [(1-q)\mathscr{A}(1-q)]^{\mathsf{qnil}}$. From Lemma 1.6 we can write

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_q.$$

Moreover, from $a^3b = ba$ and $b^3a = ab$ we obtain $a_i^3b_i = b_ia_i$ and $b_i^3a_i = a_ib_i$ for $i = 1, 2$.

(i) and (ii): It is enough to prove: (a) $a_1[b_1{}^{-1}]_{q\mathscr{A}q} = ([b_1{}^{-1}]_{q\mathscr{A}q})^3a_1$; (b) $a_1[b_1{}^{-1}]_{q\mathscr{A}q} = [b_1{}^{-1}]_{q\mathscr{A}q}a_1^3$; and (c) $[b_1{}^{-1}]_{q\mathscr{A}q}a = b_1^2a_1[b_1{}^{-1}]_{q\mathscr{A}q}$. Expressions (a) and (c) follow from $b_1^3a_1 = a_1b_1$. Expression (b) follows from $b_1a_1 = b_1a_1^3$.

(iii): By Theorem 2.1, and the invertibility of b_1 (in $q\mathscr{A}q$) we get $a_1b_1 \in [q\mathscr{A}q]^{\mathsf{d}}$ and $(a_1b_1)^{\mathsf{d}} = [b_1{}^{-1}]_{q\mathscr{A}q}a_1^{\mathsf{d}}$. Since b_2 is quasinilpotent, from Lemma 1.5,

a_2b_2 is quasinilpotent, $a_2b_2 \in [(1-q)\mathscr{A}(1-q)]^{\mathsf{d}}$ and $(a_2b_2)^{\mathsf{d}} = 0$. From Lemma 1.3, we have $ab \in \mathscr{A}^{\mathsf{d}}$ and

$$(ab)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{q\mathscr{A}q} a_1^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_q = \begin{pmatrix} [b_1{}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} a_1^{\mathsf{d}} & 0 \\ 0 & a_2^{\mathsf{d}} \end{pmatrix}_q = b^{\mathsf{d}} a^{\mathsf{d}}.$$

□

Remark. Since the hypothesis of Theorem 2.2 are symmetric on the elements a and b, we have also valid formulas by changing the roles of a and b.

Lemma 2.1. *Let $\mathscr{A}$ be a Banach algebra and let $x, y \in \mathscr{A}$ be such that $x^3y = yx$ and $y^3x = xy$. If y is quasinilpotent, then $xy = yx = 0$.*

Proof. By Lemma 1.4 (iv), for an arbitrary positive integer n, we have

$$\|xy\|^{1/n} = \|x^{26n}(xy)(y)^{2n}\|^{1/n} \le \|x\|^{26}\|xy\|^{1/n}\left[\|y^n\|^{1/n}\right]^2.$$

Making $n \to \infty$ and using that y is quasinilpotent we get $\lim_{n\to\infty} \|xy\|^{1/n} = 0$, i.e., $xy = 0$. Now, $yx = x^3y = x^2xy = 0$. □

Theorem 2.3. *Let $\mathscr{A}$ be a unital Banach algebra. Let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $a^3b = ba$ and $b^3a = ab$. Then*

(i) $(a+b)^{\mathsf{d}} = \frac{1}{8}bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}$.

(ii) $(1 - aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} = b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}$.

(iii) $(a+ab)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + 3ab\right]aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$.

(iv) $(a+ba)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + ab\right]aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$.

Proof. If $ab = 0$, then $ba = a^2ab = 0$. It is known [12, Theorem 5.7] that $ab = ba = 0$ and $a \in \mathscr{A}^{\mathsf{d}}$ and $b \in \mathscr{A}^{\mathsf{d}}$ imply $a+b \in \mathscr{A}^{\mathsf{d}}$ and $(a+b)^{\mathsf{d}} = a^{\mathsf{d}} + b^{\mathsf{d}}$. Similarly, $ba = 0$ leads to $a+b \in \mathscr{A}^{\mathsf{d}}$ and $(a+b)^{\mathsf{d}} = a^{\mathsf{d}} + b^{\mathsf{d}}$. Therefore, if $ab = 0$ or $ba = 0$, then the four formulas of this theorem hold.

(i) In the following we will assume $ab \ne 0$ and $ba \ne 0$. As in the proof of Theorem 2.1, we can represent a and b as follows:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, \qquad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, \tag{2.3}$$

where $p = aa^{\mathsf{d}}$, $a_1 \in [p\mathscr{A}p]^{-1}$ and $a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}$. The hypotheses $a^3b = ba$ and $b^3a = ab$ imply that $a_i^3b_i = b_ia_i$ and $b_i^3a_i = a_ib_i$ for $i = 1, 2$. Moreover, a_2b_2 is quasinilpotent since a_2 is quasinilpotent. In the following we will prove the theorem depending on b_1 as in Theorem 2.1.

(1) If b_1 is neither invertible nor quasinilpotent, then we represent

$$b_1 = \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q, \tag{2.4}$$

where $q = b_1 b_1^{\mathsf{d}}$, $b_1' \in [q\mathscr{A}q]^{-1}$ and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$. If we set

$$a_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_q,$$

by Lemma 1.6 we get $a_{12} = 0$ and $a_{21} = 0$. Thus

$$a_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}_q. \tag{2.5}$$

From $a_1^3 b_1 = b_1 a_1$ and $b_1^3 a_1 = a_1 b_1$ we obtain $a_{11}^3 b_1' = b_1' a_{11}$, $(b_1')^3 a_{11} = a_{11} b_1'$, $a_{22}^3 b_1'' = b_1'' a_{22}$, and $(b_1'')^3 a_{22} = a_{22} b_1''$. Moreover, since $a_1 \in [p\mathscr{A}p]^{-1}$ we get $a_{11} \in [q\mathscr{A}q]^{-1}$ and $a_{22} \in [(p-q)\mathscr{A}(p-q)]^{-1}$.

Let us define

$$x = \frac{1}{8}\left(3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1'\right). \tag{2.6}$$

By the definition of the generalized Drazin inverse, we shall prove that $a_{11} + b_1'$ is generalized Drazin invertible and $(a_{11} + b_1')^{\mathsf{d}} = x$. Before doing this, let us simplify some powers of $a_{11} + b_1'$, which will help us to prove $(a_{11} + b_1')^{\mathsf{d}} = x$. Evidently we have

$$(a_{11} + b_1')^2 = a_{11}^2 + a_{11} b_1' + b_1' a_{11} + (b_1')^2.$$

Observe that

$$a_{11} b_1' = (b_1')^3 a_{11} = (b_1')^2 b_1 a_{11} = (b_1')^2 a_{11}^3 b_1 = (b_1')^2 a_{11}^2 a_{11} b_1'.$$

The invertibility of $a_{11} b_1'$ yields

$$q = (b_1')^2 a_{11}^2 \tag{2.7}$$

(recall that the unity of the subalgebra $q\mathscr{A}q$ is q). Now we have

$$\begin{aligned} a_{11}^4 b_1' &= a_{11}(a_{11}^3 b_1') = a_{11} b_1' a_{11} = (a_{11} b_1') a_{11} = ((b_1')^3 a_{11}) a_{11} \\ &= b_1'((b_1')^2 a_{11}^2) = b_1'. \end{aligned} \tag{2.8}$$

The invertibility of b_1' leads to

$$a_{11}^4 = q. \tag{2.9}$$

Postmultplying (2.7) by a_{11}^2 and by using (2.9) we get

$$a_{11}^2 = (b_1')^2. \tag{2.10}$$

By inserting (2.10) into (2.7) we obtain

$$(b_1')^4 = q. \tag{2.11}$$

Moreover, from (2.10) we get

$$a_{11}^2 b_1' = (b_1')^3 = b_1' a_{11}^2, \qquad (b_1')^2 a_{11} = a_{11}^3 = a_{11}(b_1')^2. \tag{2.12}$$

Observe that the computations made (2.8) imply

$$a_{11} b_1' a_{11} = b_1'. \tag{2.13}$$

From (2.9) and (2.10) we have $b_1' a_{11} b_1' = (b_1' a_{11}) b_1' = (a_{11}^3 (b_1')) b_1' = a_{11}^3 (b_1')^2 = a_{11}^5 = a_{11}$. Thus,

$$b_1' a_{11} b_1' = a_{11}. \tag{2.14}$$

Expressions (2.10), (2.13), and (2.14) lead to

$$\begin{aligned}(a_{11} + b_1')^3 &= a_{11}^3 + a_{11} b_1' a_{11} + b_1' a_{11}^2 + (b_1')^2 a_{11} + a_{11}^2 b_1' + b_1' a_{11} b_1' \\ &\quad + a_{11}(b_1')^2 + (b_1')^3 = 3a_{11}^3 + 3(b_1')^3 + a_{11} + b_1'.\end{aligned} \tag{2.15}$$

Employing (2.9), (2.11), and recalling $a_{11}^3 b_1' = b_1' a_{11}$ and $(b_1')^3 a_{11} = a_{11} b_1'$ we have

$$\begin{aligned}(a_{11} + b_1')^4 &= (a_{11} + b_1')^3 (a_{11} + b_1') \\ &= (3a_{11}^3 + 3(b_1')^3 + (a_{11} + b_1'))(a_{11} + b_1') \\ &= 3a_{11}^4 + 3a_{11}^3 b_1' + 3(b_1')^3 a_{11} + 3(b_1')^4 + (a_{11} + b_1')^2 \\ &= 6q + 3a_{11} b_1' + 3b_1' a_{11} + (a_{11} + b_1')^2.\end{aligned}$$

Furthermore

$$\begin{aligned}(a_{11} + b_1')^5 &= (a_{11} + b_1')^4 (a_{11} + b_1') \\ &= \left[6q + 3a_{11} b_1' + 3b_1' a_{11} + (a_{11} + b_1')^2\right](a_{11} + b_1') \\ &= 6a_{11} + 6b_1' + 3a_{11} b_1' a_{11} + 3a_{11}(b_1')^2 + 3b_1' a_{11}^2 \\ &\quad + 3b_1' a_{11} b_1' + (a_{11} + b_1')^3,\end{aligned}$$

which by using (2.10), (2.13), (2.14), and (2.15) reduces to

$$\begin{aligned}(a_{11} + b_1')^5 &= 9a_{11} + 9b_1' + 3a_{11}^3 + 3b_1'^3 + (a_{11} + b_1')^3 \\ &= 2(a_{11} + b_1')^3 + 8(a_{11} + b_1').\end{aligned} \tag{2.16}$$

Now we have

$$\begin{aligned}(a_{11} + b_1')^7 &= [2(a_{11} + b_1')^3 + 8(a_{11} + b_1')](a_{11} + b_1')^2 \\ &= 2(a_{11} + b_1')^5 + 8(a_{11} + b_1')^3.\end{aligned} \tag{2.17}$$

In view of (2.6) and (2.15) we get

$$x = \frac{1}{8}(a_{11} + b_1')^3 - \frac{1}{4}(a_{11} + b_1'), \tag{2.18}$$

which trivially yields $x(a_{11} + b_1') = (a_{11} + b_1')x$. Furthermore, from (2.16) and (2.18)

$$\begin{aligned} x(a_{11}+b_1')x &= \left[\frac{1}{8}(a_{11}+b_1')^3-\frac{1}{4}(a_{11}+b_1')\right](a_{11}+b_1')\left[\frac{1}{8}(a_{11}+b_1')^3-\frac{1}{4}(a_{11}+b_1')\right] \\ &= \frac{1}{8}\left[(a_{11} + b_1')^5 - 2(a_{11} + b_1')^3\right]\left[\frac{1}{8}(a_{11} + b_1')^2 - \frac{1}{4}q\right] \\ &= (a_{11} + b_1')\left[\frac{1}{8}(a_{11} + b_1')^2 - \frac{1}{4}q\right] \\ &= \frac{1}{8}(a_{11} + b_1')^3 - \frac{1}{4}(a_{11} + b_1') = x. \end{aligned}$$

Now, by using (2.17) and (2.18), we have

$$\begin{aligned} (a_{11} + b_1')^6 x &= (a_{11} + b_1')^6\left[\frac{1}{8}(a_{11} + b_1')^3 - \frac{1}{4}(a_{11} + b_1')\right] \\ &= \frac{1}{8}(a_{11} + b_1')^2\left[(a_{11} + b_1')^7 - 2(a_{11} + b_1')^5\right] = (a_{11} + b_1')^5. \end{aligned}$$

Hence the expression $(a_{11} + b_1')^{5+1}x = (a_{11} + b_1')^5$ holds, and thus $a_{11} + b_1'$ is generalized Drazin invertible (in fact, is Drazin invertible) and $(a_{11} + b_1')^{\mathsf{d}} = x$.

Now, we shall study the generalized Drazin invertibility of $a_{22} + b_1''$. Recall that $a_{22}^3 b_1'' = b_1'' a_{22}$ and $(b_1'')^3 a_{22} = a_{22} b_1''$, $a_{22} \in [(p-q)\mathscr{A}(p-q)]^{-1}$, and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$. By Lemma 2.1 we get $a_{22}b_1'' = 0$. The invertibility of a_{22} leads to $b_1'' = 0$. Thus, $a_{22} + b_1'' = a_{22}$ and therefore, $(a_{22} + b_1'')^{\mathsf{d}} = a_{22}^{\mathsf{d}}$.

From (2.4), (2.5), $a_{12} = a_{21} = 0$, $a_{11} + b_1' \in [q\mathscr{A}q]^{\mathsf{d}}$, $a_{22} + b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$, and by Lemma 1.3, we have $a_1 + b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$\begin{aligned} (a_1 + b_1)^{\mathsf{d}} * &= \begin{pmatrix} (a_{11} + b_1')^{\mathsf{d}} & 0 \\ 0 & (a_{22} + b_1'')^{\mathsf{d}} \end{pmatrix}_q \\ &= \begin{pmatrix} \frac{1}{8}\left(3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1'\right) & 0 \\ 0 & a_{22}^{\mathsf{d}} \end{pmatrix}_q . \end{aligned}$$

By (2.5), (2.4), and by observing $q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q$, we have

$$q(3a_1^3 + 3b_1^3 - a_1 - b_1) = \begin{pmatrix} 3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1' & 0 \\ 0 & 0 \end{pmatrix}_q .$$

Recall that q is an idempotent in the subalgebra $p\mathscr{A}p$ (whose unity is p), and thus, $pq = qp = q$. So, $p - q = \begin{pmatrix} 0 & 0 \\ 0 & p-q \end{pmatrix}_q$, and therefore, from (2.5)

$$(p-q)[a_1{}^{-1}]_{p\mathscr{A}p} = \begin{pmatrix} 0 & 0 \\ 0 & a_{22}^{\mathsf{d}} \end{pmatrix}_q .$$

But $[a_1{}^{-1}]_{p\mathscr{A}p} \in p\mathscr{A}p$ and p is the unity of $p\mathscr{A}p$, thus $p[a_1{}^{-1}]_{p\mathscr{A}p} = [a_1{}^{-1}]_{p\mathscr{A}p}$ and $(p-q)[a_1{}^{-1}]_{p\mathscr{A}p} = (1-q)[a_1{}^{-1}]_{p\mathscr{A}p}$.

From the above computations we have

$$(a_1+b_1)^{\mathsf{d}} = \frac{1}{8}q(3a_1^3+3b_1^3-a_1-b_1)+(1-q)[a_1{}^{-1}]_{p\mathscr{A}p}.$$

From (2.3) we get $a_2^3b_2 = b_2a_2^3$ and $b_2^3a_2 = a_2b_2^3$. Recall that a_2 is quasinilpotent, hence by Lemma 2.1 we have that $a_2b_2 = b_2a_2 = 0$. Moreover, a_2 and b_2 are generalized Drazin invertible because a and b are generalized Drazin invertible (recall Lemma 1.3). By [12, Theorem 5.7] we get that a_2+b_2 is generalized Drazin invertible and $(a_2+b_2)^{\mathsf{d}} = a_2^{\mathsf{d}}+b_2^{\mathsf{d}}$, but $a^{\mathsf{d}} = 0$ since a_2 is quasinilpotent. Therefore, $(a_2+b_2)^{\mathsf{d}} = b_2^{\mathsf{d}}$. By Lemma 1.3 we get $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$\begin{aligned}(a+b)^{\mathsf{d}} &= \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{pmatrix}_p \\ &= \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p = \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_p + \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \\ &= \begin{pmatrix} \frac{1}{8}q(3a_1^3+3b_1^3-a_1-b_1)+(1-q)[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p + \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p. \end{aligned} \tag{2.19}$$

Observe that q, a_1, b_1, and $[a_1{}^{-1}]_{p\mathscr{A}p}$ belong to the subalgebra $p\mathscr{A}p$ (whose unity is p), so $q(3a_1^3+3b_1^3-a_1-b_1)p = q(3a_1^3+3b_1^3-a_1-b_1)$ and $(1-q)[a_1{}^{-1}]_{p\mathscr{A}p}p = (1-q)[a_1{}^{-1}]_{p\mathscr{A}p}$. Hence by using (2.3) and recalling that $q = b_1b_1^{\mathsf{d}}$

$$\begin{aligned} & bb^{\mathsf{d}}(3a^3+3b^3-a-b)p \\ &= \begin{pmatrix} b_1b_1^{\mathsf{d}} & 0 \\ 0 & b_2b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 3a_1^3+3b_1^3-a_1-b_1 & 0 \\ 0 & 3a_1^3+3b_1^3-a_1-b_3 \end{pmatrix}_p \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} q(3a_1^3+3b_1^3-a_1-b_1) & 0 \\ 0 & 0 \end{pmatrix}_p \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} b^{\pi}a^{\mathsf{d}} &= a^{\mathsf{d}} - bb^{\mathsf{d}}a^{\mathsf{d}} \\ &= \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p - \begin{pmatrix} b_1b_1^{\mathsf{d}} & 0 \\ 0 & b_2b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} (1-q)[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p. \end{aligned} \tag{2.21}$$

Since $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ we get $b_2^{\mathsf{d}} \in (1-p)\mathscr{A}(1-p)$ and $b_2^{\mathsf{d}}(1-p) = b_2^{\mathsf{d}}$ because the unity of the subalgebra $(1-p)\mathscr{A}(1-p)$ is $1-p$. Thus

$$b^{\mathsf{d}}a^{\pi} = b^{\mathsf{d}}(1-aa^{\mathsf{d}}) = b^{\mathsf{d}}(1-p) = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p. \tag{2.22}$$

From (2.19–2.22) we obtain

$$(a+b)^{\mathsf{d}} = \frac{1}{8} bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}.$$

(2) If b_1 is invertible, in the representation (2.4) we have $b_1 = b_1'$, $b_1'' = 0$, and $q = p$. The above computations work.

(3) If b_1 is quasinilpotent, in (2.4) we have $q = b_1 b_1^{\mathsf{d}} = 0$ (observe that in this case, the subalgebra $q\mathscr{A}q$ becomes the subalgebra $\{0\}$), $b_1 = b_1''$, and $b_1' = 0$. Also, the above computations are valid.

Therefore, (i) is proved.

(ii) Notice that $b^{\mathsf{d}}b^{\pi} = 0$. Furthermore,

$$aa^{\mathsf{d}}bb^{\mathsf{d}}b^{\mathsf{d}}a^{\pi} = pb^{\mathsf{d}}(1-p) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p = 0.$$

By (2.3) and (2.20)

$$(aa^{\mathsf{d}}bb^{\mathsf{d}}) \left[bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} \right] = bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}}.$$

Thus, we obtain

$$\begin{aligned}(1 - aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} &= (1 - aa^{\mathsf{d}}bb^{\mathsf{d}}) \left[\frac{1}{8} bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi} \right] \\ &= b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}.\end{aligned}$$

(iii) Let us decompose a and b as in (2.3), a_1 as in (2.5), and b_1 as in (2.4). We shall study the generalized Drazin invertibility of $a_{11} + a_{11}b_1'$, $a_{22} + a_{22}b_1''$, and $a_2 + a_2 b_2$.

We will apply properties (2.9), (2.10), (2.11), (2.13), (2.14) to simplify some powers of $a_{11} + a_{11}b_1'$. It is simple to see that

$$\begin{aligned}
(a_{11} + a_{11}b_1')^2 &= 2a_{11}^2 + (b_1')^3 + b_1', \\
(a_{11} + a_{11}b_1')^3 &= 3a_{11}^3 + 3b_1'a_{11} + a_{11} + a_{11}b_1', \\
(a_{11} + a_{11}b_1')^4 &= 6q + 3b_1' + 3(b_1')^3 + (a_{11} + a_{11}b_1')^2, \\
(a_{11} + a_{11}b_1')^5 &= 8(a_{11} + a_{11}b_1') + 2(a_{11} + a_{11}b_1')^3, \\
(a_{11} + a_{11}b_1')^7 &= 8(a_{11} + a_{11}b_1')^3 + 2(a_{11} + a_{11}b_1')^5.
\end{aligned}$$

As we did in the proof of (i), we have $(a_{11} + a_{11}b_1')^{\mathsf{d}} = \frac{1}{8}(a_{11} + a_{11}b_1')^3 + \frac{1}{4}(a_{11} + a_{11}b_1')$. Simplifying and using the above expression for the cube of $a_{11} + a_{11}b_1'$ we get

$$(a_{11} + a_{11}b_1')^{\mathsf{d}} = \frac{3}{8}\left[a_{11}^3 + a_{11} + b_1'a_{11} + 3a_{11}b_1' \right]. \tag{2.23}$$

Since b_1'' and a_2 are quasinilpotent and $a^3b = ba$, $b^3a = ab$ hold, by Lemma 2.1, we get $a_{22}b_1'' = 0$ and $a_2b_2 = 0$. Thus $a_{22}+a_{22}b_1''$ and $a_2+a_2b_2$ are generalized Drazin invertible and

$$(a_{22} + a_{22}b_1'')^{\mathsf{d}} = a_{22}^{\mathsf{d}} \qquad \text{and} \qquad (a_2 + a_2b_2)^{\mathsf{d}} = a_2^{\mathsf{d}} = 0, \tag{2.24}$$

the last equation be guaranteed by the quasinilpotency of a_2. By (2.23) and (2.24) and by the same argument as in (i), we get

$$(a + ab)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + ab\right] aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$$

i.e., (iii) is proved. The proof of (iv) is similar to (iii). $\square$

Theorem 2.4. *Let $\mathscr{A}$ be a unital Banach algebra and let a, b be Drazin invertible elements of $\mathscr{A}$. If $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1 - aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$, then $a + b$ is generalized Drazin invertible and*

(i) $(a + b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^na^{\mathsf{D}} + z + \sum_{n=0}^{\infty} a^{\pi}(a + b)^nb^{\pi}abb^{\mathsf{D}}z^{n+2}$.

(ii) $a^{\pi}bb^{\mathsf{D}}(a + b)^{\mathsf{d}} = z$,

where $z = a^{\pi}\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^nb^{\mathsf{D}}$.

Proof. If $ab = ba = 0$, the formulas of the Theorem hold by Theorem 5.7 of [12]. Thus, we can assume $ab \neq 0$ and $ba \neq 0$.

(i) Setting $p = aa^{\mathsf{D}}$, we represent a as in (1.3), where a_2 is nilpotent (since a is Drazin invertible instead of generalized Drazin invertible, we can get a stronger condition than the quasinilpotency). By Lemma 1.6 we have $pb = bp$, and thus we can represent b as follows

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p. \tag{2.25}$$

First we study the Drazin invertibility of $a_1 + b_1$. Since

$$a^{\mathsf{D}}b = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}b_1 & 0 \\ 0 & 0 \end{pmatrix}_p,$$

we get $\|[a_1{}^{-1}]_{p\mathscr{A}p}b_1\| = \|a^{\mathsf{D}}b\| < 1$, and thus, $p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1 \in [p\mathscr{A}p]^{-1}$. From $a_1 + b_1 = a_1(p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1)$ and $a_1 \in [p\mathscr{A}p]^{-1}$ we obtain $a_1 + b_1 \in [p\mathscr{A}p]^{-1}$ and

$$\begin{aligned} [(a_1 + b_1)^{-1}]_{p\mathscr{A}p} &= [(p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1)^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} \\ &= \sum_{n=0}^{\infty}(-1)^n([a_1{}^{-1}]_{p\mathscr{A}p}b_1)^n[a_1{}^{-1}]_{p\mathscr{A}p}. \end{aligned}$$

From this, we have $a_1+b_1\in\mathscr{A}^{\mathsf{d}}$ and

$$(a_1+b_1)^{\mathsf{d}}=\sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}}. \tag{2.26}$$

Now, we study the Drazin invertibility of a_2+b_2. To this end, let us define $q=b_2b_2^{\mathsf{D}}$ and represent b_2 as follows:

$$b_2=\begin{pmatrix} b_{11} & 0\\ 0 & b_{22}\end{pmatrix}_q,$$

where $b_{11}\in[q\mathscr{A}q]^{-1}$ and b_{22} is nilpotent. Since $a^3b=ba$ implies $a_2^3b_2=b_2a_2$, by Lemma 1.4

$$qa_2-qa_2q=qa_2(1-q)=q^na_2(1-q)=(b_2^{\mathsf{D}})^n a_2^{3^n}b_2^n(1-b_2b_2^{\mathsf{D}})^n.$$

Since a_2 is nilpotent we get $qa_2=qa_2q$. Hence,

$$a_2=\begin{pmatrix} a_{11} & 0\\ a_{21} & a_{22}\end{pmatrix}_q,\qquad a_2+b_2=\begin{pmatrix} a_{11}+b_{11} & 0\\ a_{21} & a_{22}+b_{22}\end{pmatrix}_q. \tag{2.27}$$

An evident induction argument shows that there exists $x_1,x_2,\ldots\in\mathscr{A}$ such that

$$a_2^n=\begin{pmatrix} a_{11}^n & 0\\ x_n & a_{22}^n\end{pmatrix}_q,\qquad \forall\, n\in\mathbb{N}.$$

Since a_2 is nilpotent, we get that a_{11} and a_{22} are nilpotent.

In order to study the Drazin invertibility of a_2+b_2, we use Lemma 1.2: We need prove that $a_{11}+b_{11}$ and $a_{22}+b_{22}$ are generalized Drazin invertible.

First we prove that $a_{11}+b_{11}$ is generalized Drazin invertible. Let us observe that

$$(1-aa^{\mathsf{D}})b^{\mathsf{D}}a=\begin{pmatrix} 0 & 0\\ 0 & 1-p\end{pmatrix}_p\begin{pmatrix} b_1^{\mathsf{D}} & 0\\ 0 & b_2^{\mathsf{D}}\end{pmatrix}_p\begin{pmatrix} a_1 & 0\\ 0 & a_2\end{pmatrix}_p=\begin{pmatrix} 0 & 0\\ 0 & b_2^{\mathsf{D}}a_2\end{pmatrix}_p$$

and

$$b_2^{\mathsf{D}}a_2=\begin{pmatrix} [b_{11}{}^{-1}]_{q\mathscr{A}q} & 0\\ 0 & 0\end{pmatrix}_q\begin{pmatrix} a_{11} & 0\\ a_{21} & a_{22}\end{pmatrix}_q=\begin{pmatrix} [b_{11}{}^{-1}]_{q\mathscr{A}q}a_{11} & 0\\ 0 & 0\end{pmatrix}_q,$$

which yields $\|[b_{11}{}^{-1}]_{q\mathscr{A}q}a_{11}\|=\|(1-aa^{\mathsf{D}})b^{\mathsf{D}}a\|<1$. Now

$$a_{11}+b_{11}=b_{11}\left([b_{11}{}^{-1}]_{q\mathscr{A}q}a_{11}+q\right)$$

and $b_{11}\in[q\mathscr{A}q]^{-1}$ lead to $a_{11}+b_{11}\in[q\mathscr{A}q]^{-1}$ and

$$\begin{aligned}[(a_{11}+b_{11})^{-1}]_{q\mathscr{A}q}&=\sum_{n=0}^{\infty}(-1)^n([b_{11}{}^{-1}]_{q\mathscr{A}q}a_{11})^n[b_{11}{}^{-1}]_{q\mathscr{A}q}=\sum_{n=0}^{\infty}(-1)^n(b_2^{\mathsf{D}}a_2)^n b_2^{\mathsf{D}}\\ &=(1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}.\end{aligned}$$

Since $q\mathscr{A}q$ is a subalgebra of $\mathscr{A}$ with unity, $a_{11}+b_{11}$ is generalized Drazin invertible and $(a_{11}+b_{11})^{\mathsf{d}} = [(a_{11}+b_{11})^{-1}]_{q\mathscr{A}q}$. Thus,

$$(a_{11}+b_{11})^{\mathsf{d}} = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}. \tag{2.28}$$

In the following, we study the Drazin invertibility of $a_{22}+b_{22}$. In fact, we are going to prove that $a_{22}+b_{22}$ is nilpotent. Since a_{22} and b_{22} are nilpotent, there exists $k\in\mathbb{N}$ such that $a_{22}^k = b_{22}^k = 0$. It is clear that if $a_{22}=b_{22}$, then $a_{22}+b_{22}$ is nilpotent; so we can assume $a_{22}\neq b_{22}$. Let us remark that for any $n\in\mathbb{N}$, we can write $(a_{22}+b_{22})^n = \sum_{j=1}^{2^n} c_{1j}\cdots c_{nj}$, where $c_{ij}\in\{a_{22},b_{22}\}$. Now, we are ready to prove $(a_{22}+b_{22})^{2k}=0$. To this end, we will prove that if $d_1,\dots,d_{2k}\in\{a_{22},b_{22}\}$, then $d_1\cdots d_{2k}=0$. Let $A=\{r\in\{1,\dots,2k\} : d_r = a_{22}\}$ and $B=\{r\in\{1,\dots,2k\} : d_r=b_{22}\}$ (observe that $A\cup B=\{1,\dots,2k\}$ and $A\cap B=\varnothing$). If $|\cdot|$ denotes the cardinal of a set, it is clear than $|A|\geq k$ or $|B|\geq k$. If $|A|\geq k$, by using $a_{22}^3b_{22}=b_{22}a_{22}$, then there exists $x\in\mathscr{A}$ such that $d_1\cdots d_{2k} = a_{22}^k x = 0$. If $|B|\geq k$, by using again $a_{22}^3b_{22}=b_{22}a_{22}$, there exists $y\in\mathscr{A}$ such that $d_1\cdots d_{2k} = yb_{22}^k = 0$. Thus, $a_{22}+b_{22}$ is nilpotent, and therefore, $a_{22}+b_{22}$ is Drazin invertible and $(a_{22}+b_{22})^{\mathsf{D}}=0$.

From the second expression of (2.27) and Lemma 1.2, we get a_2+b_2 is generalized Drazin invertible and

$$(a_2+b_2)^{\mathsf{d}} = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & (a_{22}+b_{22})^{\mathsf{d}} \end{pmatrix}_q,$$

where

$$\begin{aligned} u \;=\; & \sum_{n=0}^{\infty}\left[(a_{22}+b_{22})^{\mathsf{d}}\right]^{n+2} a_{21}(a_{11}+b_{11})^n(a_{11}+b_{11})^{\pi} + \\ & + \sum_{n=0}^{\infty}(a_{22}+b_{22})^{\pi}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2} \\ & - (a_{22}+b_{22})^{\mathsf{d}}a_{21}(a_{11}+b_{11})^{\mathsf{d}}, \end{aligned}$$

which, having in mind that $(a_{22}+b_{22})^{\mathsf{d}}=0$, reduces to

$$(a_2+b_2)^{\mathsf{d}} = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & 0 \end{pmatrix}_q,$$

$$u = \sum_{n=0}^{\infty}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2}. \tag{2.29}$$

From (2.29) we get

$$\begin{aligned} (a+b)^{\mathsf{d}} &= (a_1+b_1)^{\mathsf{d}} + (a_2+b_2)^{\mathsf{d}} \\ &= (a_1+b_1)^{\mathsf{d}} + (a_{11}+b_{11})^{\mathsf{d}} + \sum_{n=0}^{\infty}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2}. \end{aligned} \tag{2.30}$$

In (2.26) and (2.28) we have expressed $(a_1+b_1)^{\mathsf{d}}$ and $(a_{11}+b_{11})^{\mathsf{d}}$, respectively, in terms of a, a^{D}, b, and b^{D}. We will express the remaining terms of (2.30) in terms of a, a^{D}, b, and b^{D}. To this end, let us remark that $q = b_2 b_2^{\mathsf{D}}$ is an element of the subalgebra $(1-p)\mathscr{A}(1-p)$ (whose unity is $1-p$) and thus, $(1-p)q = q(1-p) = q$. In other words, we have

$$pq = qp = 0.$$

Furthermore, recall that an element $x \in (1-p)\mathscr{A}(1-p)$ has the representation

$$x = \begin{pmatrix} qxq & qx(1-p-q) \\ (1-p-q)xq & (1-p-q)x(1-p-q) \end{pmatrix}_q .$$

Observe that from the expression of $a_2 + b_2$ in (2.27) we can write

$$(a_2+b_2)^n = \begin{pmatrix} (a_{11}+b_{11})^n & 0 \\ y_n & (a_{22}+b_{22})^n \end{pmatrix}_q ,$$

where $y_1, y_2, \ldots$ are elements in $\mathscr{A}$ (we are not interested in their explicit expressions). We have

$$\begin{aligned} &(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}}) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)^n & 0 \\ 0 & (a_2+b_2)^n \end{pmatrix}_p \begin{pmatrix} p-b_1b_1^{\mathsf{D}} & 0 \\ 0 & 1-p-b_2b_2^{\mathsf{D}} \end{pmatrix}_p \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (a_2+b_2)^n(1-p-b_2b_2^{\mathsf{D}}) \end{pmatrix}_p . \end{aligned} \tag{2.31}$$

Having in mind that $(a_{22}+b_{22})^n \in (1-p-q)\mathscr{A}(1-p-q)$,

$$\begin{aligned} (a_2+b_2)^n(1-p-q) &= \begin{pmatrix} (a_{11}+b_{11})^n & 0 \\ y_n & (a_{22}+b_{22})^n \end{pmatrix}_q \begin{pmatrix} 0 & 0 \\ 0 & 1-p-q \end{pmatrix}_q \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (a_{22}+b_{22})^n \end{pmatrix}_q . \end{aligned}$$

Therefore,

$$(a_{22}+b_{22})^n = (1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}}). \tag{2.32}$$

Now, we are going to express a_{21} in terms of a, a^{D}, b and b^{D}. By $1-p$ is the unity of the subalgebra $(1-p)\mathscr{A}(1-p)$, one has

$$\begin{aligned} &(1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} p-b_1b_1^{\mathsf{D}} & 0 \\ 0 & 1-p-b_2b_2^{\mathsf{D}} \end{pmatrix}_p \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p \begin{pmatrix} b_1b_1^{\mathsf{D}} & 0 \\ 0 & b_2b_2^{\mathsf{D}} \end{pmatrix}_p \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (1-p-q)a_2q \end{pmatrix}_p . \end{aligned}$$

Since

$$\begin{aligned}(1-p-q)a_2q &= \begin{pmatrix} 0 & 0 \\ 0 & 1-p-q \end{pmatrix}_q \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q \\ &= \begin{pmatrix} 0 & 0 \\ (1-p-q)a_{21}q & 0 \end{pmatrix}_q\end{aligned}$$

and $a_{21} \in (1-p-q)\mathscr{A}q$ (this latter subset is not a subalgebra of $\mathscr{A}$) we have $(1-p-q)a_{21}q = a_{21}$. Thus, from the above computations, we have

$$a_{21} = (1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}}.$$

From (2.26), (2.28), (2.32), and (2.30) we get

$$\begin{aligned}(a+b)^{\mathsf{d}} &= \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z \\ &+ \sum_{n=0}^{\infty}(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})(1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}}z^{n+2},\end{aligned} \tag{2.33}$$

where $z = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. We can simplify expression of $(a+b)^{\mathsf{d}}$. By observing the expression of $(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})$ obtained in (2.31), easily we can see $(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})(1-aa^{\mathsf{D}}) = (1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})$ and using that $1-bb^{\mathsf{D}}$ is an idempotent, (2.33) reduces to

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z + \sum_{n=0}^{\infty}(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})abb^{\mathsf{D}}z^{n+2}.$$

(ii) By using (1.3) and (2.25) we have

$$(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} b_1b_1^{\mathsf{D}} & 0 \\ 0 & b_2b_2^{\mathsf{D}} \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{pmatrix}_p$$

and using that the unity of $(1-p)\mathscr{A}(1-p)$ is $1-p$ we get

$$(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = b_2b_2^{\mathsf{D}}(a_2+b_2)^{\mathsf{d}} = q(a_2+b_2)^{\mathsf{d}}.$$

From (2.29) and using $q(a_{11}+b_{11})^d = (a_{11}+b_{11})^d$ (because $(a_{11}+b_{11})^d \in q\mathscr{A}q$ and q is the unity of the subalgebra $q\mathscr{A}q$),

$$q(a_2+b_2)^d = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & 0 \end{pmatrix}_q = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_q.$$

Now, (2.28) leads to $(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. □

If b is a special perturbation of a, then we have the following theorem.

Theorem 2.5. *Let $\mathscr{A}$ be a unital Banach algebra. If $a, b \in \mathscr{A}^{\mathsf{d}}$, $a^3 b = ba$, and $b^3 a = ab$, then*

(i) $\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| \le \frac{1}{8}\left(\|bb^{\mathsf{d}}\|\|aa^{\mathsf{d}}\|\|3a^3 + 3b^3 - a - b\|\right) + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\|$.

(ii) $\|b^{\mathsf{d}}a^{\pi}\| \le \|1 - aa^{\mathsf{d}}bb^{\mathsf{d}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\|$.

Proof. (i) From Theorem 2.3, we get

$$\begin{aligned} (a+b)^{\mathsf{d}} - a^{\mathsf{d}} &= \frac{1}{8} bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi} - a^{\mathsf{d}} \\ &= \frac{1}{8} bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} - bb^{\mathsf{d}}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}. \end{aligned}$$

Thus

$$\begin{aligned} \|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| &\le \frac{1}{8}\|bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}}\| + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\| \\ &\le \frac{1}{8}\left(\|bb^{\mathsf{d}}\|\|aa^{\mathsf{d}}\|\|3a^3 + 3b^3 - a - b\|\right) + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\|. \end{aligned}$$

(ii) By the proof of Theorem 2.3 we get $aa^{\mathsf{d}}bb^{\mathsf{d}} = bb^{\mathsf{d}}aa^{\mathsf{d}}$.

By Theorem 2.3, we have

$$\begin{aligned} b^{\mathsf{d}}a^{\pi} &= (1 - aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} - b^{\pi}a^{\mathsf{d}} \\ &= (1 - bb^{\mathsf{d}}aa^{\mathsf{d}})(a+b)^{\mathsf{d}} - (1 - bb^{\mathsf{d}})a^{\mathsf{d}} \\ &= (1 - bb^{\mathsf{d}}aa^{\mathsf{d}})[(a+b)^{\mathsf{d}} - a^{\mathsf{d}}]. \end{aligned}$$

Hence; $\|b^{\mathsf{d}}a^{\pi}\| \le \|1 - bb^{\mathsf{d}}aa^{\mathsf{d}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\|$. □

In the following result we have a bound for another kind of perturbation.

Theorem 2.6. *Let $\mathscr{A}$ be a unital algebra and let $a, b \in \mathscr{A}$ be Drazin invertible satisfying $a^3 b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|a^{\pi}b^{\mathsf{D}}a\| < 1$. Then*

(i) $\|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\| \le \frac{\|a^{\mathsf{D}}\|\|a^{\mathsf{D}}b\|}{1-\|a^{\mathsf{D}}b\|} + \frac{\|b^{\mathsf{D}}\|}{1-\|a^{\pi}b^{\mathsf{D}}a\|} + \frac{\|abb^{\mathsf{D}}\|\|b^{\mathsf{D}}\|^2}{(1-\|a^{\pi}b^{\mathsf{D}}a\|)^2}\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\|$.

(ii) $\|z\| \le \|a^{\pi}bb^{\mathsf{D}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\|$, *where z is giving as in Theorem 2.4.*

Proof. (i) From Theorem 2.4, we have

$$(a+b)^{\mathsf{d}} - a^{\mathsf{D}} = \sum_{n=1}^{\infty}(-1)^n (a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z + \sum_{n=0}^{\infty} a^{\pi}(a+b)^n b^{\pi} abb^{\mathsf{D}} z^{n+2},$$

where $z = a^{\pi}\sum_{n=0}^{\infty}(-1)^n (b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. Hence, (in Theorem 2.4 we had proved that $\|a^{\mathsf{D}}b\| < 1$ and in (2.31) we have shown, by using the nilpotency of $a_{22} + b_{22}$, that there exists $k \in \mathbb{N}$ such that $a^{\pi}(a+b)^k b^{\pi} = 0$)

$$\begin{aligned} \|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\| &\le \|a^{\mathsf{D}}\|\sum_{i=1}^{\infty}\|a^{\mathsf{D}}b\|^n + \|z\| + \|abb^{\mathsf{D}}\|\|z^2\|\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\| \\ &= \|a^{\mathsf{D}}\|\frac{\|a^{\mathsf{D}}b\|}{1 - \|a^{\mathsf{D}}b\|} + \|z\| + \|abb^{\mathsf{D}}\|\|z\|^2\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\|. \end{aligned}$$

Now, we will find an upper bound for $\|z\|$. Observe that the proof of Theorem 2.4 distil $a^{\pi}(b^{\mathsf{D}}a)^n = (a^{\pi}b^{\mathsf{D}}a)^n$. Thus, $z = \sum_{n=0}^{\infty}(-1)^n(a^{\pi}b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. By using that $\|a^{\pi}b^{\mathsf{D}}a\| < 1$, one has $\|z\| \leq \|b^{\mathsf{D}}\| \sum_{n=0}^{\infty} \|a^{\pi}b^{\mathsf{D}}a\|^n = \|b^{\mathsf{D}}\| \frac{1}{1-\|a^{\pi}b^{\mathsf{D}}a\|}$. Hence, (i) is proved.

(ii) From Theorem 2.4 we have $a^{\pi}bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = z$. Also, the from the proof of Theorem 2.4 we easily get $a^{\pi}bb^{\mathsf{D}}a^{\mathsf{D}} = 0$; hence $a^{\pi}bb^{\mathsf{D}}\left[(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\right] = z$. The conclusion is obtained. □

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SOME RESULTS ON HYPERVECTOR SPACES

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Abstract. The aim of this paper is to study hypervector spaces. In this regard at first some new nontrivial examples of hypervector spaces are introduced. Then the notions of linearly span, linearly independence, basis, ordered basis, coordinates and linear transformation are investigated and some related results are obtained. Especially, it is proved that for a linear transformation $T : V \to W$ between two hypervector spaces, $\dim \ker T + \dim T(V) = \dim V$, and under certain conditions $\dim L(V, W) = \dim V \times \dim W$.

Keywords: hypervector space, linearly span, linearly independent set, basis, linear transformation, coordinate.

1. Introduction

The theory of hyperstructuers was born in 1934, when Marty [6] defined hypergroups. Since then many researches have worked on hyperalgberaic structures and developed this theory (for more see [3], [4], [5], [13]). In 1990, M. Scafati Tallini introduced the notion of hypervector spaces [8], and studied ba-

*. Corresponding author

sic properties of them (for more see [9], [10], [11]). Recently Ameri [1], [2], Vaezpoor [7] and Taghavi [12] in Iran have developed this concept. In this paper we follow [2] and study more properties of hypervector spaces. In this regard at first some new interesting nontrivial examples of hypervector spaces are introduced. Then their basic notions are investigated and some related results are obtained. Especially, it is proved that for a linear transformation $T : V \to W$, $\dim \ker T + \dim T(V) = \dim V$, and under certain conditions, $\dim L(V,W) = \dim V \times \dim W$.

2. Preliminaries

In this section we present some definitions and simple properties of hypervector spaces that we shall use in later.

A map $\circ : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B \}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.1. ([8]) Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K to be the quadruplet $(V, +, \circ, K)$, Where "$\circ$" is a mapping:

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law,

(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law,

(H_3) $a \circ (b \circ x) = (ab) \circ x$, associative law,

(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,

(H_5) $x \in 1 \circ x$.

Remark 2.2 ([8])**.** (i) In the right hand of (H_1) the sum is meant in the sense of Frobenius, i.e.

$$a \circ x + a \circ y = \{p + q : p \in a \circ x, q \in a \circ y\}.$$

Similarly we have in (H_2). Moreover, the left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in a \circ x} a \circ y.$$

(ii) ([9]) We say that $(V,+,\circ,K)$ is anti-left distributive, if

$$\forall a,b\in K,\forall x\in V,(a+b)\circ x\supseteq a\circ x+b\circ x,$$

and strongly left distributive if equality holds. In a similar way we define anti-right distributive and strongly right distributive. V is called strongly distributive if it is both strongly left and right distributive.

(iii) The mapping "$\circ$" in Definition 2.1, is called external hyperoperation.

Example 2.3 ([8])**.** In $(\mathbb{R}^2,+)$ we define the product times a scalar in $\mathbb{R}$ by setting:

$$a\circ x=\begin{cases}\text{line pass origin and point } x, & \text{if } x\neq \underline{0},\\ \{\underline{0}\}, & \text{if } x=\underline{0}.\end{cases}$$

Then $(\mathbb{R}^2,+,\circ,\mathbb{R})$ is a hypervector space.

Example 2.4 ([12])**.** $(\mathbb{C},+,\circ,\mathbb{R})$ is a hypervector space, where "$+$" is the usual sum and the mapping "$\circ:\mathbb{R}\times\mathbb{C}\longrightarrow P_*(\mathbb{C})$" is defined by the following:

$$a\circ z=\begin{cases}\left\{re^{i\theta}:0<r\leq |a||z|,\theta=arg(az)\right\}, & \text{if } a\neq 0 \text{ and } z\neq\underline{0},\\ \{\underline{0}\}, & \text{if } a=0 \text{ or } z=\underline{0}.\end{cases}$$

Lemma 2.5 ([8])**.** *Let $(V,+,\circ,K)$ be a hypervector space and $\Omega_V=0\circ\underline{0}$, where $\underline{0}$ is the zero of $(V,+)$. Then*

1. *If V is either strongly right or left distributive, then Ω_V is a subgroup of $(V,+)$.*
2. *If V is anti-left distributive, then for all $x\in V$ the set $0\circ x$ is a subgroup of $(V,+)$.*
3. *If V is strongly left distributive, then $a\circ\underline{0}=\Omega_V=a\circ\Omega_V$, for all $a\in K$.*

Proposition 2.6. *Let $(V,+,\circ,K)$ be a strongly left distributive hypervector space such that $|1\circ\underline{0}|=1$ and for all $x\in V$, $-x\neq x$, unless $x=\underline{0}$. Then for any $x\in V$, the following holds:*

$$x=\underline{0}\Longleftrightarrow\forall 0\neq a\in K; a\circ x+a\circ x=\Omega.$$

Proof. ($\Longrightarrow$) Let $x=\underline{0}$ and $0\neq a\in K$. Then by Lemma 2.5, it follows that:

$$a\circ x+a\circ x=a\circ\underline{0}+a\circ\underline{0}=\Omega+\Omega=\Omega.$$

($\Longleftarrow$) By Definition 2.1, $a\circ 2x=a\circ(x+x)\subseteq a\circ x+a\circ x=\Omega=a\circ\underline{0}$. So $a^{-1}\circ(a\circ 2x)\subseteq a^{-1}\circ(a\circ\underline{0})$. Thus $1\circ 2x\subseteq 1\circ\underline{0}=\{\underline{0}\}$. Hence $2x=\underline{0}$ and so $x=-x$. Therefore $x=\underline{0}$. □

3. New examples of hypervector spaces

In this section we present some new nontrivial examples of hypervector spaces.

Example 3.1. $(\mathbb{Z}, +, \circ, \mathbb{Q})$ is a hypervector space, where " $+$ " is the usual sum and the mapping " $\circ$ " is defined by the following:

$$\begin{cases} \circ : \mathbb{Q} \times \mathbb{Z} \longrightarrow P_*(\mathbb{Z}) \\ \dfrac{r}{s} \circ n = \{m(rn) : m \in \mathbb{Z}\}. \end{cases}$$

Example 3.2. If the external hyperoperation "$\circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2)$" for all $r, a, b \in \mathbb{R}$ is defined by each of the followings, then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a hypervector space:

(i) $r \circ (a, b) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq ra, 0 \leq y \leq rb\}$,

(ii) $r \circ (a, b) =$ Environment of the rectangle bounded with lines $x = 0, x = ra, y = 0, y = rb$,

(iii) $r \circ (a, b) = \{(x, y) \in \mathbb{R}^2 \mid -ra \leq x \leq ra, -rb \leq y \leq rb\}$,

(iv) $r \circ (a, b) =$ Environment of the rectangle bounded with lines $x = -ra, x = ra, y = -rb, y = rb$,

(v) $r \circ (a, b) =$ Environment of the circle with origin $(0, 0)$ and radius $|r|\sqrt{a^2 + b^2}$.

Proposition 3.3. *If $(V, +, \circ, K)$ is a hypervector space such that $-x \in 1 \circ x$ for all $x \in V$, then $(V, \oplus, \odot, K)$ construct a hypervector space with the following mappings:*

$$\begin{cases} \oplus : V \times V \longrightarrow V \\ x \oplus y = x + y, \end{cases} \qquad \textit{and} \qquad \begin{cases} \odot : K \times V \longrightarrow P_*(V) \\ a \odot x = -a \circ x. \end{cases}$$

Proof. Straightforward. $\square$

Proposition 3.4. *Let $(V, +, \circ, K)$ be a hypervector space. Suppose*

$$V^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in V \right\}.$$

Then $(V^n, \oplus_n, \odot_n, K)$ is a hypervector space, where "$\oplus_n$" and "$\odot_n$" are defined by the followings:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \textit{and} \quad a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \left\{ \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} : x'_i \in a \circ x_i, 1 \leq i \leq n \right\}.$$

Proof. Let $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in V^n$ and $a \in K$. Then

$$\begin{aligned}
a \odot_n \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) &= a \odot_n \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \\
&= \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i \in a \circ (x_i + y_i) \right\} \\
&\subseteq \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i \in a \circ x_i + a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i = \acute{x}_i + \acute{y}_i, \acute{x} \in a \circ x_i, \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 + \acute{y}_1 \\ \vdots \\ \acute{x}_n + \acute{y}_n \end{bmatrix} : \acute{x} \in a \circ x_i,\ \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 \\ \vdots \\ \acute{x}_n \end{bmatrix} \oplus_n \begin{bmatrix} \acute{y}_1 \\ \vdots \\ \acute{y}_n \end{bmatrix} : \acute{x} \in a \circ x_i,\ \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 \\ \vdots \\ \acute{x}_n \end{bmatrix} : \acute{x} \in a \circ x_i \right\} \oplus_n \left\{ \begin{bmatrix} \acute{y}_1 \\ \vdots \\ \acute{y}_n \end{bmatrix} : \acute{y} \in a \circ y_i \right\} \\
&= \left(a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \oplus_n \left(a \odot_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right).
\end{aligned}$$

Thus

$$a \odot_n \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) \subseteq \left(a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \oplus_n \left(a \odot_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right).$$

In a similar way, it is easy to verify the other parts of Definition 2.1. □

Remark 3.5. In Proposition 3.4, if V is strongly left, Strongly right or strongly distributive, then V^n is strongly left, Strongly right or strongly distributive, respectively.

Proposition 3.6. *Let $(V,+,\circ,K)$ be a hypervector space. Suppose*

$$M^V_{m\times n}=\left\{\begin{bmatrix} x_{11}\cdots x_{1n}\\ \vdots \ \ \vdots \ \ \vdots\\ x_{m1}\cdots x_{mn}\end{bmatrix} : x_{ij}\in V, 1\le i\le m, 1\le j\le n\right\}.$$

Then $(M^V_{m\times n},\oplus,\odot,K)$ is a hypervector space, where $\oplus$ is the usual sum of matrices and external hyperoperation $\odot : K\times M^V_{m\times n}\longrightarrow P_*(M^V_{m\times n})$ is defined by the following:

$$a\odot\begin{bmatrix} x_{11}\cdots x_{1n}\\ \vdots \ \ \vdots \ \ \vdots\\ x_{m1}\cdots x_{mn}\end{bmatrix}=\left\{\begin{bmatrix} x'_{11}\cdots x'_{1n}\\ \vdots \ \ \vdots \ \ \vdots\\ x'_{m1}\cdots x'_{mn}\end{bmatrix} : x'_{ij}\in a\circ x_{ij}, 1\le i\le m, 1\le j\le n\right\}.$$

Corollary 3.7. *Let $(V,+,\circ,K)$ be a hypervector space. Then the set*

$$\left\{\begin{bmatrix} x & -y\\ y & x\end{bmatrix} : x,y\in V\right\}$$

together operation $\oplus$ and external hyperoperation $\odot$ is a hypervector space over the field K, where $\oplus$ is the usual sum of matrices and $\odot$ is defined by the following:

$$a\odot\begin{bmatrix} x & -y\\ y & x\end{bmatrix}=\left\{\begin{bmatrix} w & -z\\ z & w\end{bmatrix} : w\in a\circ x,\ z\in a\circ y\right\}.$$

Definition 3.8 ([9])**.** A nonempty subset W of V is called a subhyperspace of V, if W is itself a hypervector space with the external hyperoperation on V, i.e.

$$\begin{cases} W\neq\phi,\\ \forall x,y\in W\Longrightarrow x-y\in W,\\ \forall a\in K,\ \ \forall x\in W\Longrightarrow a\circ x\subseteq W.\end{cases}$$

In this case we write $W\leqslant V$.

Example 3.9. (i) Let $L=\{(a,0)\mid a\in\mathbb{R}\}$ and $R=\{(0,b)\mid b\in\mathbb{R}\}$. Then L and R are subhyperspaces of the hypervector space $(\mathbb{R}^2,+,\circ,\mathbb{R})$ in Example 2.3.

(ii) In Example 3.2, the hypervector spaces are defined in parts (ii) and (iv) are subhyperspaces of the hypervector spaces are defined in parts (i) and (iii), respectively.

(iii) The hypervector space is defined in part (v) of Example 3.2, is a subhyperspace of the hypervector space in Example 2.4.

Proposition 3.10. *If V is either strongly left or right distributive hypervector space, then Ω_V is the smallest subhyperspace of V.*

Proof. By Lemma 2.5, Ω_V is a subgroup of V and $a \circ \Omega_V = \Omega_V$, for all $a \in K$. Thus Ω_V is a subhyperspace of V. Now if H is a subhyperspace of V, then for any $x \in \Omega_V$, we have:

$$x \in 0 \circ \underline{0} \subseteq 0 \circ H \subseteq H.$$

So $\Omega_V \subseteq H$. □

Proposition 3.11. *If V is a hypervector space over the field K and $H \leqslant V$, then for any $x, x_1, \ldots, x_n \in H$ and $a, a_1, \ldots, a_n \in K$, the following conditions hold:*

1. $(a_1 + \cdots + a_n) \circ x \subseteq a_1 \circ x + \cdots + a_n \circ x$,
2. $a \circ (-x) \subseteq H$,
3. $a_1 \circ x_1 + \cdots + a_n \circ x_n \subseteq H$.

Proof. Straightforward. □

4. Basis and dimension

In the sequel of this note, unless otherwise specified, we assume that $V = (V, +, \circ, K)$ is a hypervector space over the field K.

Definition 4.1 ([2])**.** Let S be a subset of V. Then the linear span of S is the smallest subhyperspace of V containing S and denoted by $SP(S)$. If S is a nonempty subset of V, then

$$\begin{aligned} SP(S) &= \{x \in V : x \in \textstyle\sum_{i=1}^{n} a_i \circ y_i, a_i \in K, y_i \in S, 1 \leq i \leq n, n \in \mathbb{N}\} \\ &= \{x_1 + \cdots + x_n : x_i \in a_i \circ y_i, a_i \in K, y_i \in S, 1 \leq i \leq n, n \in \mathbb{N}\}. \end{aligned}$$

We say that S spans V, if $SP(S) = V$. It is easy to verify that if A and B are subsets of V such that $A \subseteq B$, then $SP(A) \subseteq SP(B)$.

Proposition 4.2. *Let V be strongly left distributive and $x_1, \ldots, x_n, y_1, \ldots, y_m \in V$, such that $\{y_1, \ldots, y_m\}$ spans V and $y_1, \ldots, y_m \in SP(x_1, \ldots, x_n)$. Then $\{x_1, \ldots, x_n\}$ spans V.*

Proof. Let $x \in V$. Then $x \in b_1 \circ y_1 + \cdots + b_m \circ y_m$, for some $b_i \in K, 1 \leq i \leq m$. On the other hand,

$$\begin{cases} y_1 \in a_{11} \circ x_1 + \cdots + a_{1n} \circ x_n \\ y_2 \in a_{21} \circ x_1 + \cdots + a_{2n} \circ x_n \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ y_m \in a_{m1} \circ x_1 + \cdots + a_{mn} \circ x_n \end{cases}$$

for some $a_{ij} \in K, 1 \leq i \leq m, 1 \leq j \leq n$. Thus

$$\begin{aligned}
x &\in b_1 \circ (a_{11} \circ x_1 + \cdots + a_{1n} \circ x_n) + \cdots + b_m \circ (a_{m1} \circ x_1 + \cdots + a_{mn} \circ x_n) \\
&\subseteq b_1 \circ (a_{11} \circ x_1) + \cdots + b_1 \circ (a_{1n} \circ x_n) + \cdots \\
&+ b_m \circ (a_{n1} \circ x_1) + \cdots + b_m \circ (a_{mn} \circ x_n) \\
&= (b_1 a_{11}) \circ x_1 + \cdots + (b_1 a_{1n}) \circ x_n + \cdots + (b_m a_{n1}) \circ x_1 + \cdots + (b_m a_{mn}) \circ x_n \\
&= (b_1 a_{11} + \cdots + b_m a_{n1}) \circ x_1 + \cdots + (b_1 a_{1n} + \cdots + b_m a_{mn}) \circ x_n \\
&\subseteq SP\ (x_1, \ldots, x_n).
\end{aligned}$$

Hence $SP(x_1, \ldots, x_n) = V$. □

Definition 4.3. (i) A hypervector space V over the field K is said to be K-weak invertible or shortly weak invertible if and only if

$$\forall a \in K, \forall u, v \in V, u \in a \circ v \text{ implies that } v \in a' \circ u, \text{ for some } a' \in K.$$

(ii) ([2]) A hypervector space V over the field K is said to be K-invertible or shortly invertible if and only if

$$\forall a \neq 0 \in K, \forall u, v \in V, u \in a \circ v \text{ implies that } v \in a^{-1} \circ u.$$

Proposition 4.4. *Let V be weak invertible and $\{x_1, x_2, y_1, y_2\} \subseteq V$, such that $x_1 \in b_1 \circ y_1$ and $x_2 \in b_2 \circ y_2$, for some $b_1, b_2 \in K$. Then $SP(x_1, x_2) = SP(y_1, y_2)$.*

Proof. Let $z \in SP(x_1, x_2)$. Then $z \in a_1 \circ x_1 + a_2 \circ x_2$, for some $a_1, a_2 \in K$. So

$$\begin{aligned}
z &\in a_1 \circ (b_1 \circ y_1) + a_2 \circ (b_2 \circ y_2) \\
&= (a_1 b_1) \circ y_1 + (a_2 b_2) \circ y_2 \\
&\subseteq SP\ (y_1, y_2).
\end{aligned}$$

Thus $SP(x_1, x_2) \subseteq SP(y_1, y_2)$. On the other hand, $y_1 \in b'_1 \circ x_1$ and $y_2 \in b'_2 \circ x_2$, for some $b'_1, b'_2 \in K$. Now let $w \in SP(y_1, y_2)$. Then $w \in c_1 \circ y_1 + c_2 \circ y_2$, for some $c_1, c_2 \in K$. So

$$\begin{aligned}
w &\in c_1 \circ (b'_1 \circ x_1) + c_2 \circ (b'_2 \circ x_2) \\
&= (c_1 b'_1) \circ x_1 + (c_2 b'_2) \circ x_2 \\
&\subseteq SP\ (x_1, x_2).
\end{aligned}$$

Thus $SP(y_1, y_2) \subseteq SP(x_1, x_2)$. Therefore $SP(x_1, x_2) = SP(y_1, y_2)$. □

Proposition 4.5. *Let V be strongly left distributive and $x_1, \ldots, x_n, y \in V$. Then $SP(x_1, \ldots, x_n, y) = SP(x_1, \ldots, x_n)$ if and only if $y \in SP(x_1, \ldots, x_n)$.*

Proof. ($\Longleftarrow$) Let $y \in SP(x_1, \ldots, x_n)$. Then $y \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_1, \ldots, a_n \in K$. Now if $x \in SP(x_1, \ldots, x_n, y)$, then $x \in a'_1 \circ x_1 + \cdots + a'_n \circ x_n + b \circ y$, for some $a'_1, \ldots, a'_n, b \in K$. Thus

$$\begin{aligned}
x &\in a'_1 \circ x_1 + \cdots + a'_n \circ x_n + b \circ (a_1 \circ x_1 + \cdots + a_n \circ x_n) \\
&\subseteq a'_1 \circ x_1 + \cdots + a'_n \circ x_n + b \circ (a_1 \circ x_1) + \cdots + b \circ (a_n \circ x_n) \\
&= a'_1 \circ x_1 + \cdots + a'_n \circ x_n + (ba_1) \circ x_1 + \cdots + (ba_n) \circ x_n \\
&= (a'_1 + ba_1) \circ x_1 + \cdots + (a'_n + ba_n) \circ x_n \\
&\subseteq SP(x_1, \ldots, x_n).
\end{aligned}$$

Consequently, $SP(x_1, \ldots, x_n, y) \subseteq SP(x_1, \ldots, x_n)$. On the other hand, by Definition 4.1, $SP(x_1, \ldots, x_n) \subseteq SP(x_1, \ldots, x_n, y)$, and so the equality holds.

($\Longrightarrow$) It is clear that $y \in SP(x_1, \ldots, x_n, y)$. Thus $y \in SP(x_1, \ldots, x_n)$. $\square$

Definition 4.6 ([2])**.** A subset S of V is called linearly independent if for every vectors $x_1, \ldots, x_n \in S$ and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ x_1 + \cdots + c_n \circ x_n$ implies that $c_1 = \cdots = c_n = 0$. Note that some hypervector spaces V (some set W of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. S is called linearly dependent if it is not linearly independent. A basis for V is a linearly independent subset of V such that spans V. We say that V has finite dimensional if it has a finite basis. If V has a basis with n vectors, then every basis for V has n vectors. In this case the number n is called the dimension of V, denoted by $\dim V = n$.

Proposition 4.7. *If V is strongly distributive, then for any linearly independent subset $\{x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ of V, the set*

$$S = \left\{ \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \ \cdots \ \underline{0} \end{bmatrix}, \ldots, \begin{bmatrix} \underline{0} \ \cdots \ \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix} \right\}$$

is a basis for the hypervector space $(M^V_{m\times n}, \oplus, \odot, \mathbb{R})$ is defined in Proposition 3.6, and so $\dim M^V_{m\times n} = mn$.

Proof. Let $\begin{bmatrix} x_{11} \cdots x_{1n} \\ \vdots \ \vdots \ \vdots \\ x_{m1} \cdots x_{mn} \end{bmatrix} \in M^V_{m\times n}$. Then

$$\begin{aligned}
\begin{bmatrix} x_{11} \cdots x_{1n} \\ \vdots \ \vdots \ \vdots \\ x_{m1} \cdots x_{mn} \end{bmatrix} &= \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \ \cdots \ \underline{0} \end{bmatrix} + \cdots + \begin{bmatrix} \underline{0} \ \cdots \ \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix} \\
&\in 1 \odot \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \ \cdots \ \underline{0} \end{bmatrix} + \cdots + 1 \odot \begin{bmatrix} \underline{0} \ \cdots \ \underline{0} \\ \vdots \ \vdots \ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix}.
\end{aligned}$$

Thus S spans $M^V_{m\times n}$. Now we show that S is linearly independent. For this, let

$$\begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} \in a_{11} \circ \begin{bmatrix} x_{11} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} + a_{12} \circ \begin{bmatrix} \underline{0} & x_{12} \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} + \cdots + a_{mn} \circ \begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & x_{mn} \end{bmatrix},$$

for some $a_{11}, a_{12}, \ldots, a_{mn} \in K$. Then

$$\begin{aligned} \begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} &\in \left\{ \begin{bmatrix} x^{11}_{11} & \cdots & x^{11}_{1n} \\ \vdots & \vdots & \vdots \\ x^{11}_{m1} & \cdots & x^{11}_{mn} \end{bmatrix} : x^{11}_{11} \in a_{11} \circ x_{11}, x^{11}_{ij} \in a_{11} \circ \underline{0}, ij \neq 11 \right\} + \cdots + \\ &\quad + \left\{ \begin{bmatrix} x^{mn}_{11} & \cdots & x^{mn}_{1n} \\ \vdots & \vdots & \vdots \\ x^{mn}_{m1} & \cdots & x^{mn}_{mn} \end{bmatrix} : x^{mn}_{mn} \in a_{mn} \circ x_{mn}, x^{mn}_{ij} \in a_{mn} \circ \underline{0}, ij \neq mn \right\} \\ &= \left\{ \begin{bmatrix} \sum_{i=1}^{mn} x^i_{11} & \cdots & \sum_{i=1}^{mn} x^i_{1n} \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{mn} x^i_{m1} & \cdots & \sum_{i=1}^{mn} x^i_{mn} \end{bmatrix} : x^{rs}_{ij} \in \begin{cases} a_{ij} \circ x_{ij} & rs = ij \\ a_{ij} \circ \underline{0} & rs \neq ij \end{cases} \right\}. \end{aligned}$$

Thus $\underline{0} \in \sum_{i=1}^{mn} x^i_{11}, \ldots, \underline{0} \in \sum_{i=1}^{mn} x^i_{1n}, \ldots, \underline{0} \in \sum_{i=1}^{mn} x^i_{m1}, \ldots, \underline{0} \in \sum_{i=1}^{mn} x^i_{mn}$. Hence

$$\begin{cases} \underline{0} \in a_{11} \circ x_{11} + \sum_{i=1}^{mn} a_i \circ \underline{0} \Longrightarrow \underline{0} \in a_{11} \circ x_{11} \\ \underline{0} \in a_{12} \circ x_{12} + \sum_{i=1}^{mn} a_i \circ \underline{0} \Longrightarrow \underline{0} \in a_{12} \circ x_{12} \\ \vdots \\ \underline{0} \in \sum_{i=1}^{mn} a_i \circ \underline{0} + a_{mn} \circ x_{mn} \Longrightarrow \underline{0} \in a_{mn} \circ x_{mn} \end{cases}$$

Consequently, $\underline{0} \in a_{11} \circ x_{11} + a_{12} \circ x_{12} + \cdots + a_{mn} \circ x_{mn}$ and so $a_1 = a_2 = \cdots = a_{mn} = 0$. Therefore S is linearly independent and forms a basis for $M^V_{m\times n}$. □

Lemma 4.8. *Let V be strongly left distributive and $\{x, y, z\}$ be a linearly independent subset of V. Then $\{x + y, y + z, x + z\}$ is a linearly independent subset of V.*

Proof. Let $\underline{0} \in a_1 \circ (x + y) + a_2 \circ (y + z) + a_3(x + z)$, for some $a_1, a_2, a_3 \in K$. Then

$$\begin{aligned} \underline{0} &\in a_1 \circ x + a_1 \circ y + a_2 \circ y + a_2 \circ z + a_3 \circ x + a_3 \circ z \\ &= (a_1 + a_3) \circ x + (a_1 + a_2) \circ y + (a_2 + a_3) \circ z. \end{aligned}$$

Hence $a_1 + a_3 = 0, a_1 + a_2 = 0, a_2 + a_3 = 0$, and so $a_1 = a_2 = a_3 = 0$. □

Proposition 4.9. *Let V be strongly left distributive and x, y be linearly independent vectors of V. If $a_1, a_2, b_1, b_2 \in K$, such that $b_1 \neq 0, b_2 \neq 0$ and $a_1b_2 - b_1a_2 \neq 0$, then for all $t_1 \in a_1 \circ x, t_2 \in a_2 \circ y, s_1 \in b_1 \circ x$ and $s_2 \in b_2 \circ y$, the vectors $t_1 + t_2$ and $s_1 + s_2$ are linearly independent in V.*

Proof. Let $\underline{0} \in c \circ (t_1 + t_2) + d \circ (s_1 + s_2)$, for some $c, d \in K$. Then $\underline{0} \in c\circ(a_1\circ x+a_2\circ y)+d\circ(b_1\circ x+b_2\circ y) \subseteq c\circ(a_1\circ x)+c\circ(a_2\circ y)+d\circ(b_1\circ x)+d\circ(b_2\circ y) = (ca_1)\circ x+(ca_2)\circ y+(db_1)\circ x+(db_2)\circ y = (ca_1+db_1)\circ x+(ca_2+db_2)\circ y$. Thus $ca_1 + db_1 = 0, ca_2 + db_2 = 0$. Hence $c = d = 0$. Therefore $t_1 + t_2$ and $s_1 + s_2$ are linearly independent. □

Proposition 4.10. *If V is strongly left distributive, then any subset of V containing zero is linearly dependent.*

Proof. Let $H = \{\underline{0}, x_1, x_2, \ldots, x_n\} \subseteq V$. Then $\underline{0} \in 1\circ\underline{0}+0\circ x_1+0\circ x_2+\cdots+0\circ x_n$. Thus H is linearly dependent. □

Theorem 4.11. *Let V be anti-left distributive. Then $\beta = \{x_1, \ldots, x_n\}$ is a basis for V if and only if every element $x \in V$ belongs to a unique sum in the form $c_1 \circ x_1 + \cdots + c_n \circ x_n$, with $c_i \in K$.*

Proof. ($\Longrightarrow$) [2, Lemma 3.4.]
($\Longleftarrow$) By hypothesis β spans V. Now let $\underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_1, \ldots, a_n \in K$. By Lemma 2.5, $0 \circ x_i \leqslant V, 1 \le i \le n$. So $\underline{0} \in 0 \circ x_i, 1 \le i \le n$. Hence $\underline{0} \in 0\circ x_1+\cdots+0\circ x_n$. From uniqueness it follows that $a_i = 0, 1 \le i \le n$. Therefore β is linearly independent and consequently it is a basis for V. □

Proposition 4.12. *If $(V, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space over the field $\mathbb{R}$ and $\{x, y, z\}$ is a basis for V, then the set $\{x + y, y + z, x + z\}$ is another basis for V.*

Proof. By Lemma 4.8, the set $\{x+y, y+z, x+z\}$ is linearly independent. Now let $w \in V$. Then there exist $a, b, c \in \mathbb{R}$, such that $w \in a\circ x+b\circ y+c\circ z$. Suppose

$$a' = \frac{a-b+c}{2}, b' = \frac{c-a+b}{2}, c' = \frac{b-c+a}{2}.$$

Then it is easy to verify that $w \in a'\circ(x+y)+b'\circ(y+z)+c'\circ(x+z)$. Therefore $\{x + y, y + z, x + z\}$ spans V and so it is a basis for V. □

Theorem 4.13. *Let V be invertible and H be a subhyperspace of V with basis β. Then $\beta\cup\{y\}$ is linearly independent, for all $y \in V\setminus H$, such that $0\circ y = \{\underline{0}\}$.*

Proof. Let $\beta = \{x_1, \ldots, x_n\}$ and $\beta \cup \{y\}$ be linearly dependent. Then $\underline{0} \in a'_1 \circ x_1 + \cdots + a'_n \circ x_n + b \circ y$, for some $a'_1, \ldots, a'_n, b \in K$, such that at least one of the coefficients is nonzero. Thus $\underline{0} = t_1 + \cdots + t_n + c$, for some $t_i \in a_i \circ x_i, c \in b \circ y$. Now if $b \neq 0$, then $y \in b^{-1} \circ c$. Hence $y \in b^{-1} \circ (-t_1 - \cdots - t_n) \subseteq b^{-1} \circ (-t_1) + \cdots + b^{-1} \circ (-t_n) \subseteq b^{-1} \circ (-a'_1 \circ x_1) + \cdots + b^{-1} \circ (-a'_n \circ x_n) = (-b^{-1}a'_1) \circ x_1 + \cdots + (-b^{-1}a'_n) \circ x_n \subseteq H$. Which is contradiction. Also if $b = 0$, then $\underline{0} \in a'_1 \circ x_1 + \cdots + a'_n \circ x_n + 0 \circ y$, such that at least one of a'_i's is nonzero. Thus $\underline{0} \in a'_1\circ x_1+\cdots+a'_n\circ x_n$, Which is a contradiction, too. Therefore $\beta\cup\{y\}$ is linearly independent. □

Theorem 4.14 ([2]). *Let V be strongly left distributive and invertible. If V has a finite basis with n elements, then every linearly independent subset of V has no more than n elements.*

Theorem 4.15. *Let V be strongly left distributive and invertible such that* $\dim V = n$ *and* $0 \circ y = \{\underline{0}\}$*, for all* $y \in V$*. Then any linearly independent subset S of V with n vectors is a basis for V.*

Proof. Let $S = \{x_1, \ldots, x_n\}$ and $H = SP(S)$. If $H \neq V$, then there exists $y \in V \setminus H$. Thus by Theorem 4.13, $\{x_1, \ldots, x_n, y\}$ is linearly independent with $n+1$ elements, which is in contradiction with the Theorem 4.14. Therefore $H = V$ and so $\{x_1, \ldots, x_n\}$ is a basis for V. $\square$

Theorem 4.16. *Let X be a finite spanning set for V. Then $X \cup \{y\}$ is linearly dependent, for any $y \in V \setminus X$.*

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $y \in V \setminus X$. Then $y \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_1, \ldots, a_n \in K$. Thus $y = t_1 + \cdots + t_n$, for some $t_i \in a_i \circ x_i$, $1 \leq i \leq n$. So $\underline{0} = y - t_1 - \cdots - t_n \in 1 \circ y - a_1 \circ x_1 - \cdots - a_n \circ x_n$. Therefore $\{x_1, \ldots, x_n, y\}$ is linearly dependent. $\square$

Proposition 4.17. *Let V be invertible and $0 \circ y = \{\underline{0}\}$, for all $y \in V$. If the set $\{x_1, \ldots, x_n\}$ is linearly independent in V, such that $\{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$ is linearly dependent, for any $m > n$, then $\{x_1, \ldots, x_n\}$ is a basis for V.*

Proof. Let $H = SP(x_1, \ldots, x_n)$ and $V \neq H$. Then there exists $y \in V \setminus H$ such that by Theorem 4.13, the set $\{x_1, \ldots, x_n, y\}$ is linearly independent, which is a contradiction. Therefore $V = H$. $\square$

Proposition 4.18. *Let W_1 and W_2 be strongly left distributive and invertible subhyperspaces of V such that $W_1 \subseteq W_2$ and* $\dim W_1 = \dim W_2$*. Then $W_1 = W_2$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis for W_1 and $W_1 \neq W_2$. Then there exists $y \in W_2 \setminus W_1$. Thus by Theorem 4.13, $\{x_1, \ldots, x_n, y\}$ is a linearly independent subset of W_2 with $n+1$ vectors, which is in contradiction with the Theorem 4.14. Therefore $W_1 = W_2$. $\square$

Proposition 4.19. *Let V be strongly left distributive and invertible and $\{x_1, \ldots, x_n\}$ be a linearly independent subset of V. If $x \in V$ such that $0 \circ x = \{\underline{0}\}$ and $x \notin SP(x_1, \ldots, x_n)$, then $\{x_1 + x, \ldots, x_n + x\}$ is linearly independent in V.*

Proof. Let $\underline{0} \in a_1 \circ (x_1 + x) + \cdots + a_n \circ (x_n + x)$, for some $a_1, \ldots, a_n \in K$. Then $\underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n + (a_1 + \cdots + a_n) \circ x$. So $\underline{0} = t_1 + \cdots + t_n + b$, for some $t_i \in a_i \circ x_i, 1 \leq i \leq n, b \in (a_1 + \cdots + a_n) \circ x$. Now if $a_1 + \cdots + a_n \neq 0$, then $x \in (a_1 + \cdots + a_n)^{-1} \circ b = (a_1 + \cdots + a_n)^{-1} \circ (-t_1 - \cdots - t_n) \subseteq (a_1 + \cdots + a_n)^{-1} \circ (-a_1 \circ x_1 - \cdots - a_n \circ x_n) \subseteq \big(-a_1(a_1 + \cdots + a_n)^{-1}\big) \circ x_1 + \cdots + \big(-a_n(a_1 + \cdots + a_n)^{-1}\big) \circ x_n \subseteq SP(x_1, \ldots, x_n)$. Which is a contradiction. Also if $a_1 + \cdots + a_n = 0$, and $a_j \neq 0$ for some $1 \leq j \leq n$, then $\underline{0} \in a_1 \circ x_1 + \cdots + a_j \circ x_j + \cdots + a_n \circ x_n + 0 \circ x = a_1 \circ x_1 + \cdots + a_j \circ x_j + \cdots + a_n \circ x_n$. Which is a contradiction, too. Thus $a_1 + \cdots + a_n = 0$ and $a_i = 0$ for all $1 \leq i \leq n$, which it means that $\{x_1 + x, \ldots, x_n + x\}$ is linearly independent in V. □

Definition 4.20. Let V be anti-left distributive and finite dimensional with order basis $\beta = \{x_1, \ldots, x_n\}$. (The basis β is called ordered basis, if the order of it's vectors is important). Then by Theorem 4.11, every vector $x \in V$ belongs to a unique sum in the form $a_1 \circ x_1 + \cdots + a_n \circ x_n$, with $a_i \in K$. The scalars $a_1, \ldots, a_n$ are called the coordinates of x relative to the basis β. The coordinate matrix (or coordinate vector) of x relative to β is the column matrix in K^n whose components are the coordinates of x, i.e.

$$[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

It is clear that the coordinate matrix of x relative to β is unique.

Theorem 4.21. *Let V be strongly left distributive and finite dimensional. Let $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{\acute{x}_1, \ldots, \acute{x}_n\}$ be two ordered basis for V, such that*

$$\forall 1 \leq i \leq n,\ [\acute{x}_i]_\beta = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{1n} \end{bmatrix}.$$

Suppose $P = [a_{ij}]_{n\times n}$. Then $[x]_\beta = P\,[x]_{\acute{\beta}}$, for all $x \in V$.

Proof. Let $x \in V$ and

$$[x]_{\acute{\beta}} = \begin{bmatrix} \acute{b}_1 \\ \vdots \\ \acute{b}_n \end{bmatrix}.$$

Then

$$x \in \sum_{i=1}^{n} \acute{b}_i \circ \acute{x}_i \subseteq \sum_{i=1}^{n} \acute{b}_i \circ \left(\sum_{j=1}^{n} a_{ij} \circ x_i \right) \subseteq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b'_j \circ x_i.$$

Thus

$$[x]_\beta = \begin{bmatrix} \sum\limits_{j=1}^{n} a_{1j}\acute{b}_j \\ \vdots \\ \sum\limits_{j=1}^{n} a_{nj}\acute{b}_j \end{bmatrix} = P \begin{bmatrix} \acute{b}_1 \\ \vdots \\ \acute{b}_n \end{bmatrix} = P[x]_{\acute{\beta}}.$$

□

Remark 4.22. The matrix P in Theorem 4.21, is called the transitive matrix from basis $\acute{\beta}$ to basis β.

5. Linear transformations between hypervector spaces

In this section we investigate the concept of linear transformation between two hypervector spaces. Let V and W be hypervector spaces over the field K. Then a function $T : V \longrightarrow W$ is called

1. linear transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) \subseteq a \circ T(x), \forall x, y \in V, a \in K$,
2. good linear transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) = a \circ T(x), \forall x, y \in V, a \in K$.

The kernel of linear transformation $T : V \longrightarrow W$ is denoted by $\ker T$ and defined by $\ker T = \{x \in V : T(x) \in \Omega_V\}$.

Proposition 5.1. *Let $T : V \longrightarrow W$ be a good linear transformation. Then*

$$T(\sum_{i=1}^{n} a_i \circ x_i) = \sum_{i=1}^{n} a_i \circ T(x_i).$$

Proof. Straightforward. □

Proposition 5.2. *Let $(V^m, \oplus_m, \odot_m, K)$ and $(V^n, \oplus_n, \odot_n, K)$ be two strongly distributive hypervector spaces is defined in Proposition 3.4, such that $m < n$. Then $T : V^m \longrightarrow V^n$ with the following rule is a good linear transformation.*

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ \vdots \\ x_{m-1} + x_m \\ x_1 + x_2 \\ \vdots \\ x_1 + x_2 \end{bmatrix}$$

Proof. Straightforward. □

Example 5.3 ([2])**.** Let $(V, +, ., K)$ be a classical vector space, P be a subspace of V and the external hyperoperation $\circ : K \times V \longrightarrow P_*(V)$ is defined by $a \circ x = a.x + P$, for all $a \in K, x \in V$. Then $(V, +, \circ, K)$ is a strongly distributive hypervector space.

Example 5.4. Let $(\mathbb{R}^2, +, \cdot, \mathbb{R})$ be the classical vector space and $(\mathbb{R}^2, +, \circ, \mathbb{R})$ be the strongly distributive hypervector space is constructed in Example 5.3, with $P = \mathbb{R} \times \{0\}$. Then

$$\left[\begin{array}{ll} T: & \left((\mathbb{R}^2)^2, \oplus_2, \odot_2, \mathbb{R}\right) \longrightarrow \left((\mathbb{R}^2)^3, \oplus_3, \odot_3, \mathbb{R}\right) \\ & T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ x+y \end{bmatrix}. \end{array}\right.$$

is a linear transformation, where $((\mathbb{R}^2)^2, \oplus_2, \odot_2, \mathbb{R})$ and $((\mathbb{R}^2)^3, \oplus_3, \odot_3, \mathbb{R})$ are the hypervector spaces defined in Proposition 3.4.

Proposition 5.5. *Let $T : V \longrightarrow W$ be a linear transformation and $x_1, \ldots, x_n \in V$ such that $T(x_1), \ldots, T(x_n)$ be linearly independent in W. Then $x_1, \ldots, x_n$ are linearly independent in V.*

Proof. Let $\underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_i \in K$. Then $\underline{0} = T(\underline{0}) \in T(a_1 \circ x_1 + \cdots + a_n \circ x_n) \subseteq a_1 \circ T(x_1) + \cdots + a_n \circ T(x_n)$. Thus $a_1 = \cdots = a_n = 0$, and so $x_1, \ldots, x_n$ are linearly independent. $\square$

Theorem 5.6. *Let W be a hypervector space such that $\Omega_W = \{\underline{0}_W\}$ and $T : V \longrightarrow W$ be a linear transformation. Then T is injective if and only if* $\ker T = \{\underline{0}_V\}$.

Proof. Let T be injective and $x \in \ker T$. Then $T(x) \in \Omega_W = \{\underline{0}\}$. Thus $T(x) = \underline{0} = T(\underline{0})$, and So $x = \underline{0}$. Hence $\ker T \subseteq \{\underline{0}\}$. On the other hand, $\underline{0}_V \in \ker T$, because $T(\underline{0}_V) = \underline{0}_W \in \Omega_W$. Therefore $\ker T = \{\underline{0}_V\}$. Conversely, let $\ker T = \{\underline{0}\}$ and $x_1, x_2 \in V$, such that $T(x_1) = T(x_2)$. Then $T(x_1 - x_2) = T(x_1) - T(x_2) = \underline{0} \in \Omega_w$. Thus $x_1 - x_2 \in \ker T$, and so $x_1 = x_2$. Consequently T is injective. $\square$

Theorem 5.7 ([2]). *Let V be strongly left distributive, invertible and finite dimensional. If W is a subhyperspace of V, then* $\dim W \leq \dim V$ *and* $\dim V/W = \dim V - \dim W$, *where the external hyperoperation $\star : K \times V/W \longrightarrow P_*(V/W)$ is defined by $a \star (v + W) = a \circ v + W$.*

Theorem 5.8 ([2]). *Let V and W be strongly left distributive hypervector spaces over the field K, and $T : V \longrightarrow W$ be a linear transformation. Then*

$$\frac{V}{\ker T} \cong \frac{T(V)}{\Omega_W}.$$

Proposition 5.9 ([10]). *(i) If V is strongly left distributive, then* $\dim \Omega_V = 0$ *and $SP(\emptyset) = \Omega_V$.*

(ii) If $W \leqslant V$ such that $\dim W = 0$, *then $SP(\emptyset) = W$.*

Theorem 5.10. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces. If $T : V \longrightarrow W$ is a linear transformation, then*

$$\dim \ker T + \dim T(V) = \dim V.$$

Proof. By Theorem 5.8, $\frac{V}{\ker T} \cong \frac{T(V)}{\Omega}$. Thus $\dim \frac{V}{\ker T} = \dim \frac{T(V)}{\Omega}$. Hence by Theorem 5.7, $\dim V - \dim \ker T = \dim T(V) - \dim \Omega$. Then by Proposition 5.9, $\dim V - \dim \ker T = \dim T(V)$. Therefore $\dim \ker T + \dim T(V) = \dim V$. $\square$

Corollary 5.11. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces such that* $\dim W < \dim V$. *If $T : V \longrightarrow W$ is a linear transformation, then* $\dim \ker T > 0$.

Proof. It is clear that $T(V) \leqslant W$, so by Theorem 5.7, $\dim T(V) \leq \dim W$. Hence $\dim T(V) < \dim V$. Thus by Theorem 5.10, it follows that $\dim \ker T = \dim V - \dim T(V) > 0$. □

Corollary 5.12. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces, such that $\dim V = \dim W$, $\Omega_V = \{\underline{0}_V\}$ and $\Omega_W = \{\underline{0}_W\}$. If $T : V \longrightarrow W$ is a linear transformation, then T is injective if and only if T is surjective.*

Proof. Let T be surjective. Then $T(V) = W$ and so by Theorem 5.10, $\dim \ker T = \dim V - \dim T(V) = \dim V - \dim W = 0$. Thus by Proposition 5.9, $\ker T = SP(\emptyset) = \Omega_V = \{\underline{0}_V\}$. Hence by Theorem 5.6, T is injective. Conversely, if T is injective, then by Theorem 5.6, $\ker T = \{\underline{0}_V\}$. Thus by Proposition 5.9, $\ker T = \Omega_V = SP(\emptyset)$ and so $\dim \ker T = 0$. Thus by Theorem 5.10, $\dim T(V) = \dim V = \dim W$. Hence by Proposition 4.18, $T(V) = W$. □

Corollary 5.13. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces, such that $\dim V = \dim W$, $\Omega_V = \{\underline{0}_V\}$ and $\Omega_w = \{\underline{0}_W\}$. If $T : V \longrightarrow W$ is a linear transformation, then the followings are equivalent:*

1. *T is isomorphism;*
2. *T is injective;*
3. *T is surjective.*

Theorem 5.14. *Let V and W be hypervector spaces and $T : V \longrightarrow W$ be an isomorphism (i.e. T is a good linear transformation, which is injective and surjective). Then the followings hold:*

1. *If $\{x_1, \dots, x_n\}$ is linearly independent in V, then $\{T(x_1), \dots, T(x_n)\}$ is linearly independent in W.*
2. *If $\{x_1, \dots, x_n\}$ spans V, then $\{T(x_1), \dots, T(x_n)\}$ spans W.*
3. *If $\{x_1, \dots, x_n\}$ is a basis for V, then $\{T(x_1), \dots, T(x_n)\}$ is a basis for W.*
4. $\dim V = \dim W$.

Proof. 1. Let $\underline{0} \in a_1 \circ T(x_1) + \cdots + a_n \circ T(x_n)$, for some $a_i \in K$. Then $\underline{0} = T(\underline{0}) \in T(a_1 \circ x_1 + \cdots + a_n \circ x_n)$ and so $\underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n$. Hence $a_i = 0$, $1 \leq i \leq n$. Therefore $\{T(x_1), \dots, T(x_n)\}$ is linearly independent.

2. Let $y \in W$. Then $y = T(x)$, for some $x \in V$. Thus $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_i \in K$. Hence $y = T(x) \in T(a_1 \circ x_1 + \cdots + a_n \circ x_n) = a_1 \circ T(x_1) + \cdots + a_n \circ T(x_n)$. Therefore $\{T(x_1), \dots, T(x_n)\}$ spans W.

3. It is obtained from (1) and (2).

4. It is obtained from (3).

□

Theorem 5.15. *Let V and W be finite dimensional hypervector spaces with ordered bases $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{y_1, \ldots, y_m\}$, respectively. If $T : V \to W$ is a linear transformation such that*

$$\forall 1 \leq j \leq n,\ [T(x_j)]_{\acute{\beta}} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Then the $m \times n$ matrix $A_T = [a_{ij}]$ is such that $\forall x \in V$, $[T(x)]_{\acute{\beta}} = A_T\, [x]_\beta$.

Proof. Proof is similar to the proof of Theorem 4.21. □

Remark 5.16. The matrix A_T in Theorem 5.15, is called the matrix of T relative to the bases β and $\acute{\beta}$.

Proposition 5.17 ([7])**.** *Let V and W be hypervector spaces over the field $\mathbb{R}$. Assume that $L(V, W)$ denotes the set of all good linear transformations from V to W. For every $T, S \in L(V, W), a \in \mathbb{R}$, and $x \in V$, suppose that:*

i $(T + S)(x) = T(x) + S(x)$,

ii $a \odot T = \{T' \in L(V, W) : T'(x) \in T(a \circ x), for\ every\ x \in V\}$.

Then $(L(V, W), +, \odot, \mathbb{R})$ is a hypervector space over the field $\mathbb{R}$. If V and W are strongly left distributive, then $L(V, W)$ is strongly left distributive, too.

Theorem 5.18. *Let V and W be hypervector spaces over the field $\mathbb{R}$, with bases $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{y_1, \ldots, y_m\}$, respectively. Let W be strongly left distributive, $a \circ \underline{0} = \{\underline{0}_W\}$, for all $a \in \mathbb{R}$ and $0 \circ y = \{\underline{0}_W\}$, for all $y \in W$. Then the mapping*

$$\left[\begin{array}{c} G : L(V, W) \longrightarrow M_{m\times n} \\ T \longmapsto A_T \end{array} \right.$$

is an injective and good linear transformation, where $M_{m\times n} = (M_{m\times n}, +, ., \mathbb{R})$ is the ordinary vector space of matrices and A_T is the matrix of T relative to the bases β and $\acute{\beta}$.

Proof. Let $T, S \in L(V, W)$ and

$$\forall 1 \leq j \leq n, \quad [T(x_j)]_{\acute{\beta}} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad [S(x_j)]_{\beta'} = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix}.$$

Then $G(T) = A_T = [a_{ij}]_{mn}$ and $G(S) = A_S = [b_{ij}]_{mn}$. Also,

$$(T + S)(x_j) = T(x_j) + S(x_j) \in \sum_{i=1}^{m} a_{ij} \circ y_i + \sum_{i=1}^{m} b_{ij} \circ y_i = \sum_{i=1}^{m} (a_{ij} + b_{ij}) \circ y_i.$$

Hence

$$G(T+S) = A_{T+S} = [a_{ij} + b_{ij}]_{mn} = [a_{ij}]_{mn} + [b_{ij}]_{mn} = A_T + A_S = G(T) + G(S).$$

Therefore $G(T+S) = G(T) + G(S)$. Now if $a \in K$ and $T \in L(V, W)$, then

$$\begin{aligned} G(a \odot T) &= \{G(S) : S \in a \odot T\} = \{G(S) : \forall x \in V,\ S(x) \in a \circ T(x)\} \\ &= \{A_S : \forall x \in V,\ S(x) \in a \circ T(x)\}, \end{aligned}$$

such that for all $A_S \in G(a \odot T)$ and $1 \leq j \leq n$,

$$\begin{aligned} S(x_j) &\in a \circ T(x_j) \subseteq a \circ (a_{1j} \circ y_1 + \cdots + a_{mj} \circ y_m) \\ &\subseteq aa_{1j} \circ y_1 + \cdots + aa_{mj} \circ y_m, \end{aligned}$$

so

$$[S(x_j)]_{\acute{\beta}} = \begin{bmatrix} aa_{1j} \\ \vdots \\ aa_{mj} \end{bmatrix}.$$

Thus $A_S = [aa_{ij}]_{mn} = a \cdot [a_{ij}]_{mn} = a \cdot A_T = a \cdot G(T)$. Therefore $G(a \odot T) = a \cdot G(T)$. Consequently G is a good linear transformation. Now let T be a good linear transformation such that $G(T) = A_T = [0]_{mn}$. Then $T(x_j) \in 0 \circ y_1 + \cdots + 0 \circ y_n = \{\underline{0}\}$, for all $1 \leq j \leq n$. Thus $T = 0$, because $a \circ \underline{0} = \{\underline{0}_W\}$, for all $a \in K$. Therefore G is injective. $\square$

Corollary 5.19. *Let V and W be hypervector spaces over the field $\mathbb{R}$, with bases $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{y_1, \ldots, y_m\}$, respectively. Let W be strongly left distributive, $a \circ \underline{0} = \{\underline{0}_W\}$, for all $a \in \mathbb{R}$ and $0 \circ y = \{\underline{0}_W\}$, for all $y \in W$. Then $L(V, W) \cong M_{m \times n}$.*

Proof. Let G be the good linear transformation defined in Theorem 5.18. Then By Theorem 5.8,

$$\frac{L(V, W)}{\ker G} \cong \frac{M_{m \times n}}{\Omega_{M_{m \times n}}}.$$

It is easy to verify that $\Omega_{M_{m \times n}} = \{0_{M_{m \times n}}\}$. Also by Theorem 5.6, $\ker G = \{\underline{0}\}$. Therefore $L(V, W) \cong M_{m \times n}$. $\square$

Corollary 5.20. *Let V and W be hypervector spaces over the field $\mathbb{R}$, with bases $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{y_1, \ldots, y_m\}$, respectively. Let W be strongly left distributive, $a \circ \underline{0} = \{\underline{0}_W\}$, for all $a \in \mathbb{R}$ and $0 \circ y = \{\underline{0}_W\}$, for all $y \in W$. Then* $\dim L(V, W) = \dim V \times \dim W$.

Proof. By Corollary 5.19, $L(V, W) \cong M_{m \times n}$. Thus by Theorem 5.14,

$$\dim L(V, W) = \dim M_{m \times n} = m \times n = \dim V \times \dim W.$$

$\square$

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NEW IMPROVEMENT OF HEINZ INEQUALITIES FOR MATRICES

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Abstract. In this paper we mainly discuss Heinz inequalities involving unitarily invariant norms. By using the Hermite-Hadamard inequality, we get some refinements of the Heinz inequalities, thus new improvements of Heinz Inequalities for matrices are obtained. Our results are different from those of [7, 1, 3, 6, 8].
Keywords: refinements, Heinz inequality, convex function, Hermite-Hadamard inequality, unitarily invariant norm.

1. Introduction

On a complex separable Hilbert space, let A and B are positive operations, X is an operator, then the function

$$f(v) = \left|\left|\left|A^vXB^{1-v} + A^{1-v}XB^v\right|\right|\right|,$$

where $|||\cdot|||$ denotes unitarily invariant norm, is convex on the interval $[0, 1]$. $f(v)$ get its minimum at $v = \frac{1}{2}$, and gets its maximum at $v = 0$ and $v = 1$. Moreover, $f(v) = f(1 - v)$ for $0 \leq v \leq 1$. Thus , from [4] we know that the Heinz inequalities are valid

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leq \left|\left|\left|A^vXB^{1-v} + A^{1-v}XB^v\right|\right|\right| \leq |||AX + XB|||. \tag{1.1}$$

The convexity of the function

$$f(v) = \left|\left|\left|A^vXB^{1-v} + A^{1-v}XB^v\right|\right|\right|$$

on $[0, 1]$ is obvious.

Similar to the methods of [5] we begin our main results.

2. Main results

A real-valued function f on the interval $[a,b]$ is convex, if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2},$$

where $x, y \in [a,b]$.

Let f be a convex real-valued function on the interval $[a,b]$, then the Hermite-Hadamard integral inequality [2] is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

In [5], Kittaneh gave several refinements of the Heinz inequalities by using the Hermite-Hadamard integral inequality. Feng improved the results of [5] in the paper [3]. Wang [6] also improved the results of Kittaneh. In [8], Yan and Feng refined Kittaneh again. Abbas and Mourad [1] used a parameter to generalize the results of [3, 6, 8]. Xue [7] also generalized the results and gave a formula shown as follows,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{2n}\left((n-1)f(a)+2f\left(\frac{a+b}{2}\right)+(n-1)f(b)\right) \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $n \geq 2$ is an integer. It is easy to know that results of [7] is more general than of [1]. Other kinds of improvement of Heinz Inequalitie can be seen, for example , see [9].

In this paper, we give other refinements of Heinz inequalities for matrices which are different from [5, 3, 6, 8, 7]. We will use the following lemma to obtain several better improved Heinz inequalities.

Let's first give a lemma.

Lemma 1. *If f is a convex real-valued function on the interval $[a,b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{12}\left(f(a)+10f\left(\frac{a+b}{2}\right)+f(b)\right) \\ &\leq \frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right) \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

Proof. From the Hermite-Hadamard integral inequality we know

$$2f\left(\frac{a+b}{2}\right) \leq f(a)+f(b).$$

So

$$12f\left(\frac{a+b}{2}\right) \le f(a) + 10f\left(\frac{a+b}{2}\right) + f(b).$$

Thus

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right).$$

Next, the following inequality will be proved.

$$\frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right).$$

From

$$2f\left(\frac{a+b}{2}\right) \le f(a) + f(b),$$

we conclude that

$$f(a) + 10f\left(\frac{a+b}{2}\right) + f(b) \le 3f(a) + 6f\left(\frac{a+b}{2}\right) + 3f(b).$$

And so

$$\frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right).$$

From

$$f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \le 2f(a) + 2f(b)$$

we get that

$$\frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

So, we finish the proof. □

Set $f(v) = \left|\left|\left|A^v X B^{1-v} + A^{1-v} X B^v\right|\right|\right|$, then the following theorems can be obtained.

Theorem 1. *Let A, B are positive operators, X be an operator, then for $0 \le \mu \le 1$, we obtain*

$$\begin{aligned}
&2\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right| \\
&\le \frac{1}{6}\left(\left|\left|\left|A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu\right|\right|\right| + 5\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right|\right) \\
&\le \frac{1}{2}\left(\left|\left|\left|A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu\right|\right|\right| + 2\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right|\right) \\
&\le \left|\left|\left|A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu\right|\right|\right|.
\end{aligned} \tag{2.1}$$

Proof. Applying the previous lemma on $f(v)$ on the interval $[\mu, 1-\mu]$ when $0 \le \mu \le \frac{1}{2}$, then

$$\begin{aligned} f\left(\frac{1-\mu+\mu}{2}\right) &\le \frac{1}{12}\left(f(\mu)+10f\left(\frac{1-\mu+\mu}{2}\right)+f(1-\mu)\right) \\ &\le \frac{1}{4}\left(f(\mu)+2f\left(\frac{1-\mu+\mu}{2}\right)+f(1-\mu)\right) \\ &\le \frac{f(\mu)+f(1-\mu)}{2}, \end{aligned}$$

and thus

$$f\left(\frac{1}{2}\right) \le \frac{1}{6}\left(f(\mu)+5f\left(\frac{1}{2}\right)\right) \le \frac{1}{2}\left(f(\mu)+f\left(\frac{1}{2}\right)\right) \le f(\mu).$$

So,

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{6}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+5\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\,. \end{aligned} \tag{2.2}$$

Applying the previous lemma on $f(v)$ on the interval $[1-\mu, \mu]$ when $\frac{1}{2} \le \mu \le 1$, then

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{6}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+5\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\,. \end{aligned} \tag{2.3}$$

By combining (2.2) and (2.3) we obtain (2.1). $\square$

Theorem 2. *Let A, B are positive operators, X be an operator, then for $0 \le \mu \le 1$, we have*

$$\begin{aligned} &\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \le \frac{1}{12}\Big(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||| \\ &+10\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\Big) \\ &\le \frac{1}{4}\Big(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||| \\ &+2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\Big) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right). \end{aligned} \tag{2.4}$$

Proof. Applying the lemma to the function $f(v)$ on $[\mu, \frac{1}{2}]$, where $0 \le \mu \le \frac{1}{2}$, and on the $[\frac{1}{2}, \mu]$, where $\frac{1}{2} \le \mu \le 1$, respectively, we can finish the proof. $\square$

From (2.4) and the first part of (1.1) we obtain the refinement of first part of (1.1).

Corollary 1. *Let A, B are positive operators, X be an operator, then for $0 \leq \mu \leq 1$, we get*

$$\begin{aligned}
&2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leq \left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right. \\
&\left.+10\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq \frac{1}{4}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right. \\
&\left.+2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.
\end{aligned} \tag{2.5}$$

Applying the lemma to $f(v)$ on $[0,\mu]$, where $0 \leq \mu \leq \frac{1}{2}$, and $[\mu, 1]$, where $\frac{1}{2} \leq \mu \leq 1$, respectively, we get the following theorem.

Theorem 3. *Let A, B are positive operators, X be an operator, then*

(1) *for $0 \leq \mu \leq \frac{1}{2}$, we have*

$$\begin{aligned}
&\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||AX+XB|||+10\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{4}\left(|||AX+XB|||+2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{2}\left(|||AX+XB|||+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right).
\end{aligned} \tag{2.6}$$

(2) for $\frac{1}{2} \leq \mu \leq 1$, we have

$$\begin{aligned}
&\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||AX+XB|||+10\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{4}\left(|||AX+XB|||+2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{2}\left(|||AX+XB|||+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right).
\end{aligned} \tag{2.7}$$

Because on the interval $[0, \frac{1}{2}]$, the function $f(v)$ is decreasing and on the interval $[\frac{1}{2}, 1]$, it is increasing, then by using the inequalities (2.6) and (2.7), we get the refinement of the second inequality in (1.1).

Corollary 2. *Let A, B are positive operators, X be an operator, then for $0 \le \mu \le 1$, we have*

(1) *for $0 \le \mu \le \frac{1}{2}$ and for every unitarily norm,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\\
&\le \left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\\
&\le \frac{1}{12}\left(|||AX + XB||| + 10\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right.\\
&\left. + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right.\\
&\left. + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le |||AX + XB|||.
\end{aligned} \tag{2.8}$$

(2) *for $\frac{1}{2} \le \mu \le 1$,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\\
&\le \left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\\
&\le \frac{1}{12}\left(|||AX + XB||| + 10\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right.\\
&\left. + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{4}\left(|||AX + XB||| + 2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right.\\
&\left. + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right)\\
&\le |||AX + XB|||.
\end{aligned} \tag{2.9}$$

3. Results and discussion

By refining the Hermite-Hadamard Integral Inequality, we get a new refined Heinz Inequalities of matrices.

Can we add a parameter to get a family of refined Heinz Inequalities of matrices as [7] and [1]? We are going to consider this case in next papers.

4. Conclusion

Xue [7], Abbas and Mouradand [1] have generalized the results of [3, 6, 8]. In a different way, this paper gives a new generalization of Heinz Inequalities for matrices.

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RESEARCH ON A CLUSTERING ANALYSIS ALGORITHM FOR OPTIMAL ALLOCATION OF HUMAN RESOURCES IN COLLEGES AND UNIVERSITIES

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Abstract. With the development of digitalization, more and more colleges and universities have applied information management technology into the optimal allocation of human resources. How to manage the growing number of human resource management data and mine their potential laws to realize the optimal allocation of human resources in colleges and universities has become a top priority. In this paper, an improved k-means algorithm was introduced and its superiority was verified through an experiment by comparing the results before and after improvement. The result showed that the calculation time and complexity of the improved algorithm decreased greatly, suggesting that it could be applied for the optimal allocation of human resources in colleges and universities.
Keywords: university human resources, optimal configuration management, k-means algorithm.

Introduction

Nowadays, with the rapid development of computer technology, information management technology has been applied to the university's human resources management, resulting in a large number of data in university databases [1]. Without effective data mining technologies, these data can not play their roles. As an effective data mining algorithm, the clustering analysis algorithm has been studied by scholars all over the world. Xu et al. [2] applied the k-means algorithm based on Web user log data to perform clustering of the Web users, studied their historical Web usage data and behavioral characteristics and found that the algorithm was feasible and effective in data mining and could provide useful knowledge for Web user cluster. RJ Kuo et al. [3] integrated particle swarm optimization algorithm and k-means algorithm to cluster data and found that the particle swarm optimization algorithm could be applied to find the cluster centroid with user specified number. Besides, they used four data sets

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to evaluate the proposed particle swarm optimization algorithm and found that the algorithm had great potential and could complete the mining of data. In this paper, an improved k-means algorithm was introduced and its superiority was verified through an experiment by comparing the results before and after improvement. The result showed that the calculation time and complexity of the improved algorithm decreased greatly, suggesting that it could be applied for the optimal allocation of human resources in colleges and universities, which provided some reference for the application of clustering analysis algorithm in the optimization of human resources in colleges and universities.

1. Clustering analysis algorithm

1.1 Clustering analysis

Clustering analysis refers to the grouping of a collection of physical or abstract objects into a number of classes that consist of similar objects, whose aim is to collect data on a similar basis for classification [4-5]. As there are many types of clustering analysis algorithms, appropriate ones should be chosen based on the specific data type and clustering purpose [6]. The common clustering algorithms are as follows:

1.1.1 Classification method

A database containing j objects is divided into k categories, with each category representing a cluster, where all the objects are similar, $k \leq j$. Two conditions must be met using this method. Firstly, each object can belong to only one group, rather than multiple groups. Secondly, there must be one object in each group. The method is carried out as follows: the group number k is given; iteration positioning is used to move the objects between partitions and divide them; make the objects within the same group to be as similar as possible while those of different groups to be as diverse as possible. Currently, the k-means algorithm is a popular heuristic method, where each group can be represented with the average value of the objects in the group. In this paper, we use the improved k-means algorithm [7] to excavate human resource data.

1.1.2 Hierarchy method

The hierarchy method is to perform hierarchical decomposition of objects, which includes two types. One is the coagulation method, which includes all the similar objects into one group until all the groups are merged into one. The other is the splitting method, which splits s a big group into small groups until each object is in a separate group [8].

1.1.3 Density method

The density method means that the clustering behavior is continued as long as the number of objects within a neighbor region exceeds a threshold value [9].

1.1.4 Grid method

In the grid method, the space where the objects lie is divided into several grids, where clustering of the objects is performed so as to improve the clustering speed. STING is one of the commonly used grid methods [10].

1.2 k-means algorithm

The k-means algorithm divides j objects into k groups based on similarity, with the average value of the objects in each group as its center and significant differences between groups. The detailed algorithm is as follows:

(1) k objects are selected from j objects to be the center of k groups.

(2) Repeat step (1).

(3) According to the distance between each object to the center object of each group, they are divided into corresponding groups.

(4) Recalculate the average value of each group.

(5) Take the renewed average value of each group as new centers.

(6) Repeat step (3) and (4) until the group centers do not change any longer.

Generally, square error criterion is taken as distance calculation function, with its formula as follows:

Where is the average value of group and a is a data in group.

This algorithm calculates the square error of the objects, according to which j objects are divided. If the sum of the squared differences between them is large, group centers must be redefined to continue clustering until the sum of the square errors reaches the minimum. When the data is relatively large, the algorithm can efficiently complete the data mining work. Besides, most cases using the algorithm for data mining are ended with local optimization.

Because the algorithm needs to divide the starting center of each group, it is necessary to determine the number of groups and the initial center of each group firstly. Therefore, the k-means algorithm does not apply to groups where the size of the objects in the database is too different. Also, it is susceptible to isolated point data, which can exert great impact on the clustering analysis results.

1.3 Improved k-means algorithm

In order to avoid the impact of extreme differences, we improved the k-means algorithm by removing x maximum values and x minimum values respectively. The specific algorithm is as follows:

(1) Rank the j objects from large to small, removing x maximum values and x minimum values.

(2) Calculate the average value F of all the remaining $j-2x$ objects and take $(0-2)$ times that of the value of F to be the initial center of each group.

(3) Repeat the above step.

(4) According to the distance between each object and each group center, they are divided into corresponding groups.

(5) Recalculate the average value of each group.

(6) Take the recalculated average value as the new group center.

(7) Repeat step (5) and (6) until the group centers do not change any longer.

2. Clustering analysis of human resource management in colleges and universities

2.1 Application of clustering analysis in post setting in colleges and universities

In this paper, both the k-means algorithm before and after improvement were applied to extract the human resource data of School of Economics and Management, Yanshan University, which was compared to verify the significance of clustering analysis algorithm in human resource data mining. Before the clustering analysis, the preliminary statistics on the human resources data in the university was performed and the mining objects were determined.

This paper studied the university professional and technical personnel database. The class information table and scientific research information table of teachers between 2015-2016 were obtained from the office of academic affairs, including 1,524 records on in-service staff appointment time, standard class hours and student ratings. As the research objects are the teachers, 855 related records were selected while others were omitted. Then, after screening based on the deputy senior title, 192 records were kept for clustering analysis.

2.2 Application of k-means algorithm in human resource management in colleges and universities

In this design, the 192 included records were divided into three groups, i.e., $k=3$. Firstly, 3 objects were taken as the center of clustering. Then, according to Euclidean distance, each object was assigned to the group to which it is close in its average value. Besides, the mean vector of these objects to each cluster point was calculated and the total mean value was used as the center to perform clustering again. All the screened data were stored in an Excel table and the Excel built-in functions were used to perform clustering analysis on the data.

In the table, lines 5 to 196 are the 192 records and lines 2-4 are the new average values. A and B stand for the job number and workload respectively. G5 to G196 are the groups each point was assigned to through calculation. H1, H2 and H3 represent three groups; C1-G1 are numbers of iterations; lines 194-199 are the numbers of objects in each group after each iteration.

	A	B	C	D	E	F	G
1	Job number	workload	1	2	3	4	5
2	k1		69.3	56.25012	57.25348	43.52451	67.25254
3	k2		523.0	553.5045	582.8552	623.5457	687.0125
4	k3		236.0	623.787	785.257	1022.458	1250.852
5	200211004	69.3	H1	H2	H1	H3	H2
6	200411009	523.0	H2	H3	H2	H1	H1
...	...	...	...	...	...	...	...
195	200212007	72.0	H2	H2	H2	H1	H1
196	200421079	359.0	H1	H3	H3	H2	H2
197	Number of samples in group 1		77	65	53	38	58
198	Number of samples in group 2		24	52	86	82	98
199	Number of samples in group 3		91	75	53	72	36

Table 1. Clustering analysis in universities and colleges

2.2.1 Determination of iteration group center

Based on the k-means algorithm, k objects were randomly taken as the initial group centers, which are C2-C4 in table 1. Afterwards, new average values were generated after each iteration as new group centers, noted by D2, whose function is D2 = AVERAGEIF (C 5 : C 196, ”H1”, B 5: B 196). As an average function, AVERAGEIF is mainly used to calculate the average value of multiple table cells. In column D, the function was applied to calculate the average value of the workload data which meet the H1 condition between D5-D196.

2.2.2 Grouping of records after each iteration

In the data area, this paper made an iterative grouping of all the probabilities, and registered the value of each record and the European distance of the center of three groups [11]. According to the distance, they were assigned to the neighbor groups. For example, the function of D5 is: suppose $(\$B5 - D\$2)^2$ to be m, $(\$B5 - D\$3)$ to be n, and $(\$B5 - D\$4)^2$ to be p, then: $D5 = if((m \leq n, ifa \leq c,$ $"H1", "H2"))$, if $(n \leq p, "H2", "H3")$.

Where m refers to the European distance between record D and group 1; n refers to the European distance between record D and group 2; p refers to the European distance between record D and group 3.

When $m \leq n$, if $m \leq p$, then D belongs to group 1, denoted by $H1$; if $m > p$, then D belongs to group 3, denoted by $H3$. When $m > n$, if $n \leq p$, then D belongs to group 2, denoted by $H2$; if $n > p$, then D belongs to group 3, denoted by $H3$.

2.2.3 Calculation of the number of samples in each group

In the data area, the number of samples in each group after each iteration is recorded. For example, the function of C197 is:

C197 = COUNTIF (C5: C197, "H1")

COUNTIF is the function to calculate the number of records which meet the H1 condition.

2.2.4 Iteration

After one iteration is completed, the content in D2-D196 is used to continue the following iterations.

2.2.5 End

The iteration is stopped when the group centers and the number of samples in each group do not change any longer.

2.3 Improved k-means algorithm

As mentioned in section 2.2, we set the number of groups to be 3, i.e., $k = 3$. Then, the 192 records were ranked, the maximum vale 487.0 and minimum value 3.4 were selected and the average value of the remaining samples was calculated, denoted by Q. Afterwards, 0.5Q and 1.5Q were used as the initial center of the clustering and each object was assigned to the group to which it was close in its average value. Finally, the mean vectors of each object to each clustering point were calculated and the whole mean value was taken as the new center to perform the clustering again. The Excel table used in this section was the same as table 1, with the specific steps as follows:

2.3.1 Determination of the iteration group center

Using the improved algorithm, the ranked samples were calculated. After removing the maximum value and the minimum value, the average value Q was calculated. Then, $0.5Q$ and $1.5Q$ was used as the initial group center of the iteration and the value of $C2-4$, with the calculation formula as follows:

$C2 = AVERAGE(B\$6 : B\$195) * 0.5$

$C3 = AVERAGE(B\$6 : B\$195)$

$C4 = AVERAGE(B\$6 : B\$195) * 1.5$

After each iteration, the average value of the group of the previous iteration was calculated. For example, the function of D2 is as follows:

$D2 = AVERAGEIF(C\$5 : C\$196, "H1" \$B\$5 : \$B\$196)$

The following steps were as the same as mentioned in the above sections. When the group centers and the number of samples in groups did not change any longer, the iteration ended.

3. Results

The clustering analysis on the data of scientific research work of university teachers was performed using the above methods, with the results shown in table 2 and 3.

Workload analysis	Group center	Number of samples	Maximum value	Minimum value	Number of iteration
Group 1	186.2568	156	506.3	3.3	26
Group 2	869.3516	30	1615.4	521.6	
Group 3	3658.2615	6	4515.9	2214.3	

Table 2. k-means algorithm analysis results

Workload analysis	Group center	Number of samples	Maximum value	Minimum value	Number of iteration
Group 1	186.2568	156	506.3	3.3	13
Group 2	869.3516	30	1615.4	521.6	
Group 3	3658.2615	6	4515.9	2214.3	

Table 3. Improved k-means algorithm analysis results

As shown in table 2, the center of the three groups was 186.2568, 869.3516 and 3658.2615 respectively, with great differences between groups and small differences within each group. The minimum values of the groups are the boundary points of university teacher recruitment, i.e., the minimum standards of recruitment. Based on the workload, we can assign the recruitment conditions. The results suggest that the k-means algorithm can adapt to the excavation of human resource data in colleges and universities and complete the optimal allocation of human resources.

By comparing table 2 with table 3, it can be seen that the group centers, number of samples in each group, the maximum values and minimum values of groups were the same in the two tables, suggesting that the improved k-means algorithm also adapted to the excavation of human resource data in colleges and universities. Moreover, the number of iteration of the improved algorithm was 13 times less than that before improvement, which reduced the calculation complexity to a large degree. Therefore, the improved k-means algorithm was more convenient and can realize better optimal allocation of university human resources.

4. Conclusion

As one of clustering analysis methods, the k-means algorithm can well complete the clustering analysis of human resource data [12]. HM Hussain et al. [13] proposed a highly parallel hardware design that accelerates the k-means clustering of microarray data by implementing the k-means algorithm in a field programmable gate array. Q Ren et al. [14] improved the k-means algorithm using the kruskai algorithm, and obtained the minimum spanning tree of the clustering object by kruskai algorithm and proved that the improved algorithm was more efficient than the traditional algorithm through an experiment. In this paper, the k-means algorithm before and after improvement were both applied to carry out clustering analysis on university human resource data and the results showed that the improved algorithm was more convenient, which provide references for the application of clustering analysis in the optimization of human resources in colleges and universities.

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THE NUMERICAL RANGE OF AN ELEMENT OF A CLASS OF TOPOLOGICAL ALGEBRAS

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Abstract. Kinani, Oubbi and Oudadess (1998) show that every unital and commutative locally convex algebra with a jointly continuous product is β - subadditive, for which β is the boundedness radius. In this paper we obtain some results on numerical range of an element in β - subadditive algebras. To do this, at first, we study the dual space of topological algebras for which the boundedness radius is finite. Furthermore, we prove some new results for linear and multiplicative linear functionals on a class of topological algebras.
Keywords: multiplicative linear functional, numerical range, boundedness radius, β-subadditive algebras.

1. Introduction

Allan [1] provides the definition of the radius of boundedness β to develop the spectral theory for locally convex topological algebras. After that, the radius of boundedness β is extended for general topological algebras (see for example [11]). T.Husain [9] introduces the concepts of strongly sequential and infrasequential topological algebras and the first author [3] introduces the concept of fundamental topological algebra to generalize the famous Cohen factorization theorem. Kinani, Oubbi and Oudadess [10] show that every unital and

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commutative locally convex algebra with a jointly continuous product is β - subadditive. Also Oubbi [12] extended β - subadditive algebras. In this note, at first we study the dual space of topological algebras for which the boundedness radius is finite; and provide a norm on a subspace of the algebraic dual space of topological algebras. Next, we try to study numerical range in β - subadditive algebras. Here, we have supposed all algebras are complex unital complete metrizable topological algebra.

2. Definitions and related results

At first we begin with the previous definitions and related results.

Definition 2.1. Let x be an element of a topological algebra (A, τ). We will say that x is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of x with respect to (A, τ) is denoted by $\beta(x)$ and defined by

$$\beta(x) = \inf\{r > 0 \ : \ (\frac{x^n}{r^n}) \to 0\} \tag{2.1}$$

with the convention : $\inf \phi = +\infty$. We also say A is a $\beta-$ finite topological algebra if all elements of A are bounded.

Definition 2.2. Let A be a topological algebra.

(i) A is said to be strongly sequential if there exists a neighborhood U of 0 such that for all $x \in U$, $x^n \to 0$ as $n \to \infty$.

(ii) A is said to be infrasequential if for each bounded set $B \subseteq A$ there exists $\lambda > 0$ such that for all $x \epsilon B, (\lambda x)^n \to 0$ as $n \to \infty$.

Proposition 2.3 ([9]). *With reference to the above definitions, (i) implies (ii), and (ii) implies that* β *is finite.*

Definition 2.4 ([3]). A topological algebra A is said to be fundamental if there exists $b > 1$ such that for every sequence (a_n) of A, the convergence of $b^n(a_{n+1} - a_n)$ to zero in A implies that (a_n) is a Cauchy sequence.

Proposition 2.5 ([5]). *Let A be a strongly sequential topological algebra that* $\rho \leq \beta$. *Then A is a $Q-algebra$.*

Corollary 2.6 ([8]). *Let A be a $Q-algebra$. Then every multiplicative linear functional is automatically continuous.*

3. A norm on a subspace of the algebraic dual space of a β-finite topological algebra

Let A be a topological algebra, A' be the space of all linear functionals on A, and $f \in A'$. Define $v(f) = sup\ \{|f(x)| \ : \ \beta(x) < 1\}$. Then we have the following theorem.

Theorem 3.1. *Suppose β is finite. Taking the above notations, one has:*

(i) $v(f) = sup\ \{|f(x)| : \beta(x) \leq 1\} = sup\ \{|f(x)| : x \in \beta^{-1}\{0,1\}\}$.

(ii) If $\beta(x) \neq 0$ then we have

$$|f(x)| \ \leq\ v(f)\ \beta(x) \tag{3.1}$$

and if $\beta(x) = 0$, then we have

$$|f(x)| \ \leq\ v(f). \tag{3.2}$$

(iii) $f = 0$ if and only if $v(f) = 0$.

(iv) If $f, g \in\ A'$, $\alpha\ \in\ \mathbb{C}$ then $v(f+g)\ \leq\ v(f) +\ v(g)$, $v(\alpha f) =\ |\alpha|\ v(f)$.

(v) If $\hat{A}\ = \{f \in\ A'\ :\ v(f) < \infty\}$ then $v(.)$ is a norm on $\hat{A}$ and $(\hat{A}, v(.))$ is a Banach space.

(vi) If $f \in \hat{A}$, $\beta(x) = 0$, then, $f(x) = 0$ and therefore we have:

$$f \in \hat{A}\ \Rightarrow\ (|f(x)|\ \leq\ v(f)\ \beta(x)\ for\ all\ x \in A). \tag{3.3}$$

(vii) If A is an infrasequential algebra, then $\hat{A}\ \subseteq\ A^*$ where A^* is the set of all continuous linear functionals on A.

Proof. (i) Let $r = sup\{|f(x)| : \beta(x) <\ 1\}$, $r' = sup\{|f(x)| : \beta(x) \leq\ 1\}$. Then obviously, we have $r\ \leq\ r'$. If $n\ \in\ \mathbb{N}$, $\beta(x)\ \leq\ 1$ then $\beta((1 - \frac{1}{n})x)\ <\ 1$, that implies $|f((1 - \frac{1}{n})x)| \leq r$, i.e $(1 - \frac{1}{n})|f(x)| \leq r$. Since n is arbitrary, it follows $r' \leq\ r$. On the other hand, if we suppose $r'' =\ sup\ \{|f(x)| : x \in \beta^{-1}\{0,1\}\}$, obviously $r'' \leq\ r'$. If suppose $0 \neq \beta(x) \leq 1$, We have $|f(\frac{x}{\beta(x)})| \leq r''$ that implies $r' \leq r''$.

(ii) Let $x \in A$, $\beta(x) \neq 0$, then $\beta(\frac{x}{\beta(x)}) = 1$, that implies $|f(\frac{x}{\beta(x)})| \leq\ v(f)$ as desired.

(iii) By (3.1)and (3.2) the result is clear.

(iv) It is clear.

(v) It suffices to show that $(\hat{A}, v(.))$ is complete. Let$(f_n)_n$ be a Cauchy sequence in $(\hat{A}, v(.))$ and $x \in A$. By (3.1) and (3.2) , the sequence $(f_n(x))_n$ is a Cauchy sequence of $\mathbb{C}$ and so, there exists a function $f : A \to \mathbb{C}$ such that $f_n(x)\ \to\ f(x)$ for all $x \in A$. Let $\varepsilon >\ 0$, $x \in A$ for which $\beta(x) < 1$. There exists $N \in\ \mathbb{N}$ such that $m, n\ \geq\ N$ implies that $|f_n(x) - f_m(x)|\ \leq\ v(f_n - f_m)\ <\ \frac{\varepsilon}{2}$. Now, we let m tends to infinity, then we have $|f_n(x) - f(x)| < \varepsilon$; therefore, if $n \geq\ N$ then $v(f_n - f)\ <\ \varepsilon$.

(vi) Let $n \in \mathbb{N}$. By (3.2), $|f(nx)|\ \leq\ v(f)$ that implies $|f(x)|\ \leq\ \frac{v(f)}{n}$ for all $n \in \mathbb{N}$; i.e $f(x) = 0$.

(vii) Let $f \in \hat{A}$ and E be a bounded subset of A. Since A is an infrasequential algebra, then there exists $M > 0$ such that $\beta(x) \leq M\ for\ all\ x \in E$. By (3.3) we get $f(E)$ is bounded; that implies $f \in\ A^*$. □

We define $\hat{weak}$-topology on $\hat{A}$, the same as the definition of $weak^*$-topology on A^*.

Definition 3.2. We call the A-topology of $\hat{A}$ the $\hat{weak}$-topology on $\hat{A}$. (pronunciation: weak hat topology).

According to this definition, the next theorem is clear.

Theorem 3.3. *Let A be β-finite. We have:*

(i) The $\hat{weak}$-topology is a locally convex vector topology on $\hat{A}$.

(ii) Every linear functional φ on $\hat{A}$ is $\hat{weak}$-continuous if and only if there exists $a \in A$ such that $\varphi(\Lambda) = \Lambda a$ for every $\Lambda \in \hat{A}$.

Theorem 3.4. *(Banach- Alaoglu theorem for $\hat{weak}$-topology) We suppose A is β-finite and $D(A) = \{x \in A : \beta(x) < 1\}$ if*

$$K = \{\Lambda \in \hat{A} : \ |\Lambda x| \leq 1 \text{ on } D(A)\} \tag{3.4}$$

then $K = \{\Lambda \in \hat{A} : \ v(\Lambda) \leq 1\}$ and K is $\hat{weak}$-compact.

Proof. The proof is similar to ([13], theorem 3.15).

□

Proposition 3.5. *Let A be a strongly sequential algebra for which $\rho \leq \beta$ and Φ_A be the carrier space of A. We have*

(i) If f is multiplicative linear functional, then $v(f) = 1$ and therefore $\Phi_A \subseteq \ \hat{A} \ \subseteq A^*$.

(ii) The carrier space Φ_A is weak*-compact.

(iii) Suppose A is a Banach algebra, then for every continuous linear functional f, $\|f\| \leq v(f)$.

Proof. (i) Let $x \in A$, $\beta(x) < 1$; then there exists $b > 1$ such that $b^n \, x^n \to 0$ and by propsition 2.5 and corollary 2.6, since f is continuous, we have $b^n(f(x))^n = f(b^n x^n) \to 0$. Therefore $|f(x)| < 1$; and since $\beta(1) = 1$, we get the result.

(ii) By (i) and (3.4), it is sufficient to observe that $\Phi_A^\infty = \Phi_A \bigcup \{0\}$ is a $\hat{weak}$-closed subset of $\hat{A}$ (see [7] chapter2, section17, proposition 2).

(iii) It is sufficient to note that, $\beta(x) \leq \|x\|$. □

4. Numerical range on β- subadditive algebras

In this section, moreover, we suppose that every algebra A is β- subadditive and β-finite, that is, $\beta(x+y) \leq \ \beta(x) + \beta(y)$, $\beta(x) < \infty$ for all $x, y \in A$.

Lemma 4.1. *Suppose A is an algebra and $x_0 \in A$. Then there exists $\Lambda \in \hat{A}$ such that $\Lambda(x_0) = \beta(x_0)$ and $|\Lambda(x)| \leq \beta(x)$ for all $x \in A$.*

Proof. We apply the Hahn- Banach theorems where β is indicated seminorm on A. □

In [7 section 10] a suitable discussion is given on the numerical range of elements of normed algebras. Here we use the similar notations and extend the ideas for topological algebras. Suppose $E(a,r)=\{x\in X:\|x-a\|\leq r\}, X_1=\{x\in X:\|x\|\leq 1\}$ and $S(X)=\{x\in X:\|x\|=1\}$ for which X is a normed vector space. We define the following concepts.

Definition 4.2. We define sets $D_\beta(A;1), V_\beta(A;a)$ by
$D_\beta(A;1)=\{f\in S(\hat{A}):f(1)=1\}$,
$V_\beta(A;a)=\{f(a):f\in D_\beta(A;1)\}(a\in A)$.

The elements of $D_\beta(A;a)$ are called β- normalized states on A, $V_\beta(A;a)$ is called the β- numerical range of a. We write $D(1), V(a)$ for $D_\beta(A;1)$, $V_\beta(A;a)$ when no confusion can occur.

Proposition 4.3. *Let B be a subalgebra of A such that $1\in B, b\in B$. Then $V_\beta(B;b)=V_\beta(A;b)$.*

Proof. Let $\lambda\in V_\beta(A;b)$; therefore, there exists $f\in D(A;1), \lambda=f(b)$. If g is the restriction mapping f to B, Then $g\in D_\beta(B;1), \lambda=g(b)$ that implies $\lambda\in V_\beta(B;b)$. On the other hand, we suppose $\lambda\in V_\beta(B;b)$, therefore, there exists $f\in B':|f(x)|\leq\beta(x)\ (x\in B), \lambda=f(b), f(1)=1$; and by the Hahn- Banach theorems, there exists $F\in A'$ *such that* $|F(x)|\leq\beta(x)$, $(x\in A)$, *and* $F=f$ *on* B, which implies that, $F\in D(A;1), \lambda=F(b)$; and we get the result. □

Lemma 4.4. *$D(1)$ is non-void weak$\hat{k}$-compact convex subset of $\hat{A}$.*

Proof. Since $D(1)=(\hat{A})_1 \bigcap 1^{-1}\{1\}$, if, in lemma 4.1, we put $x_0=1$, we get the result. □

Proposition 4.5. *(i) $V(a)$ is non-void compact convex subset of $\mathbb{C}$.*
(ii) $V(\gamma+\lambda b)=\gamma+\lambda V(b),\ V(a+b)\subseteq V(a)+V(b)\ (a,b\in A,\ \lambda,\gamma\in\mathbb{C})$.
(iii) $|z|\leq\ \beta(a)\ (z\in V(a))$.

Proof. (i) By the lemma 4.4, it is clear.
(ii) It is clear.
(iii) Let $z\in V(a)$. Then, there exists $f\in D(1)$ such that $z=f(a)$, and by 3.3 we have, $|z|=|f(a)|\leq v(f)\beta(a)\leq\beta(a)$. □

Lemma 4.6. $V(a)=\bigcap_{z\in\mathbb{C}}E(z,\beta(z-a)).$

Proof. If $\lambda\in V(a)$, then $\lambda=f(a)$ for some $f\in D(1)$, and for all $z\in\mathbb{C}$ we have

$$|\lambda-z|=|f(z-a)|\leq\beta(z-a), \tag{4.1}$$

i.e.

$$\lambda\in E(z,\beta(z-a))\ (z\in\mathbb{C}). \tag{4.2}$$

Suppose on the other hand that $\lambda \in \bigcap_{z\in\mathbb{C}} E(z, \beta(z-a))$. If $a = z_0 1_A$, then $V(a) = \{f(a) : f \in D(1)\} = \{f(z_0 1_A) : f \in D(1)\} = \{z_0\}$; also $|\lambda - z_0| \leq \beta(z_0 - z_0) = 0$. Therefore $\lambda = z_0 \in V(a)$. Suppose then that $1, a$ are linearly independent, and define f_0 on their linear span by

$$f_0(w + w'a) = w + w'\lambda \ (w, w' \in \mathbb{C}). \tag{4.3}$$

Since λ satisfies (4.2), we have $|f_0(w + w'a)| \leq \beta(w + w'a)$, $v(f_0) \leq 1$. By the Hahn-Banach theorem, f_0 can be extended to $f \in \hat{A}$ with $v(f) \leq 1$. Then $f \in D(1)$ and $f(a) = f_0(a) = \lambda$. □

We remember by [2] corollary 2.2 in every fundamental algebra, we have $\rho \leq \beta$.

Proposition 4.7. *If $\rho \leq \beta$, in particular, in fundamental algebras, we have $Sp(a) \subseteq V(a)$.*

Proof. Let $\lambda \in \mathbb{C} \setminus V(a)$. Then by lemma 4.6, there exists $z \in \mathbb{C}$ such that $|z - \lambda| > \beta(z - a)$. Therefore $\beta((z-\lambda)^{-1}(z-a)) < 1$, and so

$$1 - (z-\lambda)^{-1}(z-a) \in Inv(A). \tag{4.4}$$

It follows that $\lambda - a \in Inv(A)$, and as a result $\lambda \in \mathbb{C} \setminus Sp(a)$. □

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A SHORT NOTE ON IDEMPOTENT RINGS

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Abstract. In this paper we introduce a new class of rings that we say idempotent rings. We call a ring R is idempotent, if every ideal of R is generated by an idempotent element. In this paper we prove some properties of this rings, where one of the important results is the following:

Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1 p_2 \ldots p_n$, where all of the p_i are distinct prime numbers.

Keywords: idempotent, artinian ring, noetherian ring.

1. Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital. Let M be a submodule of the R-module L. We say that L is an essential extension of M precisely when $B \cap M \neq 0$ for every non-zero submodule B of L. We say that L is an injective envelope (or injective hull) of M precisely when L is an injective R-module which is also an essential extension of M. We denote by $E(M)$ the injective envelope of M. For any unexplained notation and terminology we refer the reader to [2] and [3].

2. Main results

Definition 2.1. Let R be a ring. We say that R is idempotent if every ideal of R is generated by an idempotent element.

Lemma 2.2. *Every idempotent ring is Artinian ring.*

Proof. Suppose that $\mathfrak{m} \in \operatorname{Max}(R)$. Then there exist an element $e \in R$ such that $\mathfrak{m} = \langle e \rangle$ and $e^2 = e$. Now in local Noetherian ring $R_\mathfrak{m}$, we have $(\mathfrak{m}R_\mathfrak{m})^2 = \mathfrak{m}R_\mathfrak{m}$, and so by $Nakayama's$ lemma, $\mathfrak{m}R_\mathfrak{m} = 0$. Hence $\dim R_\mathfrak{m} = 0$. Sinc $\mathfrak{m}$ is an arbitrary maximal ideal, it follows that $\dim R = 0$ and so R is Artinian ring. $\square$

Lemma 2.3. *In idempotent ring, the Jacobson radical is zero.*

Proof. We denote the Jacobson radical of R by $J(R)$, so we show that $J(R) = 0$. There exists an element $e \in R$ such that $J(R) = \langle e \rangle$ and $e^2 = e$. Therefore $J(R) = J(R) \cdot J(R)$ and by *Nakayama's* lemma, $J(R) = 0$. $\square$

Theorem 2.4. *Let R be an idempotent ring and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be all of the maximal ideals of R. Then $R \approx \frac{R}{\mathfrak{m}_1} \oplus \ldots \oplus \frac{R}{\mathfrak{m}_n}$.*

Proof. Let $\operatorname{Max} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. By induction on n we prove the theorem.

If $n = 1$, then $J(R) = \mathfrak{m}_1 = 0$ and so $R \approx \frac{R}{\mathfrak{m}_1}$.

Now we suppose that $n \geq 0$ and consider the following exact sequence,

$$0 \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \longrightarrow \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2} \longrightarrow \frac{R}{\mathfrak{m}_1 + \mathfrak{m}_2} \longrightarrow 0$$

Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, it follow from the above exact sequence that $\frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \approx \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2}$. Now consider the following exact sequence

$$0 \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3} \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3} \longrightarrow \frac{R}{(\mathfrak{m}_1 \cap \mathfrak{m}_2) + \mathfrak{m}_3} \longrightarrow 0.$$

Again similar the above argument, since $(\mathfrak{m}_1 \cap \mathfrak{m}_2) + \mathfrak{m}_2 = R$, it follows that

$$\frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3} \approx \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3} \approx \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3}.$$

By reapiting this argument we have

$$R \approx \frac{R}{(0)} \approx \frac{R}{J(R)} = \frac{R}{\cap_{i=1}^{n} \mathfrak{m}_i} \approx \oplus_{i=1}^{n} \frac{R}{\mathfrak{m}_i}.$$

$\square$

Theorem 2.5. *Every idempotent ring R as an R-module is injective.*

Proof. Let I be an ideal of R and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"] & R \\ & R & \end{tikzcd}$$

There exists an element $e \in R$ such that $I = \langle e \rangle$ and $e^2 = e$. Let $f(e) = x$. We define the function $g : R \to R$ with $g(r) = rx$. Then we have the following relations for all $r \in R$.

$$goi(re) = g(re) = rex = ref(e) = rf(e^2) = rf(e) = f(re),$$

and so R is an injective R-module. $\square$

Corollary 2.6. *If R is an idempotent ring, then every simple R-module is injective.*

Proof. Let N be a simple R-module. Then there exists $\mathfrak{m} \in \mathrm{Max}(R)$, such that $N \approx \frac{R}{\mathfrak{m}}$. Therefore N is a direct summand of an injective R-module R and so N is injective. $\square$

Corollary 2.7. *Let R be an idempotent ring. Then for every $\mathfrak{m} \in \mathrm{Max}(R)$, $E_R\left(\frac{R}{\mathfrak{m}}\right) \approx \frac{R}{\mathfrak{m}}$.*

Proof. Follows from the above corollary. $\square$

Lemma 2.8. *Let R be an idempotent ring and M be a finitely generated R-module. Then M is injective.*

Proof. Since R is Artinian ring, it follows that $l(M) < \infty$. We prove the assersion by induction on $l(M)$. If $l(M) = 1$, then M is a simple and so the assersion follows from Corollary 2.6. Now suppose that $l(M) = n \geq 2$ and the assertion holds for $n - 1$.

Since M is Artinian R-module, it follows that M has a simple submodule such as N. Consider the following exact sequence.

$$0 \longrightarrow N \longrightarrow M \longrightarrow \frac{M}{N} \longrightarrow 0$$

$l(N) = 1$ and so N is injective. $l\left(\frac{M}{N}\right) = n - 1$ then $\frac{M}{N}$ is injective. Therefore M is also injective. $\square$

Lemma 2.9. *Let R be a Noetherian ring and $\{E_i\}_{i\in A}$ be a family of injective R-modules. Then $\varinjlim_{i\in A} E_i$ is injective.*

Proof. Is simple. $\square$

Theorem 2.10. *Let R be an idempotent ring. Then every R-module is injective.*

Proof. Let T be an R-module. Then T is a direct limit of its finitely generated submodules. $\square$

Theorem 2.11. *Let R be an idempotent ring. Then every R-module is projective and so is flat.*

Proof. By the above theorem every R-module is injective. Let T be an R-module. Then T is injective. By Matlis theorem

$$T = \oplus_{p\in \mathrm{Spec}(R)} E\left(\frac{R}{p}\right).$$

On the other hand $\mathrm{Max}(R) = \mathrm{Spec}(R)$. Therefore,

$$T = \oplus_{\mathfrak{m}\in \mathrm{Max}(R)} \frac{R}{\mathfrak{m}}.$$

Also for any $\mathfrak{m} \in \mathrm{Max}(R)$, $\frac{R}{\mathfrak{m}}$ is a direct summand of R and so is projective. Consequently T is projective. □

Theorem 2.12. *Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1p_2 \dots p_n$, where all of the p_i are distinct prime numbers.*

Proof. Let $\mathbb{Z}_t$ be an idempotent ring. Suppose on the contrary that there exists a prime number p such that $p^2 \mid t$.

In this case, we set $J = \langle \bar{p} \rangle$. Then $J^2 = \langle \bar{p^2} \rangle \neq J$ and so J is not an idempotent ideal of $\mathbb{Z}_t$ and therefore $\mathbb{Z}_t$ is not idempotent ring which is a contradiction.

Conversely, let $t = p_1 \dots p_n$, where p_i are distinct prime numbers. If $n = 1$, then all of the ideals of $\mathbb{Z}_t$ are $I = \langle 0 \rangle$ and $J = \mathbb{Z}_{p_1} = Z_t = \langle \bar{1} \rangle$ and so $\mathbb{Z}_t$ is idempotent ring.

Now let $n \geq 2$ and we set $\mathfrak{m}_i = \langle \bar{p_i} \rangle$. Then $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are all the maximal ideals of R. We claim that each $\mathfrak{m}_i$ is generated by an idempotent element. It is enough to show that for $\mathfrak{m}_1$. Since $(p_1, p_2p_3 \dots p_n) = 1$, then there exist $r, s \in \mathbb{Z}$, such that $rp_1 + sp_2 \dots p_n = 1$. Therefore $p_2 \dots p_n \mid rp_1 - 1$ and $p_1 \mid rp_1$. Hence $p_1 \dots p_n \mid rp_1(rp_1 - 1) = (rp_1)^2 - rp_1$ and so in the ring $\mathbb{Z}_t$, we have $\overline{rp_1}^2 = \overline{rp_1}$. Set $e = \overline{rp_1}$ and we claim that $\mathfrak{m}_1 = \langle e \rangle$. $rp_1 + sp_2 \dots p_n = 1$ implies that $rp_1^2 + sp_1p_2 \dots p_n = p_1$. Hence in the ring $\mathbb{Z}_t$, $\overline{rp_1^2} = \overline{p_1}$ and so $\overline{p_1e} = \overline{p_1}$ and $\mathfrak{m}_1 = \langle \overline{p_1} \rangle = \langle \overline{p_1e} \rangle \subseteq \langle e \rangle \subseteq \mathfrak{m}_1$. Therefore $\mathfrak{m}_1 = \langle e \rangle$ and every maximal ideal of $\mathbb{Z}_t$ is generated by an idempotent element. Now let I be an arbitrary ideal of $\mathbb{Z}_t$. Then there exists an element $p_{i_1}p_{i_2} \dots p_{i_k}$ such that $I = \langle p_{i_1}p_{i_2} \dots p_{i_k} \rangle$, where $p_{i_1}, \dots, p_{i_k}$ are different elements of the set $\{p_1, \dots, p_n\}$. Also we have,

$$I = \mathfrak{m}_{i_1}\mathfrak{m}_{i_2} \dots \mathfrak{m}_{i_k} = \langle e_{i_1} \rangle \dots \langle e_{i_k} \rangle = \langle e_{i_1} \dots e_{i_k} \rangle$$

where all of the e_{i_j} are idempotent and so the element $e_{i_1}e_{i_2} \dots e_{i_k}$ is also idempotent and the assertion follows. □

Remark 2.13. It is well known, in a Noetherian ring R, for any ideal I of R and any injective R-module E, $0 :_E (0 :_R I) = IE$.

Theorem 2.14. *Let R be a Noetherian ring and every R-module be an injective R-module. Then R is idempotent.*

Proof. Let I be an ideal of R. Then I is injective R-module and by Remark 2.13, we have

$$I \subseteq 0 :_I (0 :_R I) = II = I^2 \subseteq I$$

Hence $I = I^2$ and by [1, Corollary 2.5], there exists $a \in I$ such that $(1 - a)I = 0$ and so $I = \langle a \rangle$ and $a^2 = a$. □

Corollary 2.15. *The Noetherian ring R is idempotent iff every R-module is an injective R-module.*

Corollary 2.16. *If $p_1, \ldots, p_n$ are distinct prime numbers and $R = \mathbb{Z}_{p_1 \ldots p_n}$. Then every R-module is injective and projective.*

Corollary 2.17. *Let R be an idempotent ring and M be an R-module. Then the following are equivalent:*

i) There exists an exact sequence $0 \longrightarrow R \longrightarrow M$

ii) Ann $M = 0$

Proof. $i \to ii)$ is clear.

$ii \to i)$ Since M is injective, it follows by Matlis theorem $M = \oplus_{\gamma \in A} E\left(\frac{R}{\mathfrak{m}_\gamma}\right) = \oplus_{\gamma \in A} \frac{R}{\mathfrak{m}_\gamma}$. Set $T = \{\mathfrak{m}_\gamma \mid \gamma \in A\}$ and we prove that $T = \mathrm{Max}(R)$. Suppose on the contrary that $T \neq \mathrm{Max}(R)$. Let $\mathfrak{m} \in \mathrm{Max}(R) \backslash T$.

$$0 = \mathrm{Ann}\, M = \bigcap_{\gamma \in A} \mathfrak{m}_\gamma \Rightarrow \bigcap_{\gamma \in A} \mathfrak{m}_\gamma = 0 \subseteq \mathfrak{m}$$

and so there exist $\gamma \in A$ such that $\mathfrak{m}_\gamma \subseteq \mathfrak{m}$ which implies that $\mathfrak{m} = \mathfrak{m}_\gamma \in T$, which is a contradiction. Therefore $M = \oplus_{\mathfrak{m}_\gamma \in \mathrm{Max}\, R} \frac{R}{\mathfrak{m}_\gamma} \approx R$ and so the sequence $0 \longrightarrow R \longrightarrow M$ is exact. $\square$

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ERROR ESTIMATES OF FINITE VOLUME ELEMENT METHOD FOR NONLINEAR HYPERBOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. The goal of this paper is to investigote the error estimates of the fininte volume element approximation of optimal control problems governed by nonlinear hy-

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perbolic equations. By using optimize-then-discretize, variational discretization and the finite volume method to solve the distributed optimal control problems. A semi-discrete optimal system is obtained. Meanwhile, we obtain the optimal order error estimates in $L^\infty(J; L^2)$ and $L^\infty(J; H^1)$-norm.

Keywords: error estimates, variational discretization, hyperbolic optimal control problems, finite volume element method.

1. Introduction

As is known to all, optimal control problems are widely used in science and engineering. Over the past decade, a large number of numerical methods have been applied to approximate the solutions of these optimal control problems, such as finite element method, mixed finite element method, spectral method, and finite volume method, see, e.g., [25, 24, 22, 11, 9, 13, 10]. For the finite element method, some error estimates for the finite element approximation of a class of nonlinear optimal control problems can be found in [28, 29]. The error estimates of mixed finite element approximation for optimal control problems are investigated in [7, 27, 23]. Furthermore, in [26], the finite volume element method is applied to solve the distributed optimal control problems governed by hyperbolic equation, and a priori error estimates were presented. There are plenty of others studies of the numerical methods for the optimal control problems, see, e.g., [19, 30, 31, 32, 33, 34, 20].

Finite volume method, as a type of important numerical tool for solving differential equations, has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. Perhaps the most important property of Finite volume method is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field. There have been a lot of studies of the mathematical theory for finite volume element methods, see, e.g., [4, 5, 6, 8, 14, 16] and the references cited therein.

In this paper, we use the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces and their associated norms $||v||_{m,p}$ (see, e.g., [1, 3]). To simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the index $p = 2$ and Ω whenever possible, i.e., $||u||_{m,2,\Omega} = ||u||_{m,2} = ||u||_m,\ ||u||_0 = ||u||$. Let $H_0^1(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$. As usual, we use $(\cdot, \cdot)$ to denote the $L^2(\Omega)$-inner product. We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $||v||_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T ||v||^s_{W^{m,p}(\Omega)} dt)^{1/s}$ for $s \in [1, \infty)$ and the standard modification for $s = \infty$.

Now, we consider the following optimal control problem:

$$\min_{u \in U_{ad}} \frac{1}{2} \int_0^T (||y - y_d(x,t)||^2_{L^2(\Omega)} + ||u(x,t)||^2_{L^2(\Omega)}) dt, \tag{1.1}$$

$$y_{tt}(x,t) - \nabla \cdot (A \nabla y(x,t)) + \phi(y(x,t)) = Bu(x,t) + f(x,t),\ t \in J, x \in \Omega, \tag{1.2}$$

$$y(x,t) = 0, \quad t \in J, x \in \Gamma, \tag{1.3}$$

$$y(x,0) = y_0(x), \; y_t(x,0) = g(x), \quad x \in \Omega, \tag{1.4}$$

where

$$\nabla \cdot (A\nabla y) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right),$$

$\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain and Γ is the boundary of Ω, $f(\cdot,t)$, $y_d(\cdot,t) \in L^2(\Omega)$ or $H^1(\Omega)$, $J = (0,T]$, $A = (a_{i,j})_{2\times 2}$ is a symmetric, smooth enough and uniformly positive definite matrix in Ω, $B : L^2(J;L^2(\Omega)) \to L^2(J;L^2(\Omega))$ is a bounded continuous linear operator, $y_0(x) = 0, x \in \Gamma$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$. ϕ is of class C^2 with respect to the variable y, for any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq 0$. U_{ad} is a set defined by

$$U_{ad} = \{u : \; u \in L^2(J;L^2(\Omega)), \; u(x,t) \geq 0, \; a.e. \text{ in } \Omega, t \in J, \; a,b \in \mathbb{R}\}.$$

The rest of this paper is organized as follows. In Section 2, we present some notations. In Section 3, we apply finite volume method and variational discretization concept to the problem (1.1)-(1.4) and obtain the discretized optimal system. In Section 4, we analyze the error estimates between the exact solution and the finite volume element approximation.

2. Notations and preliminaeies

For a convex polygonal domain Ω, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of closed triangle elements K such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of Kas is shown in Figure 1.

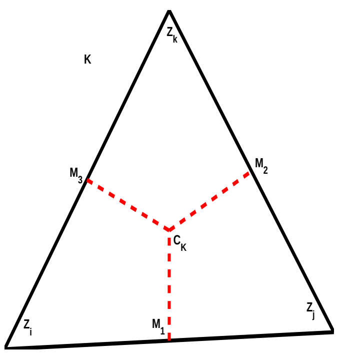

Figure 1. The dual partition of a triangular K.

The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Figure 2.

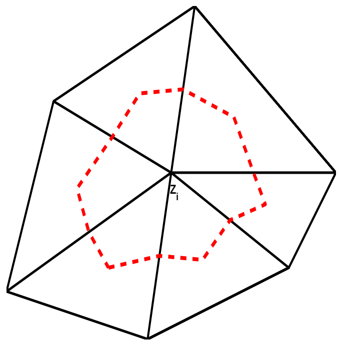

Figure 2. The control volume V_i sharing the same vertex z_i.

The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h = \max\{h_K\}$, where h_K is the diameter of the triangle K. As is shown in [17], the dual partition $\mathcal{T}_h^*$ is also quasi-uniform. Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h and the time step k.

We define the finite dimensional space V_h (i.e. trial space) associated with $\mathcal{T}_h$ for the trial functions by $V_h = \{v : v \in C(\Omega),\ v|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h,\ v|_\Gamma = 0\}$ and define the finite dimensional space Q_h (i.e. test space) associated with the dual partition $\mathcal{T}_h^*$ for the test functions by $Q_h = \{q : q \in L^2(\Omega),\ q|_V \in P_0(V),\ \forall\ V \in \mathcal{T}_h^*;\ q|_{V_z} = 0,\ z \in \Gamma\}$, where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V.

To connect the trial space and test space, we define a transfer operator $I_h : V_h \to Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i, \quad I_h v_h|_{V_i} = v_h(z_i), \quad \forall\ V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i. For the operator I_h, it is well known that there exists a positive constant C such that for all $v \in V_h$

$$||v - I_h v|| \leq Ch||v||_1. \tag{2.1}$$

Let $a(w, v) = \int_\Omega A\nabla w \cdot \nabla v dx$. We define the standard Ritz projection $R_h : H^2 \cap H_0^1 \to V_h$ by

$$a(R_h u, \chi) = a(u, \chi),\ \forall \chi \in V_h. \tag{2.2}$$

And let

$$a_h(\phi, I_h\psi) = -\sum_{z_i \in N_h} \psi(z_i) \int_{\partial V_i} A\nabla\phi \cdot \mathbf{n} ds,$$

where $\mathbf{n}$ is the unit outward normal vector to ∂V_i. π_h is defined as the linear interpolation on the triangulation $\mathcal{T}_h$.

3. Finite volume method for optimal control problems

In this section, we will use the optimize-then-discretize approach to obtain the finite volume element approximation for nonlinear hyperbolic optimal control problems.

It is well known (see, e.g.,[29]) that the optimal control problem (1.1)-(1.4) has a solution $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$, and that if a triplet $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$ is the solution of (1.1)-(1.4), then there is a co-state $p(\cdot,t) \in H_0^1(\Omega)$ such that $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$ satisfies the following optimality conditions:

$$\begin{aligned}
&(3.1) \qquad (y_{tt}, w) + (A\nabla y, \nabla w) + (\phi(y), w) = (Bu + f, w), \ \forall\ w \in H_0^1(\Omega),\\
&\qquad\qquad y(x,0) = y_0(x), \quad y_t(x,0) = g(x);\\
&(3.2) \qquad (p_{tt}, q) + (A\nabla p, \nabla q) + (\phi'(y)p, q) = (y - y_d, q), \ \forall\ q \in H_0^1(\Omega),\\
&\qquad\qquad p(x,T) = 0, \quad p_t(x,T) = 0;\\
&(3.3) \qquad \int_0^T (u + B^*p, v - u) d\tau \geq 0, \ \forall\ v \in U_{ad}.
\end{aligned}$$

If $y(\cdot,t) \in H_0^1(\Omega) \cap C^2(\Omega)$ and $p(\cdot,t) \in H_0^1(\Omega) \cap C^2(\Omega)$, then the optimal system (3.1)-(3.3) can be written by

$$\begin{aligned}
&(3.4) \quad y_{tt} - \nabla \cdot (A\nabla y) + \phi(y) = Bu + f, t \in J, x \in \Omega, y(x,t) = 0, t \in J, x \in \Gamma,\\
&\qquad\quad y(x,0) = y_0(x), \quad y_t(x,0) = g(x), \ x \in \Omega;\\
&(3.5) \quad p_{tt} - \nabla \cdot (A\nabla p) + \phi'(y)p = y - y_d, t \in J, x \in \Omega, \ p(x,t) = 0, t \in J, x \in \Gamma,\\
&\qquad\quad p(x,T) = 0, \quad p_t(x,T) = 0, \quad x \in \Omega;\\
&(3.6) \quad \int_0^T (u + B^*p, v - u) d\tau \geq 0, \ \forall\ v \in U_{ad}.
\end{aligned}$$

We use the finite volume method to discretized the state and costate equations directly. Then the optimal control problem (3.4)-(3.6) again has a solution $(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$, and that if a triplet $(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$ is the solution of (3.4)-(3.6), then there is a co-state $p_h(\cdot,t) \in V_h$ such that

$(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$ satisfies the following optimality conditions:

$$(3.7)\quad (y_{h,tt}, I_h w_h)+a_h(y_h, I_h w_h) + (\phi(y_h), I_h w_h)=(Bu_h + f, I_h w_h), \forall w_h \in V_h,$$
$$y_h(x,0) = \pi_h y_0(x), \quad y_{h,t}(x,0) = \pi_h g(x), \quad x \in \Omega;$$
$$(3.8)\quad (p_{h,tt}, I_h q_h) + a_h(p_h, I_h q_h) + (\phi'(y_h)p_h, I_h q_h) = (y_h - y_d, I_h q_h), \forall q_h \in V_h,$$
$$p_h(x,T) = 0, \quad p_{h,t}(x,T) = 0, \quad x \in \Omega;$$
$$(3.9)\quad \int_0^T (u_h + B^* p_h, v - u_h)d\tau \geq 0, \ \forall\ v \in U_{ad}.$$

In order to express the control in a concise form, we introduce a projection (see, e.g., [18])

$$P_{[a,b]}(f(x)) = \max\left(a, \min(b, f(x))\right),$$

we can denote the variational inequality (3.6) by

$$(3.10)\qquad u(x) = P_{[a,b]}(-B^* p(x,t)).$$

And the variational inequality (3.9) is equivalent to

$$(3.11)\qquad u_h(x) = P_{[a,b]}(-B^* p_h(x,t)).$$

Then the discrete optimality condition can be rewritten by: find $(y_h(\cdot,t), p_h(\cdot,t), u_h) \in V_h \times V_h \times U_{ad}$ such that

$$(3.12)\quad (y_{h,tt}, I_h w_h)+a_h(y_h, I_h w_h)+(\phi(y_h), I_h w_h)=(Bu_h + f, I_h w_h), \forall w_h \in V_h,$$
$$y_h(x,0) = \pi_h y_0(x), \quad y_{h,t}(x,0) = \pi_h g(x), \quad x \in \Omega;$$
$$(3.13)\quad (p_{h,tt}, I_h q_h)+a_h(p_h, I_h q_h)+(\phi'(y_h)p_h, I_h q_h)=(y_h - y_d, I_h q_h), \forall q_h \in V_h,$$
$$p_h(x,T) = 0, \quad p_{h,t}(x,T) = 0, \quad x \in \Omega;$$
$$(3.14)\quad u_h(x) = P_{[a,b]}(-B^* p_h(x,t)).$$

This is our finite volume method for the problems (1.1)-(1.4) which the variational concept is used for the variational inequality (3.6).

For $\varphi \in V_h$, we shall write

$$(3.15)\qquad \phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2,$$

where

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + s(\rho - \varphi))ds,$$
$$\tilde{\phi}''(\varphi) = \int_0^1 (1-s)\phi''(\rho + s(\varphi - \rho))ds$$

are bounded functions in $\bar{\Omega}$, more details can be found in [12].

4. Error estimates

In this section, to begin with, we present some useful results. Then we obtain two Lemmas to deduce the error estimates. At last, we derive some error estimates for the finite volume element approximation of the problems (1.1)-(1.4).

To describe error estimates for the finite volume methods, we will give some useful results. As shown in [15, 17], for all $w_h, v_h \in V_h$, there exist positive constants C and $h_0 > 0$ such that for all $0 < h < h_0$

$$|a_h(w_h, I_h v_h) - a_h(v_h, I_h w_h)| \le Ch||w_h||_1 \, ||v_h||_1, \tag{4.1}$$

$$a_h(v_h, I_h v_h) \ge C||v_h||_1^2, \tag{4.2}$$

$$a_h(w_h, I_h v_h) \le C||w_h||_1 \, ||v_h||_1. \tag{4.3}$$

Let $\varepsilon_a(\varphi, \chi) = a(\varphi, \chi) - a_h(\varphi, I_h \chi)$, we have (see, e.g., [21])

$$|\varepsilon_a(\varphi, \chi)| \le Ch \, ||\varphi||_1 \, ||\chi||_1, \quad \varphi, \chi \in V_h. \tag{4.4}$$

Then, we present two auxiliary problems to deduce the error estimates. Let $y_h(u)$ be the solution of

$$\begin{aligned} &(y_{h,tt}(u), I_h w_h) + a_h(y_h(u), I_h w_h) + (\phi(y_h(u)), I_h w_h) = (Bu + f, I_h w_h), \\ &y_h(u)(x, 0) = \pi_h y_0, \quad y_{h,t}(u)(x, 0) = \pi_h g, \quad x \in \Omega, \end{aligned} \tag{4.5}$$

and $p_h(y)$ be the solution of

$$\begin{aligned} &(p_{h,tt}(y), I_h q_h) + a_h(p_h(y), I_h q_h) + (\phi'(y_h(u)) p_h(u), I_h q_h) = (y - y_d, I_h q_h), \\ &p_h(y)(x, T) = 0, \quad p_{h,t}(y)(x, T) = 0, \quad x \in \Omega, \end{aligned} \tag{4.6}$$

where $w_h, q_h \in V_h$, and note that $y_h = y_h(u_h)$, $p_h = p_h(y_h)$. We have the following lemmas for $y_h(u)$, $p_h(y)$.

Lemma 4.1. *Assume that $y_h(u), p_h(y)$ are the solutions of (4.5) and (4.6), respectively. Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$*

$$||y_h(u) - y_h||_{L^\infty(J;H^1)} \le C||u - u_h||_{L^2(J;L^2(\Omega))}, \tag{4.7}$$

$$||p_h(y) - p_h||_{L^\infty(J;H^1)} \le C||y - y_h||_{L^2(J;L^2(\Omega))}. \tag{4.8}$$

Proof. Subtracting (3.7) from (4.5), we have

$$\begin{aligned} &(y_{h,tt}(u) - y_{h,tt}, I_h w_h) + a_h(y_h(u) - y_h, I_h w_h) + (\phi(y_h(u)) - \phi(y_h), I_h w_h) \\ &\quad = (B(u - u_h), I_h w_h), \ \forall w_h \in V_h. \end{aligned}$$

A direct calculation using (3.15) shows

$$\begin{aligned} &(y_{h,tt}(u) - y_{h,tt}, I_h w_h) + a_h(y_h(u) - y_h, I_h w_h) + (\tilde{\phi}'(y_h(u))(y_h(u) - y_h), I_h w_h) \\ &\quad = (B(u - u_h), I_h w_h). \end{aligned}$$

For convenience, let $\theta = y_h(u) - y_h$. We get $(\theta_{tt}, I_h w_h) + a(\theta, w_h) = \varepsilon_a(\theta, w_h) + (B(u-u_h), I_h w_h) - (\tilde{\phi}'(y_h(u))\theta, I_h w_h),\ \forall w_h \in V_h$. Choosing $w_h = \theta_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}[(\theta_t, I_h\theta_t) + a(\theta,\theta)] = \varepsilon_a(\theta,\theta_t) + (B(u-u_h), I_h\theta_t) - (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t).$$

Integrating both sides from 0 to t and noticing that $\theta(x,0) = 0$ and $\theta_t(x,0) = 0$, we have

$$\begin{aligned}(\theta_t, I_h\theta_t) + a(\theta,\theta) = {}& 2\int_0^t \varepsilon_a(\theta,\theta_t)d\tau + 2\int_0^t (B(u-u_h), I_h\theta_t)d\tau \\ & - 2\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau.\end{aligned}$$

The coercive property of $a(\cdot,\cdot)$ implies

$$\begin{aligned}(\theta_t, I_h\theta_t) + ||\theta||_1^2 \leq{}& C\int_0^t \varepsilon_a(\theta,\theta_t)d\tau \\ & + C\int_0^t (B(u-u_h), I_h\theta_t)d\tau - C\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau.\end{aligned} \tag{4.9}$$

Using (4.4) and the inverse estimate, we derive

$$\begin{aligned}\int_0^t \varepsilon_a(\theta,\theta_t)d\tau &\leq \int_0^t Ch||\theta||_1\ ||\theta_t||_1 d\tau \\ &\leq \int_0^t C||\theta||_1\ ||\theta_t||d\tau \\ &\leq C\int_0^t ||\theta||_1^2 d\tau + C\delta\int_0^t ||\theta_t||^2 d\tau.\end{aligned} \tag{4.10}$$

Using linear bound properties of B and I_h, we can write the inequality as

$$\begin{aligned}\int_0^t (B(u-u_h), I_h\theta_t)d\tau &\leq \int_0^t C||u-u_h||\ ||\theta_t||d\tau \\ &\leq C\int_0^t ||u-u_h||^2 d\tau + C\delta\int_0^t ||\theta_t||^2 d\tau.\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau &\leq \int_0^t C\|\theta\|\cdot\|\theta_t\|d\tau \\ &\leq C\int_0^t \|\theta\|^2 d\tau + C\delta\int_0^t \|\theta_t\|^2 d\tau \\ &\leq C\int_0^t \|\theta\|_1^2 d\tau + C\delta\int_0^t \|\theta_t\|^2 d\tau.\end{aligned} \tag{4.12}$$

Using (4.10)-(4.12) and for δ sufficiently small, note that $(\theta_t, I_h\theta_t)$ is equivalent to (θ_t, θ_t)(see, e.g., [17]) , we can obtain from (4.9) that

$$||\theta_t||^2 + ||\theta||_1^2 \leq C\int_0^t ||\theta||_1^2 d\tau + C\int_0^t ||u - u_h||^2 d\tau.$$

The Gronwall's lemma implies that

$$||\theta_t||^2 + ||\theta||_1^2 \leq C\int_0^T ||u - u_h||^2 d\tau = C||u - u_h||^2_{L^2(J;L^2)},$$

which completes the proof of (4.7). In a similar way, (4.8) can be verified easily. $\square$

We consider the following problem

$$\begin{cases} w_{tt}(x,t) - \nabla \cdot (A\nabla w(x,t)) = f(x,t), \quad t \in J, x \in \Omega, \\ w(x,t) = 0, \quad t \in J, x \in \Gamma, \\ w(x,T) = w_0(x), \quad w_t(x,T) = w_1(x), \ x \in \Omega, \end{cases} \tag{4.13}$$

where A, J, Ω are as described as in (1.1)-(1.4). The finite volume method for the problem (4.13) is to find $w_h(\cdot, t) \in V_h$ such that

$$\begin{cases} (w_{h,tt}, I_h\chi) + a_h(w_h, I_h\chi) = (f, I_h\chi), \ \forall \chi \in V_h, \\ w_h(x,T) = u_0(x), \quad w_{h,t}(x,T) = u_1(x). \end{cases} \tag{4.14}$$

For the finite volume method, we have the following results.

Lemma 4.2. *Let w_h, w be the solutions of* (4.14) *and* (4.13) *respectively. Assume that $f_t, f_{tt} \in L^2(J; L^2(\Omega))$, $f \in L^2(J; H^1(\Omega))$, If $u_h(0) = R_h u_0$ and $u_{h.t}(0) = R_h u_1$. Then there exists a constant C independent of h such that for all* $0 < h < h_0$

$$||w_h(t) - w(t)|| \leq Ch^2, \tag{4.15}$$

$$||w_h(t) - w(t)||_1 \leq Ch. \tag{4.16}$$

Proof. The proofs of (4.15) and (4.16) are similar to the proof of Theorem 1 in [7] and Theorem 2 in [7], respectively. $\square$

Let $(p(y), y(u))$ and $(p_h(y), y_h(u))$ be the solutions of (3.7)-(3.8) and (3.12)-(3.13), respectively. Let $J(\cdot) : U_{ad} \to \mathbb{R}$ be a G-differential convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2}(||y(u) - y_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}).$$

Then we have a sequence of convex functional $J_h : U_{ad} \to \mathbb{R}$:

$$J_h(u) = \frac{1}{2}(||y_h(u) - y_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}),$$
$$J_h(u_h) = \frac{1}{2}(||y_h(u_h) - y_d||^2_{L^2(\Omega)} + ||u_h||^2_{L^2(\Omega)}).$$

It can be shown that

$$(J'(u), v) = (u + B^* p, v),$$
$$(J'_h(u), v) = (u + B^* p_h(y), v),$$
$$(J'_h(u_h), v) = (u_h + B^* p_h, v).$$

In the following we estimate $\|u - u_h\|_{L^2(J;L^2)}$. We assume that the cost function J is strictly convex near the solution u, i.e., for the solution u there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c > 0$ satisfying:

$$(J'(u) - J'(v), u - v) \geq c\|u - v\|^2, \tag{4.17}$$

for all v in this neighborhood of u. The convexity of $J(\cdot)$ is closely related to the second order sufficient optimality conditions of optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many references, the authors assume the following second order sufficiently optimality condition (see [16, 28]): there is $c > 0$ such that $J''(u)v^2 \geq c\|v\|_0^2$.

From the assumption (4.17), by the proof contained in [2], there exists a constant $c > 0$ satisfying

$$(J'_h(v) - J'_h(u), v - u) \geq c\|v - u\|^2, \quad \forall v \in U_{ad}. \tag{4.18}$$

Theorem 4.1. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9)*, respectively. Assume that* $f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega))$, $f, y_d \in L^2(J; H^1(\Omega))$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$*. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$||u - u_h||_{L^2(J;L^2)} \leq Ch^2. \tag{4.19}$$

Proof. Let $v = u_h$ in (3.6) and $v = u$ in (3.9), then we have

$$\int_0^T (u + B^* p, u_h - u)d\tau \geq 0, \tag{4.20}$$

$$\int_0^T (u_h + B^* p_h, u - u_h)d\tau \geq 0. \tag{4.21}$$

From (4.20) and (4.21), it is easy to see that

$$\int_0^T (u - u_h, u - u_h)d\tau \leq \int_0^T (B^*(p - p_h), u_h - u)d\tau. \tag{4.22}$$

By using (4.18) and (4.22), we obtain

$$
\begin{aligned}
c\|u-u_h\|^2_{L^2(J;L^2)} \leq & \int_0^T (u, u-u_h)d\tau - \int_0^T (u_h, u-u_h)d\tau \\
= & \int_0^T (u + B^* p_h(y), u-u_h)d\tau - \int_0^T (u_h + B^* p_h, u-u_h)d\tau \\
= & \int_0^T (u-u_h, u-u_h)d\tau + \int_0^T (B^*(p_h(y)-p_h), u-u_h)d\tau \\
\leq & \int_0^T (B^*(p_h(y)-p_h, u-u_h)d\tau - \int_0^T (B^*(p-p_h), u-u_h)d\tau \\
= & \int_0^T (B^*(p_h(y)-p), u-u_h)d\tau. \qquad (4.23)
\end{aligned}
$$

Now, we estimate all terms at the right side of (4.23). From the continuity of the operator B and Cauchy inequality, we deduce that

$$
\begin{aligned}
\int_0^T (B^*(p_h(y)-p), u-u_h)d\tau \leq & C\|p_h(y)-p\|_{L^2(J;L^2)} \cdot \|u-u_h\|_{L^2(J;L^2)} \\
\leq & Ch^2\|u-u_h\|_{L^2(J;L^2)} \\
\leq & Ch^4 + \delta\|u-u_h\|_{L^2(J;L^2)}, \qquad (4.24)
\end{aligned}
$$

where $\|p-p_h(y)\|_{0,2} \leq Ch^2$(see the Theorem 2 in [7]). Combining (4.23) and (4.24), we conclude the result (4.19). □

Theorem 4.2. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9)*, respectively. Assume that* $f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega))$, $f, y_d \in L^2(J; H^1(\Omega))$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$. *Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$
||y-y_h||_{L^\infty(J;L^2)} + ||p-p_h||_{L^\infty(J;L^2)} \leq Ch^2. \qquad (4.25)
$$

Assume that $f, y_d, f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega)), y_0(x) \in H^3(\Omega), g(x) \in H^2(\Omega)$. *Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$
||y-y_h||_{L^\infty(J;H^1)} + ||p-p_h||_{L^\infty(J;H^1)} \leq Ch. \qquad (4.26)
$$

Proof. Using the triangle inequality, we have that

$$
\begin{aligned}
||y-y_h||_{L^\infty(J;L^2)} \leq ||y-y_h(u)||_{L^\infty(J;L^2)} + ||y_h(u)-y_h||_{L^\infty(J;L^2)}, \\
||p-p_h||_{L^\infty(J;L^2)} \leq ||p-p_h(y)||_{L^\infty(J;L^2)} + ||p_h(y)-p_h||_{L^\infty(J;L^2)}.
\end{aligned}
$$

Lemma 4.1 implies that

$$
||y-y_h||_{L^\infty(J;L^2)} \leq ||y-y_h(u)||_{L^\infty(J;L^2)} + C||u-u_h||_{L^2(J;L^2)}, \qquad (4.27)
$$

$$
||p-p_h||_{L^\infty(J;L^2)} \leq ||p-p_h(y)||_{L^\infty(J;L^2)} + C||y-y_h||_{L^\infty(J;L^2)}. \qquad (4.28)
$$

Using Lemma 4.1, Lemma 4.2, (4.27)-(4.28), and Theorem 2 of [7], we can easily obtain (4.25) from Theorem 4.1.

In a similar way, (4.26) can be proved easily. □

Theorem 4.3. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9), *respectively. Assume that* $f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega))$, $f, y_d \in L^2(J; H^1(\Omega))$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$. *Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$||u - u_h||_{L^\infty(J;L^2)} \leq Ch^2. \tag{4.29}$$

Proof. Using the definition of $P_{[a,b]}(\cdot)$ and (3.10)-(3.11), we have that

$$\begin{aligned} |u - u_h| &= |P_{[a,b]}(-B^*p) - P_{[a,b]}(-B^*p_h)| \\ &\leq C|p - p_h|. \end{aligned} \tag{4.30}$$

Which implies that

$$\|u - u_h\| \leq C\|p - p_h\|. \tag{4.31}$$

Then we can get

$$\|u - u_h\|_{L^\infty(J;L^2)} \leq C\|p - p_h\|_{L^\infty(J;L^2)}. \tag{4.32}$$

Finally, we can obtain (4.29) from (4.25) and (4.32). □

5. Conclusion and future works

In this paper, we consider a priori error estimates for the finite volume element approximation of nonlinear hyperbolic optimal control problem. Then we use finite volume method to discretize the state and adjoint equation of the system. Under some reasonable assumptions, we obtain some optimal order error estimates. To our best knowledge in the context of optimal control problems, these priori error estimates of finite volume method for general nonlinear hyperbolic optimal control problem is new.

In future, we shall consider a posteriori error estimates and superconvergence of the finite volume element solutions for hyperbolic optimal control problems.

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ON HYPER BCH-ALGEBRA

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Abstract. In this paper we initiate the concept of a hyper BCH-algebra which is a generalization of a BCH-algebra, and hyper BCK/BCI algebras and investigate some related properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal and strong hyper BCH-ideal in hyper BCH-algebras, and give a few relations among these hyper BCH-ideals. Finally we define homomorphism of hyper BCH-algebras.

Keywords: hyper BCH-algebra, hyper BCH-ideals, week hyper BCH-ideals, Strong hyper BCH-ideals, homomorphism.

1. Introduction

In (1966) the notion of BCK-algebra was first introduces by Y. Imai and K. Iseki [6]. The notion of BCK-algebra is a generalization of properties of the Set-difference. In (1975), the concept of ideal in BCK-algebra was first initiated by K. Iseki [7]. A remarkable feature of K. Iseki definition is that, its formulation is free from those of ring theoretical and lattice theoretical concepts. In same year K. Iseki initiated the concept of BCI-algebra [6, 8] which is the generalization of BCK-algebra. These algebras have been extensively studied

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since their introduction. The concept of ideals has played an important role in the study of the theory of BCI-algebras, [9]. In a BCI-algebra X, an ideal I need not be subalgebra of X. If the ideal I is also a subalgebra of X, then it has better algebraic properties. In (1983), Q. P. Hu and X. Li, introduced the concept of BCH-algebra [3, 4] and prove some motivating results. In (1990) and (1991) certain other properties have been studied by W. A. Dudek and J. Thomys [2] and M. A. Chaudhry, [1], respectively. In [1], the author also defines ideals in BCH-algebras. Hyperstructure represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty in (1934), [12]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Hyperstructures have many applications to several sectors of both pure and applied sciences. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure; the composition of two elements is a set. In (2000) Y. B. Jun et al applied the hyperoperation to BCK-algebras and introduced the concept of a hyper BCK-algebra [12] which is a generalization of a BCK-algebra, and investigated some related properties. Ideal theory of hyper BCK-algebra studied in [11]. Further in (2006), X.L. Xin initiated the concept of hyper BCI-algebras [13], which is basically a generalization of hyper BCK-algebras, and he proved that every hyper BCK-algebra is a hyper BCI-algebra. It should be pointed out that the research of hyper BCI-algebras seems to have been focused on the ideal theory. The author introduced the concepts of hyper BCI-ideals, weak hyper BCI-ideals, strong hyper BCI-ideals and reflexive hyper BCI-ideals in hyper BCI-algebras, and he gave the relations among these hyper BCI-ideals. In this paper we initiated the notion of hyper BCH-algebra which is a generalization of BCH-algebra and hyper BCI/BCK-algebras and studied some basic properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal and strong hyper BCH-ideal in hyper BCH-algebras, and give some relations among these hyper BCH-ideals. We define homomorphism in hyper BCH-algebra and then we investigate some related results.

2. Premilinaries

Let H be a non-empty set and "$\circ$" a function from $H \times H \rightarrow P(H) \setminus \{\phi\}$, where $P(H)$ denotes the power set of H. For any two non-empty subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We will use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$. Also we define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exist $b \in B$ such that $a \ll b$.

Definition 2.1 ([10]). A non-empty set H endowed with a constant 0 and a hyperoperation is called hyper BCK-algebra if it satisfies the following axioms:

$HK1)$ $(x \circ y) \circ (y \circ z) \ll x \circ y$,
$HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$HK3)$ $x \circ H \ll \{x\}$,
$HK4)$ $x \ll y$ and $y \ll x \Rightarrow x = y$.

for all $x, y, z \in H$.

Definition 2.2 ([13]). A non-empty set H endowed with a constant 0 and a hyperoperation is called hyper BCI-algebra if it satisfies the following axioms:

$HI1$) $(x \circ y) \circ (y \circ z) \ll x \circ y$,
$HI2$) $(x \circ y) \circ z = (x \circ z) \circ y$,
$HI3$) $x \circ H \ll \{x\}$,
$HI4$) $x \ll y$ and $y \ll x \Rightarrow x = y$.
$HI5$) $0 \circ (0 \circ x) \ll x$.
for all $x, y, z \in H$.

Definition 2.3 ([11]). Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is said to be a hyper BCK-ideal of H if $x \circ y \ll I$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, reflexive if $x \circ x \subseteq I$ for all $x \in H$, strong hyper BCK-ideal of H if $(x \circ y) \cap I = \phi$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, hyper subalgebra of H if $x \circ y \subseteq I$ for all $x, y \in I$.

Proposition 2.4 ([11]). *Let H be hyper BCK-algebra. Then,*

(i) *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H.*

(ii) *if I is a hyper BCK-ideal of H and A is a nonempty subset of H. Then $A \ll I$ implies $A \subseteq I$.*

(iii) *if I is a reflexive hyper BCK-ideal of H and $(x \circ y) \cap I = \phi$, then $x \circ y \subseteq I$ for all $x, y \in H$.*

(iv) *H is a BCK-algebra if and only if $H = \{x \in H : x \circ x = \{0\}\}$.*

3. Hyper BCH-algebra

In this section we introduce a notion of hyper BCH-algebra and studied some of its basic properties.

Definition 3.1. Let H be a on-empty set with a constant "0" and "$\circ$" be a hyper operation defined on H. Then $(H, \circ, 0)$ is said to be a hyper BCH-algebra if the following axioms are satisfied:

$HCH1$) $x \ll x$,
$HCH2$) $(x \circ y) \circ z = (x \circ z) \circ y$,
$HCH3$) $x \ll y$ and $y \ll x \Rightarrow x = y$

for all $x, y, z \in H$; where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, "$\ll$" is called a hyper order in H.

Example 3.2. Let $H = \{0, 1, 2\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1, 2\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Example 3.3. Let $H = \{0, 1, 2, 3\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{0,3\}$	$\{0,3\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$
3	$\{3\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Proposition 3.4. *Any hyper BCK/BCI- algebra is a hyper BCH-algebra.*

Proposition 3.5. *Let H be a hyper BCH-algebra, then for all $x, y, z \in H$ and $A \subseteq H$;the following holds.*

1) $x\circ y \ll z \Leftrightarrow x \circ z \ll y$
2) $x \circ y \ll x$
3) $0 \ll x$
4) $t \in 0 \circ 0 \Leftrightarrow t = 0$
5) $x \in x \circ 0$
6) $A \circ y \ll A$
7) $x \circ A \ll y \Leftrightarrow x \circ y \ll A$
8) $A \ll A \circ 0$
9) $x \circ x = \{x\} \Leftrightarrow x = 0.$

Proof. We only prove 1, 2, 5, 6, 7 and 9.

1) Let$x, y, z \in H$,be such that $x \circ y \ll z$.Then there exists $t \in x \circ y$ such that $t \ll z$.Thus $0 \in t \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y$ and hence there exists $w \in x \circ z$ such that $0 \in w \circ y$ that is $w \ll y$.Therefore $x \circ z \ll y$.

Conversly, let $x, y, z \in H$ be such that $x \circ z \ll y$. Then there exists $w \in x \circ z$ such that $w \ll y$.Thus $0 \in w \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z$ and hence there exists $t \in x \circ y$ such that $0 \in t \circ z$that is $t \ll z$.Therefore $x \circ y \ll z$.

2) Let $0 \in 0 \circ y \subseteq (x \circ x) \circ y = (x \circ y) \circ x$. Then there exists $t \in x \circ y$ such that $0 \in t \circ x \Rightarrow t \ll x \Rightarrow x \circ y \ll x$.

5) By (2) above we have $x \circ 0 \ll x$, so there exists $t \in x \circ 0$ such that t $\ll x$, since $t \in x \circ 0$, then $x \circ 0 \ll t$ and so by (1) $x \circ t \ll 0$. Thus there is $r \in x \circ t$ such that $r \ll 0$, so by (3) and $(HCH3)$ $r = 0$. so $0 \in x \circ t$, that is $x \ll t$ since $x \ll t$ and $t \ll x$; then by $(HCH3) \Rightarrow x = t$. Therefore $x \in x \circ 0$.

6) Let $a \in A$ be any element, then by (2) $a \circ y \ll a$ hence there is $b \in a \circ y \subseteq A \circ y$ such that $b \ll a$, that is $A \circ y \ll A$.

7) Since $x \circ A \ll y$ which implies that there exists $a \in A$ such that $x \circ a \ll y$. Hence by (1) $x \circ a \ll a \ll A$ implies that $x \circ y \ll A$. The proof of the converse is easy to prove.

9) $\{x\} = x \circ x \subseteq x \circ (x \circ 0)$. Hence by (5) $x \ll 0$; thus $x = 0$. The converse follows from (4). $\square$

Proposition 3.6. *In any hyper BCH-algebra H, $x \circ 0 = \{x\}$ for all $x \in H$.*

Proof. We have from above proposition (5) $x \in x \circ 0$, now let $t \in x \circ 0$.Since $x \circ 0 \ll \{x\}$, we have $t \ll x$. So, $0 \in t \circ t \subseteq (x \circ 0) \circ t = (x \circ t) \circ 0$. Then there exists $a \in x \circ t$ such that $0 \in a \circ 0$.Thus $a \ll 0$.Then $a = 0$; Thus $x \ll t$. We have that $x = t$. Therefore, $x \circ 0 = \{x\}$. □

It is known that every hyper BCI-algebra is a hyper BCH-algebrs, but the following example show that the converse is not true.

Example 3.7. Let $H = \{0, 1, 2, 3\}$and "$\circ$" be a hyperoperation define on H in the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{0\}$	$\{3\}$	$\{3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{0\}$	$\{0\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

$$(2 \circ 3) \circ (2 \circ 1) = \{2\} \circ \{3\} = \{2, 3\}$$

and

$$(1 \circ 3) = \{3\}.(2 \circ 3) \circ (2 \circ 1) \neq (1 \circ 3)$$

Example 3.8. Let $H = \{0, 1, 2, 3, 4\}$ and "$\circ$" be a hyperoperation defined of H in the following table:

$\circ$	0	1	2	3	4
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{0, 4\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{0, 4\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

$$(1 \circ 3) \circ (1 \circ 2) = \{1\} \circ \{2\} = \{1, 2\}$$

and $(2 \circ 3) = \{2\}$ that is $\{1, 2\} \not\ll \{2\}$.

Definition 3.9. A hyper BCH-algebra H is called proper if it is not a hyper BCI-algebra.

In above examples the hyper BCH-algebras are proper hyper BCH-algebras.

Definition 3.10. Let $(H, \circ)$ be a hyper BCH-algebra, and X a non-empty subset of H containing "0". Then X is called hypersubalgebra of H if X is a hyper BCH-algebra under the same hyperoperation "$\circ$" on H.

Example 3.11. From the above Example 3.8 if we let $X = \{0, 1, 2\}$, then X is a hypersubalgebra of H as we in the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

Also, let $X = \{0, 1, 3\}$. Then X is a hypersubalgebra of H.

Theorem 3.12. *Let X be a non-empty subset of a hyper BCH-algebra $(H, \circ)$. The X is a hypersubalgebra of H if and only if $x \circ y \subseteq X$ for all $x, y \in X$.*

Proof. Straghtfarword. □

Theorem 3.13. *Let $(H, \circ)$ be a hyper BCH-algebra and $X(H) = \{x \in H \mid 0 \circ x \ll \{0\}\}$. Then $X(H)$ is a hypersubalgebra of H.*

Proof. Let $x, y \in X(H)$, then by definition $a = 0 \circ a \ll \{0\}$ and $b = 0 \circ b \ll \{0\}$. Now

$$a \circ b = (0 \circ a) \circ (0 \circ b) \ll \{0\} \circ \{0\} = \{0\}$$

Hence, $a \circ b \ll \{0\}$.Which implies that $a \circ b \ll X(H)$. Hence $X(H)$ is a hypersubalgebra of H. The set $X(H)$ is called the hyper BCA-part of the hyper BCH-algebra H. □

4. Hyper BCH-Ideals

Definition 4.1. Let $(H, \circ)$ be a hyper BCH-algebra and I a subset of H. Then I is called a hyper BCH-ideal of H if:

i) $0 \in I$
ii) $x \circ y \ll I$ and $y \in I \Rightarrow x \in I$ for all $x, y \in I$.

Example 4.2. Let $H = \{0, 1, 2, 3, 4, 5\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3	4	5
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, 4\}$	$\{0, 5\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{0, 5\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, 5\}$
3	$\{3\}$	$\{0, 3\}$	$\{0, 3\}$	$\{0\}$	$\{0\}$	$\{0, 5\}$
4	$\{4\}$	$\{0, 4\}$	$\{0, 4\}$	$\{0, 4\}$	$\{0\}$	$\{0\}$
5	$\{5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra. Let $I = \{0, 1, 2, 3\}$ is an ideal of H.

Example 4.3. Let $H = \{0, 1, 2, 3, 4\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3	4
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,3\}$	$\{0,4\}$
1	$\{1\}$	$\{0\}$	$\{0,1\}$	$\{0,1\}$	$\{0,4\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$	$\{0,2\}$	$\{0,3\}$
3	$\{3\}$	$\{0,3\}$	$\{0,3\}$	$\{0\}$	$\{0,2\}$
4	$\{4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,1\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Let $I_1 = \{0, 1, 2\}$, then I_1 is a hyper BCH-ideal of H.

Let $I_2 = \{0, 1, 3\}$, then I_2 is a hyper BCH-ideal of H.

Let $I_3 = \{0, 2, 3\}$, then I_3 is not a hyper BCH-ideal of H. Because $(3 \circ 4) = \{0, 2\} \ll I_3$ and $4 \in I_3$ but $3 \notin I_3$.

Theorem 4.4. *Let $(H, \circ)$ be a hyper BCH-algebra and $\{I_\lambda | \ \lambda \in \Lambda\}$ a family of hyper BCH-ideals of H, then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a hyper BCH-ideal of H.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a hyper BCH-ideal of a hyper BCH-algebra H, then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $x \circ y \ll I_\lambda$ and $y \in I_\lambda$ for every $\lambda \in \Lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a hyper BCH-ideal of H. Therefore it implies that $x \circ \ y \ll I_\lambda$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$. Hence $x \circ \ y \ll \bigcap_{\lambda\in\Lambda} I_\lambda$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a hyper BCH-ideal of H. □

Remark 4.5. The union of two hyper BCH-ideals need not be hyper BCH-ideals. For this we have the following example.

Example 4.6. Let $H = \{0, 1, 2, 3, 4\}$ be a hyper BCH-algebra define in Example 4.3. Let $I_1 = \{0, 1, 3\}$ and $I_2 = \{0, 1, 4\}$ be hyper BCH-ideals of H. But, $(3 \circ 4) = \{0, 2\} \not\leq I_1 \cup I_2$, which show that union of two hyper BCH-ideals is not a hyper BCH-ideal.

Theorem 4.7. *Every hyper BCH-ideal of a hyper BCH-algebra is a hypersubalgebra.*

Proof. Let $(H, \circ)$ be a hyper BCH-algebra and I a hyper BCH-ideal of H. Let $x, y \in I$. Then since I is a hyper BCH-ideal of H, and so by definition it implies that, $x \circ y \ll I$; which shows that I is a hypersubalgebra of H. □

The convers of the above theorem is not true, that is a hypersubalgebra is not a hyper BCH-ideal. From the above example if we consider $I_3 = \{0, 2, 4\}$, then is a hypersubalgebra of H but not a hyper BCH-ideal of H.

Proposition 4.8. *Let I be a hyper BCH-ideal and A a subset of a hyper BCH-algebra H such that $A \ll I$. Then $A \subseteq I$.*

Proof. Let I be a hyper BCH-ideal of H and A a subset of H. Let $A \ll I$ implies there exists $a \in A$ and $x \in I$ such that $a \ll x \Rightarrow 0 \in a \circ x \ll I$. Since I is a hyper BCH-ideal of H it implies that $a \in I$ and so $A \subseteq I$. □

Definition 4.9. Let I be a non-empty subset of a hyper BCH-algebra H. Then I is said to be a weak hyper BCH-ideal of H, if for all $x, y \in H$

(i) $0 \in I$

(ii) $x \circ y \subseteq I$ and $y \in I \Rightarrow x \in I$.

Theorem 4.10. *The intersection of any family of weak hyper BCH-ideal of a hyper BCH-algebra is a weak hyper BCH-ideal.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a weak hyper BCH-ideal of a hyper BCH-algebra H. Then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $x \circ y \subseteq I_\lambda$and $y \in I_\lambda$for every $\lambda \in \Lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a weak hyper BCH-ideal of H. Therefore it implies that $x \circ\ y \subseteq I_\lambda$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$ for every $\lambda \in \Lambda$. Hence $x \circ\ y \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a weak hyper BCH-ideal of H. □

Proposition 4.11. *Every hyper BCH-ideal in hyper BCH-algebra H is a weak hyper BCH-ideal.*

Proof. Let I be a hyper BCH-ideal of a hyper BCH-algebra H. Let $x \circ y \subseteq I$ and $y \in I$ for some $x, y \in H$. Since $x \circ y \subseteq I$ which implies that $x \circ y \ll I$. Now since I is a hyper BCH-ideal of H, so it implies that $x \in I$. Hence I is a weak hyper BCH-ideal of H. □

Definition 4.12. Let I be a non-empty subset of a hyper BCH-algebra H. Then I is said to be a strong hyper BCH-ideal of H if for all $x, y \in H$

(i) $0 \in I$

(ii) $(x \circ y) \cap I \neq \phi$ and $y \in I \Rightarrow x \in I$.

Theorem 4.13. *The intersection of any family of strong hyper BCH-ideal of a hyper BCH-algebra is a stong hyper BCH-ideal.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a strong hyper BCH-ideal of a hyper BCH-algebra H. Then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $(x \circ y) \cap \bigcap_{\lambda\in\Lambda} I_\lambda \neq \phi$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a strong hyper BCH-ideal of H. Therefore it implies that $(x \circ y) \cap I_\lambda \neq \phi$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$. Hence $(x \circ y) \cap \bigcap_{\lambda\in\Lambda} I_\lambda \neq \phi$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a strong hyper BCH-ideal of H. □

Proposition 4.14. *Every strong hyper BCH-ideal in hyper BCH-algebra H is a hyper BCH-ideal.*

Proof. Let I be a strong hyper BCH-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then for $a \in x \circ y$ there exists $b \in I$ such that $a \ll b \Rightarrow 0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \phi \Rightarrow a \in I$. Thus $x \circ y \subseteq I$ and so $(x \circ y) \cap I \neq \phi$. Since I is a strong hyper BCH-ideal of H. It follows that $x \in I$. Hence I is a hyper BCH-ideal of H. □

5. Homomorphisms of hyper BCH-algebras

Definition 5.1. Let H_1 and H_2 be two hyper BCH-algebras. A mapping $\psi : H_1 \to H_2$ is called a homomorphism if

(i) $\psi(0) = 0$

(ii) $\psi(x \circ y) = \psi(x) \circ \psi(y)$; for all $x, y \in H_1$.

If ψ is $1-1$ (or onto) we say that ψ is a monomorphism (or epimorphism). And if ψ is both $1-1$ and onto, we say that ψ is an isomorphism.

Theorem 5.2. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. Then*

(i) *If S is a hyper BCH-subalgebra of H_1, then $\psi(S)$ is a hyper BCH-subalgebra of H_2,*

(ii) *$\psi(H_1)$ is a hyper BCH-subalgebra of H_2,*

(iii) *If S is a hyper BCH-subalgebra of H_2, then $\psi^{-1}(S)$ is a hyper BCH-subalgebra of H_1,*

(iv) *If I is a (weak) hyper BCH-ideal of H_2, then $\psi^{-1}(I)$ is a (weak) hyper BCH-ideal of H_1,*

(v) *$Ker\psi = \{x \in H_1 | \psi(x) = 0\}$ is a hyper BCH-ideal and hence a weak hyper BCH-ideal of H_1,*

(vi) *If ψ is onto and I is a hyper BCH-ideal of H_1 which contains $Ker\psi$, then $\psi(I)$ is a hyper BCH-ideal of H_2.*

Proof. (i) Let $x, y \in \psi(S)$. Then there exist $a, b \in S$ such that $\psi(a) = x$ and $\psi(b) = y$. It follows from Theorem 3.12 that $x \circ y = \psi(a) \circ \psi(b) = \psi(a \circ b) \subseteq \psi(S)$ so that $\psi(S)$ is a hyper BCH-subalgebra of H_2.

(ii) Proof of this is same as (i).

(iii) Since $0 \in S$, we have $\psi^{-1}(0) \subseteq \psi^{-1}(S)$. Since $\psi(0) = 0$, so $0 \in \psi^{-1}(0) \subseteq \psi^{-1}(S)$. Therefore $\psi^{-1}(S)$ is non-empty. Now let $x, y \in \psi^{-1}(S)$. Then $\psi^{-1}(x), \psi^{-1}(y) \in S$. Thus $\psi(x \circ y) = \psi(x) \circ \psi(y) \subseteq S$ and so $x \circ y \subseteq \psi^{-1}(S)$, which implies that $\psi^{-1}(S)$ is a hyper BCH-subalgebra of H_1.

(iv) Let I be a weak hyper BCH-ideal of H_2. Clearly $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \subseteq \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$. Then $\psi(x) \circ \psi(y) = \psi(x \circ y) \subseteq I$ and $\psi(y) \in I$. Since I is a weak hyper BCH-ideal, it follows from (Id2) that $\psi(x) \in I$, i.e., $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a weak hyper BCH-ideal of H_1. Now let I be a hyper BCH-ideal of H_2. Obviously $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \ll \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$. Then there exist $t \in x \circ y$ and $z \in \psi^{-1}(I)$ such that $t \ll z$, that is $0 \in t \circ z$. Since $\psi(z) \in I$ and $0 \in t \circ z \subseteq (x \circ y) \circ z$, it follows that $0 = \psi(0) \in \psi((x \circ y) \circ z) = \psi(x \circ y) \circ \psi(z) \subseteq \psi(x \circ y) \circ I$ so that $\psi(x) \circ \psi(y) = \psi(x \circ y) \ll I$. As $\psi(y) \in I$ and I is hyper BCH-ideal, by using (Id3) we have $\psi(x) \in I$, that is $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a hyper BCH-ideal of H_1.

(v) First we show that $\{0\} \subseteq H_2$ is a hyper BCH-ideal. To do this, let $x, y \in H_2$ be such that $x \circ y \ll \{0\}$ and $y \in \{0\}$. Then $y = 0$ and so $x \circ 0 = x \circ y \ll \{0\}$. Therefore there exists $t \in x \circ 0$ such that $t \ll 0$. Thus $t = 0$, and consequently

$0 \in x \circ 0$, that is $x \ll 0$, which implies that $x = 0$. This shows that $\{0\}$ is a hyper BCH-ideal of H_2. Now by (iv), $Kerf = \psi^{-1}(\{0\})$ is a hyper BCH-ideal of H_1.

(vii) Since $0 \in I$, we have $0 = \psi(0) \in \psi(I)$. Let x and y be arbitrary elements in H_2 such that $x \circ y \ll f(I)$ and $y \in \psi(I)$. Since $y \in \psi(I)$ and ψ is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = \psi(y_1)$ and $x = \psi(x_1)$. Thus $\psi(x_1 \circ y_1) = \psi(x_1) \circ \psi(y_1) = x \circ \circ y \ll \psi(I)$. Therefore there are $a \in x_1 \circ y_1$ and $b \in I$ such that $\psi(a) \ll \psi(b)$. So $0 \in \psi(a) \circ \psi(b) = \psi(a \circ b)$, which implies that $\psi(c) = 0$ for some $c \in a \circ b$. It follows that $c \in Ker\psi \subseteq I$ so that $a \circ b \ll I$. Now since I is a hyper BCH-ideal of H_1 and $b \in I$, we get $a \in I$. Thus $x_1 \circ y_1 \ll I$, which implies that $x_1 \in I$. Thus $x = \psi(x_1) \in \psi(I)$, and so $\psi(I)$ is a hyper BCH-ideal of H_2. □

Theorem 5.3. *Let $\psi : H_1 \to H_2$ be an epimorphism of hyper BCH-algebras. Then there is a one to one correspondence between the set of all hyper BCH-ideals of H_1 containing $Ker\psi$ and the set of all hyper BCH-ideals of H_2.*

Theorem 5.4. *Let $\psi : H_1 \to H_2$ and $\pi : H_1 \to H_3$ be two homomorphisms of hyper BCH- algebras such that ψ is onto and $Ker\psi \subseteq Ker\pi$. Then there exists a homomorphism $\tau : H_2 \to H_3$ such that $\tau \circ \psi = \pi$.*

Proof. Let $y \in H_2$ be arbitrary. Since ψ is onto, there exists $x \in H_1$ such that $y = \psi(x)$. Define $\tau : H_2 \to H_3$ by $\tau(y) = \pi(x)$, for all $y \in H_2$. Now we show that τ is well-defined. Let $y_1; y_2 \in H_2$ and $y_1 = y_2$. Since ψ is onto, there are $x_1; x_2 \in H_1$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. Therefore $\psi(x_1) = \psi(x_2)$ and thus $0 \in \psi(x_1) \circ \psi(x_2) = \psi(x_1 \circ x_2)$. It follows that there exists $t \in x_1 \circ x_2$ such that $\psi(t) = 0$. Thus $t \in Ker\psi \subseteq Ker\pi$ and so $\pi(t) = 0$. Since $t \in x_1 \circ x_2$ we conclude that $0 = \pi(t) \in \pi(x_1 \circ x_2) = \pi(x_1) \circ \pi(x_2)$ which implies that $\pi(x_1) \ll \pi(x_2)$. On the other hand since $0 \in \psi(x_2) \circ \psi(x_1) = \psi(x_2 \circ x_1)$, similarly we can conclude that $0 \in \pi(x_2) \circ \pi(x_1)$, that is $\pi(x_2) \ll \pi(x_1)$. Thus $\pi(x_1) = \pi(x_2)$, which shows that τ is well-defined. Clearly $\tau \circ \psi = \pi$. Finally we show that τ is a homomorphism. Let $y_1; y_2 \in H_2$ be arbitrary. Since ψ is onto there are $x_1, x_2 \in H_1$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. Then

$$\begin{aligned}
\tau(y_1 \circ y_2) &= \tau(\psi(x_1) \circ \psi(x_2)) \\
&= \tau(\psi(x_1 \circ x_2)) \\
&= (\tau \circ \psi)(x_1 \circ x_2) \\
&= \pi(x_1 \circ x_2) \\
&= \pi(x_1) \circ \pi(x_2) \\
&= (\tau \circ \psi)(x_1) \circ (\tau \circ \psi)(x_2) \\
&= \tau(\psi(x_1)) \circ \tau(\psi(x_2)) \\
&= \tau(y_1) \circ \tau(y_2)
\end{aligned}$$

Moreover since $\psi(0) = 0$ and $\pi(0) = 0$, we conclude that $\tau(0) = \tau(\psi(0)) = (\tau \circ \psi)(0) = \pi(0) = 0$. Thus τ is a homomorphism. □

Theorem 5.5. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. If I is a strong hyper BCH-ideal of H_2, then $\psi^{-1}(I)$ is a strong hyper BCH-ideal of H_1.*

Proof. Suppose I is a strong hyper BCH-ideal, then clearly $0 \in \psi^{-1}(I)$. Let $a, b \in H_1$ be such that $(a \circ b) \cap \psi^{-1}(I) \neq \phi$ and $b \in \psi^{-1}(I)$. Then we have $\phi \neq \psi(a \circ b) \cap \psi^{-1}(I)) \subseteq \psi(a \circ b) \cap \psi\psi^{-1}(I) \subseteq \psi(a) \circ \psi(b) \cap I$ and so $(\psi(a) \circ \psi(b)) \cap I \neq \phi$ and $\psi(a) \in \psi(\psi^{-1}(I)) \subseteq I$. Since I is a strong hyper BCH-ideal of H_2, we have $\psi(a) \in I$ and so $x \in \psi^{-1}(I)$. Therefore $\psi^{-1}(I)$ is a strong hyper BCH-ideal of H_1. □

Theorem 5.6. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. Then $ker\psi = \{x \in H_1 | \psi(x) = 0\}$ is a strong hyper BCH-ideal of H_1.*

Proof. To prove this first we show that $\{0\}$ is a strong hyper BCH-ideal of H_2. For this, let $a, b \in H_1$ be such that $(a \circ b) \cap \{0\} \neq \phi$ and $b \in \{0\}$. Then $b = 0$ and so $0 \in a \circ 0$ since $(a \circ 0) \cap \{0\} \neq \phi$. Thus we have $a \ll 0$. By (HCH3) and 3.5 3, we get $a = 0 \in \{0\}$. This shows that $\{0\}$ is a strong hyper BCH-ideal of H_2. It follows fromTheorem 5.5 that $ker\psi = \psi^{-1}(\{0\})$ is a strong hyper BCH-ideal of H_1. □

Theorem 5.7. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If $\psi$is onto and I is a strong hyper BCH-ideal of H_1 which contains $ker\psi$, then $\psi(I)$ is a strong hyper BCH-ideal of H_2.*

Proof. Suppose I is a strong hyper BCH-ideal of H_1. Clearly $0 \in \psi(I)$. Let $x, y \in H_2$ be such that $(x \circ y) \cap \psi(I) \neq \phi$ and $y \in \psi(I)$. Since $y \in \psi(I)$ and ψ is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = \psi(y_1)$ and $x = \psi(x_1)$. Thus $\phi \neq (x \circ y) \cap \psi(I) = \psi(x_1 \circ y_1) \cap \psi(I)$ and so there exists $a \in H_2$ such that $a \in \psi(x_1 \circ y_1)$ and $a \in \psi(I)$. It follows that there are $a_1 \in x_1 \circ y_1$ and $b_1 \in I$ such that $a = \psi(a_1)$ and $a = \psi(b_1)$ so that $0 \in a \circ a = \psi a_1 \circ \psi b_1 = \psi\,(a_1 \circ b_1)$ which implies that $\psi(c) = 0$ for some $c \in a_1 \circ b_1$. Hence $c \in ker\psi \subseteq I$ and so $(a_1 \circ b_1) \cap I \neq \phi$. Now since I is a strong hyper BCH-ideal of H_1 and $b_1 \in I$, we get $a_1 \in I$. Thus $(x_1 \circ y_1) \cap I \neq \phi$, which implies that $x_1 \in I$. Thereby $x = \psi(x_1) \in \psi(I)$, and so $\psi(I)$ is a strong hyper BCH-ideal of H_2. □

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ON THE p-NILPOTENCE OF FINITE GROUPS

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Abstract. In this paper, we give a new definition–a CSS subgroup(a subgroup H of a finite group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G). By this definition, we investigate the relationship between the p-nilpotence of G and the p-nilpotence of $N_G(P)$, and generalize the corresponding results to a saturated formation $\mathcal{F}$ which contains the class $\mathcal{N}_p$ of all p-nilpotent groups, where p is an odd prime factor of $|G|$, P a Sylow p-subgroup of a group G.

Keywords: CSS-subgroups, Sylow subgroups, mxiamal subgroups, p-nilpotent groups, saturated formations.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G(see [5]). Recall that a subgroup H of a group G is c-normal in G if there is a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the core of H in G(see [8]). Recently, Li[6] defined that a subgroup H of G is said to be an SS-quasinormal subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. By use of these definitions, many authors investigated such structure of a group as the nilpotence, the supersolvability and so on, see [8, 1, 6, 4, 2, 5].

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Enlightened by the above concepts, we give a new definition $--$ a CSS-subgroup, which is a generalization of C-normality and SS-quasinormality.

Definition 1.1. A subgroup H of a group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. In this case, K is called a normal CSS-supplement of H in G.

Remark. Obviously, if a subgroup H of G is c-normal in G, then there is a normal subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 \leq H_G$. In this case, writing $K = H_G K_1$, then we have $G = HK$ and $H \cap K = H_G$; Of course, $H \cap K$ is SS-quasinormal in G. Therefore c-normal subgroups of G also are CSS-subgroups of G. Besides, a SS-quasinormal subgroup H of G, by taking $K = G$, must be a CSS-subgroup.

However, the following examples show the above converse is not true.

Example 1. Let $G = A_5$ be the alternating group of degree five. We have $G = A_4 C_5$. Then A_4 is SS-quasinormal in G, and is a CSS-subgroup of G. However, A_4 is not c-normal in G.

Example 2. Consider $G = S_4$, the symmetric group of degree four. Take $\alpha = (34)$ and $\beta = (123)$. Then $G = \langle \alpha \rangle A_4$ and $\langle \alpha \rangle \cap A_4 = 1$, hence $\langle \alpha \rangle$ is c-normal in G. Of course $\langle \alpha \rangle$ is a CSS-subgroup of G. Let B be a subgroup of G satisfying $G = \langle \alpha \rangle B$. Then B is either A_4 or G. As $\langle \alpha \rangle \langle \beta \rangle \neq \langle \beta \rangle \langle \alpha \rangle$, which indicates that $\langle \alpha \rangle$ is not SS-quasinormal in G.

In Section 2 of this paper, we give some properties of CSS-subgroups. Let p be an odd prime factor of $|G|$, P a Sylow p-subgroup of a group G. In Section 3, we investigate the relationship between p-nilpotence of G and p-nilpotence of $N_G(P)$, and generalize the corresponding results to a saturated formation $\mathcal{F}$ which contains the class $\mathcal{N}_p$ of all p-nilpotent groups.

2. Preliminaries

In this section we list some known results and some properties about CSS-subgroups which are needed in the main results.

Lemma 2.1 ([6, Lemma 2.1]). *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

(3) *If $N \leq K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.*

(4) *If K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2 ([6, Lemma 2.5])**.** *If a p-subgroup P of G is SS-quasinormal, where p is a prime. Then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.3. *Let H be a CSS-subgroup of a group G. We have:*

(1) *If $H \leq M \leq G$, then H is a CSS-subgroup of M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$, then H is a CSS-subgroup of G if and only if H/N is a CSS-subgroup of G/N.*

(3) *Let π be a set of some primes and N a normal π'-subgroup of G. If H is a π-subgroup of G, then HN/N is a CSS-subgroup of G/N.*

Proof. (1) Since H is a CSS-subgroup of G, there exists a subgroup $K \trianglelefteq G$ such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. Notice that $H \cap K \leq H \leq M \leq G$ and $K \trianglelefteq G$, we have that $M = H(M \cap K)$ and $M \cap K \trianglelefteq M$. On the other hand, Lemma 2.1.1 shows that $H \cap K$ is SS-quasinormal in M. Also, $H \cap (M \cap K) = H \cap K$, so $H \cap (M \cap K)$ is SS-quasinormal in M. Therefore H is a CSS-subgroup of M.

(2) Let $K \trianglelefteq G$ be a CSS-supplement of H in G, then $G = HK$ and $H \cap K$ is SS-quasinormal in G. Consider the group KN/N, we have: $KN/N \trianglelefteq G$, $G/N = HK/N = (H/N) \cdot (KN/N)$, and $(H/N) \cap (KN/N) = (H \cap K)N/N$ is SS-quasinormal in G/N by Lemma 2.1(2). Thus H/N is a CSS-subgroup of G/N.

Conversely, if H/N is a CSS-subgroup of G/N, then there exists a normal subgroup K/N of G/N such that $G/N = (H/N) \cdot (K/N)$ and $(H/N) \cap (K/N)$ is SS-quasinormal in G/N. Obviously, $G = HK$ and $K \trianglelefteq G$. Also, by Lemma 2.1(3), $H \cap K$ is SS-quasinormal in G. Thus H is a CSS-subgroup of G.

(3) Let K be a normal CSS-supplement of H in G, then $G = HK$, and $H \cap K$ is SS-quasinormal in G. Since N is a normal π'-subgroup and H is a π-subgroup of G, then $N \leq K$. Consequently $G/N = (HN/N) \cdot (K/N)$ and $K/N \trianglelefteq G/N$. Also, $(HN/N) \cap (K/N) = (H \cap K)N/N$ is SS-quasinormal in G/N by Lemma 2.1.2. Therefore HN/N is a CSS-subgroup of G/N. □

Lemma 2.4. *Let P be a Sylow p-subgroup of a finite group G and N a normal subgroup of G. If every maximal subgroup of P is a CSS-subgroup of G, then every maximal subgroup of PN/N is a CSS-subgroup of G/N.*

Proof. Let S/N be a maximal subgroup of PN/N, and T a Sylow p-subgroup of S. Without loss of generality, we may assume that $T \leq P$. Obviously, $S/N = TN/N$. Notice that $p = |PN : T| = |PN : TN| = |P : T|$, it shows that T is a maximal subgroup of P. By hypothesis of the theorem, T is a CSS-subgroup of G, so there exists a normal subgroup K such that $G = TK$ and $T \cap K$ is SS-quasinormal in G. Let K_q be a Sylow q-subgroup of K, where q is a prime factor of $|K|$ and $q \neq p$. Obviously, K_q is also a Sylow q-subgroup of G, and $N \cap K_q$ is a Sylow q-subgroup of N. Set $D = \langle N \cap K_q \mid q \neq p \rangle$, then $D \leq K$ and $N = (T \cap N)D$. So $TN \cap KN = (TN \cap K)N = (TD \cap K)N = (T \cap K)DN =$

$(T \cap K)N$. By Lemma 2.1(2), we know $TN/N \cap KN/N = (T \cap K)N/N$ is SS-quasinormal in G/N, it is to say S/N is SS-quasinormal in G/N. □

Lemma 2.5. *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Proof. Notice that $P \trianglelefteq G$, so $M \le N_G(P \cap M)$. Also, it is clear that $P \cap M$ is not a Sylow p-subgroup of G, so $P \cap M \lneqq N_P(P \cap M)$. It follows that $M \lneqq N_G(P \cap M)$. By the maximality of M, we have that $N_G(P \cap M) = G$, which implies that $(P \cap M) \trianglelefteq G$. □

Lemma 2.6 ([3, Proposition IV.3.11])**.** *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where F_i is both an integrated and full formation function of $\mathcal{F}_i (i = 1,\ 2)$. Then the following statements are equivalent:*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,

(2) $F_1(p) \subseteq F_2(p)$ *for all* $p \in P$.

3. Main results

Theorem 3.1. *Let H be a normal subgroup of a group G, p an odd prime factor of $|H|$ and P a Sylow p-subgroup of H. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is a CSS-subgroup of G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. We proceed by the following steps:

Step 1. $O_{p'}(G) = 1$.

Otherwise, consider the quotient group $G/O_{p'}(G)$ and $HO_{p'}(G)/O_{p'}(G)$. For convenience, we write $D = O_{p'}(G)$. Obviously, PD/D is a Sylow p-subgroup of HD/D. Notice that $(|D|, p) = 1$, so we have $N_{G/D}(PD/D) = N_G(P)D/D$, which shows that $N_{G/D}(PD/D)$ is p-nilpotent. By applying Lemma 2.4, we have that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. The choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, so is G, a contradiction.

Step 2. *If $P \le L < G$, then L is p-nilpotent.*

As $N_L(P) \le N_G(P)$, so L satisfies the hypotheses of the theorem by Lemma 2.3.1. The minimality of G implies that L is p-nilpotent, as desired.

Step 3. *3.*1 $H = G$;

3.2 If K be a normal subgroup of G, then G/K is p-nilpotent. So G has a unique minimal normal subgroup N and $N \nleq \Phi(G)$.

3.1 If $H < G$, by using Step 2, we have that H is p-nilpotent. Also, Step 1 implies that H is a p-group, so $H = P$. Therefore, $N_G(P) = N_G(H) = G$ is p-nilpotent, a contradiction.

3.2 It is clear that P is a Sylow p-subgroup by 3.1, therefore $N_{G/K}(PK/K) = N_G(P)K/K$ is p-nilpotent. Applying Lemma 2.4, we have that G/K satisfies the hypotheses of the theorem. So the choice of G implies that G/K is p-nilpotent.

Since the class of all p-nilpotent groups is a saturated formation, we may assume that G has the unique minimal normal subgroup, say N and $N \nleq \Phi(G)$.

Step 4. *$O_p(G) > 1$, moreover, G is p-solvable.*

Since G is not p-nilpotent, by a result of Thompson [[7], Corollary], there exists a non-trivial character subgroup T of P such that $N_G(T)$ is not p-nilpotent. Now, T *char* $P \trianglelefteq N_G(P)$, so $N_G(P) \leq N_G(T)$. Step 2 implies that $N_G(T) = G$, so $T \trianglelefteq G$, therefore $O_p(G) > 1$. By Step 3.2, we have that $G/O_p(G)$ is p-nilpotent, so G ia p-solvable.

Step 5. *$N = O_p(G)$ and $|N| = p$.*

Since $O_p(G) > 1$, by Step 3.2, we have that $N \leq O_p(G)$. Therefore there exists a maximal subgroup M such that $G = NM$ and $N \cap M = 1$. So $G = O_p(G)M$. By Lemma 2.5, we have $O_p(G) \cap M = 1$, hence $N = O_p(G)$.

Next, we affirm that $N \nleq \Phi(P)$. Otherwise, $N \leq \Phi(G)$, a contradiction to Step 3.2. Therefore there exists a maximal subgroup P_1 of P such that $N \nleq P_1$, so $P = NP_1$. Put $N_1 = N \cap P_1$, then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. By hypotheses, there exists a normal subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is SS-quasinormal in G. By the minimality of N, we have that $N \leq K_1$ and $P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N$ is a Sylow p-subgroup of K_1. Let K_{1q} be a Sylow q-subgroup of K_1, where $q \neq p$. K_{1q} is also a Sylow q-subgroup of G. By Lemma 2.2, we have $(P_1 \cap K_1)K_{1q}$ is a subgroup. Since $N \leqslant K_1$, it follows that $N_1 = N \cap (P_1 \cap K_1)K_{1q} \trianglelefteq (P_1 \cap K_1)K_{1q}$. Therefore N_1 is normal in the subgroup $\langle N, (P_1 \cap K_1)K_{1q} | q \in \pi(G), q \neq p\rangle = K_1$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq P_1$, then $N_1 \trianglelefteq P_1K_1 = G$. The minimality of N yields $N_1 = 1$. Consequently, N is a cyclic subgroup of order p.

Step 6. *The final contradiction.*

Notice that $N = O_p(G) \trianglelefteq G$ and $|N| = p$, we have that $N \leq Z(P)$, so $P \leq C_G(N)$. On the other hand, G is p-solvable and $O_{p'}(G) = 1$, so $C_G(O_p(G)) \leq O_p(G)$. It follows that $P = O_p(G) = N$, hence $N_G(P) = G$. Therefore, by using the hypothesis of the theorem, we have that G is p-nilpotent, a contradiction. □

Remark. In Theorem 3.1, if take $H = G$, then we have the following:

Corollary 3.2. *Let G be a group, and P a Sylow p-subgroup of G such that $N_G(P)$ is p-nilpotent, where p is an odd prime factor of $|G|$. If every maximal subgroup of P is a CSS-subgroup of G, then G is p-nilpotent.*

Corollary 3.3. *Let H be a normal subgroup of a group G and p an odd prime factor of $|H|$. Also, let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{N}_p$ of all p-nilpotent groups and $G/H \in \mathcal{F}$. If $N_G(P)$ is p-nilpotent and every maximal* *subgroup of P is a CSS-subgroup of G, then $G \in \mathcal{F}$, where P is a Sylow p-subgroup of H.*

Proof. It is clear that $N_H(P)$ is p-nilpotent and every maximal subgroup of P is a CSS-subgroup of H. By Corollary 3.2, we have that H is p-nilpotent. Now let $H_{p'}$ be the normal Hall p'-subgroup of H, then $H_{p'} \trianglelefteq G$. By using similar arguments such as in the proof of Theorem 3.1, we have that G/H satisfies the hypotheses of the corollary, so $G/H \in \mathcal{F}$ by induction. Let $F_i(i = 1, 2)$ be the full and integrated formation function such that $\mathcal{N}_p = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively, then $G/C_G(K_1/K_2) \in F_1(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H_{p'}$ and every prime q dividing $|K_1/K_2|$. By Lemma 2.6, we have that $G/C_G(K_1/K_2) \in F_2(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H_{p'}$ and every prime q dividing $|K_1/K_2|$. Therefore, it follows that $G \in \mathcal{F}$. Hence, we may assume that $H_{p'} = 1$ and henceforth $H = P$ is a p-group. So, by the hypotheses of the corollary, $N_G(P) = G$ is p-nilpotent and therefore $G \in \mathcal{F}$. □

Remark. In Theorem 3.1, Corollary 3.2 and Corollary 3.3, the assumption that "$N_G(P)$ is p-nilpotent" is necessaey. For example, we consider the group $G = A_5$ and $p = 5$. In this case, since every maximal subgroup of Sylow 5-subgroup of G is 1, we see that every maximal subgroup of Sylow 5-subgroup of G is a CSS-subgroup of G, but G is not 5-nilpotent.

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ECONOMIC BENEFIT EVALUATION OF FOREST ECO-TOURISM ATTRACTIONS BASED ON FACTOR ANALYSIS

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Abstract. With the rapid development of economy, the tourism in China has flourished and eco-tourism has emerged. Eco-tourism is an inexorable trend in the sustainable development of tourism. Requirements for eco-tourism attractions are compared to ordinary tourist attractions. This study selected the forest eco-tourist attraction in Yunnan as an example because Yunnan has a warm climate, beautiful sceneries and rich forest resources. The economic benefits of the attraction were analyzed using factor analysis method. Score was calculated through extracting common factor, the index variables were processed, and the index results were then applied to the solution formulas. The obtained economic benefit results were analyzed and compared. Finally, relevant suggestions were proposed for local government, citizens and tourists. This work provides a reference for the economic construction and development of forest eco-tourism attractions in the future.
Keywords: factor analysis, forest eco-tourism, economic benefits, evaluation.

1. Introduction

Since the 21st century, forest area has greatly reduced with the rapid development of economy and the acceleration of urbanization [14]. People demand more about nature and are eager to get close to nature. With the rising of tourism,

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people tend to get closer to nature by means of tourism. But some areas have been excessively developed for pursuing economic benefits despite resource and environmental bearing capacity and moreover the uncivilized behaviors of some tourists make tourist environment worse and worse, leading to the increasing demand on eco-tourism [21].

Forest eco-tourism has captured the attention of the whole world rather than China alone. After the Second World War, countries such as America, Germany, England, Japan and France began to put emphasis on forest eco-tourism and spent heavily on building forest eco-tourism attractions. With the development of forest eco-tourism attractions, many benefits have been produced [22]. China, one of the countries with the largest number of eco-system categories, has rich forest resources [26]. Yunnan, in the southwest of China, covers an area of 19.924 million qing (1 qing = 6.6667 hectares) [24] and had a forest coverage rate of 55.7% till 2015; Yunnan has rich forest resources, beautiful sceneries and a pleasant climate.

Till December 2015, Yunnan has had 41 forest ecological parks with a total area of 150, 000 hectares and 159 natural reserve areas which covers an area of 2,840 thousand hectares [7] and a forest tourism system dominated by forest park and natural reserve area has formed preliminarily [9]. Wu B. J. [16] made a tourist satisfaction evaluation and analysis for Guangdong Dawang Mountain National Forest Park using importance-performance analysis diagram, perfected the deficiencies, and proposed strategies of Dawangshan forest eco-tourism development according to quadrant distribution. Zhan H. et al. [19] discussed the eco-tourism resources in natural conservation areas using gray cluster model and formulated an eco-tourism resources evaluation indicator system according to the environmental characteristics and research suggestions. This study made a factor analysis on the national forest parks in Yunnan. Yunnan is one of the minority enclaves and has wonderful national culture. Therefore, the forest eco-tourism attraction in Yunnan is more representative.

Introduction of scenic spot

Pudacuo national forest park, the first national park in China, locates in Shangri-La and covers an area of 1, 313 square kilometers; it is 22 kilometers away from the county [13]. The park has national sceneries such as plateau lakes; hence it is a good place for tourists. By now, the management right and operation right of the park have been separated. Besides, it has four basic functions, i.e., ecological protection, amusement, scientific research and survey and education practice [4].

Xishuangbanna forest park locating in Jinghong city, accounts for 1666.7 square kilometers [18]. It is the only tropical rain forest protection area in China. People are attracted by the park for its amazing animals and precious flowers and plants. The park relies mainly on tropical rain forest sightseeing, while relaxation and vacation are subsidiary, embellished by national customs [10].

Laojunshan national park locates in the west of Lijiang [8]. It gains its name because the local people say that TaishangLaojun (the supreme god of Taoism) is refining dan on the mountain. The park accounting for 1,324 square kilometers is famous for Danxia landform and picturesque peaks and rocks and it is called the originator of mountains in Yunnan province [23]. Centering on resource protection and tourism development, overall planning and all-round consideration and making progress while maintaining stability are the cores of the development and construction of Laojunshan.

The establishment of evaluation model based on factor analysis

Principle introduction

The indexes were grouped according to the correlation between them. The most representative common factor was selected out from each group [6]; thus we obtained several different common factors and calculated variance. To be short, the weight of evaluation was obtained by analyzing the selected factors.

Model establishment

Suppose that α scenic spots were selected and every scenic spot had β benefit indexes. Let original variable A (the mean value of $A_1, A_2, \ldots, A_\beta$) be equal to 0 and standard deviation equal to 1, then $B = (B_1, B_2, \ldots, B_\beta)$ was obtained. Index variables were supposed as $C_1, C_2, \ldots, C_\beta$ and common factors as $g_1, g_2, \ldots, g_t (t < \beta)$. Suppose that $B = (B_1, B_2, \ldots, B_\beta)$ index vector was observable, then $E(B) = 0$ and covariance matrix $cov(B) = \Sigma$. Besides, $G = (g_1, g_2, \ldots, g_t)$, $(t < \beta)$ variable was unobservable, then $E(G) = 0$ and covariance matrix $cov(G) = M$. Every component was independent from each other.

$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma^\beta)$ and G were independent from each other; $E(\gamma) = 0$; the diagonal matrix of $\Sigma(\gamma)$ was

$$(2-2-1) \qquad cov(\gamma) = \begin{pmatrix} \mu_{11}^2 & & & 0 \\ & \mu_{22}^2 & & \\ & & \cdots & \\ 0 & & & \mu_{\beta\beta}^2 \end{pmatrix}$$

If β original variables were expressed as $g_1, g_2, \ldots, g_t$, then we have:

$$(2-2-1) \qquad \begin{cases} B_1 = w_{11}g_1 + w_{12}g_2 + \ldots + w_{1t}gt + \gamma_1 \\ B_2 = w_{21}g_1 + w_{22}g_2 + \ldots + w_{2t}g_t + \gamma_2 \\ \ldots \\ B_\beta = w_{\beta 1}g_1 + w_{\beta 2}g_2 + \ldots + w_{\beta t}g_t + \gamma_\beta \end{cases}$$

i.e.,

$$= WG + \gamma, W = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1t} \\ w_{21} & w_{22} & \dots & w_{2t} \\ & & \dots & \\ w_{\beta 1} & w_{\beta 2} & \dots & w_{\beta t} \end{pmatrix}.$$

We called W as factor loading matrix and its elements as factor loading. Then the score of single common factor was calculated. Suppose original data matrix $A = (a_{ij})_{\alpha\times\beta}, (I = 1, 2, \dots, \alpha;\ j = 1, 2, \dots, \beta)$, we have $\widehat{G} = W'R^{-1}A(2-2-3)$ (W: factor loading matrix; R: original variable related matrix; A: original vector).

The selection of economic benefit indexes

Following the principles of scientificity, integrity and feasibility [1], the selected economic benefit evaluation indexes were summarized.

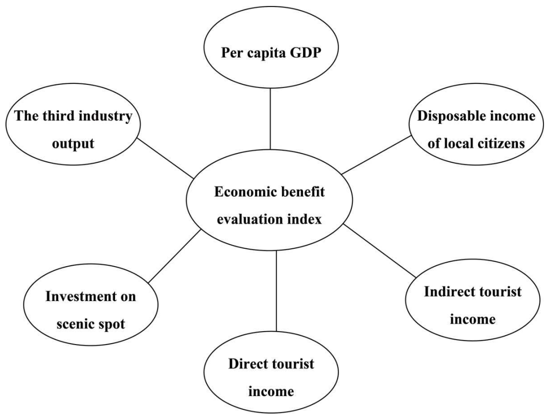

Figure 1: Economic indexes

Per capita GDP

Per capita GDP is an important index for reflecting the economic development condition and living level in an area; it can be obtained by dividing total output value in some period by population [1].

Group / Per capita GDP, permanent resident population and rank	Pudacuo national forest park	Xishuangbanna forest park	Laojunshan national park in Lijiang
Per capita GDP in 2015	39582.31	29031.98	22745.1
Rank	3	4	10
Permanent resident population (ten thousand)	40.7	115.7	127.5

Table 1. Per capita GDP of three scenic spots and their ranks in Yunnan

The comparison of the three parks suggested that, the per capita GDP of Pudacuo national forest park was the highest because it has a small permanent resident population; the per capita GDP of Xishuangbanna ranked the second among the three scenic spots, but it had the highest total output value; the per capita GDP of Laojunshan ranked low due to the large permanent resident population.

Per capita disposable income of local citizens

Per capita disposable income refers to the income spending on daily life after the deduction of tax, which can fully display the living level of citizens [1]. In 2015, the per capita income of Pudacuonational forest park, Xishuangbanna park and Laojunshan national park was 6,487 yuan, 10, 080 yuan and 6, 037 yuan respectively.

Direct tourist income

Direct tourism income refers to ticket income. Tourists need to pay a certain amount of money before entering scenic spots. The charge is different in every scenic spot, and some scenic spots are even for free.

Indirect tourist income

Indirect tourist income comes from accommodation, catering, entertainment, transportation, collocation and performance watching. The sum of direct tourist income and indirect tourist income is the total income of a scenic spot.

Group / Year	Pudacuo national forest park	Xishuangbanna forest park	Laojunshan national park in Lijiang
2015	3.14	4.13	5.84

Table 2. Total income of the three scenic spots (unit: 0.1 billion yuan)

Investment on scenic spot

The money that government or enterprise put into the construction and perfection of scenic spots is called investment on scenic spots. It can effectively reflect the emphasis of governmental departments and enterprises on scenic spots.

Appreciation of the third industry

Forest eco-tourism not only can increase local tourism income, but also can significantly promote the third industry such as real estate in local area; hence it is taken as one of the economic benefit evaluation indexes [15].

Group / Year	Pudacuo national forest park	Xishuangbanna forest park	Laojunshan national park in Lijiang
2015	5.52	5.84	4.32

Table 3. Appreciation of the third industry in the three scenic spots (unit: 0.1 billion yuan)

Analysis of economic benefit evaluation results

Data in year 2015 were selected. Per capita GDP was supposed as A_1, per capita disposable income as A_2, total income as A_3, the output of the third industry as A_4, and original matrix as A. The detailed data are shown in table 4.

Group / Economic benefit index	Pudacuo national forest park	Xishuangbanna forest park	Laojunshan national park in Lijiang
Per capita GDP	39582.31	29031.98	22745.1
Per capita disposable income (yuan)	6487	10080	6037
Total income (0.1 billion yuan)	3.14	4.13	0.94
Appreciation of the third industry output (0.1 billion yuan)	5.52	5.84	4.32

Table 4. Summary of economic indexes

$$A = \begin{pmatrix} 39582.31 & 6487 & 3.14 & 5.52 \\ 29031.98 & 10080 & 4.13 & 5.84 \\ 22745.1 & 6037 & 5.84 & 4.32 \end{pmatrix}$$

Using SPSS 17.0, relevant coefficients were obtained, as shown below.

	B_1	B_2	B_3	B_4
B_1	1	0.684	0.237	-0.796
B_2	0.684	1	-0.398	-0.951
B_3	0.237	-0.398	1	0.562
B_4	-0.796	-0.951	0.562	1

Table 5. Relevant coefficients of the original index matrix

The observation of relevant coefficients of the original matrix suggested that, per capita GDP and per capita disposable income was in a highly negative correlation, indicating there was a huge gap between per capita GDP and per capita disposable income and there was no correlation between per capita disposable income and the appreciation of the third industry output; per capita GDP was in a positive correlation to per capita disposable income, suggesting the improvement of per capita GDP was beneficial to per capita disposable income; total input was in a positive correlation to the appreciation of the third industry output and they promote each other.

Table 6 demonstrates that, the accumulated variance contribution rate of the first and third factor reached 100%, but the rate of the other factors was so low that it can be ignored. Hence the first and third factors could be taken for effectively evaluating the economic benefit of scenic spots. Through calculating loading factor matrix, we have:

	Characteristic value	Variance contribution rate	Accumulated variance contribution rate	Characteristic value	Variance contribution rate	Accumulated variance contribution rate	Characteristic value	Variance contribution rate	Accumulated variance contribution rate
1	4.845	49.337	49.337	4.845	49.337	49.337	4.839	49.116	49.116
2	1.75E-14	1.95E-15	100						
3	4.729	48.651	48.651	4.729	48.651	100	4.697	4.697	100
4	2.42E-14	2.99E-15	100						

Table 6. Variable characteristic values and variance contribution rates

	1	3
B_1	0.913	0.501
B_2	-0.795	-0.416
B_3	0.436	0.891
B_4	0.893	0.124

Table 7. Factor loading matrix

After substituting the above values into expression $(2-2-2)$, we have:

(1) $$B_1 = 0.913g_1 + 0.501g_3$$

(2) $$B_2 = 0.795g_1 - 0.416g_3$$

(3) $$B_3 = -0.436g_1 + 0.891g_3$$

(4) $$B_4 = -0.893g_1 + 0.124g_3.$$

Referring to the expression, we found the absolute value of factor loading capacity was smaller or equal to 0.5. There was overlapped information, suggesting the correlation still could be reduced. After rotation, we obtained the matrix after transformation.

	1	3
B_1	0.999	0.483
B_2	-0.665	-0.283
B_3	0.998	0.756
B_4	0.463	0.084

Table 8. Factor loading matrix obtained after rotation

After substituting the above data into the expression $(2-2-2)$, we have:

(5) $$B_1 = 0.999g_1 + 0.483g_3$$

(6) $$B_2 = -0.665g_1 - 0.283g_3$$

(7) $$B_3 = 0.998g_1 + 0.765g_3$$

(8) $$B_4 = 0.463g_1 + 0.084g_3$$

After rotation, we found that, the absolute value of factor loading capacity was smaller or equal to 0.552; hence the rotation method was not available. Thus we explained g_2 and g_4 by taking g_1 and g_3 as common factors.

Coefficient matrix obtained by regression method was as follows.

	1	3
B_1	0.203	0.025
B_2	-0.092	-0.185
B_3	-0.074	-0.114
B_4	0.161	0.098

Table 9. Factor score coefficient matrix

Then the values of g_1 and g_3 of the three parks were calculated and ranked, as shown below.

Group	g_1		g_3			
	Score	Rank	Score	Rank	Total score	Total rank
Pudacuo national forest park	0.8513	1	0.1533	2	1.0046	1
Xishuangbanna national park	-0.394	3	0.1619	1	-0.2321	3
Laojunshan national park	0.395	2	0.1206	3	0.5156	2

Table 10. Ranking of common factor scores of the three scenic spots

g1 of Pudacuo national forest park ranked the first, g_3 of Xishuangbanna national park ranked the first, the total score of Pudacuo national park was 1.0046, suggesting the economic benefit of Pudacuo national forest park was the highest, followed by Laojunshan and Xishuangbanna.

Besides, we found two problems. The first problem is that, insufficient investment can inhibit the development of the forest park. Though the scale of scenic spots is continuously enlarged, the mode of input being larger than output remains unchanged and economic benefit is not as expected [17]. The fund invested on forest park come from the government and enterprises. However, the fund assigned to every scenic spot is low because Yunnan province is a major tourism province and has many scenic spots, resulting in no increase of economic benefit. The second problem is that the management model is affected by problems left over by history. For example, Pudacuopark was not a national park when it was established. At first, it was managed by enterprises, but then thoroughly managed by the government; currently, the management right and operation right of the park are separated [20]. But the defects existing when the park was managed by enterprises still have influence. Diqing state government as the main part has the power of administrative jurisdiction and departments at different levels play a supervision role; the insufficient agent authorization of scenic spot authority blocks the joint development of scenic spots and communities. The working efficiency of scenic spots is low due to the inconsistent attitudes of investment companies and authority to policies. Moreover, there are no definite laws for the construction and management of Pudacuo national forest park locating in minority habitation in China and the enforcement system of the park originated from foreign countries, which is inconsistent with the national condition.

Suggestions for the development of forest eco-tourism in Yunnan
Suggestions to government

The leading function of government should be combined with the pushing action of market to improve management efficiency [5]. In the initial construction period of forest national park, the government should play a leading role and emphatically promote the establishment of regulations of scenic spots. Moreover, the market allocates resources. What is more, the assistance of enterprises and the public is indispensable [25]. Only in this way can we promote the establishment of forest parks, improve per capita GDP of scenic spots and drive the harmonious development of the third industry.

Suggestions to scenic spots

The first suggestion is to optimize the existing products and develop new products. The characteristics of forest parks should be put into full play and combine with the advantages of local area. Besides, network should be internationalized to keep up with the trend of age and products need to be refined by considering the demand of tourists [11]. As shown in figure 2, there are totally three categories of tourism products and several subcategories.

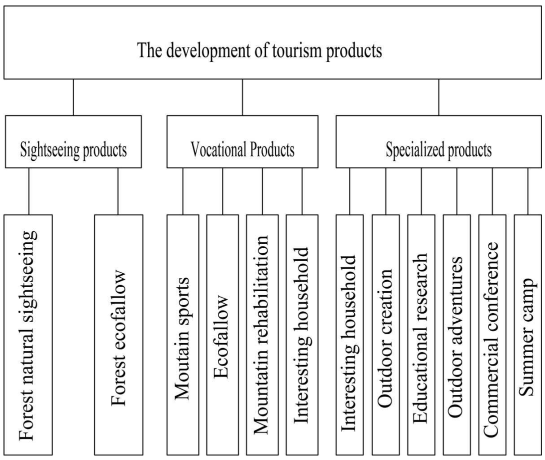

Figure 2: Product development

Secondly, scenic spots should establish good public image and improve popularity [2], i.e., strengthen publicity, cultivate special culture and do cultural transmission.

Lastly, the introduction of talents should be attached great importance. On the one hand, scenic spots should enroll students from relevant professions. On the other hand, scenic spots can cultivate those students to carry out researches to guide practice [3].

Suggestions to local citizens

Local citizens should pay attention to or utilize culture and form a hospitable atmosphere spontaneously to make guests feel at home. Yunnan is a province with the most minorities; besides Han nationality, there were another 25 nationalities living in Yunnan. Every nationality has its own special culture. Local citizens and the other nationalities can learn from each other and seek common points while reserving difference to make local culture yield unusually brilliant results [12].

Conclusion

There are many ways for evaluating the economic benefits of a tourist attraction, and factor analysis method is one of them. Firstly, factors were selected from economic field; then common factor was selected after multiple factors were analyzed and calculated; the demonstration of benefits using common factor provided the evaluation of economic benefit of scenic spots with data support and finally the economic benefits of forest ecological scenic spots were systemically assessed. Based on it, some suggestions were proposed to the establishment of scenic spots, aiming to improve the benefits of scenic spots.

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ON THE GENERALIZATION OF $(\in, \in \vee q)$-INTUITIONISTIC FUZZY BI-IDEALS OF SEMIGROUPS

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Abstract. In this article, we introduce the notion of $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideals, $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals and study some of its properties. We study the related properties of the $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals, $(1, 2)$-ideals and in particular, an $(\in, \in \vee q_k)$-fuzzy bi-ideals and $(1, 2)$-ideals in semigroups will be investigated.

Keywords: $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideals,$(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals, $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals.

1. Introduction

The idea of a fuzzy set was first originated by Zadeh in 1965 [1]. Fuzzy set theory has been shown to be a useful tool to define conditions in which the data

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are inexact or vague. Fuzzy sets theory handle such conditions by attributing a degree to which a certain object belongs to a set. The concept of fuzzy group was first proposed by Rosenfeld [2]. The notion of fuzzy semigroups was first studied by Kuroki in his standard paper [3]. The concepts of fuzzy ideals, bi-ideals, semi-prime ideals, quasi-ideals of semigroups are initiated by Kuroki in [4, 5, 6, 7, 8, 9, 10]. A logical account of fuzzy semigroup was specified by Mordeson et.al. [11], and they have found theoretical results on fuzzy semigroups and their use in fuzzy languages, fuzzy finite state machines, and fuzzy coding. The book of Mordeson and Malik treaties with the application of fuzzy method to the notions of formal languages and automata [12]. Newly, fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. Ameri and Noari in [13] introduced fuzzy hyperalgebras and investigated some vital results. In [14], Davvaz et.al., initiated fuzzy Hv-ideals in Γ-Hv-rings. The concept of fuzzy Γ-hypernearrings was initiated by Davvaz in [15]. In [16], Davvaz originated fuzzy Krasner (m, n)-hyperrings. Sun et al., consider fuzzy hypergraphs on fuzzy relations in [17]. The notion of "belong to" relation $(\in)$ was intitiated by Pu and Lia in [18]. The concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset was proposed by Morali in [19]. Bhakat and Das in [20], initiated the ideas of (α, β)-fuzzy subgroups by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) concerning with a fuzzy point and a fuzzy subgroup, and defined an $(\in, \in \vee q)$-fuzzy subgroup of a group. Kazanci and Yamak in [21], studied generalized types fuzzy bi-ideals of semigroups and defined $(\in, \in \vee q)$-fuzzy bi-ideals of semigroups. The concept of generalized fuzzy interior ideals of semigroups was studied by Jun and Song in [22]. Shabir et. al. in [23], characterized regular semigroups by the properties of (α, β)-fuzzy ideals, bi-ideals and quasi-ideals. S In [24], Shabir et. al. originated the notion of $(\in, \in \vee q_k)$-fuzzy ideals of semigroups and characterized regular semigroups by these ideals. Shabir and Mehmood in [25], initiated the notion of $(\in, \in \vee q_k)$-fuzzy h-ideals of hemirings and characterized different classes of hemirings by the using the concept of $(\in, \in \vee q_k)$-fuzzy h-ideals. Aslam et al. in [26], originated the notion of (α, β)-fuzzy Γ-ideals of Γ-LA-semigroups and given some characterization of Γ-LA-semigroups by (α, β)-fuzzy Γ-ideals:

In 1986, the notion of intuitionistic fuzzy set (IFS) was premised by Atanassov in [27]. An Atanassov intuitionistic fuzzy set is considered as a generalization of fuzzy set [1]. In the sense of Atanassov an IFS is characterized by a pair of functions valued in $[0, 1]$: the membership function and the non-membership function. The evaluation degrees of membership and non-membership are independent. Thus, an Atanassov intuitionistic fuzzy set is most substantial and brief to designate the spirit of fuzziness, and Atanassov intuitionistic fuzzy set theory may be more appropriate than fuzzy set theory for dealing with imperfect knowledge in many problems. Biswas in [28], use the idea of intuitionistic fuzzy set and initiated the the notion of intuitionistic fuzzy subgroup of a group. Kim and Jun in [29], originated intuitionistic fuzzy ideals of semigroups. In [30], Kim and Lee initiated the notion of intuitionistic fuzzy bi-ideals of semigroups. The concepts of intuitionistic fuzzy interior ideals of semigroups was initiated by Kim and Jun in [31]. The concept of intuitionistic fuzzy point was initiated by Coker and Demirci in [32]. Jun in [33], introduced the concept of (Φ, Ψ)-intuitionistic fuzzy subgroups. Aslam and Abdullah in [34], initiated the concept of (Φ, Ψ)-intuitionistic fuzzy ideals of semigroups. Abdullah et.al., in [35], initiated the concept of (α, β)-intuitionistic fuzzy ideals of hemirings by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) between an intuitionistic fuzzy point and an intuitionistics fuzzy set, and they defined prime (semi-prime) (α, β)-intuitionistic fuzzy ideals of hemirings. In [36], Khan et. al. initiated the notion of $(\in, \in q_k)$-intuitionistic fuzzy bi-ideals in ordered semigroups.

In this article, we introduce the notion of $(\in, \in q_k)$-intuitionistic fuzzy bi-ideal, $(\in, \in q_k)$-intuitionistic fuzzy $(1,2)$-ideal of semigroup, and studied related properties. We also prove that in regular semigroup, every $(\in, \in q_k)$-intuitionistic fuzzy $(1,2)$- ideal of semigroup S is an $(\in, \in q_k)$-intuitionistic fuzzy bi-ideal of semigroup S.

2. Preliminaries

In this section we give some basic definitions and results which are use in this note. Throughout in this article S will denote semigroup unless otherwise stated.

An algebraic system $(S, \cdot)$ consisting of a non-empty set S together with an associative binary operation " $\cdot$ " is called a semigroup. A subsemigroup of S is a non-empty set A such that $A^2 \subseteq A$. A left (resp. right) ideal of S is a non-empty set if $SA \subseteq A (AS \subseteq A)$. It is called two sided ideal of S if it is both left and right ideal of S. A quasi-ideal Q of S is a non-empty subset of S if $QS \cap SQ \subseteq Q$. A bi-ideal of S is a subsemigroup B of S if $BSB \subseteq B$. A generalized bi-ideal B of S is a non-empty subset of S if $BSB \subseteq B$. An interior ideal A of S is a subsemigroup of S if $SAS \subseteq A$. An element "x" of S is called a regular element if there exists an element $a \in S$ such that $x = xax$. "S" is called regular if every element of S is regular.

Definition 2.1 ([27])**.** Suppose X is a non-empty set. An intuitionistic fuzzy set (briefly, IFS) F is object having the form

$$F = \{\langle x, \lambda_F(x), \mu_F(x)\rangle : x \in X\}$$

where the functions $\lambda_F : X \to [0,1]$ and $\mu_F : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set F, respectively, and $\lambda_F(x) + \mu_F(x) \leq 1$ for all $x \in S$ for simplicity, we use the symbol $F = \langle \lambda_F, \mu_F\rangle$ for the IFS $F = \{\langle x, \lambda_F(x), \mu_F(x)\rangle : x \in X\}$.

Definition 2.2 ([28])**.** An intuitionistic fuzzy subsemigroup of S is an IFS $F = \langle \lambda_F, \mu_F\rangle$ in S if the satisfy the following conditions:

(IF1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y)$,

(IF2) $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y)$,
$\forall\ x, y \in S$.

Definition 2.3 ([28])**.** An intuitionistic fuzzy left (resp. right) ideal of S is an IFS $F = \langle \lambda_F, \mu_F \rangle$ in S if it satisfy $\lambda_F(xy) \geq \lambda_F(y)\,(\lambda_F(xy) \geq \lambda_F(x))$ and $\mu_F(xy) \leq \mu_F(y)\,(\mu_F(xy) \leq \mu_F(x))$ for all $x, y \in S$.

Definition 2.4 ([29])**.** An intuitionistic fuzzy subsemigroup of S is an IFS $F = \langle \lambda_F, \mu_F \rangle$ in S if the following conditions hold:

(IF1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y), \lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z)$
(IF2) $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y), \mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z)$
$\forall\ x, y, z \in S$.

Definition 2.5 ([36])**.** Let c be a point in a non-empty set X. If $t_1, t_2 \in (0, 1]$ are two real numbers such that $0 \leq t_1 + t_2 \leq 1$, then the IFS $\langle x; (t_1, t_2) \rangle = \langle a, x_{t_1}, 1 - x_{1-t_2} \rangle$ is said to be an intuitionistic fuzzy point (IFP for short) in X, where t_1 (resp, t_2) is the degree of membership (resp, non-membership) of $\langle x; (t_1, t_2) \rangle$ and $x \in X$ is the support of $\langle x; (t_1, t_2) \rangle$. Let $\langle x; (t_1, t_2) \rangle$ be an IFP in X and let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS in X. Then, $\langle x; (t_1, t_2) \rangle$ is said to belong to F, written $\langle x; (t_1, t_2) \rangle \in F$, if $\lambda_F(x) \geq t$ and $\mu_F(x) \leq t_2$. We say that $\langle x; (t_1, t_2) \rangle$ is quasi-coincident with F, written $\langle x; (t_1, t_2) \rangle\, q_k F$, if $\lambda_F\,(x) + t_1 + k > 1$ and $\mu_F(x) + t_2 + k < 1$. To say that $\langle x; (t_1, t_2) \rangle \in \vee q_k F$ (resp, $\langle x; (t_1, t_2) \rangle \in \wedge q_k F$) means that $\langle x; (t_1, t_2) \rangle \in F$ or $\langle x; (t_1, t_2) \rangle\, q_k F$ (resp, $\langle x; (t_1, t_2) \rangle \in F$ and $\langle x; (t_1, t_2) \rangle\, q_k F$) and $\langle x; (t_1, t_2) \rangle\, \overline{\in \vee q_k} F$ means that $\langle x; (t_1, t_2) \rangle \in \vee q_k F$ does not hold and $t_1 \wedge t_2 = \min\{t_1, t_2\}$; $r_1 \vee r_2 = \max\{r_1, r_2\}$.

3. $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals

In this section, we initiated the notion of $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left(resp. right, two sided) ideal, $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals in semigroups and investigated some of its properties.

Definition 3.1. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S if satisfy the following condition:

$\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$, $\forall$ $x, y \in S$, $k \in [0, 1$, $t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1)$

Definition 3.2. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if satisfy the following condition:

$(IFI1)$ $\langle y; (t, r) \rangle \in F \Rightarrow \langle xy; (t, r) \rangle \in \vee q_k F$, (resp. $\langle x; (t, r) \rangle \in F \Rightarrow \langle xy; (t, r) \rangle \in \vee q_k F), \forall x, y \in S$, $k \in [0, 1)$, $t \in (0, 1]$ and $r \in [0, 1)$.

Definition 3.3. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy ideal of S, if it is an $(\in, \in \vee q_k)$-intutionistic fuzzy left ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy right ideal of S.

Definition 3.4. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if satisfy the following conditions:

$(IFB1)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(IFB2)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle z; (t_2, r_2) \rangle \in F \Rightarrow \langle xyz; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(\forall x, y, z \in S$ and $k \in [0, 1))(t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1))$

Definition 3.5. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S if satisfy the following conditions:

$(\forall a, x, y, z \in S$ and $k \in [0, 1))(t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1))$

$(IF1)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(IF2)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle z; (t_2, r_2) \rangle \in F \Rightarrow \langle xa\,(yz)\,; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

Theorem 3.6. *Let B be a left (resp. right) ideal of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash R)\,(\lambda_F(x) = 0$ and $\mu_F\,(x) = 1)\,,$

2) $(\forall x \in S \backslash R)\,\left(\lambda_F(x) \geq \frac{1-k}{2}\right.$ and $\left.\mu_F\,(x) \leq \frac{1-k}{2}\right)\,,$

Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S.

Proof. Let $x, y \in S$ and $t \in (0, 1]$, and $r \in [0, 1)$ be such that $\langle y; (t, r) \rangle\, qF$. Then, $\lambda_F\,(y) + t > 1$ and $\mu_F\,(y) + r < 1$. So, $y \in B$. Therefore, $xy \in B$. Thus, if $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$, then $\lambda_F\,(xy) \geq \frac{1-k}{2} \geq t$ and $\mu_F\,(xy) \leq \frac{1-k}{2} \leq r$. Hence $\langle xy; (t, r) \rangle \in F$. If $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$, then $\lambda_F\,(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F\,(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus $\langle xy; (t, r) \rangle\, q_k F$. Hence $\langle xy; (t, r) \rangle \in \vee q_k F$. Since $t + r \leq 1$, the case $t > \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$ does not occur. From the fact that $\langle y; (t, r) \rangle\, q_k F$, if implies that the case $t \leq \frac{1-k}{2}$ and $r < \frac{1-k}{2}$ does not occur. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy left ideal of S. □

Theorem 3.7. *Let B be a subsemigroup of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash B)\,(\lambda_F\,(x) = 0$ and $\mu_F\,(x) = 1)\,,$

2) $(\forall x \in S \backslash B)\,\left(\lambda_F\,(x) \geq \frac{1-k}{2}\right.$ and $\left.\mu_F\,(x) \leq \frac{1-k}{2}\right)\,.$

Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S.

Proof. Proof of the Theorem follows from Theorem 3.6. □

Theorem 3.8. *Let B be a bi-ideal of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash B)\left(\lambda_F(x)=0 \text{ and } \mu_F(x)=1\right)$,
2) $(\forall x \in S \backslash B)\left(\lambda_F(x) \geq \frac{1-k}{2} \text{ and } \mu_F(x) \leq \frac{1-k}{2}\right)$,
Then, $F=\langle\lambda_F, \mu_F\rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.

Proof. Let $x, y \in S$ and $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be such that $\langle x;(t_1, r_1)\rangle\, qF$ and $\langle y;(t_2, r_2)\rangle\, qF$. Then $\lambda_F(x)+t_1>1$ and $\mu_F(y)+r_1<1$, and $\lambda_F(x)+t_2>1$ and $\mu_F(y)+r_2<1$. Hence $x, y \in B$. since B is a subsemigroup, therefore $xy \in B$ and so, $\lambda_F(xy) \geq \frac{1-k}{2}$ and $\mu_F(xy) \leq \frac{1-k}{2}$. If $t_1 \wedge t_2>\frac{1-k}{2}$ and $r_1 \vee r_2<\frac{1-k}{2}$, then $\lambda_F(xy)+t_1 \wedge t_2+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and $\mu_F(xy)+r_1 \vee r_2+k<\frac{1-k}{2}+\frac{1-k}{2}+k=1$. Hence $\langle xy;(t_1 \wedge t_2, r_1 \vee r_2)\rangle\, q_k F$. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xy) \geq t_1 \wedge t_2$ and $\mu_F(xy) \leq r_1 \vee r_2$ and so, $\langle xy;(t_1 \wedge t_2, r_1 \vee r_2)\rangle \in F$. Since $t_1+r_1 \leq 1$ and $t_2+r_2 \leq 1$, the case $t_1 \wedge t_2>\frac{1-k}{2}, r_1 \vee r_2 \geq \frac{1-k}{2}$ does not hold. From the fact that $\langle x;(t_1, r_1)\rangle\, qF$ and $\langle y;(t_2, r_2)\rangle\, qF$, it implies that $t_1 \wedge t_2 \leq \frac{1-k}{2}, r_1 \vee r_2<\frac{1-k}{2}$ does not hold. Hence, $F=\langle\lambda_F, \mu_F\rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Now let, $a, b, c \in S$ and $t_1, t_2 \in(0,1]$ and $r_1, r_2 \in[0,1)$ be such that $\langle x;(t_1, r_1)\rangle\, qF$ and $\langle z;(t_2, r_2)\rangle\, qF$. Then $\lambda_F(x)+t_1>1$ and $\mu_F(x)+r_1<1$, and $\lambda_F(z)+t_2>1$ and $\mu_F(z)+r_2<1$. Hence $x, z \in B$. since B is a bi-ideal. Therefore $xyz \in B$. Hence $\lambda_F(xyz) \geq \frac{1-k}{2}$ and $\mu_F(xyz) \leq \frac{1-k}{2}$. If $t_1 \wedge t_2>\frac{1-k}{2}$ and $r_1 \vee r_2<\frac{1-k}{2}$, then $\lambda_F(xyz)+t_1 \wedge t_2+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and $\mu_F(xyz)+r_1 \vee r_2+k<\frac{1-k}{2}+\frac{1-k}{2}+k=1$. Hence, $\langle xyz;(t_1 \wedge t_2, r_1 \vee r_2)\rangle\, q_k F$. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xyz) \geq t_1 \wedge t_2$ and $\mu_F(xyz) \leq r_1 \vee r_2$ implies that $\langle xyz;(t_1 \wedge t_2, r_1 \vee r_2)\rangle \in F$. Since $t_1+r_1 \leq 1$ and $t_2+r_2 \leq 1$, the case $t_1 \wedge t_2>\frac{1-k}{2}, r_1 \vee r_2 \geq \frac{1-k}{2}$ does not hold. From the fact that $\langle x;(t_1, r_1)\rangle\, qF$ and $\langle y;(t_2, r_2)\rangle\, qF$, it implies that $t_1 \wedge t_2 \leq \frac{1-k}{2}, r_1 \vee r_2<\frac{1-k}{2}$ does not hold. Hence, $F=\langle\lambda_F, \mu_F\rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. □

Theorem 3.9. *Let B be a $(1,2)$-ideal of S and $F=\langle\lambda_F, \mu_F\rangle$ be an IFS such that:*
1) $(\forall x \in S \backslash B)\left(\lambda_F(x)=0 \textit{ and } \mu_F(x)=1\right)$,
2) $(\forall x \in S \backslash B)\left(\lambda_F(x) \geq \frac{1-k}{2} \textit{ and } \mu_F(x) \leq \frac{1-k}{2}\right)$.
Then, $F=\langle\lambda_F, \mu_F\rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S.

Proof. Proof of the Theorem follows from Theorem 3.8. □

Theorem 3.10. *Let $F=\langle\lambda_F, \mu_F\rangle$ is an intuitionistic fuzzy set in S. Then $F=\langle\lambda_F, \mu_F\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if the following conditions satisfied:*

1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.
2) $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Proof. Let $F=\langle\lambda_F, \mu_F\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.
1) Suppose $x, y \in S$. We consider the following two cases:
i) $\lambda_F(x) \wedge \lambda_F(y)<\frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(y)>\frac{1-k}{2}$
ii) $\lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2}$

Case i. Suppose that $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Then, $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y)$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y)$. We choose $t \in (0,1]$ and $r \in [0,1)$ in such a way that $\lambda_F(xy) < t < \lambda_F(x) \wedge \lambda_F(y)$ and $\mu_F(xy) > r > \mu_F(x) \vee \mu_F(y)$. Then, $\langle x; (t,r) \rangle \in F$ and $\langle y; (t,r) \rangle \in F$, but $\langle xy; (r,s) \rangle \overline{\in \vee q_k} F$, which is a contradiction.

Case ii. Suppose that, $\lambda_F(xy) < \frac{1-k}{2}$ and $\mu_F(xy) > \frac{1-k}{2}$. Then, $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in F$ and $\langle y; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in F$, but $\langle xy; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \overline{\in \vee q_k} F$, which is a contradiction. Hence, $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.

2) Now suppose $x, y, z \in S$. We consider the following two cases:

i) $\lambda_F(x) \wedge \lambda_F(z) < \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(z) > \frac{1-k}{2}$

ii) $\lambda_F(x) \wedge \lambda_F(z) \geq \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(z) \leq \frac{1-k}{2}$

Case i. Suppose that $\lambda_F(xyz) < \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) > \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$, then $\lambda_F(xyz) < \lambda_F(a) \wedge \lambda_F(c)$ and $\mu_F(xyz) > \mu_F(x) \vee \mu_F(z)$. We choose $t \in (0,1]$ and $r \in [0,1)$ in such a way that $\lambda_F(xyz) < t < \lambda_F(x) \wedge \lambda_F(z)$ and $\mu_F(xyz) > r > \mu_F(x) \vee \mu_F(z)$. Then, $\langle x; (t,r) \rangle \in F$ and $\langle z; (t,r) \rangle \in F$, but $\langle xyz; (t,r) \rangle \overline{\in \vee q_k} F$, which is a contradiction.

Case ii. Suppose that $\lambda_F(xyz) < \frac{1-k}{2}$ and $\mu_F(xyz) > \frac{1-k}{2}$. Then, $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in F$ and $\langle z; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in F$. But $\langle xyz; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \overline{\in \vee q_k} F$, which is a contradiction. Hence $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Conversely, suppose that $F = \langle \lambda_F, \mu_F \rangle$ satisfy (i) and (ii). Let $x, y \in S$, $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be in a way that $\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F$. Then, $\lambda_F(x) \geq t_1$ and $\mu_F(x) \leq r_1$, $\lambda_F(y) \geq t_2$ and $\mu_F(y) \leq r_2$. Now we have $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. It Implies that $\lambda_F(xy) \geq t_1 \wedge t_2 \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq r_1 \vee r_2 \vee \frac{1-k}{2}$. Then, we have the following two cases.

i) $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$.

ii)$t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, the other cases does not occurs.

Case i. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xy) \geq t_1 \wedge t_2$ and $\mu_F(xy) \leq r_1 \vee r_2$, which implies that $(xy)(t_1 \wedge t_2, r_1 \vee r_2) \in F$.

Case ii. If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then, $\lambda_F(xy) \geq \frac{1-k}{2}$ and $\mu_F(xy) \leq \frac{1-k}{2}$, which implies that $\lambda_F(xy) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xy) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus, $\langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \, q_k F$. Hence, $\langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

Now, let $x, y, z \in S$ and $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be in a way that $\langle x; (t_1, r_1) \rangle \in F$ and $\langle z; (t_2, r_2) \rangle \in F$. Then, $\lambda_F(x) \geq t_1$ and $\mu_F(x) \leq r_1, \lambda_F(z) \geq t_2$ and $\mu_F(z) \leq r_2$. Now we have $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$. It Implies that $\lambda_F(xyz) \geq t_1 \wedge t_2 \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq r_1 \vee r_2 \vee \frac{1-k}{2}$. Then, we have the following two cases.

i) $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$

ii) $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$

Case i. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then $\lambda_F(xyz) \geq t_1 \wedge t_2$ and $\mu_F(xyz) \leq r_1 \vee r_2$, which implies that $(xyz)(t_1 \wedge t_2, r_1 \vee r_2) \in F$.

Case ii: If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then $\lambda_F(xyz) \geq \frac{1-k}{2}$ and $\mu_F(xyz) \leq \frac{1-k}{2}$, which implies that $\lambda_F(xyz) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xyz) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus, $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2)\rangle\, q_k F$. Hence, $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2)\rangle \in \vee q_k F$. $\square$

Every intuitionistic fuzzy bi-ideal and an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of a semigroup S. But the converse is not true. For this we have the following example.

Example 3.11. Let $S = \{a, b, c, d, e\}$ be a semigroup with the following table.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	c
d	a	a	c	d	e
e	a	a	c	c	e

Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS is a semigroup S, defined by $\lambda_F(a) = \lambda_F(c) = 0.3, \lambda_F(b) = \lambda_F(e) = 0.6$, $\lambda_F(d) = 0.5$ and $\mu_F(a) = \mu_F(c) = 0.2$, $\mu_F(b) = \mu_F(e) = 0.3$, $\mu_F(d) = 0.5$. Take $\frac{1-k}{2} = 0.2$. Thus by simple calculation $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_{0.6})$-intuitionistic fuzzy bi-ideal of S. But $F = \langle \lambda_F, \mu_F \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S nor an fuzzy bi-ideal of S. i.e,

$$\lambda_F(d \cdot e \cdot d) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(d) \wedge \lambda_F(d) \wedge 0.5$$
$$\text{and } \lambda_F(d \cdot e \cdot d) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(d) \wedge \lambda_F(d)$$

Remark 3.12. From above example we say that an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is a generalization of an$(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal and fuzzy bi-ideal of S.

Theorem 3.13. *Let $F = \langle \lambda_F, \mu_F \rangle$ is an intuitionistic fuzzy set in S. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S if and only if the following conditions satisfied:*

i) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.

ii) $\lambda_F(xa(yz)) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xa(yz)) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Proof. Proof of the Theorem follows from Theorem 3.10. $\square$

Lemma 3.14. *Every $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

The converse of the above Lemma not true. For this we have the following example.

Let $S = \{a, b, c, d\}$ be a semigroup with the following table:

$\cdot$	a	b	c	d
a	a	c	a	a
b	a	a	a	a
c	b	a	a	d
d	a	a	d	a

(a) : Let $F = \langle\lambda_F, \mu_F\rangle$ be an IFS defined by, $\lambda_F(a) = 0.6$, $\lambda_F(b) = 0.5$, $\lambda_F(c) = 0.3$, $\lambda_F(d) = 0.2$ and $\mu_F(a) = \mu_F(b) = 0.6$, $\mu_F(c) = \mu_F(d) = 0.4$. Then $F = \langle\lambda_F, \mu_F\rangle$ is an $(\in, \in\vee q_{0.3})$-intuitionistic fuzzy bi-ideal of S, where $\frac{1-k}{2} = 0.3$. Clearly it is not an $(\in, \in\vee q)$-intuitionistic fuzzy bi-ideal nor intuitionistic fuzzy bi-ideal of S. Because, $\lambda_F(a \cdot b) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(a) \wedge \lambda_F(b) \wedge 0.5$ and $\lambda_F(a \cdot b) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(a) \wedge \lambda_F(b)$. Also $\lambda_F(c \cdot d) = \lambda_F(d) = 0.2 \ngeq 0.3 = \lambda_F(c) \wedge \frac{1-k}{2}$. Which shows that $F = \langle\lambda_F, \mu_F\rangle$ is not $(\in, \in\vee q_{0.3})$-intuitionistic fuzzy right ideal of S.

(b) : Let $F = \langle\lambda_F, \mu_F\rangle$ be an IFS defined by, $\lambda_F(a) = 0.7$, $\lambda_F(b) = 0.3$, $\lambda_F(c) = 0.4$, $\lambda_F(d) = 0.2$ and $\mu_F(a) = \mu_F(b) = 0.2$, $\mu_F(c) = \mu_F(d) = 0.32$.Then $F = \langle\lambda_F, \mu_F\rangle$ is an $(\in, \in\vee q_{0.3})$-intuitionistic fuzzy bi-ideal of S. Clearly it is not an $(\in, \in\vee q)$-intuitionistic fuzzy bi-ideal nor intuitionistic fuzzy bi-ideal of S. Because, $\lambda_F(c \cdot a) = \lambda_F(b) = 0.3 \ngeq 0.5 = \lambda_F(c) \wedge \lambda_F(a) \wedge 0.5$ and $\lambda_F(c \cdot a) = \lambda_F(b) = 0.3 \ngeq 0.5 = \lambda_F(c) \wedge \lambda_F(a)$. Also $\lambda_F(d \cdot c) = \lambda_F(d) = 0.2 \ngeq 0.3 = \lambda_F(c) \wedge \frac{1-k}{2}$. Which shows that $F = \langle\lambda_F, \mu_F\rangle$ is not an $(\in, \in\vee q_{0.3})$-intuitionistic fuzzy left ideal of S.

Lemma 3.15. *(i) Every $(\in\vee q_k, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

(ii) Every $(\in, \in\vee q)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S.

(iii) Every $(\in, \in)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S.

Proof. Straightforward. □

Examples 3.11 and 3 shows that the converse of the above Lemma 3.15 is not true in general.

Lemma 3.16. *Let $\{F_i\}_{i\in I}$ be a family of an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $\bigcap_{i\in I} F_i$ is an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S, where $\bigcap_{i\in I} F_i = \left\langle \bigwedge_{i\in I} \lambda_F, \bigvee_{i\in I} \mu_F \right\rangle$.*

Proof. Straightforward. □

Lemma 3.17. *Let $\{F_i\}_{i\in I}$ be a family of an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $\bigcup_{i\in I} F_i$ is not an $(\in, \in\vee q_k)$-intuitionistic fuzzy bi-ideal of S, where $\bigcup_{i\in I} F_i = \left\langle \bigwedge_{i\in I} \lambda_{F_i}, \bigvee_{i\in I} \mu_{F_i} \right\rangle$. For this we have the following example.*

Example 3.18. Let $S = \{a, b, c, d\}$ be a semigroup with the following table:

$\cdot$	a	b	c	d
a	a	c	a	a
b	a	a	a	a
c	b	a	a	d
d	a	a	d	a

Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $IFS's$ of semigroup S defined by $\lambda_E(a) = 0.7$, $\lambda_E(b) = 0.5$, $\lambda_E(c) = \lambda_E(d) = 0.3$ and $\mu_E(a) = \mu_E(c) = 0.5$, $\mu_E(b) = \mu_E(d) = 0.2$, and $\lambda_F(a) = 0.8$, $\lambda_F(b) = \lambda_F(d) = 0.3$, $\lambda_F(c) = 0.4$ and $\mu_F(a) = \mu_F(b) = 0.5$, $\mu_F(c) = \mu_F(d) = 0.3$. Then, both $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ are an $(\in, \in \vee q_{0.4})$-intuitionistic fuzzy bi-ideals of S, where $\frac{1-k}{2} = 0.4$. But $E \cup F$ is not an $(\in, \in \vee q_{0.4})$-intuitionistic fuzzy bi-ideal of S. i.e, $(\lambda_E \vee \lambda_F)(bc) = (\lambda_E \vee \lambda_F)(d) = \lambda_E(d) \vee \lambda_F(d) = 0.3 \vee 0.3 = 0.3$ and $(\lambda_E \vee \lambda_F)(b) \wedge (\lambda_E \vee \lambda_F)(c) \wedge \frac{1-k}{2} = 0.5 \wedge 0.4 \wedge \frac{1-k}{2} = 0.4$. Hence $(\lambda_E \vee \lambda_F)(bc) = 0.3 \ngeq 0.4 = (\lambda_E \vee \lambda_F)(b) \wedge (\lambda_E \vee \lambda_F)(c) \wedge \frac{1-k}{2}$.

Theorem 3.19. *Let $\{F_i\}_{i \in I}$ be a family of an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S such that $F_i \subseteq F_j$ or $F_j \subseteq F_i$ for all $i, j \in I$. Then $\bigcup_{i \in I} F_i$ is not an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, where $\bigcup_{i \in I} F_i = \left\langle \bigvee_{i \in I} \lambda_F, \bigwedge_{i \in I} \mu_F \right\rangle$.*

Proof. For all $x, y \in S$, we have

$$\begin{aligned} \left(\bigvee_{i \in I} \lambda_{F_i}(xy) \right) &= \bigvee_{i \in I} (\lambda_{F_i}(xy)) \geq \bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \\ &= \bigvee_{i \in I} \lambda_{F_i}(x) \wedge \bigvee_{i \in I} \lambda_{F_i}(y) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}. \end{aligned}$$

It is clear that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \leq \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Suppose that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \neq \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Then there exists t such that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) < t < \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Since $\lambda_{F_i} \subseteq \lambda_{F_j}$ or $\lambda_{F_j} \subseteq \lambda_{F_i}$ for all $i, j \in I$, thus there exists $k \in I$ such that $t < \lambda_{F_k}(x) \wedge \lambda_{F_k}(y) \wedge \frac{1-k}{2}$. On the other hand $t > \lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2}$ for all $i \in I$, a contradiction. Hence

$$\bigvee_{i\in I}\left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2}\right) = \left(\bigvee_{i\in I}\lambda_{F_i}\right)(x) \wedge \left(\bigvee_{i\in I}\lambda_{F_i}\right)(y) \wedge \frac{1-k}{2}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I}\mu_{F_i}(xy)\right) &= \bigwedge_{i\in I}(\mu_{F_i}(xy)) \\ &\leq \bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \\ &= \bigwedge_{i\in I}\mu_{F_i}(x) \vee \bigwedge_{i\in I}\mu_{F_i}(y) \vee \frac{1-k}{2} \\ &= \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.\end{aligned}$$

It is clear that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \geq \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Suppose that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \neq \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Then there exists r such that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) > r > \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Since $\mu_{F_i} \subseteq \mu_{Fj}$ or $\mu_{Fj} \subseteq \mu_{Fi}$ for all $i, j \in I$. Thus there exists $k \in I$ such that $r > \mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}$. On the other hand, $r < \mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}$ for all $i \in I$, which is a contradiction. Hence,

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) = \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Let $a, b, c \in S$, we have

$$\begin{aligned}\left(\bigvee_{i\in I} \lambda_{F_i}(xyz)\right) &= \bigvee_{i\in I} \left(\lambda_{F_i}(xyz)\right) \geq \bigvee_{i\in I}\left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(z) \wedge \frac{1-k}{2}\right) \\ &= \bigvee_{i\in I} \lambda_{F_i}(x) \wedge \bigvee_{i\in I} \lambda_{F_i}(z) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{i\in I} \lambda_{F_i}\right)(x) \wedge \left(\bigvee_{i\in I} \lambda_{F_i}\right)(z) \wedge \frac{1-k}{2}\end{aligned}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I} \mu_{F_i}(xyz)\right) &= \bigwedge_{i\in I} \left(\mu_{F_i}(xyz)\right) \leq \bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(z) \vee \frac{1-k}{2}\right) \\ &= \bigwedge_{i\in I} \mu_{F_i}(x) \vee \bigwedge_{i\in I} \mu_{F_i}(z) \vee \frac{1-k}{2} \\ &= \left(\bigwedge_{i\in I} \mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I} \mu_{F_i}\right)(z) \vee \frac{1-k}{2}.\end{aligned}$$

Hence, $\bigcup_{i\in I} F_i = \left\langle \bigvee_{i\in I} \lambda_F, \bigwedge_{i\in I} \mu_F \right\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. $\square$

Definition 3.20. Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ of S. Then, the $\frac{1-k}{2}$-product of E and F is defined by:

$$\begin{aligned}E \circ_{\frac{1-k}{2}} F &= \left\langle \lambda_E \circ_{\frac{1-k}{2}} \lambda_F, \mu_E \circ_{\frac{1-k}{2}} \mu_F \right\rangle \\ \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(a) &= \left\{ \begin{array}{ll} \bigvee_{a=xy} \left(\lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right) & \text{if } a = xy \\ 0 & \text{if } a \neq xy \end{array} \right\} \\ \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a) &= \left\{ \begin{array}{ll} \bigwedge_{a=xy} \left(\mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2}\right) & \text{if } a = xy \\ 0 & \text{if } a \neq xy \end{array} \right\} \\ E \cap_{\frac{1-k}{2}} F &= \left\langle \lambda_E \wedge_{\frac{1-k}{2}} \lambda_F, \mu_E \vee_{\frac{1-k}{2}} \mu_F \right\rangle \\ \left(\lambda_E \wedge_{\frac{1-k}{2}} \lambda_F\right)(a) &= \lambda_E(a) \wedge \lambda_F(a) \wedge \frac{1-k}{2} \\ \left(\mu_E \vee_{\frac{1-k}{2}} \mu_F\right)(a) &= \mu_E(a) \vee \mu_F(a) \vee \frac{1-k}{2}.\end{aligned}$$

Remark 3.21. Let E, F, G, H are $IFS's$ of S such that $E \subseteq F$ and $G \subseteq H$. Then $E \circ_{\frac{1-k}{2}} F \subseteq G \circ_{\frac{1-k}{2}} H$.

Lemma 3.22. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $E \cap_{\frac{1-k}{2}} F$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

Definition 3.23. An $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal $F = \langle \lambda_F, \mu_F \rangle$ of S is said to be $\frac{1-k}{2}$-idempotent if $F \circ_{\frac{1-k}{2}} F = F$.

Lemma 3.24. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Then $F \circ_{\frac{1-k}{2}} F \subseteq F$.*

Lemma 3.25. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals of S. Then $F \circ_{\frac{1-k}{2}} F \subseteq 1 \circ_{\frac{1-k}{2}} F$.(resp. $F \circ_{\frac{1-k}{2}} F \subseteq E \circ_{\frac{1-k}{2}} 1$).*

Theorem 3.26. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$, where $S = \langle 1, 0 \rangle$, $1(a) = 1$ and $0(a) = 0$ for all $a \in S$.*

Proof. Let $x \in S$. Then we have the following two cases:
(i) If $x \neq ab\ \forall a, b \in S$.
(ii) If $x = ab$ for some $a, b \in S$.
Case (i). If $a \neq xy$, then

$$\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) = 0 \leq \lambda_F(x) \wedge \frac{1-k}{2}$$

and

$$\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) = 1 \geq \mu_F(x) \vee \frac{1-k}{2}.$$

Thus, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Case (ii). If $x = ab$ for some $x, y \in S$, then

$$\begin{aligned}
\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) &= \bigvee_{x=ab} \left\{\lambda_F(a) \wedge \left(1 \circ_{\frac{1-k}{2}} \lambda_F\right)(b) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=ab} \left\{\lambda_F(a) \wedge \left\{\bigvee_{b=pq} 1(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=ab} \bigvee_{b=pq} \left\{\lambda_F(x) \wedge 1 \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=apq} \left\{\lambda_F(x) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\}.
\end{aligned}$$

Since $x = ab = a(pq) = apq$ and $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, therefore we have $\lambda_F(apq) \geq \lambda_F(a) \wedge \lambda_F(q) \wedge \frac{1-k}{2}$. Hence,

$$\begin{aligned}
\bigvee_{x=apq} \left\{\lambda_F(a) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} &\leq \bigvee_{x=apq} \{\lambda_F(apq)\} = \lambda_F(x) \\
\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) &\leq \lambda_F(x)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) \\
&= \bigwedge_{x=ab} \left\{\mu_F(a) \vee \left(0 \circ_{\frac{1-k}{2}} \mu_F\right)(b) \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=ab} \left\{\mu_F(a) \vee \left\{\bigwedge_{y=pq} \left\{1(p) \vee \mu_F(q) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=ab} \bigwedge_{y=pq} \left\{\mu_F(a) \vee 0 \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=apq} \left\{\mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}\right\}.
\end{aligned}$$

Since $x = ab = a(pq) = apq$ and $F = \langle \lambda_F, \mu_F \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, therefore we have $\mu_F(apq) \leq \mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}$. Hence,

$$\begin{aligned}
&\bigwedge_{x=apq} \left\{\mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \leq \bigwedge_{a=xpq} \{\mu_F(apq)\} = \mu_F(x) \\
&\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) \leq \mu_F(x).
\end{aligned}$$

Thus, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$. □

Theorem 3.27. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S if and only if $F \circ_{\frac{1-k}{2}} F \subseteq F$.*

Theorem 3.28. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if the following condition satisfied.*

(i) $F \circ_{\frac{1-k}{2}} F \subseteq F$,

(ii) $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Proof. Let $F = \langle \lambda_F, \mu_F \rangle$ be an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.Then, by Lemma 3.24 and Theorem 3.26, we have $F \circ_{\frac{1-k}{2}} F \subseteq F$ and $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Conversely, assume that condition (i) and (ii) satisfied. Let $x, y \in S$ be such that $a = xy$. Then, we have

$$\begin{aligned}
\lambda_F(xy) &= \lambda_F(a) \geq \left(\lambda_F \circ_{\frac{1-k}{2}} \lambda_F\right)(a) \\
&= \bigvee_{a=xy} \left\{\lambda_F(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mu_F(xy) &= \mu_F(a) \leq \left(\mu_F \circ_{\frac{1-k}{2}} \mu_F\right)(a) \\
&= \bigwedge_{a=pq} \left\{\mu_F(p) \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \\
&\leq \mu_F(p) \vee \mu_F(q) \vee \frac{1-k}{2}.
\end{aligned}$$

Now, let $x, y, z \in S$ such that $a = xyz$. Then, we have

$$\begin{aligned}
\lambda_F(xyz) &= \lambda_F(a) \geq \left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(a) \\
&= \bigvee_{a=pq} \left\{\lambda_F(p) \wedge \left(1 \circ_{\frac{1-k}{2}} \lambda_F\right)(q) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{a=pq} \left\{\lambda_F(p) \wedge \left(\bigvee_{q=st} \left\{1(t) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{a=pq} \left\{\lambda_F(p) \wedge \left(\bigvee_{q=st} \left\{1 \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2}\right\} \\
&\geq \bigvee_{a=pq} \bigvee_{q=st} \left\{\lambda_F(p) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\} \\
&\geq \bigvee_{a=pst} \left\{\lambda_F(p) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\} \\
&\geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mu_F(xyz) &= \mu_F(a) \leq \left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(a) \\
&= \bigwedge_{a=pq} \left\{\mu_F(p) \vee \left(0 \circ_{\frac{1-k}{2}} \mu_F\right)(q) \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{a=pq} \left\{\mu_F(p) \vee \left\{\bigwedge_{q=st} \left\{0(s) \vee \mu_F(t) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{a=pq} \left\{\mu_F(p) \vee \left\{\bigwedge_{q=st} \left\{0 \vee \mu_F(t) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{a=pq} \bigwedge_{q=st} \left\{\mu_F(p) \vee \mu_F(t) \vee \frac{1-k}{2}\right\} \\
&\leq \bigwedge_{a=pst} \left\{\mu_F(p) \vee \mu_F(t) \vee \frac{1-k}{2}\right\} \\
&\leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}.
\end{aligned}$$

Hence, $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. $\square$

Theorem 3.29. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right, two sided) ideal of S if and only if it satisfied:*
$S \circ_{\frac{1-k}{2}} F \subseteq F$ $\left(F \circ_{\frac{1-k}{2}} S \subseteq F,\ S \circ_{\frac{1-k}{2}} F \subseteq F \text{ and } F \circ_{\frac{1-k}{2}} S \subseteq F\right)$.

Proof. Straightforward. $\square$

Theorem 3.30. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $A = E \circ_{\frac{1-k}{2}} F$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

Proof. Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S and $a \in S$. Then we have two cases:

(i) If $x \neq ab$ for any $a, b \in S$. (ii) If $x = ab$ for some $a, b \in S$.

Case i. If $x \neq ab$ for any $a, b \in S$, then

$$\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)\right)(x) = 1 \leq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(x)$$

and

$$\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right) \circ_{\frac{1-k}{2}} \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)\right)(x) = 0 \geq \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(x).$$

Thus, $A \circ_{\frac{1-k}{2}} A \subseteq A$.

Case ii. If $x = ab$ for some $a, b \in S$, then

$$\begin{aligned}
&\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)\right)(a) \\
= &\bigvee_{a=xy} \left\{\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(x) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(y) \wedge \frac{1-k}{2}\right\} \\
= &\bigvee_{a=xy} \left\{ \begin{array}{c} \left\{\bigvee_{x=pq} \left\{\lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\}\right\} \wedge \\ \left\{\bigvee_{y=st} \left\{\lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2}\right\}\right\} \end{array} \right\} \\
= &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{\left\{\lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \wedge \lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2}\right\} \\
= &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{\lambda_E(p) \wedge \lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2}\}\right\} \\
\leq &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{\lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t)\}\right\}.
\end{aligned}$$

Since $a = xy$, $x = pq$ and $y = st$. So, $a = (xy)(st) = (pqs)t$ and we have

$$\begin{aligned}
&\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{\lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t)\}\right\} \\
\leq &\bigvee_{a=(xys)t} \left\{\lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t)\right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(xys) \geq \lambda_E(x) \wedge \lambda_E(s) \wedge \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{a=(xys)t} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t) \right\} \\
&\leq \bigvee_{a=(xys)t} \left\{ \lambda_E(xys) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{a=uv} \left\{ \lambda_E(u) \wedge \lambda_E(v) \wedge \frac{1-k}{2} \right\} \\
&= \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a).
\end{aligned}$$

Therefore, $\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \right)(a) \leq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a)$. Now,

$$\begin{aligned}
&\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \circ_{\frac{1-k}{2}} \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \right)(a) \\
&= \bigwedge_{a=xy} \left\{ \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(x) \vee \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(y) \vee \frac{1-k}{2} \right\} \\
&= \bigwedge_{a=xy} \left\{ \bigwedge_{x=pq} \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \bigwedge_{y=st} \left\{ \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \right\} \vee \frac{1-k}{2} \right\} \\
&= \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \left\{ \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \right\} \right\} \\
&= \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \} \right\} \\
&\geq \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\}.
\end{aligned}$$

Since $a = xy$, $x = pq$ and $y = st$. So, $a = (xy)(st) = (pqs)t$ and we have

$$\begin{aligned}
&\bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\} \\
&\geq \bigwedge_{a=(xys)t} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\mu_E(xys) \geq \mu_E(x) \vee \mu_E(s) \vee \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigwedge_{a=(xys)t} \left\{\mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t)\}\right\} \\
&\geq \bigwedge_{a=(xys)t} \left\{\mu_E(xys) \vee \mu_E(s) \vee \frac{1-k}{2}\}\right\} \\
&\geq \bigwedge_{a=uv} \left\{\mu_E(u) \vee \mu_F(v) \vee \frac{1-k}{2}\right\} = \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a).
\end{aligned}$$

Therefore, $\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right) \circ_{\frac{1-k}{2}} \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)\right)(a) \geq \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a)$ and hence $A \circ_{\frac{1-k}{2}} A \subseteq A$. Thus, $A = E \circ_{\frac{1-k}{2}} F$ is an intutionistic $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S.

Now, let $a, b, c \in S$. Then,

$$\begin{aligned}
&\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(a) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)\right)(c) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=xy} \left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right\} \wedge \bigvee_{c=pq} \left\{\lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2} \\
&= \bigvee_{a=xy} \bigvee_{c=pq} \left\{\left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right\} \wedge \left\{\lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{a=xy} \bigvee_{c=pq} \left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{a=xy} \bigvee_{c=pq} \left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\}.
\end{aligned}$$

Since $a = xy$ and $c = pq$, so $abc = (xy)\, b\, (pq) = (x\, (yb)\, p)\, q$ and we have

$$\begin{aligned}
&\bigvee_{a=xy} \bigvee_{c=pq} \left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{abc=(x(yb)p)q} \left\{\left\{\lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right\} \wedge \lambda_F(q)\right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(x\,(yb)\,p) \geq \lambda_E(x) \wedge \lambda_E(yb) \wedge \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{abc=(x(yb)p)q}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\lambda_F(q)\right\}\\
&\leq\bigvee_{abc=(x(yb)p)q}\left\{\lambda_E(x(yb)p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\\
&=\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(abc).
\end{aligned}$$

Thus,

$$\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(abc)\geq\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(a)\wedge\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(c)\wedge\frac{1-k}{2}$$

and

$$\begin{aligned}
&\left(\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(a)\wedge\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)\right)(c)\wedge\frac{1-k}{2}\\
&=\left\{\bigvee_{a=xy}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\right\}\wedge\left\{\bigvee_{c=pq}\lambda_E(p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\wedge\frac{1-k}{2}\\
&=\bigvee_{a=xy}\bigvee_{c=pq}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\left\{\lambda_E(p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{a=xy}\bigvee_{c=pq}\left\{\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_E(p)\wedge\lambda_F(q)\}\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{a=xy}\bigvee_{c=pq}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}.
\end{aligned}$$

Since $a=xy$ and $c=pq$, so $abc=(xy)b(pq)=(x(yb)p)q$ and we have

$$\begin{aligned}
&\bigvee_{a=xy}\bigvee_{c=pq}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{abc=(x(yb)p)q}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\lambda_F(q)\right\}.
\end{aligned}$$

Since $E=\langle\lambda_E,\mu_E\rangle$ is an interval valued $(\in,\in\vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(x(yb)p)\geq\lambda_E(x)\wedge\lambda_E(yb)\wedge\frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{abc=(x(yb)p)q}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\lambda_F(q)\right\}\\
&\le\bigvee_{abc=(x(yb)p)q}\left\{\lambda_E(x(yb)p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\\
&=\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(abc).
\end{aligned}$$

Thus,

$$\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(abc)\ge\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(a)\wedge\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(c)\wedge\frac{1-k}{2}$$

and

$$\begin{aligned}
&\left(\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)(a)\vee\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)\right)(c)\vee\frac{1-k}{2}\\
&=\begin{array}{c}\bigwedge_{a=xy}\left\{\mu_E(x)\vee\mu_F(y)\vee\frac{1-k}{2}\right\}\vee\\ \bigwedge_{c=pq}\left\{\mu_E(p)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}\vee\frac{1-k}{2}\end{array}\\
&=\bigwedge_{a=xy}\bigwedge_{c=pq}\left\{\begin{array}{c}\left\{\mu_E(x)\vee\mu_F(y)\vee\frac{1-k}{2}\right\}\vee\\ \left\{\mu_E(p)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}\vee\frac{1-k}{2}\end{array}\right\}\\
&\le\bigwedge_{a=xy}\bigwedge_{c=pq}\left\{\mu_E(x)\vee\mu_F(y)\vee\mu_E(p)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}\\
&\le\bigwedge_{a=xy}\bigwedge_{c=pq}\left\{\mu_E(x)\vee\lambda_F(y)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}
\end{aligned}$$

Since $a=xy$ and $c=pq$, so $abc=(xy)\,b\,(pq)=(x\,(yb)\,p)\,q$ and we have

$$\begin{aligned}
&\bigwedge_{a=xy}\bigwedge_{c=pq}\left\{\mu_E(x)\vee\mu_F(y)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}\\
&\le\bigwedge_{abc=(x(yb)p)q}\left\{\left\{\mu_E(x)\vee\mu_F(y)\vee\frac{1-k}{2}\right\}\vee\lambda_F(q)\right\}.
\end{aligned}$$

Since $E=\langle\lambda_E,\mu_E\rangle$ is an $(\in,\in\vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\mu_E(x(yb)p)\ge\mu_E(x)\vee\mu_E(yb)\vee\frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigwedge_{abc=(x(yb)p)q}\left\{\left\{\mu_E(x)\vee\mu_F(y)\vee\frac{1-k}{2}\right\}\vee\mu_F(q)\right\}\\
&\le\bigwedge_{abc=(x(yb)p)q}\left\{\mu_E(x(yb)p)\vee\mu_F(q)\vee\frac{1-k}{2}\right\}=\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)(abc).
\end{aligned}$$

Thus,

$$\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(abc) \leq \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a) \vee \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(c) \vee \frac{1-k}{2}.$$

Hence, $A = E \circ_{\frac{1-k}{2}} F$ is an intutionistic $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. $\square$

For any intutionstic fuzzy set $F = \langle \lambda_F, \mu_F \rangle$ in S and $t \in (0,1], r \in [0,1)$, we denote $F_{(t,r)} = \{x \in S : \langle x; (t,r) \rangle\, q_k F\}$ and $[F]_{(t,r)} = \{x \in S : \langle x; (t,r) \rangle \in \vee q_k E\}$.

Obviously, $[F]_{(t,r)} = F_{(t,r)} \cup U_{(t,r)}$, where $U_{(t,r)}$, $F_{(t,r)}$ and $[F]_{(t,r)}$ are called $\in$-level set, q_k-level set and $\in \vee q_k$-level set of $F = \langle \lambda_F, \mu_F \rangle$ respectively.

Theorem 3.31. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a left (resp. right) ideal of S.*

Proof. Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy left ideal of S and $U_{(t,r)} \neq \emptyset$ for all $t \in (0,1]$ and $r \in [0,1)$. Let $y \in U_{(t,r)}$ and $x \in S$. Then, $\lambda_F(x) \geq t$ and $\lambda_F(x) \leq r$. Since

$$\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(b) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} \geq t$$

and

$$\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \leq r \vee \frac{1-k}{2} \leq r.$$

So, $xy \in U_{(t,r)}$. Hence $U_{(t,r)} \neq \emptyset$ is a left ideal of S.

Conversely, Let $F = \langle \lambda_F, \mu_F \rangle$ be an intutionistic fuzzy set in a way that $U_{(t,r)} \neq \emptyset$ is a left ideal of S. Assume that there exists $x, y \in S$ such that $\lambda_F(xy) < \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \frac{1-k}{2}$. We choose $t \in (0,1]$ and $r \in [0,1)$, then $\lambda_F(xy) < t < \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > r > \mu_F(x) \vee \frac{1-k}{2}$. Then $y \in U_{(t,r)}$, but $xy \notin U_{(t,r)}$, which is a contradiction. Hence, $\lambda_F(xy) \leq \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \geq \mu_F(x) \vee \frac{1-k}{2}$. $\square$

Theorem 3.32. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a bi-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.31. $\square$

Theorem 3.33. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a $(1,2)$-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.31. $\square$

Theorem 3.34. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S if and only if for all $t \in (0,1]$ and $r \in D[0,1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a subsemigroup of S.*

Proof. Let $x, y \in [F]_{(t,r)}$. Then, $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$ or $\lambda_F(x) + t + k > 1$ and $\mu_F(x) + r + k < 1$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$ or $\lambda_F(y) + t + k > 1$ and $\mu_F(y) + r + k < 1$, thus we have the following four cases:

(i) $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$,

(ii) $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$ and $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$,

(iii) $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$,

(iv) $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$, and $\lambda_F(y) + t + k > 1$ and $\mu_F(y) + r + k < 1$.

For the first case, by Theorem 3.10 (i), it implies that

$$\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} = t \wedge \frac{1-k}{2} = \begin{cases} \frac{1-k}{2}, & \text{if } t > \frac{1-k}{2} \\ t, & \text{if } t \leq \frac{1-k}{2} \end{cases}$$

and

$$\begin{aligned} \mu_F(xy) &\leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \\ &= r \vee \frac{1-k}{2} = \begin{cases} \frac{1-k}{2}, & \text{if } s < \frac{1-k}{2} \\ r, & \text{if } r \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

and hence, $\lambda_F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xy) + r + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ which implies that $(xy)(t,r) q_k F$ or $xy \in F_{(t,r)}$. Hence $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. For the second case we assume that $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$. Then $1-t < \frac{1-k}{2}$ and $1-r > \frac{1-k}{2}$. If $\lambda_F(x) \geq \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(x) \leq \mu_F(y) \vee \frac{1-k}{2}$, then $\lambda_F(x) \geq \lambda_F(y) \wedge \frac{1-k}{2} > 1-t$ and $\mu_F(y) \leq \mu_F(y) \vee \frac{1-k}{2} < 1-r$ and if $\lambda_F(y) > \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(x) < \mu_F(y) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(x) \geq t$ and $\mu_F(xy) \leq \mu_F(x) \leq r$. Hence, $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. Now suppose that $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$. Then $1-t \geq \frac{1-k}{2}$ and $1-r \leq \frac{1-k}{2}$. If $\lambda_F(y) \geq \lambda_F(x) \wedge \frac{1-k}{2}$ and $\mu_F(y) \leq \mu_F(x) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(y) \wedge \frac{1-k}{2} \geq t$ and $\mu_F(x) \leq \mu_F(y) \vee \frac{1-k}{2} \leq r$ and if $\lambda_F(y) < \lambda_F(x) \wedge \frac{1-k}{2}$ and $\mu_F(y) > \mu_F(x) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(y) \geq 1-t$ and $\mu_F(xy) \leq \mu_F(y) \leq 1-r$. Hence, $ab \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. We have similar result for the case (iii). For the case four, if $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$. Then$1-t < \frac{1-k}{2}$ and $1-r > \frac{1-k}{2}$. Hence,

$$\begin{aligned} \lambda_F(xy) &\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2} \\ \lambda_F(x) \wedge \lambda_F(y) > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) < \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mu_F(xy) &\leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} < 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2} \\ \mu_F(x) \vee \mu_F(y) > 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) > \frac{1-k}{2} \end{cases} \end{aligned}$$

and hence $xy \in F_{(t,r)} \subseteq [F]_{(t,r)}$. If $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$, then $\frac{1-k}{2} \leq 1-t$ and $\frac{1-k}{2} \geq 1-r$. Thus,

$$\begin{aligned} \lambda_F(xy) &\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} \geq t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2} \\ \lambda_F(x) \wedge \lambda_F(y) > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) < \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mu_F(xy) &\geq \mu_F(x) \wedge \mu_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} \leq r, & \text{if } \mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2} \\ \mu_F(x) \vee \mu_F(y) < 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) > \frac{1-k}{2} \end{cases} \end{aligned}$$

which implies that $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$.

Conversely, assume that $F = \langle \lambda_F, \mu_F \rangle$ is not an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Then, there exists $x, y \in S$ such that $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Let

$$t = \frac{1}{2}\left[\lambda_F(xy) + \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right]$$

and

$$r = \frac{1}{2}\left[\mu_F(xy) + \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}\right].$$

Then,

$$\lambda_F(xy) < t < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$$

and

$$\mu_F(xy) > r > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}.$$

Which implies that $x, y \in [F]_{(t,r)}$ and $xy \in [F]_{(t,r)}$. Hence, $\lambda_F(xy) \geq t$ and $\mu_F(xy) \leq r$ or $\lambda_F(xy) + t + k > 1$ and $\mu_F(xy) + r + k < 1$, which is contradiction. Therefore, we have $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Thus, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. $\square$

Theorem 3.35. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a left (resp. right) ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.36. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a bi-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.37. *An $IFSF = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a $(1,2)$-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.38. *Every $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S.*

Proof. Straightforward. □

Theorem 3.39. *In a regular semigroup every $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S.*

Proof. Suppose S is regular, then every bi-ideal of S is left (resp. right) ideal of S. Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S and $x, y \in S$, xSx is a bi-ideal of S. Then aSa is a right ideal of S. Since S is regular, thus we have $ab \in (aSa)\, S \subseteq aSa$, this implies that $xy = xyx$ for some $y \in S$. Also since $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. It implies that

$$\lambda_F\,(xy) = \lambda_F\,(xyx) \geq \lambda_F\,(x) \wedge \lambda_F\,(x) \wedge \frac{1-k}{2} \geq \lambda_F\,(x) \wedge \frac{1-k}{2}$$

and

$$\mu_F\,(xy) = \mu_F\,(xyx) \leq \mu_F\,(x) \vee \mu_F\,(x) \vee \frac{1-k}{2} \leq \mu_F\,(x) \vee \frac{1-k}{2}.$$

Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy right ideal of S. □

Theorem 3.40. *In a regular semigroup every $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left bi-ideal of S.*

Proof. Let S be a regular semigroup and $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S. Let $a, b, x \in S$. Since S is regular, we have $ax \in (aSa)\, S \subseteq aSa$, this implies that $ax = asa$ for some $s \in S$. Thus

$$\lambda_F(axb) = \lambda_F((asa)b) = \lambda_F(ax(ab)) \geq \lambda_F(a) \wedge \lambda_F(a) \wedge \lambda_F(b) \wedge \frac{1-k}{2}$$

and $\mu_F(axb) = \mu_F((asa)b) = \mu_F(ax(ab)) \leq \mu_F(a) \vee \mu_F(a) \vee \mu_F(b) \vee \frac{1-k}{2}$. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. □

Theorem 3.41. *Let $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. If S is completely regular and $\lambda_F(a) \leq \frac{1-k}{2}$, and $\mu_F(a) \geq \frac{1-k}{2}$ for all $a \in S$, then $F(a) = F(a^2)$ for all $a \in S$.*

Proof. Straightforward. □

4. Conclusion

It is recognized that semigroups are basic algebraic structures in several applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these prospects of applications, semigroups are now widely studied in fuzzy setting. An intuitionistic fuzzy set is more substantial and brief to designate the essence of fuzziness. Intuitionistic fuzzy set theory is more appropriate than the fuzzy set theory for dealing with imperfect knowledge in several problems. In the structural study of semigroup, we notified that intuitionistic fuzzy ideals with superior properties continuously play an significant role. The intuitionistic fuzzy point of a semigroup S is basic tools to define the algebraic subsystems of S. So, we combined the above notions and initiated new types of intuitionistic fuzzy bi-ideals and $(1,2)$-ideals of semigroups which are said to be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal. The results in the paper are generalizations of results about ordinary $(\in, \in \vee q)$- intuitionistic fuzzy ideals in semigroups. In future, we will focus on the following topics:

(1) Characterizations of regular semigroups by the properties of $(\in, \in \vee q_k)$-intuitionistic fuzzy ideals

(2) We will define $(\in, \in \vee q_k)$-intuitionistic fuzzy (interior, prime, generalized bi, prime bi) ideals of a semigroup and characterize different classes of semigroups by the properties of $(\in, \in \vee q_k)$-intuitionistic-fuzzy ideals. In future we will extend our study to other algebraic structures like ring theory, module theory, soft semigroups etc.

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THE FORM OF AUTOMORPHISMS OF AN ABELIAN GROUP HAVING THE WEAK EXTENSION PROPERTY

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Abstract. Let A be an abelian group and let α be an automorphism of A. In this paper we show that if the restriction of α to any p-component A_p of A is of the form: $\alpha_{\restriction A_p} = \pi id_{A_p} + \rho$, where p is a prime number, π a p-adic invertible number and $\rho \in Hom(A_p, A^1)$ with A^1 is the first subgroup Ulm of the group A. Then α satisfies the weak extension property.

Keywords: abelian groups, p-groups, torsion groups, automorphism group.

1. Introduction

The study of the characterization of automorphisms having the property of extension had begun by P. E Schupp showed, in [12], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives in, [10], a simpler proof of Schupp's result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. In [8] M. Dugas and R. Gobel gave another simpler proof of Schupp's result, using only the elementary theory of groups. In [2] L. Ben Yakoub shows that the result of Schupp is not valid in general for algebras on a

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commutative ring. It is not yet known whether this result holds true for algebras (of finite dimensions) on a commutative field. In [3] L. Ben Yakoub and M. P. Malliavin show that the property of extension also characterizes derivations in associative algebras for some algebras quantum properties. In this article, we will define the property of the weak extension by:

An automorphism α of an abelian group A has the weak extension property if for all abelian group B for all monomorphism $\lambda : A \longrightarrow B$ and if there exists an element $m \in \mathbb{N}^*$ such that the restriction of λ to mA is an isomorphism from mA to mB, then there exists $\tilde{\alpha} \in Aut(B)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \alpha \downarrow & & \downarrow \tilde{\alpha} \\ A & \xrightarrow{\lambda} & B \end{array}$$

In other words: $\tilde{\alpha}\lambda = \lambda\alpha$. By way of example, any automorphism of an abelian group without torsion possesses the property of the weak extension (see [1], [16]).

2. Main result

Theorem 2.1. *Let A be an abelian group and let α be an automorphism of A.*

If the restriction of α to any p-component A_p of A is of the form: $\alpha_{\restriction A_p} = \pi id_{A_p} + \rho$, where p is a prime number, π a p-adic invertible number and $\rho \in Hom(A_p, A^1)$ with A^1 is the first subgroup Ulm of A. Then α satisfies the weak extension property.

Before giving proof of this theorem, we will need certain results.

Lemma 2.2. *Let T_A be the torsion part of A. If α_1 is the restriction of α to T_A then α_1 is an automorphism of T_A. Moreover, α_1 satisfies the weak extension property.*

Proof. Let $x \in T_A$. There exists $n \in \mathbb{N}^*$ such that $nx = 0$, whence $n\alpha(x) = \alpha(nx) = 0$, we deduce that $\alpha_1(T_A) \subseteq T_A$.

On the other hand, $\forall y \in T_A$, there exists $x \in T_A$ such that $y = \alpha(x)$; If $0 = my = m\alpha(x)$ for some $m \in \mathbb{N}^*$, then $\alpha(mx) = 0$ implies $mx = 0$, then $x \in T_A$. We conclude that $\alpha(T_A) = T_A$, consequently $\alpha_1 = \alpha_{\restriction T_A}$ is an automorphism of T_A. Let $(T_A)_p$ be the p-component of the torsion group T_A. From the above assumptions, ${\alpha_1}_{\restriction (T_A)_p} = \pi \, id_{(T_A)_p} + \rho$ where π is an invertible p-adic number and $\rho \in Hom((T_A)_p, (T_A)^1_p)$ with $(T_A)^1_p$ the first subgroup Ulm of the group $(T_A)_p$. Therefore, according to the characterization of the automorphisms possessing the weak extension property in the category of torsion abelian groups see, [15], we deduce that $\alpha_1 = \alpha_{\restriction T_A}$ satisfies the weak extension property.

Proposition 2.3. *Let $\lambda : A \to A'$ be a monomorphism of abelian groups and let $\lambda_{\restriction m_0 A}$ be the restriction of λ to $m_0 A$ such that $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$*

where $m_0 \in \mathbb{N}^*$. If T_A and $T_{A'}$ are respectively the torsion parts of A and A'. So:

(i) $\lambda_1 : T_A \to T_{A'}$ is a monomorphism.

(ii) There exists an automorphism α' of $T_{A'}$ which makes switch the following diagram:

$$\begin{array}{ccc} T_A & \xrightarrow{\lambda_1} & T_{A'} \\ \alpha_1 \downarrow & & \downarrow \alpha' \\ T_A & \xrightarrow{\lambda_1} & T_{A'} \end{array}$$

Proof. (i) It suffices to prove that: $\lambda(T_A) \subseteq T_{A'}$.

Let $a \in T_A$; There exists $m \in \mathbb{N}^*$ such that $ma = 0$, hence $m\lambda(a) = 0$, we deduce that $\lambda(T_A) \subseteq T_{A'}$.

(ii) From the assumptions we have: $m_0 A \simeq m_0 A'$ where $m_0 \in \mathbb{N}^*$.

The proposition 8.37 (see [18], p: 295) shows that for $m_0 \in \mathbb{N}^*$: $T(m_0 A) \simeq T(m_0 A')$, hence $m_0 T_A \simeq m_0 T_{A'}$, so for some $m_0 \in \mathbb{N}^*$: $\lambda_{1\restriction m_0 T_A} \in Isom(m_0 T_A; m_0 T_{A'})$ and since the automorphism $\alpha_1 \in Aut(T_A)$ satisfies the weak extension property, then there exists an automorphism $\alpha' \in T_{A'}$ such that $\alpha' \lambda_1 = \lambda_1 \alpha_1$.

Lemma 2.4. *Let A' be an abelian group and $\lambda : A \to A'$ an monomorphism. If $\lambda(A) = A_1$ and if $\alpha_2 = \lambda\alpha\lambda^{-1}$, then, α_2 is an automorphism of A_1 which makes switch the following diagram:*

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A_1 \\ \alpha \downarrow & & \downarrow \alpha_2 \\ A & \xrightarrow{\lambda} & A_1 \end{array}$$

Proof. Since $\lambda : A \to A'$ is a monomorphism and $\lambda(A) = A_1$, then, $\lambda : A \to A_1$ is an isomorphism, consequently, $\alpha_2 = \lambda\alpha\lambda^{-1}$ is an automorphism of A_1 and we have: $\alpha_2 \lambda = \lambda \alpha$.

Proposition 2.5. *Let $\lambda : A \to A'$ be a monomorphism of abelian groups and let $\lambda_{\restriction m_0 A}$ be the restriction of λ to $m_0 A$ such that $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$ where $m_0 \in \mathbb{N}$. If T_A and $T_{A'}$ are respectively the torsion parts of A and A'; So:*

1. *$A' = A_1 + T_{A'}$;*

2. *$A_1 \cap T_{A'} = T_{A_1}$;*

3. *$\lambda(T_A) = T_{A_1}$.*

Proof. 1) Since $\lambda : A \to A'$ is a monomorphism and $\lambda(A) = A_1$, then $\lambda : A \to A_1$ is an isomorphism.

Hence $\lambda(m_0 A) = m_0 \lambda(A) = m_0 A_1$. And since $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$. So $\lambda(m_0 A) = m_0 A' = m_0 A_1$. Let $x \in A'$, hence $m_0 x \in m_0 A' = m_0 A_1$ which implies that there exists $a_1 \in A_1 \subset A'$ such that $m_0 x = m_0 a_1$. Therefore

$m_0(x - a_1) = 0$ and consequently $x - a_1 \in T_{A'}$. And since $x = a_1 + x - a_1$ with $a_1 \in A_1$ and $x - a_1 \in T_{A'}$ therefore $A' \subset A_1 + T_{A'}$. On the other hand $A_1 \subset A'$ and $T_{A'} \subset A'$ therefore $A_1 + T_{A'} \subset A'$. We conclude that $A' = A_1 + T_{A'}$.

2) Since $A_1 \subset A'$, then $T_{A_1} \subset T_{A'}$; of plus $T_{A_1} \subset A_1$, we deduce that $T_{A_1} \subset A_1 \cap T_{A'}$. On the other hand, for all $x \in A_1 \cap T_{A'}$, then, $x \in T_{A'}$; There exists $n \in \mathbb{N}^*$ such that $nx = 0$. Since $x \in A_1$, therefore $x \in T_{A_1}$, consequently, $A_1 \cap T_{A'} = T_{A_1}$.

3) Let $x \in T_A$; There exists $m \in \mathbb{N}^*$ such that $mx = 0$, whence $0 = \lambda(mx) = m\lambda(x)$, so $\lambda(x) \in T_{A_1}$, we deduce that $\lambda(T_A) \subset T_{A_1}$. Now either $x \in T_{A_1}$; There exists $m \in \mathbb{N}^*$ such that $mx = 0$. Since $T_{A_1} \subset A_1 = \lambda(A)$, then $x = \lambda(a)$ where $a \in A$. So $0 = mx = m\lambda(a) = \lambda(ma)$. Thus $ma = 0$, hence $a \in T_A$ which implies that $x \in \lambda(T_A)$. It is concluded that $\lambda(T_A) = T_{A_1}$.

3. The proof of theorem 2.1

Let A be an abelian group and let α be an automorphism of A.

Let A' be an abelian group and $\lambda : A \to A'$ a monomorphism.

T_A, T_{A_1} and $T_{A'}$ are Respectively the torsion parts of A, A_1 and A'.

We define the endomorphism α_3 of the group A' by: $(\alpha_3)_{\restriction A_1} = \alpha_2$ and $(\alpha_3)_{\restriction T_{A'}} = \alpha'$.

The endomorphism α_3 is well defined. Indeed, if $a_1 + b_1 = a_2 + b_2$ where $a_1, a_2 \in A_1$ and $b_1, b_2 \in T_{A'}$ then $a_1 - a_2 = b_2 - b_1 \in A_1 \cap T_{A'} = T_{A_1} = \lambda(T_A)$. This implies that there exists $a \in T_A$ such that $a_1 - a_2 = b_2 - b_1 = \lambda(a)$. Hence,

$$\begin{cases} \alpha_2(a_1 - a_2) = \alpha_2\lambda(a) = \lambda\alpha(a), \\ \alpha'(b_2 - b_1) = \alpha'\lambda(a) = \lambda\alpha_1(a) = \lambda\alpha(a). \end{cases}$$

Therefore, $\alpha_2(a_1 - a_2) = \alpha'(b_2 - b_1)$, i.e, $\alpha_2(a_1) + \alpha'(b_1) = \alpha_2(a_2) + \alpha'(b_2)$. Hence $\alpha_3(a_1 + b_1) = \alpha_3(a_2 + b_2)$. Moreover α_3 is an automorphism of A'. Indeed:

Injection. Let $a' = a_1 + b_1 \in A'$ where $a_1 \in A_1$ and $b_1 \in T_{A'}$ such that $a' \in Ker(\alpha_3)$.

First case. If we have $a_1 \in T_{A'}$. Then $a_1 + b_1 \in T_{A'}$, consequently $0 = \alpha_3(a_1 + b_1) = \alpha'(a_1 + b_1)$. And since $\alpha' \in Aut(T_{A'})$ then $a' = a_1 + b_1 = 0$.

Second case. If we have $a_1 \notin T_{A'}$. Then since $b_1 \in T_{A'}$, Whence there exists $m \in \mathbb{N}^*$ such that $mb_1 = 0$, so $0 = m\alpha_3(a') = \alpha_3(ma') = \alpha_3(ma_1 + mb_1) = \alpha_3(ma_1) = \alpha_2(ma_1)$ which implies that $ma_1 = 0$ because $\alpha_2 \in Aut(A_1)$, consequently $a_1 \in T_{A_1} \subset T_{A'}$ which is absurd.

Surjection. Let $a' = a_2 + b_2 \in A'$ where $a_2 \in A_1$ and $b_2 \in T_{A'}$, since $\alpha_2 \in Aut(A_1)$ and $\alpha' \in Aut(T_{A'})$, then

$$\begin{cases} \exists a_1 \in A_1 : \alpha_2^{-1}(a_1) = a_2, \\ \exists b_1 \in T_{A'} : \alpha'^{-1}(b_1) = b_2. \end{cases}$$

So $\alpha_3(a_2+b_2)=\alpha_2(a_2)+\alpha'(b_2)=\alpha_2\alpha_2^{-1}(a_1)+\alpha'\alpha'^{-1}(b_1)=a_1+b_1$. Therefore $\alpha_3 \in Aut(A')$. In addition the automorphism α_3 of group A' commutes the following diagram:

$$\begin{array}{ccc} A & \stackrel{\lambda}{\longrightarrow} & A' \\ \alpha\downarrow & & \downarrow\alpha_3 \\ A & \stackrel{\lambda}{\longrightarrow} & A' \end{array}$$

Indeed, according to proposition 0.3 and lemma 0.4 we have: $\alpha_2\lambda=\lambda\alpha$ and $\alpha'\lambda=\lambda\alpha$, so

$$\begin{aligned} \alpha_3\lambda(a) &= \alpha_3\lambda(a)=(\alpha_2+\alpha')\lambda(a)=\alpha_2\lambda(a)+\alpha'\lambda(a)=\lambda\alpha_2(a)+\lambda\alpha'(a) \\ &= \lambda(\alpha_2+\alpha')(a)=\lambda\alpha_3(a). \end{aligned}$$

We conclude that α satisfies the weak extension property.

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SOME NEW PROPERTIES ON λ-COMMUTING OPERATORS

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Abstract. In this paper, we study the operator equation $AB = \lambda BA$ for a bounded linear operators A, B on a complex Hilbert space. We focus on algebraic relations between different operators that include normal, M-hyponormal, quasi $*$-paranormal and other classes.

Keywords: Hilbert space, λ-commute, binormal, M-hyponormal, isometry, k-paranormal, quasi $*$-paranormal.

1. Introduction

Throughout, we will denote by $\mathcal{B}(\mathcal{H})$ the complex Banach algebra of all bounded linear operators on a infinite dimensional complex Hilbert space $\mathcal{H}$. We denote the range and the kernel of $A \in \mathcal{B}(\mathcal{H})$ by $R(A)$ and $N(A)$ respectively.
Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be:

- positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$
- self-adjoint if $A = A^*$

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- isometry if $A^*A = I$, which equivalent to the condition $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$
- normal if $A^*A = AA^*$
- unitary $A^*A = AA^* = I$ (i.e. A is an onto isometry)
- quasinormal if $A(A^*A) = (A^*A)A$
- binormal if $(A^*A)(AA^*) = (AA^*)(A^*A)$ [3]
- subnormal if A has a normal extension
- hyponormal if $A^*A \geq AA^*$, which equivalent to the condition $\|A^*x\| \leq \|Ax\|$ for all $x \in \mathcal{H}$ [15]
- M-hyponormal if $A^*A \geq MAA^*$, where $M \in \mathbb{R}$ and $M \geq 1$ which equivalent to the condition $\|A^*x\| \leq M\|Ax\|$ for all $x \in \mathcal{H}$ [20]
- p-hyponormal if $(A^*A)^p \leq (AA^*)^p$, where $0 < p \leq 1$ [1]
- class $\mathcal{A}$ if $|A|^2 \leq |A^2|$, where $|A| = (A^*A)^{\frac{1}{2}}$
- paranormal if $\|Ax\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [4]
- k-paranormal if $\|Ax\|^k \leq \|A^kx\|\|x\|^{k-1}$ for all $x \in \mathcal{H}$ and $k \geq 2$
- $*$-paranormal if $\|A^*x\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [10]
- quasi $*$-paranormal if $\|A^*Ax\|^2 \leq \|A^3x\|\|Ax\|$ for all $x \in \mathcal{H}$ [12]
- log-hyponormal if A invertible and satisfies $\log(A^*A) \geq \log(AA^*)$ [16]
- p-quasihyponormal if $A^*[(A^*A)^p - (AA^*)^p]A \geq 0$, where $0 < p \leq 1$ [2]
- normoloid if $\|A\| = r(A)$
- quasinilpotent if $r(A) = 0$, where $r(A) = \lim \|A^n\|^{\frac{1}{n}}$.

We can notice that A is hyponormal if A is p-hyponormal with $p = 1$. By Löwner-Heinz inquality p-hyponormal is q-hyponormal for every $0 < q \leq p \leq 1$ [14]. Also we can notice that A is paranormal if A is k-paranormal with $k = 2$. It known that invertible p-hyponormal is log-hyponormal. We can consider log-hyponormal operator as 0-hyponormal [16]. It is well known that for any operators A, B and C we have

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \,\forall\, \lambda > 0 \;\Leftrightarrow\; ||Bx||^2 \leq ||Ax||||Cx|| \,\text{for all}\, x \in H.$$

Thus we have

- A is quasi $*$-paranormal if and only if $A^*[(A^*)^2A^2 - 2\lambda AA^* + \lambda^2]A \geq 0$ for all $\lambda > 0$.
- A is $*$-paranormal if and only if $(A^*)^2A^2 - 2\lambda AA^* + \lambda^2 \geq 0$ for all $\lambda > 0$.

We have also the following inclusions:

- quasinormal $\subseteq$ binormal
- class $\mathcal{A}$ $\subseteq$ paranormal
- hyponormal $\subseteq$ *-paranormal $\subseteq$ quasi $*$-paranormal
- invertible p-hyponormal $\subseteq$ log-hponormal $\subseteq$ paranormal.
- self-adjoint $\subseteq$ normal $\subseteq$ quasinormal $\subseteq$ subnormal $\subseteq$ hyponormal
- hyponormal $\subseteq$ p-hyponormal $\subseteq$ p-quasihyponormal $\subseteq$ class $\mathcal{A}$.

For a scalar λ, two operators A and B in $\mathcal{B}(\mathcal{H})$ are said be λ-commute if $AB = \lambda BA$. Recently many authors have studied this equation for several classes of operators, for example:

- In [11] the authors have proved that if an operator in $\mathcal{B}(\mathcal{H})$ λ-commutes with a compact, then this operator has a non-trivial hyperinvariant subspace.
- In [8] Conway and Prajitura characterized the closure and the interior of the set of operators that λ-commute with a compact operator.
- In [19] Zhang ,Ohawada and Cho have studied the properties of an operator λ-commutes with a paranormal.
- In [5] Brooke, Busch and Pearson showed that if AB is not quasinilpotent, then $|\lambda| = 1$, and if A or B is self-adjoint then $\lambda \in \mathbb{R}$.
- In [18] Yang and Du gave a simple proofs and generalizations of this results, particulary if AB is bounded below if and only if both A and B are bounded below.
- In [14] Schmeger generalized this results to hermitian or normal elements of a complex Banach algebra.
- In [6] Cho, Duggal, Harte and Ota generalized some Schmeger's results.

The aim of this paper is to study the situation for binormal, M-hyponormal, quasi $*$-paranormal operators. Again other related results are also given.

2. Main results

We begin with the following result.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal. If A is quasinilpotent, then $A = 0$.*

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal, then we have

$$\|A^*Ax\| = \|A^3x\|^2\|Ax\|^2 \text{ for all } x \in \mathcal{H}.$$

Therefore $\|Ax\|^4 = \langle A^*Ax, x\rangle^2 \leq \|A^*Ax\|^2\|x\|^2 \leq \|A^3x\|\|Ax\|\|x\|^2$.

Thus $\|Ax\|^3 \leq \|A^3x\|\|x\|^2$ for all $x \in \mathcal{H}$, whence A is 3-paranormal. By [17, Lemma 1], then every k-paranormal is normaloid. Thus we conclude that A is normaloid and hence $r(A) = \|A\|$. On the other hand A is quasinilpotent, then we obtain $\|A\| = r(A) = 0$. Therefore $A = 0$. □

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be $*$-paranormal.*

If A is quasinilpotent, then $A = 0$.

Proof. By Lemma 2.1 and since every $*$-paranormal is also quasi $*$-paranormal. □

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$, A is quasinormal and B is normal. If $|\lambda| = 1$, then AB is quasinormal.*

Proof. Assume that $AB = \lambda BA \neq 0$, then $B^*A^* = \bar{\lambda}A^*B^*$. Since B and λB are normal operators and by Fuglede-Putnam Theorem, then $BA^* = \lambda A^*B$ and $AB^* = \bar{\lambda}B^*A$. Moreover we have

$$\begin{aligned}
AB[(AB)^*AB] &= [AB][B^*A^*AB] \\
&= [\lambda BA]B^*A^*AB \\
&= \lambda B[AB^*]A^*AB \\
&= \lambda B[\bar{\lambda}B^*A]A^*AB \\
&= |\lambda|^2[BB^*][AA^*A]B \\
&= [B^*B][A^*AA]B \\
&= B^*[BA^*]AAB \\
&= B^*[\lambda A^*B]AAB \\
&= B^*A^*[\lambda BA]AB \\
&= B^*A^*[AB]AB \\
&= [(AB)^*AB]AB.
\end{aligned}$$

Therefore AB is quasinormal. □

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$, A is binormal and B is normal. If $|\lambda| = 1$, then AB is binormal.*

Proof. Since B and λB are normal operators and by Fuglede-Putnam Theorem, then we have $BA^* = \lambda A^* B$ and $AB^* = \bar{\lambda} B^* A$. Therefore we obtain

$$\begin{aligned}
AB(AB)^*(AB)^*AB &= A[BB^*]A^*B^*A^*AB \\
&= A[B^*B]A^*B^*A^*AB \\
&= [AB^*]BA^*[B^*A^*]AB \\
&= [\bar{\lambda}B^*A]BA^*[\bar{\lambda}A^*B^*]AB \\
&= (\bar{\lambda})^2 B^*[AB]A^*A^*[B^*A]B \\
&= (\bar{\lambda})^2 B^*[\lambda BA]A^*A^*[\frac{1}{\bar{\lambda}}AB^*]B \\
&= |\lambda|^2 B^*B[AA^*A^*A]B^*B \\
&= B^*B[A^*AAA^*]B^*B \\
&= B^*[BA^*]AA[A^*B^*]B \\
&= B^*[\lambda A^*B]AA[\frac{1}{\bar{\lambda}}B^*A^*]B \\
&= \frac{\lambda}{\bar{\lambda}}B^*A^*[BA]AB^*[A^*B] \\
&= \lambda^2 B^*A^*[\frac{1}{\lambda}AB]AB^*[\frac{1}{\lambda}BA^*] \\
&= B^*A^*ABA[B^*B]A^* \\
&= B^*A^*ABA[BB^*]A^* \\
&= (AB)^*ABAB(AB)^*,
\end{aligned}$$

then AB is binormal. □

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.*

Suppose that A is k-paranormal and B is isometry, then the following statements are equivalent:

1. *AB is k-paranormal*
2. *$\sigma(AB) \neq \{0\}$*
3. *$|\lambda| = 1$.*

Proof. Suppose that A is k-paranormal and B is isometry with $AB = \lambda BA \neq 0$.

We first show that (1) ⇒ (2). Suppose that AB is k-paranormal.

If AB is quasinilpotent ($\sigma(AB) = \{0\}$). Since every k-paranormal is isometry, then we obtain $\|AB\| = r(AB) = 0$ and hence $AB = 0$ and this is a contradiction with $AB \neq 0$. Therefore AB is not quasinilpotent and hence $\sigma(AB) \neq \{0\}$.

We prove that (2) ⇒ (3). Suppose that $\sigma(AB) \neq \{0\}$, then

$$r(AB) \neq 0. \tag{1}$$

Since $AB = \lambda BA \neq 0$ and by [5, Proposition 1], then $\sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$. Hence

$$r(AB) = |\lambda| r(AB). \tag{2}$$

Therefore by (1) and (2) we obtain $|\lambda| = 1$. Finally we show that (3) $\Rightarrow$ (1). Suppose that $|\lambda| = 1$, for any unit vector $x \in \mathcal{H}$ we have

$$\begin{aligned}
\|(AB)x\|^k &= \|A(Bx)\|^k \\
&\leq \|A^k(Bx)\| \|Bx\|^{k-1} \quad (A \,\text{is}\; k-\text{paranormal}) \\
&\leq \|A^k Bx\| \qquad (B \,\text{is isometry}).
\end{aligned}$$

Hence

$$\|(AB)x\|^k \leq \|A^k Bx\|. \tag{3}$$

On the other hand by induction we show that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$ for every $k \in \mathbb{N}^*$. For $k = 1$ we have $(AB)^1 = \lambda^{\frac{1(1-1)}{2}} B^{1-1} A^1 B$. Assume that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$ for $k \geq 2$. Finally we have

$$\begin{aligned}
(AB)^{k+1} = AB(AB)^k &= (\lambda BA)(\lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B) \\
&= \lambda^{\frac{k(k-1)}{2}+1} BAB^{k-1} A^k B \\
&= \lambda^{\frac{k(k-1)}{2}+1} B(AB)B^{k-2} A^k B \\
&= \lambda^{\frac{k(k-1)}{2}+1} B(\lambda BA)B^{k-2} A^k B \\
&= \lambda^{\frac{k(k-1)}{2}+2} B^2 AB^{k-2} A^k B \\
&\vdots \\
&\vdots \\
&\vdots \\
&= \lambda^{\frac{k(k-1)}{2}+k} B^k AB^{k-k} A^k B \\
&= \lambda^{\frac{(k+1)k}{2}} B^k A^{k+1} B.
\end{aligned}$$

We conclude that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$, for every $k \in \mathbb{N}^*$. Then for every unit vector $x \in \mathcal{H}$ we obtain

$$\begin{aligned}
\|(AB)^k x\| &= \|\lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k Bx\| \\
&= |\lambda|^{\frac{k(k-1)}{2}} \|B^{k-1} A^k Bx\| \\
&= \|A^k Bx\| \qquad (B^{k-1} \text{ is isometry and} \, |\lambda| = 1).
\end{aligned}$$

Hence

$$\|(AB)^k x\| = \|A^k Bx\| \text{ for any unit vector}\, x. \tag{4}$$

Finally by (3) and (4) we conclude that $\|(AB)x\|^k \leq \|A^k Bx\| = \|(AB)^k x\|$, for any unit vector x. Therefore AB is k-paranormal. $\square$

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then*

1. *if A^* is M_1-hyponormal and B is M_2-hyponormal, then $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$*
2. *if A is M_1-hyponormal and B^* is M_2-hyponormal, then $|\lambda| \geq (M_1M_2)^{-\frac{1}{2}}$.*

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.

1. Since we have

$$\begin{aligned}
|\lambda|\|BA\| &= \|\lambda BA\| \\
&= \|AB\| \\
&= \|B^*A^*AB\|^{\frac{1}{2}} \quad (\|T\| = \|TT^*\|^{\frac{1}{2}} \\
&\leq M_1^{\frac{1}{2}}\|B^*AA^*B\|^{\frac{1}{2}} \ \ (A^* \text{ is } M_1 - \text{hyponormal}: \ A^*A \leq M_1AA^*) \\
&\leq M_1^{\frac{1}{2}}\|A^*B\| \ \ (\|T^*T\|^{\frac{1}{2}} = \|T\|) \\
&\leq M_1^{\frac{1}{2}}\|A^*BB^*A\|^{\frac{1}{2}} \quad (\|T\| = \|TT^*\|^{\frac{1}{2}}) \\
&\leq (M_1M_2)^{\frac{1}{2}}\|A^*B^*BA\|^{\frac{1}{2}} \ \ (B \text{ is } M_2 - \text{hyponormal: } BB^* \leq M_2B^*B) \\
&\leq (M_1M_2)^{\frac{1}{2}}\|BA\| \quad (\|T^*T\|^{\frac{1}{2}} = \|T\|).
\end{aligned}$$

   Therefore $|\lambda|\|BA\| \leq (M_1M_2)^{\frac{1}{2}}\|BA\|$ Hence $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$.

2. Since $AB = \lambda BA$ and $\lambda \neq 0$, then $BA = \lambda^{-1}AB$ and by first implication we obtain $|\lambda^{-1}| \leq (M_2M_1)^{\frac{1}{2}}$ and hence $|\lambda| \geq (M_2M_1)^{-\frac{1}{2}}$.

□

Corollary 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then*

1. *if A^* and B are hyponormal, then $|\lambda| \leq 1$*
2. *if A and B^* are hyponormal, then $|\lambda| \geq 1$.*

Proof. By Theorem 2.4 and we take $M_1 = M_2 = 1$. □

Theorem 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.*
If A^* is M_1-hyponormal and B is M_2-hyponormal, then A^*B and BA^* are $M_1M_2|\lambda|^2$-hyponormal.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then

$$\begin{aligned}
(A^*B)^*A^*B &= B^*AA^*B \\
&\geq M_1B^*A^*AB \\
&\geq M_1\bar{\lambda}A^*B^*\lambda BA \\
&\geq M_1|\lambda|^2A^*B^*BA \\
&\geq M_1|\lambda|^2A^*M_2BB^*A \\
&\geq M_1M_2|\lambda|^2(B^*A)^*B^*A.
\end{aligned}$$

Therefore A^*B is $M_1M_2|\lambda|^2$-hyponormal.

In the same way we obtain BA^* is $M_1M_2|\lambda|^2$-hyponormal. □

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LIE IDEALS WITH SYMMETRIC LEFT BI-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring and U be a nonzero lie ideal of R. A symmetric bi-additive mapping $D(.,.) : R \times R \to R$ is called a symmetric bi-derivation and d is a trace of D. In this paper we shall show that $U \subseteq Z(R)$ such that R admitting the trace d satisfying the several conditions of symmetric left bi-derivation.

Keywords: prime ring, symmetric mapping, trace, derivation, symmetric bi-derivation, symmetric bi-additive mapping, symmetric left bi-derivation.

1. Introduction

The concept of a symmetric bi-derivation has been introduced by Maksa.Gy in [5, 6]. A classical result in the theory of centralizing mappings is a theorem first proved by E. Posner [8] which stated that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Vukman.J [9, 10] has studied some results concerning symmetric bi-derivations on prime and semi prime rings. In [1] Argac, Yenigul and in [7] Muthana obtained the similar type of results on lie ideals of R. In this paper we proved some results in symmetric left bi-derivations in prime rings.

Throughout this paper R will be associative. We shall denote by $Z(R)$ the center of a ring R. Recall that a ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. We shall write $[x, y]$ for $xy - yx$. The symbol $x \circ y$ stands for anti commutator $xy + yx$. An additive map $d : R \to R$ is called

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derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x,\ y \in R$.A mapping $B(.,\ .) : R \times R \to R$ is said to be symmetric if $B(x,\ y) = B(y,\ x)$ holds for all pairs $x,\ y \in R$. A mapping map $f : R \to R$ defined by $f(x) = B(x,\ x)$, where $B(.,\ .) : R \times R \to R$ is a symmetric mapping, is called a trace of B. It is obvious that, in case $B(.,\ .) : R \times R \to R$ is symmetric mapping which is also bi-additive (i. e. additive in both arguments) the trace of B satisfies the relation $f(x+y) = f(x) + f(y) + 2B(x,\ y)$ for all $x,\ y \in R$. We shall use also the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping $D(.,\ .) : R \times R \to R$ is called a symmetric bi-derivation if $D(xy,\ z) = D(x,\ z)y + xD(y,\ z)$ is fulfilled for all $x,\ y,\ z \in R$.Obviously, in this case also the relation $D(x,\ yz) = D(x,\ y)z + yD(x,\ z)$ $x,\ y,\ z \in R$. A symmetric bi-additive mapping $D(.,\ .) : R \times R \to R$ is called a symmetric left bi-derivation if $D(xy,\ z) = xD(y,\ z) + yD(x, z)$ for all $x,\ y,\ z \in R$. Obviously, in this case also the relation $D(x,\ yz) = yD(x,\ z) + zD(x, y)$ for all $x,\ y,\ z \in R$. A mapping $f : R \to R$ is said to be commuting on R if $[f(x),\ x] = 0$ holds for all $x \in R$. A mapping $f : R \to R$ is said to be centralizing on R if $[f(x),\ x] \in Z(R)$ is fulfilled for all $x \in R$. A ring R is said to be n-torsion free if whenever $na = 0$ with $a \in R$ then $a = 0$, where n is nonzero integer.

We shall frequently use the following identities and several well known facts about the semiprime rings without specific mention.

$$\begin{aligned}
[xy, z] &= x[y, z] + [x, z]y;\\
[x,\ yz] &= y[x,\ z] + [x,\ y]z;\\
x \circ yz &= (x \circ y)z - y[x,\ z] = y(x \circ z) + [x,\ y]z;\\
xy \circ z &= x(y \circ z) - [x,\ z]y = (x \circ z)y + x[y, z].
\end{aligned}$$

Remark 1. Let U be a square closed lie ideal of R. Notice that $xy + yx = (x+y)^2 - x^2 - y^2$, for all $x,\ y \in U$. Since $x^2 \in U$, for all $x \in U$ $xy + yx \in U$ for all $x,\ y \in U$. Hence we find that $2xy \in U$ for all $x \in U$. Therefore, for all $r \in R$, we get $2r[x,\ y] = 2[x,\ ry] - 2[x,\ r]y \in U$ and $2[x,\ y]r = 2[x,\ ry] - 2[y,\ r]y \in U$ so that $2R[U, U] \subseteq U$ and $2[U, U]R \subseteq U$.

This remark will be freely used in the whole paper without specific reference.

Lemma 1 (4, Corollary 2.1)**.** *Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \nsubseteq Z(R)$ and $a, b \in U$.*

(i) if $aUa = \{0\}$, then $a = 0$;

(ii) if $aUa = \{0\}$ $(Ua = \{0\})$, then $a = 0$;

(iii) if U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.

Lemma 2 (1, Theorem 3)**.** *Let R be 2-torsion free prime ring and U be a nonzero Lie ideal of R. Let $B : R \times R \to R$ be a symmetric bi-derivation and f be the trace of B be such that:*

(i) $f(U) = 0$, then $U \subseteq Z(R)$ or $f = 0$;

(ii) $f(U) \subseteq Z(R)$ *and* U *be a square closed Lie ideal, then* $U \subseteq Z(R)$ *or* $f = 0$.

Lemma 3 (3, Lemma 1)**.** *Let* R *be a 2-torsion free semiprime ring and* U *be a Lie ideal of* R*. Suppose that* $[U, U] \subseteq Z(R)$*, then* $U \subseteq Z(R)$*.*

Lemma 4 (2, Lemma 4)**.** *Let* R *be a 2-torsion free prime ring and* $U \nsubseteq Z(R)$ *be a Lie ideal of* R *and* $a, b \in R$*, if* $aUb = \{0\}$ *then* $a = 0$ *and* $b = 0$*.*

Lemma 5. *Let* R *be a 2-torsion free prime ring and* U *be a square closed lie ideal of* R*. Suppose that* $D : R \times R \to R$ *is a symmetric left bi-derivation and* d *the trace of* D *such that* $[d(x), y] \in Z(R)$*, for all* $x, y \in U$*, then either* $U \subseteq Z(R)$ *or* $d = 0$*.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$

(1) We have $[d(x), y] \in Z(R)$, for all $x, y \in U$.

We replace y by $2yz$ in (1), we get

$$\begin{aligned}
&[d(x), 2yz] \in Z(R),\\
&2y[d(x), z] + 2[d(x), y]z \in Z(R),\\
&y[d(x), z] + [d(x), y]z \in Z(R), \text{ for all } x, y, z \in U.
\end{aligned}$$

This implies that $[[d(x), y]z + y[d(x), z], r] = 0$, for all $x, y, z \in U$ and $r \in R$

(2) $[d(x), y][z, r] + [y, r][d(x), z] = 0$, for all $x, y, z \in U$ and $r \in R$.

We replacing r by $2yt$ in (2), we get

(3) $[y, z][d(x), z] = 0$ for all $x, y, z \in U$.

We replacing y by $2yt$ in (3), we get

$$\begin{aligned}
&[2yt, z][d(x), z] = 0,\\
&2[y, z]t[d(x), z] + 2y[t, z][d(x), z] = 0,\\
&[y, z]t[d(x), z] = 0, \text{ for all } x, y, z, t \in U,\\
&[y, z]U[d(x), z] = 0, \text{ for all } x, y, z, t \in U.
\end{aligned}$$

Thus in view of Lemma 4 we find that for each pair of $x, y, z \in U$ either $[y, z] = 0$ or $[d(x), z] = 0$ For each $z \in U$, let $A^1 = \{y \in U / [y, z] = 0\}$ and $B^1 = \{x \in U / [f(x), z] = 0\}$. Hence A^1 and B^1 are the additive subgroups of U whose unionis U. By Brauer's trick, we have either $U = A^1$ or $U = B^1$. If $U = A^1$, then $[y, z] = 0$ for all $y, z \in U$ and have $U \subseteq Z(R)$ a contradiction. On the other hand if $U = B^1$ then $[d(x), z] = 0$, for all $x, z \in U$ and hence $f(U) \subseteq C_R(U) = Z(R)$ then by Lemma 2, we get $d = 0$. This completes the proof of the lemma. □

Theorem 6. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D: R \times R \to R$ is a symmetric left bi-derivation and d the trace of D. If $[d(x), x] = 0$, for all $x \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$

(4) Since we have given that $[d(x), x] = 0$, for all $x, y \in U$

We replacing x by $x + y$ in (4), we get $d(x + y), x + y] = 0$, $[d(x) + d(y) + 2D(x, y), x + y] = 0$, $[d(x), x] + [d(x), y] + [d(y), x] + [d(y), y] + 2[D(x, y), x] + 2[D(x, y), y] = 0$. By using (4), in the above equation we get

$$[d(x), y]+[d(y), x]+2[D(x, y), x]+2[D(x, y), y] = 0 \text{ for all } x, y \in U \tag{5}$$

We replacing x by $-x$ in (5), we get

$$\begin{aligned} &[d(-x), y]+[d(y), -x]+2[D(-x, y), -x]+2[D(-x, y), y] = 0, \\ &[d(x), y] - [d(y), x] + 2[D(x, y), x] - 2[D(x, y), y] = 0, \text{ for all } x, y \in U. \end{aligned} \tag{6}$$

By adding (5) and (6), we get

$$[d(x), y] + 2[D(x, y), x] = 0, \text{ for all } x, y \in U. \tag{7}$$

We replacing y by $2yz$ in (7), we get

$$\begin{aligned} &[d(x), 2yz] + 2[D(x, 2yz), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[yD(x, z) + zD(x, y), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[yD(x, z), x] + 4[zD(x, y), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[y, x]D(x, z) + 4y[D(x, z), x] \\ &+ 4[z, x]D(x, y) + 4z[D(x, y), x] = 0, \\ &2y[d(x), z] + 2z[d(x), y] + 4[y, x]D(x, z) + 4y[D(x, z), x] \\ &+ 4[z, x]D(x, y) + 4z[D(x, y), x] = 0, \\ &2y([d(x), z] + 2[D(x, z), x]) + 2z([d(x), y] + 2[D(x, y), x]) \\ &+ 4[y, x]D(x, z) + 4[z, x]D(x, y) = 0. \end{aligned}$$

By using (4) in the above equation we get

$$\begin{aligned} &4[y, x]D(x, z) + 4[z, x]D(x, y) = 0, \\ &[y, x]D(x, z) + [z, x]D(x, y) = 0, \quad \text{for all } x, y, z \in U. \end{aligned} \tag{8}$$

We replace z by x in (8) we get

$$\begin{aligned} &[y, x]D(x, x) + [x, x]D(x, y) = 0, \\ &[y, x]D(x, x) = 0, \text{ for all } x, y \in U. \end{aligned} \tag{9}$$

We replacing y by $2yz$ in (9), we get $[2yz,\ x]\,D\,(x,\ x) = 0$, $2\,[y,\ x]\,zD\,(x,\ x) + 2y\,[z,\ x]\,D\,(x,\ x) = 0$.

By using (9) in the above equation we get $2[y,x]zD(x,x) = 0$, $[y,x]zD(x,x) = 0$, for all $x, y, z \in U$, this gives $[y,x]UD(x,x) = 0$, for all $x, y \in U$. By Lemma 4 for each $x \in U$ either $[y,x] = 0$ or $D(x,x) = 0$, for all $x, y \in U$. In the first case it fallows that by Lemma 3, $x \in Z(R)$ for all $x \in U$. Thus if $x \notin Z(R)$ then $D(x,x) = 0$. Let $x, z \in U$ such that $x \in Z(R)$ and $z \notin Z(R)$. Hence $x+z \notin Z(R)$ and $x - z \notin Z(R)$. Thus $D(x+z, x+z) = 0$ and $D(x-z, x-z) = 0$. Adding the above two relations, we get $2D(x,x) = 0$, since R is 2-torsion free ring, we get $D(x,x) = 0$. Thus for all $x \in U$, $D(x,x) = 0$ and by Lemma 2, $d = 0$. □

Theorem 7. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d\,([x,\ y]) - [d\,(x)\,,\ y] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d\ = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d\,([x,\ y]) - [d\,(x)\,,\ y] \in Z(R),\ \text{for all } x,\ y \in U. \tag{10}$$

We replace y by $y + z$ in (10), we get $d([x, y + z]) - [d(x), y + z] \in Z(R)$, $d([x,y]+[x,z]) - [d(x), y] - [d(x), z] \in Z(R)$, $d([x,y]) + d([x,z]) + 2D([x,y],[x,z]) - [d(x), y] - [d(x), z] \in Z(R)$. By using (10) in the above equation we get

$$D\,([x,\ y]\,,\ [x,\ z]) \in Z(R),\ \text{for all } x, y, z \in U \tag{11}$$

We replace z by y in (11), we get $D\,([x,\ y]\,,\ [x,\ y]) \in Z(RD\,([x,\ y]\,,\ [x,\ y]) \in Z(R)$, for all $x,\ y \in U$

$$d\,([x,\ y]) \in Z\,(R)\,,\ \text{for all } x,\ y \in U \tag{12}$$

By subtracting (10) from (12) we get $[d\,(x)\,,\ y] \in Z(R)$, for all $x,\ y \in U$ By using Lemma 5, we get the required result. □

Theorem 8. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d\,(x \circ y) - [d\,(x)\,,\ y] \in Z\,(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z\,(R)$ or $d\ =\ 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z\,(R)$. We have

$$d\,(x \circ y) - [d\,(x)\,,\ y] \in Z(R),\ \text{for all } x,\ y \in U. \tag{13}$$

We replace y by $y + z$ in (13), we get, $d\,(x \circ y + z) - [d\,(x)\,,\ y + z] \in Z(R)$, $d\,(x \circ y) + d\,(x \circ z) + 2D\,(x \circ y,\ x \circ z) - [d\,(x)\,,\ y] - [\mathrm{d\,(x)\,,\ z}] \in Z(R)$. By using (13) in the above equation we get $2D\,(x \circ y,\ x \circ z) \in Z(R)$

$$D\,(x \circ y,\ x \circ z) \in Z(R),\ \text{ for all } x, y, z \in U \tag{14}$$

We replace z by y in (14), we get $D(x \circ y,\ x \circ y) \in Z(R)$, for all $x,\ y \in U$

$$d(x \circ y) \in Z(R), \text{ for all } x,\ y \in U. \tag{15}$$

By subtracting (13) from (15), we get $[d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. By using Lemma 5, we get the required result. □

Theorem 9. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d(x) \circ y - [d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d(x) \circ y - [d(x),\ y] \in Z(R), \text{ for all } x,\ y \in U. \tag{16}$$

$d(x)y + yd(x) - d(x)y + yd(x) \in Z(R)$, $2yd(x) \in Z(R)$, $yd(x) \in Z(R)$, for all $x,\ y \in U$, $[yd(x),\ r] = 0$, for all $x,\ y \in U$ and $r \in R$.

$$y[d(x),\ r] + [y,\ r]d(x) = 0, \text{ for all } x,\ y \in U \text{ and } r \in R. \tag{17}$$

We replace y by $2yt$ in (17), we get $2ty[d(x),\ r] + [2ty,\ r]d(x) = 0$, $2ty[d(x),\ r] + 2t[y,\ r]d(x) + 2[t,\ r]yd(x) = 0$. By using (17) in the above equation we get $[t,\ r]yd(x) = 0$, for all $x,\ y,\ t \in U$ and $r \in R$, $[t,\ r]Ud(x) = 0$, for all $x,\ t \in U$ and $r \in R$. By using Lemma 4 we get either $[t,\ r] = 0$ or $d(x) = 0$, for all $x,\ t \in U$ and $r \in R$. If $[t,\ r] = 0$ then $U \subseteq Z(R)$ a contradiction. Hence if $d(x) = 0$ for all $x \in U$, then by Lemma 2, we get $d = 0$. □

Theorem 10. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ is any mapping such that $[d(x),\ y] - [x,\ g(x)] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$[d(x),\ y] - [x,\ g(y)] \in Z(R), \text{ for all } x,\ y \in U. \tag{18}$$

We replace x by $x + z$ in (18), we get $[d(x+z),\ y] - [x+z,\ g(y)] \in Z(R)$

$$[d(x),\ y] + [d(z),\ y] + 2[D(x,\ z),\ y] - [x,\ g(y)] - [z,\ g(y)] \in Z(R)$$

By using (18) in the above equation we get $2[D(x,\ z),\ y] \in Z(R)$

$$[D(x,\ z),\ y] \in Z(R) \text{ for all } x, y, z \in U \tag{19}$$

We replace z by x in (19), we get $[D(x,\ x),\ y] \in Z(R)$, for all $x,\ y \in U$, $[d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. Hence by Lemma 3, we get the required result. □

Theorem 11. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ is any mapping such that $d(x) \circ d(y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

(20) $$d(x) \circ d(y) - [d(x), y] \in Z(R), \text{ for all } x, y \in U.$$

We replace y by $y+z$ in (20), we get $d(x) \circ d(y+z) - [d(x), y+z] \in Z(R)$, $(x) \circ d(y) + d(x) \circ d(z) + 2d(x) \circ D(y, z) - [d(x), y] - [d(x), z] \in Z(R)$, for all $x, y, z \in U$. By using (20) in the above equation we get $2d(x) \circ D(y, z) \in Z(R)$

(21) $$d(x) \circ D(y, z) \in Z(R)$$

We replace z by y in (21), we get $d(x) \circ D(y, y) \in Z(R)$

(22) $$d(x) \circ d(y) \in Z(R), \text{ for all } x, y \in U$$

By subtracting (20) from (22), we get $[d(x), y] \in Z(R)$, for all $x, y \in Z(R)$. Thus by using Lemma 1, we get the required result. □

Theorem 12. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ be any mapping such that $d(x)y - xg(y) \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

(23) $$d(x)y - xg(y) \in Z(R), \text{ for all } x, y \in U$$

We replace x by $x+z$ (23), we get $d(x+z)y - (x+z)g(y) \in Z(R)$

(24) $$d(x)y + d(z)y + 2D(x, z)y - xg(y) - zg(y) \in Z(R) \text{ for all } x, y, z \in U.$$

By using (23) in (24), we get $2D(x, z)y \in Z(R)$

(25) $$D(x, z)y \in Z(R), \text{ for all } x, y, z \in U.$$

We replace z by x in (25), we get $D(x, x)y \in Z(R)$, $d(x)y \in Z(R)$, for all $x, y \in U$

(26) $$[d(x)y, r] = 0, \text{ for all } x, y \in U \text{ and } r \in R$$

We replace y by $2yt$ in (26), we get $[d(x)2yt, r] = 0$, $2[d(x)y, r]t + 2d(x)y[t, r] = 0$. By using (26) in the above equation, we get $2d(x)y[t, r] = 0$, $2d(x)y[t, r] = 0$, for all $x, y \in U$ and $r \in R$, $2d(x)U[t, r] = 0$, for all $x, y, t \in U$ and $r \in R$, $d(x)U[t, r] = 0$, for all $x, t \in U$ and $r \in R$.

By using Lemma 4, we get either $[t, r] = 0$ or $d(x) = 0$ for all $x, t \in U$ and $r \in R$.

If $[t, r] = 0$ then $U \subseteq Z(R)$ a contradiction. Hence if $d(x) = 0$, for all $x \in U$, then by Lemma 2, we get $d = 0$. □

Theorem 13. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d(x)y - xg(y) \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d(xy) - d(x)y - xd(y) \in Z(R). \tag{27}$$

We replace x by $x+z$ in (27), we get $d((x+z)y) - d(x+z)y - (x+z)d(y) \in Z(R)$, $d(xy+zy) - d(x+z)y - (x+z)d(y) \in Z(R)$, $d(xy) + d(zy) + 2D(xy, zy) - d(x)y - d(z)y - 2D(x, z)y - xd(y) - zd(y) \in Z(R)$, for all $x, y, z \in U$.

By using (27) in the above equation, we get $2D(xy, zy) - 2D(x, z)y \in Z(R)$,

$$D(xy, zy) - D(x, z)y \in Z(R), \text{ for all } x, y, z \in U. \tag{28}$$

We replace z by x in (28), we get $D(xy, xy) - D(x, x)y \in Z(R)$,

$$d(xy) - d(x)y \in Z(R), \text{ for all } x, y \in U. \tag{29}$$

We replace y by $y+z$ in (29), we get $d(x(y+z)) - d(x)(y+z) \in Z(R)$, $d(xy+xz) - d(x)(y+z) \in Z(R)$, $d(xy) + d(xz) + 2B(xy, xz) - d(x)y - d(x)z \in Z(R)$, for all $x, y, z \in U$.

By using (29) in the above equation, we get

$$B(xy, xz) \in Z(R), \text{ for all } x, y, z \in U. \tag{30}$$

We replace z by y in (30), we get $D(xy, xy) \in Z(R)$,

$$d(xy) \in Z(R), \text{ for all } x, y \in U. \tag{31}$$

By subtracting (29), from (31), we get $d(x)y \in Z(R)$, for all $x, y \in Z(R)$, $[d(x)y, r] = 0$, for all $x, y, z \in U$ and $r \in R$

$$[d(x), r]y + d(x)[y, r] = 0, \text{ for all } x, y, z \in U \text{ and } r \in R. \tag{32}$$

We replace r by $d(x)$ in (32), we get $[d(x), d(x)]y + d(x)[y, d(x)] = 0$,

$$d(x)[y, d(x)] = 0. \tag{33}$$

We replace y by $2yz$ in (33), we get $d(x)[2yz, d(x)] = 0$, $2d(x)[y, d(x)] + 2d(x)y[z, d(x)] = 0$. By using (33) in the above equation we get $2d(x)y[z, d(x)] = 0$.

$$d(x)y[z, d(x)] = 0, \text{ for all } x, y, z \in U. \tag{34}$$

Multiplying (34) left by z we get

$$zd(x)y[z, d(x)] = 0 \text{ for all } x, y, z \in U. \tag{35}$$

We replace y by $2zy$ in (34), we get $d(x)\,2zy\,[z,\ d(x)] = 0$

$$d(x)\,zy\,[z,\ d(x)] = 0, \text{ for all } x,\ y,\ z \in U. \tag{36}$$

By combining (35) and (36), we get $[z,\ d(x)]y[z,\ d(x)] = 0$, $[z,\ d(x)]U[z,\ d(x)] = \{0\}$. By using Lemma 1, we get $[z,\ d(x)] = 0$, for all $x,\ z \in U$ and by Lemma 5, we get $d = 0$. □

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DETERMINATION OF THE SHORTEST PATH IN VAGUE NETWORKS

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Abstract. We propose a new approach to determine the shortest path in a vague network(VN), a network in which vertices and edges remain crisp but each edge $(i, i+1)$ has an associated weight, which is a vague number of the form $[R_{it}, R_{if}]$ for each i. For each VN, we associate two vague networks called true and false limit fuzzy networks having the same set of vertices and edges but each edge $(i, i+1)$ is attached with a vague weight R_{it} and R_{if} respectively. We exhibit that the shortest path of weight $w = [w_t, w_f]$ an vague number in VN, the path for which the shortest path of weight w_t in the true fuzzy network coincides with the shortest path of weight w_t in the true limit vague network. The concept is illustrated with the help of a simple situation and the validation of mathematical verification is provided.

Keywords: fuzzy network, vague network (VN), shortest path, vague shortest path.

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1. Introduction

Graph theory has found its importance in many real time problems. Recent applications in graph theory is quite interesting analysing any complex situations and moreover in engineering applications. It has got numerous applications on operations research, system analysis, network routing, transportation and many more. In 1975, Rosenfeld [22] discussed the concept of fuzzy graphs whose ideas are implemented by Kauffman [16] in 1973. The fuzzy relation between fuzzy sets were also considered by Rosenfeld who developed the structure of fuzzy graphs, obtaining various analagous results of several graph theoretical concepts. Bhattacharya [4] gave some remarks of fuzzy graphs. The complement of fuzzy graphs was introduced by Mordeson [17]. Atanassov introduced the concept of intuitionistic fuzzy relation and intuitionistic fuzzy graphs [2, 3, 28, 29]. Talebi and Rashmanlou [41] studied the properties of isomorphism and complement of interval-valued fuzzy graphs. They defined isomorphism and some new operations on vague graphs [42, 43]. Borzooei and Rashmalou analysed new concepts of vague graphs [5], degree of vertices in vague graphs [6], more results on vague graphs [7], semi global domination sets in vague graphs with application [8] and degree and total degree of edges in bipolar fuzzy graphs with application [9]. Rashmanlou *et.al.* defined the complete interval-valued fuzzy graphs [23]. Rashmanlou and Pal studied intuitionistic fuzzy graphs with categorical properties [28], some properties of highly irregular interval-valued fuzzy graphs [27], more results on highly irregular bipolar fuzzy graphs [29], balanced interval-valued fuzzy graphs [25] and antipodal interval-valued fuzzy graphs [24]. Samanta and Pal investigated fuzzy k-competition and p-competition graphs, and concept of fuzzy planar graphs in [20, 21, 30]. Also they introduced fuzzy tolerance graph [39], bipolar fuzzy hypergraphs [40] and given several properties on it. Pal and Rashmanlou [19] defined many properties of irregular interval-valued fuzzy graphs. Ganesh *et al.* [12, 13] analysed the properties of Regular product vague graphs and product vague line graphs.

In graph theory the shortest path problem is the problem of finding a path between two vertices such that sum of the weight of its constituent edges is minimized. An example is finding the shortest way to get from one location to another on a road map. The vertices(or nodes) represents the locations and are weighted by the time needed to travel that segment and the edges(or links) represents the roads leading to various places connected through out the destination point. The shortest path problem has transportation, communication routing and scheduling. Now, in any network path the arc length may represent time or cost. Therefore in the real world, it can be considered to be a fuzzy set.To analyse any complete information we make intensive use of graphs and its properties. For working on partial information or incomplete information or to handle the systems containing the elements of uncertainty we understand that fuzzy logic and its involvement in graph theory is applied.

We analyse the shortest path of any source to destination using vague networks(VN). We consider the directed network consisting of a finite set of vertices and finite set of edges. It is assumed that there is only one edge between any two vertices. The fuzzy shortest problem was first analysed by Dubois and Prade [11] . They used Floyd's algorithm and Ford's algorithm to treat the fuzzy shortest path problem. Although in their method of shortest length could be obtained but sometimes the corresponding path in the network does not exist. [15]Klein proposed a dynamical programming recursion based fuzzy algorithm [10] and later developed by many researchers. Recently, the concept of Interval valued fuzzy matrices(IVFM) as a generalization of fuzzy matrix was introduced by Shyamal and Pal [38], by extending the max-min operations on Fuzzy algebra $F = [0, 1]$ for elements $a, b \in F, a + b = \max\{a, b\}$ and $a.b = \min\{a, b\}$. Let F_{mn} be the set of all $m \times n$ fuzzy matrices over the fuzzy algebra with support $[0, 1]$, ie., the matrices whose entries are intervals and all the intervals are subintervals of the interval $[0, 1]$, then $\max\{a_i, b_i\} = [\max\{a_{iL}, b_{iL}\}, \max\{a_{iU}, b_{iU}\}]$. In earlier works, represented Interval Valued Fuzzy Matrices $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ where each a_{ij} is a subinterval of the interval $[0, 1]$ as the interval matrix $A = [A_L, A_U]$ whose ij^{th} entry is the interval $[a_{ijL}, a_{ijU}]$, where the lower limit $A_L = (a_{ijL})$ and the upper limit $A_U = (a_{ijU})$ are fuzzy matrices such that the $A_L \leq A_U$ that is $a_{ijL} \leq a_{ijU}$ under the usual ordering of real numbers. In this paper, we adopt a similar technique to determine the shortest path for an vague network (VN), that is the path in which the sum of the weight of its constituent edges is minimized, by way of constructing two vague networks corresponding to the true and false limits for an VN as a generalisation of fuzzy shortest path technique presented in [15].Meenakshi *et al.* [18] determined the shortest path in interval-valued fuzzy networks. Sahoo *et al.* [31, 32, 33, 34, 35] analysed about different types of product on intuitionistic fuzzy graphs, intuitionistic fuzzy competition graph, intuitionistic fuzzy tolerance graph with application, product on intuitionistic fuzzy graphs and degree, covered and paired domination in intuitionistic fuzzy graphs. We propose a new approach to determine the shortest path in VN in which the edges representing the roads connecting the cities and each edge $(i, i+1)$ has an associated weight representing the traffic on the road connecting the cities i and $i+1$, which is an vague number of the form $R_i = [R_{iT}, R_{iF}]$ for each i and we apply the technique used in [15] to determine the shortest path in true and false limits of the fuzzy networks. We have defined the shortest path of VN as the path for which the shortest path in true limit vague network coincides with the shortest path in false limit vague networks and weight $[w_T, w_F]$ where w_T and w_F are the weights of the shortest path for true and false networks respectively.In this work we analyse the shortest path of vague networks using DP recursion algorithm. For further terminologies, the readers are referred to [1-6,14,15].

2. Preliminaries

A graph (V, E) be a set of points V and a set of pairs of these points E. The set V refers to the vertices of the graph and the set E refers to the edges of the graph. An edge is denoted by a pair of vertices $\{i, j\}$. If E is changed to a set of ordered pairs of distinct elements of V, then $G : (V, E)$ is a directed graph and E is the set of ordered pairs (i, j) . The ordered pairs (i, j) are referred to as arcs or edges and an arc goes from vertex i to vertex j. An arc (i, i) is referred to as a loop. A path from a vertex s to a vertex t is a sequence of arcs of the form $(p, i_1), (i_1, i_2)...(i_k, q)$.

If each arc (i, j) has an associated weight or length C_{ij} , then an (p, q) path has an associated weight or length equal to sum of the weights of the constituent arcs in the path. This in turn gives rise to the shortest path problem, which is to find the path with minimal weight between two vertices p and q. There are different ways to find one shortest path for a network. Some of the more general methods such as the labeling algorithm follow from dynamic progamming. It is assumed that graphs for the models to be presented or directed graphs, that is graph without cycles.

For an acyclic directed graphs $G : (V, E)$ with N vertices numbered from 1 to N such that '1' is the source and 'N' is the sink, a dynamic programming(DP) formulation for the shortest path problem is given as in

$$f_i(S_{i+1}) = \min_{x_i}(R_i(X_i, S_{i+1}) + f_{i-1}(S_1)), \tag{2.1}$$

where $f_{i-1}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages and S_i is the input to the stage $i-1$, X_i denotes the vector of decision variable at stage i, $R_i(X_i, S_{i+1})$ is the return function of the stage i and $f_i(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i. Throughout the algorithm, vertex i is labeled with $f(i)$, and labels allow the determination of the path.

Through Belman's principle of optimality this recursion is very flexible and has many applications. One obvious flexibility is that the sum in can be replaced by almost any binary operator and the recursion will hold in . for the fuzzy optimization problems under that max-min composition, the sum in is the fuzzy addition and is reformulated as

$$f_i(S_{i+1}) = \min_{x_i}(R_i(X_i, S_{i+1}) + f_{i-1}(S_1)). \tag{2.2}$$

3. Shortest path of an VN

A vague network includes nodes and directed links. Each node represents a city. Each directed links $(i, i + 1)$ connects city i to $i + 1$. Let $X_i = \{X_1, X_2, X_3, ..., X_{i-1}\}$ denotes the vector of decision variable at stage i and $S_i = \{S_1, S_2, ..., S_{i+1}\}$ is the input to the stage $i - 1$. f_{i-1} denotes the fuzzy optimal value of the objective function corresponding to the last $i - 1$ stages.

If $X_i : R_i \to S_{i+1}$, then it indicates that the degree of relevance from stage i to stage $i+1$ is R_i , where R_i is a sub interval of $[0, 1]$. Let $R_i = [R_{iT}, R_{iF}]$.

Since R_i is an interval of $[0, 1]$, $R_{it,R_{if}}$, $R_i(X_i, S_{i+1})$ is the weight of the corresponding arc $(i, i+1)$. For this vague network(VN), let us construct two networks which we call as true limit vague network $(VN)_T$ and false limit vague network $(VN)_F$ with the same set of nodes and links, the weight of the corresponding arc $(i, i+1)$ in the lower limit vague network is R_{iT} and in the upper limit vague network in R_{iF}.

The vague shortest path networks can also be viewed in terms of the Dynamic programming (DP) recursion given in equation (2.1) . This recursion is very close to Ford's Algorithm and is easily extended to vague numbers as in equation (2.2). Then the DP recursion for lower vague network is

$$f_{iT}(S_{i+1}) = \min_{X_i}\{\max[R_{iT}(X_i, S_{i+1}), f_{(i-1)T}(S_i)]\}, \tag{3.1}$$

where $f_{(i-1)T}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages S_i is the input to the stage $i-1$ of lower vague networks $(VN)_t$, X_i denotes the vector of decision variable at stage i, $R_{iT}(X_i, S_{i+1})$ is the return function of the stage i and $f_{iT}(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of lower vague networks $(VN)_T$. DP recursion for upper vague network is

$$f_{iF}(S_{i+1} = \min_{X_i}\{\max[R_{iF}(X_i, S_{i+1}), f_{(i-1)F}(S_i)]\}. \tag{3.2}$$

Let us define DP recursion for Interval valued fuzzy network as,

$$f_{i-1}(S_i) = [f_{(i-1)T}(S_i), f_{(i-1)F}(S_i)]. \tag{3.3}$$

Then by recursion

$$f_i(S_{i+1}) = [f_{(i-1)T}(S_{i+1}), f_{(i-1)F}(S_{i+1})]. \tag{3.4}$$

By previous equations we get the equation

$$\begin{aligned} f_i(S_{i+1}) = {} & [\min_{X_i}\{\max[R_{iT}(X_i, S_{i+1}), f_{(i-1)T}(S_i)]\}, \\ & \min\{\max[R_{iF}(X_i, S_{i+1}), f_{(i-1)F}(S_i)]\} \\ = {} & [\min_{X_i}\{\max\{R_{iT}(X_i, S_{i+1}), R_{iF}(X_i, S_{i+1})], [f_{(i-1)T}(S_i), f_{(i-1)F}(S_i)\}\}] \\ = {} & [\min_{X_i}\{\max[R_iF(X_i, S_{i+1}), f_{i-1}(S_i)]\}], \end{aligned} \tag{3.5}$$

where $f_i(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of vague networks (VN) , $f_{(i-1)}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages and S_i is the input to the stage $i-1$ of vague networks (VN) , X_i denotes the vector of decision variable at stage i , $R_i(X_i, S_{i+1})$ is the return function of the stage i of vague networks (VN).

Definition 3.1. *Shortest path in VN = Shortest path in true limit vague network* $(VN)_T$ *= Shortest path in false limit vague network* $(VN)_F$*. Weight of the shortest path of* $VN = [W_T, W_F]$ *where* W_T *and* W_F *are weights of the fuzzy shortest path in* $(VN)_T$ *and* $(VN)_F$ *respectively.*

Algorithm.

Step 1: Identify the decision variables and specify objective function to be optimized for vague networks.

Step 2: Decompose the network into a number of smaller sub intervals. Identify the stage variable at each stage and write down the vague transformation function as a function of the state variable and decision variable at the next stage.

Step 3: Write down a general recursive relationship for completing the vague optimal policy of VN by using the interval valued fuzzy dynamic programming recursion in (3.4) and (3.7).

Step 4: Construct appropriate stage to show the required values of the return function at each Stage in VN.

Step 5: Determine the overall fuzzy optimal decision or policy and its value at each stage of an VN.

Step 6: We get the shortest path of IVFN.

Now, A_N^t be the vague networks, representing the weight of N during time interval t.

$$A_N^t = [A_{NT}^t, A_{NF}^T], \tag{3.6}$$

where A_{NT}^t is the true limit (R_{iT}) of the vague network and A_{NF}^T is false limit (R_{iF}) of the vague network. Then,

$$\text{shortest path in } A_N^t = \text{ shortest path in } A_{NT}^t = \text{shortest path in } A_{NF}^t. \tag{3.7}$$

Weight of the shortest path of VN =

$$\begin{aligned} &[\text{Weight of the shortest path in } A_{NT}^t,\\ &\ \text{Weight of the shortest path in } A_{NF}^t]. \end{aligned} \tag{3.8}$$

We shall illustrate the technique with a simple example and provide the mathematical verification.

Example 3.1. We consider a network $N = (V, E)$ consisting n nodes (cities) and m edges (roads) connecting the cities of a country. If we measure the crowdness that is traffic of the roads of the network for particular time duration. It is quite tough to measure the crowdness in a duration as it is not fixed, but varies from time to time. So, appropriate technique to grade the crowdness deals with an interval and not a fixed point. Hence we use the concept of vague measures true and false limits to analyse the crowdness range.

The network N is a vague network in which the weight of each arc $(i, i+1)$ depends upon the crowdness.

Suppose that we want to select the shortest highway route(path) between two cities. The following route network provices the possible routes between the starting city at node 1 and the destination city at node 7. The routes pass through intermediate cities designated by nodes 2 to 6. By using our

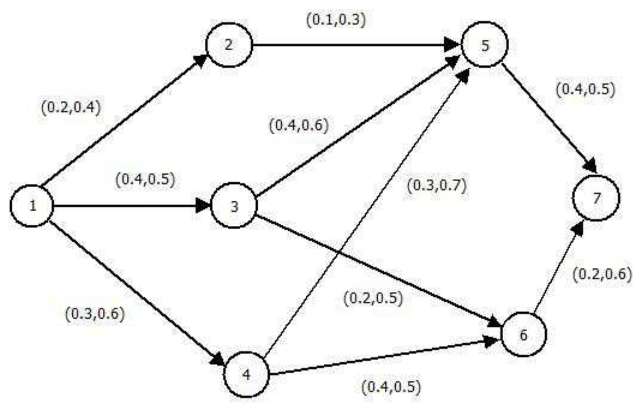

Fig 1: Vague graph with true and false limits

representation , $A^t_N = [A^t_{NT}, A^t_{NF}]$ Now we apply the algorithm to find a path

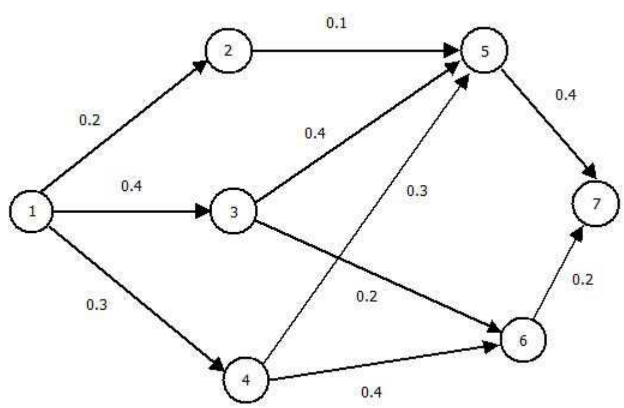

Fig 2 : Vague graph with true limits

between city 1 to city 7 which is minimum among all the paths between city 1 to city 7.

(i) Shortest path for the true limit vague network.

First we decompose the true limit vague network into sub networks or stages as Now S_1 is the state in which the node 1 lies also, S_1 has only state value $S_1 = 1$. State S_2 has only three possible values say 2,3 and 4 corresponding to stage 1 and so on. Possible alternative paths from one stage

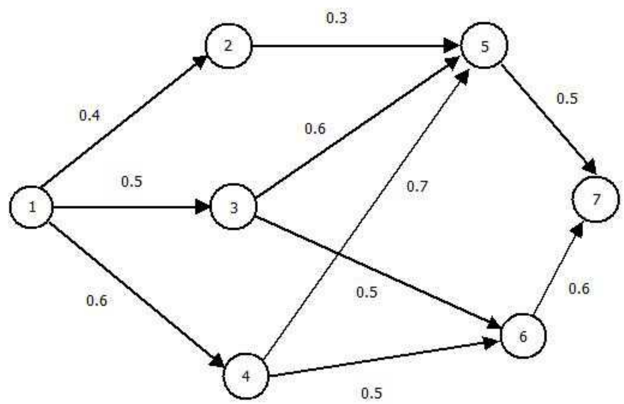

Fig 4: Vague graph with false limits

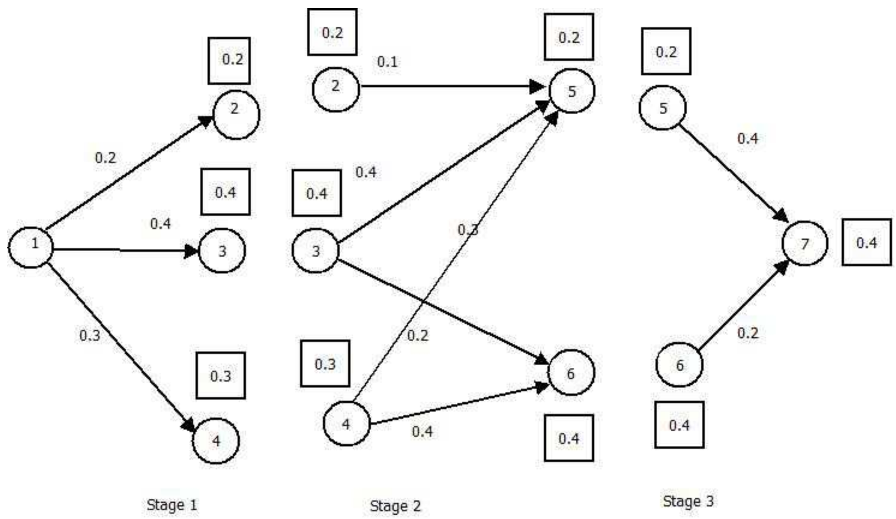

Fig 3 : Vague true network with stages

to the other will be called decision variables by X_i the decision which takes from S_{i-1} to S_i. The return or the gain which obviously being the function of decision will be denoted by $R_{iT}(X_i, S_{i+1})$. Here $R_{iT}(X_i, S_{i+1})$ can be identified with the true limit of the corresponding arc. By equation we have $f_{iT}(S_{i+1}) = \min_{X_i}\{\max[R_{iT}(X_i, S_{i+1}), f_{(i-1)T}(S_i)]\}$. Now initially for $i = 0$, $f_i(S_{i+1}) = f_0(S_1) = f_0(1) = 0$.

For Stage 1, (i=1), $f_1(S_2) = \min_{X_1}\{\max[R_{1T}(X_1, S_2), f_0(S_1)]\}$ $= \min_{X_1}[R_{1T}(X_1, S_2)]$.

Now tabulating the date for $f_1(S_2)$

S_1	S_2	X_i	$R_{iT}(X_1, S_2)$	$f_1(S_2)$	fuzzy optimal policy
1	2	1-2	0.2	0.2	1-2
	3	1-3	0.4	0.4	1-3
	4	1-4	0.3	0.3	1-4

For Stage 2 ($i = 2$), $f_2(S_3) = \min_{X_2}\{\max[R_{2T}(X_2, S_3), f_1(S_2)]\}$.

S_2	S_3	X_2	$R_{2T}(X_2, S_3)$	$\max(R_2, f_1)$	$f_2(S_3)$	fuzzy optimal policy
2		2-5	0.1	0.2	0.2	2-5
3	5	3-5	0.4	0.4	0.4	3-5
		3-6	0.3	0.4	0.4	3-6
4	6	4-5	0.3	0.3	0.3	4-5
		4-6	0.4	0.4	0.4	4-6

For last stage 3 ($i = 3$), $f_3(S_4) = \min_{X_3}\{\max[R_{3L}(X_3, S_4), f_2(S_x)]\}$.

S_2	S_3	X_2	$R_{2T}(X_2, S_3)$	$\max(R_2, f_1)$	$f_2(S_3)$	fuzzy optimal policy
5		5-7	0.2	0.4	0.4	5-7
6	7	6-7	0.4	0.4	0.4	6-7

Therefore, for the true limit vague network of the shortest path from city 1 to city 7 is $1 \to 4 \to 6 \to 7$

Weight of the shortest path $W_T = (0.2, 0.1, 0.4)$.

(ii) Shortest path for the false limit fuzzy matrices. Decompose the false limit fuzzy network into sub network or stage as follows

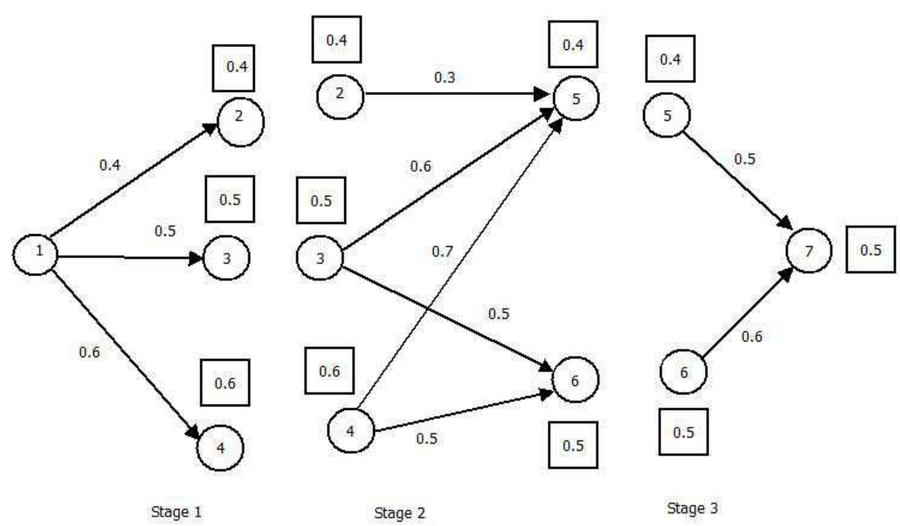

Fig 5 : Vague false network with stages

Similarly we have to find the false limit of the shortest path. Here $R_{iF}(X_i, S_{i+1})$ can be defined with the false limit of the corresponding arc.

By equation we have, $f_{iF}(S_{i+1}) = \min_{X_i}\{\max[R_{iF}(X_i, S_{i+1}), f_{(i-1)F}(S_i)]\}$.
Now, initially for $i = 0$, $f_i(S_{i+1}) = f_0(S_1) = f_0(1) = 0$.
For Stage $1(i = 1)$, $f_1(S_2) = \min_{X_1}\{\max[R_{1F}(X_1, S_2), f_0(S_1)]\}$
$= \min_{X_1}[R_{1F}(X_1, S_2)]$.

Now tabulating the data for $f_1(S_2)$

S_1	S_2	X_i	$R_{iT}(X_1, S_2)$	$f_1(S_2)$	fuzzy optimal policy
1	2	1-2	0.4	0.4	1-2
	3	1-3	0.5	0.5	1-3
	4	1-4	0.6	0.6	1-4

For stage 2 $(i = 2)$, $f_2(S_3) = \min_{X_2}\{\max[R_2(X_2, S_3), f_1(S_2)]\}$.

S_2	S_3	X_2	$R_{2T}(X_2, S_3)$	$\max(R_2, f_1)$	$f_2(S_3)$	fuzzy optimal policy
2		2-5	0.3	0.4	0.4	2-5
3	5	3-5	0.6	0.5	0.6	3-5
		3-6	0.5	0.5	0.5	3-6
4	6	4-5	0.7	0.6	0.7	4-5
		4-6	0.4	0.6	0.6	4-6

For last stage 3 $(i = 3)$, $f_2(S_4) = \min_{X_3}\{\max[R_3(X_3, S_4), f_2(S_3)]\}$.

S_2	S_3	X_2	$R_{2T}(X_2, S_3)$	$\max(R_2, f_1)$	$f_2(S_3)$	fuzzy optimal policy
5		5-7	0.5	0.4	0.5	5-7
6	7	6-7	0.6	0.5	0.6	6-7

Therefore the shortest path from city 1 to city 7 for the false limit vague network is $1 \to 2 \to 5 \to 7$. Weight of the shortest path $W_F = (0.4, 0.3, 0.6)$.

Now we conclude by equation. Shortest path in A^t_N = Shortest path in A^t_{NT} = Shortest path in $A^t_{NF} = 1 \to 2 \to 5 \to 7$, ie.., $W = [W_T, W_F] = [(0.2, 0.1, 0.4), (o.4, 0.3, 0.6)] = ([0.2, 0.4], [0.1, 0.3], [0.4, 0.6])$.

Therefore the shortest path of VN is $1 \to 2 \to 5 \to 7$

4. Conclusion

In this work we construct two vague networks namely $(FN)_T$ and $(FN)_F$ with the associated weight R_{iT} and R_{iF} respectively. Since the vertex sets and edge sets are same for VN, $(FN)_T$ and $(FN)_F$ and weight of the each node $(i, i+1)$ in VN is an interval of the form $W_i = [W_{iT}, W_{iF}]$. We conclude that the shortest path for an VN is the path for which the shortest path in true limit vague network coincides with the shortest path in false limit vague network and weight is $[W_T, W_F]$ where W_T andW_F are the weights of the shortest path for true and false limit vague networks.

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NEW CONCEPTS IN INTERVAL-VALUED INTUITIONISTIC FUZZY GRAPHS

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Abstract. Intuitionistic fuzzy graphs is a highly growing research area as it is the generalization of the fuzzy graphs. In this paper, we introduce the concept of Interval-valued Intuitionistic fuzzy graphs(IVIFG), we also analyse some properties of IVIFG based on morphism such as weak isomorphism, co-weak isomorphism and some concepts on automorphism.

Keywords: IVIFG, weak isomorphism of IVIFG, co-weak isomorphism of IVIFG.

1. Introduction

Graph theory has found its importance in many real time problems. Recent applications in graph theory is quite interesting analysing any complex situations and moreover in engineering applications. It has got numerous applications

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on operations research, system analysis, network routing, transportation and many more. To analyse any complete information we make intensive use of graphs and its properties. For working on partial information or incomplete information or to handle the systems containing the elements of uncertainty we understand that fuzzy logic and its involvement in graph theory is applied. In 1975, Rosenfeld [21] discussed the concept of fuzzy graphs whose ideas are implemented by Kauffman [18] in 1973. The fuzzy relation between fuzzy sets were also considered by Rosenfeld who developed the structure of fuzzy graphs, obtaining various analagous results of several graph theoretical concepts. Bhattacharya [4] gave some remarks of fuzzy graphs. The complement of fuzzy graphs was introduced by Mordeson [19]. Atanassov introduced the concept of intuitionistic fuzzy relation and intuitionistic fuzzy graphs [2, 3, 32, 33]. Talebi and Rashmanlou [36] studied the properties of isomorphism and complement of interval-valued fuzzy graphs. They defined isomorphism and some new operations on vague graphs [37, 38]. Borzooei and Rashmalou analysed new concepts of vague graphs [5], degree of vertices in vague graphs [6], more results on vague graphs [7], semi global domination sets in vague graphs with application [8] and degree and total degree of edges in bipolar fuzzy graphs with application [9]. Rashmanlou et.al., defined the complete interval-valued fuzzy graphs [24]. Rashmanlou and Pal studied intuitionistic fuzzy graphs with categorical properties [29], some properties of highly irregular interval-valued fuzzy graphs [28], more results on highly irregular bipolar fuzzy graphs [30], balanced interval-valued fuzzy graphs [26] and antipodal interval-valued fuzzy graphs [25]. Samanta and Pal investigated fuzzy k-competition and p-competition graphs, and concept of fuzzy planar graphs in [21, 22, 31] . Also they introduced fuzzy tolerance graph [34], bipolar fuzzy hypergraphs [35] and given several properties on it. Pal and Rashmanlou [20] defined many properties of irregular interval-valued fuzzy graphs. Ganesh *et al.* [10, 11] analysed the properties of Regular product vague graphs and product vague line graphs. The article has been composed of four sections. Ganesh *et al.* [12, 13, 14, 15] has analysed some concepts on faces and dual of m-polar fuzzy graphs, regular bipolar fuzzy graphs, isomorphic properties of m-polar fuzzy graphs and novel concepts on strongly edge irregular m-polar fuzzy graphs.In section 1,we introduce the survey of Interval-valued intuitionistic fuzzy graphs. In section 2 we define the preliminaries of Intuitionistic fuzzy graphs and basic definitions, definition of IVIFG. In section 3 we define automorphic IVIFG and analyse the concepts of weak and co-weak isomorphic properties of IVIFG.For further terminologies, the readers are referred to [1-6,12,13].

2. Preliminaries

A fuzzy graph G=(V,σ,μ) where V is the vertex set, σ is a fuzzy subset of V and μ is a membership value on σ such that μ(u,v) $\leq$ σ(u) $\wedge$ σ(v) for every u,v $\in$ V.The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where

$\sigma = \sup\rho(\sigma) = \{x \in V : \sigma(x) > 0\}and\mu = \sup\rho(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$. H = $(\sigma^{\shortmid}; \mu^{\shortmid})$ is a fuzzy subgraph of G if there exists X $\subseteq$ V such that, $\sigma^{\shortmid} : X \to [0,1]$ is a fuzzy subset and$\mu^{\shortmid} : X \times X \to [0,1]$ is a fuzzy relation on $\sigma^{\shortmid}$ such that μ(u,v) $\leq \sigma$(u) $\wedge$ σ(v), for all x,y $\in$ X.

Definition 2.1. *By an interval-valued fuzzy graph of a graph G we mean a pair $G^* = (A, B)$ where $A = [\mu_A^-, \mu_A^+]$ and $\mu_B : V \times V \to [0,1]$ are bijective such that membership value of nodes and edges are distinct and $\mu_B(x,y) \leq \mu_v(x) \wedge \mu_v(y) \forall x, y \in V$*

Definition 2.2. *An interval $[\mu - \epsilon, \mu + \epsilon]$ is said to be an ϵ-neighborhood of any membership value(ie., corresponding to any nodes or edges) μ for any ϵ satisfying the following conditions.*

(i) $\epsilon \ngtr min\{\mu_v(v_i), \mu_e(e_{ij})\}$;

(ii) $\epsilon \ngtr 1 - max\{\mu_v(v_i), \mu_e(e_{ij})\}$;

(iii) $\epsilon \neq d(\mu(x), \mu(y)) or \frac{1}{2} d(\mu(x), \mu(y))$ where $d(\mu(x), \mu(y)) = |\mu(x) - \mu(y)|$ and $\mu(x), \mu(y)$ are the membership or nodes or edges.

Definition 2.3. *By an interval-valued intuitionistic fuzzy graph of a graph G we mean a pair $G^* = (A, B)$ where $A = [(\mu_A^-, \mu_A^+), (\nu_A^-, \nu_A^+)]$ and $\mu_e : V \times V \to [0,1]$ and $\nu_e : V \times V \to [0,1]$ are bijective such that true and false membership value of nodes and edges are distinct and $\mu_e(x,y) \leq \mu_v(x) \wedge \mu_v(y) \forall x, y \in V$, $\nu_e(x,y) \geq \nu_v(x) \vee \nu_v(y) \forall x, y \in V$*

Definition 2.4. *An interval-valued intuitionistic fuzzy graph(IVIFG) is said to be strong for the lower and upper bounds (μ^-, μ^+) and (ν^-, ν^+) of the edges and vertices satisfying the following conditions $\mu_e(x,y) = \mu_v(x) \wedge \mu_v(y) \forall x, y \in V$, $\nu_e(x,y) = \nu_v(x) \vee \nu_v(y) \forall x, y \in V$*

Definition 2.5. *Let $G = (V, E)$ be an IVIFG. Then the degree of a vertex v is defined by $d(v) = (d_\mu(v), d_\nu(v))$ where $d_\mu(v) = \sum_{u \neq v} (\mu_e^-(v,u), \mu_e^+(v,u))$ and $d_\nu(v) = \sum_{u \neq v} (\nu_e^-(v,u), \nu_e^+(v,u))$*

Definition 2.6. *Let $G = (V, E)$ be an IVIFG. Then the total degree of a vertex v is defined by $td(v) = (td_\mu(v), td_\nu(v))$ where $td_\mu(v) = \sum_{u \neq v} (\mu_e^-(v,u) + \mu_v^-(v), \mu_e^+(v,u) + \mu_v^+(v))$ and $td_\nu(v) = \sum_{u \neq v} (\nu_e^-(v,u) + \nu_v^-(v), \nu_e^+(v,u) + \nu_v^+(v))$*

Definition 2.7. *Let $G = (V, E)$ be an IVIFG. If all the vertices of G have same degree then G is said to be regular IVIFG.*

Definition 2.8. *Let $G = (V, E)$ be an IVIFG. Then the order of G is defined as $O(G) = [\sum_{v \in V} \mu_v^-(v), \sum_{v \in V} \mu_v^+(v)],\ [\sum_{v \in V} \nu_v^-(v), \sum_{v \in V} \nu_v^+(v)]$*

Definition 2.9. *Let $G = (V, E)$ be an IVIFG. Then the size of G is defined as $S(G) = [\sum_{u \neq v} \mu_e^-(v,u), \sum_{u \neq v} \mu_e^+(v,u)],\ [\sum_{u \neq v} \nu_e^-(v,u), \sum_{u \neq v} \nu_e^+(v,u)]$*

Remark 2.1. In any IVIFG G, we have

$$\sum_{v \in V} d_G(v) = 2\{(\sum_{u \neq v} \mu_e^-(v,u), \sum_{u \neq v} \mu_e^+(v,u)]), (\sum_{u \neq v} \nu_e^-(v,u), \sum_{u \neq v} \nu_e^+(v,u))\} = 2S(G).$$

Definition 2.10. *Let $G = (V, E)$ be an IVIFG. Let $e_ij \in B$ be an edge of G where e_ij has its lower and upper bounds μ_e^-, ν_e^- and μ_e^+, ν_e^+. Then the degree of an edge e_ij defined as $d_\mu(e_ij) = d_\mu(v_i) + d_\mu(v_j) - 2\mu_e(e_ij)$ and $d_\nu(e_ij) = d_\nu(v_i) + d_\nu(v_j) - 2\nu_e(e_ij)$, for all its the vertices having the lower and upper bounds μ^-, ν^- and μ^+, ν^+ respectively.*

Definition 2.11. *Let $G = (V, E)$ be an IVIFS. Let $e_ij \in B$ be an edge of G where e_ij has its lower and upper bounds μ_e^-, ν_e^- and μ_e^+, ν_e^+. Then the total degree of an edge e_ij defined as $td_\mu(e_{ij}) = d_\mu(e_ij) + \mu(e_{ij})$ and $td_\nu(e_{ij}) = d_\nu(e_ij) + \nu(e_{ij})$, for all the lower and upper bounds μ^-, ν^- and μ^+, ν^+ respectively.*

3. Automorphic IVIFG

In this section we introduce the isomorphic properties of IVIFG.

Example 3.1. The below figure represents the IVIFG G of a crisp graph G^*

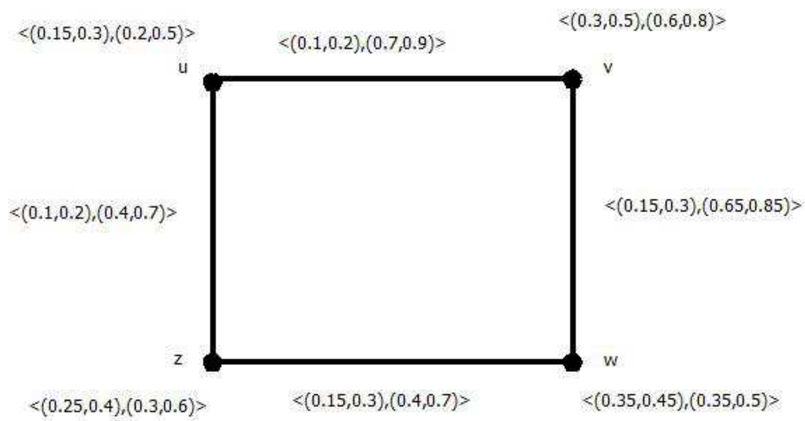

Fig 1 : Interval-valued Intuitionistic Fuzzy Graph

Throughout this work G^* is a crisp graph and G is a IVIFG.

Definition 3.1. *Let G_1 and G_2 be the IVIFGs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(i) $\mu_{A_1}^-(x_1) \leq \mu_{A_2}^-(f(x_1))$, $\mu_{A_1}^+(x_1) \leq \mu_{A_2}^+(f(x_1))$;

(ii) $\nu_{A_1}^-(x_1) \geq \nu_{A_2}^-(f(x_1))$, $\nu_{A_1}^+(x_1) \geq \nu_{A_2}^+(f(x_1))$;

(iii) $\mu_{B_1}^- x_1y_1 \leq \mu_{B_2}^-(f(x_1 f(y_1)), \mu_{B_1}^+ x_1y_1 \leq \mu_{B_2}^+(f(x_1 f(y_1))$;

(iv) $\nu_{B_1}^- x_1y_1 \geq \nu_{B_2}^-(f(x_1 f(y_1)), \nu_{B_1}^+ x_1y_1 \geq \nu_{B_2}^+(f(x_1 f(y_1))$, *for all* $x_1 \in V_1, x_1y_1 \in E_1$.

Definition 3.2. *Let G_1 and G_2 be the IVIFGs. An isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(i) $\mu_{A_1}^-(x_1) = \mu_{A_2}^-(f(x_1))$, $\mu_{A_1}^+(x_1) = \mu_{A_2}^+(f(x_1))$;

(ii) $\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1))$, $\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1))$;

(iii) $\mu^-_{B_1}(x_1y_1) = \mu^-_{B_2}(f(x_1f(y_1)), \mu^+_{B_1}(x_1y_1) = \mu^+_{B_2}(f(x_1f(y_1))$;

(iv) $\nu^-_{B_1}(x_1y_1) = \nu^-_{B_2}(f(x_1f(y_1)), \nu^+_{B_1}(x_1y_1) = \nu^+_{B_2}(f(x_1f(y_1))$, *for all* $x_1 \in V_1$, $x_1y_1 \in E_1$.

Definition 3.3. *Let G_1 and G_2 be the IVIFGs. Then a weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(i) f is a homomorphism;

(ii) $\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1))$, $\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1))$;

(iii) $\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1))$, $\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1))$.

It is clear that a weak isomorphism maintains only the weights of the nodes.

Example 3.2. Consider the IVIFGs G_1 and G_2 of G^*_1 and G^*_2 respectively,

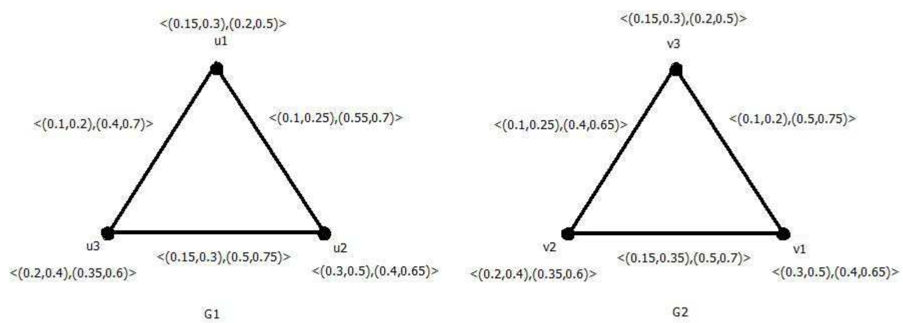

Fig 2 : Weak Isomorphism of IVIFG

A map $f : V_1 \to V_2$ defined by $f(u_1) = v_3, f(u_2) = v_1$ and $f(u_3) = v_2$. Then we have:

$$\mu^-_{A_1}(u_1) = \mu^-_{A_2}(v_3), \mu^+_{A_1}(u_1) = \mu^+_{A_2}(v_3),$$
$$\nu^-_{A_1}(u_1) = \nu^-_{A_2}(v_3), \nu^+_{A_1}(u_1) = \nu^+_{A_2}(v_3).$$
$$\mu^-_{A_1}(u_2) = \mu^-_{A_2}(v_1), \mu^+_{A_1}(u_2) = \mu^+_{A_2}(v_1),$$
$$\nu^-_{A_1}(u_2) = \nu^-_{A_2}(v_1), \nu^+_{A_1}(u_2) = \nu^+_{A_2}(v_1).$$
$$\mu^-_{A_1}(u_3) = \mu^-_{A_2}(v_2), \mu^+_{A_1}(u_3) = \mu^+_{A_2}(v_2),$$
$$\nu^-_{A_1}(u_3) = \nu^-_{A_2}(v_2), \nu^+_{A_1}(u_3) = \nu^+_{A_2}(v_2).$$

But we see that:

$$\mu^-_{B_1}(u_1u_2) = \mu^-_{B_2}(v_3v_1), \mu^+_{B_1}(u_1u_2) \neq \mu^+_{B_2}(v_3v_1),$$
$$\nu^-_{B_1}(u_1u_2) \neq \nu^-_{B_2}(v_3v_1), \nu^+_{B_1}(u_1u_2) \neq \nu^+_{B_2}(v_3v_1)$$
$$\mu^-_{B_1}(u_1u_3) = \mu^-_{B_2}(v_3v_2), \mu^+_{B_1}(u_1u_3) \neq \mu^+_{B_2}(v_3v_2),$$
$$\nu^-_{B_1}(u_1u_3) = \nu^-_{B_2}(v_3v_2), \nu^+_{B_1}(u_1u_3) \neq \nu^+_{B_2}(v_3v_2)$$
$$\mu^-_{B_1}(u_3u_2) = \mu^-_{B_2}(v_2v_1), \mu^+_{B_1}(u_3u_2) \neq \mu^+_{B_2}(v_2v_1),$$
$$\nu^-_{B_1}(u_3u_2) = \nu^-_{B_2}(v_2v_1), \nu^+_{B_1}(u_3u_2) \neq \nu^+_{B_2}(v_2v_1).$$

Hence the map is a weak isomorpism but not an isomorphism.

Definition 3.4. *Let G_1 and G_2 be the IVIFGs. Then a co-weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:*

(i) f is a homomorphism;

(ii) $\mu^-_{B_1}(x_1y_1) = \mu^-_{B_2}(f(x_1)f(y_1)), \mu^+_{B_1}(x_1y_1) = \mu^+_{B_2}(f(x_1)f(y_1))$;

(iii) $\nu^-_{B_1}(x_1y_1) = \nu^-_{B_2}(f(x_1)f(y_1)), \nu^+_{B_1}(x_1y_1) = \nu^+_{B_2}(f(x_1)f(y_1))$, for all $x_1 \in V_1, x_1y_1 \in E_1$.

It is clear that a co-weak isomorphism maintains only the weights of the arcs.

Example 3.3. Consider the IVIFGs G_1 and G_2 of G_1^* and G_2^* respectively,

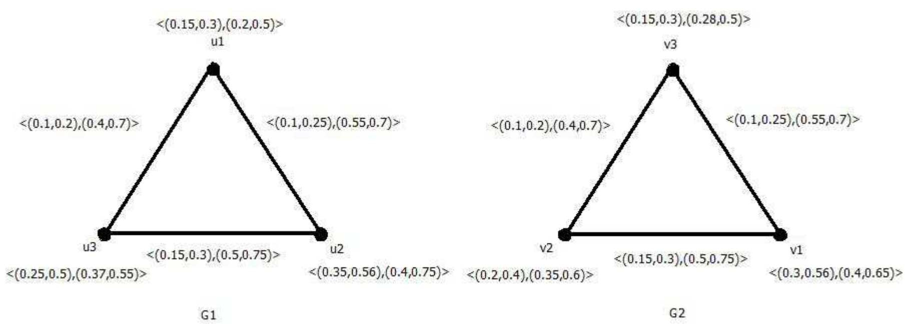

Fig 3 : co-weak Isomorphism of IVIFG

A map $f : V_1 \to V_2$ defined by $f(u_1) = v_3, f(u_2) = v_1$ and $f(u_3) = v_2$. Then we have:

$$\mu^-_{A_1}(u_1) = \mu^-_{A_2}(v_3), \mu^+_{A_1}(u_1) = \mu^+_{A_2}(v_3),$$
$$\nu^-_{A_1}(u_1) \neq \nu^-_{A_2}(v_3), \nu^+_{A_1}(u_1) = \nu^+_{A_2}(v_3).$$
$$\mu^-_{A_1}(u_2) \neq \mu^-_{A_2}(v_1), \mu^+_{A_1}(u_2) = \mu^+_{A_2}(v_1),$$
$$\nu^-_{A_1}(u_2) = \nu^-_{A_2}(v_1), \nu^+_{A_1}(u_2) \neq \nu^+_{A_2}(v_1).$$
$$\mu^-_{A_1}(u_3) \neq \mu^-_{A_2}(v_2), \mu^+_{A_1}(u_3) \neq \mu^+_{A_2}(v_2),$$
$$\nu^-_{A_1}(u_3) \neq \nu^-_{A_2}(v_2), \nu^+_{A_1}(u_3) \neq \nu^+_{A_2}(v_2).$$

But we see that

$$\begin{aligned}
&\mu^-_{B_1}(u_1u_2) = \mu^-_{B_2}(v_3v_1), \mu^+_{B_1}(u_1u_2) = \mu^+_{B_2}(v_3v_1),\\
&\nu^-_{B_1}(u_1u_2) = \nu^-_{B_2}(v_3v_1), \nu^+_{B_1}(u_1u_2) = \nu^+_{B_2}(v_3v_1)\\
&\mu^-_{B_1}(u_1u_3) = \mu^-_{B_2}(v_3v_2), \mu^+_{B_1}(u_1u_3) = \mu^+_{B_2}(v_3v_2),\\
&\nu^-_{B_1}(u_1u_3) = \nu^-_{B_2}(v_3v_2), \nu^+_{B_1}(u_1u_3) = \nu^+_{B_2}(v_3v_2)\\
&\mu^-_{B_1}(u_3u_2) = \mu^-_{B_2}(v_2v_1), \mu^+_{B_1}(u_3u_2) = \mu^+_{B_2}(v_2v_1),\\
&\nu^-_{B_1}(u_3u_2) = \nu^-_{B_2}(v_2v_1), \nu^+_{B_1}(u_3u_2) = \nu^+_{B_2}(v_2v_1).
\end{aligned}$$

Hence the map is a co-weak isomorpism but not an isomorphism.

Remark 3.1.

1. If $G_1 = G_2 = G$, then the homomorphism f over itself is called an endomorphism. An Isomorphism f over G is called an automorphism.
2. Let $A = (\mu^-_A, \mu^+_A, \nu^-_A, \nu^+_A)$ be an IVIFG with an underlying set V. Let $Aut(G)$ be the set of all bipolar intuitionistic automorphism of G. Let $e : G \to G$ be a map defined by $e(x) = x$, for all $x \in V$ clearly $e \in Aut(G)$.
3. If $G_1 = G_2$, then the weak and co-weak isomorphishms actually become isomorphic.
4. If $f : V_1 \to V_2$ is a bijective map then $f^{-1} : V_1 \to V_2$ is also a bijective map.

Definition 3.5. *An Interval-valued intuitionistic fuzzy set $A = (\mu^-_A, \mu^+_A, \nu^-_A, \nu^+_A)$ in a semigroup S is called a interval-valued intuitionistic subsemigroup of S if it satisfies the following conditions:*

$\mu^-_B(xy) \le (\mu^-_A(x) \wedge \mu^-_A(y))$, $\mu^+_B(xy) \le (\mu^+_A(x) \wedge \mu^+_A(y))$
$\nu^-_B(xy) \ge (\nu^-_A(x) \vee \mu^-_A(y))$, $\nu^+_B(xy) \ge (\nu^+_A(x) \vee \nu^+_A(y))$, *for all* $x, y \in S$.

Definition 3.6. *An Interval-valued intuitionistic fuzzy set $A = (\mu^-_A, \mu^+_A, \nu^-_A, \nu^+_A)$ in a group G is called a interval-valued intuitionistic fuzzy subgroup of a group G if it is a interval-valued intuitionistic fuzzy sub-semigroup of G and satisfies* $\mu^-_A(x^{-1}) = \mu^-_A(x)$, $\mu^+_A(x^{-1}) = \mu^+_A(x)$, $\nu^-_A(x^{-1}) = \nu^-_A(x)$, $\nu^+_A(x^{-1}) = \nu^+_A(x)$

We now show how to associate an interval-valued intuitionistic fuzzy group with a interval-valued intuitionistic fuzzy graph in a natural way.

Proposition 3.1. *Let $G = (A, B)$ be an IVIFG and let $Aut(G)$ be the set of all automorphisms of G. Then $(Aut(G), \circ)$ forms a group.*

Proof. We have the following conditions:

$$\begin{aligned}
\mu^-_A((\phi \circ \psi)(x)) &= \mu^-_A(\phi(\psi(x))) \le \mu^-_A(\phi(x)) \ge \mu^-_A(x),\\
\mu^+_A((\phi \circ \psi)(x)) &= \mu^+_A(\phi(\psi(x))) \le \mu^+_A(\phi(x)) \ge \mu^+_A(x),\\
\nu^-_A((\phi \circ \psi)(x)) &= \nu^-_A(\phi(\psi(x))) \ge \nu^-_A(\phi(x)) \ge \nu^-_A(x),\\
\nu^+_A((\phi \circ \psi)(x)) &= \nu^+_A(\phi(\psi(x))) \ge \nu^+_A(\phi(x)) \ge \nu^+_A(x),
\end{aligned}$$

$$\mu_B^-((\phi\circ\psi)(x))(\phi\circ\psi)(y))) = \mu_B^-(\phi(\psi(x)))\phi(\psi(y)) \le \mu_B^-(\phi(x)\phi(y)) \le \mu_B^-(xy),$$
$$\mu_B^+((\phi\circ\psi)(x))(\phi\circ\psi)(y))) = \mu_B^+(\phi(\psi(x)))\phi(\psi(y)) \le \mu_B^+(\phi(x)\phi(y)) \le \mu_B^+(xy),$$
$$\nu_B^-((\phi\circ\psi)(x))(\phi\circ\psi)(y))) = \nu_B^-(\phi(\psi(x)))\phi(\psi(y)) \ge \nu_B^-(\phi(x)\phi(y)) \ge \nu_B^-(xy),$$
$$\nu_B^+((\phi\circ\psi)(x))(\phi\circ\psi)(y))) = \nu_B^+(\phi(\psi(x)))\phi(\psi(y)) \ge \nu_B^+(\phi(x)\phi(y)) \ge \nu_B^+(xy).$$

Thus $\phi\circ\psi \in Aut(G)$. Clearly, $Aut(G)$ satisfies associativity under the operation $\circ$, $\phi\circ e = e\circ\phi$.

$\mu_A^-(\phi^{-1}) = \mu_A^-(\phi)$,$\mu_A^+(\phi^{-1}) = \mu_A^+(\phi)$,$\nu_A^-(\phi^{-1}) = \nu_A^-(\phi)$,$\nu_A^+(\phi^{-1}) = \nu_A^+(\phi)$, for all $\phi \in Aut(G)$.

Hence $(Aut(G), \circ)$ forms a group. □

Now we state some propositions without their proofs as follows.

Proposition 3.2. *Let $G = (A, B)$ be an IVIFG and let $Aut(G)$ be the set of all automorphisms of G. Let $g = (\mu_g^-, \mu_g^+, \nu_g^-, \nu_g^+)$ be an interval-valued intuitionistic fuzzy set in $Aut(G)$ defined by*

$$\mu_g^-(\phi) = \inf\{\mu_B^-(\phi(x), \phi(y)) : (x, y) \in V \times V\},$$
$$\mu_g^+(\phi) = \inf\{\mu_B^+(\phi(x), \phi(y)) : (x, y) \in V \times V\},$$
$$\nu_g^-(\phi) = \sup\{\nu_B^-(\phi(x), \phi(y)) : (x, y) \in V \times V\},$$
$$\nu_g^+(\phi) = \sup\{\nu_B^+(\phi(x), \phi(y)) : (x, y) \in V \times V\},$$

for all $\phi \in Aut(G)$. Then $g = (\mu_g^-, \mu_g^+, \nu_g^-, \nu_g^+)$ is an interval-valued intuitionistic fuzzy group on $Aut(G)$.

Proposition 3.3. *Every interval-valued intuitionistic fuzzy group has an embedding into the interval-valued intuitionistic fuzzy group of the group of automorphisms of some IVIFG.*

We now prove that the isomorphism (weak isomorphism) between IVIFG is an equivalence solution (partial order relation).

Proposition 3.4. *Let G_1, G_2, G_3 be IVIFGs. Then the isomorphism between these IVIFGs is an equivalence relation.*

Proof. Reflexivity property is obvious. To prove the symmetry, let $f : V_1 \to V_2$ be an isomorphism of G_1 onto G_2. Then f is bijective map defined by

$$f(x_1) = x_2, \quad \forall x_1 \in V_1 \tag{3.1}$$

satistying the following conditions:

(i) $\mu_{A_1}^-(x_1) = \mu_{A_2}^-(f(x_1))$, $\mu_{A_1}^+(x_1) = \mu_{A_2}^+(f(x_1))$;

(ii) $\nu_{A_1}^-(x_1) = \nu_{A_2}^-(f(x_1))$, $\nu_{A_1}^+(x_1) = \nu_{A_2}^+(f(x_1))$;

(iii) $\mu_{B_1}^-(x_1y_1) = \mu_{B_2}^-(f(x_1)f(y_1))$, $\mu_{B_1}^+(x_1y_1) = \mu_{B_2}^+(f(x_1)f(x_2))$;

(iv) $\nu_{B_1}^-(x_1y_1) = \nu_{B_2}^-(f(x_1)f(y_1))$, $\nu_{B_1}^+(x_1y_1) = \nu_{B_2}^+(f(x_1)f(x_2))$, for all $x_1 \in V_1, x_1y_1 \in E_1$.

Since f is bijective from 3.1 it follows that: $f^{-1}(x_2) = x_1$, for all $x_2 \in V_2$. Thus:

(i) $\mu^-_{A_1}(f^{-1}(x_2)) = \mu^-_{A_2}(x_2)$, $\mu^+_{A_1}(f^{-1}(x_2)) = \mu^+_{A_2}(x_2)$;

(ii) $\nu^-_{A_1}(f^{-1}(x_2)) = \nu^-_{A_2}(x_2)$, $\nu^+_{A_1}(f^{-1}(x_2)) = \nu^+_{A_2}(x_2)$, for all $x_2 \in V_2$;

(iii) $\mu^-_{B_1}(f^{-1}(x_2y_2)) = \mu^-_{A_2}(x_2y_2)$, $\mu^+_{A_1}(f^{-1}(x_2y_2)) = \mu^+_{A_2}(x_2y_2)$;

(iv) $\nu^-_{B_1}(f^{-1}(x_2y_2)) = \nu^-_{A_2}(x_2y_2)$, $\nu^+_{A_1}(f^{-1}(x_2y_2)) = \nu^+_{A_2}(x_2y_2)$, for all $x_2y_2 \in E_2$.

Hence a bijective map $f^{-1} : V_2 \to V_1$ is an isomorphism from G_2 onto G_2.

To prove the transitivity, let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be the isomorphisms of G_1 onto G_2 and G_2 onto G_3, respectively. Then $g \circ f : V_1 \to V_3$ is a bijective map from V_1 and V_3, where $(g \circ f)(x_1) = g(f(x_1))$, for all $x_1 \in V_1$. Since a map $f : V_1 \to V_2$ defined by $f(x_1) = x_2$, for all $x_1 \in V_1$ is an isomorphism, so we have

$$
\begin{aligned}
&\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)) = \mu^-_{A_2}(x_2),\\
&\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1)) = \mu^+_{A_2}(x_2),\\
&\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1)) = \nu^-_{A_2}(x_2),\\
&\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1)) = \nu^+_{A_2}(x_2), \forall x_1 \in V_1.
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\mu^-_{B_1}(x_1y_1) = \mu^-_{B_2}(f(x_1)f(y_1)) = \mu^-_{B_2}(x_2y_2),\\
&\mu^+_{B_1}(x_1y_1) = \mu^+_{B_2}(f(x_1)f(y_1)) = \mu^+_{B_2}(x_2y_2),\\
&\nu^-_{B_1}(x_1y_1) = \nu^-_{B_2}(f(x_1)f(y_1)) = \nu^-_{B_2}(x_2y_2),\\
&\nu^+_{B_1}(x_1y_1) = \nu^+_{B_2}(f(x_1)f(y_1)) = \nu^+_{B_2}(x_2y_2), \forall x_1y_1 \in E_2.
\end{aligned}
\tag{3.3}
$$

Since a map $g : V_2 \to V_3$ defined by $g(x_2) = x_3$ for $x_2 \in V_2$ is an isomorphism, We have

$$
\begin{aligned}
&\mu^-_{A_2}(x_2) = \mu^-_{A_3}(g(x_2)) = \mu^-_{A_3}(x_3),\\
&\mu^+_{A_2}(x_2) = \mu^+_{A_3}(g(x_2)) = \mu^+_{A_3}(x_3),\\
&\nu^-_{A_2}(x_2) = \nu^-_{A_3}(g(x_2)) = \nu^-_{A_3}(x_3),\\
&\nu^+_{A_2}(x_2) = \nu^+_{A_3}(g(x_2)) = \nu^+_{A_3}(x_3), \forall x_2 \in V_2.
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\mu^-_{B_2}(x_2y_2) = \mu^-_{B_3}(g(x_2)g(y_2)) = \mu^-_{B_3}(x_3y_3),\\
&\mu^+_{B_2}(x_2y_2) = \mu^+_{B_3}(g(x_2)g(y_2)) = \mu^+_{B_3}(x_3y_3),\\
&\nu^-_{B_2}(x_2y_2) = \nu^-_{B_3}(g(x_2)g(y_2)) = \nu^-_{B_3}(x_3y_3),\\
&\nu^+_{B_2}(x_2y_2) = \nu^+_{B_3}(g(x_2)g(y_2)) = \nu^+_{B_3}(x_3y_3).
\end{aligned}
\tag{3.5}
$$

From 3.2 and 3.4 and $f(x_1) = x_2$,$x_1 \in V_1$, we have

$$\begin{aligned} &\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)) = \mu^-_{A_3}(g(x_2)) = \mu_{A_3}(g(f(x_1)), \\ &\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1)) = \mu^+_{A_3}(g(x_2)) = \mu_{A_3}(g(f(x_1)), \\ &\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1)) = \nu^-_{A_3}(g(x_2)) = \nu_{A_3}(g(f(x_1)), \\ &\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1)) = \nu^+_{A_3}(g(x_2)) = \nu_{A_3}(g(f(x_1)), \forall x_1 \in V_1. \end{aligned} \tag{3.6}$$

From 3.3 and 3.5, we have

$$\begin{aligned} \mu^-_{B_1}(x_1y_1) &= \mu^-_{B_2}(f(x_1)f(y_1)) = \mu^-_{B_2}(x_2y_2) = \mu^-_{B_3}(g(x_2)g(y_2)) \\ &= \mu^-_{B_3}(g(f(x_1))g(f(y_1)), \\ \mu^+_{B_1}(x_1y_1) &= \mu^+_{B_2}(f(x_1)f(y_1)) = \mu^+_{B_2}(x_2y_2) = \mu^+_{B_3}(g(x_2)g(y_2)) \\ &= \mu^+_{B_3}(g(f(x_1))g(f(y_1)), \\ \nu^-_{B_1}(x_1y_1) &= \nu^-_{B_2}(f(x_1)f(y_1)) = \nu^-_{B_2}(x_2y_2) = \nu^-_{B_3}(g(x_2)g(y_2)) \\ &= \nu^-_{B_3}(g(f(x_1))g(f(y_1)), \\ \nu^+_{B_1}(x_1y_1) &= \nu^+_{B_2}(f(x_1)f(y_1)) = \nu^+_{B_2}(x_2y_2) = \nu^+_{B_3}(g(x_2)g(y_2)) \\ &= \nu^+_{B_3}(g(f(x_1))g(f(y_1)), \forall x_1y_1 \in E_1. \end{aligned} \tag{3.7}$$

Thus, we prove that $g \circ f$ is an isomorphism between G_1 and G_3.

Hence the proof. □

Proposition 3.5. *Let G_1, G_2, G_3 be IVIFGs. Then the weak isomorphism between these IVIFGs is a partial order relation.*

Proof. Reflexive property is obvious.

To prove the antisymmetry, let $f : V_1 \to V_2$ be a weak isomorphism of G_1 onto G_2. Then f is a bijective map defined by $f(x_1) = x_2$, for all $x_1 \in V_1$ satisfying the following

$$\begin{aligned} &(i) \quad \mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)), \mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_2)), \\ &(ii) \quad \nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1)), \nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_2)), \\ &(iii) \quad \mu^-_{B_1}(x_2y_2) \le \mu^-_{B_2}(f(x_1)f(y_1)), \mu^+_{B_1}(x_1y_1) \le \mu^+_{B_1}(f(x_1)f(y_1), \\ &(iv) \quad \nu^-_{B_1}(x_2y_2) \ge \nu^-_{B_2}(f(x_1)f(y_1)), \nu^+_{B_1}(x_1y_1) \\ &\qquad \ge \nu^+_{B_1}(f(x_1)f(y_1).x_1 \in V_1, \forall x_1y_1 \in E_1. \end{aligned} \tag{3.8}$$

Let $g : V_2 \to V_1$ be a weak isomorphism of G_2 onto G_1. Then g is a bijective map defined by $g(x_2) = x_1$, for all satisfying

$$\begin{aligned} &\mu^-_{A_2}(x_2) = \mu^-_{A_1}(g(x_2)), \mu^+_{A_2}(x_2) = \mu^+_{A_1}(g(x_2)), \\ &\nu^-_{A_2}(x_2) = \nu^-_{A_1}(g(x_2)), \nu^+_{A_2}(x_2) = \nu^+_{A_1}(g(x_2)), \forall x_1 \in V_2 \\ &\mu^-_{B_2}(x_2y_2) \le \mu^-_{B_1}(g(x_2)g(y_2)), \mu^+_{B_1}(x_2y_2) \le \mu^+_{B_1}(g(x_2)g(y_2), \\ &\nu^-_{B_2}(x_2y_2) \ge \nu^-_{B_1}(g(x_2)g(y_2)), \nu^+_{B_1}(x_2y_2) \ge \nu^+_{B_1}(g(x_2)g(y_2), \forall x_2y_2 \in E_2. \end{aligned} \tag{3.9}$$

The inequalities 3.8 and 3.9 holds on the finite sets V_1 and V_2 only when G_1 and G_2 have the same number of edges and the corresponding edges have weight. Hence G_1 and G_2 are identical.

To prove the transitivity, let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be the isomorphisms of G_1 onto G_2 and G_2 onto G_3, respectively. Then $g \circ f : V_1 \to V_3$ is a bijective map from V_1 and V_3, where $(g \circ f)(x_1) = g(f(x_1))$, for all $x_1 \in V_1$. Since a map $f : V_1 \to V_2$ defined by $f(x_1) = x_2$, for all $x_1 \in V_1$ is a weak isomorphism, so we have

$$\begin{aligned}
&\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)) = \mu^-_{A_2}(x_2),\\
&\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1)) = \mu^+_{A_2}(x_2),\\
&\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1)) = \nu^-_{A_2}(x_2),\\
&\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1)) = \nu^+_{A_2}(x_2), \forall x_1 \in V_1.
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\mu^-_{B_1}(x_1y_1) \le \mu^-_{B_2}(f(x_1)f(y_1)) = \mu^-_{B_2}(x_2y_2)\\
&\mu^+_{B_1}(x_1y_1) \le \mu^+_{B_2}(f(x_1)f(y_1)) = \mu^+_{B_2}(x_2y_2)\\
&\nu^-_{B_1}(x_1y_1) \ge \nu^-_{B_2}(f(x_1)f(y_1)) = \nu^-_{B_2}(x_2y_2)\\
&\nu^+_{B_1}(x_1y_1) \ge \nu^+_{B_2}(f(x_1)f(y_1)) = \nu^+_{B_2}(x_2y_2), \forall x_1y_1 \in E_1.
\end{aligned} \tag{3.11}$$

Since a map $g : V_2 \to V_3$ defined by $g(x_2) = x_3$ for $x_2 \in V_2$ is a weak isomorphism, We have

$$\begin{aligned}
&\mu^-_{A_2}(x_2) = \mu^-_{A_3}(g(x_2)) = \mu^-_{A_3}(x_3),\\
&\mu^+_{A_2}(x_2) = \mu^+_{A_3}(g(x_2)) = \mu^+_{A_3}(x_3),\\
&\nu^-_{A_2}(x_2) = \nu^-_{A_3}(g(x_2)) = \nu^-_{A_3}(x_3),\\
&\nu^+_{A_2}(x_2) = \nu^+_{A_3}(g(x_2)) = \nu^+_{A_3}(x_3), \forall x_2 \in V_2.
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\mu^-_{B_2}(x_2y_2) \le \mu^-_{B_3}(g(x_2)g(y_2)) = \mu^-_{B_3}(x_3y_3),\\
&\mu^+_{B_2}(x_2y_2) \le \mu^+_{B_3}(g(x_2)g(y_2)) = \mu^+_{B_3}(x_3y_3),\\
&\nu^-_{B_2}(x_2y_2) \ge \nu^-_{B_3}(g(x_2)g(y_2)) = \nu^-_{B_3}(x_3y_3),\\
&\nu^+_{B_2}(x_2y_2) \ge \nu^+_{B_3}(g(x_2)g(y_2)) = \nu^+_{B_3}(x_3y_3), \forall x_1y_1 \in E_1.
\end{aligned} \tag{3.13}$$

From 3.10 and 3.12 and $f(x_1) = x_2$,$x_1 \in V_1$, we have

$$\begin{aligned}
&\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1)) = \mu^-_{A_3}(g(x_2)) = \mu_{A_3}(g(f(x_1)),\\
&\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1)) = \mu^+_{A_3}(g(x_2)) = \mu_{A_3}(g(f(x_1)),\\
&\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1)) = \nu^-_{A_3}(g(x_2)) = \nu_{A_3}(g(f(x_1)),\\
&\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1)) = \nu^+_{A_3}(g(x_2)) = \nu_{A_3}(g(f(x_1)), \forall x_1 \in V_1.
\end{aligned} \tag{3.14}$$

From 3.11 and 3.13, we have

$$
\begin{aligned}
\mu^-_{B_1}(x_1y_1) &\leq \mu^-_{B_2}(f(x_1)f(y_1)) = \mu^-_{B_2}(x_2y_2) = \mu^-_{B_3}(g(x_2)g(y_2)) \\
&= \mu^-_{B_3}(g(f(x_1))g(f(y_1))), \\
\mu^+_{B_1}(x_1y_1) &\leq \mu^+_{B_2}(f(x_1)f(y_1)) = \mu^+_{B_2}(x_2y_2) = \mu^+_{B_3}(g(x_2)g(y_2)) \\
&= \mu^+_{B_3}(g(f(x_1))g(f(y_1))), \\
\nu^-_{B_1}(x_1y_1) &\geq \nu^-_{B_2}(f(x_1)f(y_1)) = \nu^-_{B_2}(x_2y_2) \\
&= \nu^-_{B_3}(g(x_2)g(y_2)) \\
&= \nu^-_{B_3}(g(f(x_1))g(f(y_1))), \\
\nu^+_{B_1}(x_1y_1) &\geq \nu^+_{B_2}(f(x_1)f(y_1)) = \nu^+_{B_2}(x_2y_2) = \nu^+_{B_3}(g(x_2)g(y_2)) \\
&= \nu^+_{B_3}(g(f(x_1))g(f(y_1))), \forall x_1y_1 \in E_1.
\end{aligned} \tag{3.15}
$$

Thus, we prove that $g \circ f$ is a weak isomorphism between G_1 and G_3. Hence the proof. □

4. Conclusion

Interval-valued intuitionistic fuzzy graph have numerous application in the real life systems and real life applications where the level of information inherited in the system varies with respect to time and have different level of precision. Most of the actions in real life situations are time dependent and also ambigous in partial information, symbolic models in expert system are more effective than traditional methods to identify the upper and lower bounds of the true and false membership values in an interval. In this paper, we introduced the concept of automorphism on IVIFG. Also we investigate the properties of morphism on IVIFG.

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LINEAR CODES ON UNITARY SPACE

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Abstract. Linear codes are an important class of codes. They are the most studied codes from a mathematical point of view. In this work, we propose linear codes in unitary space, then describe a way for finding a new parity check matrix of linear codes in unitary space. In the end, we give a decoding procedure for linear codes in unitary space.

Keywords: linear codes, generator matrix, parity check matrix, unitary space.

1. Introduction

Among all types of codes, linear codes are studied the most. Since linear codes are vector spaces, their algebraic structure often make them easier to describe, encode, and decode than nonlinear codes. The code alphabet for linear codes is a finite field, although sometimes other algebraic structures can be used to define codes that are also called linear.

Let $\mathbb{F}_q^n$ denote the vector space of all n-tuples over the finite field $\mathbb{F}_q$. An (n, M)-code $\mathcal{C}$ over $\mathbb{F}_q$ is a subset of $\mathbb{F}_q^n$ of size M. If $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$, then $\mathcal{C}$ will be called an $[n, k]$-linear code over $\mathbb{F}_q$. The linear code $\mathcal{C}$ has q^k codewords. If $\mathcal{C}$ has the minimum distance d, then $\mathcal{C}$ is an $[n, k, d]$-linear code over $\mathbb{F}_q$. The Hamming weight of a vector V is the number of its non-zero entries and is denoted by $\mathrm{w}_H(v)$. We have $\mathrm{w}_H(v) = d_H(v, 0)$. The minimum weight of the code $\mathcal{C}$ is the minimum non-zero weight among all codewords of $\mathcal{C}$, $\mathrm{w}_{\min}(\mathcal{C}) = \min_{0\neq x\in\mathcal{C}}(\mathrm{w}_H(x))$.

It is customary to put the codewords of a basis for a linear code $\mathcal{C}$ into a matrix. A generator matrix for an $[n, k]$-code $\mathcal{C}$ is any $k \times n$ matrix G whose rows form a basis for $\mathcal{C}$. Also a parity check matrix of a linear code $\mathcal{C}$ is a matrix H whose columns form a basis for the dual code $\mathcal{C}^{\perp}$.

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Moreover, a code is of practical use only if an efficeint decoding scheme can be applied to it. For linear codes, syndrome decoding is an efficient way to decode them. In fact, let $\mathcal{C}$ be an $[n,k,d]$-linear code over $\mathbb{F}_q$, then for any $v \in \mathbb{F}_q^n$, the syndrome of v is the word $S(v) = vH^t$.

Now, we want to describe linear codes in unitary space. This paper is organized as follows: In the second section, we mention some notes about linear codes and their minimum distance. In the third section, we introduce unitary space, linear codes in unitary space, then find a new parity-check matrix of linear codes in unitary space. Also we assert syndrome decoding of linear codes in unitary space. Finally we conclude with the summary of the main results of this paper.

2. Preliminaries

A linear code $\mathcal{C}$ of lenght n and dimension k over $\mathbb{F}_q$ is often called a q-ary $[n,k]$-code or an $[n,k]$-code. It is also an (n,q^k)-linear code. If the distance d of $\mathcal{C}$ is known, it is also sometimes referred to as an $[n,k,d]$-linear code.

Definition 2.1. Let x be a word in $\mathbb{F}_q^n$. The (Hamming) weight of x, denoted by $\mathrm{w}(x)$, is defined to be the number of non-zero coordinates in x, i.e. $\mathrm{w}(x) = d(x,0)$, where 0 is the zero word.

An important invariant of a code is the minimum distance between codewords. The (Hamming) distance $d(x,y)$ between two vectors $x,y \in \mathbb{F}_q^n$ is defined to the number of coordinates in which x and y differ.

Lemma 2.2 (5, Lemma 5.1)**.** *If $x,y \in \mathbb{F}_q^n$, then $d(x,y) = \mathrm{w}(x-y)$.*

The (minimum) distance of a code $\mathcal{C}$ is the smallest distance between distinct codewords and is important in determining the error-correcting capability of $\mathcal{C}$. In fact, the (minimum) distance of $\mathcal{C}$ is denoted by $d(\mathcal{C})$, where

$$d(\mathcal{C}) = \min\{d(x,y) \mid x,y \in \mathcal{C},\ x \neq y\}.$$

Theorem 2.3 (7, Theorem 4.3.8)**.** *Let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$. Then $d(\mathcal{C}) = \mathrm{w}(\mathcal{C})$.*

Theorem 2.4 (5, Theorem 5.5)**.** *Let G be a generator matrix of an $[n,k]$-code, then G can be transformed to the standard form $[I_k|A]$, where I_k is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix.*

Since the generator matrix of a linear code has full row rank, it is quite obvious that any linear code is equivalent to a linear code that has a generator matrix in standard form.

Definition 2.5. Let $\mathcal{C}$ be an $[n,k]$-code over $GF(q)$ and a is any vector in V. Then the set $a+\mathcal{C}$ is defined by $a+\mathcal{C} = \{a+x \mid x \in \mathcal{C}\}$, is called a coset of $\mathcal{C}$.

Theorem 2.6 (5, Theorem 6.4). *Suppose that $\mathcal{C}$ is an $[n,k]$-code over $GF(q)$, then:*

1. *every vector of $V(n,q)$ is in some coset of $\mathcal{C}$,*
2. *every coset contains exactly q^k vectors,*
3. *two cosets either are disjoint or coinside.*

H is the generator matrix of some codes, called the dual or orthogonal of $\mathcal{C}$ and denoted $\mathcal{C}^{\perp}$. Notice that $\mathcal{C}^{\perp}$ is an $[n,n-k]$-code. In fact,

$$\mathcal{C}^{\perp} = \{x \in \mathbb{F}_q^n \mid x.c = 0,\ \forall c \in \mathcal{C}\}.$$

If G is in standard form $[I_k|A]$, one can take $H = [-A^t|I_{n-k}]$. Also we have $HG^t = 0$.

Theorem 2.7 (5, Theorem 7.5). *Let $\mathcal{C}$ be an $[n,k]$-code on $V(n,q)$, then $(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$.*

3. Linear codes in unitary space

In this section, we want to introduce linear codes in unitary space and some of their properties.

Definition 3.1. A bilinear form on a vector space V over a field $\mathbb{F}$ is a function $<,>$: $V \times V \longrightarrow \mathbb{F}$ that satisfies

$$< \lambda u + \mu v, w >= \lambda < u,w > +\mu < v,w >,$$
$$< u, \lambda v + \mu w >= \lambda < u,v > +\mu < u,w >,$$

for all $u,v,w \in V$ and $\lambda, \mu \in \mathbb{F}$.

It is symmetric, if $< u,v >=< v,u >$ for all $u,v \in \mathbb{F}$. It is skew-symmetric, if $< u,v >= - < v,u >$, for all $u,v \in \mathbb{F}$. It is also an alternating, if $< v,v >= 0$ for all $v \in \mathbb{F}$.

Definition 3.2. A conjugate symmetric sesquilinear form on V over a field $\mathbb{F}$ that has an automorphism σ of order 2, is a function $<,>$: $V \times V \longrightarrow \mathbb{F}$ that satisfies $< \lambda u + \mu v, w >= \lambda < u,w > +\mu < v,w >$, and $< u,v >=< v,u >^{\sigma}$, for all $u,v,w \in V$ and $\lambda, \mu \in \mathbb{F}$.

If $X \subseteq V$, then define the subspace $X^{\perp} = \{v \in V \mid\ < x,v >= 0, \forall x \in X\}$. The set $X^{\perp}$ is called the radical of $<,>$ and is denoted $\mathrm{Rad}(<,>)$. We say that $<,>$ is non-degenerate if $\mathrm{Rad}(<,>) = 0$.

A non-degenerate conjugate symmetric sesquilinear form is called a unitary form. A vector space V together with a unitary form is called unitary space and is denoted by $(V, <,>_U)$.

A vector $v \in V$ is called isotropic if $< v,v >_U = 0$. A subspace W of V is called totally isotropic if $<,>_U$ restricted to W is zero.

Theorem 3.3 (3, Theorem 18.1). *All automorphisms $GF(q^m)$ over $GF(q)$ are $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$, where $\sigma_j(\alpha) = \alpha^{q^j}$, $\alpha \in GF(q^m)$, $0 \leq j \leq m-1$. As a result*

$$\mathrm{Aut}(\frac{GF(q^m)}{GF(q)}) \cong \mathbb{Z}_m.$$

By Theorem 3.3, for non-degenerate unitary space $(V, <,>_U)$ with $\dim(V) = n$ over $\mathbb{F}$, we have $\mathbb{F} = GF(q^2)$ and $\sigma(a) = a^q = \overline{a}$ $(a \in \mathbb{F})$.

Definition 3.4. If V is a unitary space with form $<,>_U$, then an isometry g of V is an invertible linear transformation of V that satisfies $< ug, vg >_U = < u, v >_U$ for all $u, v \in V$.

Definition 3.5. Let $(V, <,>_U)$ be a unitary space and $B = \{v_1, \ldots, v_n\}$ be an ordered basis of V. Then $B = (b_{ij})_{1 \leq i,j \leq n}$ in which $b_{ij} = < v_i, v_j >_U$, $1 \leq i, j \leq n$ is called matrix of form $<,>_U$ related to basis B. Hence for every $x, y \in V$, we can write

$$x = \sum_{i=1}^{n} x_i v_i, \quad y = \sum_{j=1}^{n} y_j v_j.$$

such that $x_i, y_j \in \mathbb{F}$, $1 \leq i, j \leq n$. Therefore

$$\begin{aligned} < x, y >_U = &< \sum_{i=1}^{n} x_i v_i, \sum_{j=1}^{n} y_j v_j >_U = \sum_{i,j=1}^{n} x_i < v_i, v_j >_U {y_j}^{\sigma} \\ &= \sum_{i,j=1}^{n} x_i b_{ij} {y_j}^{\sigma} = x^t B y^{\sigma}. \end{aligned}$$

Proposition 3.6 (6, Proposition 2.3.2). *Let V be a $2n$-dimensional unitary vector space with unitary form $<,>_U$. Then there is a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of V such that for all i, j, we have*

$$< v_i, v_j >_U = < w_i, w_j >_U = 0, \quad < v_i, w_j >_U = \delta_{ij}.$$

Such a basis for a unitary vector space V is called a unitary basis.

Definition 3.7. Assume that $(V, <,>_U)$ is a non-degenerate unitary space with $\dim(V) = n$ over field $GF(q^2)$. With choosing unitary basis, a unitary form can be described for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $x, y \in V$ by

$$< x, y >_U = x_1 \overline{y}_1 + x_2 \overline{y}_2 + \cdots + x_n \overline{y}_n = \sum_{i=1}^{n} x_i \overline{y}_i.$$

3.1 The dual code in a non-degenerate unitary space

In a non-degenerate unitary space $V(n, q)$, if for $u, v \in V(n, q)$, $< u, v >_U = 0$, then u and v are called orthogonal.

The dual code of a linear code $\mathcal{C}$ in $V(n, q)$ is denoted by $\mathcal{C}^{\perp_U}$. In fact, it is the set of those vectors of $V(n, q)$ which are orthogonal to every codeword of $\mathcal{C}$, i.e. $\mathcal{C}^{\perp_U} = \{v \in V(n, q) | < u, v >_U = 0, \forall u \in \mathcal{C}\}$. It is easy to check that $\mathcal{C}^{\perp_U}$ is a linear code.

Let G be a generator matrix of $\mathcal{C}$. A vector v of $V(n, q)$ belongs to $\mathcal{C}^{\perp_U}$ if and only if u is orthogonal to every row of G. It means that $v \in \mathcal{C}^{\perp_U}$ if and only if $< v, u >_U = 0$, for every row u of G.

Lemma 3.8. *Let $(V, <, >_U)$ be a non-degenerate unitary space of finite dimension. If W is a subspace of V, then*

$$\dim W^{\perp_U} = \dim V - \dim W$$

Proof. Let V^* be the dual space of vector space V. The map ϕ, where

$$\begin{aligned} \phi : V &\longrightarrow V^* \\ v &\longmapsto \phi_v \end{aligned}$$

and $\phi_v(w) = < v, w >_U$, $\forall w, v \in V$ is a inversable linear transformation.

Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W. We claim that the elements of $\phi_{w_i} \in V^*$, $1 \leq i \leq k$ are independent. Assume that

$$\sum_{i=1}^{k} \lambda_i \phi_{w_i} = 0, \quad \lambda_i \in \mathbb{F},$$

then

$$\sum_{i=1}^{k} \lambda_i \phi_{w_i}(v) = \sum_{i=1}^{k} \lambda_i < w_i, v >_U = < \sum_{i=1}^{k} \lambda_i w_i, v >_U = 0, \quad \forall v \in V.$$

Since $(V, <, >_U)$ is non-degenerate, it follows that

$$\sum_{i=1}^{k} \lambda_i w_i = 0 \Longrightarrow \lambda_i = 0, \quad \text{for} \;\; 1 \leq i \leq n.$$

Therefore the elements of ϕ_{w_i}, $(1 \leq i \leq k)$ are distinct independent. As a result

$$\begin{aligned} W^{\perp_U} &= \{x \in V \mid \; < w_i, x >_U = 0, \;\; \forall w_i, \; 1 \leq i \leq k\} \\ &= \{x \in V \mid \phi_{w_i}(x) = 0, \;\; \forall w_i, \; 1 \leq i \leq k\} \\ &= \{x \in V \mid x \in \ker\phi_{w_i}, \;\; \forall w_i, \; 1 \leq i \leq k\} \\ &= \bigcap_{i=1}^{k} \ker\phi_{w_i}. \end{aligned}$$

But ϕ_{w_i} is a non-zero functional and the dimension of its kernel is $\dim V - 1$. Because ϕ_{w_i}, $(1 \leq i \leq k)$ are independent, it yields that

$$\dim\,(\bigcap_{i=1}^{k}\ker\phi_{w_i}) = \dim V - k.$$

The proof is complete. □

If $\mathcal{C}$ is an $[n,k]$-code on $V(n,q)$, then $\mathcal{C}^{\perp_U}$ is an $[n,n-k]$-code.

Theorem 3.9. *Let $\mathcal{C}$ be an $[n,k]$-code on $V(n,q)$, then $(\mathcal{C}^{\perp_U})^{\perp_U} = \mathcal{C}$.*

Proof. Since every vector of $\mathcal{C}$ is orthogonal to every vector of $\mathcal{C}^{\perp_U}$, so $\mathcal{C} \subseteq (\mathcal{C}^{\perp_U})^{\perp_U}$. On the other hand,

$$\dim\,((\mathcal{C}^{\perp_U})^{\perp_U}) = n - \dim\,(\mathcal{C}^{\perp_U}) = n-(n-k) = k = \dim\,(\mathcal{C}),$$

thus $(\mathcal{C}^{\perp_U})^{\perp_U} = \mathcal{C}$. □

For a vector $u \in V(n,q)$ and a $(l \times n)$ matrix B with rows $r_1, r_2, \ldots, r_l$, we define $u.B = (< u, r_1 >_U, < u, r_2 >_U, \ldots, < u, r_l >_U)$.

Definition 3.10. A parity check matrix H_U for an $[n,k]$-code $\mathcal{C}$ on $V(n,q)$ is a generator matrix of $\mathcal{C}^{\perp_U}$.

H_U is an $(n-k)\times n$ matrix satisfying $< h, g >_U = 0$ for every row h of H_U and row g of generator matrix G, i.e. $g.H_U = 0$.

From Lemma 3.8 and Theorem 3.9, if H_U is a parity check matrix of an $[n,k]$-code $\mathcal{C}$, then $\mathcal{C} = \{x \in V(n,q) \mid x.H_U = 0\}$

Theorem 3.11. *If $G = [I_k|A]$ is the generator matrix of an $[n,k]$-code $\mathcal{C}$, then a parity check matrix of $\mathcal{C}$ is $H_U = [B|I_{n-k}]$ where $A = (a_{ij})_{k\times(n-k)}$, $B = (b_{kt})_{(n-k)\times k}$ and $b_{kt} = -\overline{a_{tk}}$.*

Proof. H_U has the size required of a parity-check matrix and its rows are linearly independent. It is enough to show that every row of H_U is orthogonal to every row of G.

Let u be the ith row of H_U and v is the jth row of G. Then

$$v = (0,\ldots,0,\overbrace{1}^{j\text{th}},0,\ldots,0,a_{j\times 1},a_{j\times 2},\ldots,a_{j\times(n-1)},a_{j\times n}),$$
$$u = (-\overline{a_{1\times i}},-\overline{a_{2\times i}},\ldots,-\overline{a_{j\times i}},\ldots,-\overline{a_{k\times i}},0,\ldots,\overbrace{1}^{(k+i)\text{th}},0,\ldots,0).$$

Therefore $< v, u >_U = -a_{j\times i} + a_{j\times i} = 0$. For other rows of H_U and G, it is similar. □

3.2 Syndrome decoding

Let Let H_U be a parity check matrix of an $[n,k]$-code $\mathcal{C}$. Then for any vector $v \in V(n,q)$, the $1 \times (n-k)$ row vector $S(v) = vH_U$ is called the syndrome of v.

If the columns of H_U are $H_1, H_2, \ldots, H_{n-k}$, then

$$S(v) = (< v, H_1 >_U, < v, H_2 >_U, \ldots, < v, H_{n-k} >_U).$$

Also $S(v) = 0$ if and only if $v \in \mathcal{C}$.

Proposition 3.12. *Two vectors u and v are in the same coset of $\mathcal{C}$ if and only if they have the same syndrome.*

Proof. For two arbitrary vectors u and v in $V(n,q)$, u and v are in the same coset

$$\begin{aligned}
&\iff u + \mathcal{C} = v + \mathcal{C} \\
&\iff u - v \in \mathcal{C} \\
&\iff (u-v)H_U = 0 \\
&\iff (< u-v, H_1 >_U, \ldots, < u-v, H_{n-k} >_U) = 0, \\
&\qquad H_i \ (1 \le i \le n-k) \text{ is } i\text{th column of } H_U \\
&\iff (< u, H_1 >_U, \ldots, < u, H_{n-k} >_U) = (< v, H_1 >_U, \ldots, < v, H_{n-k} >_U) \\
&\iff uH_U = vH_U \\
&\iff S(u) = S(v).
\end{aligned}$$

□

Theorem 3.13. *Let $\mathcal{C}$ be an $[n,k]$-code with parity check matrix H_U. For each codeword of Hamming weight t, there exist t-columns of H_U which are linearly dependent. Conversely if there exist t-columns linearly dependent of H_U, then there exists a codeword of Hamming weight t in $\mathcal{C}$.*

Proof. Assume that the parity check matrix H_U is in the form $H_U = (H_1, H_2, \ldots, H_n)$, where H_i represents the ith column of H_U. Given a codeword $v = (v_1, v_2, \ldots, v_n) \in \mathcal{C}$, we have $0 = vH_U = (v_1\overline{H}_1, v_2\overline{H}_2, \ldots, v_n\overline{H}_n)^t$. This implies that $\mathcal{C}$ has a vector v of weight t if and only if H_U has t-columns linearly dependent. □

Corollary 3.14. *Let $\mathcal{C}$ be an $[n,k]$-code with the parity check matrix H_U, then the minimum distance of $\mathcal{C}$ is equal to the smallest number of columns of H_U that are linearly independent.*

The vector having minimum weight in a coset is called the coset leader.

From group theory, we know that if $\mathcal{C}$ is an $[n,k]$-code over $GF(q)$, then $V(n,q)$ is partitioned into disjoint cosets of $\mathcal{C}$ as follows: $V(n,q) = (0 + \mathcal{C}) \cup (e_1 + \mathcal{C}) \cup \ldots \cup (e_r + \mathcal{C})$, where $r = q^{n-k} - 1$.

We may choose $0, e_1, \ldots, e_r$ to be the coset leader.

Now we present a method to partition the 2^n vectors of $V(n,q)$ to $\frac{2^n}{2^k} = 2^{n-k}$ cosets. We place the elements of $\mathcal{C}$ in a row with the vector 0 as left most element. Select from the remaining vector, a vector e_1 of minimum weight, then form the second row by placing e_1 under 0 and e_1+x under x for each $x \in \mathcal{C}$. We continue this process untill we have used all elements of $V(n,q)$. Now we have an array of rows and columns. This is called a standard array of $\mathcal{C}$.

If $v_1 = 0, v_2, \ldots, v_{2^k}$ are the codewords of $\mathcal{C}$, in Figure 1, we show a standard array of $\mathcal{C}$.

$$\begin{array}{ccccccc}
\text{Coset Leader} & & & & & & \\
v_1{=}0 & v_2 & \ldots & v_j & \ldots & v_{2^k} \\
e_2 & e_2+v_2 & \ldots & e_2+v_j & \ldots & e_2+v_{2^k} \\
e_3 & e_3+v_2 & \ldots & e_3+v_j & \ldots & e_3+v_{2^k} \\
\vdots & \vdots & & \vdots & & \vdots \\
e_{2^{n-k}} & e_{2^{n-k}}+v_2 & \ldots & e_{2^{n-k}}+v_j & \ldots & e_{2^{n-k}}+v_{2^k}
\end{array}$$

Figure 1. Standard array for an $[n,k]-$ code.

We know that the syndrome of a vector in $V(n,q)$ is an $(n-k)$-tuple and there is one to one correspondence between a coset leader and a syndrome. We form a decoding table which consists of 2^{n-k} coset leaders and their corresponding syndromes. The steps of the decoding of a received vectors are as follows:

1. Get the syndrome of v, vH_U,
2. Determind the coset leader e_l whose its syndrome is vH_U,
3. Decode the received vector v into $v - e_l$.

Example 3.15. Write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$. Let $\mathcal{C}$ be the $\mathbb{F}_4$-linear code with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & \alpha & \alpha \\ 0 & 1 & 0 & \alpha & 1 & \alpha \\ 0 & 0 & 1 & \alpha & \alpha & 1 \end{pmatrix}.$$

Since $\sigma(\alpha) = \alpha^2 = \overline{\alpha}$, the parity check matrix of $\mathcal{C}$ is

$$H_U = \left(\begin{array}{ccc|ccc} -1 & -\alpha^2 & -\alpha^2 & 1 & 0 & 0 \\ -\alpha^2 & -1 & -\alpha^2 & 0 & 1 & 0 \\ -\alpha^2 & -\alpha^2 & -1 & 0 & 0 & 1 \end{array}\right).$$

By Corollary 3.14, the minimum distance of the code is 3, so it is possible to correct the error patterns of weight 1 or 0. Hence all vectors of weight 1 or 0 can be used as coset leader. The correcable error patterns and their corresponding syndromes are given in Table 1.

Syndrome	Coset Leader
$(1, \alpha, \alpha)$	(1,0,0,0,0,0)
$(\alpha, 1, \alpha)$	(0,1,0,0,0,0)
$(\alpha, \alpha, 1)$	(0,0,1,0,0,0)
(1,0,0)	(0,0,0,1,0,0)
(0,1,0)	(0,0,0,0,1,0)
(0,0,1)	(0,0,0,0,0,1)

Table 1: Decoding Table for the $\mathbb{F}_4$-linear code $\mathcal{C}$.

Assume that the codeword $v = (0, 1, 0, \alpha, 1, \alpha)$ is transmitted and $w = (0, 1, 1, \alpha, 1, \alpha)$ is received. We compute the syndrome of w. We have

$$wH_U = (0, 1, 1, \alpha, 1, \alpha) \left(\begin{array}{ccc|ccc} -1 & -\alpha^2 & -\alpha^2 & 1 & 0 & 0 \\ -\alpha^2 & -1 & -\alpha^2 & 0 & 1 & 0 \\ -\alpha^2 & -\alpha^2 & -1 & 0 & 0 & 1 \end{array} \right) = (\alpha, \alpha, 1).$$

From Table 1, we see that $(\alpha, \alpha, 1)$ is the syndrome of the coset leader $e = (0, 0, 1, 0, 0, 0)$. Hence $(0, 0, 1, 0, 0, 0)$ is assumed to be the error pattern and w is decoded into: $v = w - e = (0, 1, 1, \alpha, 1, \alpha) - (0, 0, 1, 0, 0, 0) = (0, 1, 0, \alpha, 1, \alpha)$.

Example 3.16. Let $\mathcal{C}$ be a code over $GF(9)$ with parity check matrix

$$H_U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

Suppose that $v = (v_1, v_2, \ldots, v_8)$ is the codeword transmitted and $w = (w_1, w_2, \ldots, w_8)$ is the received vector.

Assume that a single error has occurred such that the error magnitude is k and the error position is l. Then $(w_1, w_2, \ldots, w_8) = (v_1, \ldots, v_{l-1}, v_l + k, v_{l+1}, \ldots, v_8)$. The syndrome is

$$\begin{aligned} (s_1, s_2) = wH_U \quad &= \quad (\overline{1}w_1 + \overline{1}w_2 + \overline{1}w_3 + \overline{1}w_4 + \overline{1}w_5 + \overline{1}w_6 + \overline{1}w_7 + \overline{1}w_8, \\ &\qquad \overline{1}w_1 + \overline{2}w_2 + \overline{3}w_3 + \overline{4}w_4 + \overline{5}w_5 + \overline{6}w_6 + \overline{7}w_7 + \overline{8}w_8). \end{aligned}$$

$$\begin{aligned} s_1 &= \overline{1}w_1 + \overline{1}w_2 + \overline{1}w_3 + \overline{1}w_4 + \overline{1}w_5 + \overline{1}w_6 + \overline{1}w_7 + \overline{1}w_8 \\ &= \overline{1}v_1 + \overline{1}v_2 + \overline{1}v_3 + \overline{1}v_4 + \overline{1}v_5 + \overline{1}v_6 + \overline{1}v_7 + \overline{1}v_8 + k \equiv k \pmod 9 \\ s_2 &= \overline{1}w_1 + \overline{2}w_2 + \overline{3}w_3 + \overline{4}w_4 + \overline{5}w_5 + \overline{6}w_6 + \overline{7}w_7 + \overline{8}w_8 \\ &= \overline{1}v_1 + \overline{2}v_2 + \overline{3}v_3 + \overline{4}v_4 + \overline{5}v_5 + \overline{6}v_6 + \overline{7}v_7 + \overline{8}v_8 + \overline{l}k \equiv \overline{l}k \pmod 9. \end{aligned}$$

Since $\sigma(\alpha) = \alpha^2 = \overline{\alpha}$, we have

$$\begin{aligned} s_1 &= w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 \\ &= v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + w_8 + k \equiv k \pmod 9 \\ s_2 &= w_1 + 4w_2 + 7w_4 + 7w_5 + 7w_7 + w_8 \\ &= v_1 + 4v_2 + 7v_4 + 7v_5 + 7v_7 + v_8 + \overline{l}k \equiv \overline{l}k \pmod 9. \end{aligned}$$

With simple computation, we see that the error position l is given by $(\overline{\frac{s_2}{s_1}})$ and the error magnitude k is given by s_1.

After calculating the syndrome (s_1, s_2), the decoding scheme is as follows:

1. If $(s_1, s_2) = (0, 0)$, then w is a codeword and we assume that there is no errors,

2. Suppose $s_1 \neq 0$, $s_2 \neq 0$. We assume that a single error has occurred which is corrected by subtracting s_1 from the $(\overline{\frac{s_2}{s_1}})$th entry of w,

3. If $s_1 = 0$ or $s_2 = 0$ but not both, it follows that there are at least two errors.

For example, if received vector is $w = 21513412$, we get $s_1 = 1$ and $s_2 = 7$, so $(\overline{\frac{s_2}{s_1}}) = 4$. Thus the 4th digit should have been $1 - 1 = 0$, *i.e.* transmitted vector is 21503412.

4. Conclusion

In this work, we investigated unitary space and linear codes in unitary space. Also, we mentioned a new parity check matrix of linear codes in unitary space and syndrome decoding of linear codes. For further research, it would be natural to generalized our presented results for linear codes in orthogonal space.

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A CATASTROPHIC TEMPERATURE CHANGE IN THE HEAT EQUATION

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Abstract. In this paper we modify the heat equation with the inclusion of a delta function contribution and find the solution of such an equation. It is found that there is an abrupt rise in the temperature across the board.

1. Introduction

The heat equation [1, 2] is a type of second order partial differential equation used to study the distribution of heat, or, more precisely the variation in temperature in a specific region in accordance with time. In this paper, we examine the scenario where in a large body of water, say a lake or a flooded region, there is a sudden rise in the temperature at some instant. The temperature rises momentarily to an extreme value. This could happen if one drops a device (e.g., a bomb of a few kilotons) [3, 4]. At that instant of time, the sharp rise of temperature is beyond the usual temperature variations.

In the next section, we commence by introducing the one dimensional heat equation and modify it with a delta function contribution which represents a singularity due to it's properties. Then, we solve it by the original methodology used by Fourier himself and others [5, 6, 7] in order to find out the exact nature of the modified equation. In the third section, we discuss the opposite case of sudden and catastrophic cooling.

2. The modified heat equation

Here, we would like to observe how we can explain the phenomenon of a sudden and catastrophic temperature rise in a very small time interval, $-\epsilon < t < +\epsilon$, by

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resorting to the heat equation. It is to be borne in mind that the negative sign in the lower limit signifies the time before the instant of the blast. Let $\Theta(x,t)$ denote the temperature at a position x and instant t in a long, thin thermally conducting rod of length d that extends from $x=0$ to $x=d$ in 1D. We assume that the sides of the rod are insulated so that heat energy neither enters nor leaves the rod through its sides. Also, we assume that heat energy is neither created nor destroyed in the interior of the rod and there are no radiative losses. Then, the temperature $\Theta(x,t)$ abides by the heat equation given as [2]

$$\alpha\frac{\partial^2\Theta}{\partial x^2} = \frac{\partial\Theta}{\partial t}, \quad \forall\ 0 < x < d, \forall\ t \geq 0, \tag{1}$$

where $\alpha = \frac{k}{c_p\rho}$, k being the thermal conductivity, c_p being the specific heat capacity and ρ being the mass density of the material under consideration. Without loss of generality, we consider the one dimensional case: it's generalization to 3D is immediate. We would like to propose the following modified form of equation (1) by adding an extra term to the right hand side

$$\alpha\frac{\partial^2\Theta}{\partial x^2} = \{1-\delta(t)\}\frac{\partial\Theta}{\partial t}, \tag{2}$$

where $\delta(t)$ represents the Dirac delta as a function of time and the negative sign is due to the fact that we consider the heat equation moments before the blast and the temperature rises radically. It should be observed that in the modified equation (2) the derivative with respect to time (t) suddenly reaches a very high value because of the introduction of the delta function which in physical terms can be looked upon as a device which artificially triggers such an steep rise in the temperature.

As we stated earlier, the objective of the current work is to find out what are the new solutions due to this extra term. Using separation of variables as [5, 6]

$$\Theta(x,t) = X(x)T(t)$$

one can solve (2) in terms of x and t. Now, suppose the temperature and the boundaries of the rod are kept fixed as 0. Then, we have the following boundary conditions:

$$\Theta(0,t) = 0, \quad \forall\ t > 0,$$

$$\Theta(d,t) = 0, \quad \forall\ t > 0.$$

Now, considering the fact that

$$\Theta(x,t) = X(x)T(t)$$

is a solution for the heat equation (2), we must have

$$\alpha X(x)T'(t) = \{(1-\delta(t))\}X(x)T'(t)$$

which implies that

$$\frac{X''}{X(x)} = \frac{\{(1-\delta(t))\}}{\alpha}\frac{T'(t)}{T(t)} = \xi, \tag{3}$$

where ξ is some constant. Now, from (3) we have the following two equations which are to be solved.

$$X''(x) - \xi X(x) = 0$$

and

$$T'(t) - \frac{\alpha\xi}{1-\delta(t)}T(t) = 0.$$

It is obvious that we will have two cases depending on ξ being zero or nonzero. We shall consider both cases below.

Case-1: $\xi = 0$.
Now, for $\xi = 0$, the solution in terms of x is simply given by

$$X(x) = a_1 + a_2 x \tag{4}$$

and for the solution in terms of t is given by

$$T(t) = a_3. \tag{5}$$

Therefore the solution to the heat equation (2) is given by

$$\Theta(x,t) = a_3(a_1 + a_2 x), \tag{6}$$

where a_1, a_2 and a_3 are integration constants. Now, let us impose the boundary conditions mentioned before. The first boundary condition, $\Theta(0,t) = 0$, is satisfied when we have

$$X(0) = 0$$

which in turn is satisfied when we have

$$a_1 = 0.$$

Again, the second boundary condition, $\Theta(d,t) = 0$, is satisfied when we have

$$X(d) = 0$$

which is satisfied only when we have

$$a_1 + da_2 = 0 \Rightarrow a_2 = 0.$$

Hence, the boundary conditions are satisfied when we have

$$a_1 = a_2 = 0$$

which implies that in general

$$X(x) = 0.$$

But, this will make the whole solution of the heat equation (2) to be trivial and of no fruitful result. Thus, we discard the case of $\xi = 0$.

Case-2: $\xi \neq 0$
In this case, we have the general solution in terms of x as

$$X(x) = a_1 e^{\sqrt{\xi}x} + a_2 e^{-\sqrt{\xi}x}.$$

Now, the first boundary condition is satisfied when we have

$$X(0) = 0$$

which in turn is satisfied when

$$a_1 + a_2 = 0 \Rightarrow a_1 = -a_2.$$

Again, the second boundary condition is satisfied when we have

$$X(d) = 0$$

which again is satisfied only when

$$a_1 e^{\sqrt{\xi}d} + a_2 e^{-\sqrt{\xi}d} = 0 \Rightarrow a_1[e^{\sqrt{\xi}d} - e^{-\sqrt{\xi}d}] = 0$$

since, $a_1 = -a_2$. From this we obtain

$$e^{\sqrt{\xi}d} = e^{-\sqrt{\xi}d} \Rightarrow e^{2\sqrt{\xi}d} = 1$$

which implies that we have

$$\sqrt{\xi} = \frac{n\pi}{d} i \Rightarrow \xi = -\frac{n^2\pi^2}{d^2},$$

where n is some integer. Thus, the general solution in terms of x can be written as

$$X(x) = a_1[e^{\frac{in\pi x}{d}} - e^{-\frac{in\pi x}{d}}] = 2ia_1 \sin(\frac{n\pi x}{d}). \tag{7}$$

Now, let us delve into finding the general solution in terms of t. We have

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} - \frac{\alpha\xi}{1-\delta} T(t) = 0.$$

From this we can write

$$\int \frac{\{1-\delta(t)\}\mathrm{d}T(t)}{T(t)} = \alpha\xi \int \mathrm{d}t$$

which yields

$$\ln[T(t)] + \int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \alpha\xi t + c. \tag{8}$$

Now, let us consider the second term on the left hand side. We can integrate it by parts such that we have

$$\int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \delta(t) \int \frac{\mathrm{d}T(t)}{T(t)} - \int \frac{\mathrm{d}}{\mathrm{d}T(t)}\{\delta(t)\} \ln[T(t)]\mathrm{d}T(t)$$

which gives

$$\int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \delta(t) \ln[T(t)] - \int \frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)} \ln[T(t)]\mathrm{d}T(t).$$

It is to be noted that the integration constant will be merged with the integration constant (c) to make it c', on the right hand side of equation (8). Now, we would like to define the following terms

$$\omega(t) = \int \frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)} \ln[T(t)]\mathrm{d}T(t)$$

and

$$\sigma(t) = \delta(t) \ln[T(t)].$$

Again, it is conspicuous that the delta function depends explicitly on the variable time (t) only and hence

$$\frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)} = \frac{\partial\delta(t)}{\partial T(t)} = 0$$

and thus, $\omega(t) = 0$. Therefore, we are left with the improper function $\sigma(t)$. Now, we know that the delta function has the following property

$$\delta(t) = 0, \quad \forall\ t \neq 0,$$

$$\delta(t) = L, \quad \text{when } t = 0,$$

where, L is arbitrarily very large. Using this property of the delta function, the function $\sigma(t)$ can be defined as follows

$$\sigma(t) = 0, \quad \forall\ t \neq 0,$$

$$\sigma(t) = R, \quad \text{when } t = 0,$$

where, R is extremely large. Thus, from equation (8) we have

$$\ln[T(t)] - \sigma(t) = \alpha\xi t + c'$$

which gives the general solution in terms of t as

$$T(t) = a_3 e^{\alpha\xi t + \sigma(t)}, \tag{9}$$

where $a_3 = e^{c'}$. Thus, the partial solutions (7) and (9) finally yield the solution for the heat equation (2) as

$$\Theta(x,t) = \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2 \pi^2}{d^2} t + \sigma(t)}, \tag{10}$$

where $\chi_n = 2ia_1a_3$. Here, it is easy to see that at the moment of the blast when $t = 0$, we have

$$\Theta(x,0) = \chi_n \sin(\frac{n\pi x}{d}) e^{R} \tag{11}$$

and since R is extremely large, the value of Θ is very large, i.e., we have an abrupt and extreme rise in the temperature of the medium. Interestingly, there is a sudden change in the system due to the abrupt phenomenon of the bomb blast [3, 4]. Now, far away from the instant $t = 0$, the heat equation is the same as the original equation (1). As we already mentioned in the beginning of this section, our work encompasses a small time interval, namely, $-\epsilon < t < +\epsilon$. Now, it is to be noted that this mathematical formulation can be extended to higher dimensions too.

Now suppose, in a flooded region the spread (or length) of the water body is d and at instant $t = 0$ an explosion device (just to vaporize the water) is detonated. Then, from (10) it is obvious that the temperature $\Theta(x,t)$ will be infinite. Practically, this means that we would achieve an extremely high temperature. Also, in equation (11) we see that since the trigonometric function, $\sin(\frac{n\pi x}{d})$, is bounded, the temperature rise at the instant, $t = 0$, is independent across the position x. So, irrespective of the coordinate we have a great amount of temperature. This can be very useful to evaporate unnecessary water bodies emanating from floods or other reasons.

3. Sudden decrease

Now, let us consider a sudden temperature drop in a localized area. To take this into account, we modify equation (2) by replacing the $\{-\delta(t)\}$ with $\{+\delta(t)\}$, such that we have

$$\alpha \frac{\partial^2 \Theta}{\partial x^2} = \{1 + \delta(t)\} \frac{\partial \Theta}{\partial t}.$$

In this case, proceeding with the same methodology we would have the solution as

$$\Theta(x,t) = \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2 \pi^2}{d^2} t - \sigma(t)}.$$

Thus at the instant, $t = 0$ (in the interval, $-\epsilon < t < +\epsilon$), we have

$$\Theta(x,0) = \chi_n \sin(\frac{n\pi x}{d}) e^{-R},$$

where R is again an arbitrarily large number. Therefore, there would be a sudden precipitous localized cooling. This would, for instance, represent an extreme climate scenario or even a cryogenic device. However, we can only draw a qualitative conclusion, as of now. In the case of climate,the problem is very complex, as is well known [8, 9, 10, 11, 12, 13].

4. Discussion

Now, it is known that since (10) is the solution to the modified heat equation (2), one has

$$\Theta(x,t) = \sum_{n=1}^{\infty} \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2\pi^2}{d^2}t+\sigma(t)}. \tag{12}$$

Now, if we denote the solution at $t = 0$ as

$$\Theta(x,0) = f(x)$$

then the constants χ_n are given by

$$\chi_n = \frac{2}{d}\int_0^d f(x)\sin(\frac{n\pi x}{d})\mathrm{d}x.$$

However, the most important conclusion of the current paper is embodied in equation (11) of the preceding section. The instant, $t = 0$, is the key feature in the solution of the modified heat equation (2). As we have already mentioned in the preceding section, for all instants of time after the blast, i.e. for $t > 0$, the heat equation and it's solution presume their original form without the singularity.

Our work could have immediate practical results based on what we have shown and also the introduction of the delta function in the parabolic second order differential equation could inspire similar approaches in other cases too.

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RESEARCH ON THE DEVELOPMENT OF BUSINESS MODEL BASED ON MOBILE APPs

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Abstract. As the mobile Internet develops rapidly, it is playing an increasingly important role in people's daily life nowadays. Mobile ends such as mobile phones and pads lift the time and place limitations of applications (APPs) and APP users spend more time on mobile ends than on the PC end. This paper mainly studies the business models of mobile APPs. Firstly, the theoretical basis related to this study is described. Then, taking a reading APP-QQ reading as an example, we studies its business model from the aspects of industrial chain model, profit model, communication channels and market positioning and makes a multi - level fuzzy comprehensive evaluation on it. Finally, suggestions are put forward to solve the problems found by evaluation. The study of the business model of reading apps is conducive to the development of new functions of reading software to protect readers' loyalty.
Keywords: internet, mobile end, reading APP, business model.

1. Introduction

Today, the rapid development of mobile Internet has opened a new journey of the development of the Internet [1]. Mobile Internet brings mobile Internet operators, end manufacturers and software developers together to form a complex collaborative and competitive ecosystem which creates a large number of mobile APPs which are the research objects in this paper. On this subject, domestic and foreign scholars have carried out some researches. Luo Min et al [2] suggested that a business model was a set composed by the organization itself, employees, shareholders, customers and supply chain partners to obtain excess profits, providing explicit external assumptions and internal resources. Xia Yunfeng [3] divided products into two abstract parts of core and premium and proposed that the business model of an enterprise was composed of the potential energy model and the premium model. Magretta [4] believed that the business model and business process system was closely related and it solved the four basic questions about the business experience. After presenting the four constituent elements of a business model, Dubosson et al. [5] evaluated these quantitative indicators using a balanced scorecard. All the above researches were carried out based on the business model while this paper took a reading APP as an example to see big things from small ones. By learning form the recommendations of the QQ reading APP, a new path can be directed for the future development of mobile APPs.

2. Related theories

2.1 Overview of business model

Business model integrates all the factors required in the operation of an enterprise [6] to realize the maximum value of customers and form a complete and efficient operation system with its unique core competiveness. By satisfying the requirements and realizing the values of customers through the optimal implementation form, it also enables the system to reach the goal of obtaining sustained profits. Usually, business models include operation model, profitability model, and advertising revenue model. Ostwald and Pinnie [7] held that the business model consists of four latitudes (customer, provider, infrastructure and financial viability) and nine elements (cost structure, revenue source, important cooperation, value proposition, core resources, channel access, customer relationship and segmentation, etc). This paper takes a reading APP as an example to describe the development process of mobile reading APPs.

2.2 Multi-level fuzzy evaluation method

Fuzzy comprehensive evaluation [8] is a comprehensive evaluation method based on fuzzy mathematics to solve fuzzy problems and the ones which are difficult to be quantified, which was first put forward in 1965 by an automatic control expert Chad. Also, it is an evaluation method that integrates both accurate and inaccurate analyses which is applied in various fields. In complex systems, there are different levels of factors to consider. Hence, the evaluation factors are divided into several categories according to their attributes. Firstly, evaluation is carried out on each category of factors. Then, all the evaluation results are combined for a comprehensive evaluation. The main steps are as follows:

(1) Establishment of evaluation factor indicator system [9]. According to a certain property, the evaluation factor is divided into multiple plates.

(2) Determination of the evaluation matrix. Evaluate each small factor by a single level to obtain an evaluation matrix.

(3) Determination of factor weight. Use the expert survey method to determine the weight of the evaluation index.

(4) Evaluation implementation. Calculate the final comprehensive evaluation value according to the relevant formulas and evaluation matrixes.

3. Business model of the QQ reading APP

3.1 Industrial chain model

3.1.1 Upstream of the industry chain

In the upstream of an industry chain are normally manufacturers who provide content of the APP. In the QQ reading APP, it mainly includes authors, publishers, intermediate agents, etc. Today's domestic publishers are divided into journal publishers and electronic publishers [10]. Journal publishers integrate the content into a database and provide readers with paid resources. Electronic publishers make paper books into e-books or audio books which are then uploaded to reading software for users to read. Original literature websites publish the network literature created by network authors and provide them on the website to authors with fees charged.

3.1.2 Midstream of the industry chain

In the midstream of an industry chain often locates the content publishers, who upload the final work to the releasing platform after getting digital licenses for copyright and content resources, pricing and integration. Located between content providers and service providers and vendors, content publishers play a mainstay role in the whole industry chain [11].

3.1.3 Downstream of the industry chain

In the downstream of the industry chain lie the vendors. Connecting service providers with consumers, content vendors are mainly engaged in service sales business. With the QQ reading APP itself a vendor, some film and television enterprises are also vendors. Readers can subscribe to the books or network literature works which they are interested in via the QQ reading APP. As for the film and television enterprises, they can select the works which they like to make them into TV series or films.

3.2 Profit mode

The QQ reading APP has the following profit models [12]: (1) free reading and advertising revenue. (2) part free and part charged with original price. (3) book download charges. (4) monthly service charges. Some books or their sections are free for users to read, which attracts more readers. Excellent literary works can guide readers to support the authorized editions which charge for fees. Once readers have a good impression on the APP, they will not easily quit. Thus, the loyalty of readers is improved.

3.3 Communication channel

(1) Interpersonal propagation. Readers share their feelings or thoughts about the books they read on social platforms such as QQ or Wechat APPs to reach the purpose of recommendation.

(2) Organization propagation. Information interaction between readers can be realized on the reviewing square of the APP by bringing readers who like the same type of works together to make the readers find a sense of belonging and thus improve their loyalty [13].

(3) Social propagation. By means of the functions of sharing and commenting of the APP, more readers can be attracted, which improves the popularity of the APP.

3.4 Market positioning

To have a good sales market, market positioning and customer segmentation [14] should be first carried out for the APP. According to the survey, people with different age, gender, income and class have different preference on literature works. Table 1 is a survey of the preferences of different people.

age	group	Reading preference	Percentage (%)
15-24	students	Art type novels	30
21-28	Female white collars	Novels and human and life class books	26
26-35	male and female book lovers with discretion ability	Art type novels and human and social science books	25
36-65	Mid aged book lovers	Human and social science and historical and military books	19

Figure 1: Reader segmentation

As shown in table 1, students, female white collars and male and female book lovers with discretion ability occupy a large percentage. Therefore, the market positioning of the APP is novels, art and human and social science books.

4. Evaluation on the APP with fuzzy evaluation method

4.1 Fuzzy evaluation results

An evaluation team with 10 members was selected to carry out evaluation on each project module of the APP. The satisfaction degree of evaluation is: very satisfied (95), satisfied (85), average (75), poor (65) and very poor (55). In addition, score 55 indicates unqualified. The detailed evaluation results are shown in table 2.

Module evaluation/ weight (secondary comprehensive evaluation)	Project evaluation /weight (first comprehensive evaluation)	Degree of satisfaction				
		Very satisfied	satisfied	average	poor	Unqualified
Book town/0.21	Content quantity/0.18	1	9	0	0	0
	Book classification/0.08	1	2	7	0	0
	Updating speed/0.18	1	7	2	0	0
	Special topic recommendation/0.02	7	1	1	1	0
	Top list recommendation/0.02	1	3	4	2	0
	Search function/0.11	4	5	1	0	0
Business management /0.15	Free trial reading/0.12	6	3	1	0	0
	Payment convenience/0.09	0	1	3	6	0
	Bargain price/0.05	1	1	2	6	0
	Free in a limited time/0.15	1	1	7	1	0
Reading function/0.28	Text adjustment/0.173	3	6	1	0	0
	Night mode/0.14	1	7	2	0	0
	Brightness mode/0.14	7	1	1	1	0
	Background adjustment/0.115	3	3	2	2	0
	Page switch/0.041	7	1	2	0	0
	Screen switch/0.099	0	4	4	2	0
	Content search/0.041	6	3	0	1	0
	Note adding/0.066	3	6	1	0	0
	Audio book/0.033	1	1	7	1	0
	Eye protection mode/0.123	0	0	0	0	10
	Automatic page switch/0.024	8	1	1	0	0
bookshelf/0.08	Cover page display/0.08	0	3	6	1	0
	Book sorting/0.32	0	3	6	1	0
	Local book lead in/0.32	4	5	1	0	0
	Book searching/0.2	0	0	4	6	0
	Background change/0.08	0	2	6	2	0
Social function/0.2	Registration convenience/0.07	1	7	1	0	0
	Cloud synchronizing/0.121	0	0	0	0	10
	Personal growth system/0.111	8	2	0	0	0
	Note synchronizing/0.121	1	6	3	0	0
	Friend adding/0.09	0	0	1	9	0
	Book sharing/0.08	0	5	5	0	0
	forum/0.02	0	5	4	1	0
	Social app sharing/0.181	0	7	2	1	0
	Check in/0.181	7	3	0	0	0

Figure 2: Reader segmentation

According to the indicators, the proportion of evaluation modules was 0.21, 0.15, 0.28, 0.08, 0.2 and 0.08 respectively. Then, evaluation was performed based on the secondary evaluation on the results of first evaluation on the matrix of table 2, with the formulas as follows:

$$E_p = \frac{\Sigma E}{n}, \quad n \in 1, 2, k, 10, \tag{1}$$

$$E_{pw} = E_p * W, \tag{2}$$

$$E_z = \Sigma E_{pw}. \tag{3}$$

E refers to the evaluation value, E_p refers to the average evaluation value, E_{pw} refers to the weighted average value, W refers to the weight and E_z refers to the comprehensive evaluation value. The calculation results are as below:

$$\begin{aligned} B_1 &= [0.18, 0.08, 0.18, 0.02, 0.11] \begin{bmatrix} 0.1 & 0.9 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.7 & 0 & 0 \\ 0.1 & 0.7 & 0.2 & 0 & 0 \\ 0.7 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & 0.3 & 0.4 & 0.1 & 0 \end{bmatrix} \\ &= (0.12, 0.37, 0.18, 0.01, 0) \end{aligned} \tag{4}$$

$$\begin{aligned} B_2 &= [0.12, 0.09, 0.05, 0.15] \begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0.03 & 0.6 & 0 \\ 0.1 & 0.1 & 0.2 & 0.6 & 0 \\ 0.1 & 0.1 & 0.7 & 0.1 & 0 \end{bmatrix} \\ &= (0.09, 0.07, 0.13, 0.1, 0) \end{aligned} \tag{5}$$

$$\begin{aligned} B_3 = [&0.173, 0.14, 0.115, 0.041, 0.099, 0.041, 0.066, 0.033, 0.123, 0.024] \\ &\begin{bmatrix} 0.3 & 0.6 & 0.1 & 0 & 0 \\ 0.1 & 0.7 & 0.2 & 0 & 0 \\ 0.7 & 0.1 & 0.1 & 0.1 & 0 \\ 0.3 & 0.3 & 0.2 & 0.2 & 0 \\ 0.7 & 0.1 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0.2 & 0 \\ 0.6 & 0.3 & 0 & 0.1 & 0 \end{bmatrix} = (0.29, 0.35, 0.16, 0.06, 0.12) \end{aligned} \tag{6}$$

$$\begin{aligned} B_4 = [&0.08 \quad 0.32 \quad 0.32 \quad 0.2 \quad 0.08] \\ &\begin{bmatrix} 0 & 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0.3 & 0.6 & 0.1 & 0 \\ 0.4 & 0.5 & 0.1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0.2 & 0.6 & 0.2 & 0 \end{bmatrix} = (0.13, 0.3, 0.4, 0.18, 0) \end{aligned} \tag{7}$$

$$B_5 = [0.07, 0.121, 0.111, 0.121, 0.09, 0.08, 0.02, 0.181, 0.181, 0.02]$$

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0.8 & 0.2 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.9 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.1 & 0 \\ 0 & 0.7 & 0.2 & 0.1 & 0 \\ 0 & 0.7 & 0.3 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 \end{bmatrix} = (0.29, 0.34, 0.14, 0.1, 0.12). \tag{8}$$

$$B_6 = [1][0 \quad 0 \quad 0.7 \quad 0.30] = (0, 0, 0.7, 0.3, 0) \tag{9}$$

$B_1, B_2, B_3, B_4, B_5, B_6$ refer to evaluation quantization values of each element in each section. Then, the first and secondary comprehensive evaluation values can be obtained:

$$B = (0.21, 0.15, 0.28, 0.08, 0.2, 0.08) \begin{bmatrix} 0.12 & 0.37 & 0.18 & 0.01 & 0 \\ 0.09 & 0.07 & 0.13 & 0.1 & 0 \\ 0.29 & 0.35 & 0.16 & 0.06 & 0.12 \\ 0.13 & 0.3 & 0.4 & 0.18 & 0 \\ 0.29 & 0.34 & 0.14 & 0.1 & 0.12 \\ 0 & 0 & 0.7 & 0.3 & 0 \end{bmatrix}$$

$$= (0.18, 0.28, 0.22, 0.09, 0.06) \tag{10}$$

$$E_1 = (0.12, 0.37, 0.18, 0.01, 0)(95, 85, 75, 65, 55)^T = 57, \tag{11}$$

$$E_2 = (0.09, 0.07, 0.13, 0.1, 0)(95, 85, 75, 65, 55)^T = 31, \tag{12}$$

$$E_3 = (0.29, 0.35, 0.16, 0.06, 0.12)(95, 85, 75, 65, 55)^T = 80, \tag{13}$$

$$E_4 = (0.13, 0.3, 0.4, 0.18, 0)(95, 85, 75, 65, 55)^T = 80, \tag{14}$$

$$E_5 = (0.29, 0.34, 0.14, 0.1, 0.12)(95, 85, 75, 65, 55)^T = 80, \tag{15}$$

$$E_6 = (0, 0, 0.7, 0.3, 0)(95, 85, 75, 65, 55)^T = 72 \tag{16}$$

$E_1, E_2, E_3, E_4, E_5, E_6$ refer to the first comprehensive evaluation values.

Then, the total value of secondary comprehensive evaluation can be obtained, as follows:

$$E_z = (0.18, 0.28, 0.22, 0.09, 0.06)(95, 85, 75, 65, 55)^T = 68. \tag{17}$$

5. Fuzzy comprehensive evaluation analysis

As shown in equation (13), (14), (15) and (16), the fuzzy evaluation values of "bookshelf", "social function" and "feedback" are high, with obvious advantages. Hence, only improvement is needed to strengthen the reading function of the APP. As shown in equation (11), the "book town" function is yet to be perfected to meet the requirements of readers. As shown in equation (12), the "business management" function of the APP needs improvement on both discount scale and payment means. That is to say, there are more selections on payment means, which should be simplified as much as possible [15] to reduce the imitation on payment software. As for the discount aspect, the number of books which are free in a limited time should be increased. As shown in table 2, the "eye protection mode" function in the "reading function" module should be implemented; the "cover page display", "book assorting" and "background change" functions in the "bookshelf" part should be adjusted according to the readers' requirement; the "book searching" function should be perfected; The "cloud sync" feature can be turned on in the "social" section, and other not ideal features should be improved; as for the "feedback" section, its functions are not prominent and should be strengthened, which is helpful for the improvement of the software according to the opinions put forward by readers.

With the multi-level fuzzy comprehensive evaluation method, the advantages and disadvantages of the APP are displayed clearly. Therefore, new functions should be added to the APP to perfect it. Especially for the "business management" part, its score is only 31, which is far much lower than the average value and unqualified. The reason for this is that readers show resistance psychology [16] to paid reading because they previously enjoy free reading with no attention paid to the copyright problem. Meanwhile, the growing number of network writers has somehow caused the decline of the quality of network literature works, which makes the readers lose confidence to paid reading.

6. Conclusion

In order to study the business model of mobile APPs, this paper takes the QQ reading APP as an example and analyzes its reading characteristics and modules. Then, its business module is studied from the aspects of industrial chain model, profit model, communication channels and market positioning. For the problems existing in the APP, such as copyright issues, imperfect functions, undesirable page layout and book classification confusion, they will be solved in the near future considering Tencent's strong financial and technical strength.

In the future, the QQ reading APP will develops towards a diversified direction and rely more on advertising to profit, with perfected functions. For example, eye protection model is an inevitable trend. Under the current situation that mobile APPs are facing bottle necks, especially in profit mode aspect, the future trend of the QQ reading APP can be learnt by mobile APPs, i.e., they should rely more on advertising. Therefore, the emphasis on the studies of future business modes of mobile APPs should be put on their profit modes.

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ON THE ELEMENTARY AND BASIC CHARACTERS OF $G_n(q)$

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Abstract. We discuss in this paper the elementary and basic characters of $G_n(q)$. The paper highlights the main idea that every irreducible character of $G_n(q)$ appears uniquely in the basic characters. In particular we determine the elementary and basic characters of $G_3(q)$ for any q. As examples, the theory is also applied to $G_3(2)$, $G_3(3)$ and $G_4(2)$.

Keywords: general linear group, Sylow subgroup, Fischer matrices, irreducible characters, elementary and basic characters.

1. Introduction and notations

The study of the irreducible characters of $G_n(q)$, a Sylow p-subgroup of $GL(n, q)$ has attracted much attention over the years. By inducing linear characters of some special subgroups of $G_n(q)$, Andre in [2] introduced the notion of elementary characters. These characters were also known to Lehrer (see [8]). The basic characters are a special product of some of these elementary characters and their constituents form a partition of all the irreducible characters of $G_n(q)$ (see for example [2]).

The group $G_n(q)$ is given by

$$G_n(q) = \{(a_{ij}) \in GL(n, q) | a_{ij} = 0, j < i, a_{ii} = 1 \text{ and } a_{ij} \in \mathbb{F}_q, 1 \leq i < j \leq n\},$$

that is a group of upper triangular matrices with $1's$ in the major diagonal and other entries coming from $\mathbb{F}_q$, where F_q is the Galois field of q elements with $\mathbb{F}_q^*$ and $\mathbb{F}_q^+$ being the multiplicative and additive groups respectively. The

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group $G_n(q)$ can be written as a split extension $N{:}G$ where N is an elementary abelian p-group $p^{k(n-1)}$ and G is the group $G_{n-1}(q)$. We use this fact to apply the method of Fischer matrices in constructing the character tables. The method of Coset Analysis by Moori [10] is used for the determination of the conjugacy classes.

By $\lambda_{ij}(\alpha)^{G_n(q)}$, we will mean the induction of $\lambda_{ij}(\alpha)$ to $G_n(q)$ where $\alpha \in \mathbb{F}_q^*$ and $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$ is a linear character of G_{ij}. The G_{ij} are special subgroups of $G_n(q)$ where $(i,j) \in \Phi(n) = \{(i,j)|1 \leq i < j \leq n\}$. We identify a particular type of subsets of $\Phi(n)$ denoted by D as the basic subsets. To each basic subset D we can associate a function $\varphi : D \longrightarrow \mathbb{F}_q^*$ such that $\varphi((i,j)) \in \mathbb{F}_q^*$ (see for instance Subsection 3.2) and thereby denote the basic characters by $\xi_D(\varphi)$. This is the standard notation as used in [2].

In Section 2 we briefly describe the method of constructing the elementary and basic characters of $G_n(q)$. Using Fischer matrices we obtain the character tables of $G_3(2)$ and $G_3(3)$ in Sections 3 and 4. The elementary and basic characters of these groups are discussed in Subsections 3.1, 3.2 and 4.1. We have the main Theorem 4.2 on the elementary and basic characters of $G_3(q)$ proved in Subsection 4.2. The character table of $G_4(2)$, its elementary and basic characters are discussed in Section 5. For general notation we use ATLAS [4] and Isaacs [6].

2. Elementary and basic characters

In this section we briefly describe the elementary and basic characters of $G_n(q)$. We use the notation as in [2]. Basic characters have also been described in [7] as super characters.

Definition 2.1. *Let*

$$G_n(q) = \{(a_{ij}) \in GL(n,q)|a_{ij} = 0, j < i, a_{ii} = 1 \text{ and } a_{ij} \in \mathbb{F}_q, 1 \leq i < j \leq n\},$$

be a Sylow $p-$ subgroup of $GL(n,q)$. We set $\Phi(n) = \{(i,j)|1 \leq i < j \leq n\}$ be the set of pairs for the position of $a_{ij} \in \mathbb{F}_q$ in a matrix of $G_n(q)$. The elements of $\Phi(n)$ are called ***positive roots****.*

Definition 2.2. *By fixing i for $1 \leq i \leq n$, define the $i^{th}-$ row of $G_n(q)$ to be the set*

$$r_i(n) = \{(i,j) \in \Phi(n)|i < j \leq n\}.$$

Similarly by fixing j the $j^{th}-$ column of $G_n(q)$ is the set

$$c_j(n) = \{(i,j) \in \Phi(n)|1 \leq i < j\}.$$

Using the elements of $\Phi(n)$, we define

$$G_{ij} = \{(x_{ab}) \in G_n(q)|x_{ib} = 0 \ i < b < j\},$$

G_{ij} are subgroups of $G_n(q)$. The irreducible characters of $G_n(q)$ that were constructed by Lehrer [8] are described in Proposition 2.1 following here below.

Proposition 2.1. *Let $\alpha \in \mathbb{F}_q^*$ and ψ_o be a non-trivial irreducible character of $\mathbb{F}_q^+$ considered as an additive group. Let $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$ be a function such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha(x_{ij}))$ for all $x \in G_{ij}$ where $x = (x_{ab})$. Then $\lambda_{ij}(\alpha)$ is a linear character of G_{ij} and furthermore $\xi_{ij}(\alpha) = \lambda_{ij}(\alpha)^{G_n(q)}$ is an irreducible character of $G_n(q)$.*

Proof. See [8]. □

The following definitions are from [2].

Definition 2.3. *The characters $\lambda_{ij}(\alpha)^{G_n(q)}$, as given in Proposition 2.1, are called the (i,j)th* ***elementary characters*** *associated with α.*

Definition 2.4. *Let $D \subseteq \Phi(n)$ such that $|D \cap r_i(n)| \leq 1$ and $|D \cap c_j(n)| \leq 1$. Let $\varphi_{ij} : D \longrightarrow \mathbb{F}^*$ be a function, then*

$$\xi_D(\varphi) = \prod_{(i,j)\in D} \xi_{ij}(\varphi(i,j))$$

is a character of $G_n(q)$ called the ***basic character*** *of $G_n(q)$ and the subset D is called a* ***basic subset*** *of $\Phi(n)$.*

3. Elementary and basic characters of $G_3(2)$

We first construct the character table of $G_3(2)$ by using Fischer matrices. For details on coset analysis and Fischer matrices the readers are referred to ([1], [9], [10], [11]). The elementary and basic characters are discussed in Subsections 3.1 and 3.2. Table 1 gives the conjugacy classes of $G_3(2)$ computed using the coset analysis technique.

classes of $G_2(2)$	classes of $G_3(2)$	$\lvert C_{G_3(2)}(g)\rvert$
$(1a)$	$(1A)$	2^3
	$(2A)$	2^3
	$(2C)$	2^2
$(2a)$	$(2C)$	2^2
	$(4A)$	2^2

Table 1: The Conjugacy Classes of $G_3(2)$

The inertia factor groups are; $H_1 = H_2 = G \cong \mathbb{Z}_2$ and $H_3 = \{1\}$.

The Fischer matrices on the representatives of the classes of G are given below

$$M(1a) = \begin{array}{cc} & \begin{array}{ccc} |C_{\bar{G}}(1A)| & |C_{\bar{G}}(2A)| & |C_{\bar{G}}(2B)| \end{array} \\ \begin{array}{cc} |C_{H_1}(1a)| & 2 \\ |C_{H_2}(1a)| & 2 \\ |C_{H_3}(1a)| & 1 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \end{array},$$

$$M(2a) = \begin{array}{cc} & \\ |C_{H_1}(2a)| & 2 \\ |C_{H_2}(2a)| & 2 \end{array} \begin{array}{c} \begin{array}{cc} |C_{\bar{G}}(2C)| & |C_{\bar{G}}(4A)| \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}.$$

Note that character tables of the inertia factor groups H_1 and H_2 are 2×2 invertible matrices. Thus, by multiplying (usual matrix multiplication) the partial character tables of the inertia factor groups H_1, H_2 and H_3 by the corresponding rows of the Fischer matrices above, we obtain

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \end{bmatrix}.$$

Similarly we obtain

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The character table of $G_3(2)$ is given below as Table 2.

$[cl(g)]$	$1A$	$2A$	$2B$	$2C$	$4A$
$C_G(g)$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 2: The Character Table of $G_3(2)$

3.1 Elementary characters of $G_3(2)$

We apply in this Subsection the ideas of [2] to our group $G_3(2)$ to identify its elementary and basic characters. We will later apply the same theory to $G_3(3)$, $G_4(2)$ and $G_3(q)$ in general (see Subsections 4.1, 4.2, 4 and 5). For $\alpha = 1$ and using the character table of $\mathbb{F}_2$, we have ψ_o, the fixed non-trivial irreducible character of $\mathbb{F}_2$, given by say χ_2.

We also have that

$$\Phi(3) = \{(i,j) | 1 \leq i < j \leq 3\} = \{(1,2),(1,3),(2,3)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_3$ and

$$G_{13} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_2 \right\}$$

which is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Thus $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$, such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha x_{13})$ for all $x \in G_{13}$. We then have that

$$\begin{aligned} \lambda_{ij}(1)(x) = \psi_o(x_{13}) &= 1 \text{ if } x_{13} = 0 \\ &- 1 \text{ if } x_{13} = 1. \end{aligned}$$

The character $\lambda_{13}(1)^{G_3(2)}$ is an irreducible character of $G_3(2)$ of degree

$$[G_3(2) : G_{13}] = 2.$$

Using the character table of $G_3(2)$ in Table 2, we identify $\lambda_{13}(1)^{G_3(2)} = \chi_5$. Since we need

$$(q-1)(\frac{n(n-1)}{2}) = (2-1)(\frac{3(3-1)}{2}) = 3$$

elementary characters, we have two more elementary characters to be induced from G_{12} and G_{23} and clearly they are both of degree 1. We also have that $\lambda_{12}(1)(x),\ \lambda_{23}(1)(x) \in \{1, -1\}$. Using the character table and the structure of the conjugacy class representatives of $G_3(2)$, we easily identify $\lambda_{12}(1)^{G_3} = \chi_3$ and $\lambda_{23}(1)^{G_3} = \chi_2$. Hence the three elementary characters of $G_3(2)$ are χ_2, χ_3 and χ_5 of degrees 1, 1 and 2 respectively.

3.2 Basic characters of $G_3(2)$

To calculate the basic characters of $G_3(2)$, we first identify the basic subsets of $\Phi(3)$. Since $\Phi(3) = \{(1,2), (1,3), (2,3)\}$, there are 8 subsets of $\Phi(3)$. The following subsets are basic $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(2,3)\}$, $D_5 = \{(1,2), (2,3)\}$.

Remark 3.1. For the group $G_3(2)$, we have $r_1(3) = \{(1,2), (1,3)\}$, $r_2(3) = \{(2,3)\}$, $c_1(3) = \emptyset$, $c_2(3) = \{(1,2)\}$ and $c_3(3) = \{(1,3), (2,3)\}$. It is clear that $|D_s \cap r_i(3)| \leq 1$ and $|D_s \cap c_j(3)| \leq 1$ for all $s \in \{1,2,3,4,5\}$, $i \in \{1,2\}$ and $j \in \{2,3\}$. Thus D_s is basic.

We now use the formula

$$\xi_D(\varphi) = \prod_{(i,j)\in D} \xi_{i,j}(\varphi(i,j)) \tag{1}$$

for the basic characters where $\varphi : D \longrightarrow \mathbb{F}_q^*$. By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_3(2)$. In this way we have the basic characters of $G_3(2)$, namely

$$\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_3} = \chi_3,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_3} = \chi_5,\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_3} = \chi_2,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_2 \times \chi_3 = \chi_4.
\end{aligned}$$

Table 3 below gives the basic characters of $G_3(2)$ decomposed in terms of its irreducible characters viz. $\chi_1,\ \chi_2,\ \chi_3,\ \chi_4$ and χ_5.

basic characters	χ_1	χ_2	χ_3	χ_4	χ_5
$\xi_{\emptyset}(\varphi)$	1	0	0	0	0
$\xi_{D_2}(\phi)$	0	0	1	0	0
$\xi_{D_3}(\phi)$	0	0	0	0	1
$\xi_{D_4}(\phi)$	0	1	0	0	0
$\xi_{D_5}(\phi)$	0	0	0	1	0

Table 3: The Constituents of the Basic Characters of $G_3(2)$

We observe from Table 3 above that all the basic characters of $G_3(2)$ are irreducible.

4. Elementary and basic characters of $G_3(3)$

We have that $G_3(3) = 3^2{:}3$ and $\mathbb{F}_3 = \{0, 1, -1\}$. From the character table of $G_3(q)$ in [3], we have that

- $2q - 1 = 2(3) - 1 = 5$ inertia groups,
- $q^2 + q - 1 = 3^2 + 3 - 1 = 11$ conjugacy classes,
- $q^2 = 3^2 = 9$ irreducible characters of degree 1,
- $q - 1 = 3 - 1 = 2$ irreducible characters of degree 3.

To find the conjugacy classes of $G_3(3)$, we use the method of coset analysis. Let $G_3(3) = N{:}G$, where $G \cong \mathbb{Z}_3$ and the conjugacy class representatives of G are $\{1a, 3a, 3b\}$. Table 4 following here below, gives the conjugacy classes of $G_3(3)$.

Theorem 4.1. *The group $G_3(3)$ has the following conjugacy classes as listed in Table 4 below, where the upper cases label conjugacy classes of $G_3(3)$ and lower cases are reserved for its subgroups (inertia factor groups).*

Proof. An application of the coset analysis method. □

classes of G	classes of G_3	$\lvert C_{G_3}(g)\rvert$
$(1a)$	$(1A)$	3^3
	$(3A)$	3^3
	$(3B)$	3^3
	$(3C)$	3^2
	$(3D)$	3^2
$(3a)$	$(3E)$	3^2
	$(3F)$	3^2
	$(3G)$	3^2
$(3a)$	$(3H)$	3^2
	$(3I)$	3^2
	$(3J)$	3^2

Table 4: The Conjugacy Classes of $G_3(3)$

The structure of the inertia factor groups is $H_1 = \mathbb{Z}_3 = H_2 = H_3$ and $H_4 = \{1\} = H_5$. We obtain the following character table for $G \cong \mathbb{Z}_3$ as in Table 5, where $a = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$ and $\bar{a}$ is the complex conjugate.

$[g]$	$1a$	$3a$	$3b$
$C_G(g)$	3	3	3
χ_1	1	1	1
χ_2	1	a	$\bar{a}$
χ_3	1	$\bar{a}$	a

Table 5: The Character Table of $G \cong \mathbb{Z}_3$

Thus using the general form of the Fischer matrix in [3], we have

$$M(1a) = \begin{array}{r} \\ |C_{H_1}(1a)|\ 3 \\ |C_{H_2}(1a)|\ 3 \\ |C_{H_3}(1a)|\ 3 \\ |C_{H_4}(1a)|\ 1 \\ |C_{H_5}(1a)|\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} |C_{\bar{G}_3}(1A)| & |C_{\bar{G}}(3A)| & |C_{\bar{G}}(3B)| & |C_{\bar{G}}(3C)| & |C_{\bar{G}}(3D)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a & \bar{a} \\ 1 & 1 & 1 & \bar{a} & a \\ 3 & b & \bar{b} & 0 & 0 \\ 3 & \bar{b} & b & 0 & 0 \end{pmatrix} \end{array}.$$

In the above Fischer matrix, $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $b = \frac{-3}{2} - \frac{3\sqrt{3}}{2}i$.

The other Fischer matrices are given below, where a and $\bar{a}$ are as in the Fischer matrix above

$$M(3a) = \begin{array}{r} \\ |C_{H_1}(3a)|\ 3 \\ |C_{H_2}(3a)|\ 3 \\ |C_{H_3}(3a)|\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}_3}(3E)| & |C_{\bar{G}}(3F)| & |C_{\bar{G}}(3G)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix} \end{array},$$

$$M(3b) = \begin{array}{c} \\ |C_{H_1}(3b)|\ 3 \\ |C_{H_2}(3b)|\ 3 \\ |C_{H_3}(3b)|\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}_3}(3H)| & |C_{\bar{G}}(3I)| & |C_{\bar{G}}(3J)| \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{array} \right) \end{array}.$$

Thus, for instance, on the classes $(1A)$, $(3A)$, $(3B)$, $(3C)$ and $(3D)$, the character values are obtained by multiplying the partial character tables of the inertia factor groups by the corresponding rows of the Fischer matrices and we obtain

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & a & \bar{a} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & a & \bar{a} \\ 1 & 1 & 1 & a & \bar{a} \\ 1 & 1 & 1 & a & \bar{a} \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \bar{a} & a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \bar{a} & a \\ 1 & 1 & 1 & \bar{a} & a \\ 1 & 1 & 1 & \bar{a} & a \end{bmatrix},$$

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 & b & \bar{b} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & b & \bar{b} & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 & \bar{b} & b & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & \bar{b} & b & 0 & 0 \end{bmatrix}.$$

In this manner we can obtain the character table of $G_3(3)$ given as in Table 6.

$[g]$	$1A$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$3H$	$3I$	$3J$
$\lvert C_G(g)\rvert$	3^3	3^3	3^3	3^2	3^2	3^2	3^2	3^2	3^2	3^2	3^2
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	a	a	a	$\bar{a}$	$\bar{a}$	$\bar{a}$
χ_3	1	1	1	1	1	$\bar{a}$	$\bar{a}$	$\bar{a}$	a	a	a
χ_4	1	1	1	a	$\bar{a}$	1	a	$\bar{a}$	1	a	$\bar{a}$
χ_5	1	1	1	a	$\bar{a}$	a	$\bar{a}$	1	$\bar{a}$	1	a
χ_6	1	1	1	a	$\bar{a}$	$\bar{a}$	1	a	a	$\bar{a}$	1
χ_7	1	1	1	$\bar{a}$	a	1	$\bar{a}$	a	1	$\bar{a}$	a
χ_8	1	1	1	$\bar{a}$	a	a	1	$\bar{a}$	$\bar{a}$	a	1
χ_9	1	1	1	$\bar{a}$	a	$\bar{a}$	a	1	a	1	$\bar{a}$
χ_{10}	3	b	$\bar{b}$	0	0	0	0	0	0	0	0
χ_{11}	3	$\bar{b}$	b	0	0	0	0	0	0	0	0

Table 6: The Character Table of $\bar{G} = G_3(3)$

$[cl(g)]$	0	1	-1
$\lvert C_{\mathbb{F}_2}(g)\rvert$	3	3	3
χ_1	1	1	1
χ_2	1	a	$\bar{a}$
χ_3	1	$\bar{a}$	a

Table 7: The Character Table of $\mathbb{F}_3$

4.1 Elementary characters of $G_3(3)$

We now consider the field $\mathbb{F}_3 = \{0, 1, -1\}$. Then $\alpha \in \{1, -1\}$ and using the character table of $\mathbb{F}_3$ as in Table 7, we fix $\psi_o = \chi_2$.

In Table 7 we have $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$.

We also have that

$$\Phi(3) = \{(i, j) | 1 \leq i < j \leq 3\} = \{(1, 2), (1, 3), (2, 3)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_3$ and

$$G_{13} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_3 \right\},$$

which is the group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Thus $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$, such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha x_{13})$ for all $x \in G_{13}$.

For instance, for $\alpha = 1$ we have

$$\begin{aligned} \lambda_{ij}(1)(x) = \psi_o(x_{13}) &= 1 \text{ if } x_{13} = 0, \\ &= a \text{ if } x_{13} = 1, \\ &= \bar{a} \text{ if } x_{13} = -1, \end{aligned}$$

where $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $\bar{a}$ is its complex conjugate. The character $\lambda_{13}(\alpha)^{G_3(3)}$ is an irreducible character of $G_3(3)$ of degree

$$[G_3(3) : G_{13}] = 3.$$

Using the character table of $G_3(3)$ in Table 6, we identify $\lambda_{13}(1)^{G_3(3)} = \chi_{11}$ and $\lambda_{13}(-1)^{G_3(3)} = \chi_{10}$.

Since we need

$$(q - 1)(\frac{n(n-1)}{2}) = (3 - 1)(\frac{3(3-1)}{2}) = 6$$

elementary characters, we have four more elementary characters to be induced from G_{12} and G_{23} and they are all of degree 1. We also have that $\lambda_{12}(\alpha)(x)$, $\lambda_{23}(\alpha)(x) \in \{1, a, \bar{a}\}$. For instance, by using the character table and the structure

$[g]$	$1A$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$3H$	$3I$	$3J$
$\lambda_{12}(1)^{G_3}$	1	1	1	1	1	a	a	a	$\bar{a}$	$\bar{a}$	$\bar{a}$

Table 8: The Values of $\lambda_{12}(1)(x)$

of the conjugacy class representatives of $G_3(3)$ we have the values of $\lambda_{12}(1)^{G_3(3)}$ on the conjugacy class representatives of $G_3(3)$ as in Table 8.

We easily identify $\lambda_{12}(1)^{G_3(3)} = \chi_2$, $\lambda_{12}(-1)^{G_3(3)} = \chi_3$, $\lambda_{23}(1)^{G_3(3)} = \chi_4$ and $\lambda_{23}(-1)^{G_3(3)} = \chi_7$. Hence the six elementary characters of $G_3(3)$ are χ_2, χ_3, χ_4, χ_7, χ_{10} and χ_{11}.

4.2 Basic characters of $G_3(3)$

As in Subsection 3.2, the basic subsets of $\Phi(3)$ are $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(2,3)\}$, $D_5 = \{(1,2),(2,3)\}$. By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_3(3)$. In this way we have the basic characters of $G_3(3)$, namely

$$
\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_3} = \chi_2,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(-1) = \lambda_{12}(-1)^{G_3} = \chi_3,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_3} = \chi_{10},\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(-1) = \lambda_{13}(-1)^{G_3} = \chi_{11},\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_3} = \chi_4,\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(-1) = \lambda_{23}(1)^{G_3} = \chi_7,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_2 \times \chi_4 = \chi_5,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(-1) \times \xi_{23}(-1)\\
&= \lambda_{12}(-1)^{G_3} \times \lambda_{23}(-1)^{G_3} = \chi_3 \times \chi_7 = \chi_9,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(-1) \times \xi_{23}(1)\\
&= \lambda_{12}(-1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_3 \times \chi_4 = \chi_6,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(-1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(-1)^{G_3} = \chi_2 \times \chi_7 = \chi_8.
\end{aligned}
$$

Thus all the irreducible characters of $G_3(3)$ are basic characters. In general, we have the following Theorem 4.2.

Theorem 4.2. *All the irreducible characters of $G_3(q)$ are basic characters.*

Proof. From $\Phi(3)$, we have the groups $G_{12} = G_3(q)$, $G_{13} = \mathbb{Z}_q \times \mathbb{Z}_q$ and $G_{23} = G_3(q)$. The characters $\lambda_{13}(\alpha)^{G_3(q)}$ are irreducible characters of $G_3(q)$ of degree

$|G_3(q) : G_{13}| = q$. Thus associated with α, we have $q-1$ elementary characters of $G_3(q)$ of degree q. Since $\xi_{D_3}(\varphi) = \lambda_{13}(\alpha)^{G_3(q)}$, we have $q-1$ basic characters arising this way. This accounts for the $q-1$ irreducible characters of $G_3(q)$ of degree q. The basic subsets D_2 and D_4 both give rise to $\lambda_{12}(\alpha)^{G_3(q)} = \lambda_{12}(\alpha)$ and $\lambda_{23}(\alpha)^{G_3(q)} = \lambda_{23}(\alpha)$ respectively. Thus we have $2q-2$ basic characters of degree 1. For the basic subset D_5 we have $\xi_{D_5}(\varphi) = \lambda_{12}(\alpha)^{G_3(q)} \times \lambda_{23}(\alpha)^{G_3(q)}$. This gives us q^2-2q+1 basic characters of degree 1. The basic subset D_1 contributes 1 irreducible character that is χ_1. We then have $(q^2-2q+1)+(2q-2)+1 = q^2$, therefore accounting for the q^2 linear characters of $G_3(q)$. □

5. The group $G_4(2)$

We determine the character table of $G_4(2)$, its elementary and basic characters.

5.1 The character table of $G_4(2)$

Applying coset analysis we obtain the following conjugacy classes of $G_4(2)$ listed in Table 9.

classes of G	classes of $G_4(2)$	$\lvert C_{G_4}(g)\rvert$
$(1a)$	$(1A)$	2^6
	$(2A)$	2^6
	$(2B)$	2^5
	$(2C)$	2^4
$(2a)$	$(2D)$	2^5
	$(2E)$	2^5
	$(4A)$	2^4
$(2b)$	$(2F)$	2^4
	$(2G)$	2^4
	$(4B)$	2^3
$(2c)$	$(2H)$	2^4
	$(2I)$	2^4
	$(4C)$	2^4
	$(4D)$	2^4
$(4a)$	$(4E)$	2^3
	$(4F)$	2^3

Table 9: The Conjugacy Classes of $G_4(2)$

We now calculate the Fischer matrices. Since G has four orbits on the conjugacy classes of N, it also has four orbits on $\mathrm{Irr}(N)$. We check the lengths of these orbits on $\mathrm{Irr}(N)$. Note that the trivial character is fixed. We now have the lengths $w+u+s=7$, thus from the maximal subgroups of G, we get that $w=1$, $u=2$ and $s=4$. In this case, the lengths are the same as those on the conjugacy classes of G. Note that $N \cong V_3(2)$, the vector space of dimension

3 over 2 elements. We know that $\mathrm{Irr}(2^3)$ is the dual of $V_3(2)$ denoted by V^*. In this case $V_3(2) \cong V^*$ is a G-module. Hence the inertia factor groups are $H_1 = G = H_2$, $H_3 = \mathbb{Z}_2$, and for H_4 we note that there are two subgroups, V_4 and $\mathbb{Z}_4$ of G of index 2. Hence we ought to determine which one stabilizes the representative from the orbit with 2 elements.

By writing, $G =< a, b >= \{1, a, a^2, a^3, b, ab, a^2b, a^3b\} \cong D_8$, we have $(V_4)_1 = \{1, a^2, b, a^2b\}$ and $(V_4)_2 = \{1, a^2, ab, a^3b\}$. We identify H_4 to be $(V_4)_1$, that is the point stabilizer of the orbit with 2 elements. The character table of H_4 and the fusion of H_4 into G are given in Table 10 and Table 11 respectively.

$[g]$	1	a^2	b	a^2b
$C_G(g)$	4	4	4	4
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Table 10: The Character Table of H_4

classes of H_4	classes of $G_3(2)$
1	$1a$
a^2	$2a$
b	$2c$
a^2b	$2c$

Table 11: The Fusion of the Classes from H_4 to G

The character table of $H_1 = G = G_3(2) \cong D_8$ is given in Table 12 following below.

$[cl(g)]$	$1a$	$2a$	$2b$	$2c$	$4a$
$C_G(g)$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 12: The Character Table of H_1

Note that $H_3 =< x >\cong \mathbb{Z}_2$, where x comes from the $2b$ class of $G \cong D_8$.

In the following, we discuss the calculation of Fischer matrices. The Fischer matrix $M(1a)$ has the form:

$$M(1a) = \begin{array}{c} \\ |C_{H_1}(1a)|\ 8 \\ |C_{H_2}(1a)|\ 8 \\ |C_{H_3}(1a)|\ 2 \\ |C_{H_4}(1a)|\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\bar{G}}(1A)| & |C_{\bar{G}}(2A)| & |C_{\bar{G}}(2B)| & |C_{\bar{G}}(2C)| \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & a & d & g \\ 4 & b & e & h \\ 2 & c & f & i \end{array} \right) \end{array},$$

We use the column orthogonalities to obtain the values of a, b and c. We have $8 + 8|a|^2 + 2|b|^2 + 4|c|^2 = 64$ so that $4|a|^2 + |b|^2 + 2|c|^2 = 28$. Similarly $8 + 8a + 8b + 8c = 0$, we obtain $a = 1$, $b = -4$ and $c = 2$. we can compute the other unknown entries of $M(1a)$ in the same way and we obtain $d = 1$, $e = 0$, $f = -2$, $g = -1$, $h = 0$ and $i = 0$.

By similar computations, we obtain

$$M(2a) = \begin{array}{c} \\ |C_{H_1}(2a)|\ 8 \\ |C_{H_2}(2a)|\ 8 \\ |C_{H_4}(2a)|\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}}(2D)| & |C_{\bar{G}}(2E)| & |C_{\bar{G}}(4A)| \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{array} \right) \end{array},$$

$$M(2b) = \begin{array}{c} \\ |C_{H_1}(2b)|\ 4 \\ |C_{H_2}(2b)|\ 4 \\ |C_{H_3}(2b)|\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}}(2F)| & |C_{\bar{G}}(2G)| & |C_{\bar{G}}(4B)| \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{array} \right) \end{array},$$

$$M(2c) = \begin{array}{c} \\ |C_{H_1}(2c)|\ 4 \\ |C_{H_2}(2c)|\ 4 \\ |C_{H_4}(2c)|\ 4 \\ |C_{H_4}(2c)|\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\bar{G}}(2H)| & |C_{\bar{G}}(2I)| & |C_{\bar{G}}(4C)| & |C_{\bar{G}}(4D)| \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right) \end{array},$$

$$M(4a) = \begin{array}{c} \\ |C_{H_1}(4a)|\ 4 \\ |C_{H_2}(4a)|\ 4 \end{array} \begin{array}{c} \begin{array}{cc} |C_{\bar{G}}(4E)| & |C_{\bar{G}}(4F)| \end{array} \\ \left(\begin{array}{cc} 1 & 1 \\ 1 & \text{-1} \end{array} \right) \end{array}.$$

To obtain the character table of $G_4(2)$ we multiply the appropriate partial character tables of the inertia factor groups by the appropriate rows of the Fischer matrices. Thus for the classes $(1A)$, $(2A)$, $(2B)$ and $(2C)$ of $\bar{G} = G_4(2)$, by using rows of $M(1a)$ and first columns of the character tables of inertia factor groups, we have

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\2 & 2 & 2 & -2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}4 & -4 & 0 & 0\end{bmatrix}=\begin{bmatrix}4 & -4 & 0 & 0\\4 & -4 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}2 & 2 & -2 & 0\end{bmatrix}=\begin{bmatrix}2 & 2 & -2 & 0\\2 & 2 & -2 & 0\\2 & 2 & -2 & 0\\2 & 2 & -2 & 0\end{bmatrix}.$$

Similarly for the classes $2D$, $2E$ and $4A$, we have

$$\begin{bmatrix}1\\1\\1\\1\\-2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\\-2 & -2 & -2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\\-2\end{bmatrix}\begin{bmatrix}1 & 1 & -1\end{bmatrix}=\begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\\1 & 1 & -1\\1 & 1 & -1\\-2 & -2 & 2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}2 & -2 & 0\end{bmatrix}=\begin{bmatrix}2 & -2 & 0\\2 & -2 & 0\\-2 & 2 & 0\\-2 & 2 & 0\end{bmatrix}.$$

Thus continuing in this manner, we obtain the full character table of $G_4(2)$ as shown in Table 13.

5.2 The Elementary characters of $G_4(2)$

As in section 3.1, we take $\mathbb{F}_2 = \{0, 1\}$. Then $\alpha = 1$ and $\psi_o = \chi_2$, the non-trivial irreducible character of $\mathbb{F}_2$. We have that

$$\Phi(4) = \{(i, j) | 1 \le i < j \le 4\} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_{34} = G_4(2)$ and

$$G_{13} = \left\{ \begin{pmatrix}1 & 0 & b & c\\0 & 1 & d & e\\0 & 0 & 1 & f\\0 & 0 & 0 & 1\end{pmatrix} \mid b, c, d, e, f \in \mathbb{F}_2 \right\}. \tag{2}$$

$[g]$	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$4A$	$2F$	$2G$	$4B$	$2H$	$2I$	$4C$	$4D$	$4E$	$4F$
$\lvert C_G(g)\rvert$	2^6	2^6	2^5	2^4	2^5	2^5	2^4	2^4	2^4	2^3	2^4	2^4	2^4	2^4	2^3	2^3
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	−1	−1
χ_4	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1
χ_5	2	2	2	2	−2	−2	−2	0	0	0	0	0	0	0	0	0
χ_6	1	1	1	−1	1	1	−1	1	1	−1	1	−1	1	−1	1	−1
χ_7	1	1	1	−1	1	1	−1	1	1	−1	−1	1	−1	1	−1	1
χ_8	1	1	1	−1	1	1	−1	−1	−1	1	1	−1	1	−1	−1	1
χ_9	1	1	1	−1	1	1	−1	−1	−1	1	−1	1	−1	1	1	−1
χ_{10}	2	2	2	−2	−2	−2	2	0	0	0	0	0	0	0	0	0
χ_{11}	4	−4	0	0	0	0	0	2	−2	0	0	0	0	0	0	0
χ_{12}	4	−4	0	0	0	0	0	−2	2	0	0	0	0	0	0	0
χ_{13}	2	2	−2	0	2	−2	0	0	0	0	2	0	−2	0	0	0
χ_{14}	2	2	−2	0	2	−2	0	0	0	0	−2	0	2	0	0	0
χ_{15}	2	2	−2	0	−2	2	0	0	0	0	0	2	0	−2	0	0
χ_{16}	2	2	−2	0	−2	2	0	0	0	0	0	−2	0	2	0	0

Table 13: The Character Table of $\bar{G} = G_4(2)$

Using GAP [5], this group is isomorphic to a split extension of the form $2^4{:}2$ and using the IdSmallGroup(G_{13}) function, this is the group number 27 on the GAP list for groups of order 32.

$$G_{14} = \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid c, d, e, f \in \mathbb{F}_2 \right\} \tag{3}$$

which is the group $\mathbb{Z}_2 \times D_8$ and

$$G_{24} = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, e \in \mathbb{F}_2 \right\}, \tag{4}$$

this is of the form $(\mathbb{Z}_2 \times D_8){:}\mathbb{Z}_2$, it is the group number 49 on the GAP list for groups of order 32.

We get

$$(q-1)(\frac{n(n-1)}{2}) = (2-1)(\frac{4(4-1)}{2}) = 6$$

elementary characters. Three of these are the elementary characters $\lambda_{12}(1)(x)^{G_4(2)}$, $\lambda_{23}(1)(x)^{G_4(2)}$, $\lambda_{34}(1)(x)^{G_4(2)}$ all of degree 1. The character table and structure of the conjugacy class representatives of $G_4(2)$ allow us to identify these three elementary characters as $\lambda_{12}(1)(x)^{G_4(2)} = \chi_6$, $\lambda_{23}(1)(x)^{G_4(2)} = \chi_3$ and $\lambda_{34}(1)(x)^{G_4(2)} = \chi_2$. We now identify the last three elementary characters $\lambda_{13}(1)(x)^{G_4(2)}$, $\lambda_{14}(1)(x)^{G_4(2)}$ and $\lambda_{24}(1)(x)^{G_4(2)}$.

We consider G_{13}. The structure of G_{13} is known from the relation 2. We use GAP [5] to construct G_{13} as a subgroup of $G_4(2)$. We have the following conjugacy classes of G_{13} computed using GAP [5] in Table 14.

classes of $G_{13}(2)$	$1a$	$2a$	$2b$	$2c$	$4a$	$2d$	$2e$	$2f$	$2g$	$4b$	$2h$	$2i$	$4c$	$2j$
$C_{G_{13}}(x)$	32	8	32	16	8	32	32	16	16	8	16	16	8	16
$[cl(x)]$	1	4	1	2	4	1	1	2	2	4	2	2	4	2

Table 14: The Conjugacy Classes of G_{13}

Using GAP [5], we get the fusions of the conjugacy classes of G_{13} to the conjugacy classes of $G_4(2)$ as in Table 15 below.

classes of G_{13}	classes of $G_4(2)$	$\lambda_{13}^{G_4(2)}(y)$
$1a$	$1A$	2
$2a$	$2H$	2
$2b$ $2e$	$2D$	2
$2c$ $2i$	$2F$	0
$2d$	$2A$	2
$2f$ $2j$	$2G$	0
$2g$	$2B$	-2
$2h$	$2E$	-2
$4a$ $4c$	$4E$	0
$4b$	$4C$	-2

Table 15: The Fusion of Classes of G_{13} to $G_4(2)$

With the information about fusions as in Table 15 above, we calculate the permutation character $(1_{G_{13}})^{G_4(2)} = \chi_1+\chi_6$. Since $\lambda_{13}(1)(x) = 1$ or $\lambda_{13}(1)(x) = -1$ according as $x_{13} = 0$ or $x_{13} = 1$, we can induce the character $\lambda_{13}(1)(x)$ by the induction formula to obtain the values as in Table 15 above on the classes of $G_4(2)$. Note that for any $c \in G_4(2)$ not listed in Table 15, we have that $\lambda_{13}(1)^{G_4(2)}(c) = 0$. Using the character table of $G_4(2)$, we then identify $\lambda_{13}(1)^{G_4(2)} = \chi_{13}$. The same analysis applies in identifying $\lambda_{14}(1)^{G_4(2)} = \chi_{11}$ and $\lambda_{24}(1)^{G_4(2)} = \chi_5$. Therefore the six elementary characters of $G_4(2)$ are $\lambda_{12}(1)(x)^{G_4(2)} = \chi_6$, $\lambda_{23}(1)(x)^{G_4(2)} = \chi_3$, $\lambda_{34}(1)(x)^{G_4(2)} = \chi_2$, $\lambda_{13}(1)^{G_4(2)} = \chi_{13}$, $\lambda_{14}(1)^{G_4(2)} = \chi_{11}$ and $\lambda_{24}(1)^{G_4(2)} = \chi_5$.

5.3 Basic characters of $G_4(2)$

To calculate the basic characters of $G_4(2)$, we first identify the subsets of $\Phi(4)$ which are basic. Since

$$\Phi(4) = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\},$$

there are 64 subsets of $\Phi(4)$. The following subsets are basic $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(1,4)\}$, $D_5 = \{(2,3)\}$, $D_6 = \{(2,4)\}$, $D_7 = \{(3,4)\}$. In addition, we have 7 basic 2-subsets of $\Phi(4)$, namely $D_8 = \{(1,2), (2,3)\}$, $D_9 = \{(1,2),(2,4)\}$, $D_{10} = \{(1,2),(3,4)\}$, $D_{11} = \{(1,3),(2,4)\}$, $D_{12} = \{(1,3),(3,4)\}$, $D_{13} = \{(1,4),(2,3)\}$, $D_{14} = \{(2,3),(3,4)\}$ and only one 3-subset, namely $D_{15} = \{(1,2),(2,3),(3,4)\}$. Note that none of the 4-subsets and 5-subsets is basic.

By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_4(2)$. In this way, we have the basic characters of $G_4(2)$, namely

$$\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_4} = \chi_6,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_4} = \chi_{13},\\
\xi_{D_4}(\varphi) &= \xi_{14}(\varphi(1,4)) = \xi_{14}(1) = \lambda_{14}(1)^{G_4} = \chi_{11},\\
\xi_{D_5}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_4} = \chi_3,\\
\xi_{D_6}(\varphi) &= \xi_{24}(\varphi(2,4)) = \xi_{24}(1) = \lambda_{24}(1)^{G_4} = \chi_5,\\
\xi_{D_7}(\varphi) &= \xi_{12}(\varphi(3,4)) = \xi_{34}(1) = \lambda_{34}(1)^{G_4} = \chi_2.\\
\xi_{D_8}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \lambda_{12}(1)^{G_4} \times \lambda_{23}(1)^{G_4} = \chi_6 \times \chi_3,\\
\xi_{D_9}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{24}(\phi(2,4)) = \lambda_{12}(1)^{G_4} \times \lambda_{24}(1)^{G_4} = \chi_6 \times \chi_5,\\
\xi_{D_{10}}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{34}(\varphi(3,4)) = \lambda_{12}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_6 \times \chi_2,\\
\xi_{D_{11}}(\varphi) &= \xi_{13}(\varphi(1,3)) \times \xi_{24}(\varphi(2,4)) = \lambda_{13}(1)^{G_4} \times \lambda_{24}(1)^{G_4} = \chi_{13} \times \chi_5,\\
\xi_{D_{12}}(\varphi) &= \xi_{13}(\varphi(1,3)) \times \xi_{34}(\phi(3,4)) = \lambda_{13}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_{13} \times \chi_2,\\
\xi_{D_{13}}(\varphi) &= \xi_{14}(\varphi(1,4)) \times \xi_{23}(\varphi(2,3)) = \lambda_{14}(1)^{G_4} \times \lambda_{23}(1)^{G_4} = \chi_{11} \times \chi_3,\\
\xi_{D_{14}}(\varphi) &= \xi_{23}(\varphi(2,3)) \times \xi_{34}(\varphi(3,4)) = \lambda_{23}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_3 \times \chi_2,\\
\xi_{D_{15}}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) \times \xi_{34}(\varphi 1`(3,4))\\
&= \lambda_{12}(1)^{G_4} \times \lambda_{23}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_6 \times \chi_3 \times \chi_2.
\end{aligned}$$

Table 16 below gives a summary of the basic characters of $G_4(2)$ decomposed in terms of its irreducible characters. We observe from Table 16 that except $\xi_{D_{11}}(\phi)$, all the other basic characters are its irreducible characters.

basic characters	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
$\xi_{\emptyset}(\varphi)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_2}(\varphi)$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\xi_{D_3}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$\xi_{D_4}(\varphi)$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\xi_{D_5}(\varphi)$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_6}(\varphi)$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_7}(\varphi)$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_8}(\varphi)$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\xi_{D_9}(\varphi)$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\xi_{D_{10}}(\varphi)$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\xi_{D_{11}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$\xi_{D_{12}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\xi_{D_{13}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\xi_{D_{14}}(\varphi)$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_{15}}(\varphi)$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

Table 16: The Constituents of The Basic Characters of $G_4(2)$

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THE CONDITIONS OF EXISTENCE OF A SOLUTION OF THE TWO-POINT IN TIME PROBLEM FOR NONHOMOGENEOUS PDE

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Abstract. The existence of a solution of the problem with local homogeneous two-point in time conditions for nonhomogeneous PDE of the second order in time and generally infinite order in spatial variables was investigated in the classes of entire functions. The case when the characteristic determinant of the problem is identically zero was studied. We proposed the differential-symbol method of constructing the solution of the problem.
Keywords: characteristic determinant of a problem, two-point local conditions, differential-symbol method.

1. Introduction

The problems with n-point in time conditions ($n \in \mathbb{N}\backslash\{1\}$) for PDE are generalization of multipoint problems for ODE which are known in the literature as the Vallee-Poussin problems [1]. The multipoint problems for PDE are ill-posed and their solvability is connected with problem of small denominators (below estimation of the so-called characteristic determinant). Papers (see [2, 3, 4] and bibliography in them) are devoted to research of the multipoint in time problems in the bounded domains based on the metric approach.

The spaces of functions which allow exponential growth as the classes of unique solvability of the multipoint problem for PDE in unbounded layer are studied in the papers [5, 6, 7].

The differential-symbol method of solving the problem with initial and two-point in time conditions for PDE is proposed in the works [8, 9, 10]. In these papers, classes of entire functions and classes of quasipolynomials as the classes of unique solvability of the problems are dedicated.

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The work [11], in particular, is devoted to constructing the polynomial solutions of the system of PDE with constant coefficients.

This paper is the continuation of researches [12, 13, 14, 15]. It is devoted to research of existence of solutions of the problem with local homogeneous two-point in time conditions for nonhomogeneous PDE of the second order with respect to time variables when the characteristic determinant is identically zero.

2. Problem statement

In the domain $(t,x) \in \mathbb{R}^{1+s}$, $x=(x_1,\dots,x_s)$, $s \in \mathbb{N}$, we investigate a solvability of the problem

$$L\Big(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)U(t,x) \equiv \frac{\partial^2 U}{\partial t^2} + 2\,a\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t} + b\Big(\frac{\partial}{\partial x}\Big)U = f(t,x), \tag{2.1}$$

$$\begin{aligned} &l_{0\partial}U(t,x) \equiv A_1\Big(\frac{\partial}{\partial x}\Big)U(0,x) + A_2\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t}(0,x) = 0,\\ &l_{1\partial}U(t,x) \equiv B_1\Big(\frac{\partial}{\partial x}\Big)U(h,x) + B_2\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t}(h,x) = 0, \quad h>0. \end{aligned} \tag{2.2}$$

In equation (2.1) $f(t,x)$ is given nonzero function, $a\big(\frac{\partial}{\partial x}\big)$ and $b\big(\frac{\partial}{\partial x}\big)$ are the following differential expressions

$$a\Big(\frac{\partial}{\partial x}\Big) = \sum_{|k|=0}^{\infty} a_k \frac{\partial^k}{\partial x^k}, \quad b\Big(\frac{\partial}{\partial x}\Big) = \sum_{|k|=0}^{\infty} b_k \frac{\partial^k}{\partial x^k},$$

where $a_k, b_k \in \mathbb{C}$, $k=(k_1,\dots,k_s) \in \mathbb{Z}_+^s$, $|k| = k_1+\dots+k_s$, $\frac{\partial^k}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1}\dots\partial x_s^{k_s}}$, moreover their symbols $a(\nu)$ and $b(\nu)$ are entire functions (in particular, they can be polynomials), $\nu=(\nu_1,\dots,\nu_s) \in \mathbb{C}^s$.

The differential polynomials with complex coefficients $A_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$ and $B_2\left(\frac{\partial}{\partial x}\right)$ in local two-point conditions (2.2) are presented, moreover the corresponding symbols $A_1(\nu)$, $A_2(\nu)$, $B_1(\nu)$ and $B_2(\nu)$ for each $\nu \in \mathbb{C}^s$ satisfy the inequality

$$\big(|A_1(\nu)|^2 + |A_2(\nu)|^2\big)\big(|B_1(\nu)|^2 + |B_2(\nu)|^2\big) \neq 0.$$

The solution of problem (2.1), (2.2) is understood as entire function of the following form

$$U(t,x) = \sum_{\widetilde{k}\in\mathbb{Z}_+^{1+s}} u_{\widetilde{k}}\, t^{k_0} x^k, \quad \widetilde{k} = (k_0,k), \quad u_{\widetilde{k}} \in \mathbb{C},$$

of variables t and x, which satisfy equation (2.1) in $\mathbb{R}^{1+s}$ and conditions (2.2) in $\mathbb{R}^s$.

For ODE

$$L\Big(\frac{d}{dt},\nu\Big)T(t,\nu)=0,\quad \nu\in\mathbb{C}^s,$$

we consider the fundamental system of solutions $\big\{T_0\,(t,\nu)\,,\,T_1\,(t,\nu)\,\big\}$ normal at the point $t=0$ and write the determinant:

$$\Delta\,(\nu)=\begin{vmatrix} l_{0\nu}T_0(t,\nu) & l_{0\nu}T_1(t,\nu) \\ l_{1\nu}T_0(t,\nu) & l_{1\nu}T_1(t,\nu) \end{vmatrix}, \tag{2.3}$$

where

$$l_{0\nu}T_0(t,\nu)\equiv A_1(\nu),\qquad l_{0\nu}T_1(t,\nu)\equiv A_2(\nu),$$

$$l_{1\nu}T_j(t,\nu)\equiv B_1\,(\nu)\,T_j(h,\nu)+B_2\,(\nu)\,\frac{dT_j}{dt}(h,\nu),\quad j\in\{0,1\}.$$

The determinant $\Delta\,(\nu)$ is the characteristic determinant of problem (2.1), (2.2).

Let's establish the solvability of problem (2.1), (2.2) in the class of entire functions when characteristic determinant (2.3) of the problem is identically zero.

3. The conditions of existence of solution of the problem

Since $a(\nu)$ and $b(\nu)$ are entire functions then [16] the functions $T_0(t,\nu)$ and $T_1(t,\nu)$ are entire functions in vector-parameter $\nu\in\mathbb{C}^s$ for all $t\in\mathbb{R}$. So the function $\Delta(\nu)$ (as superposition of entire functions) is entire function too.

Let's consider the function

$$\Phi\,(t,\lambda,\nu)=\frac{e^{\lambda t}-T_0\,(t,\nu)-\lambda T_1\,(t,\nu)}{L\,(\lambda,\nu)}, \tag{3.1}$$

which is the solution of Cauchy problem

$$L\left(\frac{d}{dt},\nu\right)\Phi=e^{\lambda t},\quad \Phi\,(0,\lambda,\nu)=0,\quad \left.\frac{\partial\Phi\,(t,\lambda,\nu)}{\partial t}\right|_{t=0}=0.$$

The function (3.1) is the quasipolynomial of variable t, besides $\Phi\,(t,\lambda,\nu)$ is entire function of the first order in parameter λ and entire function of order $\bar p$ in the set of parameters $\nu_1,\ldots,\nu_s$. Here $\bar p=\max\{p_a,\ p_b/2\}$, where p_a and p_b are degrees of $a\,(\nu)$ and $b\,(\nu)$ accordingly if $a\,(\nu)$ and $b\,(\nu)$ are polynomials, and $\bar p=\infty$ if $a\,(\nu)$ or $b\,(\nu)$ is not polynomial.

We introduce some classes of entire functions. These classes depend on value p where $p=\max\{\bar p,\,1\}\in[1;+\infty]$.

$A_{p'}$ is the class of entire functions $\varphi(x)$ the order of which is less than p', where $1/p+1/p'=1$, if $1<p<+\infty$;

$A_{p'}=A_1$ is the class of entire functions $\varphi(x)$ of exponential type if $p=\infty$;

$A_{p'}=A_\infty$ is the class of entire functions $\varphi(x)$ if $p=1$.

By $\mathbb{A}_{p'}$, denote the class of entire functions $U(t,x)$ which for each fixed $t \in \mathbb{R}$ belong to $A_{p'}$.

Let's show that in the case $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ the solution of problem (2.1), (2.2) exists under some conditions on the function $f(t,x)$ and it can be found by the formula

$$U(t,x) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O}, \tag{3.2}$$

in which $\nu\cdot x = \nu_1 x_1 + \ldots + \nu_s x_s$.

We consider the function $\Phi_1(\lambda,\nu) = l_{1\nu}\Phi(t,\lambda,\nu)$, where

$$l_{1\nu}\Phi(t,\lambda,\nu) \equiv B_1(\nu)\Phi(h,\lambda,\nu) + B_2(\nu)\frac{\partial\Phi}{\partial t}(h,\lambda,\nu).$$

Theorem 3.1. *Let for two-point problem* (2.1), (2.2), *in which* $\Delta(\nu) \equiv 0$ *in* $\mathbb{C}^s$ *the following condition is satisfied:*

for all $x \in \mathbb{R}^s$ *and* $f \in \mathbb{A}_{p'}$ *the identity*

$$f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O} \equiv 0 \tag{3.3}$$

is fulfilled.

Then the solution of problem (2.1), (2.2) *in the class* $\mathbb{A}_{p'}$ *exists and it can be obtained by formula* (3.2).

Proof. First we note that the result of action of the differential expression $f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ onto the function $e^{\nu\cdot x}\Phi(t,\lambda,\nu)$ in formula (3.2) is entire function of the first order in λ and of the order p in the set of variables $\nu_1,\ldots,\nu_s$.

Further we define the differential expression $f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ for entire function $f(t,x)$ of the class $\mathbb{A}_{\bar{p}'}$ by differential expression of infinite order by replacing in the Maclaurin expansion of the function $f(t,x)$ the variables t and the vector-parameter x by $\frac{\partial}{\partial\lambda}$ and $\frac{\partial}{\partial\nu}$ accordingly. Then the expression in the right side of formula (3.2) is the series that defines after setting $\lambda = 0$ and $\nu = O$ entire function $U(t,x)$ which belongs to the class $A_{p'}$ for each fixed t [17], i. e. $U(t,x) \in \mathbb{A}_{p'}$.

Let's prove that the function (3.2) satisfy the equation (2.1):

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)U(t,x) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}L\left(\frac{d}{dt},\nu\right)\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O}$$
$$= f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\lambda t+\nu\cdot x}\right\}\Big|_{\lambda=0,\,\nu=O} = f(t,x).$$

In addition, from the condition $\Phi(0,\lambda,\nu)=\dfrac{\partial\Phi}{\partial t}(0,\lambda,\nu)=0$ we get

$$l_{0\partial}U(t,x)=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(A_1\left(\frac{\partial}{\partial x}\right)\{\Phi(0,\lambda,\nu)\,e^{\nu\cdot x}\}\right)\Big|_{\lambda=0,\nu=O}$$
$$+f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(A_2\left(\frac{\partial}{\partial x}\right)\left\{\frac{\partial\Phi}{\partial t}(0,\lambda,\nu)\,e^{\nu\cdot x}\right\}\right)\Big|_{\lambda=0,\nu=O}\equiv 0.$$

Since the identity (3.3) is fulfilled, we show that function (3.2) satisfy the second condition in (2.2):

$$\begin{aligned}
l_{1\partial}U(t,x)&=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_1\Big(\frac{\partial}{\partial x}\Big)\Big\{\Phi(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O}\\
&\quad+f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_2\Big(\frac{\partial}{\partial x}\Big)\Big\{\frac{\partial\Phi}{\partial t}(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O}\\
&=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_1(\nu)\Big\{\Phi(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O}\\
&\quad+f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_2(\nu)\Big\{\frac{\partial\Phi}{\partial t}(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O}\\
&=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\Big\}\Big|_{\lambda=0,\,\nu=O}\equiv 0.
\end{aligned}$$

The theorem is proved. □

Remark 3.2. Solution (3.2) of problem (2.1), (2.2) in the class $\mathbb{A}_{p'}$ is nonunique, because null-space of the problem in the same class is nontrivial [12].

4. Examples

Let's establish the conditions of solvability of two-point problem (2.1), (2.2) for the specific examples.

Example 4.1. In the domain $(t,x)\in\mathbb{R}^2$ we investigate the problem of finding the solutions of the equation

$$\left[\frac{\partial^2}{\partial t^2}+2\frac{\partial^2}{\partial t\partial x}+1+\frac{\partial^2}{\partial x^2}\right]U(t,x)=f(t,x),\tag{4.1}$$

that satisfy local two-point conditions

$$\frac{\partial U}{\partial x}(0,x)+\frac{\partial U}{\partial t}(0,x)=0,\quad \frac{\partial U}{\partial x}(\pi,x)+\frac{\partial U}{\partial t}(\pi,x)=0.\tag{4.2}$$

♦ This problem is the problem (2.1), (2.2), in which $a(\nu)=\nu$, $b(\nu)=1+\nu^2$, $h=\pi$, $A_1(\nu)=B_1(\nu)=\nu$, $A_2(\nu)=B_2(\nu)=1$, $\bar{p}=p=1$.

The fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2}+2\nu\frac{d}{dt}+1+\nu^2\right]T\left(t,\nu\right)=0$$

normal at the point $t=0$ has the form

$$T_0\left(t,\nu\right)=e^{-\nu t}\left[\nu\sin t+\cos t\right],\quad T_1\left(t,\nu\right)=e^{-\nu t}\sin t.$$

The characteristic determinant of problem (4.1), (4.2) yields

$$\Delta\left(\nu\right)=\begin{vmatrix}\nu & 1\\ -\nu e^{-\pi\nu} & -e^{-\pi\nu}\end{vmatrix}\equiv 0.$$

The condition of existence (3.3) of solution of problem (4.1), (4.2) according to Theorem 3.1 is following:

for $f\in\mathbb{A}_\infty$ such identity

$$f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{e^{-\nu\pi}\left(\lambda+\nu\right)\frac{e^{(\lambda+\nu)\pi}+1}{\left(\lambda+\nu\right)^2+1}e^{\nu x}\right\}\Bigg|_{\lambda=\nu=0}\equiv 0$$

holds in $\mathbb{R}$.

For example, this identity is satisfied for the function of form $f\left(t,x\right)=e^{x-t}$. The solution of problem (4.1), (4.2) for this function can be found by formula (3.2):

$$U\left(t,x\right)=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{e^{\nu x}\Phi\left(t,\lambda,\nu\right)\right\}\Big|_{\lambda=\nu=0}=\left\{e^{\nu x}\Phi\left(t,\lambda,\nu\right)\right\}\Big|_{\lambda=-1,\,\nu=1}$$
$$=\frac{e^{-t}-T_0\left(t,1\right)+T_1\left(t,1\right)}{L\left(-1,1\right)}e^{x}=e^{x-t}-e^{x-t}\cos t.$$

Note that obtained solution of problem (4.1), (4.2) is nonunique. For example, the solution of problem (4.1), (4.2) is function of the form $U\left(t,x\right)=e^{x-t}$, and it is also the sum of this function with arbitrary elements of null-space of the problem. Let's note that elements of the null-space of problem (4.1), (4.2) have the form

$$U\left(t,x\right)=\varphi\left(x-t\right)\cos t,$$

where φ is arbitrary twice continuously differentiable function in $\mathbb{R}$. □

Example 4.2. Let's investigate the existence conditions of the solution of the two-point problem in domain $t\in\mathbb{R}$, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ for nonhomogeneous differential-functional equation

$$\frac{\partial^2}{\partial t^2}U\left(t,x\right)+2\frac{\partial}{\partial t}U\left(t,x+\omega\right)+2U\left(t,x+\omega\right)-U\left(t,x\right)=f\left(t,x\right)\tag{4.3}$$

with homogeneous local conditions

$$U(0,x)+\frac{\partial U}{\partial t}(0,x)=0,\ \ U(1,x)+\frac{\partial U}{\partial t}(1,x)=0,\tag{4.4}$$

where $\omega=(1,1,-1)$ is the displacement vector in spatial coordinates.

♦ Differential-functional equation (4.3) we can write as the differential equation of infinite order

$$\left[\frac{\partial^2}{\partial t^2}+2\,e^{\omega\cdot\frac{\partial}{\partial x}}\frac{\partial}{\partial t}+2\,e^{\omega\cdot\frac{\partial}{\partial x}}-1\right]U\,(t,x)=f(t,x).$$

For this problem, we have $a\,(\nu)=e^{\omega\cdot\nu}$, $b\,(\nu)=2\,e^{\omega\cdot\nu}-1$, $\nu=(\nu_1,\nu_2,\nu_3)$, $A_1\,(\nu)=A_2\,(\nu)=B_1\,(\nu)=B_2\,(\nu)=1$, $s=3$, $h=1$, $\bar{p}=p=\infty$.

The fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2}+2\,e^{\omega\cdot\nu}\frac{d}{dt}+2\,e^{\omega\cdot\nu}-1\right]T\,(t,\nu)=0$$

normal at the point $t=0$ has the form

$$\begin{aligned}&T_0(t,\nu)=e^{-te^{\omega\cdot\nu}}\left\{e^{\omega\cdot\nu}\frac{\sinh\left[t(e^{\omega\cdot\nu}-1)\right]}{e^{\omega\cdot\nu}-1}+\cosh\left[t(e^{\omega\cdot\nu}-1)\right]\right\},\\&T_1(t,\nu)=e^{-te^{\omega\cdot\nu}}\frac{\sinh\left[t(e^{\omega\cdot\nu}-1)\right]}{e^{\omega\cdot\nu}-1}\end{aligned}\tag{4.5}$$

(in particular, if $e^{\omega\cdot\nu}=1$ we obtain $T_0(t,\nu)=e^{-t}(t+1),\ T_1(t,\nu)=te^{-t}$).

For problem (4.3), (4.4), we have:

$$\Delta\,(\nu)=\begin{vmatrix}1 & 1\\ e^{-e^{\omega\cdot\nu}}\dfrac{\sinh\left[e^{\omega\cdot\nu}-1\right]}{e^{\omega\cdot\nu}-1} & e^{-e^{\omega\cdot\nu}}\dfrac{\sinh\left[e^{\omega\cdot\nu}-1\right]}{e^{\omega\cdot\nu}-1}\end{vmatrix}\equiv 0.$$

The condition of existence of solutions of problem (4.3), (4.4) according to theorem 3.1 is following:

for $f\in\mathbb{A}_1$ such identity

$$f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{\frac{e^{\lambda}-e^{-2e^{\omega\cdot\nu}+1}}{\lambda-1+2e^{\omega\cdot\nu}}e^{\nu\cdot x}\right\}\Big|_{\lambda=0,\,\nu=O}\equiv 0\tag{4.6}$$

holds in $\mathbb{R}^3$.

Condition (4.6) is satisfied, in particular, if the right-hand side of equation (4.3) has the form:

$$f(t,x)=\cos[2\pi t]e^{-t+x_2+x_3}.$$

Really,

$$\begin{aligned}&\cos\left[2\pi\frac{\partial}{\partial\lambda}\right]e^{-\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\nu_2}+\frac{\partial}{\partial\nu_3}}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O}\\&=\frac{1}{2}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=2\pi i-1,\,\nu=(0,1,1)}+\frac{1}{2}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=-2\pi i-1,\,\nu=(0,1,1)}\\&=\frac{1}{2}e^{x_2+x_3}\left\{\frac{e^{\lambda}-e^{-1}}{\lambda+1}\right\}\Big|_{\lambda=2\pi i-1}+\frac{1}{2}e^{x_2+x_3}\left\{\frac{e^{\lambda}-e^{-1}}{\lambda+1}\right\}\Big|_{\lambda=-2\pi i-1}\equiv 0.\end{aligned}$$

So for function $f(t,x) = \cos[2\pi t]e^{-t+x_2+x_3}$ problem (4.3), (4.4) has solution in $\mathbb{A}_1$, which can be found by formula (3.2):

$$\begin{aligned} U(t,x) &= f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O} \\ &= \frac{1}{2}\Big\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\Big\}\Big|_{\lambda=2\pi i-1,\nu=(0,1,1)} + \frac{1}{2}\Big\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\Big\}\Big|_{\lambda=-2\pi i-1,\,\nu=(0,1,1)} \\ &= \frac{1}{4\pi^2}e^{-t+x_2+x_3}\big\{1-\cos[2\pi t]\big\}. \end{aligned}$$

Let's note that obtained solution of problem (4.3), (4.4) is only partial solution, because it is found to within elements of the null-space of the problem of form

$$U(t,x) = \varphi(x)\, e^{-t},$$

where φ is arbitrary continuously function in $\mathbb{R}^3$. □

5. Conclusions

We found the condition of existence of solution of the problem in the class of entire functions for nonhomogeneous PDE of second order with respect to time variable, in which homogeneous local two-point conditions are imposed, and infinite order with respect to spatial variables in the case when the characteristic determinant identically equals to zero. We showed examples for which the solutions of two-point problems exist. These solutions are constructed by using the differential-symbol method.

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MULTIGROUPS AND MULTICOSETS

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Abstract. The present Cantor's set theory has limitations. In various ways, it cannot well represent realities because an element x, in Cantor's sense, is either in or not in X. Even, when $x \in X$, it can only occur once; no repetition is allowed. But so many real life problems are only well represented by sets which allow repetition(s), such as multiset. Such cases arise in, though not limited to, database query, chemical structures and computer proramming.

In this paper, we have some results on the algebraic structure of multisets and some properties of their multicosets.

Keywords: multisets, multigroups, submultiset, multicoset.

1. Introduction

So many real life problems are only well represented by sets which allow repetition(s), such as multiset. Such cases occur very frequently in chemical sciences and computer programming.

The notion can be traced to Dedekind[4]. In the recent time, Nazmul *et al* has put algebraic group structure on multisets[8]. Related algebraic properties as in the classical group can now be studied.

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In this paper, we have some results on the algebraic structure of multisets and some characterisations.

2. Preliminaries

In this paper, we shall use X to denote a non-empty set.

Definition 2.1. A multiset M drawn from a set X is denoted by the count function $C_M : X \longrightarrow I\!N \cup \{0\}$ defined by $C_M(x) = n \in I\!N$, the (multiplicity) or number of occurrence of x in M, where $I\!N$ is the set of positive integers.

Example 2.2. Let set $X = \{1, 2, 3, 4\}$. Then $M = \{1, 1, 1, 2, 2, 3, 3, 3\}$ is a multiset over X with $C_M(1) = 3$, $C_M(2) = 2$, $C_M(3) = 3$ and $C_M(4) = 0$.

Definition 2.3. Let multisets A and B be drawn from X. A is said to be a submultiset of B and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x)$.

Definition 2.4. The root set or support of a multiset M, which is denoted by M^*, is the set which contains the distinct elements in the multiset. Hence, M^* is the set of $x \in M$ such that $C_M(x) > 0$.

Definition 2.5. Consider a multiset M over a set X.

(1) It is simple if the cardinality of its root is 1;

(2) It is regular if $C_M(x) = C_M(y),\ \forall x, y \in M$;

(3) The peak element $x \in M$ is such that $C_M(x) \geq C_M(y),\ \forall y \in M$;

(4) The intersection of two multisets A and B is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and their union is denoted by

$$C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\};$$

(5) A and B are equal if and only if $C_A(x) = C_B(x)$.

$[X]^\alpha$ is the set of all the multisets whose elements have the multiplicity of not more than α. $MS(X)$ is the set of all multisets over X. An empty multiset ϕ is such that $C_\phi(x) = 0,\ \forall x \in X$. Cardinality of a multiset M is denoted by $|M| = \sum C_M(x),\ \forall x \in M$.

Definition 2.6. Let X be a group and $e \in X$ its identity. Then, $\forall x, y \in X$, a multiset M drawn from X is called a multigroup if

(1) $C_M(xy) \geq C_M(x) \wedge C_M(y)$;

(2) $C_M(x^{-1}) \geq C_M(x)$.

The implication of Definition 2.6 is that $C_M(x) = C_M((x^{-1})^{-1}) \geq C_M(x^{-1})$. Thus, $C_M(x) = C_M(x^{-1})$. So, $C_M(e) \geq C_M(x) \wedge C_M(x^{-1}) = C_M(x)$. We shall call $MG(X)$ the set of all multigroups over X.

Example 2.7. Let $G = \{e, a, b, ab\}$ be Klein 4 group. Then

(1) $A = \{e, e, e, a, a\}$ is a multigroup;

(2) $B = \{e, e, e, a, a, b, b\}$ is not a multigroup since $0 = C_B(ab) \ngeq C_B(a) \wedge C_B(c) = 2$.

Definition 2.8. A multigroup M over X is called abelian if $C_M(xy) = C_M(yx)$, $\forall x, y \in X$.

Definition 2.9. Let $A, B \in MS(X)$.

(1) $A \circ B$ is a multiset associated with

$$C_{A\circ B}(x) = \bigvee\{C_A(y) \wedge C_B(z) : y, z \in X, x = yz\};$$

(2) A^{-1} is a multiset associated with

$$C_{A^{-1}}(x) = C_A(x^{-1}),\ \forall x \in X;$$

(3) $A_n = \{x : C_A(x) \geq n\}$;

(4) $[n]_x$ is a multiset containing only x in n times;

(5) The complement of the multiset $M \in [X]^\alpha$ denoted by M' is such that

$$C_{M'}(x) = \alpha - C_M(x);$$

(6) $nA = \{x^n : x \in A\}$, where n is the multiplicity of each element that appears from A.

Proposition 2.10 ([8])**.** *Let $A, B \in MS(X)$ and $m, n \in \mathbb{N}$.*

(1) *If $A \subseteq B$, then $A_n \subseteq B_n$;*

(2) *If $m \leq n$, then $A_m \supseteq A_n$;*

(3) *$(A \cap B)_n = A_n \cap B_n$;*

(4) *$(A \cup B)_n = A_n \cup B_n$;*

(5) *$A = B$ if and only if $A_n = B_n$, $\forall n \in \mathbb{N}$.*

Proposition 2.11 ([8])**.** *Let X be a group and $A \in MG(X)$. Then, A_n, for $n \in \mathbb{N}$, is a subgroup of X.*

Proposition 2.12 ([8])**.** *Let X be a group and $A \in MS(X)$. Then, $A \in MG(X)$ if and only if the following conditions are satisfied:*

(1) $A \circ A \subseteq A$ *and* $A^{-1} \subseteq A$ (*or* $A \subseteq A^{-1}$ *or* $A^{-1} = A$) *or*

(2) $A \circ A^{-1} \subseteq A$.

Remark 2.13. It has been established in [8] that the intersection of multigroups is again a multigroup. It was also illustrated that the union of multigroups needs not be a multigroup. It is later shown in this paper the condition necessary and sufficient for the union of multigroups to be a multigroup.

We shall state the following proposition by Nazmul et al in [8] and make some comments on its inadequacy. Later in this work, we shall state the correct form.

Proposition 2.14 ([8])**.** *Let $A \in MS(X)$. Then the following are equivalent:*

(1) $C_A(xy) = C_A(yx)$;

(2) $C_A(xyx^{-1}) = C_A(y)$;

(3) $C_A(xyx^{-1}) \geq C_A(y)$;

(4) $C_A(xyx^{-1}) \leq C_A(y)$.

Remark 2.15. Note that with Proposition 2.14(2), properties (3) and (4) are very trivial. So it is only necessary to state (1) and (2) and devise a better proof.

Definition 2.16 ([8])**.** Let X be a group and e its identity. Also, let $H \in MG(X)$ and $x \in X$.

(1) $[C_H(e)]_x \circ H$ is called a left multicoset of H in X denoted by xH;

(2) $H \circ [C_H(e)]_x$ is called a right multicoset of H in X denoted by Hx.

Proposition 2.17 ([8])**.** *Let $A \in MS(X)$. The following assertions are equivalent.*

(1) $C_A(xy) = C_A(yx)$, $\forall x, y \in X$;

(2) $A \circ B = B \circ A$, $\forall A, B \in MS(X)$.

Proposition 2.18 ([8])**.** *Let $H \in AMG(X)$ and define $X/H = \{xH : x \in X\}$. Then the following assertions hold:*

(1) $(xH) \circ (yH) = (xy)H$, $\forall x, y \in X$;

(2) *If* $xH = x_1H$ *and* $yH = y_1H$ *then* $(xy)H = (x_1y_1)H$;

(3) $(X/H, \circ)$ *is a group;*

(4) $X/H \simeq X/H^*$.

Proposition 2.19 ([8])**.** *Let* $H \in AMG(X)$*. If* $xH = yH$*, then* $C_H(x) = C_H(y)$, $\forall x, y \in X$.

Remark 2.20. It will be shown later that the results in Propositions 3.25 and 3.33 of [8] fail.

3. Some properties of multigroup and abelian multigroup

We now begin to introduce some new results in the following sections.

Proposition 3.1. *Let* $A \in MS(X)$*. Then,* $A \in MG(X)$ *if and only if* $A \circ A^{-1} = A$.

Proof. Let $A \in MG(X)$. Then for any $x, y \in X$,

$$C_A(yz) \geq \{C_A(y) \wedge C_A(z)\} \Rightarrow C_A(yz) = \bigvee\{C_A(y) \wedge C_A(z)\}.$$

Note that

$$\begin{aligned}
C_{A\circ A^{-1}}(x) &= \bigvee\{C_A(y) \wedge C_{A^{-1}}(z) : x = yz\} \\
&= \bigvee\{C_A(y) \wedge C_A(z^{-1}) : x = yz\} \\
&= \bigvee\{C_A(y) \wedge C_A(z) : x = yz\} \\
&= C_A(yz) \\
&= C_A(x).
\end{aligned}$$

Conversely, let $A = A \circ A^{-1}$.

$$\begin{aligned}
C_A(xy^{-1}) &= C_{A\circ A^{-1}}(xy^{-1}) \\
&= \bigvee\{C_A(x) \wedge C_{A^{-1}}(y^{-1})\} \\
&= \bigvee\{C_A(x) \wedge C_A((y^{-1}))^{-1}\} \\
&= \bigvee\{C_A(x) \wedge C_A((y)\}.
\end{aligned}$$

But

$$\bigvee\{C_A(x) \wedge C_A((y)\} \geq \{C_A(x) \wedge C_A((y)\}.$$

□

Nazmul *et al* in [8] have shown that the intersection of multigroups is also a multigroup and have also illustrated with an example that the union of a multigroup is not a multigroup. In what follows, the condition necessary for the union of two multigroups to be a multigroup is stated and proved.

Proposition 3.2. *Let $A, B \in MG(X)$, Then $A \cup B \in MG(X)$ if $C_A(a) = C_A(b) = C_A(ab)$ or $C_B(a) = C_B(b) = C_B(ab)$, $\forall a \in A, b \in B$.*

Proof.

$$\begin{aligned} C_{A\cup B}(ab) &= C_A(ab) \vee C_B(ab) \\ &\geq [(C_A(a) \wedge C_A(b)) \vee (C_B(a) \wedge C_B(b))] \\ &= C_A(a) \vee (C_B(a) \wedge C_B(b)) \\ &= C_{A\cup B}(a) \wedge C_{A\cup B}(b). \end{aligned}$$

□

In what follows, a revision of Proposition 3.24 of [8] is made as earlier remarked in Remark 2.15.

Proposition 3.3. *Let $A \in MG(X)$. Then $C_A(xy) = C_A(yx)$ if and only if $C_A(xyx^{-1}) = C_A(y)$.*

Proof. Assume $C_A(xy) = C_A(yx)$. Then

$$C_A(xyx^{-1}) = C_A(x(yx^{-1})) = C_A((x^{-1})xy) = C_A(y).$$

Also, assume $C_A(xyx^{-1}){=}C_A(y)$. Then $C_A(yx){=}C_A(x(yx)x^{-1}){=}C_A(xy)$. □

Recall the definition of an *abelian multigroup* from Definition 2.8. Aslo, Example 3.27 in [8] has shown that every multigroup over an abelian group X is an abelian multigroup. The following example shows that the group X needs not be an abelian for it to have an abelian multigroup over it.

Example 3.4. Let $X = S_3 = \{e, (12), (13), (23), (123), (132)\}$, where e is the identity of X.

$$A = \{e, e, e, (12), (12), (13), (13), (23), (23), (123), (123), (132), (132)\}$$

is a multigroup. Indeed, it is an abelian multigroup since

$$C_A((12)(13)) = C_A((123)) = C_A((132)) = C_A((13)(12)).$$

But

$$(123) = (12)(13) \neq (13)(12) = (132).$$

4. Some properties of multicosets

In this section, X is a group, e its identity and $H \in MG(X)$. Also, an alternative and a rather easier approach is provided to the concept of multicoset. Note that the multiset $[C_H(e)]_x$ is a *simple* multiset in which x is counted as much as e is counted in $H \in MG(X)$. Recalling the Definition 2.9 of $A \circ B$, the left multicoset $[C_H(e)]_x \circ H$ can be worked out, though a bit tedious than the alternative presented in this section.

Definition 4.1. Let $(X, *)$ be a group and $H = \{y_1, y_2, y_3 \cdots, y_n\} \in MG(X)$. Then

$$x * H = \{x * y_1, x * y_2, x * y_3, \cdots, x * y_n\}$$

is called the left multicoset $[C_H(e)]_x \circ H$.

Example 4.2. Consider the multiplicative group of units. Let $H = \{1, 1, 1, -1, -1, i, i, -i, -i\}$. $[C_H(e)]_i = \{i, i, i\}$ and $[C_H(e)]_i \circ H = x * H = \{1, 1, -1, -1, -i, -i, i, i, i\}$. Obviously, $x * H$ needs not be a multigroup.

Remark 4.3. Note that

$$C_{xH}(x) = C_H(e) = C_H(x^{-1}x).$$

Then,

$$C_{xH}(y) = C_H(x^{-1}y).$$

We simply use xH for $x * H$ henceforth except otherwise is necessary.

Proposition 4.4. *$H = yH$ if and only if $C_{xH}(x) = C_H(y)$, $\forall x, y \in X$.*

Proof. Assume that $H = yH$. Then

$$C_{xH}(x) = C_H(e) = C_H(y^{-1}y) = C_{yH}(y) = C_H(y).$$

Conversely, assume that $C_{xH}(x) = C_H(y)$. Then

$$C_H(y) = C_{xH}(x) = C_H(e) = C_H(y^{-1}y) = C_{yH}(y).$$

□

Example 4.5. Let $X = S_3$. If $y = (12)$ and

$$H = \{e, e, e, (12), (12), (12), (13), (13), (123), (123)\},$$

it is obvious that $H = yH$ and Proposition 4.4 can be verified, noting that the choice of x and y can be varied.

We recall that Proposition 3.32 of [8] (herein as Proposition 2.19) requires that H should be an abelian multigroup and that $xH = yH$ for it to be regular. But the following result shows that H needs not be abelian.

Proposition 4.6. *$xH = yH$ if and only if $C_H(x^{-1}y) = C_{xH}(x)$, $\forall x, y \in X$, in which case, xH, and indeed H, is regular.*

Proof. Assume that $xH = yH$. Then, $H = x^{-1}yH$ and let $z = x^{-1}y$. $C_{xH}(x) = C_H(z) = C_H(x^{-1}y)$ by Proposition 4.4. Hence, $C_{xH}(x) = C_{xH}(y)$. Thus, both xH and H are regular.

Conversely, assume $C_{xH}(x) = C_H(x^{-1}y)$, $\forall x, y \in X$. $C_{yH}(y) = C_H(e) = C_{xH}(x) = C_H(x^{-1}y) = C_{xH}(y)$. □

Remark 4.7. With the foregoing properties, relation can be defined on the elements of multigroups over x.

Proposition 4.8. *If we define $x \sim y$ as $C_H(x^{-1}y) = C_{xH}(x)$, for any $x, y \in X$, then $\sim$ is an equivalence relation on X.*

Proof. (1) *Reflexivity:* $x \sim x$ since $C_H(x^{-1}x) = C_H(e) = C_{xH}(x)$.

(2) *Symmetricity:* Let $x \sim y$. Then, $C_H(x^{-1}y) = C_{xH}(x) = C_H(e) = C_H(y^{-1}y)C_{yH}(y) = C_H(y^{-1}x)$. Thus, $y \sim x$.

(3) *Transitivity:* Let $x \sim y$ and $y \sim z$. Then, $C_H(x^{-1}y) = C_{xH}(x)$ and $C_H(y^{-1}z) = C_{yH}(y)$. Let $x^{-1}y = h_1, y^{-1}z = h_2 \in H$, then, $y = xh_1, z = yh_2 = xh_1h_2 = xh \in xH$. Thus, $x^{-1}z = h \in H$. Hence, $C_H(x^{-1}z) = C_{xH}(x) \Rightarrow x \sim z$. □

The following result shows that multicosets are invariant for regular multigroup H over a group X the support of H is X.

Proposition 4.9. *Let $H \in MG(X)$ be regular such that $H^* = X$. Then, for any $x, y \in X$, the following are equivalent:*

(1) *$xH = H = Hx$;*

(2) *$xH = Hy$;*

(3) *$xHy = xyH = yxH = H$.*

Proof. (1) $\Rightarrow$ (2): $C_{xH}(x) = C_H(x) = C_{yH}(x)$.

(2) $\Rightarrow$ (3): Since $x, y \in H, xy, yx \in H$. By (2), $xyH = yxH$ and by (1), $xyH = H = yxH$. $C_{xHy}(x) = C_H(x^{-1}xy^{-1}) = C_H(y^{-1}) = C_{yH}(e) = C_H(e) = C_{xH}(x) = C_H(x)$.

(3) $\Rightarrow$ (1): Since $xyH = yxH = H$, $xH = H$. $C_{xH}(x) = C_H(e) = C_H(xx^{-1}) = C_{xH}(x) = C_{Hx}(x)$, where e is the identity of X. □

Remark 4.10. If $xH = Hx$, then H is normal.

Proposition 4.11. *Every regular multigroup is abelian but the converse is not true in general.*

Proof. Let H be regular multiset over a group X. Since x and y are arbitrary in H, if $xy = z \in H$ so is $yx = w \in H$. Then, $C_H(xy) = C_H(z) = C_H(w) = C_H(yx)$. H is abelian. But if $H = \{e, e, e, (132), (132), (123), (123)\}$, where e is the identity of X, it is abelian but $C_H(x) \neq C_H(y)$, $\forall x, y \in H$. □

Proposition 4.12. *Let $H \in MG(X)$ and $X = \{x_i\}_{i=1}^k$. Then,*

$$\cup x_i H = \cup n\{x_i\}, \ \forall x_i \in X,$$

where $n = C_H(e)$, e is the identity of X and $|\cup x_i H| = n|X|$. Besides, $\cup x_i H$ is regular.

Proof. $\forall x \in X, C_{\cup x_iH}(x) = \vee C_{x_iH}(x) = C_H(e) = n = C_{n\{x\}}(x) = \vee C_{n\{x_i\}}(x) = C_{\cup n\{x_i\}}(x)$. Note that since $C_{\cup x_iH}(x) = n\ \forall x \in X$, $\cup x_iH$ is regular. Furthermore,

$$|\cup x_iH| = \sum_{j=1}^{k} C_{\cup x_iH}(z_j) = \sum_{j=1}^{k} \vee C_{\cup x_iH}(w_i) = C_H(e)|X| = n|X|.$$

□

Example 4.13. Let $X = S_3$ and $H = \{e, e, (12), (12)\} = eH$, where e is the identity of X.

$$(13)H = \{(13), (13), (132), (132)\};$$

$$(23)H = \{(23), (23), (123), (123)\};$$

$$(12)H = \{e, e, (12), (12)\};$$

$$(132)H = \{(132), (132), (13), (13)\}$$

and $(123)H = \{(123), (123), (23), (23)\}$. Then,

$$\cup xH = \{e, e, (12), (12), (13), (13), (23), (23), (132), (132), (123), (123)\}.$$

In classical group theory, $xH \neq yH \Rightarrow xH \cap yH = \emptyset$, but, in multigroup theory, $xH \neq yH \nRightarrow xH \cap yH = \emptyset$ but rather implies that $C_H(x) \neq C_H(y)$. In what follows, we show by means of counter examples that the result of Proposition 3.25 of [8] fails. Subsequently, we also show that the result of Proposition 3.33 of [8] fails, since its proof is based on the former.

Example 4.14. Consider a multiset

$$A = \{e, e, (12), (12), (12), (13), (13), (123), (123), (132), (132)\}.$$

This is such that $C_A(xy) = C_A(yx)$. Then Consider a multiset $B = \{(13), (13)\}$. $A \circ B = \{e, e, (13), (13), (123), (123)\}$ and $B \circ A = \{e, e, (13), (13), (132), (132)\}$. Thus, $A \circ B \neq B \circ A$ as claimed by [8].

It can be observed that the result can be true if X is an abelian group. Alternatively, if X is not an abelian group, $\forall b \in B$, $C_A(b) \neq 0$.

Example 4.15. Let $H = \{e, e, (12), (12)\}$ as in Example 4.13. This is regular and also abelian. But $\{(12), (12), (23), (23), (123), (123)\} = (13)H \circ (23)H \neq [(13)(23)]H = (123)H = \{(23), (23), (123), (123)\}$. Thus, $(xH) \circ (yH) \neq (xy)H$ as claimed by [8]. Furthermore, $(13)H = (132)H$ and $(23)H = (123)H$ but $(13)(23)H = (123)H \neq eH = (132)(123)H$. Also, $(X/H, \circ)$ is not a group since

$$(13)H \circ (23)H = \{(12), (12), (23), (23), (123), (123)\} \notin \{xH : x \in X\}.$$

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REVERSES OF THE TRIANGLE INEQUALITY FOR ABSOLUTE VALUE IN HILBERT C^*-MODULES

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Abstract. In this paper we obtain some inequalities related to the reverse triangle inequalities for vectors in the framework of Hilbert C^*-modules. Also we improve a celebrated reverse triangle inequality due to Diaz and Metcalf. As a consequence, we apply our results to get some operator inequalities.

Keywords: triangle inequality, Hilbert C^*-module, C^*-algebra, positive element.

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1. Introduction and preliminaries

If $(\mathcal{X}; \|\cdot\|)$ is a normed linear space, then

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} \|x_i\|, \tag{1.1}$$

for any vectors $x_i \in \mathcal{X}, i \in \{1, ..., n\}$. Inequalities of this kind have been called triangle inequality. A number of mathematicians have investigated the inequality (1.1) in various settings. Farenick [13] have investigated the triangle inequality over matrix algebras in Hilbert C^*-modules. We also refer to interesting papers by Shrawan et al. [15] and Dadipour et al. [6]. Some versions of the triangle inequality with simple conditions for the case of equality are presented in [5, 14].

The first to consider the problem of obtaining reverses for the triangle inequality in the more general case of Hilbert and Banach spaces were Diaz and Metcalf [7] who showed that in an inner product space $\mathcal{H}$ over the real or complex number field, the following reverse of the triangle inequality holds

$$r\sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.2}$$

provided

$$0 \leq r \leq \|x_i\| \leq \operatorname{Re}\langle x_i, e\rangle$$

for $k \in \{1, \ldots, n\}$, where $e \in \mathcal{H}$ is a unit vector, i.e. $\|e\| = 1$.

Another reverse of the generalized triangle inequality in Hilbert space was given in [10, Theorem 5] as follows:

Theorem 1.1. *Let $(\mathcal{H}; \langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in \mathcal{H}$, for all $i \in \{1, ..., n\}$ and $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ (probability distribution). If there exists constants $r_i > 0, i \in \{1, ..., n\}$, so that*

$$\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \leq r_i$$

for all $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^{n} p_i\|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{1.3}$$

Some other interesting reverses of the triangle inequality for the case of Hilbert space can be found in [12]. For related results, see also [1, 2, 3, 4, 8, 16].

The motivation of this paper is to extend some generalizations of the reverse triangle inequality like (1.3), in the framework of Hilbert C^*-modules (see Theorem 2.1). We also improve inequality (1.2) in a similar framework (this will be considered in Theorem 3.1).

At the end of this section, we would like to recall some notions, which will be used in the forthcoming sections. Let $\mathcal{A}$ be a C^*-algebra. A *pre-Hilbert $\mathcal{A}$-module* is a linear space $\mathcal{X}$ which is a right $\mathcal{A}$-module together with an $\mathcal{A}$-valued mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ with following properties:

(a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(b) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;

(c) $\langle x, ya \rangle = \langle x, y \rangle a$;

(d) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in \mathcal{X}, a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It is straightforward that a C^*-algebra valued inner product is conjugate-linear in the first variable. We can define a norm on $\mathcal{X}$ by $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If $\mathcal{X}$ is complete with respect to this norm, then $\mathcal{X}$ is called a *Hilbert $\mathcal{A}$-module*. The absolute value of $x \in \mathcal{X}$ is defined as the square root of $\langle x, x \rangle$, and it is denoted by $|x|$. It is worthwhile to point out that this is not actually an extension of a norm, in general, since it may happen that the triangle inequality does not hold.

Throughout the article, $\mathcal{A}$ and $\mathcal{X}$ are C^*-algebra and Hilbert $\mathcal{A}$-module respectively. A C^*-algebra is called unital if $\mathcal{A}$ has a unit $\mathit{1}_{\mathcal{A}}$ and for each $a \in \mathcal{A}$ we have $a.\mathit{1}_{\mathcal{A}} = a$. For convenience, in unital C^*-algebra $\mathcal{A}$ we write a instead of $a.\mathit{1}_{\mathcal{A}}$.

2. On the generalized reverses of the triangle inequality

We start our work by presenting a reverse of the triangle inequality for Hilbert C^*-modules.

Theorem 2.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are positive elements in real number field such that $\sum_{i=1}^n p_i = 1$. If there exist positive elements r_i, $i \in \{1, ..., n\}$ in $\mathcal{A}$, so that*

$$\left| x_i - \sum_{j=1}^{n} p_j x_j \right|^2 \leq r_i^2 \tag{2.1}$$

for $i \in \{1, .., n\}$, then

$$\sum_{i=1}^{n} p_i |x_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \right|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{2.2}$$

Proof. According to (2.1) we have

$$\langle x_i, x_i \rangle - 2\mathrm{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq r_i^2. \tag{2.3}$$

Multiply (2.3) by $p_i \geq 0$, and sum over i from 1 to n, to get

$$\sum_{i=1}^{n} p_i \langle x_i, x_i \rangle - 2\mathrm{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq \sum_{i=1}^{n} p_i r_i^2.$$

This says that

$$\sum_{i=1}^{n} p_i |x_i|^2 - 2\mathrm{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left| \sum_{j=1}^{n} p_j x_j \right|^2 \leq \sum_{i=1}^{n} p_i r_i^2,$$

this inequality is equivalent with

$$\sum_{i=1}^{n} p_i |x_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \right|^2 \leq \sum_{i=1}^{n} p_i r_i^2,$$

which is inequality (2.2). □

As a consequence of Theorem 2.1 we have the following generalization of the reverse triangle inequality in the framework of Hilbert C^*-modules.

Proposition 2.1. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem* 2.1*, then*

$$\mathrm{Re} \left(\sum_{i=1}^{n} p_i |x_i| \right) \left| \sum_{j=1}^{n} p_j x_j \right| \leq \left| \sum_{i=1}^{n} p_i x_i \right|^2 + \frac{1}{2} \sum_{i=1}^{n} p_i r_i^2. \tag{2.4}$$

Proof. From (2.3) we obviously have

$$|x_i|^2 + \left| \sum_{j=1}^{n} p_j x_j \right|^2 \leq 2\,\mathrm{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + r_i^2, \tag{2.5}$$

for all $i \in \{1, ..., n\}$. Whence

$$2\,\mathrm{Re}\,|x_i| \left| \sum_{j=1}^{n} p_j x_j \right| \leq |x_i|^2 + \left| \sum_{j=1}^{n} p_j x_j \right|^2.$$

Here we exploited the fact that for each $a, b \in \mathcal{A}$, $2\,\mathrm{Re}\,ab^* \leq |a|^2 + |b|^2$. Therefore

$$2\,\mathrm{Re}\,|x_i| \left| \sum_{j=1}^{n} p_j x_j \right| \leq 2\,\mathrm{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + r_i^2$$

for all $i \in \{1, ..., n\}$. Arguments similar to the ones used in the proof of Theorem 2.1 give us (2.4). □

Remark 2.1. In particular, if $\mathcal{A}$ be a commutative C^*-algebra, by utilizing the inequality $2\,|a|\,|b| \le |a|^2 + |b|^2$, we can obtain from (2.5) the following result:

$$\sum_{i=1}^{n} p_i\,|x_i|\left|\sum_{j=1}^{n} p_j x_j\right| \le \left|\sum_{i=1}^{n} p_i x_i\right|^2 + \frac{1}{2}\sum_{i=1}^{n} p_i r_i^2.$$

One more consequence of Theorem 2.1 is the following result:

Proposition 2.2. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem* 2.1 *with the additional assumption that $\mathcal{A}$ is commutative. Then*

$$\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\,|x_i|\left|\sum_{j=1}^{n} p_j x_j\right|\right) \le 2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2.$$

Proof. If we multiply (2.5) by $p_i > 0$ and sum over i from 1 to n, we get

$$\sum_{i=1}^{n} p_i|x_i|^2 + \left|\sum_{j=1}^{n} p_j x_j\right|^2 \le 2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2.$$

We now use the fact that $2\,|a|\,|b| \le |a|^2 + |b|^2$. Thus,

$$\begin{aligned}\sum_{i=1}^{n} p_i|x_i|^2 + \sum_{i=1}^{n}\frac{1}{n}\left|\sum_{j=1}^{n} p_j x_j\right|^2 &= \sum_{i=1}^{n}\left(p_i|x_i|^2 + \frac{1}{n}\left|\sum_{j=1}^{n} p_j x_j\right|^2\right)\\ &\ge \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\,|x_i|\left|\sum_{j=1}^{n} p_j x_j\right|\right)\end{aligned}$$

for all $i \in \{1, ..., n\}$. This is the same as saying that

$$2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2 \ge \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\,|x_i|\left|\sum_{j=1}^{n} p_j x_j\right|\right).$$

□

3. The case of a unit vector

The following refinement of the Diaz-Metcalf result may be stated as well:

Theorem 3.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. Suppose that $x_i \in \mathcal{X}$ for all $i \in \{1, \ldots, n\}$ satisfy the condition*

(3.1)
$$\left(\sum_{i=1}^{n} r_1\,|x_i|\right)^2 \le \left(\sum_{i=1}^{n} \operatorname{Re}\langle e, x_i\rangle\right)^2, \quad \left(\sum_{i=1}^{n} r_2\,|x_i|\right)^2 \le \left(\sum_{i=1}^{n} \operatorname{Im}\langle e, x_i\rangle\right)^2,$$

for each $i \in \{1, \ldots, n\}$*, where* e *be a unit vector in* $\mathcal{X}$ *and* r_1, r_2 *are positive elements in* C^**-algebra* $\mathcal{A}$*. Then*

$$\sqrt{r_1^2 + r_2^2} \sum_{i=1}^{n} |x_i| \leq \left| \sum_{i=1}^{n} x_i \right| . \tag{3.2}$$

Proof. We can simply exploit the Cauchy-Schwarz inequality and find the upper bound

$$\left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right|^2 \leq \|e\|^2 \left| \sum_{i=1}^{n} x_i \right|^2 = \left| \sum_{i=1}^{n} x_i \right|^2 . \tag{3.3}$$

We can rewrite the first term as

$$\begin{aligned} \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right|^2 &= \left| \sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle + i \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right) \right|^2 \\ &= \left(\sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle \right)^2 + \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right)^2 . \end{aligned}$$

On the other hand, from (3.1) we infer that

$$r_1^2 \left(\sum_{i=1}^{n} |x_i| \right)^2 \leq \left(\sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle \right)^2$$

and

$$r_2^2 \left(\sum_{i=1}^{n} |x_i| \right)^2 \leq \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right)^2 .$$

Adding these two inequalities to inequality (3.3), we deduce the desired inequality (3.2). □

Remark 3.1. If $\mathcal{A}$ is a commutative C^*-algebra, then we can replace conditions (3.1) with

$$0 \leq r_1 |x_i| \leq \operatorname{Re} \langle e, x_i \rangle , \ 0 \leq r_2 |x_i| \leq \operatorname{Im} \langle e, x_i \rangle .$$

We can apply Theorem 3.1 to derive some new operator inequalities. We only give the following such results. Notice that, if $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, then $\mathbb{B}(\mathcal{H})$ becomes a $\mathbb{B}(\mathcal{H})$-module if the inner product of elements $A, B \in \mathbb{B}(\mathcal{H})$ is defined by $\langle A, B \rangle = A^* B$.

Corollary 3.1. *Let* $A_i \in \mathbb{B}(\mathcal{H})$ *for all* $i \in \{1, \ldots, n\}$ *satisfy the condition*

$$0 \leq B_1 |A_i| \leq \operatorname{Re} A_i, \ 0 \leq B_2 |A_i| \leq \operatorname{Im} A_i,$$

for each $i \in \{1, \ldots, n\}$ *and* B_1, B_2 *are positive operators in* $\mathbb{B}(\mathcal{H})$*, then*

$$\sqrt{B_1^2 + B_2^2} \sum_{i=1}^{n} |A_i| \le \left| \sum_{i=1}^{n} A_i \right|.$$

In particular, for $i \in \{1, 2\}$ *we have*

$$\sqrt{B_1^2 + B_2^2} \left(|A_1| + |A_2| \right) \le |A_1 + A_2|. \tag{3.4}$$

The following reverse of the generalized triangle inequality also holds. Before we proceed, we need the following lemma:

Lemma 3.1. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $a \in \mathcal{A}$*.*

(a) *If* a *is self adjoint, then* $a \le |a|$*.*

(b) *If* a *is normal, then* $|\mathrm{Re}\, a| \le |a|$*.*

Theorem 3.2. *Let* $\mathcal{X}$ *be a Hilbert* $\mathcal{A}$*-module, and* e *be a unit vector in* $\mathcal{X}$*. If* $\langle e, \sum_{i=1}^{n} x_i \rangle$ *and* r_i *are normal and positive elements in* $\mathcal{A}$ *for* $i \in \{1, ..., n\}$ *respectively, and* $x_i \in \mathcal{X}$ *for all* $i \in \{1, ..., n\}$*, such that*

$$|x_i| - \mathrm{Re} \langle e, x_i \rangle \le r_i, \tag{3.5}$$

for each $i \in \{1, ..., n\}$*, then*

$$\sum_{i=1}^{n} |x_i| - \left| \sum_{i=1}^{n} x_i \right| \le \sum_{i=1}^{n} r_i. \tag{3.6}$$

Proof. If we sum in (3.5) over i from 1 to n, then we get

$$\sum_{i=1}^{n} |x_i| \le \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle + \sum_{i=1}^{n} r_i. \tag{3.7}$$

A little calculation shows that

$$\begin{aligned} \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle &\le \left| \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| && \text{(by Lemma 3.1 (a))} \\ &\le \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| && \text{(by Lemma 3.1 (b))} \\ &\le \|e\| \left| \sum_{i=1}^{n} x_i \right| && \text{(by Cauchy-Schwarz inequality)} \\ &= \left| \sum_{i=1}^{n} x_i \right|. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we get (3.6). □

Theorem 3.2 immediately yields:

Corollary 3.2. *If we consider $\mathcal{H}$ as a $\mathbb{C}$-module, then from* (3.6) *we can obtain the following reverse trangle inequality*

$$\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\sum_{i=1}^{n}r_i,$$

where r_i are positive elements in $\mathbb{R}$ for $\{1,\ldots,n\}$ (see [11] and also [9, Theorem 44]).

Remark 3.2. If $\mathcal{A}$ is a commutative C^*-algebra, then the assumption $\langle e,\sum_{i=1}^{n}x_i\rangle$ are normal is not necessary.

Another consequence of our discussion is the following.

Corollary 3.3. *Let $A_i\in\mathbb{B}(\mathcal{H})$, for each $i\in\{1,\ldots,n\}$ and $\sum_{i=1}^{n}A_i$ be normal. If B_i are positive operators in $\mathbb{B}(\mathcal{H})$ for all $i\in\{1,\ldots,n\}$ such that*

$$|A_i|-\operatorname{Re}A_i\leq B_i,$$

for each $i\in\{1,\ldots,n\}$, then

$$\sum_{i=1}^{n}|A_i|-\left|\sum_{i=1}^{n}A_i\right|\leq\sum_{i=1}^{n}B_i.$$

In particular, for $i\in\{1,2\}$ we have

$$|A_1|+|A_2|-|A_1+A_2|\leq B_1+B_2.$$

Now we present a useful lemma, which is applied in the next theorem.

Lemma 3.2. *Let $\mathcal{A}$ be a C^*-algebra and a,b in $\mathcal{A}$ be positive elements and $ab=ba$, then*

$$\sqrt{ab}\leq\frac{a+b}{2}. \tag{3.9}$$

The next theorem is known; see [9, Theorem 50]. The proof given here is different, and in the spirit of our discussion.

Theorem 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $e\in\mathcal{X}$ be such that $|e|=1$ and $x_i\in\mathcal{X}$, $i\in\{1,...,n\}$. If $M_i>m_i>0$ for all $i\in\{1,...,n\}$, are such that*

$$\left|x_i-\frac{M_i+m_i}{2}e\right|^2\leq(M_i+m_i)^2, \tag{3.10}$$

then

$$\sum_{i=1}^{n}|x_i|-\left|\sum_{i=1}^{n}x_i\right|\leq\sum_{i=1}^{n}\frac{(M_i-m_i)^2}{M_i+m_i}.$$

Proof. It follows from left side of inequality (3.10) that

$$\left\langle x_i - \frac{M_i + m_i}{2} e, x_i - \frac{M_i + m_i}{2} e \right\rangle$$
$$= |x_i|^2 - (M_i + m_i) \operatorname{Re} \langle x_i, e \rangle + \left| \frac{M_i + m_i}{2} \right|^2 .$$

Using the substitutions $a = |x_i|^2$ and $b = \left| \frac{M_i + m_i}{2} \right|^2$ in (3.9), this can be rewritten as

$$2 |x_i| \left| \frac{M_i + m_i}{2} \right| \leq |x_i|^2 + \left| \frac{M_i + m_i}{2} \right|^2$$

or, after rearranging terms,

$$|x_i| - \operatorname{Re} \langle x_i, e \rangle \leq \frac{(M_i - m_i)^2}{M_i + m_i} .$$

Hence by Theorem 3.2 we obtain

$$\sum_{i=1}^{n} |x_i| - \left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i} .$$

The validity of this inequality is just Theorem 3.3. □

Another result of this type is the following one:

Theorem 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $e \in \mathcal{X}$ be such that $|e| = 1$ and $x_i \in \mathcal{X}$, $i \in \{1, ..., n\}$. If $M_i \geq 0$ for all $i \in \{1, ..., n\}$, are such that*

$$\left| x_i - \frac{M_i}{2} e \right|^2 \leq M_i^2, \tag{3.11}$$

then

$$\sum_{i=1}^{n} |x_i|^2 - \operatorname{Re} \left\langle \sum_{i=1}^{n} M_i x_i, e \right\rangle \leq \frac{3}{4} \sum_{i=1}^{n} M_i^2 . \tag{3.12}$$

Proof. A short calculation reveals that

$$\left\langle x_i - \frac{M_i}{2} e, x_i - \frac{M_i}{2} e \right\rangle = |x_i|^2 + \left| \frac{M_i}{2} \right|^2 |e| - 2 \operatorname{Re} \left\langle x_i, \frac{M_i}{2} e \right\rangle . \tag{3.13}$$

According to (3.13) validity of (3.11) implies

$$|x_i|^2 + \left| \frac{M_i}{2} \right|^2 |e| - 2 \operatorname{Re} \left\langle x_i, \frac{M_i}{2} e \right\rangle \leq M_i^2$$

which on simplification reduces to

$$|x_i|^2 - \operatorname{Re} \langle M_i x_i, e \rangle \leq \frac{3}{4} M_i^2 .$$

Summing over all terms then yields (3.12). □

The following particular case is of interest:

Theorem 3.5. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $e_1, e_2, ..., e_n$ be a sequence of unit vectors in $\mathcal{X}$ such that $\langle e_i, e_j \rangle = 0$ for $i \neq j \leq n$, and let $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are positive elements in real number field such that $\sum_{i=1}^{n} p_i = 1$. If there exist constants positive elements r_i in $\mathcal{A}$ so that*

$$\left| x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j \rangle e_j \right|^2 \leq r_i^2,$$

for all $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^{n} p_i |x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{3.14}$$

Proof. A straightforward computation shows that

$$\begin{aligned}
&\left\langle x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j \rangle e_j, x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j \rangle e_j \right\rangle \\
&= \langle x_i, x_i \rangle + \left\langle \sum_{i=1}^{n} p_i e_i \langle e_i, x_i \rangle, \sum_{j=1}^{n} p_j e_j \langle e_j, x_j \rangle \right\rangle - 2 \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \\
&= \langle x_i, x_i \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \langle e_i, x_i \rangle^* \langle e_j, e_j \rangle \langle e_j, x_j \rangle - 2 \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \\
&= |x_i|^2 + \sum_{i=1}^{n} p_j^2 \langle e_j, x_j \rangle^* \langle e_j, e_j \rangle \langle e_j, x_j \rangle - 2 \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \\
&= |x_i|^2 + \sum_{j=1}^{n} p_j^2 \langle e_j, x_j \rangle^* \langle e_j, x_j \rangle - 2 \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \\
&= |x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2.
\end{aligned}$$

Using this one can see that

$$|x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \leq r_i^2. \tag{3.15}$$

If we multiply (3.15) by $p_i \geq 0$ and sum over i from 1 to n, we obtain

$$\sum_{i=1}^{n} p_i |x_i|^2 - \sum_{i=1}^{n} |p_i \langle e_i, x_i \rangle|^2 \leq \sum_{i=1}^{n} p_i r_i^2$$

which finishes the proof. $\square$

Corollary 3.4. *With the substitution* $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$, (3.14) *becomes*

$$\sum_{i=1}^{n} |x_i|^2 - \frac{1}{n} \sum_{i=1}^{n} |\langle e_i, x_i \rangle|^2 \leq \sum_{i=1}^{n} r_i^2.$$

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THE BICLIQUE PARTITION NUMBER OF SOME IMPORTANT GRAPHS

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Abstract. The biclique partition number of a graph G, $bp(G)$ is the minimum number of complete bipartite subgraphs needed to partition the edge set of G. Let $r(G) = \max\{n_+(G), n_-(G)\}$ where $n_+(G), n_-(G)$ are the number of positive and the number of negative eigenvalues of the adjacency matrix of G, respectively. A graph G satisfying, $bp(G) = r(G)$ is called an eigensharp graph. In this paper we apply Pollak and Graham Theorem to find the biclique partition number of the line graph of complete graph and its complement, the line graph of complete bipartite graph and its complement and the line graph of a tree graph and we discuss the eigensharp property of these graphs. Also we identify the biclique partition number of the $kth-$power graph of paths and cycles.
Keywords: graph, clique, biclique, biclique partition number, line graph, complete graph, complete bipartite graph, tree graph, kth-power of a path, kth-power of a cycle.

1. Introduction

All graphs in this paper are finite undirected simple graphs. For a graph $G = (V(G), E(G))$, the set $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. A clique is a complete subgraph. A biclique is a complete bipartite subgraph. The set of eigenvalues of $A(G)$, the adjacency matrix of G, is called the spectrum of G and is written spec(G). If λ_i ,$1 \leq i \leq k$, are the distinct eigenvalues of $A(G)$ with multiplicity m_i, then we write

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \\ m_1 & m_2 & ... & m_k \end{pmatrix}.$$

Covering of a graph by a certain type of subgraphs is an important concept. It has been studied by different authors. There are several types of graph covering, including path covering, tree covering, clique covering and biclique

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covering. In this paper, we'll focus on the biclique partition covering of a graph. The biclique partition covering of a graph has been studied by several authors, see [2], [3], [4], [6] and [7].

A biclique partition of a graph G is a collection of bicliques of G that partition the edge set of G. Similarly, a biclique cover of G is a collection of bicliques that cover the edge set of G. The minimum cardinality of a biclique partition of a graph G is called the biclique partition number, denoted by $bp(G)$. Note that $bp(G) \leq n-1$. This holds because for any graph G with n vertices stars on $n-1$ vertices form a biclique partition for G. In 1971, Graham and Pollak [6] proved that $bp(K_n) = n-1$. Witsenhausen ($cf.$ [6]), showed that for a graph G with n vertices, the biclique partition number $bp(G)$ is bounded below as follows:

$$bp(G) \geq \max\{n_+(G), n_-(G)\}$$

where $n_+(G), n_-(G)$ are the number of positive and the number of negative eigenvalues of the adjacency matrix of G, respectively. Suppose that $\max\{n_+(G), n_-(G)\} = r(G)$. A graph G satisfying, $bp(G) = r(G)$ is called an eigensharp graph, see [7].

Given a graph G, an independent set is a subset of the vertex set of G such that no two vertices are adjacent. The independence number $\alpha(G)$ is the cardinality of a largest set of independent vertices. A maximum independent set with the largest number of vertices in a given graph G is denoted by $I(G)$, (i.e. $\alpha(G) = |I(G)|$). A star S_n is a tree with $(n+1)$ vertices with one vertex having degree n, and the other n vertices having degree 1. In S_n the vertex of degree n is called the center of the star. The star graph S_n is therefore isomorphic to the complete bipartite graph $K_{1,n}$. Because every star is a complete bipartite graph, the vertex cover number of G is an upper bound of $bp(G)$, and moreover one can easily prove that if G is a graph on n vertices then $bp(G) \leq n - \alpha(G)$. For a graph G when the biclique partition covering of minium cardinality is a collection of stars we use star center to represent the star, $K_{1,n}$. In this paper we will consider the edges of K_n partitioned into $n-1$ bicliques using edge-disjoint stars.

In this paper, Graham and Pollak Theorem is the cornerstone of our results. We study the biclique partition number for several classes of graphs. In section 2, we study the biclique partition number of the line of any complete graph and its complement, the line graph of any complete bipartite graph and its complement. We characterize when these graphs are eigensharp depending on the set of edges incident with v generate a clique in $L(G)$ of order degree(v). The cliques of $L(G)$ in this way partition the edge set of $L(G)$, and we apply Graham and Pollak Theorem to calculate the biclique partition number of these graphs. In section 3, we discuss the biclique partition number of the line graph of a tree graph, by noting that the line graph of a tree graph is a connected block graph in which each cutpoint is on exactly two blocks. Also we applied Graham and Pollak Theorem on these blocks to calculate the biclique partition number of the line graph of a tree graph.

Finally, the biclique partition number of the $kth-$power of paths and cycles are characterized.

2. The biclique partition number for some families of line graphs

In this section, the biclique partition number of the line graphs of complete graphs and their complements, and that of line graphs of complete bipartite graphs and their complements are completely characterized. Moreover, the biclique partition number of the line graph of trees is calculated.

Definition. *Let G be a simple graph with n* vertices *and m edges. The line graph $L(G)$ of G is the simple graph whose vertex set is the set of edges of G and for any $a, b \in V(L(G))$* the *vertices a and b are adjacent in $L(G)$ if and only if a and b have a common vertex in G.*

For a graph G and a vertex v in G with degree$(v) \geq 2$, the set of edges incident to v generate a clique in $L(G)$ of order degree(v). The cliques of $L(G)$ in this way partition the edges of $L(G)$, and this partition helps us to find the biclique partition number of the line graph using Graham and Pollak Theorem. Each vertex of $L(G)$ belongs to exactly two cliques which are the two cliques corresponding to the two endpoints of the corresponding edge in G. We will study the biclique partition number of line graph of some families of graphs and investigate when they have the eigensharp property. It is easy to show that the line graph of a star graph, $L(S_n) = K_n$, and therefore $bp(L(S_n)) = n - 1$, and so $L(S_n)$ is eigensharp. Trees are eigensharp, see [7]. Also paths P_n are eigensharp and the complements of paths are eigensharp too, see [2]. We note that $L(P_n) = P_{n-1}$, so line graph of path and its complement are eigensharp. The line graph of a cycle C_n is isomorphic to C_n, which is eigensharp when $n \neq 4k$ with $k > 2$, see [7]. Moreover, its complement is eigensharp, see [2].

2.1 The graphs $L(K_n)$ and their complement $\overline{L(K_n)}$

The line graph of K_n, $L(K_n)$ is a graph with $\binom{n}{2}$ vertices and $n\binom{n-1}{2}$ edges. Suppose that the vertices $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. In K_n, we denote the edge between u_i and u_j by $u_{_{i,j}}$. Therefore $V(L(K_n)) = \{u_{_{i,j}} : 1 \leq i < j \leq n\}$. Then two distinct vertices in $L(K_n)$are adjacent if their labels share exactly one digit. So, when n is even, then the set $I(L(K_n)) = \{u_{_{1,2}}, u_{_{3,4}}, u_{_{5,6}}, \ldots, u_{_{n-3,n-2}}, u_{_{n-1,n}}\}$ is a maximum independent set in $L(K_n)$, and if n is odd then the set $I(L(K_n)) = \{u_{_{1,2}}, u_{_{3,4}}, u_{_{5,6}}, \ldots, u_{_{n-2,n-1}}\}$ is a maximum independent set in $L(K_n)$, which is of order $\left\lfloor \frac{n}{2} \right\rfloor$. The vertex u_i has degree $(n-1)$ in K_n, so we have in $L(K_n)$, n cliques of order $(n-1)$, say $B_i : 1 \leq i \leq n$, where $V(B_i) = \{u_{_{i,j}} : 1 \leq j \leq n$, $i \neq j, \}$. Take any edge in $L(K_n)$, then this edge comes from two adjacent edges in K_n, suppose that the common vertex is u_i and the ends of these two edges in K_n are u_j and u_k respectively, the edge in $L(K_n)$ that is incident with vertex $u_{i,j}$ and $u_{i,k}$, so this edge belongs to exactly one member from the family $\{B_i\}_{i=1}^n$. So the family $\{B_i\}_{i=1}^n$ forms a partition of the edges of $L(K_n)$, and clearly $V(B_i) \cap V(B_j) = \{u_{i,j}\}$.

To get a better understanding of the computation of $bp(L(K_n))$, we give the following example of the characterization of $L(K_4)$ and $L(K_5)$.

Example 1. As we explained earlier, the family $\{B_i\}_{i=1}^4$ forms a partition of the edge set of $L(K_4)$. Each B_i is a clique of order 3, and also $V(B_1) = \{u_{1,2}, u_{1,3}, u_{1,4}\}, V(B_2) = \{u_{2,1}, u_{2,3}, u_{2,4}\}, V(B_3) = \{u_{3,1}, u_{3,2}, u_{3,4}\}$ and $V(B_4) = \{u_{4,1}, u_{4,2}, u_{4,3}\}$. It is clear that $I(L(K_4)) = \{u_{1,2}, u_{3,4}\}$. By Graham and Pollak Theorem, $bp(B_i) = 2$. So we need two vertices to cover each B_i by biclique partiton of minimum cardinality. Let H be the induced subgraph of $L(K_4)$ such that $V(H) = V(L(K_4)) - I(L(K_4)) = \{u_{1,3}, u_{1,4}, u_{2,3}, u_{2,4}\}$. Then for B_1, the number of vertices having label 1 is 2, which we need to cover $B_1$by stars, and the same for other cliques B_2, B_3 and B_4, and therefore $bp(L(K_4)) = |H| = 4$.

Example 2. The family $\{B_i\}_{i=1}^5$ forms a partition of the edge set of $L(K_5)$. Each B_i is a clique of order 4, and also $V(B_1) = \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}\}, V(B_2) = \{u_{2,1}, u_{2,3}, u_{2,4}, u_{2,5}\}, V(B_3) = \{u_{3,1}, u_{3,2}, u_{3,4}, u_{3,5}\}, V(B_4) = \{u_{4,1}, u_{4,2}, u_{4,3}, u_{4,5}\}$ and, $V(B_5) = \{u_{5,1}, u_{5,2}, u_{5,3}, u_{5,4}\}$. It is clear that $I(L(K_5)) = \{u_{1,2}, u_{3,4}\}$. By Graham and Pollak Theorem, $bp(B_i) = 3$. So we need three vertices to cover each B_i by biclique partiton of minimum cardinality. Let H be the induced subgraph of $L(K_5)$ such that $V(H) = V(L(K_5)) - I(L(K_5)) = \{u_{1,3}, u_{1,4}, u_{1,5}, u_{2,3}, u_{2,4}, u_{2,5}, u_{3,5}, u_{4,5}\}$. Then for $B_1$the number of vertices having label 1 is 3 vertices, which covers $B_1$by stars, and the same for other cliques B_2, B_3, B_4. But for B_5 there are 4 vertices having label 5, it is not possible to dispense with any one of them, because we need $u_{1,5}, u_{2,5}, u_{3,5}$ and $u_{4,5}$to cover B_1, B_2, B_3 and B_4 respectively by stars covering and therefore, $bp(L(K_5))$=$|H|$=8.

In general for $L(K_n)$, we will get the following theorem.

Theorem 3. *For the graph* $L(K_n)$, $bp(L(K_n)) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. In $L(K_n)$ the family $\{B_i\}_{i=1}^n$ forms a partition of $E(L(K_n))$, where B_i is clique of order $(n-1)$. Let H be the induced subgraph of $L(K_n)$ such that $V(H) = V(L(K_n)) - I(L(K_n))$. We claim that the vertices of H give a biclique partition of minimum cardinality. Take any B_i in $L(K_n)$, by Graham and Pollak Theorem, $bp(B_i) = n-2$, so we need exactly $(n-2)$ vertices from B_i (i.e. the vertices have label i) to cover it by biclique partition of minimum cardinality. Since $V(B_i) = \{u_{i,j} : 1 \le j \le n,\, i \ne j\}$. There are four cases for $B_i, 1 \le j \le n$.

Case 1. If i is odd, and $i < n$, then $u_{i,i+1} \in I(L(K_n))$, and therefore $u_{i,i+1} \notin H$.

Case 2. If i is even, and $i < n$, then $u_{i-1,i} \in I(L(K_n))$, and therefore $u_{i-1,i} \notin H$.

Case 3. If n is even, and $i = n$, then $u_{n-1,n} \in I(L(K_n))$, and therefore $u_{n-1,n} \notin H$.

Thus for the three cases we conclude that for any B_i in $L(K_n)$, only remained exactly $(n-2)$ vertices have label i that belong to H and so B_i is covered by stars which is a biclique partition of minium cardinality.

Case 4. If n is odd, and $i = n$, then H contains the vertices $u_{n,1}, u_{n,2}, \dots, u_{n,n-1}$ having label n, and it is not possible to dispense with any one of them, because we need $u_{n,1}, u_{n,2}, \dots, u_{n,n-1}$ to cover $B_1, B_2, \dots, B_{n-1}$respectively by biclique partition of minimum cardinality. Hence

$$bp(L(K_n)) = |H| = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor .$$

□

On the other hand,

$$spec(L(K_n)) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see [1], and hence when $n = 5$, then $r(L(K_n)) = n$ and if $n \geq 6$, then $r(L(K_n)) = \frac{n(n-3)}{2}$. So, $L(K_n)$ is not eigensharp for $n \geq 4$ and it is eigensharp when $n < 4$.

Now we show that the complement of $L(K_n)$ is eigensharp, consider the following theorem.

Theorem 4. *The complement of* $L(K_n)$, $\overline{L(K_n)}$, *is eigensharp.*

Proof. We will show that $bp(\overline{L(K_n)}) = r(\overline{L(K_n)})$, we know that

$$spec(L(K_n)) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see [1]. Since $L(K_n)$ is $(2n-4)-$regular,

$$spec(\overline{L(K_n)}) = \begin{pmatrix} \binom{n}{2} - (2n-4) - 1 & 3-n & 1 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see[1]. Hence $r(\overline{L(K_n)}) = n_+(\overline{L(K_n)}) = \binom{n-1}{2}$, and therefore $bp(\overline{L(K_n)}) \geq \binom{n-1}{2}$. In fact equality is achieved. The graph $\overline{L(K_n)}$ has maximum independent set of order $n-1$ (i.e. $\alpha(\overline{L(K_n)}) = n-1$), since in K_n each vertex is incident to $n-1$ edges, and these edges are pairwise adjacent vertices in $L(K_n)$ and so they are nonadjacent vertices in $\overline{L(K_n)}$. These edges are the largest set of pairwise nonadjacent vertices in $\overline{L(K_n)}$. And so $bp(\overline{L(K_n)}) \leq \binom{n}{2} - (n-1) = \binom{n-1}{2}$. Thus $bp(\overline{L(K_n)}) = \binom{n-1}{2}$ and therefore $\overline{L(K_n)}$ is eigensharp. □

2.2 The graph $L(K_{n,m})$ and their complement $\overline{L(K_{n,m})}$

First, we will characterize the graph $L(K_{n,m})$ and we assume that $2 \leq n \leq m$. The complete bipartite graph $K_{n,m}$, has two maximal independent sets X_n, Y_m where $X_n = \{x_1, \dots, x_n\}$ and $Y_m = \{y_1, \dots, y_m\}$. In $K_{n,m}$ each edge is incident with one vertex from X_n and one vertex from Y_m, we may say that the vertex set is $V(L(K_{n,m})) = \{u_{i,j} : 1 \leq i \leq n \ , 1 \leq j \leq m \ \}$ and $u_{i,j}$ represents the edge in $K_{n,m}$ between x_i and y_j. Now, in $L(K_{n,m})$ the vertex $u_{i,j}$ is adjacent to $u_{k,h}$ if and only if $i = k$ or $j = h$. In X_n each x_i has degree m, so we have in $L(K_{n,m})$, n cliques of order m, denoted by $B(H_i) : 1 \leq i \leq n$, induced by $H_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,m}\}$. Since X_n is an independent set in $K_{n,m}$, we have $H_i \cap H_j = \varphi$, for all $i \neq j$, $1 \leq i, j \leq n$. On the other hand Y_m is an independent set, so we have m cliques of order n say $B(V_i)$, induced by $V_i = \{u_{1,i}, u_{2,i}, \dots, u_{n,i}\}$ and $V_i \cap V_j = \varphi$, $i \neq j$, $1 \leq i, j \leq m$. Since each edge connects one vertex from X_n and one vertex from Y_m, we have $H_i \cap V_j = u_{i,j}$.

Now we want to compute the biclique partition number of $L(K_{n,m})$.

Theorem 5. *For the graph $L\left(K_{n,m}\right)$, $bp(L\left(K_{n,m}\right)) = n\left(m-1\right)$, where $m \geq n \geq 2$.*

Proof. In $L\left(K_{n,m}\right)$ the family $\{B(H_i)\}_{i=1}^{n}$ is a family of disjoint cliques each of order m. Suppose $A_i : 1 \leq i \leq n$ is a biclique partition of $B(H_i)$, we need $(m-1)$ distinct vertices for each member of $\{B(H_i)\}_{i=1}^{n}$to cover it by stars, therefore $|A_i| = m-1$. Taking A_i in this way, $A_1 = \{u_{1,2}, u_{1,3}, \dots, u_{1,m}\}, A_2 = \{u_{2,1}, u_{2,3}, \dots, u_{2,m}\}, \dots,$ $A_i = \{u_{i,1}, u_{i,2}, u_{i,3}, \dots, u_{i,i-1}, u_{i,i+1}, \dots, u_{i,m}\}$, $\dots$, $A_n = \{u_{n,1}, u_{n,2}, \dots, u_{n,n-1}, u_{n,n+1}, \dots, u_{n,m}\}$, (i.e. we take all vertices of $B(H_i)$ and we leave the vertex $u_{i,i}$). We will prove that $\{A_i\}_{i=1}^{n}$ is a biclique partition of minium cardinality for $L(K_{n,m})$. First, A_i is biclique partition of $B(H_i)$ for each $1 \leq i \leq n$, because $|A_i| = m-1$ and $A_i \cap A_j = \varphi : 1 \leq i, j \leq n$, $i \neq j$. Second, for any $B(V_j) : 1 \leq j \leq m$, we have $bp\left(B(V_j)\ \right) = n-1$. Take the induced subgraph by the vertices $\{u_{i,j} : 1 \leq i \leq n, i \neq j\} \subseteq B(V_j)$ which contains $n-1$ vertices, so it covers all $B(V_j)$ by stars covering, and for all $1 \leq i \leq n$, $u_{i,j} \in A_i$, so $\{u_{i,j} : 1 \leq i \leq n, i \neq j\} \subseteq \bigcup_{i=1}^{n} A_i$. This gives us $\{A_i\}_{i=1}^{n}$ is a family of stars which cover $B(V_j)$: $1 \leq j \leq m$. So, we have $\{A_i\}_{i=1}^{n}$ is a biclique partition of minimum cardinality for $L(K_{n,m})$. Therefore we have $bp\left(L(K_{n,m})\right) = \sum_{i=1}^{n}\ |A_i| = n\left(m-1\right) = n\left(m-1\right).$ □

Question. Is $L(K_{n,m})$ eigensharp?

To answer this question we will use the spectrum of the cartesian product $K_n \times K_m$, see [1]. In fact, $K_n \times K_m$ is isomorphic to the line graph of the complete bipartite graph $K_{n,m}$, $L(K_{n,m})$. Now, the spectrum of $K_n \times K_m$ is equal to

$$spec\left(L\left(K_{n,m}\right)\right) = \begin{pmatrix} m+n-2 & m-2 & n-2 & -2 \\ 1 & n-1 & m-1 & \left(m-1\right)\left(n-1\right) \end{pmatrix}.$$

Thus $r\left(L\left(K_{n,m}\right)\right)=(m-1)(n-1)$. So, $L\left(K_{n,m}\right)$ is not eigensharp.

In the following theorem we determine the biclique partition number of the graph $\overline{L\left(K_{n,m}\right)}$.

Theorem 6. *For the graph* $\overline{L\left(K_{n,m}\right)}$*,* $bp(\overline{L\left(K_{n,m}\right)}) = m\left(n-1\right)$*, where* $m \geq n \geq 2$.

Proof. Recall that for the graph $L\left(K_{n,m}\right), V(L\left(K_{n,m}\right)) = \{u_{_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq m\}$, it easy to show that in $\overline{L\left(K_{n,m}\right)}$ the vertices $u_{_{i,j}}$ and $u_{_{k,l}}$ are adjacent if and only if $k \neq i$ and $l \neq j$. Also $E(\overline{L\left(K_{n,m}\right)}) = \{u_{_{i,j}}u_{_{k,l}} : k \neq i \text{ and } l \neq j\}$, therefore the sets $H_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,m}\} : 1 \leq i \leq n$, is an independent set of vertices in $\overline{L\left(K_{n,m}\right)}$.

Now to calculate $bp(\overline{L\left(K_{n,m}\right)})$, consider the pairwise disjoint family $S = \{H_i\}_{i=1}^{n}$. Let F be the induced subgraph in $L\left(K_{n,m}\right)$ such that $V(F) = \digamma = H_k \cup H_j$ for some $k \neq j$,$1 \leq j, k \leq n$. The adjacency matrix of $\digamma$ is $A\left(\digamma\right) = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix}$, it is clear that B is the adjacenay matrix of K_m, (i.e. $B = A(K_m)$). Since $\digamma$ is a bipartite graph, λ is an eigenvalue of multiplicity p if and only if $-\lambda$ is an eigenvalue of multiplicity p. So,

$$spec(\digamma) = \begin{pmatrix} m-1 & -1 & 1 & 1-m \\ 1 & m-1 & m-1 & 1 \end{pmatrix}$$

and therefore $r(\digamma) = m$. So, $bp(\digamma) \geq m$. The induced subgraph $\digamma$ is partitioned into $K_{1,m-1}, K_{2,m-1}, \ldots, K_{m,m-1}$. Take one set of H_k or H_j it will cover the induced subgraph $\digamma$ by m stars. So, $bp(F) = m$. Since any induced subgraph of $\overline{L\left(K_{n,m}\right)}$ that contains two disjoint members of S, the biclique partition number that covers by stars must be m. Therefore, it can be proved by induction on $H_i : 1 \leq i \leq n$ that $bp(\overline{L\left(K_{n,m}\right)}) = \sum_{i-1}^{n-1} |H_i| = m(n-1)$, by showing that we need $\{H_i\}_{i=1}^{n-1}$ to cover $\overline{L\left(K_{n,m}\right)}$ by stars. Hence we get the result. $\square$

For eigensharpness property of $\overline{L\left(K_{n,m}\right)}$, we find the spectrum of $\overline{L\left(K_{n,m}\right)}$. Since $L\left(K_{n,m}\right)$ is $(m+n-2)$-regular graph,

$$spec(\overline{L\left(K_{n,m}\right)}) = \begin{pmatrix} mn-m-n+1 & 1-m & 1-n & 1 \\ 1 & n-1 & m-1 & \left(m-1\right)\left(n-1\right) \end{pmatrix},$$

therefore $r\left(\overline{L\left(K_{n,m}\right)}\right) = (m-1)(n-1)+1$, and this shows that $\overline{L\left(K_{n,m}\right)}$ is not eigensharp.

3. The biclique partition number of the line graph of a tree graph $L(T)$

In this section, we obtain the biclique partition number of the line graph of a tree of order n, $L(T)$.

Definition. *A block graph is a graph in which every biconnected component (block) is a maximal clique.*

Theorem 7 ([5])**.** *A graph is the line graph of a tree if and only if it is a connected block graph in which each cutpoint is exactly on two blocks.*

Suppose that $B_1, B_2, \ldots, B_k$ are the blocks of $L(T)$ of orders $n_i : 1 \leq i \leq k$, and $n_i \geq 2$. Let H be an induced subgraph containing two blocks from $L(T)$ say B_i and B_j such that B_i and B_j are not disjoint. The block B_i and B_j have a unique cut vertex in common, say h. Then to compute $bp(H)$, we will look to $bp(B_i)$ and $bp(B_j)$. We know that $bp(B_i) = n_i - 1$ and $bp(B_j) = n_j - 1$, suppose that the vertex set of biclique partition of minium cardinality for $B_i = \{u_1, \ldots, u_{n-1}\}$ and the vertex set of biclique partition of minium cardinality for $B_j = \{v_1, \ldots, v_{n-1}\}$, so to cover H by biclique partition, we can take star cover of $\{u_1, \ldots, u_{n-1}\} \cup \{v_1, \ldots, v_{n-1}\}$. To find a biclique partition of minium number of H. Take$V(B_i) \cap V(B_j) = \{h\}$, $h \in \{u_1, \ldots, u_{n-1}\}$ and $h \in \{v_1, \ldots, v_{n-1}\}$; because h covers all edges incident with h in B_i and all edges incident with h in B_j. Therefore the biclique partition number of H equal to $(n_i - 1) + (n_j - 1) - 1$. The result easily extended to all blocks in $L(T_n)$, because each cutpoint is exactly in two blocks. Thus we have the following theorem.

Theorem 8. *If $L(T)$ has r cutpoints, such that degree $(r) > 3$, then $bp(L(T)) = bp(B_1) + ... + bp(B_k) - r = \sum_{i=1}^{k}(n_i - 1) - r$.*

We do not know whether the graph $L(T)$ is an eigensharp graph or not.

4. The biclique partition number of kth-power of path and cycle graphs

This section focuses on the biclique partition numbers of $kth-$power graph of paths and cycles. For any *positive* integer k and a connected graph G, a new graph, G^k, called the $kth-$power of G, can be defined as follows: $V(G^k) = V(G)$ and two distinct vertices u and v in G^k are adjacent if the distance between u and v in G is less than or equal to k. In this section, we will calculate the biclique partition numbers of the $kth-$power of a path, P_n^k, and the $kth-$power of a cycle, C_n^k.

The kth-power of path graph P_n^k. First, we describe the $kth-$power graph of a path, P_n^k. Then we will determine the biclique partition number of the $kth-$power graph of a path P_n^k, using Graham and Pollak Theorem. Suppose that the vertices of $V(P_n^k) = \{i : 1 \leq i \leq n\}$. The vertices i and j are adjacent if and only if $d(i, j) \leq k$. Let H be the induced subgraph containing $(k + 1)$ consecutive vertices. Clearly, the induced subgraph of these vertices in P_n^k is a maximal clique of order $k + 1$.

First, we will assume that $(k + 1)$ divides n. Suppose $\frac{n}{k+1} = m$, so we can divide the vertices of P_n^k into pairwise disjoint family of subgraphs $\{H_i\}_{i=1}^{m}$, each of them has $(k+1)$ consecutive vertices $H_1 = \{1, 2, \ldots, k+1\}, H_2 = \{k+2, \ldots, 2(k+1)\}, \ldots, H_m = \{(m-1)(k+1)+1, \ldots, m(k+1)\}$. So, $\{H_i\}_{i=1}^{m}$ is a family of pairwise disjoint cliques each of them has order $(k+1)$. We rename the vertices of H_i in order to be used to calculate $bp\left(P_n^k\right)$. Suppose that $H_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k+1}\}$, and we call $v_{i,1}$ the first element. Now we will study the relation between H_i and $H_{i+1} : 1 \leq i < m$. Let $v_{i,j}$ be a vertex in H_i, then $v_{i,j}$ is adjacent to all vertices in H_i, and $v_{i+1,p} \in H_{i+1} \quad : p < j \leq k+1$, because $d\left(v_{i,j}, v_{i+1,p}\right) \leq k$. So, the first element $v_{i,1}$ is not adjacent to any vertex in H_{i+1}.

In the following theorem we determine the biclique partition number of, P_n^k, assuming that $(k+1)$ divides n.

Theorem 9. *For the graph P_n^k, if $(k+1) \mid n$, then $bp\left(P_n^k\right) = \frac{k}{k+1}n$.*

Proof. Suppose $\frac{n}{k+1} = m$. So, we can divide the vertices of P_n^k into disjoint family, $\{H_i\}_{i=1}^{m}$ of cliques of order $(k+1)$. By Graham and Pollak Theorem, $bp(H_i) = k$. Let $X_i = \{v_{i,2}, \ldots, v_{i,k+1}\}$ be a biclique partition of minimum cardinality of H_i. Now we claim that $\bigcup_{i=1}^{m} X_i$ is a biclique partition of minium cardinality for P_n^k. Since $\{H_i\}_{i=1}^{m}$ is a family of disjoint cliques. Clearly, we need X_i to partition the edges of each H_i as star covering. Also these stars cover the edges between H_i and H_{i+1}. Then $\bigcup_{i=1}^{m} X_i$ partition the edges in $E(P_n^k)$. Also, using Graham and Pollak Theorem this is the least number needed. Thus $bp(P_n^k) = \sum_{i=1}^{m} |X_i| = \frac{kn}{k+1}$. □

On the other hand when n is not a multiple of $k+1$, by division algorithm $n = (k+1)m + r$, $0 < r < k+1$. Hence we get the following theorem.

Theorem 10. *For the graph P_n^k, $bp\left(P_n^k\right) = km + (r-1)$, where $n = (k+1)m + r$, $1 \leq r \leq k$.*

Proof. From the assumption, $\left\lfloor \frac{n}{k+1} \right\rfloor = m$. So, as in the previous theorem the first $m(k+1)$ vertices can be covered by disjoint family of m cliques each of order $k+1$. The edges can be covered by $\bigcup_{i=1}^{m} X_i$ as a star covering. While the last r vertices induce a clique, R, of order r. By, Graham and Pollak Theorem, we need at least $r-1$ vertices to partition the edges of this clique. The edges between the clique H_m and R are covered by stars of H_m. Hence we get the result. □

The kth-power of cycle graph C_n^k. First, we describe the $kth-$power of a cycle, C_n^k. Then we will determine the biclique partition number of $kth-$power of cycle, C_n^k, using Graham and Pollak Theorem. Let O be the induced subgraph containing $(k+1)$ consecutive vertices. Clearly, the induced subgraph of these vertices in C_n^k is a maximal clique of order $k+1$.

Using division algorithm $n = (k+1)m + r$, $0 \leq r < k+1$. Therefore we can divide the vertices of C_n^k into disjoint family of vertices each consists of

$k+1$ except the last one with r vertices. These member of this family induce a maximal cliques. So, we have $(m+1)$ disjoint cliques. By Graham and Pollak Theorem we need at least km vertices to cover all edges in the first m cliques, we need $r-1$ vertices to cover the last clique of order r. But stars with these vertices as their centers do not cover all edges in C_n^k, when $r \neq 0$; because edges between O_1 and the last clique are not covered. So, we need all vertices of the last clique, (i.e. r vertices), to cover all edges. Hence $bp\left(C_n^k\right) = km + r$. Thus we get the following theorem

Theorem 11. *For the graph C_n^k, $bp\left(C_n^k\right) = km + r$, where $n = (k+1)m + r$, $1 \leq r \leq k$.*

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A CONCISE PROOF OF A DOUBLE INEQUALITY INVOLVING THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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Abstract. In this note we provide a concise proof for a double inequality involving the exponential and logarithmic functions, our method is based on the usage of the majorization inequalities and the Schur-convexity of a function.

Keywords: inequality, exponential function, logarithmic function, Schur-convexity, majorization inequality, Stolarsky mean.

1. Introduction

In [4], Guo and Qi presented a double inequality involving the exponential and logarithmic functions, as follows:

$$\ln\frac{e^x-e^y}{x-y}<\frac{(x-1)e^x-(y-1)e^y}{e^x-e^y}<\ln\frac{e^x+e^y}{2}, \tag{1}$$

where x and y are arbitrary real numbers with $x \neq y$.

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In [4], the authors also mentioned that the idea of establishing inequality (1) was motivated by the following two inequalities [1, p. 352]:

$$e^{\frac{x+y}{2}} < \frac{e^x - e^y}{x-y} < \frac{e^x + e^y}{2}, \tag{2}$$

$$\frac{x+y}{2} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}, \tag{3}$$

where x and y are arbitrary real numbers with $x \neq y$.

Guo and Qi [4] proved the inequality (1) by the method of mathematical analysis. In this paper, we give a new proof of inequality (1) using the majorization inequalities (introduced by Hardy et al. [2]) and Schur-convexity (introduced by Schur [3]). As an application of inequality (1), we establish a comparison result of the Stolarsky means for different parameters.

2. Definitions and Lemmas

In this section, we need to introduce some definitions and lemmas relating to the theory of majorization inequalities.

Definition 1. [5, 6] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

1. $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \preceq \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order. Furthermore, $\boldsymbol{x}$ is said to be strictly majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\boldsymbol{x}$ is not permutation of $\boldsymbol{y}$.

2. Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ is said to be a strictly Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $f(\boldsymbol{x}) < f(\boldsymbol{y})$. f is said to be a strictly Schur-concave function on Ω if and only if $-f$ is strictly Schur-convex function on Ω.

Definition 2 ([5, 6])**.** Let $\Omega \subseteq \mathbb{R}^n$. Ω is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

Lemma 1 ([5, 6])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $f : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then f is the strictly Schur-convex (Schur-concave) function, if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) > 0 (< 0, respectively) \tag{4}$$

holds for any $\boldsymbol{x} \in \Omega^0$ and $x_1 \neq x_2$.

3. The proof of inequality (1)

Let us first deal with the left-hand inequality of (1), which reads as follows:

Proposition 1. *For arbitrary real numbers x, y with $x \neq y$, we have*

$$\ln \frac{e^x - e^y}{x - y} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}. \tag{5}$$

Proof. By the L'Hospital rule, it is easy to find that

$$\lim_{x \to y} \left[\ln \frac{e^x - e^y}{x - y} - \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y} \right] = 0.$$

Thus, we define a function $f(x, y)$ by

$$f(x,y) = \begin{cases} \ln \frac{e^x - e^y}{x-y} - \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}, & x \neq y \\ 0, & x = y. \end{cases}$$

Note that the inequality (5) is symmetrical with respect to variables x and y. To prove inequality (5), it is sufficient to prove that $f(x, y) < 0$ for $x > y$.

Let us now discuss the Schur-convexity of $f(x, y)$ on $\Omega = \{(x, y) : x > y, x, y \in \mathbb{R}\}$.

Differentiating $f(x, y)$ with respect to x gives

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{e^x}{e^x - e^y} - \frac{1}{x-y} - \frac{xe^x(e^x - e^y) - e^x[(x-1)e^x - (y-1)e^y]}{(e^x - e^y)^2} \\ &= \frac{e^{2x} - e^{x+y} - e^{2x} - (y - x - 1)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{x-y} \\ &= \frac{(x-y)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{x-y}. \end{aligned}$$

Similarly to the above, we have

$$\frac{\partial f}{\partial y} = \frac{(y-x)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{y-x}.$$

Hence,

$$\Delta_1 := (x-y)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = 2\left[\frac{(x-y)^2 e^{x+y}}{(e^x - e^y)^2} - 1\right].$$

It is easy to observe that

$$\frac{(x-y)^2 e^{x+y}}{(e^x - e^y)^2} - 1 < 0 \Longleftrightarrow e^{x+y} < \left(\frac{e^x - e^y}{x-y}\right)^2,$$

which is equivalent to, a known result, the left-hand side inequality of (2). Hence, we obtain $\Delta_1 < 0$. By Lemma 1, we conclude that $f(x,y)$ is strictly Schur-concave on Ω. Further, from an evident majorization relationship

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right) \prec (x,y),$$

along with the definition of Schur-concave function, we deduce that

$$0 = f\left(\frac{x+y}{2},\frac{x+y}{2}\right) > f(x,y),$$

which implies the desired inequality (5). The proof of Proposition 1 is complete. □

Let us now verify the validity of the right-hand inequality of (1), which is stated by Proposition 2 below.

Proposition 2. *For arbitrary real numbers x,y with $x \neq y$, we have*

$$\frac{(x-1)e^x-(y-1)e^y}{e^x-e^y} < \ln\frac{e^x+e^y}{2}. \tag{6}$$

Proof. By using the L'Hospital rule, it is not difficult to verify that

$$\lim_{x\to y}\left[\ln\frac{e^x+e^y}{2}-\frac{(x-1)e^x-(y-1)e^y}{e^x-e^y}\right]=0.$$

Thus, we define a function $g(x,y)$ by

$$g(x,y)=\begin{cases}\ln\frac{e^x+e^y}{2}-\frac{(x-1)e^x-(y-1)e^y}{e^x-e^y}, & x\neq y\\ 0, & x=y.\end{cases}$$

Because the inequality (6) is symmetrical with respect to variables x and y, in order to prove inequality (6), it is enough to prove that $g(x,y)>0$ for $x>y$.

In the following we discuss the Schur-convexity of $g(x,y)$ on $\Omega=\{(x,y): x>y, x,y\in\mathbb{R}\}$.

Direct computation gives

$$\begin{aligned}\frac{\partial g}{\partial x}&=\frac{e^x}{e^x+e^y}-\frac{xe^x(e^x-e^y)-e^x[(x-1)e^x-(y-1)e^y]}{(e^x-e^y)^2}\\&=\frac{e^x}{e^x+e^y}-\frac{e^{2x}-(x-y+1)e^{x+y}}{(e^x-e^y)^2}\end{aligned}$$

and

$$\frac{\partial g}{\partial y}=\frac{e^y}{e^x+e^y}-\frac{e^{2y}-(y-x+1)e^{x+y}}{(e^x-e^y)^2}.$$

Therefore,

$$\begin{aligned}\Delta_2 :&= (x-y)\left(\frac{\partial g}{\partial x}-\frac{\partial g}{\partial y}\right)\\ &= (x-y)\left[\frac{e^x-e^y}{e^x+e^y}+\frac{-e^{2x}+e^{2y}+2(x-y)e^{x+y}}{(e^x-e^y)^2}\right]\\ &= \frac{4e^{x+y}(x-y)^2}{(e^x+e^y)(e^x-e^y)^2}\left(\frac{e^x+e^y}{2}-\frac{e^x-e^y}{x-y}\right)\\ &= \frac{4e^{x+y}(x-y)^2}{(e^x+e^y)(e^x-e^y)^2}\left(\frac{e^x+e^y}{2}-\frac{1}{x-y}\int_y^x e^t dt\right).\end{aligned}$$

Recall the well-known Hermite-Hadamard inequality for a convex function ψ on the interval $[x, y]$:

$$\frac{1}{x-y}\int_y^x \psi(t)dt \le \frac{\psi(x)+\psi(y)}{2}. \tag{7}$$

If we take $\psi(t) = e^t$, then we have a strict inequality of (7), that is,

$$\frac{1}{x-y}\int_y^x e^t dt < \frac{e^x+e^y}{2}.$$

Hence, we obtain $\Delta_2 > 0$, this implies that $g(x, y)$ is strictly Schur-convex on Ω. Then, from

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right) \prec (x,y),$$

it follows that

$$0 = g\left(\frac{x+y}{2},\frac{x+y}{2}\right) < g(x,y),$$

which implies the required inequality (6). The Proposition 2 is proved. □

4. An application to the Stolarsky mean

In order to demonstrate the application of inequality (1), we establish a comparison result of the Stolarsky means for different parameters.

Let $(x, y) \in \mathbb{R}^2_+$. The Stolarsky mean of (x, y) is defined in [7] as

$$E(a,b;x,y)=\begin{cases}\left(\dfrac{b}{a}\cdot\dfrac{y^a-x^a}{y^b-x^b}\right)^{1/(a-b)}, & ab(a-b)(x-y)\neq 0,\\ \left(\dfrac{1}{a}\cdot\dfrac{y^a-x^a}{\ln y-\ln x}\right)^{1/a}, & a(x-y)\neq 0, b=0;\\ \dfrac{1}{e^{1/a}}\left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a-y^a)}, & a(x-y)\neq 0, a=b;\\ \sqrt{xy}, & a=b=0, x\neq y;\\ x, & x=y.\end{cases}$$

We have the following inequalities for the Stolarsky mean $E(a,b;x,y)$.

Proposition 3. *Let u,v be arbitrary positive numbers with $u \neq v$. Then*

$$E(1,0;u,v) < E(1,1;u,v) < E(2,1;u,v). \tag{8}$$

Proof. Taking $e^x = u$ and $e^y = v$ in the inequality (1), we obtain

$$\ln\left(\frac{u-v}{\ln u - \ln v}\right) < \frac{(\ln u - 1)u - (\ln v - 1)v}{u-v} < \ln\frac{u+v}{2}$$

$$\Longleftrightarrow \frac{u-v}{\ln u - \ln v} < \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{1/(u-v)} < \frac{u+v}{2}. \tag{9}$$

Obviously, the inequality (9) can be equivalently transformed to the desired inequality (8) according to the definition of $E(a,b;x,y)$ described above. This proves Proposition 3.

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CRITERION FOR NONEXISTENCE HORSESHOE-LIKE IN C^1 TOPOLOGY

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Abstract. In this paper we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition A with real and positive eigenvalues, then Λ is not horseshoe-like.
Keywords: hyperbolic set, partially hyperbolic set, horseshoe.

1. Introduction

Bowen in his remarkable survey on Anosov diffeomorphism has proved that C^{1+}-diffeomorphisms do not have fat horseshoes, these are horseshoes of positive Lebesgue measure. In contrast, he gave an example of a totally disconnected horseshoe on sphere $\mathbb{S}^2$ of positive volume. On the other hand, Bowen has proved that a basic set (locally maximal hyperbolic set with a dense orbit) of a C^2 diffeomorphism which attracts a set with positive volume, necessarily attracts a neighborhood of itself [3 Theorem 4.11]. In particular, the unstable manifolds through points of this set must be contained in it, and consequently C^2 diffeomorphisms have no horseshoes with positive volume. In this context A.Fakhari and M.Soufi proved that any partially hyperbolic horseshoe-like attractor of a C^1 -generic diffeomorphism has zero volume [4]. As well they constructed a C^1–diffeomorphism with a partially hyperbolic horseshoe-like attractor of positive volume. In this paper we show that under some conditions there is no horseshoe-like in the context of C^1-diffeomorphisms. Indeed we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition A with real and positive eigenvalues, then Λ is not horseshoe-like.

Let $f : M \to M$ be a diffeomorphism of a compact connected Riemannian manifold M. A set Λ is said to be invariant relative to f if $f(\Lambda) = \Lambda$.

For a point $x \in M$ the stable set of x is

$$W^s(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to +\infty\}$$

and the unstable of x is

$$W^u(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to -\infty\}.$$

Let $\mathcal{O}(p)$ be a hyperbolic periodic orbit of f, then the dimension of unstable manifold of p is called index of p.

A compact invariant set Λ is said to be *horseshoe-like* if there are local stable and local unstable manifolds through all its points which intersect Λ in a Cantor set.

A splitting $T_\Lambda M = E \oplus F$ of the tangent bundle restricted to an invariant set Λ is dominated splitting if there is a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E_x\|.\|Df^{-1} \mid F_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

Λ is *partially hyperbolic*, if additionally E is *uniformly contracting* or F is *uniformly expanding*, i.e there exists $0 < \lambda < 1$ such that

$$\|Df \mid E_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid F_{f(x)}\| \leq \lambda.$$

A compact invariant set Λ is called *hyperbolic* if there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to Λ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on M) for every $x \in \Lambda$

$$\|Df \mid E^s_x\| < \lambda$$

and

$$\|Df^{-1} \mid E^u_x\| < \lambda.$$

Alves and Pinheiro have studied nonuniformly expanding partially hyperbolic sets for C^{1+} diffeomorphisms [1]. They have proved that if non-uniformly expanding condition holds for a positive Lebesgue set of points, then Λ contains some local unstable disk. As a corollary, they deduced the non-existence of partially hyperbolic horseshoe like sets of positive volume. Also, Pacifico *et al.* have tried to construct Lorenz attractor of positive volume in the C^1-topology. The same result have obtained in the context of the volume preserving diffeomorphism. Indeed Xia proved in [2] that if an invariant set Λ of a volume-preserving C^{1+}-diffeomorphism f with positive volume has a dominated splitting $E \oplus F$, with E is uniformly contractive, then Λ contains stable leaves of almost every point. This argument leads to another proof of the classical result toward the ergodicity of C^{1+} volume-preserving Ansosov diffeomorphisms without using the Hopf argument.

2. Main theorems

In this section we present a condition that an invariant set satisfying it, is not horseshoe-like.

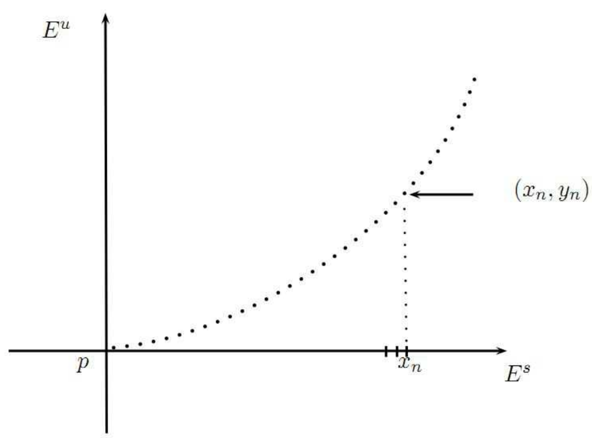
Definition. Let Λ be an invariant set. We say that a point $p \in M$ satisfies condition A if there are a local chart h at p and sequences $\{x_n\}$ and $\{w_n\} \subset T_pM$, $w_n = \sum_{i=1}^m \lambda_i^{w_n} v_i$ such that for $1 \le i \le m$,

$$\begin{cases} \lim_{n\to\infty} \frac{\lambda_i^{w_n} - \lambda_i^{w_{n+1}}}{\lambda_i^{w_{n+1}}} = 0 \\ \lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\ \lim_{n\to\infty} \lambda_i^{w_n} = 0 \\ h^{-1}(w_n) = x_n \in \Lambda, \end{cases}$$

where $\{v_1, v_2, ..., v_m\}$ is a basis of T_pM.

Remark. In the above definition $\lambda_i^{w_n}$ is a notation relative to w_n as a scaler. Indeed for any $\alpha \in T_pM$, since $\{v_1, v_2, ..., v_m\}$ is a basis of T_pM, so one can write $\alpha = \sum_{i=1}^m \lambda_i^{\alpha} v_i$ where λ_i^{α} for $1 \le i \le m$, are scalers.

Example 1. Let $f : M \to M$ be a C^1-diffeomorphism on a C^∞-manifold M with $dimM = 2$ and $p \in M$ be a hyperbolic fixed point of f. Let f at p in local chart be as $f(x, y) = (4x, \frac{1}{8}y)$ and $\{(x_n, y_n)\}$ be a sequence such that $x_n = \frac{1}{n}$, $y_n \to 0$ as the following figure.



If Λ is a closed invariant set containing p and $\{(x_n, y_n)\} \subset \Lambda$, then p satisfies condition A, since $x_{n+1} < x_n$, $\frac{x_n - x_{n+1}}{x_{n+1}} \to 0$ and $x_n \to 0$.

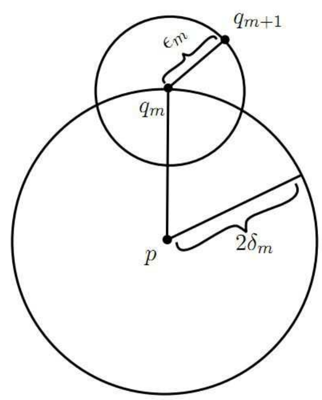
Definition. Let Λ be a close invariant subset of the compact manifold M. A point $p \in \Lambda$ is said to be *topologically dense point* if

$$\lim_{\delta\to 0} \frac{1}{\delta} \max\{\epsilon > 0 \mid B_\epsilon(x) \cap \Lambda = \emptyset,\ \forall x \in B_\delta(p)\} = 0$$

where $B_r(z) = \{x \in M \mid d(z, x) < r\}$.

Example 2. Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a perodic point containing a saddle fixed point $p \in \Lambda$ which is topologically dense and whose eigenvalues are real and positive.

p is topologically dense therefore for every $m \in \mathbb{N}$ there are positive integers ϵ_m, δ_m such that $\frac{\epsilon_m}{2\delta_m} \to 0$ as $m \to +\infty$ and $B_{\delta_m}(x) \cap \Lambda \neq \emptyset$ for every $x \in B_{2\delta_m}(p)$. Thus by induction we find sequence $\{q_m\} \subseteq \Lambda$ such that $d(q_m, q_{m+1}) = \epsilon_m$ and $d(q_m, p) = 2\delta_m$ (see the following Figure). So by taking suitable charts we can suppose that $p = 0$ and $\lim_{m\to\infty} \frac{q_m - q_{m+1}}{q_m} = 0$. Therefore p satisfies condition A.



The following theorem shows that Example 2 is a prototype structures for an invariant set to be not horseshoe-like.

Theorem 1. *Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a periodic point. Suppose Λ contains a saddle fixed point p satisfying condition A with real and positive eigenvalues. Then Λ is not horseshoe-like.*

Proof. We show that there is a connected component in Λ which is not consist of a single point. So Λ is not a Cantor set and hence it is not horseshoe-like. Since p is a hyperbolic point, there is an $\epsilon_0 > 0$ and a homeomorphism $h : B_{\epsilon_0}(p) \to T_pM$ such that

$$h(p) = 0 \tag{1}$$

$$D_p f o h = h o f. \tag{2}$$

There exists $\epsilon' > 0$ such that

$$\left\{v \in T_pM \mid \|v\| < \epsilon'\right\} = h(B_{\epsilon_0}(p)).$$

Let $\{\tilde{\lambda}_i \mid 1 \leq i \leq s\}$ be the set of all eigenvalues of D_pf which norm greater than 1. Denote by $\{\tilde{\lambda}_i \mid s+1 \leq i \leq m\}$ the set of all eigenvalues of D_pf which norm less than 1 and let $\{v_1, \ldots, v_s\}$ and $\{v_{s+1}, \ldots, v_m\}$ be the set of eigenvectors of $\{\tilde{\lambda}_i \mid 1 \leq i \leq s\}$ and $\{\tilde{\lambda}_i \mid s+1 \leq i \leq m\}$ respectively. Put

$$\bar{L} = \left\{ \sum_{i=1}^{m} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2m} \right\}$$

and

$$L = \left\{ \sum_{i=1}^{s} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2s} \right\}.$$

We can see that $h^{-1}(\bar{L})$ and $h^{-1}(L) \subset B_{\epsilon_0}(p)$. Since p is a saddle point with condition A, there are sequences $\{x_n\}$ and $\{w_n\} \subset T_pM$, $w_n = \sum_{i=1}^{m} \lambda_i^{w_n} v_i$ such that for $1 \le i \le s$, we have

$$(1.3) \qquad \begin{cases} \lim_{n\to\infty} \frac{\lambda_i^{w_n} - \lambda_i^{w_{n+1}}}{\lambda_i^{w_{n+1}}} = 0 \\ \lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\ \lim_{n\to\infty} \lambda_i^{w_n} = 0 \\ h^{-1}(w_n) = x_n \in \Lambda \end{cases}$$

Let $z \in L$. For every $s+1 \le i \le m$ there exists $N_i \in \mathbb{N}$ such that for any $n \ge N_i$, $0 < \lambda_i^{w_n} < \frac{\delta}{2M(m-s)}$ where

$$M = \max\{\tilde{\lambda}_i \mid 1 \le i \le m\}.$$

Put $N_0 = \max\{N_i \mid s+1 \le i \le m\}$. Since $z \in L$, we can consider $z = \sum_{i=1}^{s} \lambda_i^z v_i$ such that $0 < \lambda_i < \frac{\epsilon'}{2s}$. For every $k > N_0$ put

$$m_k = \min \left\{ m \mid \lambda_i^{w_{m+1}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \right\}$$

for $1 \le i \le s$. So

$$\lambda_i^{w_{m_{k+1}}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \le \lambda_i^{w_{m_k}}$$

for $1 \le i \le s$.

We claim that there exists $k_0 > N_0$ such that for $1 \le i \le s$,

$$\lambda_i^{w_{m_{k_0+1}}} > \frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^{k_0}}.$$

Suppose our claim is not true. Hence for every $k > N_0$ and some $1 \le i \le s$

$$\lambda_i^{w_{m_{k+1}}} \le \frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^k} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \le \lambda_i^{w_{m_k}}.$$

So

$$\frac{\lambda_i^{w_{m_k}} - \lambda_i^{w_{m_{k+1}}}}{\lambda_i^{w_{m_{k+1}}}} \ge \frac{\frac{\frac{2\delta}{2s\tilde{\lambda}_i^k}}{\lambda_i^z + \frac{\delta}{2s}}}{2s\tilde{\lambda}_i^k} = \frac{2\delta}{\lambda_i^z + \frac{\delta}{2s}} > 0$$

that contradicts (1.3). Hence there exists $n_0 > N_0$ such that

$$\frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^{n_0}} < \lambda_i^{w_{m_{n_0}+1}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^{n_0}}$$

for $1 \leq i \leq s$. This shows that

$$\sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{y_{m_{n_0}}+1} - \lambda_i^z \right| < \frac{\delta}{2}.$$

Hence we have

$$\begin{aligned}
&\left\| Df^{n_0} \left(\sum_{i=1}^{m} \lambda_i^{w_{m_{n_0}}+1} v_i \right) - z \right\| \\
&= \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i + \sum_{i=s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i - \sum_{i=1}^{s} \lambda_i^z v_i \right\| \\
&\leq \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda^{w_{m_{n_0}}+1} v_i - \sum_{i=1}^{s} \lambda_i^z v_i \right\| + \left\| \sum_{s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i \right\| =: B
\end{aligned}$$

Since for $s+1 \leq i \leq m$, $\tilde{\lambda}_i^{n_0} < 1$ and $0 < \lambda^{w_{m_{n_0}}+1} < \frac{\delta}{2M(m-s)}$. Hence

$$\begin{aligned}
B &\leq \sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} - \lambda_i^z \right| + \sum_{i=s+1}^{m} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} \right| \\
&< \frac{\delta}{2} + \frac{(m-s)\delta}{2(m-s)M} < \delta.
\end{aligned}$$

This shows that $Df^{n_0} \left(\sum_{i=1}^{m} \lambda_i^{w_{m_{n_0}}+1} v_i \right) \in B_\delta(z)$. This shows that for every $x \in h^{-1}(L)$ there is sequence such that

$$\begin{aligned}
&D_p f^m(v_{n_m}) \longrightarrow h(x) \\
&\{h(v_{n_m}) = x_{n_m}\} \subset \Lambda
\end{aligned}$$

since $h^{-1} \circ D_p f \circ h = f$ so

$$f^m(x_{n_m}) \longrightarrow x.$$

Λ is closed and invariant so we have $x \in \Lambda$. Hence $h^{-1}(L) \subset \Lambda$. Note that $h^{-1}(L)$ is connected component. Hence Λ is not like horseshoe. $\square$

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NON-LINEAR STABILITY OF L_4 IN THE R3BP WHEN THE SMALLER PRIMARY IS A HETEROGENEOUS TRIAXIAL RIGID BODY WITH N LAYERS

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Abstract. In the present paper, the non-linear stability of the triangular libration point (L_4) in the restricted three-body problem(R3BP) when less massive primary is a heterogeneous triaxial rigid body has been studied with the assumption that the primary has N layers having different densities. Following the procedure of Birkhoff's normalization, we normalized the Hamiltonian up to second order and the co-ordinates (x, y) are expanded in double D'Alembert series. The non-linear stability of the triangular libration point is discussed by applying Moser's modified version of Arnold's theorem (1961) as well as following the procedure as adopted by Bhatnagar and Hallan (1983). It is observed that Moser's theorem is applicable in the range of linear stability, except for three mass ratios depending upon heterogeneous triaxial rigid body.

Keywords: R3BP, libration point L_4, non-linear stability, heterogeneous triaxial rigid body, densities.

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1. Introduction

The restricted three-body problem (R3BP) concerned the motion of the infinitesimal mass under the gravitational influence of two finite bodies: There are two masses, called primaries, moving in circular orbits around their common center of mass. There is another mass, which is infinitesimal, moving in the plane of motion of the primaries such that it does not influence the motion of the primaries but is influenced by them. To describe the motion of the third mass is called the circular restricted problem. In case the primaries move in elliptical orbits instead of circular orbits, the problem is known as pseud restricted problem. R3BP has been stated by many authors. In the restricted three-body problem, distinct particular solutions exist. These particular solutions are known as Lagrangian points and in the bi-dimensional system, five Lagrangian points exist. Three of these are the collinear points and the other two are the triangular points. Several authors have investigated the stability of these points in linear sense and found that the collinear points are unstable where as non collinear points are stable in the some range.

Many mathematicians and astronomers have discussed non-linear stability by taking different aspects of the restricted three-body problem and made valuable contributions. Deprit and Deprit (1967) discussed the stability of the triangular Lagrangian points. Bhatnagar and Hallan(1983) studied the effect of perturbations in Coriolis and Centrifugal force on the nonlinear stability of the equilibrium points in the restricted problem of three bodies . Gyorgyey (1985) investigated on the non-linear stability of motions around L_5 in the elliptic restricted problem of three bodies. Krzysztof et al. (1991) studied about the libration points in the restricted photo-gravitational three-body problem. Sharma et al. (1997) discussed on the effect of oblateness on the non-linear stability of L_4 in the restricted three-body problem. Esteban et al. (2001) analyzed the rotating stratified heterogeneous oblate spheroid in Newtonian physics. Jain et al. (2001) studied on the non-linear stability of L_4 in the restricted three-body problem when the primaries are triaxial rigid bodies. Andres et al.(2001) studied the non-linear stability of the equilibria in the gravity field of a finite straight segment. Chandra et al. (2004) discussed the effect of oblateness on the non-linear stability of the triangular liberation points of the restricted three-body problem in the presence of resonances. Aggarwal et al. (2006) investigated the Non-linear stability of L_4 restricted three-body problem for radiated axes symmetric primaries with resonances. Kushvah et al.(2007) studied the non-linear stability in the generalized photo-gravitational restricted three-body problem with Poynting-Robertson drag. Singh (2011) examined the non-linear stability in the restricted three-body problem with oblate variable mass. Ishwar et al. (2012) investigated the non-linear stability in photo-gravitational non-planer R3BP with oblate smaller primary. Jain et al. (2014) studied the non-linear stability of L_4 in the restricted problem when the primaries are finite straight segment under resonances. Ansari (2017 a, b) investigated the dynamical behavior in the restricted three-body problem with perturbations. Shalini et al. (2016, 2017) studied the stability of L_4 in the R3BP by taking the smaller primary is a heterogeneous spheroid with layers.

In the present work, we propose to discuss the R3BT with the assumption that the massive primary is a point mass and less massive primary is a heterogeneous triaxial rigid body with N layers, having different densities ρ_i and axes (a_i, b_i, c_i), $(i = 1, 2, 3, \ldots, N)$ respectively. The main objective of this paper is to study of the stability of libration point L_4 in non-linear sense. For this we will apply Moser's modified version of Arnold's theorem (1961) and procedure as adopted by Bhatnagar and Hallan (1983).

This paper should be read in conjunction with the papers by Bhatnagar and Hallan (1983) and Shalini (2017). As to save space, we are not mentioning the values of various variables given in those papers, although they are used in this paper.

This paper is organized as follows: In section Introduction we have reviewed the literature related to R3BP under different perturbations. In section Equations of motion we have derived the potential of heterogeneous triaxial rigid body with N layers and mean motion of the primaries and further, formulated the equations of motion of the proposed system. Section Location of Triangular Points,we have obtained the coordinates of non-collinear libration points. Section First order normalization, deals with the first order normalization. In section Second order normalization, we have determined the second order normalization. In section Second order coefficients in the frequencies, we have found the second order frequencies. In section Stability, we have checked the non-linear stability of triangular libration points. Section Conclusion, contains the conclusion of the obtained results.

2. Equations of motion

Let m_1 and m_2 $(m_1 > m_2)$ be the two masses of the primaries at P_1 and P_2 respectively as shown in Fig. (1) are moving in circular orbits around their common center of mass O which is taken as the origin, $OP_1 = 1-\mu$ and $OP_2 = \mu$, as the distances of primaries from the center of mass. Let m_1 be point mass and m_2 a heterogeneous triaxial rigid body with N layers having axes (a_i, b_i, c_i) $(a_i > b_i > c_i)$ and densities ρ_i. An infinitesimal mass m_3 which is much less than masses of the primaries is moving in the plane of motion of m_1 and m_2. Let $\overrightarrow{r}$, $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ as the distances of infinitesimal mass, first primary and second primary from center of mass respectively. $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are the gravitational forces acting on m_3 due to m_1 and m_2 respectively. Also let us consider that the principal axes of heterogeneous triaxial rigid body remain parallel to the synodic axes $Oxyz$ throughout the motion and the equatorial plane of m_2 is coincide with the plane of motion of m_1 and m_2.$(Fig.1(a),(b))$ The equation of motion of m_3 in the vector form is

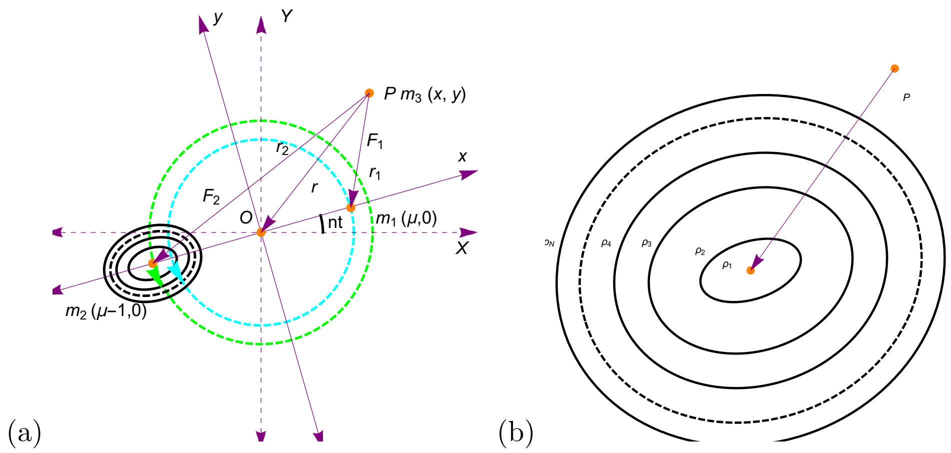

Figure 1: (a): Configuration of the restricted three-body problem with m_2 as heterogeneous triaxial rigid body, (b): Heterogeneous triaxial rigid body with N Layers

$$m_3\left(\frac{\partial^2\overrightarrow{r}}{\partial t^2}+2\overrightarrow{\omega}\times\frac{\partial\overrightarrow{r}}{\partial t}+\frac{\partial\overrightarrow{\omega}}{\partial t}\times\overrightarrow{r}+\overrightarrow{\omega}\times(\overrightarrow{\omega}\times\overrightarrow{r})\right)=\overrightarrow{F},\tag{1}$$

where $\overrightarrow{r}=\overrightarrow{OP}$, $\overrightarrow{\omega}=\mathrm{n}\,\widehat{k}=$ angular velocity = constant, $\overrightarrow{F}=$ total force acting on m_3.

The gravitational potential of the heterogeneous triaxial rigid body of mass m_2, with N layers of densities ρ_i and axes (a_i, b_i, c_i), $\rho_i < \rho_{i+1}$, $a_i < a_{i+1}$, $b_i < b_{i+1}$, $c_i < c_{i+1}$ at the point P is

$$V_2 = V_{NN} + V_{(N-1)N} + \ldots + V_{2N} + V_{1N}(say),\tag{2}$$

where $V_{NN}, V_{(N-1)N}, \ldots, V_{2N}, V_{1N}$, are the potential of the triaxial rigid body of densities $\rho_{NN}, \rho_{(N-1)N}, \ldots\ldots, \rho_{2N}, \rho_{1N}$, for the regions $N, N-1, \ldots, 1$, respectively. Here, $V_{NN} = V'_{NN} - V'_{N(N-1)}(say)$, where $V'_{NN}=$ potential of the triaxial rigid body of axes (a_N, b_N, c_N) with homogeneous density ρ_N throughout at $P, = \frac{-4\pi\rho_N G}{3r_2}a_N b_N c_N[1+\frac{1}{10r_2^2}(2a_N^2-b_N^2-c_N^2-\frac{3}{r_2^2}(a_N^2-b_N^2)y^2)]$, and $V'_{N(N-1)}=$ potential of the triaxial rigid body of axes $(a_{N-1}, b_{N-1}, c_{N-1})$ with homogeneous density ρ_N throughout at $P=\frac{-4\pi\rho_N G}{3r_2}a_{N-1}b_{N-1}c_{N-1}[1+\frac{1}{10r_2^2}(2a_{N-1}^2-b_{N-1}^2-c_{N-1}^2-\frac{3}{r_2^2}(a_{N-1}^2-b_{N-1}^2)y^2)]$.

Thus,

$$\begin{cases} V_{NN}=\frac{-4\pi\rho_N G}{3r_2}[a_N b_N c_N\{1+\frac{1}{10r_2^2}(2a_N^2-b_N^2-c_N^2-\frac{3}{r_2^2}(a_N^2-b_N^2)y^2)\} \\ -a_{N-1}b_{N-1}c_{N-1}\{1+\frac{1}{10r_2^2}(2a_{N-1}^2-b_{N-1}^2-c_{N-1}^2-\frac{3}{r_2^2}(a_{N-1}^2-b_{N-1}^2)y^2)\}]. \end{cases}\tag{3}$$

Similarly, we can find $V_{(N-1)N}, \ldots, V_{2N}, V_{1N}$, and substituting in Eq.(2), we have

$$V_2 = -\frac{m_2G}{r_2} - \frac{k_1'G}{2r_2^3} + \frac{k_2'Gy^2}{2r_2^5}. \tag{4}$$

where

$$k_1' = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i - \rho_{i+1})a_ib_ic_i\sigma_{i,1}), \quad k_2' = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i - \rho_{i+1})a_ib_ic_i\sigma_{i,2}),$$

$\sigma_{i,1} = \frac{(2a_i^2-b_i^2-c_i^2)}{5}$, $\sigma_{i,2} = \frac{(a_i^2-b_i^2)}{5}$, $\sigma_{N+1} \neq 0$.

Hence, the total potential at P due to m_1 and m_2 is given by $V = -\frac{m_1G}{r_1} - \frac{m_2G}{r_2} - \frac{k_1'G}{2r_2^3} + \frac{k_2'Gy^2}{2r_2^5}$.

Let us fix the units of mass,length and time with the assumption that the gravitational constant G, the sum of the masses and the distance between both primaries be equal to unity.

Then the equations of motion in Synodic co-ordinates system and dimensionless variables are

$$\begin{cases} \ddot{x} - 2n\dot{y} = \frac{\partial\Omega}{\partial x} \\ \ddot{y} + 2n\dot{x} = \frac{\partial\Omega}{\partial y} \end{cases} \tag{5}$$

where $\Omega = n(\frac{x^2+y^2}{2}) + \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + \frac{k_1}{2r_2^3} - \frac{3k_2y^2}{r_2^5}$, $r_1^2 = (x-\mu)^2 + y^2$, $r_2^2 = (x-\mu+1)^2 + y^2$,

$$k_1 = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i' - \rho_{i+1}')a_i'b_i'c_i'\sigma_{i,1}'), \quad k_2 = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i' - \rho_{i+1}')a_i'b_i'c_i'\sigma_{i,2}'),$$

$\sigma_{i,1}' = \frac{(2a_i'^2-b_i'^2-c_i'^2)}{5R^2}$, $\sigma_{i,2}' = \frac{(a_i'^2-b_i'^2)}{5R^2}$, $a_i' = \frac{a_i}{R}$, $b_i' = \frac{b_i}{R}$, $c_i' = \frac{c_i}{R}$, $\rho_i' = \frac{\rho_i}{M}$, $\rho_{N+1}' \neq 0$, $M = m_1 + m_2$, $k_1, k_2 << 1$, R = dimensional distance between the primaries.

2.1 Mean motion

The potential of the triaxial rigid body is $-(\frac{m_2G}{R} + \frac{k_1G}{2R^3})$. Let the distances of m_1 and m_2 from the center of mass O be a' and b' respectively. Since m_1 and m_2 are moving in circular orbits about O, we have $m_1a'n^2 = (\frac{m_2G}{R^2} + \frac{3k_1G}{2R^4})m_1$ and $m_2b'n^2 = (\frac{m_2G}{R^2} + \frac{3k_1G}{2R^4})m_1$.

Adding these equations, we have $n^2 = (\frac{m_2G}{(a'+b')^3} + \frac{3k_1G}{2(a'+b')^5})(\frac{m_1+m_2}{m_2})$. Using the dimensionless variables, we get the mean motion as

$$n = 1 + ck_1, \tag{6}$$

where $c = \frac{3}{4\mu}$.

3. Locations of triangular points

The locations of triangular libration points are solutions of the Eq. (5) obtained by making all the derivatives equals to zero (i.e.$\Omega_x = 0$ and $\Omega_y = 0$). i.e.:

$$\begin{aligned} & nx - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{(3k_1)(x+1-\mu)}{2r_2^5} \\ & + \frac{(15k_2)(x+1-\mu)y^2}{2r_2^7} = 0, \end{aligned} \tag{7}$$

And

$$(n - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5} - \frac{3k_2}{r_2^5} + \frac{15k_2y^2}{2r_2^7})y = 0. \tag{8}$$

From equation (8), we have two cases, either $y = 0$, or

$$(n - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5} - \frac{3k_2}{r_2^5} + \frac{15k_2y^2}{2r_2^7}) = 0. \tag{9}$$

The collinear libration points are the solution of the equation (7), when $y = 0$. The non-collinear libration points are the solutions of the equations (7) and (9).

If we put $k_1 = k_2 = 0$ in equations (7) and (8), we get the classical case of the R3BP and consequently. $r_1 = r_2 = 1$ is the required solution. Now, we discuss only the location of libration point L_4. For this, we suppose that the solution of the above equations when $y \neq 0$ are $r_1 = 1 + \pi$, $r_2 = 1 + \pi'$, $\pi, \pi' \ll 1$. Putting the values of r_1 and r_2 in the equations $r_1^2 = (x-\mu)^2 + y^2$, $r_2^2 = (x-\mu+1)^2 + y^2$ and solving, we get $x = \mu - \frac{1}{2} - (\pi - \pi')$, $y = \sqrt{\frac{3}{2} - \frac{1}{\sqrt{3}}(\pi + \pi')}$.

Now, we substitute the values of r_1, x, y in equations (7) and (9) and rejecting the second and higher order terms of π and π', we get the co-ordinates of the stationary points $L_4(x, y)$ and $L_5(x, -y)$ as $x = \mu - \frac{1}{2} + p_1k_1 + p_2k_2$, $y = \frac{\sqrt{3}}{2} + p_3k_1 + p_4k_2$, respectively, where $p_1 = \frac{-1}{2\mu}$, $p_2 = \frac{7\mu-11}{8\mu(\mu-1)}$, $p_3 = \frac{3-4c\mu}{6\sqrt{3}\mu}$, $p_4 = \frac{11-5\mu}{8\sqrt{3}\mu(\mu-1)}$.

4. First order normalization

Following the procedure as adopted by Bhatnagar and Hallan (1983) to derived the first order normalization. The Lagrangian function of the equation(5) is given by

$$\Gamma = \frac{1}{2}\{\dot{x}^2 + \dot{y}^2 + n^2(x^2 + y^2) + 2n(x\dot{y} - y\dot{x})\} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{k_1}{2r_2^3} - \frac{3k_2y^2}{2r_2^5}. \tag{10}$$

Now shifting the origin to $L_4(x, y)$, and expanding Γ in power series of x and y, it can be expressed as $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + ...$, where

$$\begin{aligned}
\Gamma_0 &= \frac{1}{8}(11 + \gamma^2 + t_1k_1 - 9k_2),\\
\Gamma_1 &= \frac{-\dot{x}}{2}(\sqrt{3} + t_2k_1 + 2p_4k_2) - \frac{\dot{y}}{2}(\gamma + t_3k_1 - 2p_2k_2)\\
&\quad + x(t_4k_1 + t_5k_2) + y(t_6k_1 + t_7k_2),\\
\Gamma_2 &= \frac{1}{2}(\dot{x^2} + \dot{y^2}) + n(x\dot{y} - y\dot{x}) - \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2)xy\\
&\quad + \frac{1}{64}(72 + t_{10}k_1 + t_{11}k_2)y^2 + \frac{1}{64}(24 + t_{12}k_1 + t_{13}k_2)x^2,\\
\Gamma_3 &= \frac{3}{32}(-2\sqrt{3} + t_{14}k_1 + t_{15}k_2)y^3 + \frac{x^3}{32}(-14\gamma + t_{16}k_1 + t_{17}k_2)y^2\\
&\quad - \frac{3xy^2}{32}(-22\gamma + t_{18}k_1 + t_{19}k_2) + \frac{3x^2y}{64}(2\sqrt{3} + t_{20}k_1 + t_{21}k_2)x^2,\\
\Gamma_4 &= \frac{x^4}{512}(-148 + t_{22}k_1 + t_{23}k_2) + \frac{3y^4}{256}(2 + t_{24}k_1 + t_{25}k_2)y^2\\
&\quad - \frac{15xy^3}{64}(6\sqrt{3}\gamma + t_{26}k_1 + t_{27}k_2) + \frac{5x^3y}{64}(10\sqrt{3}\gamma + t_{28}k_1 + t_{29}k_2)\\
&\quad + \frac{1}{64}(1 + t_{30}k_1 + t_{31}k_2)x^2y^2,
\end{aligned}$$

all the values of t_i, $(i = 1, 2, ...31)$ are refer in Appendix.

Corresponding to the Lagrangian function Γ given by equation (5), the Hamiltonian function is given by $H = -\Gamma + p_x\dot{x} + p_y\dot{y}$, where p_x and p_y are the momenta coordinates and given by $p_x = \frac{\partial\Gamma}{\partial\dot{x}} = \dot{x} - ny$, $p_y = \frac{\partial\Gamma}{\partial\dot{y}} = \dot{y} + nx$. Finally, the Hamiltonian function becomes $H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + n(yp_x - xp_y) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{k_1}{2r_2^3} + \frac{k_2y^2}{2r_2^5}$. Applying the following translation $x \to (x - \frac{\gamma}{2}p_1k_1 + p_2k_2)$, $y \to (y + \frac{\sqrt{3}}{2}p_3k_1 + p_4k_2)$, $p_x \to p_x - n(\frac{\sqrt{3}}{2} + p_3k_1 + p_4k_2)$, $p_y \to p_y + n(\frac{-\gamma}{2} + p_1k_1 + p_2k_2)$, One can find the Hamiltonian H as $H = \sum_{k=0}^{\infty} H_k$, where $H_k =$ the sum of the terms of k^{th} degree homogenous in variables x, y, p_x, p_y.

Now

$$\begin{aligned}
H_0 &= -\Gamma_0,\\
H_1 &= \frac{1}{64}(-40 - 12(11 + 11\gamma p_1 - 3\sqrt{3}p_3)k_1\\
&\quad + (141 - 132\gamma p_2 + 36\sqrt{3}p_4)k_2),\\
H_2 &= \frac{1}{2}(p_x^2 + p_y^2) + n(yp_x - xp_y) + Ex^2 + Fy^2 + 2Gxy, \qquad (11)\\
H_3 &= -\Gamma_3,\\
H_4 &= -\Gamma_4,
\end{aligned}$$

where $E = \frac{1}{64}(8 + c_1k_1 + c_2k_2)$, $F = \frac{1}{64}(-40 + c_3k_1 + c_4k_2)$, $G = \frac{3}{16}(8\sqrt{3}\gamma + c_5k_1 + c_6k_2)$, $c_1 = 12(-1 + 7\gamma p_1)$, $c_2 = 3(45 + 28\gamma p_2 + 4\sqrt{3}p_4)$, $c_3 = -12(11 +$

$11\gamma p_1 - 3\sqrt{3}p_3)$, $c_4 = 141 - 132\gamma p_2 + 36\sqrt{3}p_4$, $c_5 = 4(-5\sqrt{3} + \sqrt{3}p_1 - 11\gamma p_3)$, $c_6 = 65\sqrt{3} + 4\sqrt{3}p_2 - 44\gamma p_4)$.

To investigate the stability of motion as in Whittaker (1965), we consider the following set of linear equations in the variables x and y

$$-\lambda p_x = \frac{\partial H_2}{\partial x} = 2Ex + Gy - np_y,$$

$$-\lambda p_y = \frac{\partial H_2}{\partial y} = 2Fy + Gx - np_x, \tag{12}$$

$$\lambda x = \frac{\partial H_2}{\partial p_x} = p_x + ny,$$

$$\lambda y = \frac{\partial H_2}{\partial p_y} = p_y - nx, \quad i.e. AX = 0,$$

where

$$A = \begin{pmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}.$$

The equation (12) will have a non-zero solution if and only if $Det(A) = 0$, which implies that $\lambda^4 + 2\lambda^2(E + F + n^2) + FE - G^2 - 2n^2(E + F) + n^4 = 0$.

and so the characteristic equation corresponding to Hamiltonian H_2 given in equation(11) is given by $16\lambda^4 + (16 + 2(-36 + 32c - 12\gamma p_1 + 9\sqrt{3}p_3)k_1 - 6(-23 + 4\gamma p_2 - 4\sqrt{3}p_4)k_2)\lambda^2 + 27(1-\gamma^2) + 4(252 + 384c + 540\gamma - 288\gamma p_1 + 18\sqrt{3}(-3 + 22\gamma^2)p_3)k_1 + 4(-819 - 1755\gamma - 288\gamma p_2 + 36\sqrt{3}(-3 + 11\gamma^2)p_4)k_2 = 0$,

The stability of Libration point L_4 is assured only when the discriminant of the characteristic equation is greater than zero, implying that $\mu < \mu_c = \mu_0 - (3.76183..)k_1 + (10.825...)k_2$, where $\mu_0 = 0.0385208965\ldots$ When $D > 0$, the roots $\pm i\omega_1'$ and $\pm i\omega_2'$ (ω_1' and ω_2' being long/short-periodic frequencies) are related to each other as $\omega_1'^2 + \omega_2'^2 = 1 + p_5k_1 + p_6k_2$, $\omega_1'^2\omega_2'^2 = \frac{27}{16}(1-\gamma^2) + p_7k_1 + p_8k_2$, $(0 < \omega_1' < \omega_2' < \frac{1}{\sqrt{2}})$, where $p_5 = \frac{1}{8}(-36 + 32c - 12\gamma p_1 + 9\sqrt{3}p_3)$, $p_6 = \frac{-3}{8}(-23 + 4\gamma p_2 - 4\sqrt{3}p_4)$, $p_7 = \frac{1}{64}(252 + 384c + 540\gamma - 288\gamma p_1 + 18\sqrt{3}(-3 + 22\gamma^2)p_3)$, $p_8 = \frac{1}{64}(-819 - 1755\gamma - 288\gamma p_2 + 36\sqrt{3}(-3 + 11\gamma^2)p_4)$.

It is observed that the perturbed frequencies (ω_1', ω_2') are related to the unperturbed one (ω_1, ω_2) as $\omega_1' = \omega_1(1 + pk_1 + p'k_2)$, $\omega_2' = \omega_2(1 + qk_1 + q'k_2)$, where $p = \frac{27p_5 - 27\gamma^2 p_5 - 16p_7\omega_2^2}{54(1-\gamma^2)k^2}$, $p' = \frac{27p_6 - 27\gamma^2 p_6 - 16p_8\omega_2^2}{54(1-\gamma^2)k^2}$, $q = \frac{27p_5 - 27\gamma^2 p_5 - 16p_7\omega_1^2}{54(-1+\gamma^2)k^2}$, $q' = \frac{27p_6 - 27\gamma^2 p_6 - 16p_8\omega_1^2}{54(-1+\gamma^2)k^2}$, $k^2 = 2\omega_1^2 - 1 = 1 - 2\omega_2^2$,

Following the method given in Whittaker (1965), we use a canonical transformation from the phase space (x, y, p_x, p_y) into the phase space of the angles (ϕ_1, ϕ_2) and the actions (I_1, I_2), so that the Hamiltonian H_2 be normalized.

$$X = JT, \tag{13}$$

where

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix},$$

$$J = (a'_{ij})_{1\le i,j\le 4},\ Q_i = (\frac{2I_i}{\omega'_i})^{\frac{1}{2}} \sin\phi_i,\ P_i = (2I_i\omega'_i)^{\frac{1}{2}} \cos\phi_i,\ \ (i = 1, 2).$$

Now, we have calculated all the elements of J and we obtain $a'_{ij} = a_{ij}(1 + \alpha_{ij}k_1 + \alpha'_{ij}k_2)$, $i, j = 1, 2, 3, 4$, where $a_{11} = a_{12} = 0$, $a_{13} = \frac{l_1}{2\omega_1 k_1}$, $a_{14} = \frac{l_2}{2\omega_2 k_2}$, $a_{21} = \frac{-4\omega_1}{l_1 k}$, $a_{22} = \frac{-4\omega_2}{l_2 k}$, $a_{23} = \frac{3\sqrt{3}\gamma}{2\omega_1 l_1 k}$, $a_{24} = \frac{3\sqrt{3}\gamma}{2\omega_2 l_2 k}$, $a_{31} = \frac{-\omega_1 m_1}{2l_1 k}$, $a_{32} = \frac{-\omega_2 m_2}{2l_2 k}$, $a_{33} = \frac{3\sqrt{3}\gamma}{2\omega_1 l_1 k}$, $a_{34} = \frac{3\sqrt{3}\gamma}{2\omega_2 l_2 k}$, $a_{41} = \frac{3\sqrt{3}\gamma\omega_1}{2l_1 k}$, $a_{42} = \frac{3\sqrt{3}\gamma\omega_2}{2l_2 k}$, $a_{43} = \frac{n_1}{2\omega_1 l_1 k}$, $a_{44} = \frac{n_2}{2\omega_2 l_2 k}$, and all the values of α_{ij} and α'_{ij}, $i, j = 1, 2, 3, 4$, are given in Appendix.

The transformation changes the second order part of the Hamiltonian into the normal form $H_2 = \omega'_1 I_1 - \omega'_2 I_2$ and the general solutions of the corresponding equations of motion are $I_i =$ Constant $(i = 1, 2)$, $\phi_1 = \omega'_1 t+$ Constant, $\phi_2 = \omega'_2 t+$ Constant.

5. Second order normalization

Moser's conditions are utilized for transforming the Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. Here we wish to perform Birkhoff's normalization for which the co-ordinates (x, y) are to be expanded in double D'Alembert series:

$$x = \sum_{n\ge 1} B_n^{1,0}, \qquad y = \sum_{n\ge 1} B_n^{0,1},$$

where the homogeneous components $B_n^{1,0}$ and $B_n^{0,1}$ of degree n in $\sqrt{I_1}$, $\sqrt{I_2}$ are of the form

$$\sum_{0\le m\le n} I_1^{\frac{1}{2(n-m)}} I_2^{\frac{1}{2(n-m)}} \sum_{(i,j)} (C_{n-m,m,i,j} \cos(i\phi_1 + j\phi_2) + S_{n-m,m,i,j} \sin(i\phi_1 + j\phi_2)). \tag{14}$$

The double summation over the indices i and j is such that (a) i runs over those integers in the interval $0 \le i \le n - m$ that have the same parity as $n - m$ (b) j runs over those integers in the interval $-m \le j \le m$ that have the same parity as m. I_1 and I_2 are to be regarded as constants of integration and ϕ_1, ϕ_2 are to be determined as linear functions of time such that

$$\dot{\phi_1} = \omega'_1 + \sum_{n\ge 1} f_{2n}(I_1, I_2),\ \ \dot{\phi_2} = -\omega'_2 + \sum_{n\ge 1} g_{2n}(I_1, I_2),$$

where f_{2n} and g_{2n} are of the form

$$f_{2n} = \sum_{0\le m\le n} f'_{2(n-m),2m} I_1^{n-m} I_2^m, \quad g_{2n} = \sum_{0\le m\le n} g'_{2(n-m),2m} I_1^{n-m} I_2^m.$$

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y given by equation(14). The second order components $B_2^{1,0}$ and $B_2^{0,1}$ are solutions of the partial differential equations $\Delta_1\Delta_2 B_2^{1,0} = \Phi_2$ and $\Delta_1\Delta_2 B_2^{0,1} = \Psi_2$, where $\Delta_i = (D^2 + \omega_i'^2)$, $(i = 1, 2)$, $D = \omega_1' \frac{\partial}{\partial\phi_1} - \omega_2' \frac{\partial}{\partial\phi_2}$.

$\Phi_2 = X_2(D^2 - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) + Y_2(2nD + \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2))$, $\Psi_2 = Y_2(D^2 - \frac{1}{32}(24 + t_{12}k_1 + t_{13}k_2)) - X_2(2nD - \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2))$, and X_2 and Y_2 are homogeneous components of order 2 obtained on substituting $x = B_1^{1,0} + B_2^{1,0}$, $y = B_1^{0,1} + B_2^{0,1}$, in $\frac{\partial\Gamma_3}{\partial x}$, $\frac{\partial\Gamma_3}{\partial y}$.

6. Second order coefficients in the frequencies

To make use of Moser's modified version of Arnold's theorem (1961), it is necessary to reduce the Hamiltonian to its normalized form. So, we performed the first and second order normalization. We have found the second order coefficients in the frequencies. For this we have obtained the partial differential equations which are satisfied by the third order homogeneous components of the fourth order part of Hamiltonian H_4 and second order polynomials in the frequencies. Following the iterative procedure of Bhatnagar and Hallan (1983), we note that the third order components $B_3^{0,1}$ and $B_3^{1,0}$ can be obtained by solving the partial differential equations

$$\begin{cases} \Delta_1\Delta_2 B_3^{1,0} = \Phi_3 - 2f_2P - 2g_2Q, \\ \Delta_1\Delta_2 B_3^{0,1} = \Psi_3 - 2f_2U - 2g_2V, \end{cases} \tag{15}$$

where

$$\begin{aligned}
\Phi_3 &= X_3(D^2 - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) + Y_3(2nD + \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2)), \\
P &= \frac{\partial}{\partial\phi_1}\{(\omega_1' \frac{\partial B_1^{1,0}}{\partial\phi_1} - nB_1^{0,1})(\omega_1'^2 \frac{\partial^2}{\partial\phi_1} - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) \\
&\quad + (\omega_1' \frac{\partial B_1^{0,1}}{\partial\phi_1} - nB_1^{1,0})(2n\omega_1' \frac{\partial}{\partial\phi_1} + \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2))\}, \\
Q &= \frac{\partial}{\partial\phi_2}\{(-\omega_2' \frac{\partial B_1^{1,0}}{\partial\phi_2} - nB_1^{0,1})(\omega_2'^2 \frac{\partial^2}{\partial\phi_2} - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) \\
&\quad + (-\omega_2' \frac{\partial B_1^{0,1}}{\partial\phi_2} - nB_1^{1,0})(-2n\omega_2' \frac{\partial}{\partial\phi_2} + \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2))\},
\end{aligned}$$

$$\Psi_3 = Y_3(D^2 - \frac{3}{32}(24 + t_{12}k_1 + t_{13}k_2)) - X_3(2nD - \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2)),$$

$$U = \frac{\partial}{\partial\phi_1}(\omega_1'\frac{\partial B_1^{0,1}}{\partial\phi_1} - nB_1^{1,0})(D^2 - (\frac{3}{4} + \frac{3k_1}{8} + (\frac{3}{2} + \frac{\sqrt{3}}{8} - \frac{21\gamma}{16})k_2))$$
$$+ (\omega_1^{'}\frac{\partial^2 B_1^{1,0}}{\partial\phi_1} - nB_1^{0,1})(2nD - (\frac{3\sqrt{3}\gamma}{4} - \frac{15\sqrt{3}k_1}{8} + \frac{11k_1\gamma}{8} + \frac{3\sqrt{3}k_2}{16})),$$

$$V = \frac{\partial}{\partial\phi_2}(\omega_2'\frac{\partial B_1^{1,0}}{\partial\phi_1} - nB_1^{0,1})(2nD - (\frac{3\sqrt{3}\gamma}{4} - \frac{15\sqrt{3}k_1}{8} + \frac{11k_1\gamma}{8} + \frac{3\sqrt{3}k_2}{16}))$$
$$- (\omega_2^{'}\frac{\partial^2 B_2^{1,0}}{\partial\phi_2} - nB_1^{1,0})(D^2 - (\frac{3}{4} + \frac{3k_1}{8} + (\frac{3}{2} + \frac{\sqrt{3}}{8} - \frac{21\gamma}{16})k_2)),$$

and X_3 and Y_3 are homogeneous components of order 3 obtained on substituting $x = B_1^{1,0} + B_2^{1,0}$, $y = B_1^{0,1} + B_2^{0,1}$ in $\frac{\partial\Gamma_3}{\partial x} + \frac{\partial\Gamma_4}{\partial x}$ and $\frac{\partial\Gamma_3}{\partial y} + \frac{\partial\Gamma_4}{\partial y}$. The components $B_3^{0,1}$ and $B_3^{1,0}$ are not required to be found out. We find the coefficients of $Cos\phi_1$, $Sin\phi_1$, $Cos\phi_2$ and $Sin\phi_2$ in the right-hand sides of equation(15), they are the critical terms. We eliminate these terms by properly choosing the coefficients in the polynomials $f_2 = f_{2,0}^{'}I_1 + f_{0,2}^{'}I_2$, $g_2 = g_{2,0}^{'}I_1 + g_{0,2}^{'}I_2$, where

$$f_{2,0}^{'} = \frac{\text{Coefficient of Cos } \phi_1 in\Phi_3}{2(\text{Coefficient of Cos}\phi_1\text{in}P)} = A(\text{say}),$$
$$f_{0,2}^{'} = g_{2,0}^{'} = \frac{\text{Coefficient of Cos}\phi_2 in\Phi_3}{2(\text{Coefficient of Cos}\phi_2 inQ)} = B(\text{say}),$$
$$g_{0,2}^{'} = \frac{\text{Coefficient of Cos}\phi_2 in\Psi_3}{2(\text{Coefficient of Cos}\phi_2\text{in}Q)} = C(\text{say}).$$

7. Non-linear stability

Now the condition is $K_1\omega_1' + K_2\omega_2' \neq 0$, for all pairs (K_1, K_2) of rational integers such that $|K_1| + |K_2| \leq 4$.

We calculate, $K_1\omega_1' + K_2\omega_2' = 0$, $\Leftrightarrow \frac{\omega_1^{'}}{\omega_2^{'}} = -\frac{K_1}{K_2}$.

Here, we have, $0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1$, and so $0 < \omega_2' < \frac{1}{\sqrt{2}} < \omega_1' < 1$ ($|K_1| << 1$, $|K_2| << 1$).

So, we have

$$\frac{\omega_1^{'}}{\omega_2^{'}} > 1 \tag{16}$$

equation (16) to be true, K_1 and K_2 are of opposite signs and $-\frac{K_1}{K_2} > 1$.

Therefore, K_1, K_2 can have the following values, $K_1 = 1, K_2 = -2$; $K_1 = -1, K_2 = 2$. $K_1 = 1, K_2 = -3$; $K_1 = -1, K_2 = 3$.

Case-I. When $K_1 = 1, K_2 = -2$; $K_1 = -1, K_2 = 2$. Equation (16) gives

$$\frac{\omega_1^{'}}{\omega_2^{'}} = 2, i.e.\omega_1^{'} - 2\omega_2^{'} = 0. \tag{17}$$

Solving equations (14) and (17) and putting $\gamma = 1 - 2\mu$, we get

$$\mu_1' = (0.024293897\ldots) + (2.08929\ldots)k_1 + (4.69455\ldots)k_2.$$

Case-II. When $K_1 = 1, K_2 = -3$; $K_1 = -1, K_2 = 3$.
Equation (16) gives

$$\frac{\omega_1^{'}}{\omega_2^{'}} = 3, i.e. \omega_1^{'} - 3\omega_2^{'} = 0. \tag{18}$$

Solving equations (14) and (18) and putting $\gamma = 1 - 2\mu$, we get

$$\mu_2' = (0.013516016\ldots) + (2.01103\ldots)k_1 + (4.70867\ldots)k_2.$$

Hence for the values μ_1' and μ_2' of the mass ratio condition (a) of Moser's theorem is not satisfied.

The normalized Hamiltonian up to fourth order is written as $H = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \ldots$

The determinant D occurring in condition (a) of Moser's theorem is

$$D = \begin{vmatrix} A & B & \omega_1' \\ B & C & -\omega_2' \\ \omega_1' & -\omega_2' & 0 \end{vmatrix} = -(A\omega_2^{'2} + 2B\omega_1^{'} + C\omega_1^{'2}).$$

Following the iterative procedure of Bhatnagar and Hallan (1983), we observed that Moser's second condition is violated for the unperturbed problem (i.e. for $k_1 = k_2 = 0$) when $\mu_3 = 0.0109137\ldots$ When $k_1, k_2 \neq 0$, we take $\mu_3^{'} = \mu_3 + Xk_1 + X'k_2$ such that $D = 0$. It is also observed that the condition (b) of Moser's theorem is satisfied i.e. $D \neq 0$, if in the interval$0 < \mu < \mu_c$, the mass ratio does not take the value $\mu_3^{'} = \mu_3 + Xk_1 + X'k_2$, where $X = 40.917\ldots$, $X' = 607.324\ldots$.

8. Conclusion

E. P. Esteban and S. Vazquez have studied the rotating stratified heterogeneous oblate spheroid in Newtonian Physics by taking three layers. But we have taken the smaller primary with mass m_2 a heterogeneous triaxial rigid body with N layers having different densities ρ_i and axes $(a_i, b_i, c_i), (i = 1, 2, 3, 4, \ldots, N)$ respectively in the restricted three-body problem. We have found that there exist five stationary solutions (called libration points) of the equations of motion. Three of them are collinear and two are triangular equilibrium points.

We also observed that in the non-linear sense, collinear points are unstable for all mass ratios μ, and triangular points are stable in the range of linear stability $0 < \mu < \mu_c$, $\mu < \mu_c = \mu_0 - (3.76183\ldots)k_1 + (10.825\ldots)k_2$, Where, $\mu_0 = 0.0385208965\ldots$ except for three mass ratios

$$\mu_1' = (0.0242939\ldots) + (2.08929\ldots)k_1 + (4.69455\ldots)k_2,$$
$$\mu_2' = (0.013516016\ldots) + (2.01103\ldots)k_1 + (4.70867\ldots)k_2,$$
$$\mu_3' = (0.0109366\ldots) + (40.917\ldots)k_1 + (607.324\ldots)k_2,$$

at which Moser's theorem does not apply.

Here, if we take $k_1 = k_2 = 0$, then the values of $\mu_1^{'}$, $\mu_2^{'}$ and $\mu_3^{'}$ agree with those found by Deprit and Deprit (1967).

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Appendix.

$t_1 = 2(2 + 3c + c\gamma^2)$,
$t_2 = \sqrt{3}c + 2p_3$,
$t_3 = c\gamma - 2p_1$,
$t_4 = 3 + 4c\gamma - 3p_1 + 3\sqrt{3}\gamma p_3$,
$t_5 = 3(15 + 4p_2 - 4\sqrt{3}\gamma p_4)$,
$t_6 = (-3\sqrt{3} + 4\sqrt{3}c - 4\sqrt{3}\gamma p_1 + 9p_3)$,
$t_7 = 3(7\sqrt{3} - 4\sqrt{3}\gamma p_2 + 12p_4)$,
$t_8 = 4(-5\sqrt{3} + \sqrt{3}p_1 - 11\gamma p_3)$,
$t_9 = 4\sqrt{3}p_2 - 44\gamma p_4 + 65\sqrt{3}$,
$t_{10} = 33 + 16c + 33\gamma p_1 - 9\sqrt{3}p_3$,
$t_{11} = -47 + 44\gamma p_2 - 12\sqrt{3}p_4$,
$t_{12} = 3 + 16c - 21\gamma p_1 - 3\sqrt{3}p_3$,
$t_{13} = -3(45 + 28\gamma p_3 + 4\sqrt{3}p_4)$,
$t_{14} = -15\sqrt{3} - 15\sqrt{3}\gamma p_1 + p_3)$,
$t_{15} = 40\sqrt{3} - 15\sqrt{3}\gamma p_2 + p_4$,
$t_{16} = 25 - 37p_1 + 25\sqrt{3}\gamma p_3$,
$t_{17} = -37p_2 + 25\sqrt{3}\gamma p_4$,
$t_{18} = 85 - 41p_1 + 45\sqrt{3}\gamma p_3$,
$t_{19} = -40 - 41p_2 + 45\sqrt{3}\gamma p_4$,
$t_{20} = -15\sqrt{3} + 25\sqrt{3}\gamma p_1 + 41p_3)$,
$t_{21} = 25\sqrt{3}\gamma p_2 + 41p_4$,
$t_{22} = 10(-57 + 23\gamma p_1 + 57\sqrt{3}p_3)$,
$t_{23} = 10(23\gamma p_2 + 57\sqrt{3}p_4)$,
$t_{24} = 5(37 + 37\gamma p_1 + 11\sqrt{3}p_3)$,
$t_{25} = 5(-272 + 37\gamma p_2 + 11\sqrt{3}p_4)$,
$t_{26} = -35\sqrt{3} + 23\sqrt{3}p_1 - 37\gamma p_3$,
$t_{27} = 56\sqrt{3} + 23\sqrt{3}p_2 - 37\gamma p_4$,

$t_{28} = -21\sqrt{3} + 57\sqrt{3}p_1 + 43\gamma p_3,$
$t_{29} = 57\sqrt{3}p_2 - 43\gamma p_4,$
$t_{30} = -60(-93 + 43\gamma p_1 + 69\sqrt{3}p_3,$
$t_{31} = 15(1965 + 172\gamma p_2 + 276\sqrt{3}p_4,$

$\alpha_{13} = \frac{-1}{32k^2 l_2(3+4\omega_1^2)}\{-6336c - 864p + 45c_1 + 832p_7 + (1536c + 2304p - 16c_1 - 832p_5 - 256p_7)\omega_1^2 + (5376p - 3072c - 16c_1 + 256p_5)\omega_1^4 + (39 - 12\omega_1^2)c_5c_7\}$, $\alpha'_{13} = \frac{-1}{32k^2 l_1(3+4\omega_1^2)}\{864q + 45c_2 + 832p_8 + (2304q - 16c_2 - 832p_6 - 256p_6)\omega_1^2 + (5376q - 16c_2 + 256p_6)\omega_1^4 + 1024\omega_1^6 + (39 - 12\omega_1^2)c_5c_7\}$,

$\alpha_{21} = \frac{-\omega_1}{8k^3 l_1^2(3+4\omega_1^2)}\{3168c - 864p - 9c_1 + 320p_7 + (6336c - 1152p - 40c_1 + 320p_5 - 727p_7)\omega_1^2 + (-768p - 512c - 16c_1 + 768p_5)\omega_1^4 + (1024c - 1024p)\omega_1^6 - (15 + 36\omega_1^2)c_6c_7\}$,

$\alpha'_{21} = \frac{-\omega_1}{8k^3 l_1^2(3+4\omega_1^2)}\{-864q - 9c_2 - 320p_8 + (-1152q - 40c_2 + 320p_6 - 728p_8)\omega_1^2 + (-768q - 16c_2 + 768p_6)\omega_1^4 - 1024q\omega_1^6 - (15 - 36\omega_1^2)c_6c_7\}$,

$\alpha_{23} = \frac{-1}{64k^3 l_1^3 c_7}\{108864c + 23328p + (101376c - 41472p + 15552p_5)\omega_1^2 - (151296p + 116736c - 14080p_5)\omega_1^4 + (147456c - 48128p - 27648p_5)\omega_1^6 + (-49152c + 20480p + 4096p_5)\omega_1^8 - 16384p\omega_1^{10} + (-729 + 660\omega_1^2 + 1296\omega_1^4 - 192\omega_1^6)c_5c_7 - 67p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_1(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{23} = \frac{-1}{64k^3 l_1^3 c_7}\{-23328q + (41472q - 15552p_6)\omega_1^2 + (151296q - 14080p_6)\omega_1^4 + (48128q + 27648p_6)\omega_1^6 - (20480q + 4096p_6)\omega_1^8 + 16384q\omega_1^{10} - (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_6c_7 + 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{31} = \frac{-\omega_1^2}{64k^3 l_1^3}\{38592c - 864p + (-40704c - 11520p + 4928p_5)\omega_1^2 + (5120c - 3840p - 51200p_5)\omega_1^4 + (-4096c + 16384p + 1024p_5)\omega_1^6 + 4096\omega_1^8 + 16384q\omega_1^{10} + (-231 + 240\omega_1^2 + 48\omega_1^4)c_5c_7 + 64p_7(-70 + 80\omega_1^2 + 16\omega_1^4) + c_1(-333 + 284\omega_1^2 + 336\omega_1^4 + 64\omega_1^6)\}$,

$\alpha'_{31} = \frac{-\omega_1^2}{64k^3 l_1^3}\{-864q - (11520q - 4928p_6)\omega_1^2 + (-3840q - 3120p_6)\omega_1^4 + (16384q - 1024p_6)\omega_1^6 + 4096\omega_1^8 + (-231 + 240\omega_1^2 + 48\omega_1^4)c_6c_7 + 64p_8(-77 + 80\omega_1^2 + 16\omega_1^4) + c_2(-333 + 284\omega_1^2 + 336\omega_1^4 + 64\omega_1^6)\}$,

$\alpha_{33} = \frac{1}{64k^3 l_1^3 c_7}\{-85536c - 23328p + (-120384c + 41472p - 15552p_5)\omega_1^2 + (151296p + 64512c - 14080p_5)\omega_1^4 + (-137216c + 48128p + 27648p_5)\omega_1^6 + (24576c - 20480p - 4096p_5)\omega_1^8 + 16384(p - c)\omega_1^{10} + (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_5c_7 - 64p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{33} = \frac{-1}{64k^3 l_1^3 c_7}\{-23328q + (41472q - 15552p_6)\omega_1^2 - (151296q - 14080p_6)\omega_1^4 + (48128q + 27648p_6)\omega_1^6 + (-20480q - 4096p_6)\omega_1^8 + 16384q\omega_1^{10} + (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_6c_7 + 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{41} = \frac{-\omega_1^2}{64k^3 l_1^3 c_7}\{108864c - 23328p + (101376c - 3456p + 15552p_5)\omega_1^2 + (-116736c - 46848p + 14080p_5)\omega_1^4 + (147456c - 68608p - 27648p_5)\omega_1^6 + (-49152c + 69632p + 4096p_5)\omega_1^8 + 16384p\omega_1^{10} + (-729 - 660\omega_1^2 + 1296\omega_1^4 - 192\omega_1^6)c_5c_7 - 64p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_1(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{41} = \frac{-\omega_1^2}{64k^3 l_1^3 c_7}\{23328q + (-3456q + 15552p_6)\omega_1^2 + (-46848q + 14080p_6)\omega_1^4 + (-68608q - 27648p_6)\omega_1^6 + (69632q + 4096p_5)\omega_1^8 + 16384q\omega_1^{10} + (-729 - 660\omega_1^2 +$

$1296\omega_1^4 - 192\omega_1^6)c_6c_7 - 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{43} = \frac{1}{64k^3l_1^3}\{4928c + 7776p + (-15552c - 10368p + 7488p_5)\omega_1^2 + (-5376c - 48384p - 1536p_5)\omega_1^4 + (35840c - 24576p - 7168p_8)\omega_1^6 - 4096(c-p)\omega_1^8 + (-351 + 72\omega_1^2 + 336\omega_1^4)c_5c_7 + 64p_7(-117 + 24\omega_1^2 + 112\omega_1^4) + 3c_1(-135 + 12\omega_1^2 + 176\omega_1^4 + 64\omega_1^6)\}$,

$\alpha'_{43} = \frac{1}{64k^3l_1^3}\{7776q - (10368q - 7488p_6)\omega_1^2 + (48384q - 1536p_5)\omega_1^4 + (24576q - 7168p_6)\omega_1^6 + 4096q\omega_1^8 + (-351 + 72\omega_1^2 + 336\omega_1^4)c_6c_7 + 64p_8(-117 + 24\omega_1^2 + 112\omega_1^4) + 3c_2(-135 + 12\omega_1^2 + 176\omega_1^4 + 64\omega_1^6)\}$,

The values of α_{ij} and α'_{ij} for $j = 1, 2$, can be obtained from those for $j = 1, 3$, respectively by replacing ω_1 by ω_2, l_1 by l_2, m_1 by m_2 and n_1 by n_2 whenever they occur, keeping k unchanged.

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WEAKLY θ_I-PREOPEN SETS AND DECOMPOSITION OF CONTINUITY

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Abstract. In this paper we introduce and study the notion of weakly θ_I-preopen sets and weakly θ_I-precontinuous functions to obtain a decomposition of continuity. We also investigate their fundamental properties.
Keywords: ideal,ideal topological spaces, θ_I-pre open sets, weakly θ_I-pre open sets, weakly θ_I-precontinuous functions.

1. Introduction

The concept of ideals in general topological spaces was introduced and studied by Hamlett and Jankovic [9] (see also [10], [11]) and Vaidyanathaswamy [33]and other papers. Newcomb [27], Rancin [29], Samuels [31] and Hamlet et al. ([9], [10], [11]) motivated the research by applying topological ideals to generalize the most basic properties in general topology. Jankovic and Hamlet [18] introduced the notion of I-open sets in ideal topological space. El-Monsef et al.[25] further investigated I-open sets and I-continuous functions in ideal topological space. Some new forms of I-open sets are introduced in [7] (see [15]) and other papers. Yuksel et al. [35] and Acikgoz et al.[1] have investigated some new classes of functions in ideal topological spaces. Hatir and Noiri [15] introduced the

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notions of weakly semi-I-open sets and weakly semi-I-continuous functions in ideal topological space. Q.L.Shi [30] initiated and elaborated the notion of weakly α-I-open sets, weakly α-I-continuous, weakly α-I-open, weakly α-I-closed functions and weakly α-I-paracompact spaces in ideal topological spaces. In 2013 Mustafa and Al-Ghour [26] defined the notion of weakly b-I-open sets, weakly b-I-continuous, weakly b-I-open and weakly b-I-closed functions in ideal topological spaces. Quite recently in [4]some new forms of θ_I-open sets have introduced and studied and a new decomposition of continuity is obtained by Al-Omari and Noiri. The concept of θ_I-open sets is based on θ-open sets due to Veličko [34]. A set A is said to be θ-open [34], if every point of A has an open neighborhood whose closure is contained in A.

This new concept of θ_I-preopen sets motivated me to generalize this notion as weakly θ_I-preopen sets. The main theme of the present paper is to devise and elaborate the concept of weakly θ_I-preopen sets and to obtain new decomposition of continuity in ideal topological spaces. This paper is organized as follows, in section 3 we define weakly θ_I-preopen sets and establish its interrelationships with some other generalized open sets and also study its characterizations. In section 4 we define and study strong θ_{pre}-t-I sets, strong θ_{pre}-B-I sets and θ^B-sets. In section 5 we introduce and investigate weakly θ_I-precontinuous and weakly θ_I-preirresolute functions in ideal topological spaces.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) will denote topological spaces with no separation properties assumed. $Cl(V)$ and $Int(V)$ will denote the closure and the interior of V in X, respectively, for a subset V of a topological space (X, τ). $C(X)$ denotes the collection of closed subsets of X. An ideal I on a nonempty set X is a nonempty collection of subsets of X which satisfies the following:

1. $V \in I$ and $U \subset V$ implies $U \in I$,
2. $V \in I$ and $U \in I$ implies $V \cup U \in I$.

The pair (X, τ, I) of a topological space (X, τ) and an ideal I on X is called an ideal topological space or simply an ideal space. It is important that a family of sets is a filter if and only if the family of the complements of these sets is an ideal. One connection between an ideal and the topology on a given ideal space arises through the concept of the local function on a subset. Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the collection of all subsets of X, a set operator $(.)$:$P(X) \to P(X)$ called a local function of A with respect to τ and I, is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X$: $(U \cap A) \notin I$, for every $U \in \tau(x)$ $\}$, where $\tau(x) = \{U \in \tau : x \in U\}$ [33] (c.f. [18], [19]). A Kuratowski closure operator $Cl^*(A) = A \cup A^*(I, \tau)$ induces a topology $\tau^*(I, \tau)$ called the $*$-topology which is finer than τ. It is generated by the base $\beta(I, \tau)$ $= \{U \setminus I : U \in \tau$ and $I \in I$ $\}$. In general $\beta(I, \tau)$ is not always a topology as shown in [18]. We will write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. In general X^* is a proper subset of X. Hayashi [17] used the hypothesis $X = X^*$ and Samuels [31] used the hypothesis $\tau \cap I = \phi$.

Although these two conditions are equivalent due to [18] and therefore the ideal topological spaces satisfying this hypothesis are called as Hayashi-Samuels spaces ([19], [33], [17]).

Now we recall some definitions and results which are used in this paper.

Definition 1. A subset A of a topological space (X, τ) is said to be

1. preopen [22] if $A \subset Int(Cl(A))$,
2. semi-open [20] if $A \subset Cl(Int(A))$,
3. α-open [23] if $A \subset Int(Cl(Int(A)))$,
4. β-open [24] if $A \subset Cl(Int(Cl(A)))$,
5. b-open [5] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,
6. a t-set [32] if $Int(A) = Int(Cl(A))$,
7. a B-set [32] if $A = U \cap V$, where $U \in \tau$ and V is a t-set,
8. a t-I-set [12] if $Int(A) = Int(Cl^*(A))$,
9. a B_I-set [12] if $A = U \cap V$, where $U \in \tau$ and V is a t-I-set,
10. a strong t-I-set [14] if $Int(A) = sCl(Int(Cl^*(A)))$,
11. a strong B_I-set [14] if $A = U \cap V$, where U is an open set and V is a strong t-I-set.

The θ-interior [34] of A in X is the union of all θ-open subsets contained in A and is denoted by $Int_\theta(A)$. The complement of a θ-open set is said to be θ-closed. The θ-closure of A is defined as $Cl_\theta(A) = \{x \in X\colon (Cl(U) \cap A) \neq \phi$, for all $U \in \tau(x)\}$ and a set A is θ-closed if and only if $A = Cl_\theta(A)$. All θ-open sets form a topology on X which is coarser than τ and denoted by τ_θ. A topological space (X, τ_θ) is regular if and only if $\tau = \tau_\theta$. The θ-closure of a given set need not be a θ-closed set. A point $x \in X$ is called a θ-I-closure point of A if $(Cl^*(U) \cap A) \neq \phi$ for each open set U containing x. The set of all θ_I-closure points of A is called the θ_I-closure of A and denoted by $Cl_{\theta_I}(A) = \{x \in X : (Cl^*(U) \cap A) \neq \phi$ for all $U \in \tau(x)$ $\}$. A subset A is said to be θ_I-closed if $Cl_{\theta_I}(A) = A$. The complement of a θ_I-closed set is called a θ_I-open set. In other words A is said to be θ_I-open if $Cl_{\theta_I}(X \setminus A) = X \setminus A$.

Definition 2 ([3]). Let (X, τ, I) be an ideal topological space. A point $x \in X$ is called a θ_I-interior point of A if there exists an open set containing x such that $U \subseteq Cl^*(U) \subseteq A$. The set of all θ_I-interior points of A is called the θ_I-interior of A and denoted by $Int_{\theta_I}(A)$. A is θ_I-open if and only if A=$Int_{\theta_I}(A)$.

The following results are useful in the sequel:

Lemma 1 ([18]). *Let (X, τ, I) be an ideal topological space and A, B be any two subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $A^* \subseteq B^*$;*
2. *If $A^* = Cl(A^*) \subseteq Cl(A)$;*
3. *$(A^*)^* \subseteq A^*$;*
4. *$(A \cup B)^* = A^* \cup B^*$.*
5. *If $U \in \tau$, then $U \cap \ A^* \subset (U \cap A)^*$.*

Lemma 2 ([4]). *Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties hold.*

1. *If A is open, then $Cl(A) = Cl_{\theta_I}(A) = Cl_\theta(A)$.*
2. *If A is closed, then $Int(A) = Int_{\theta_I}(A) = Int_\theta(A)$.*

Definition 3. A subset A of an ideal topological space (X, τ, I) is said to be

1. I-open [25] if $A \subset Int(A^*)$.
2. α-I-open [12] if $A \subseteq Int(Cl^*(Int(A)))$
3. pre-I-open [7] if $A \subseteq Int(Cl^*(A))$.
4. semi-I-open,[12] if $A \subseteq Cl^*(Int(A))$.
5. β-I-open [15] if $A \subseteq Cl(Int(Cl^*(A)))$.
6. θ_I-preopen [4] if $A \subseteq Int(Cl_{\theta_I}(A)))$.
7. θ_I-semi-open [4] if $A \subseteq Cl(Int_{\theta_I}(A))$,
8. θ_I-β-open [4] if $A \subseteq Cl(Int(Cl_{\theta_I}(A)))$.
9. θ_I-α-open [4] if $A \subseteq Int(Cl(Int_{\theta_I}(A)))$.
10. weakly semi-I-open [16] if $A \subset Cl^*(Int(Cl(A)))$.
11. weakly pre-I-open [14] if $A \subset sCl(Int(Cl^*(A))$.
12. weakly b-I-open [26] if $A \subseteq Cl^*(Int(Cl(A))) \cup Cl(Int(Cl^*(A)))$.

Lemma 3 ([13]). *For a subset A of a topological space (X, τ), the following properties hold:*

1. *$sCl(A) = A \cup Int(Cl(A))$,*
2. *If A is open then $sCl(A) = Int(Cl(A))$.*

3. Weakly θ_I-preopen sets

Definition 4. A subset A of an ideal topological space (X, τ, I) is said to be weakly θ_I-preopen if $A \subseteq sCl(Int(Cl_{\theta_I}(A)))$.

The family of all weakly θ_I-preopen sets of the space (X, τ, I) will be denoted by Wθ_IPO(X, τ).

Theorem 1. *For any subset A of an ideal topological space (X, τ, I), the following properties hold:*

1. *Every θ_I-preopen set is weakly θ_I-preopen.*
2. *Every weakly θ_I-preopen set is θ_I-β-open.*
3. *Every preopen set is θ_I-preopen and hence weakly θ_I-preopen .*

Proof. Let A be any subset of an ideal topological space (X, τ, I).

1. Suppose A is θ_I-preopen. By using the definition of a θ_I-preopen set, we have $A \subseteq Int(Cl_{\theta_I}(A)) \subseteq sCl(Int(Cl_{\theta_I}(A)))$.This shows that A is weakly θ_I-preopen. This shows that A is weakly θ_I-preopen.

2. Suppose A is weakly θ_I-preopen then we have $A \subseteq sCl(Int(Cl_{\theta_I}(A))) \subseteq Cl(Int(Cl_{\theta_I}(A))))$. This implies that A is θ_I-β-open.

3. Suppose A is preopen then we have $A \subseteq Int(Cl(A)) \subseteq Int(Cl_{\theta_I}(A)))$ and therefore A is θ_I-preopen and hence A is weakly θ_I-preopen. $\square$

3.1 Interrelationship

The following diagram will describe the interrelations among a weakly θ_I-preopen set and some other existing open sets in an ideal topological space. None of these implications is reversible as shown by examples given below.

$$\begin{tikzcd}
\alpha - open \arrow[r] & preopen \arrow[r] \arrow[dl] & \beta - open \arrow[d] \\
\theta_I - preopen \arrow[r] & weakly\ \theta_I - preopen \arrow[r] & \theta_I - \beta - open
\end{tikzcd}$$

Example 1. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b\}, \{b, c, d\}\}$ and $I = P(X)$, then (X, τ, I) is an ideal topological space.

$C(X) = \{X, \phi, \{c, d\}, \{a, d\}, \{d\}, \{a, c, d\}, \{a\}\}$. Let $A = \{b, d\}$ be any subset of X, then $Cl_{\theta_I}(\{b, d\}) = \{b, d\}$ and $Int(\{b, d\}) = \{b\}$ and $A = \{b, d\} \subsetneq \{b\}$. This implies that A is not a θ_I-preopen set. But $sCl(\{b\}) = \{b\} \cup X = X$, consequently $A = \{b, d\} \subseteq X$. This shows that A is weakly θ_I-preopen.

Example 2. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b\}, \{b, c, d\}, \{c, d\}\}$ and $I = P(X)$, then (X, τ, I) is an ideal topological space. $C(X) = \{X, \phi, \{a, c, d\}, \{a, b\}, \{a\}\}$. Let $A = \{a, b\}$ is β-open and hence θ_I-β-open. Because $cl(int(cl(A))) = int(cl(A)) = Cl(Int(Cl_{\theta_I}(\{a, b\}))) = \{a, b\} = A$. But $A = \{a, b\}$ is not weakly θ_I-preopen, since $sCl(Int(Cl_{\theta_I}(\{a, b\}))) = \{b\}$, which is not containing $\{a, b\}$.

Example 3. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b, c\}, \{a, c, d\}, \{a\}, \{a, b\}, \{a, c\}\}$ and $I = \{\phi, \{d\}\}$, then (X, τ, I) is an ideal topological space.

$C(X) = \{X, \phi, \{d\}, \{c, d\}, \{b, c, d\}, \{b\}, \{b, d\}\}$. Let $A = \{c\}$ be any subset of X, which is weakly θ_I-preopen, as $Cl_{\theta_I}(\{c\}) = X$ and $sCl(Int(Cl_{\theta_I}(\{c\})) = X$, which contains $\{c\}$. But it is not preopen, since $A \nsubseteq Int(Cl(\{c\})) = \phi$.

Theorem 2. *Let (X, τ, I) be an ideal topological space. Let V, A and A_α be the subsets of X. Then*

1. *If A_α is weakly θ_I-preopen for each $\alpha \in \wedge$, then $\cup_{\alpha\in\wedge} A_\alpha$ is weakly θ_I-preopen.*
2. *If A is weakly θ_I-preopen and V is α-open, then $A \cap V$ is weakly θ_I-preopen.*

Proof. 1. Since A_α is weakly θ_I-preopen for each $\alpha\in\wedge$, $A_\alpha \subseteq sCl(Int(Cl_{\theta_I}(A_\alpha)))$ for each $\alpha \in \wedge$. Therefore $A_\alpha \subseteq sCl(Int(Cl_{\theta_I}(\cup_{\alpha\in\wedge} A_\alpha)))$ for each $\alpha \in \wedge$ and $\cup_{\alpha\in\wedge} A_\alpha \subseteq sCl(Int(Cl_{\theta_I}(\cup_{\alpha\in\wedge} A_\alpha)))$.

Hence $\cup_{\alpha\in\wedge} A_\alpha$ is weakly θ_I-preopen.

2. $A \cap V \subseteq sCl(Int(Cl_{\theta_I}(A))) \cap Int(Cl(Int(V)))$
$= Int(Cl(Int(Cl_{\theta_I}(A))) \cap Int(Cl(Int(V)))$
$= Int[Cl(Int(Cl_{\theta_I}(A)) \cap Cl(Int(V))]$
$= Int[Cl[Int(Cl_{\theta_I}(A)) \cap (Int(V))]]$
$= sCl[Int(Cl_{\theta_I}(A) \cap Int(V))]$
$\subseteq sCl[Int(Cl_{\theta_I}(A \cap Int(V))] \subseteq sCl[Int(Cl_{\theta_I}(A \cap V))]$.

Therefore $(A \cap V) \subseteq sCl(Int(Cl_{\theta_I}(A \cap V))$. This shows that $A \cap V$ is weakly θ_I-preopen. □

Theorem 3. *For an ideal topological space (X, τ, I) and $A \subseteq X$, we have:*

1. *If $I = \phi$, then A is θ_I-open if and only if A is θ-open.*
2. *If $I = P(X)$, then A is θ_I-preopen if and only if A is preopen.*

Proof. 1. *Sufficiency*- It follows directly from [4].

Necessity-If $I = \phi$, then $A^* = Cl(A)$ and therefore $Cl_{\theta_I}(A) = Cl_\theta$.

2. *Sufficiency*- It follows from the Theorem 1.

Necessity-If $I = P(X)$, then $A^* = \phi$, therefore $Cl_{\theta_I}(A) = Cl(A)$, which implies the preopeness of A. □

Definition 5. A subset of an ideal topological space (X, τ, I) is said to be weakly θ_I-preclosed if its complement is weakly θ_I-preopen.

Theorem 4. *A subset A of an ideal topological space (X, τ, I) is said to be weakly θ_I-preclosed if and only if $sInt(Cl(Int_{\theta_I}(A))) \subset A$.*

Proof. Let A be a weakly θ_I-preclosed subset of the ideal topological space (X, τ, I), then $X \setminus A$ is weakly θ_I-preopen and hence $(X \setminus A) \subseteq sCl(Int(Cl_{\theta_I}(X \setminus A))) = X \setminus sInt(Cl(Int_{\theta_I}(A)))$. This implies $(X \setminus A) \subseteq (X \setminus (sInt(Cl(Int_{\theta_I}(A))))$.

Hence we have $sInt(Cl(Int_{\theta_I}(A))) \subseteq A$. Conversely, suppose $sInt(Cl(Int_{\theta_I}((A)))$ $\subseteq A$, then $X \setminus A \subseteq X \setminus sInt(Cl(Int_{\theta_I}(A))) = sCl(Int(Cl_{\theta_I}(X \setminus A)))$ and hence $(X \setminus A)$ is weakly θ_I-preopen. Therefore A is weakly θ_I-preclosed. □

Remark 1. The finite intersection of weakly θ_I-preopen sets need not be weakly θ_I-preopen.

Example 4. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b, c\}, \{a, b\}\}$ and

$I = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ then (X, τ, I) is an ideal topological space. The family of closed subsets of X, $C(X) = \{X, \phi, \{d\}, \{c, d\}\}$. Then $A = \{a, d\}$ and $B = \{b, d\}$ are weakly θ_I-preopen, but their intersection $A \cap B = \{d\}$ is not weakly θ_I-preopen. Since $A \nsubseteq sCl(Int(Cl_{\theta_I}(\{d\}))) = \phi$.

Lemma 4. *For two subsets A and U of an ideal topological space (X, τ, I), the following is true: $U \cap Cl_{\theta_I}(A) \subseteq Cl_{\theta_I}(U \cap A)$ if U is θ_I-open.*

Proof. Let $x \in U \cap Cl_{\theta_I}(A)$. Then for every θ_I-open set V containing x, $V \cap U$ is a θ_I-open set containing x and hence $(V \cap U) \cap A \neq \phi$. This implies that $x \in Cl_{\theta_I}(U \cap A)$ and therefore we get the desired result. □

Lemma 5 ([18]). *Let (X, τ, I) be an ideal topological space and B be any subset of X such that $B \subset A \subset X$. Then $B^*(\tau|A, I|A) = B^*(\tau, I) \cap A$.*

If (X, τ, I) is an ideal topological space and A is subset of X; we denote by $\tau|A$ the relative topology on A and $I|A = \{A \cap I : I \in I\}$ is an ideal on A.

Lemma 6 ([16]). *Let (X, τ, I) be an ideal topological space, $A \subset X$ and $U \in \tau$. Then $Cl^*(A) \cap U = Cl^*_U(A \cap U)$.*

Theorem 5. *Let (X, τ, I) be an ideal topological space and $A \subseteq U \in \tau$ then A is weakly θ_I-preopen in (X, τ, I)if and only if A is weakly θ_I-preopen in (U, $\tau|U$, $I|U$).*

Proof. *Necessity.* Let A be any weakly θ_I-preopen set in (X, τ, I), then we have $A \subseteq sCl(Int(Cl_{\theta_I}(A))) = Int(Cl(Int(Cl_{\theta_I}(A))))$ as $Int(Cl_{\theta_I}(A))$ is an open set.

Now $A = U \cap A \subseteq U \cap Int(Cl(Int(Cl_{\theta_I}(A))))$
$= Int(U \cap Int(Cl(Int(Cl_{\theta_I}(A))))$, being an open set.
$= Int_U(U \cap Int(Cl(Int(Cl_{\theta_I}(A))))$
$\subset Int_U(U \cap Cl(U \cap Int(Cl_{\theta_I}(A)))))$
$\subseteq Int_U(Cl_U(U \cap Int(Cl_{\theta_I}(A)))$
$= sCl_U(Int(U \cap (Cl_{\theta_I}(A))))$
$\subset sCl_U(Int_U(U \cap (Cl_{\theta_I}(A)))$
$= sCl_U(Int_U((Cl_{\theta_I})_U)(A))))$ by Lemma 6. This shows that A is weakly θ_I-preopen in $(U, \tau \mid U, I \mid U)$.

Sufficiency. Let A be weakly θ_I-preopen in $(U, \tau \mid U, I \mid U)$. Then we have $A \subset sCl_U(Int_U((Cl_{\theta_I})_U)(A))))$
$= sCl_U(Int_U(Cl_{\theta_I}(A) \cap U)))$

$= sCl_U(U \cap Int(Cl_{\theta_I}(A) \cap U)))$, by Lemma 6
$= sCl_U(Int(Cl_{\theta_I}(A) \cap U))) = Int_U(Cl_U(Int(U \cap (Cl_{\theta_I}(A))))$
$= Int_U(U \cap Cl(Int(U \cap (Cl_{\theta_I}(A))))$
$= U \cap Int(Cl(Int(Cl_{\theta_I}(A))))$
$\subseteq Int(Cl(Int(Cl_{\theta_I}(A))))$
$= sCl(Int(Cl_{\theta_I}(A))))$.

This implies that A is weakly θ_I-preopen in (X, τ, I) □

Corollary 1. *Let(X, τ, I) be an ideal topological space. If $U \in \tau$ and A is weakly θ_I-preopen, then $U \cap A$ is weakly θ_I-preopen in $(U, \tau|U, I|U)$.*

Proof. Since $U \in \tau$ and A is weakly θ_I-preopen in (X, τ, I).

Since every open set is alpha-open, therefore by by Theorem 2, $U \cap A$ is weakly θ_I-preopen in (X, τ, I). Since $U \in \tau$ and by Theorem 5, $U \cap A$ is weakly θ_I-preopen in $(U, \tau|U, I|U)$. □

Definition 6. [8] A space (X, τ) is called submaximal if every dense subset of X is open.

Lemma 7. *[21] If (X, τ) is submaximal, then $PO(X, \tau) = \tau$.*

Corollary 2. *If (X, τ) is submaximal, then for any ideal I on X, $\theta_I PO(X) = \tau$.*

Proof. It follows directly from the fact that every preopen set is θ_I-preopen. □

Remark 2. If (X, τ) is submaximal, then for any ideal I on X, Wθ_IPO(X) $= \tau$.

Theorem 6 ([4])**.** *Let (X, τ, I) be an ideal topological space. The following are equivalent;*

1. *The θ_I-closure of every θ_I-open subset of X is θ_I-open;*
2. *$Cl(Int_{\theta_I}(A)) \subseteq Int(Cl_{\theta_I}(A))$ for every subset A of X;*
3. *$\theta_I PO(X) \subseteq \theta_I SO(X)$;*
4. *The θ_I-closure of every θ_I-β-open subset of X is θ_I-open;*
5. *$\theta_I \beta O(X) \subseteq \theta_I PO(X)$.*

Definition 7. A subset A of an ideal topological space (X, τ, I) is called θ_I-dense if $Cl_{\theta_I}(A) = X$.

Remark 3. Every θ_I-dense subset of an ideal topological space (X, τ, I) is θ_I-preopen.

Proof. It is obvious. □

Theorem 7. *For a subset A of an ideal topological space (X, τ, I), the following properties are equivalent:*

1. $\theta_I PO(X) \subseteq \theta_I SO(X)$,
2. *Every* θ_I*-dense subset of X is* θ_I*-semiopen,*
3. $Int_{\theta_I}(A)$ *is* θ_I*-dense for every* θ_I*-dense subset A,*
4. $Int_{\theta_I}[\theta_I\text{-}F_r(A)] = \phi$ *for every subset A,*
5. $\theta_I \beta O(X) \subseteq \theta_I SO(X)$,

Proof. (1) ⇒ (2)It follows directly from the Remark 3.

(2) ⇒ (3) Let A be θ_I-dense, then A is θ_I-semiopen. Therefore $A \subseteq Cl(Int_{\theta_I}(A)) \subseteq Cl_{\theta_I}(Int_{\theta_I}(A)) \subseteq Cl_{\theta_I}(Cl_{\theta_I})(A) = Cl_{\theta_I}(A) = X$, as A is θ_I-dense, we have $Cl_{\theta_I}(Int_{\theta_I}(A)) = X$. Thus $Int_{\theta_I}(A)$ is θ_I-dense.

(3) ⇒ (4) Suppose A be any subset of X, we have $X = Cl_{\theta_I}(A) \cup (X \setminus Cl_{\theta_I}(A)) = Cl_{\theta_I}(A) \cup Int_{\theta_I}(X \setminus A) \subseteq Cl_{\theta_I}(A) \cup Cl_{\theta_I}(Int_{\theta_I}(X \setminus A)) = Cl_{\theta_I}(A \cup Int_{\theta_I}(X \setminus A))$. This shows that $A \cup Int_{\theta_I}(X \setminus A)$ is θ_I-dense and therefore $Int_{\theta_I}(A \cup Int_{\theta_I}(X \setminus A))$ is θ_I-dense. $Int_{\theta_I}[(A \cup Int_{\theta_I}(X \setminus A)) \cap ((X \setminus A) \cup Int_{\theta_I}(A))] = X \setminus \theta_I\text{-}F_r(A)$. Since $X \setminus (\theta_I\text{-}F_r(A))$ is the intersection of the two θ_I-dense sets therefore $X \setminus (\theta_I\text{-}F_r(A))$ is θ_I-dense.

(4) ⇒ (5) Let $A \in \theta_I \beta O(X)$. Then by (4) and Theorem 3.15 of [4] $A \in \theta_I SO(X)$.

(5) ⇒ (1) It is obvious. □

Definition 8. A space (X, τ) is extremally disconnected [36] if the closure of every open set in X is open.

Theorem 8. *If a topological space* (X, τ) *is extremally disconnected and* $A \in \theta_I SO(X)$*, then* $A \in \theta_I \alpha O(X)$.

Proof. Let $A \in \theta_I SO(X)$, then we have $A \subseteq Cl(Int_{\theta_I}(A))$. Since X is extremally disconnected, we have $Cl(Int_{\theta_I}(A)) = Int(Cl(Int_{\theta_I}(A)))$. Hence $A \subseteq Cl(Int_{\theta_I}(A)) = Int(Cl(Int_{\theta_I}(A)))$. □

Theorem 9. *If a topological space* (X, τ) *is extremally disconnected and* $A \in \theta_I \beta O(X)$*, then* $A \in W\theta_I PO(X)$.

Proof. Let $A \in \theta_I \beta O(X)$, then we have $A \subseteq Cl(Int(Cl_{\theta_I}(A)))$. Since X is extremally disconnected, we have $Cl(Int(Cl_{\theta_I}(A))) = Int[Cl(Int(Cl_{\theta_I}(A)))]$. Therefore $A \subseteq Cl(Int(Cl_{\theta_I}(A))) = Int[Cl(Int(Cl_{\theta_I}(A)))] = sCl(Int(Cl_{\theta_I}(A)))$. This implies that $A \in W\theta_I PO(X)$. □

4. Strong θ_{pre}-t-I-sets

Definition 9. A subset of an ideal topological space is called a

1. Strong θ_{pre}-t-I-set if $sCl(sInt(Cl_{\theta_I}(A))) = Int(A)$.

2. θ_{pre}-t-I-set [4] if $Int(Cl_{\theta_I}(A))) = Int(A)$.

Theorem 10. *Let A and B be subsets of an ideal topological space (X, τ, I). If A and B are strong θ_{pre}-t-I-sets, then $A \cap B$ is a strong θ_{pre}-t-I-set.*

Proof. Since A and B are strong θ_{pre}-t-I-sets, then we have $sCl(sInt(Cl_{\theta_I}(A))) = Int(A)$ and $sCl(sInt(Cl_{\theta_I}(B))) = Int(B)$.

Now $Int(A \cap B) \subseteq Int(Cl_{\theta_I}(A \cap B)))$
$\subseteq sInt(Cl_{\theta_I}(A \cap B)))$
$\subseteq sCl(sInt(Cl_{\theta_I}(A \cap B)))$
$\subseteq sCl(sInt[(Cl_{\theta_I}(A)) \cap (Cl_{\theta_I}(B))]))$
$\subseteq sCl[(sInt(Cl_{\theta_I}(A)))) \cap (sInt(Cl_{\theta_I}(B)))]$
$\subseteq sCl(sInt(Cl_{\theta_I}(A)))) \cap sCl(sInt(Cl_{\theta_I}(B)))$
$= Int(A) \cap Int(B) = Int(A \cap B)$.

Therefore $sCl(sInt(Cl_{\theta_I}(A \cap B))) = Int(A \cap B)$ and hence $A \cap B$ is a strong θ_{pre}-t-I-set. □

Theorem 11. *Every strong θ_{pre}-t-I-set is a θ_{pre}-t-I-set.*

Proof. Let A be any strong θ_{pre}-t-I-set, then we have $sCl(sInt(Cl_{\theta_I}(A)))) = Int(A)$.

Therefore $Int(Cl_{\theta_I}(A))) \subseteq sCl(sInt(Cl_{\theta_I}(A)))) = Int(A) \subseteq Int(Cl_{\theta_I}(A)))$ and hence $Int(Cl_{\theta_I}(A))) = Int(A)$. □

Theorem 12. *For a subset A of an ideal topological space (X, τ, I), the following properties are equivalent:*

1. *A is regular open.*

2. *$sCl(Int(Cl_{\theta_I}(A))) = A$ and A is open.*

3. *A is a strong θ_{pre}-t-I-set and weakly θ_I-preopen.*

Proof. (1) ⇒ (2) Since A is regular open, we have $Int(Cl(A)) = A$ and A is open. Therefore by Lemma 2, we have $Cl_{\theta_I}(A) = Cl(A)$.

Hence $sCl(Int(Cl_{\theta_I}(A))) = sCl(lnt(Cl(A))) = sCl(A) = A \cup Int(Cl(A)) = A \cup A = A$.

(2) ⇒ (3) It is direct from the definition.

(3) ⇒ (1) Let A be strong θ_{pre}-t-I-set and weakly θ_I-preopen, then we have $A \subseteq sCl(Int(Cl_{\theta_I}(A)))$ and $sCl(Int(Cl_{\theta_I}(A))) = Int(A)$. We have $A \subseteq sCl(Int(Cl_{\theta_I}(A))) = Int(A) \subseteq A$, then A is open. Therefore by Lemma 3, $A = sCl(Int(Cl(A))) = Int(Cl(Int(Cl(A)))) = Int(Cl(A)))$. Hence A is regular open. □

5. Strong θ_{pre}-*B*-*I*-sets and θ^B sets

Definition 10. A subset A of an ideal topological space (X, τ, I) is called

1. a strong θ_{pre}-B-I set if $A = U \cap V$, where $U \in \tau$ and and V is a strong θ_{pre}-t-I-set.

2. a θ_{pre}-B-I set [4] if there exist $U \in \tau$ and a θ_{pre}-t-I-set V in X such that $A = U \cap V$.

Theorem 13. *For a subset A of an ideal topological space (X, τ, I), the following properties hold:*

1. *If A is a strong θ_{pre}-t-I-set, then it is a strong θ_{pre}-B-I set.*

2. *If A is a strong θ_{pre}-B-I set, then it is a θ_{pre}-B-I set.*

Proof. 1. Let A be a strong θ_{pre}-t-I-set, then we have $sCl(sInt(Cl_{\theta_I}(A))) = Int(A)$. $A = A \cap X$ and X is open. This implies that A is a strong θ_{pre}-B-I set.

2. Let A be a strong θ_{pre}-B-I set, then we have $A = U \cap V$, where U is an open set and V is strong θ_{pre}-t-I-set. By Theorem 11, V is θ_{pre}-t-I-set and hence A is a θ_{pre}-B-I set. □

Definition 11. [4] A subset A of an ideal topological space (X, τ, I) is called a θ^Aset if $A = U \cap V$, where $U \in \tau$ and V is strongly θ_I-semi-closed i.e. V is θ_I-semi-closed and $Int(Cl_{\theta_I}(A)) = Cl(Int_{\theta_I}(A))$.

Definition 12. [4] A subset A of an ideal topological space (X, τ, I) is said to be θ_I-β-closed if $Int(Cl(Int_{\theta_I}(A)))) \subseteq A$.

Definition 13. A subset A of an ideal topological space (X, τ, I) is called a θ^B-set if $A = U \cap V$, where $U \in \tau$ and V is θ_I-β-closed.

Theorem 14. *Every θ^A-set is θ^B-set.*

Proof. Let V be strongly θ_I-semi-closed, then $Int(Cl_{\theta_I}(V)) \subseteq V$ and $Int(Cl_{\theta_I}(V)) = Cl(Int_{\theta_I}(V))$. Now $Int(Cl(Int_{\theta_I}(V)))$
$= Int(Int(Cl_{\theta_I}(V)))$
$\subseteq Int(Cl_{\theta_I}(V)) \subseteq V$. We get $Int(Cl(Int_{\theta_I}(V)))) \subseteq V$. Therefore V is θ_I-α-closed. This implies that every θ^A-set is θ^B-set. □

But the converse of Theorem 14 need not be true as shown by the following example.

Example 5. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b, c\}, \{a, c\}\}$ and $I = \{\phi, \{a, c\}\}$ then (X, τ, I) is an ideal topological space. $C(X) = \{X, \phi, \{d\}, \{b, d\}\}$. If $A = \{a, b\}$ then we can write A as $A = X \cap A$, where X is an open set and A is θ_I-β-closed, since $Int(Cl(Int_{\theta_I}(A))) = \phi \subseteq A$. Hence A is θ^B-set. But A is not a θ^A-set, since A is not a θ_I-semi-closed, as $Int(Cl_{\theta_I}(A)) = X \nsubseteq A$.

Theorem 15. *For a subset A of an ideal topological space (X, τ, I), the following properties are equivalent:*

1. *A is open.*
2. *A is preopen and a θ^B-set.*
3. *A is θ_I-preopen and a θ^B-set.*
4. *A is weakly θ_I-preopen and a θ^B-set.*
5. *A is θ_I-β-open and a θ^B-set.*
6. *A is weakly θ_I-preopen and a strong θ_{pre}-B-I-set.*

Proof. Here (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) follows from Theorem 1.

(5) $\Rightarrow$ (1) It follows directly from Theorems 1, 6 and 14.

(1) $\Rightarrow$ (6): It is obvious.

(6) $\Rightarrow$ (1): Let A be weakly θ_I-preopen and a strong θ_{pre}-B-I-set, then we have $A \subseteq sCl(Int(Cl_{\theta_I}(A)))$ and $A = U \cap V$, where U is open and V is a strong θ_{pre}-t-I-set so that $sCl(sInt(Cl_{\theta_I}(V))) = Int(V)$. Hence we get
$A = A \cap U \subseteq sCl(Int(Cl_{\theta_I}(A))) \cap U = \{(sCl(Int(Cl_{\theta_I}(U \cap V)))) \cap U\}$
$\subseteq \{sCl(Int(Cl_{\theta_I}(U)))\} \cap \{sCl(Int(Cl_{\theta_I}(V)))\} \cap U$
$= \{sCl(Int(Cl_{\theta_I}(V)))\} \cap U$
$\subseteq \{sCl(sInt(Cl_{\theta_I}(V)))\} \cap U$
$= (Int(V) \cap U) \subseteq (V \cap U) = A.$
Thus A is an open set. □

6. Weakly θ_I-precontinuous

Definition 14. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

1. precontinuous [22] if preimage of every open set in Y is preopen in X.
2. θ_I-α-continuous if the preimage of every open set in Y is θ_I-α-open in X.
3. θ_I-precontinuous [4] if the preimage of every open set in Y is θ_I-preopen in X.
4. θ_I-β-continuous [4] if the preimage of every open set in Y is a θ_I-β-open in X.
5. θ_{pre}-B-I-continuous [4] if the preimage of every open set in Y is a θ_{pre}-B-I-set.
6. θ^A-continuous [4] if the preimage of every open set in Y is a θ^A set.

Definition 15. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

1. weakly θ_I-precontinuous if the preimage of every open set in Y is weakly θ_I-preopen in X.

2. strong θ_{pre}-t-I-continuous if the preimage of every open set in Y is a strongly θ_{pre}-t-I-set.

3. strong θ_{pre}-B-I-continuous if the preimage of every open set in Y is a strongly θ_{pre}-B-I-set.

4. θ^B-continuous if the preimage of every open set in Y is a θ^B-set.

Theorem 16. *1. Every θ_I-precontinuous function is weakly θ_I-precontinuous.*

2. Every precontinuous function is weakly θ_I-precontinuous.

3. Every weakly θ_I-precontinuous function is θ_I-β-continuous.

4. Every θ^A-continuous function is θ^B-continuous.

Proof. It follows directly from the Theorems 1 and 14. □

The converse of (1)-(3) in Theorem 16 need not be true as shown in the following three examples.

Example 6. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $I = P(X)$ then (X, τ, I) is an ideal topological space. $C(X) = \{\phi, X, \{b, c, d\}, \{c, d\}, \{b\}\}$. Let $Y = \{1, 2, 3, 4\}$ and $\sigma = \{Y, \phi, \{1, 2\}, \{1, 2, 3\}\}$ then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be the function defined as $f(a) = 2, f(b) = 3, f(c) = 4, f(d) = 1$. Then f is weakly θ_I-precontinuous function but it is not θ_I-precontinuous. Since the preimage of every open set in Y is weakly θ_I-preopen but it is not θ_I-preopen in X. For, let $A = \{1, 2\}$, then the preimage $f^{-1}(\{1, 2\}) = \{a, d\}$ is weakly θ_I-preopen but it is not θ_I-open.

Example 7. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}, \{c\}\}$ and $I = \{\phi, \{a\}\}$ then (X, τ, I) is an ideal topological space. $C(X)=\{\phi, X, \{b, c, d\}, \{a, b, d\}, \{b, d\}\}$. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{2\}, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}\}$ then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function defined as $f(a) = 2$, $f(b) = 3$, $f(c) = 1$, $f(d) = 4$. Then f is weakly θ_I-precontinuous function but it is not precontinuous, since the preimage of every open set in Y is weakly θ_I-preopen but it is not preopen in X. For, let $A = \{2, 3\}$, then its preimage $f^{-1}(\{2, 3\}) = \{a, b\}$ is not a preopen in X.

Example 8. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{b\}, \{a, b, d\}\}$ and $I = \{\phi, \{a\}, \{b\}, \{a, b, \}\}$ then (X, τ, I) is an ideal topological space. $C(X) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{c\}\}$. Let $Y = \{1, 2, 3, 4\}$ and $\sigma = \{Y, \phi, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ then (Y, σ) is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be the function defined as $f(a) = 4$, $f(b) = 1$, $f(c) = 2$, $f(d) = 3$. Then f is θ_I-β-continuous but it is not weakly θ_I-precontinuous. Since the preimage of every open set in Y is θ_I-β-continuous but it is not weakly θ_I-preopen in X. For, let $A = \{1, 2\}$, then its preimage $f^{-1}(\{1, 2\}) = \{b, c\}$ is not a weakly θ_I-preopen set but it is θ_I-β-open in X.

Example 9. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b, c\}, \{a, c\}\}$ and $I = \{\phi, \{a, c, \}\}$ then $(X,\ \tau,\ I)$ is an ideal topological space. $C(X) = \{\phi, X, \{d\}, \{b, d\}\}$. Let $Y = \{1, 2, 3, 4\}$ and $\sigma = \{Y, \phi, \{2\}, \{2, 3\}\}$ then $(Y,\ \sigma)$ is a topological space. Let $f : (X, \tau, I) \to (Y, \sigma)$ be the function defined as $f(a) = 3$, $f(b) = 2$, $f(c) = 4$, $f(d) = 1$. Then f is θ^B-continuous but it is not θ^A-continuous, since the preimage of every open set in Y is a θ^B-set but it is not a θ^A-set. For, let $A = \{2, 3\}$, its preimage $f^{-1}(\{2, 3\} = \{a, b\}$ is θ_I-β-closed but it is not θ_I-semiclosed in X as $X = X \cap A$ where X is open and A is θ_I-β-closed(for θ^B-set) or strong θ_I-semiclosed in X(for θ^A-set).

Definition 16. Let A be a subset of the space $(X,\ \tau,\ I)$ and let $x \in X$. Then A is called a weakly θ_I-preneighborhood of x if there exists a weakly θ_I-preopen set V containing x such that $V \subseteq A$.

Theorem 17. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is weakly θ_I-precontinuous.*
2. *For each $x \in X$ and for each $U \in \sigma$ containing $f(x)$, $f^{-1}(U)$ is weakly θ_I-preneighborhood of x.*

Proof. (1) $\Rightarrow$ (2) Suppose $x \in X$ and U is any open set in Y such that $f(x) \in U$. By Theorem 18 there exists a weakly θ_I-preopen set M containing x in X such that $f(M) \subseteq U$; hence $x \in M \subseteq f^{-1}(U)$. Therefore $f^{-1}(U)$ is weakly θ_I-preneighborhood of x.

(2) $\Rightarrow$ (1) Let U be any open set in Y and $x \in f^{-1}(U)$. Since $f^{-1}(U)$ is weakly θ_I-preneighborhood of x, therefore there exists a weakly θ_I-preopen set M_x such that $x \in M_x \subset f^{-1}(U)$. Thus we have $f^{-1}(U) = \bigcup\{M_x : x \in f^{-1}(U)\}$ and hence $f^{-1}(U)$ is weakly θ_I-preopen in X. $\square$

Theorem 18. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is is weakly θ_I-precontinuous.*
2. *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists a weakly θ_I-preopen set U containing x such that $f(U) \subset V$.*
3. *For each $x \in X$ and each $V \in \sigma$ containing f(x), $Cl_{\theta_I}(f^{-1}(V))$ is weakly θ_I-preneighborhood of x.*
4. *The inverse image of each closed set in Y is weakly θ_I-preclosed.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and let V be any open set in Y such that $f(x) \in V$. Set $P = f^{-1}(V)$. By (1) P is weakly θ_I-preopen and therefore $x \in P$ implies that $f(P) \subset V$.

(2) $\Rightarrow$ (3) Since V is open in Y and $f(x) \in V$, then by (2) there exists a weakly θ_I-preopen set P containing x such that $f(P) \subset V$. Therefore $x \in P \subseteq sCl(Int(Cl_{\theta_I}(P))) \subseteq sCl(Int(Cl_{\theta_I}(f^{-1}(V)))) \subseteq Cl_{\theta_I}(Cl_{\theta_I}(f^{-1}(V))) = Cl_{\theta_I}(f^{-1}(V))$. This shows that $Cl_{\theta_I}(f^{-1}(V))$ is a weakly θ_I-preneighborhood of x.

(3) $\Rightarrow$ (1) Let V be any open set in Y and $x \in f^{-1}(V)$. By (3), $Cl_{\theta_I}(f^{-1}(V)$ is weakly θ_I-preneighborhood of x, there exists a weakly θ_I-preopen set U_x in X such that $x \in U_x \subseteq Cl_{\theta_I}(f^{-1}(V)$. Hence $Cl_{\theta_I} f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} U_x$. This implies that $f^{-1}(V)$ is weakly θ_I-preopen in X and therefore f is weakly θ_I-precontinuous.

(1) $\Leftrightarrow$ (4) It is obvious. $\square$

Theorem 19. *If $f : (X, \tau, I) \to (Y, \sigma)$ is any weakly θ_I-precontinuous function and $U \in \tau$, then the restriction $f \mid U : (U, \tau \mid U, I \mid U) \to (Y, \sigma)$ is weakly θ_I-precontinuous.*

Proof. Let V be any open set in Y. Since f is weakly θ_I-precontinuous, $f^{-1}(V)$ is weakly θ_I-preopen. Since U is open, by Corollary 1, $U \cap f^{-1}(V)$ is weakly θ_I-preopen in $(U, \tau \mid U, I \mid U)$. Since $(f \mid U)^{-1}(V) = U \cap f^{-1}(V)$ and $(f \mid U)^{-1}(V)$ is weakly θ_I-preopen in $(U, \tau \mid U, I \mid U)$. This implies that $(f \mid U)$ is weakly θ_I-precontinuous function. $\square$

Theorem 20. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function and $\{U_\alpha : \alpha \in \tau\}$ be an open cover of X. Then f is weakly θ_I-precontinuous if and only if the restriction $(f \mid U_\alpha)$:$(U_\alpha,\ \tau \mid U_\alpha,\ I \mid U_\alpha)$ is weakly θ_I-precontinuous for each $\alpha \in \triangle$.*

Proof. *Necessity.* It follows directly from the Theorem 19.

Sufficiency. Let V be any open set in Y. Since $(f \mid U_\alpha)$ is a weakly θ_I-precontinuous for each $\alpha \in \triangle$, $(f \mid U_\alpha)^{-1}(V)$ is a weakly θ_I-preopen set in $(U_\alpha,\ \tau \mid U_\alpha,\ I \mid U_\alpha)$. Hence by Theorem 5 $(f \mid U_\alpha)^{-1}(V)$ is weakly θ_I-preopen in $(X,\ \tau,\ I)$. Moreover we consider $f^{-1}(V) = X \cap f^{-1}(V) = \bigcup_{\alpha \in \triangle}(\ U_\alpha \cap f^{-1}(V)) = \bigcup_{\alpha \in \triangle}(f \mid U_\alpha)^{-1}(V)$. By using Theorem 2, $\bigcup_{\alpha \in \triangle}(f \mid U_\alpha)^{-1}(V)$ is weakly θ_I-preopen in $(X,\ \tau,\ I)$.

Therefore $f^{-1}(V)$ is weakly θ_I-preopen in $(X,\ \tau,\ I)$. Hence f is weakly θ_I-precontinuous. $\square$

Theorem 21. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is weakly θ_I-precontinuous if and only if the function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is weakly θ_I-precontinuous.*

Proof. *Necessity.* Let f be weakly θ_I-precontinuous. Let $x \in X$ and R be any open neighborhood of $g(x)$ in $X \times Y$. Then there exists an open set $P \times Q$ in $X \times Y$ such that $g(x) = (x, f(x)) \in (P \times Q) \subseteq R$. By assumption f is weakly

θ_I-precontinuous and there exists a weakly θ_I-preopen set P_0 in X containing x such that $f(P_0) \subset Q$. By Theorem 2 $P \cap P_0$ is weakly θ_I-preopen and $g(P \cap P_0) \subset (P \times Q) \subset R$. This implies that g is weakly θ_I-precontinuous.

*Sufficiency.*Suppose that the function g is weakly θ_I-precontinuous Let $x \in X$ and Q be any open set in Y containing $f(x)$. Then $X \times Q$ is open in $X \times Y$. Since g is weakly θ_I-precontinuous, by hypothesis there exits a weakly θ_I-preopen set P containing x such that $g(P) \subset X \times Q$ and hence we get $f(P) \subset Q$. This shows that f is weakly θ_I-precontinuous. □

Definition 17. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be weakly θ_I-preirresolute if $f^{-1}(V)$ is weakly θ_I-preopen in (X, τ, I) for every weakly θ_J-preopen set V in (Y, σ, J).

Theorem 22. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \rho)$ be two functions, then the following properties hold:*

1. *if f is weakly θ_I-precontinuous and g is continuous, then $g \circ f$ is weakly θ_I-precontinuous.*
2. *if f is weakly θ_I-preirresolute and g is weakly θ_I-precontinuous then $g \circ f$ is weakly θ_I-precontinuous.*

Proof. It is obvious from the definitions. □

7. Decompositions of continuity

Theorem 23. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is continuous;*
2. *f is precontinuous and θ^B-continuous;*
3. *f is θ_I-precontinuous and θ^B-continuous;*
4. *f is weakly θ_I-precontinuous and θ^B-continuous;*
5. *f is θ_I-β-continuous and θ^B-continuous;*
6. *f is weakly θ_I-precontinuous and strongly θ_{pre}-B-I-continuous.*

Proof. It follows directly from Theorem 15. □

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OBJECT-ORIENTED ROAD EXTRACTION BASED ON IMPROVED FCM AND SHAPE FILTER IN HIGH RESOLUTION RS IMAGERY

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Abstract. An object-oriented method of road extraction is proposed for high-resolution remote sensing imagery aiming to its characteristics. At first, bilateral filter is used in original imagery to smooth detail information and retain road edge; then it proposes an improved Fuzzy C-Means algorithm combined with the neighborhood information in order to deal with outliers better, so the imagery is segmented to independent objects by improved FCM algorithm; and then it filters every objects by geometric feature, after that it connects road segments to get network by region growing algorithm and executes post-processes by morphology method. The experiments show that the method can extract the road target efficiently from high resolution imagery with higher accuracy.
Keywords: high-resolution remote sensing imagery, road extraction, object-oriented, FCM, shape filter.

1. Introduction

Extracting different kinds of ground objects in remote sensing imageries (RS imagery) is an important method in constructing and updating geo-database. As main artificial ground objects in modern traffic system, roads are all-important identified objects and accurate road extraction plays a key role in GIS updating, city observation and planning and so on. According to the degree of human-computer interaction, there are two kinds of extraction methods: semi-automatic and automatic method. Semi-automatic extraction obtains seed points, road width and direction by human-computer interaction. Ribbon snake model [1] and ZiplockSnake model [2] are classic semi-automatic methods which take advantage of geometric and radiation feature meanwhile. Literature [3] proposes a semi-automatic method based on mean-shift algorithm that can extract center line accurately. In this method the seed points are chose artificially.

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As a main research domain, automatic methods extract road objects intelligently and automatically through machine learning algorithms [4]. The paper [5] can extract the road from the three-dimensional transportation network including overpass by fuzzy inference with very big time costing. A new method based on Expand Kalman filter and particle filter is proposed in [6] by which all road junctions can be detected, but the result seriously depends on the choice of model parameter. In short, extracting road network accurately and quickly from RS imagery is still a very difficult task because of complex and different road circumstance and occlusion due to shadow and other objects in the road.

The above algorithms could be classified to pixel-oriented road extraction method in which pixel is considered as independent individual and pixels' relationship are separated. In contrast, object-oriented methods consider that imagery is composed by objects that include important semantic information and relationship [7]. Besides spectral feature, high resolution RS imagery owns abundant spatial structure and texture information. The paper proposes an object-oriented method based on improved FCM for road extraction in high resolution RS imagery. At first, it achieves the purpose of "retaining edges and denoising" by bilateral filter. Then the paper proposes an improved Fuzzy C-Means (FCM) algorithm which eliminates its sensibility to noise by introduce neighborhood information to distance measurement and membership degree. It extracts road segments as road seeds through shape filter in segmented objects and then connects these road seeds into road network by region growing algorithm. At last, mathematical morphology methods are used to repair road and extract center line. The experiments prove efficiency and accuracy of the method.

2. Object-oriented road extraction based on improved FCM and shape filter

The difficulty of road extraction is that road feature would be affected by sensor type, spectral resolution and spatial resolution. In general, road has geometrical feature, radiate feature, topological features and context feature which obviously know from other ground objects. The paper obtains segmented objects using cluster algorithm by taking advantage of radiate feature, and does shape filtering by taking advantage of geometrical feature, at last connects road segments according to topological features. The algorithm flow is shown in Figure 1.

2.1 Bilateral filtering pre-process

More abundant details are provided by high resolution RS imagery, but some ground objects such as pedestrians, vehicles and traffic lines would be the noise disturbance in road extraction and then decrease the extraction accuracy. Therefore, imagery pre-process is necessary in order to smooth noise. In the pre-process, full retaining of road edge information is very useful in extraction at

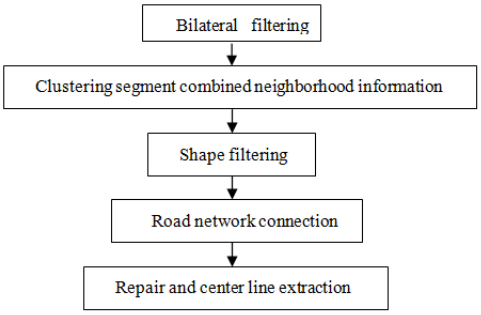

Figure 1: Flow chart of road extraction

the same time of smoothing noise. As a consequence, bilateral filter [8] is a very good choice for RS imagery pre-process. In the region with gradual change of imagery, bilateral filter will translate to Gaussian low-pass filters because of smaller intensity difference between neighborhood pixels. In the region with drastic change of imagery, original intensity value will be replaced by weighted mean of similar intensity value near the edge points. So it can achieve the purpose of smoothing the imagery and retaining edge information at meanwhile. Just like Gaussian filter, bilateral filter utilizes local weighted mean, but its weighting coefficient in 2-dimension neighborhood is a nonlinearity combination of space proximity factor ω_s and intensity similarity factor ω_r .For a center pixel $f(x,y)$, the weight $\omega(i,j)$ of pixel $f(i,j)$ in its neighborhood $N_{x,y}$ is:

(1) $$\omega_s(i,j) = e^{-\frac{|i-x|^2+|j-y|^2}{2{\sigma_s}^2}},$$

(2) $$\omega_r(i,j) = e^{-\frac{|f(i,j)-f(x,y)|^2}{2\sigma_r^2}},$$

(3) $$\omega(i,j) = \omega_r(i,j)\omega_r(i,j).$$

In format (1) and (2), σ_s and σ_r are variances of two smooth factors respective by which control degree of attenuation of smooth factors. In the region with gradual change of imagery, bilateral filter will translate to Gaussian low-pass filters because smaller intensity difference between neighborhood pixels. In the region with drastic change of imagery, original intensity value will be replaced by weighted mean of similar intensity value near the edge points. So it can achieve the purpose that smoothing the imagery and retaining edge information at meanwhile.

2.2 Imagery cluster segmentation

2.2.1 Fuzzy C-means cluster algorithm

The radiation features of road include: there are bigger gradient in two edges; gray value of road region is consistent and obvious different from non-road region such as trees and buildings and so on. Hence, Fuzzy C-Means cluster algorithm [9] could be adapted to obtain all different objects by segmenting imagery. Fuzzy C-Means cluster algorithm is unsupervised classification method that applies fuzzy theory to clustering analysis and indicates level that a sample belongs to every group by membership degree without absolute "belonging to" or "not belonging to" concept. By application of fuzzy theory, FCM cluster algorithm can get better result comparing to other "hard classification" methods. The objective function of FCM is

$$J(U,V) = \sum_{k=1}^{n}\sum_{i=1}^{c}(u_{ik})^m \|x_k - v_i\|^2, \tag{4}$$

$d_{ik} = \|x_k - v_i\|$ is Euclidean distance between cluster center v_i and sample x_k $m \in (1,\infty)$ is fuzzy weighted index number. The parameter u_{ik} describes the membership degree of sample x_k which belongs to cluster center v_i and is normalized by $\sum_{c}^{i=1} u_{ik} = 1, \forall k \in \{1,2,...,n\}$ .The algorithm searches cluster centers V and membership matrix U iteratively in order to minimize objective function J:

$$u_{ik} = \frac{1}{\sum\limits_{j=1}^{c}\left(\frac{\|x_k - v_i\|}{\|x_k - v_j\|}\right)^{\frac{2}{m-1}}} \quad 1 \le i \le c, 1 \le k \le n, \tag{5}$$

$$v_i = \frac{\sum\limits_{k=1}^{n}(u_{ik})^m x_k}{\sum\limits_{k=1}^{n}(u_{ik})^m} \quad 1 \le i \le c. \tag{6}$$

In imagery segmentation, cluster process use color values of three channels as samples feature and initialize cluster center or membership matrix.

2.2.2 Improved FCM combining neighborhood information

There are some shortages in traditional FCM algorithm: ① The choice of initial cluster centers will influence the algorithm performance; ② The algorithm is sensitive to isolated points and illumination and noise will affect segmentation result. Therefore traditional FCM method cannot eliminate noise influence from isolated points. The accuracy of later road extraction will be reduced because that objects from inefficient segmentation method are not accurate enough. Consequently, an improved FCM method is proposed in paper that combines spatial distance information and neighborhood gray difference information.

The standard FCM algorithm does not take influence from neighborhood pixels to center point into account when it computes distance between the pixel and cluster center from which FCM is sensitive to isolated points. Through analysis to standard FCM, the clustering performance depends on membership degree u_{ik} and distance d_{ik} between the cluster center and the point to be classified. For this reason, the paper introduces neighborhood information to simplex Euclidean distance and corresponding membership degree u_{ik} as well. Whether the pixel belongs to road region is relative to its spatial position in road detection of remote sensing imagery. An isolated point has bigger probability to be a road edge if all neighborhoods belong to road. The paper adjusts average weighted value to new weighted value that combines spatial neighborhood information and neighborhood gray difference information in the round, by which influence from neighborhood to center pixel becomes smaller with the increase of their spatial Euclidean distance and with the increase of their gray difference value. The distance $d_{ik}^{N_k}$ combining neighborhood information and neighborhood gray difference information is defined by format (7):

$$d_{ik}^{N_k} = \|x_k - v_i\| + \sum_{l \in N_k} w_l \, \|x_l - v_i\|. \tag{7}$$

If the overlarge weight is given to neighborhood information, the segment precision will reduce. In order to keep the balance between neighborhood information and gray difference, we split the weight equally.So w_l is defined as

$$w_l = 1 - \frac{1}{2}\left(\frac{d_{kl}}{\sum\limits_{l \in N_k} d_{kl}} + \frac{S_{kl}}{\sum\limits_{l \in N_k} S_{kl}} \right). \tag{8}$$

The parameter N_k is the selected window with $N \times N$ size. The parameter d_{kl} is Euclidean distance between center pixel k and the pixel l that fall into N_k . The parameter S_{kl} is gray difference value between l and k . We can see from (8) that the weight is equally split between distance information and gray difference information.

Therefore, cluster center v_i^N and membership degree u_{ik}^N can be defined by follow formats:

$$u_{ik}^N = \frac{1}{\sum\limits_{j=1}^{c} \left(\frac{\|x_k - v_i\| + \sum\limits_{l \in N_k} w_l \|x_k - v_i\|}{\|x_k - v_j\| + \sum\limits_{l \in N_k} w_l \|x_k - v_j\|} \right)^{\frac{2}{m-1}}} \quad 1 \le i \le c, 1 \le k \le n, \tag{9}$$

$$v_i^N = \frac{\sum\limits_{k=1}^{n} \left(u_{ik}^N\right)^m x_k}{\sum\limits_{k=1}^{n} \left(u_{ik}^N\right)^m} \quad 1 \le i \le c, 1 \le k \le n. \tag{10}$$

The new objective function is

$$J^N(U,V) = \sum_{k=1}^{n} \sum_{i=1}^{c} (u_{ik}^N)^m \, \|x_k - v_i^N\|^2. \tag{11}$$

The improved FCM algorithm can be used in handling remote sensing imagery including noise. When noise point is center pixel, it can adjust its membership automatically to avoid mistaken classification as a result of influence from a number of normal neighborhoods. At mean while, the algorithm can reduce influence from neighborhood noise points to normal pixels classification maximally so that enhance ability of noise suppression and decrease false alarm rate. The algorithm process is:

(1) Determination of parameters: classification number C , weighted index number m , termination error ε and iteration number *loop* ;

(2) Initialize membership matrix $U = \left[{u_{ik}}^N \right]_{c \times n}$;

(3) Compute objective function value $J^N(U, V)$. If difference value from last objective function value is lesser than ε or iterations times reach to threshold , the algorithm process will stop. Otherwise, the flow go to the step(4);

(4) Compute new cluster centers v_i^N , then return to step (3).

The improved FCM algorithm can be used in handling RS imagery including noise. The Figure 1(a) is original image that to be segmented with 466 $\times$ 540 size. There are some outliers in the original image. These outliers will lead to too fragmentized segment result that is not conducive to extract objects in RS imagery. The Figure 1(b) and Figure 1(c) are segment results of traditional FCM and our improved FCM method respectively. Obviously, in improved FCM algorithm the outliers can be managed better due to combination of neighborhood information.

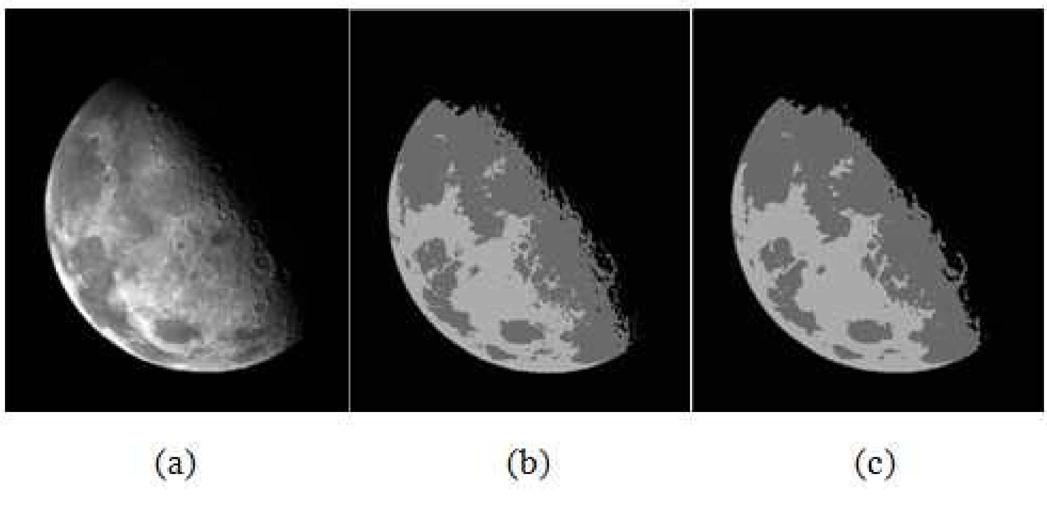

Figure 2: Segmented results of traditional FCM and our improved FCM method

2.3 Object filtering combining shape feature

The road has some distinguishing features comparing to other ground objects. For example, total area is not too small and it is always spindly and so on. Therefore, specific shape features could be used in filtering segmented objects in order to obtaining candidate road segments. The following features can be utilized in shape filtering such as Area$S$, Length-width ratio$R$, Filling degree$F$,Shape factor$e$,Length$L$ and Width$W$.

Area$S$: pixel total number of every segmented object. We can set area threshold to exclude disturbed objects that are too small. The threshold T_s can be set by combining resolution of remote sensing imagery.

Length-width ratio $R : R(R = L_{ext}/W_{ext})$ is the length-width ratio of minimum bounding rectangle of segmented objects.

Filling degree $F : F = S/S_{est}$. It is the pixels ratio in object and minimum bounding rectangle at mean while.

Shape factor$e$: It describes the perimeter of unit area of object. The bigger e value indicates more complicated shape, and vice versa. For example, e value of circle is 12.6, e value of square is 16 and triangle is 20.8.

Length L and Width W : L and W is length and width of object minimum bounding rectangle corresponding. The two features can filter other lathy ground objects such as rivers and building roof.

2.3.1 Road network connection

The topological property of road means that road segment will constitute connected road network without sudden interruption. Therefore, the road segments obtained by shape filtering should be connected to network. The region grow algorithm [10] can be used to construct road network.

At first, some seed points from edges of candidate road segments are selected randomly. Then the method searches pixels of 4-neighborhood from every seed point. The pixels whose gray difference is smaller than threshold T will be added to increased region. Then this pixel will be a new seed point and same search process will begin from it. The process is repeated until all seed points can not satisfied the condition. The method can merger similar regions furthest and then connect candidate road segments ideally. The algorithm flow is showed in figure 2.

2.3.2 Extraction of road center line

After obtaining road network, we should thin road to gain center lines. There are some holes in the extracted road surface resulted from spectral difference and they will be closed-loops after thinning. It is necessary to carry post-process to them. At first, close and dilation operation of morphology are used to fill holes. Then the thinning process will get good result. After thinning we remove short lines (burrs) by method of iterative endpoints removing [11].

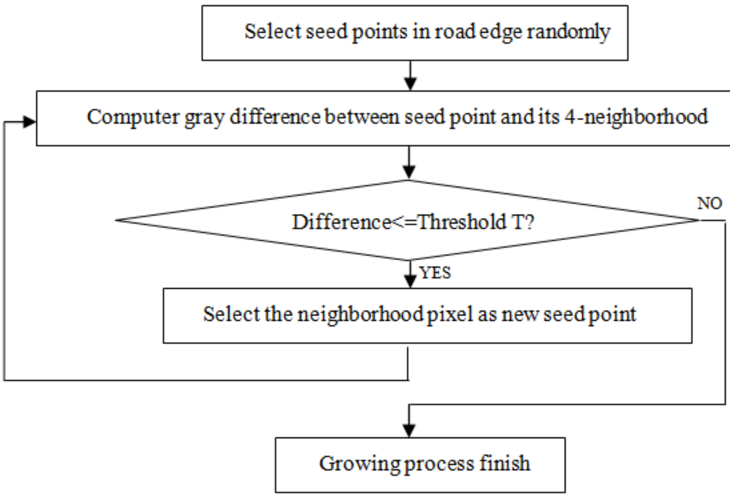

Figure 3: flow chart of region grow algorithm

3. Experiments and result analysis

At first, it choices a RS imagery of the urban area of Zhejiang from World View satellite with 0.3m spatial resolution and 280×430 size in 2012 as the first experiment data. As shown in Figure3 (a), the imagery includes different kinds ground object such as house roof, grassland, soil and roads and so on. At the same time, there is a long and narrow area in the imagery that has same spectral feature with road area. Generally, N is equal to 2 (the half width of bilateral filter). It means the size of filter window is 5×5. When $\sigma_s = 2$ and $\sigma_r = 0.1$, the imagery can get the better result of retaining edges and reducing noise by repeating experiments and summarize. The cluster number k of Improved FCM is equal to 5.The fuzzy weighted index $M = 2$. All parameters of shape filter are set by repeating experiments respectively: the area threshold=20; the Length-width ratio $R = 5$; Filling degree $F = 0.4$; the Shape factor $e = 18$; The Length $L = 50$ and Width $W = 10$.

The original imagery is handled by graying process and bilateral filter. From the pre-processed imagery in Figure 3(b), it can be seen that vehicles and pedestrians in the road are blurred and edges of road and buildings are retained well. Figure 3(c) is segmented result by improved FCM after 100 times iteration. The Figure 3(d) is the binary imagery after extracting road objects and thresholding. Then, every connected region is filtered by shape feature. The roughly extracted roads are showed in Figure 3(e). The road segments are connected to network by region growing algorithm which is showed in Figure 3(f). After final post-process by morphological method, the road center line is extracted as showed in Figure 3(g). Because of efficient pre-process by bilateral filter and more accurate segmentation by improved FCM algorithm, the method can extract main road correctly through getting rid of the ground objects that have same spectral feature to roads.

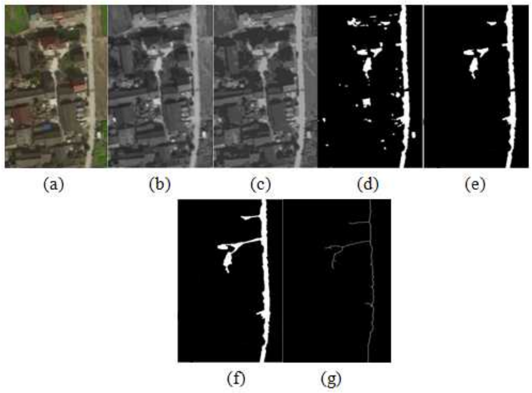

Figure 4: Results of experiment in urban road

In order to verify universality of the algorithm, the second experiment extracts the roads from RS imagery of rural circumstance. The original imagery with 660×396 size is showed in Figure 4(a). In the imagery, there are woodland and farmland with vegetational cover, bare land and roads. In the bottom right of road, a part of bare land has similar spectral feature to road. The half width of bilateral filter is 2(the size of its window is 5×5). Through repeating experiments and summarize, we get better result when $\sigma_s = 2$ and $\sigma_r = 0.1$. The cluster number k of Improved FCM is 4.In the similar way, $m$is equal to 2. All parameters of shape filter are set by repeating experiments respectively: the area threshold=60; the Length-width ratio $R = 8$; Filling degree $F = 0.4$; the Shape factor $e = 18$; The Length $L = 80$ and Width $W = 20$. The result imagery after bilateral filtering is showed in Figure4 (b). Figure4 (c) is clustered imagery and Figure4 (d) is binaryzation process result to segmented road respectively. The Figure 4(e) is obtained after shape filtering and holes filling by morphological method to binaryzation imagery. The experiment cuts the step of road network connection because that successive road network has been finished. Finally, the center line of road is showed in Figure 4(f) after thinning and eliminating burrs. The efficiency of eliminating burrs is ideal in this experiment because of the regular shape of the road. Therefore, the method can extract completed road from RS imagery in rural scene as well.

4. Conclusions and future work

The paper proposes an object-oriented road extraction method in high resolution RS imagery, which introduces the improved FCM algorithm combining neighborhood information in extraction process. The improved FCM algorithm adjusts average weighted value to new weighted value that combines spatial

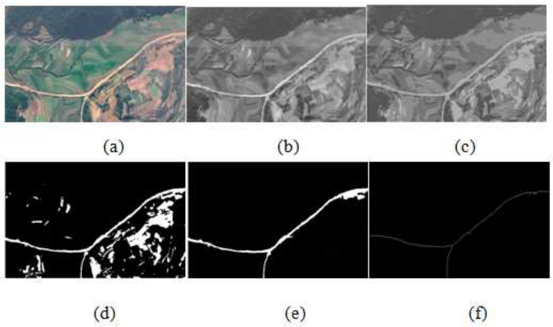

Figure 5: Results of experiment in rural road

neighborhood information and neighborhood gray difference information in the round. The experiment result indicated that it can deal with outliers better compared to traditional FCM which will avoid too fragmentized segment result that is not conducive to extract objects in RS imagery. The paper obtains independent objects by improved FCM through imagery segmentation and then generates road network combining geometric features and topology features. At last, the extracted road is repaired and thinned by morphological method. Through experiments and analysis, the proposed method can extract road goal from imagery clearly and completely. In future work, other features should be taken into account in objects segmentation and the self-adapting mechanism for threshold selection should be established combining machine learning method.

Acknowledgments

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RESEARCH ON THE OPTIMIZATION OF PATH INFORMATION IN THE PROCESS OF LOGISTICS DISTRIBUTION BY IMPROVED ANT COLONY ALGORITHM

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Abstract. Whether the logistics distribution path is reasonable determines distribution speed and distribution efficiency. In this research, the ant colony algorithm is introduced in detail, and a mathematical model of the algorithm is established for the characteristics of logistics distribution problems, and the algorithm is further improved and optimized on convergence rate and global searching ability. The experimental results showed that the improved algorithm optimized the logistics distribution path and could find the optimal path scheme quickly and effectively, proving that it is feasible and promising in the optimization of logistics distribution paths.

Keywords: ant colony algorithm, logistics distribution, path optimization.

Introduction

Ant colony algorithm was initially proposed by M. Dorigo [1], which is a heuristic search algorithm [2] based on population optimization that can find the best path to the destination through active feedback and distributed collaboration with a strong vitality. Research and optimization of ant colony algorithm are of great importance to network routing and urban transport systems.

The ant colony algorithm has been involved in many combinatorial optimization problems, from quadratic assignment problem [3], job-shop scheduling problem to protein folding problem and vehicle routing problem [4], which shows its practicability. Though playing a great role in the optimization of logistics distribution, the algorithm faces some difficulties. Qi [5] processed the routing problem of vehicles taking advantages of simulated annealing and ant colony optimization. In the first stage, simulated annealing provided a good initial solution for ant colony optimization. In the second stage, the near-optimal solution was searched in local scope using iterated local search. In this way, the routes of vehicles were optimized, which made logistics management more scientific. In the study of Guo [6], the development of robots and route planning algorithm were analyzed, and the advantages and disadvantages of the traditional intelligent route planning were emphatically studied. The route planning problem of robots was studied using ant colony algorithm, and some solutions were put forward. The largest difficulty is in logistics management actually, as there are more and more limitations and requirements on transport and distribution plans [7]. If different search options were used, the results obtained will vary. Chang et al. [8] put forward a multi-objective genetic algorithm based on greedy search which could regulate the allocation of available resources and automatically generate various feasible emergency logistics schedules for decision-makers to minimize the logistics time and cost in distribution planning and made an optimization analysis on logistics management using different algorithms.

As the algorithm is applied in practice more and more frequently, the system complexity increases [9], with more and more data to be processed, under which circumstance single or one or two intelligent methods cannot well solve problems. Hence, in this design, we improved the algorithm and carried out simulation analysis to solve the path optimization problem of logistics distribution, hoping to develop optimal transport and distribution plans, saving time and consumption.

1. Ant colony algorithm

1.1 Principle and basic model of the algorithm

In nature, ants are randomly distributed. Once an ant finds food, it will leave a pheromone trail before returning to the nest, following which other ants can find the food. In this way, the pheromones on the path will be gradually strengthened. Nevertheless, as time goes by, the pheromones begin to evaporate and the appeal falls, resulting in a longer time for other ants to follow the trail to find food. As a result, shorter paths are more favorable for ants to follow, leaving a greater pheromone density on the paths. In addition, pheromone can avoid convergence to local optimal solution [10]. If there is no evaporation at all, the path which the first ant seeks will be too attractive, which will limit the exploration of solutions.

In general, when an ant finds a good path from the nest to the food source (ie, a short path), the other ants are more likely to follow the path, eventually leading all the ants to follow the path. The idea of ant colony algorithm is to solve problems by simulating the behavior of ants [11]. The basic steps of the ant colony algorithm are as follows:

Set the number of ants in the ant colony to be Q, the distance between client i and j to be d_{ij} and degree of intimacy, visibility, between them to be x_{ij}, $\eta_{ij} = 1/d_{ij}$, the heromone concentration between them to be τ_{ij}. Then, at time point t, the probability of ant k to move from client i to client j is as follows:

$$p_{ij}^{k}(t) = \frac{\tau_{ij}(t)^{\alpha} \cdot \eta_{ij}(t)^{\beta}}{\sum_{k \in A_k} \tau_{ik}(t)^{\alpha} \cdot \eta_{ik}(t)^{\beta}}, j \in A_k, 0, j \notin A_k. \tag{1}$$

Where A_k refers to a collection of customer points that have not yet been accessed, which is changing in the evolution process. α, β refer to the roles of pheromones and heuristic factors accumulated in movement in path selection. The pheromone update rules on the relevant path are as follows:

$$\tau_{ij}(t+n) = \rho \cdot \tau_{ij}(t) + \Delta\tau_{ij}, \tag{2}$$

$$\Delta\tau_{ij}(t+x) = \sum_{k=1}^{Q} \Delta\tau_{ij}^{k}. \tag{3}$$

Where ρ refers to the information retention level.

Taking a twin bridge model as an example, suppose that the remaining pheromones of ants are proportional to the number of ants in an asymmetric bridge. Meanwhile, suppose a short bridge to be A and a long bridge to be B and A_m and B_m respectively refer to the number of ants that cross the bridges ($A_m + B_m = m$). If the ants arrive at the front of the two bridges, then the probability of crossing bridge A by the $m+1$ time is as below:

$$P_A(m) = \frac{(A_m + k)^k}{(A_m + k)^k + (B_m + k)^k}. \tag{4}$$

Where A and B are parameters that are used to match the actual data, and the probability meets the following condition:

$$P_B(m) = 1 - P_A(m). \tag{5}$$

1.2 Improvement and optimization of ant colony algorithm

The genetic algorithm begins with a solution to the group problem, and each group contains a certain number of individuals [12], and the entities of these individuals are genetically encoded. The algorithm simplifies the situation that the coding work is based on the theory of "survival of the fittest" and is repeated until an approximate optimal solution is found. Genetic algorithms are mainly used for selection, crossover and variant operations [13] and evolve and generate new generations according to optimization principles. Besides, it performs selection of functions based on the degree of fitness and crosses the parental body to produce new individuals on which mutation is realized, which is circulated until the best solution is produced.

The ant colony algorithm has a global search function and can be combined with other algorithms, with good adaptability and robustness as well as good parallel processing performance. However, the algorithm is prone to be restricted to the local best solution and tends to be affected by initial parameters [14]. Similarly, the genetic algorithm has strong adaptability and versatility, global optimization performance and parallel processing performance, it also has good scalability. Hence, the combination of the two algorithms is conductive to improve the convergence rate of the algorithm.

Symbol definition: $s_{i,j}$ refers to the distance between client i and client $j(i,j = 0,1,2,\ldots,L)$, when $i,j = 0$, it refers to the distribution center; S_k refers to the maximum travel distance of vehicle k; n_k refers to the number of customers assigned to vehicle k, when $n_k = 0$, it means that vehicle k is not involved in distribution; G_k refers to a collection of customer points of vehicle $k(k = 1,2,\ldots,K)$, when $n_k = 0, G_k = \emptyset$, when $n_k \neq 0$, $\{r_k^1, r_k^2, \ldots, r_{k^k}^n \subset \{1,2,\ldots,L\}$, where g_k^i indicates that the order of the customer point in the distribution line of vehicle k is i.

The constraint condition of the optimized algorithm is:

$$1) \sum_{i=1}^{n_k} q_{r_k^i} \leq Q_k; n_k \neq 0, \tag{6}$$

$$2) \sum_{i=1}^{n_k} S_{g_{k^{i-1}}, r_k^i} + s_{g_k^n k,0} \leq S_k; n_k \neq 0, \tag{7}$$

$$3) G_{k_1} \cap G_{k_2} = \emptyset; k_1 \neq k_2, \tag{8}$$

$$4) \sum_{k=1}^{k} G_k = \{1,2,\ldots,L\}; 0 \neq n_k \leq L, \sum_{k=1}^{K} n_k = l, \tag{9}$$

The optimization goal is as follows:

$$\min Z = \sum_{k=1}^{k} \left[\sum_{i=1}^{n_k} s_{g_k^{i-1}, r_k^i} + s_{g_k^n,0} \right] \circ sgn\, n_k, \tag{10}$$

where

$$sgn\, n_k = \begin{cases} 0, & n_k \geq 1 \\ 1, & n_k = 0. \end{cases} \tag{11}$$

The update rule is:

$$\tau_{ij}(t+s) = \rho \circ \tau_{ij}(t) + \Delta\tau_{ij}, \tag{12}$$

$$\Delta\tau_{ij} = \sum_{k=1}^{K} \Delta\tau_{ij}^k. \tag{13}$$

By combining the two algorithms to solve the problem of logistics optimization, the optimal solution of the path can be calculated, and the advantages of the two algorithms can be fully exploited to avoid some defects. The convergence curves of the two at each moment are shown in Figure 1.

The basic idea of the genetic ant colony algorithm is to calculate the minimum total convergence time first; before the most appropriate time, using the good randomness of the genetic algorithm and its faster convergence rate, the ant colony algorithm has stronger parallel processing capacity and higher efficiency and can be applied for the exploration of the optimal path in logistics and distribution.

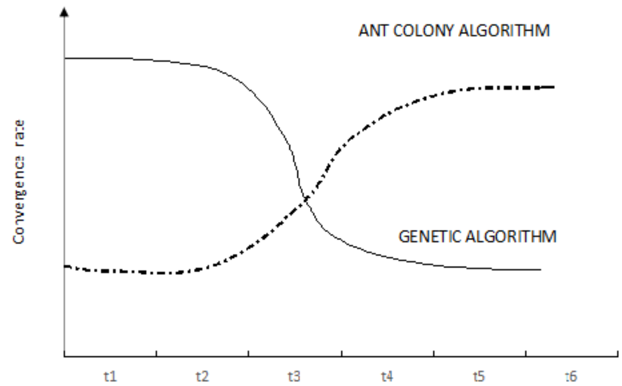

Figure 1: Curves of convergence rate

2. Overview of logistics distribution problems

Logistics needs to be organized and implemented in detail. In the general business sense, logistics and distribution is the management process between logistics [15] so as to meet the requirements of customers or enterprises. Logistics management objects cover food, materials, animals, equipment and liquids as well as abstract items such as time and information. Physical logistics usually involves the integration of information flow, material handling, production, packaging, inventory, transportation, warehousing and transport safety. Logistics distribution problem is part of the supply chain management, the distribution organizational procedures formulated to complete the distribution task and the basic content of system management [16]. The main flow of logistics is shown in Figure 2.

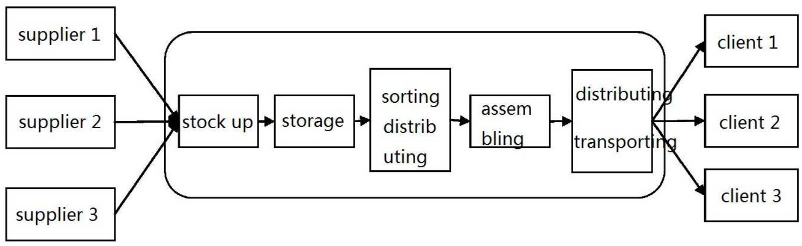

Figure 2: Logistics distribution flow chart

As shown in Figure 2, logistics distribution is systematic and orderly, and whether the distribution path is reasonable has great impact on distribution rate and cost. Therefore, taking a reasonable method to determine the distribution path is very important work in the distribution process [17].

3. Experimental simulation and analysis

To verify the practical feasibility of the optimized algorithm, this study simulates the logistics distribution process in Beijing. The distribution programs of 10 districts in Beijing is selected (specific orientation is simulated according to Google map, with variations), which are Xicheng, Dongcheng, Haidian, Chaoyang, Fengtai, Mentougou, Shijingshan, Fangshan, Tongzhou, Daxing (numbered $0, 1, 2, \ldots, 9$), with the following assumptions:

(1) The distance from the distribution center to the city where the customer is located and the amount of tasks that the city needs to deliver are known.

(2) Ignore the impact of weather, traffic and other factors on transport; there are interconnected roads between cities.

(3) All customer demand form is the same, with land transport adopted.

(4) Take Xicheng District as the center, with star connection as the starting connection mode (as shown in Figure 3).

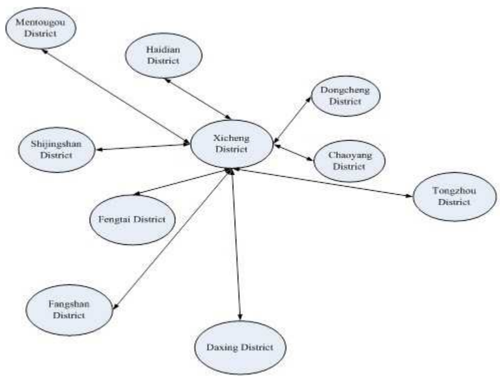

Figure 3: The logistics distribution network map in Beijing

The logistics distribution network map in Beijing

	0	1	2	3	4	5	6	7	8	9
0	0	7.3	11.2	8.1	14.2	30.5	16.1	32.4	26.6	23.3
1	7.3	0	15.1	4.6	19.7	29.7	21.7	38.9	25.7	29.1
2	11.2	15.1	0	18.1	15.2	23.7	14.2	33.7	37.5	29.9
3	8.1	4.6	18.1	0	22.9	37.1	23.9	42.7	20.4	32.3
4	14.2	19.7	15.2	22.9	0	24.6	11.2	22.7	38.7	21.1
5	30.5	29.7	23.7	37.1	24.6	0	13.9	30.3	58.9	41.9
6	16.1	21.7	14.2	23.9	11.2	13.9	0	23.7	45.8	29.9
7	32.4	38.9	33.7	42.7	22.7	30.3	23.7	0	59.9	26.3
8	26.6	25.7	37.5	20.4	38.7	58.9	45.8	59.9	0	47.1
9	23.3	29.1	29.9	32.3	21.1	41.2	29.9	26.3	47.1	0

Table 1. Distance matrix between districts in Beijing (in kilometers)

Assume that the distance between the distribution points is shown in Table 1. A total of 10 vehicles (a maximum load capacity of 2 tons) need to be deployed for distribution, all of which starts form Xicheng District and returns to Xicheng District from the original path after distribution. Therefore, the total vehicle delivery distance is:

$$L = \sum_{i-1}^{n} l_{oi},$$

where l_{oi} refers to the distance between the Xicheng District to the i-th district and the total distance is 339.4 km. In order to minimize the delivery time, shorten the distance, improve efficiency as much as possible, this paper applies the improved algorithm to optimize the actual distribution path in Beijing. First of all, the area around the Xicheng District is divided into three areas by the direction, which are then optimized accordingly, as shown in figure 4. The distance between the starting point of Xicheng District and the distribution points of other districts and the distance between districts are shown in table 1.

Target description: minimum travel distance: $\min Z$.

Description of the constraints:

(1) The maximum carrying capacity of the distribution vehicle is $q = 10$ tons.

(2) The amount of goods required in each district is $u_i = 2$ tons.

(3) Each district uses only one vehicle.

(4) After distribution, each vehicle must return to the cargo center of Xicheng District.

The other parameters are initialized as follows: $\alpha = 1, \beta = 5, \rho = 0.7, Q = 5$, maximum number of iterations $NC = 200$. Run the genetic ant colony algorithm, the solutions are: the number of vehicles = 3, the total vehicle travel distance is 289.3 km. The optimized path is shown in Figure 4.

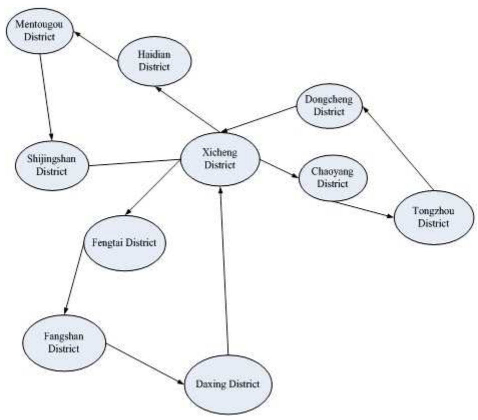

Figure 4: Logistics distribution network optimization diagram in Beijing

The optimized algorithm optimized the distribution path in Beijing, with the number of vehicles required for delivery reduced from 10 to 3 and the total distance traveled by the vehicle shorted from 339.4 kilometers to 289.3 kilometers, which effectively reduced the cost of logistics courier companies. Therefore, the algorithm has practical application value.

4. Conclusion

With the development of market economy, the logistics and distribution industry develops rapidly and more and more enterprises see the importance of logistics distribution in their production and sales process. To achieve the purpose of the vehicle's energy-saving and emission reduction, the key is to achieve the optimization of the logistics and distribution path [18]. The genetic ant colony algorithm combines the advantages of both, making it more flexible and more widely used.

In this paper, the ant colony algorithm and the genetic algorithm are effectively combined to explore the optimization of the transport vehicle path. The experimental results showed that the proposed algorithm not only solved the redundancy of the genetic algorithm which was easy to occur in the loop phenomenon, but also could solve the shortcomings of the early loop iteration of the algorithm. Still, shortcomings exist in this algorithm. For example, many actual influential factors are neglected in the experiment, which in fact play certain roles in real operation. Therefore, further studies are needed in the near future.

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THE 1-PLANARITY OF INTERSECTION GRAPH OF IDEALS OF A RING

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Abstract. A graph $G = (V(G), E(G))$ is called 1-planar if it can be drawn in the plane such that every edge of the graph is cut by at most one other edge of the graph. For any ring R, the ideal intersection graph of R, denoted by $G(R)$, is the graph whose vertices are the nontrivial proper ideals of R and two distinct vertices are adjacent if they have nontrivial intersection. In this paper, we characterize when the intersection graph $G(R)$ of a ring R, is 1-planar.

Keywords: intersection graph of ideals of a ring, 1-planar graph, artinian ring.

1. Introduction

Through out this paper a graph means a finite simple graph, i.e. a graph without loops and multiple edges. Recall that a graph is called planar, if it can be drawn in the plane with nonintersecting edges except of the ends. Always it is interesting to characterize when a graph is planar. One generalization of planar graphs is called 1-planar. A graph is called 1-planar if it can be drawn in the plane with every edge is cut in at most one point except of the ends. This class of graphs is interesting in computer sciences, especially in networks. It was studied extensively in literature, see [3], [7], and [8]. The following lemma, summarizes the 1-planarity of a graph G with seven vertices or less, see [7].

Lemma 1. *The graph $K_7 - K_3$ is the unique 7-vertex minimal non 1-planar graph.*

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All rings, R, in this paper are commutative with unity $1 \neq 0$. A local ring R, is a ring with only one maximal ideal, say M. An Artinian ring R is the direct product of a finite number of Artinian local rings, $R = \prod_{i=1}^{m} R_i$. Moreover, every ideal of R_i is finitely generated. For an Artinian local ring R with unique maximal ideal M, there exists a least positive integer k such that $M^k = 0$. In this case we say that the local ring R has nilpotency index k, see [2].

Let $G(R)$ be the ideal intersection graph of a ring R, with vertices are the proper nontrivial ideals of the ring R, and two vertices are adjacent if they intersect nontrivially. Akbari et. al. [1] proved that, R is Artinian if $G(R)$ is finite. Moreover, if $G(R)$ is finite and connected, then $\operatorname{diam}(G(R)) \leq 2$. From now on, it is enough to consider Artinian rings, to characterize when the intersection graph $G(R)$, is 1-planar.

The idea behind introducing the intersection graph, $G(R)$, of a ring R is to study the interrelationship between algebraic properties of the ring R and the graph theoretic properties of the graph $G(R)$. So, $G(R)$ was investigated by many authors in literature, see [1], [4] and [9]. Jafari and Rad in [6] characterized when the intersection graph, $G(R)$, of ideals of a ring R is planar. Here in the same spirit we characterize when the intersection graph $G(R)$, is 1-planar.

For undefined notions and terminology, the reader is referred to [2] and [5].

2. The 1-planarity of intersection graph $G(R)$

As the first result, we characterize when an Artinian ring, which is the direct product of at least three local rings, is 1-planar.

Theorem 1. *Let $R = \prod_{i=1}^{m} R_i$, for $m \geq 3$. The graph $G(R)$ is 1-planar if and only if R is the product of three fields.*

Proof. Firstly, assume that $m = 3$ and that R_1, R_2, and R_3 are fields. Then $G(R)$ has exactly six vertices. So, $G(R)$ is 1-planar. But, if at least one of R_1, R_2, and R_3 is not a field, say R_1, with nontrivial maximal ideal M_1. Then $G(R)$ has at least the following vertices $R_1 \times R_2 \times \{0\}$, $R_1 \times \{0\} \times \{0\}$, $M_1 \times \{0\} \times \{0\}$, $M_1 \times R_2 \times \{0\}$, $R_1 \times \{0\} \times R_3$, $M_1 \times \{0\} \times R_3$, $M_1 \times R_2 \times R_3$. These vertices induce the complete subgraph K_7. Hence, by Lemma 1, the result follows.

Secondly, assume that $m > 3$, then $|G(R)| \geq 7$ and K_7 is an induced subgraph of $G(R)$. Thus, by lemma 1, $G(R)$ is non 1-planar graph. □

Lemma 2. *Let R be a direct product of a field, say F_1, and a local ring, say R_2. Then $G(R)$ is 1-planar if and only if $G(R_2)$ is isomorphic to K_1, K_2, or P_3.*

Proof. Assume that $G(R_2)$ is isomorphic to K_2, then there is an ideal I_1 of R_2 such that $\{0\} \subsetneq I_1 \subsetneq M \subsetneq R_2$, and $V(G(R)) = V(G(F_1 \times R_2)) = \{F_1 \times \{0\}, F_1 \times I_1, F_1 \times M, \{0\} \times I_1, \{0\} \times M, \{0\} \times R_2\}$. Then $|G(R)| = 6$, hence $G(R)$ is 1-planar. While, if $G(R_2)$ is isomorphic to K_1 then $|G(R)| < 6$. Moreover, if $G(R_2)$ is isomorphic to P_3, then there are two ideals I_1 and I_2 of R_2 such that $\{0\} \subsetneq I_1 \subsetneq M \subsetneq R_2$, $\{0\} \subsetneq I_2 \subsetneq M \subsetneq R_2$, and $I_1 \cap I_2 = \emptyset$. Hence, $G(R) = G(F_1 \times R_2)$ is 1-planar, see Figure 1.

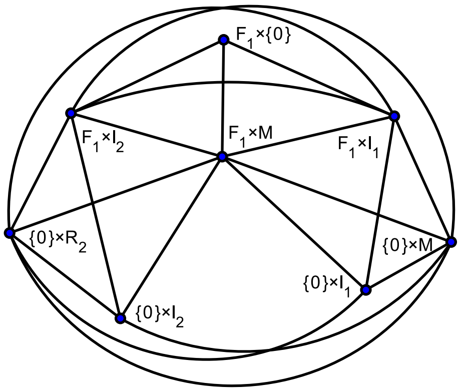

Figure 1: $G(R) = G(F_1 \times R_2)$, where $G(R_2)$ is isomorphic to P_3

On the other hand, assume that $|G(R_2)| = 3$ but not isomorphic to P_3, then $G(R_2)$ is isomorphic to K_3, where $\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq M \subsetneq R_2$. Then the set of vertices $F_1 \times M$, $F_1 \times I_1$, $F_1 \times I_2$, $\{0\} \times I_1$, $\{0\} \times I_2$, $\{0\} \times M$, $\{0\} \times R_2$, induces a subgraph isomorphic to K_7 in the graph $G(R)$. Hence, by Lemma 1, $G(R)$ is non 1-planar.

Moreover, assume that $|G(R_2)| \geq 4$, then the simplest connected intersection graph of $G(R_2)$, is a star graph with three edges, R_2 is local ring. Though, we have three mutually disjoint, say I_1, I_2, I_3, non trivial and non maximal ideals, that are subsets of maximal ideal, say M. Then the set of vertices $\{0\} \times R_2$, $\{0\} \times M$, $\{0\} \times I_1$, $F_1 \times M$, $F_1 \times I_1$, $F_1 \times I_2$, $F_1 \times I_3$, induce a subgraph isomorphic to $K_7 - P_2$ in the graph of $G(R)$. Hence, by Lemma 1, $G(R)$ is non 1-planar. □

Lemma 3. *Let R be a direct product of two local rings, $R = R_1 \times R_2$, that are not fields. Then $G(R)$ is 1-planar if and only if each R_i has only one proper nontrivial ideal.*

Proof. If both rings R_1 and R_2 has at most one nontrivial proper ideal, say M_1 and M_2 respectively. Then $G(R)$ has at most the following vertices $\{0\} \times R_2$, $\{0\} \times M_2$, $M_1 \times \{0\}$, $M_1 \times R_2$, $R_1 \times \{0\}$, $M_1 \times M_2$, and $R_1 \times M_2$. The graph of $G(R)$ is isomorphic to a subgraph of $K_7 - C_4$. Hence by Lemma 1, $G(R)$ is 1-planar.

On the other hand, if both R_1 and R_2 are not fields with one of them has at least two nontrivial proper ideals, say $I \subset M_2 \subset R_2$, and R_1 has $M_1 \neq 0$, then $G(R)$ has at least seven mutually adjacent vertices, namely $\{0\} \times M_2$, $R_1 \times$

M_2, $R_1 \times I$, $\{0\} \times I$, $M_1 \times R_2$, $M_1 \times M_2$, and $M_1 \times I$. These induce the complete subgraph K_7. Hence, by Lemma 1, $G(R)$ is non 1-planar. □

Now, we summarize the case when R is the direct product of two local rings.

Theorem 2. *For $R = R_1 \times R_2$, $G(R)$ is 1-planar if and only if one of the following holds:*

1. *R_1 is a field and $G(R_2)$ is isomorphic to a path of length at most three.*
2. *Each R_i has at most one proper non trivial ideal.*

Finally, we consider the local Artinian rings.

Theorem 3. *If a ring R is a local ring whose maximal ideal is principal, then $G(R)$ is 1-planar if and only if $|G(R)| \leq 6$*

Proof. Let R be a local ring whose maximal ideal is principal. Then R is a chained principal ideal ring. So, $G(R)$ is a complete graph. Hence, by Lemma 1, the result follows. □

Theorem 4. *Let R be a local non principal ideal ring, whose maximal ideal has exactly two generators x and y. Then $G(R)$ is 1-planar if and only if either $M^2 = 0$, or $M^4 = 0$ with $M =< x, y >$ and $x^2 = 0 = y^2$.*

Proof. If either $M^2 = 0$, or $M^4 = 0$ with $M =< x, y >$ and $x^2 = 0 = y^2$. Then $|G(R)| \leq 5$, hence by Lemma 1, $G(R)$ is 1-planar.

Conversely, assume that M has nilpotency index 4 with $M =< x, y >$ and $x^4 = 0 = y^2$. Then $G(R)$ has a subgraph, that is induced by the vertices $< x, y >$, $< x^2, y >$, $< x^2 >$, $< x^3, y >$, $< x^3 >$, $< x >$, and $< xy >$, which is isomorphic to the complete K_7. Moreover, assume that M has nilpotency index equals 3 with $M =< x, y >$ and $x^3 = 0 = y^3$. Then $G(R)$ has the subgraph, say G_1, that is induced by the vertices $< x >, < y >, < x^2 >, < x, y >, < x, y^2 >$, $< x^2, y >$, and $< x^2, y^2 >$, that is isomorphic to the graph $K_7 - P_2$. Now, assume that M has nilpotency index equals r with $r \geq 5$ and $M =< x, y >$, then $G(R)$ has a subgraph that is isomorphic to K_7. Hence, for all these cases, $G(R)$ is non 1-planar by Lemma 1. □

Theorem 5. *Let R be a local ring with maximal ideal M, that can't be generated by less than three generators. Then $G(R)$ is 1-planar if and only if $M^2 = 0$ and $M =< x, y, z >$.*

Proof. If $M =< x, y, z >$ and $M^2 = 0$, then $G(R)$ has at most the following vertices $< x >$, $< y >$, $< z >$, $< x, y >$, $< x, z >$, $< y, z >$, $< x, y, z >$. Then the graph of $G(R)$ is isomorphic to a proper subgraph of $K_7 - k_3$, hence by Lemma 1, $G(R)$ is 1-planar.

Assume that R is a local ring with maximal ideal M with three generators or more. If the nilpotency index of M equals r with $r \geq 3$. Then $|G(R)| \geq 8$ and $G(R)$ has a subgraph isomorphic to G_1, that is defined in the proof of the previous theorem. Hence the result follows. □

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RESEARCH ON PERSONALIZED RECOMMENDATION ALGORITHM OF CROSS-BORDER E-COMMERCE UNDER LARGE DATA BACKGROUND

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Abstract. With the expansion of the E-commerce industry, the coverage of goods is becoming increasingly wider. Moreover, foreign E-commerce industries have gradually expanded to the Chinese market, resulting in higher requirements of domestic consumers on the safety, variety and cost performance of foreign products. Therefore, how to make the sales of cross-border E-commerce more stable and keep the balance of cross-border E-commerce inventory and sales is an urgent problem to be solved. As traditional modes are not suitable for foreign E-commerce industries, a personalized recommendation system with favorable big data processing capacity is needed to address the problem. This paper introduced a common personalized recommendation system and applied the collaborative filtering algorithm as the main algorithm of the system to solve problems in practice. The results showed that the improved collaborative filtering recommendation system could meet the requirement of the times and was worth being promoted.

Keywords: large data, cross-border E-commerce, personalized recommendation system.

Introduction

Cross-border E-commerce refers to an international commercial activity through which transaction bodies of different countries reach deals, make payments and cross - border deliveries via e-commerce platform. Though the development and popularization of computer technology and information technology has brought convenience to peoples life in recent years, it has also caused some problems due to the imperfect cross - border E-commerce development system and the shortage of professional staffs, which have been studied by some experts. Wang W [1] believed that cross-border E-commerce was an important way to export Chinese products, an effective method to solve the logistic problems should be sought, and government investment and information exchange should be enhanced. He also proposed a new E-commerce operation mode to reduce human labor and realize the rational use of labor resources [2]. Holding that the traditional recommendation algorithm had high requirement on accuracy and was not easy to implement, Song S et al. [3] put forward a user-based Slope One algorithm.

Zhou X et al. [4] argued that in a society where information was exploded, it was difficult for users to find information of interest to them, and it was difficult for users to obtain information with low page views, which were common problems in the development of the E-commerce industry, among which information overload problem was the most important and most urgent problem to be solved. Therefore, the personalized recommendation technology was introduced to help improve users shopping experience, increase user stickiness, and promote the E-commerce site sales.

1. Cross-border E-commerce under the big data background

With the advent of the Internet age and the continuous accumulation of application system data, there are more and more digital products and Internet costumers, which produces rich big data resources [5]. In this study, the Hadoop architecture was applied to analyze massive amounts of data. Besides, the core problems of big data recommendation were considered.

At present, the personalized recommendation system is divided into collaborative filtering recommendation, content-based recommendation and recommendation based on association rules [6]. Collaborative filtering recommendation is the earliest and most widely used personalized recommendation technology [7] which mainly includes user-based collaborative filtering and item-based collaborative filtering. In addition, there are also content-based personalized recommendation system [8] and rule-based personalized recommendation system [9]. In this study, the improved user-based recommendation algorithm was used to solve the problems. The Hadoop distributed computer was used to sort and store user commodity log. Cross-border E-commerce personalized recommendation services include recommendation of the products which customers may like, new products and relevance of commodities. The specific process is as follows:

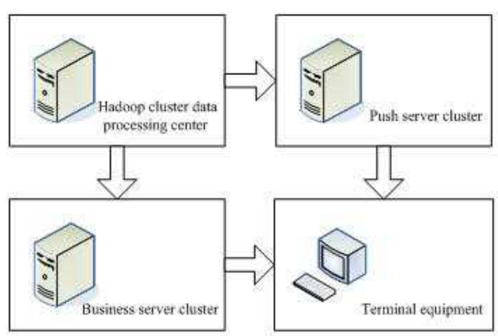

Figure 1: The architecture of the personalized recommendation system

As shown in Figure 1, Hadoop cluster data processing center is responsible for storage and processing of user feedback information; Business server cluster is the main equipment for the business management of the whole system which periodically transmits the data content of the terminal equipment to the Hadoop processing center via the network; Push server cluster transmits recommendation and private messages to the target user at regular intervals.

Recommended algorithm module

1.1 User-Based collaborative filtering algorithm

The user-based collaborative filtering algorithm is the earliest, most widely used and successful recommendation algorithm. In this algorithm, users who have similar interest as user A are found firstly. Then, the items which these users like and are not known to user A are recommended to user A:

(1) Find the user cluster which has the similar interest as the target user.

(2) Recommend items which have been evaluated by users with similar interests to the target user, expressed as:

$$p(u,m) = \sum_{v \in S(u,K) \cap N(m)},$$

where $S(u,K)$ refers to K users who have similar interest as user u, $N(m)$ refers to the user cluster which evaluates item m, w_{uv} refers to the interest similar degree between user u and user v, and r_{vm} refers to the interest of user v to item m.

1.2 Item-Based collaborative filtering algorithm

The item-based collaborative filtering algorithm is based on the assumption that the items similar to the ones favored by a user will also be favored by the user. The algorithm has the similar recommendation procedures as the user-based recommendation algorithm, including similarity degree calculation and score prediction. There are three commonly used predictive models for the item-based collaborative filtering algorithm:

(1) Weight similarity calculation method

$$P_{ui} = \frac{\sum_{j \in I_{neighbor}} sim(i,j) \cdot R_{uj}}{\sum_{j \in I_{neighbor}} sim(i,j)}.$$

(2) Return model prediction method

$$\overline{R}_j = \alpha \overline{R}_i + \beta + \varepsilon.$$

(3) Park adopted prediction method

$$P_{ui} = \overline{R}_i + \frac{\sum_{j \in I_{neighbor}} sim(i,j) \cdot (R_{uj-\overline{R}_j})}{\sum_{j \in I_{neighbor}} (|sim(i,j)|)},$$

where P_{ui} refers to the predicted score of user u to item i, $I_{neighbor}$ refers to item i's neighborhood item cluster, R_{uj} refers to the score of user u to item j, and R_i and R_j refers to the average values of the score clusters of item i and j.

1.3 Hybrid recommendation algorithm

Many mathematical algorithms can solve some professional problems in life [10, 11]. Hybrid recommendation algorithm is introduced in order to solve the cold start and data sparse problem in the traditional collaborative filtering method. It combines user-based algorithm with item-based algorithm and can calculate similarity more accurately. When predicting scores, it considers both control factor and balance factor and carries out recommendation based on the comprehensive results. To obtain more accurate predicting results, the control factor $\lambda(0 \leq \lambda \leq 1)$ was combined with the balance factors m_u and m_i.

The calculation formula of the balance factor m_u is

$$m_u = \sum_{u_m \in N(u)} \frac{(sim(u_m, u))^2}{\sum_{u_m \in N(u)} sim(u_m, u)}.$$

The calculation formula of the balance factor m_i is

$$m_i = \frac{(sim(i_n, i))^2}{\sum_{i_n \in N(i)} sim(i_n, i)}.$$

Based on the combination of the balance factors m_u and m_i, and the control factor λ, parameters a_u and a_i were added. The definitions of the two parameters were:

$$a_u = \frac{m_u \times \lambda}{m_u \times l + m_i \times (1 - \lambda)},$$
$$a_i = \frac{m_i \times (1 - l)}{m_u \times \lambda + m_i \times (1 - l)}.$$

It can be known from the above formulas that $a_u + a_i = 1$.

When neither the user neighbor cluster $N(u)$ nor the item-based neighbor cluster is an empty set, the score is:

$$P(r_{u,i}) = t_u \times P_u(r_u, i) + P_i(r_u, i).$$

1.4 Evaluation indicators of the recommendation system

To determine whether a recommendation system meets the requirements, normally three indicators, i.e. precision indicator, recall indicator and MAE (mean absolute error) indicator, are applied [12]. The precision indicator is a basic and commonly used indicator to evaluate the recommendation system currently.

The precision rate of the Top-N recommendation system can be determined by precision and recall indicators [13], and its definition was:

$$Precision = \frac{\sum_{u \in U} |Re(u) \cap Te(u)|}{\sum_{u \in U} |Re(u)|},$$

where $Re(u)$ refers to the linked list recommended based on the user training set, and $Te(u)$ refers to the linked list recommended based on the user test set

$$Recall = \frac{\sum_{u \in U} |Re(u) \cap Te(u)|}{\sum_{u \in U} |Te(u)|}.$$

The above two formulas suggested that there is mutual effect between precision and recall, one rises and the other decreases. In practical situation, a comprehensive evaluation indicator which combines both indicators should be applied.

$$F1 = \frac{2 \times Precision \times Recall}{Precision + Recall},$$

$$Emeasure = \frac{1}{\alpha(1/Precision) + (1-\alpha)(1/Recall)}.$$

The larger the $F1$, the better the recommendation effect.

MAE evaluates the precision of the predication score based on the size of difference between the prediction score and the actual score. For example, the scoring item number of user x in the test item set is T_x, the actual scoring set is $\{x_1, x_2, x_3, \ldots, x_n\}$, and the prediction scoring set is $\{p_1, p_2, p_3, \ldots, p_{r_u}\}$; then the calculation formula for MAEx is:

$$MAE_x = \frac{\sum_{i=1}^{T_x} |u_i - p_i|}{T_x}.$$

For the recommendation system, the MAE calculation result of all users in the system is:

$$MAE = \frac{\sum_{u=1}^{M} MAE_u}{M}.$$

According to the above equation, the greater the deviation of the prediction score from the actual score, the greater the MAE value.

2. Application of personalized algorithms

2.1 The improved collaborative filtering algorithm

In the multi-personalized recommendation system, the collaborative filtering algorithm is one of the most widely used algorithms. The traditional user-based algorithm recommends products to users based on the interest of similar uses on products. The application of the traditional user-based algorithm will greatly reduce the precision because of the multiple categories and wide coverage of

User/Item	Item1 (home products)	Item2 (home products)	Item3 (digital products)	Item4 (digital products)	Item5 (digital products)	Item6 (home products)
User1	3	1	1	3	3	4
User2	3	1	1	3	3	4
User3	3	1	1	3	3	4
User4	1	3	4	3	1	1
User5	2	2	4	3	2	1
User6	3	1	4	3	2	1
User7	3	2	4	4	2	**

Table 1. Cross-border E-commerce scoring table

cross-border E-commerce products. In Table 1, there are 7 users and 6 products. Item 1, 2 and 6 are home products while item 3, 4 and 5 are digital products. Firstly, the traditional collaborative filtering algorithm was applied to predict $R_{7,6}$. The users which were similar to user 7 were selected, i.e. {User4, User5, User6}. Then the interest of user 7 to item 6 was determined based on the interest of these users to item 6, $R_{7,6} = 1$.

The prediction of the interest of user 7 to home products was the target; however the similarity of interest between these users and user 7 obtained before was on digital products, which might lead to the inaccuracy of the prediction values.

To avoid this problem, a user-based multi-interest collaborative filtering algorithm was proposed. Based on the interest of a user on a product, it was known that the similar users were {User1, User2, User3}, and then $R_{7,6} = 4$ was obtained. The proposed algorithm considered the direct correlation between users and products based on different attributes of products, suggesting higher accuracy and practicability.

2.2 The application of the improved algorithm in cross-border E-commerce

2.2.1 Characteristics of cross-border E-commerce

Compared to the traditional E-commerce, cross-border E-commerce requires more effective product recommendation [14] because consumers have higher requirements on products with clearer aim. The consumer group of cross-border E-commerce is global, and the anonymous browsing of products by users can increase difficulty to information collection [15]. Therefore, the traditional E-commerce personalized recommendation method and algorithm are no longer applicable to the environment in which cross-border E-commerce is located.

2.2.2 Cross-border E-commerce user group

The user group of cross-border E-commerce is large, which includes cross-border online shopping users, pragmatism users, fashion users, entry users and potential users. The consumption levels of those users are shown in Figure 2.

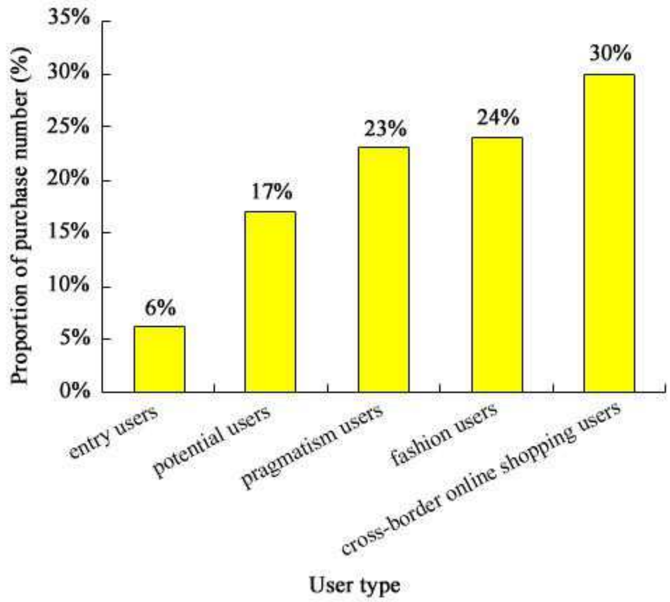

Figure 2: Proportion of consumption number of different user groups

As shown in Figure 3, the contribution of cross-border online shopping users is the highest, which is the reason why mainstream websites take them as key consumers. Besides, pragmatism users rank the second and fashion users the third. Nevertheless, potential users, though have the largest scale, have the smallest contribution. For the prevention of the cold start problem [16], the system will allow customers to choose their own interested category of goods when they log in for the first time, known as explicit interest. In addition, merchants can judge users interest on a product based on the information such as the browsing, forwarding and collecting information of the product of the users. According to different behavior of users, evaluation is made, which is called hidden interest.

To effectively determine the interests of users, we combined explicit interest with hidden interest and obtained a comprehensive calculation formula:

$$R_{u,i} = \lambda Re_{u,i} + \beta Ri_{u,i},$$

where $Re_{u,i}$ is explicit interest and $Ri_{u,i}$ is hidden interest; the range of $Re_{u,i}$ and $Ri_{u,i}$ is between 0-5 level, and the comprehensive interest range of a product is also between 0-5 level.

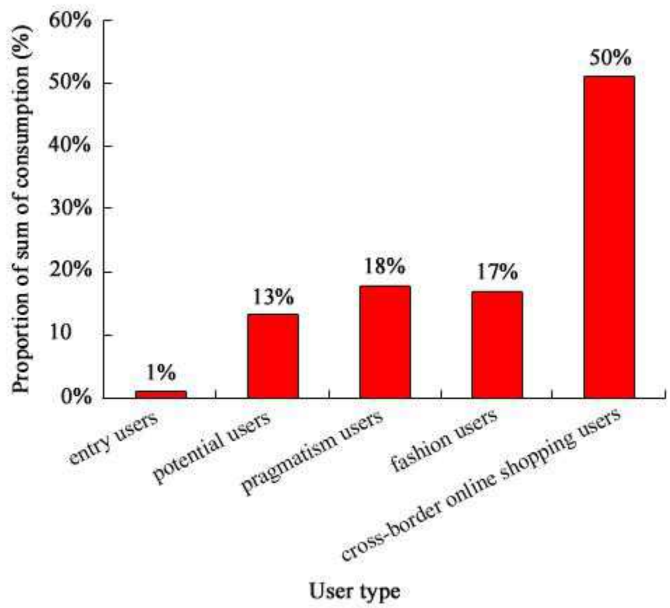

Figure 3: Proportion of sum of consumption of different user groups

According to the formula, recommendation results could be obtained. Firstly, the category of product was determined. Then, all the products belonging to the category were listed. Afterwards, the Top-N user was selected from neighbors by calculating neighbor y that has the similar interest as user u. Then, the interest of user u on the product was calculated:

$$P_{u,j} = \overline{R}_u + \sum_{i \in y} w(u,i)(R_{i,j} - \overline{R}),$$

where $w(u,i)$ refers to the similarity between user u and i. Ranking is performed based on the size of similarity and the Top-N product is recommended to user u.

2.2.3 Effect analysis of improved algorithm

In this study, the data of four months released by an E-commerce corporate was taken as the research subjects.

After processing the data in Table 2, Table 3 was obtained.

Then, the offline results of the data were tested using the aforementioned recall and precision formulas.

As shown in Table 4, the recommendation results of the improved algorithm were better than the results of the traditional algorithm.

	User-id	Brand-id	type	Visit-datetime
1	10944750	13451	0	6.4
2	10944750	13451	2	6.4
3	10944750	13451	2	6.4
4	10944750	13451	0	6.4
5	10944750	13451	0	6.4
6	10944750	13451	0	6.4
7	10944750	13451	0	6.4
8	10944750	13451	0	6.4
9	10944750	13451	0	6.4
10	10944750	21110	0	6.7
11	10944750	1131	0	7.23
12	10944750	1131	0	7.23
13	10944750	8689	0	5.2
14	10944750	8689	2	5.2

Table 2. User behavior data

	User-id	Brand-id	click	fav	addcart	buy
1	10944750	21110	5	0	1	0
2	10944750	25687	6	0	1	3
3	10944750	25372	1	1	0	1

Table 3. Processed data

3. Conclusion

This paper systematically introduced the classification of personalized recommendation systems and analyzed several collaborative filtering algorithms and their evaluation indexes. Besides, the improved algorithm was applied to the product recommendation of cross-border E-commerce. Nevertheless, there are rooms for further improvement and research, which is expected to be realized in future studies.

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Algorithm	Precision rate	Recall rate
Traditional User-based	5.35%	5.45%
Improved User-based	5.75%	5.80%

Table 4. Results of two algorithms

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ASSIMILATION OF INSAT-3D SOUNDER RETRIEVED THERMODYNAMIC PROFILES USING WRF MODEL FOR EXTREME RAINFALL EVENT OVER NORTH CENTRAL PART OF INDIA

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Abstract. The impact of Indian National Satellite-3D (INSAT-3D) retrieved thermodynamic profiles (temperature and humidity) on Weather Research Forecasting (WRF) model forecast is examined in this study. The extreme rainfall event which occurred during July 25-26, 2015 over the North central part of India is taken as the case study. The analysis obtained after assimilation is compared with the European Centre for Medium-Range Weather Forecasts (ECMWF) analysis. Obtained results show quite good improvement in humidity and temperature analysis when compared with ECMWF analysis. Positive improvements are observed in 24 h WRF model predicted rainfall on assimilation of INSAT-3D temperature and humidity profiles.
Keywords: INSAT-3D, WRF model, forecast, analysis, rainfall.

1. Introduction

Rainfall is an important parameter that changes in scales from few meters to several of kilo meters. The significance of exact rainfall delineate and forecast are broadly recognized. The precision of Numerical Weather Prediction (NWP) relies on the nature of the initial conditions. Atmospheric observation from various sources (Radar, Satellite, Aircraft, Radiosonde, etc.) are utilized to introduce operational weather prediction models. An evaluation of the nature of the precipitation outline is essential to comprehend the qualities and inadequacies of current forecast/assimilation frameworks and furthermore in perspective of future climate/weather projection. The role of satellite observations in NWP

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models have been growing rapidly due to the increase of number of weather satellites. The NWP models have been developed from the last decades with the continuous process in both data assimilation techniques and numerical model [1]. Data assimilation is the technique in which observations of the original system are incorporated into the model state of a numerical model of that system. The aim of data assimilation is to produce a model state that is as close to the 'original' state as possible, i.e. one that describes the observed reality in the optimum way, which is referred to as the analysis.

Due to their high spatial and transient determination, geostationary satellite instruments give real time data about the advancement of the climate wonders over the observing domain, when contrasted with polar-orbiting satellite instruments. Nonetheless, geostationary satellite instruments have bring down unearthly determination and are accordingly less equipped for giving vertical soundings of the climate than polar-orbiting satellite instruments [2]. India successfully launched on 26^{th} July 2013 INSAT-3D satellite at 82^{o} E. INSAT-3D carried out two meteorological instruments, that is the sounder and the imager [3]. The six channels with imager has, one within electromagnetic range and five inside the infrared (IR) region and nineteen sounder channels, one within visible band of reflected solar energy and eighteen IR channel measures emitted energy. The sounder instrument gives vertical structure of the environment. With the dispatch of INSAT-3D, barometrical soundings are workable interestingly over a moderately information scanty region, for example, the Indian Ocean from a geostationary stage. Atmospheric thermodynamic conditions over the Arabian Sea and the Bay of Bengal impact the climate frameworks over the Indian region. The INSAT-3D data, especially sounder data, can possibly contribute fundamentally to mesoscale climate guaging over the Indian region [2].

Precipitation digestion is considered as one of the essential ways to deal with enhanced the climate figures. Rainfall perception incorporates the atmospheric information as far as winds, temperature and specific humidity and furthermore adds to the model atmospheric spending plan. A few numerical modeling research have demonstrated that precipitation data enhanced the climate estimate [1, 4, 5, 6]. Various affectability examinations for rainfall assimilation have been performed at different forecast/explore focuses like in European Centre for Medium-Range Weather Forecasts (ECMWF) [7] and National Centers for Environmental Prediction (NCEP) [5, 8]. For some reasons, rainfall assimilation is an additional unpredictable issue distinguished to assimilation of convectional or on the other hand clear-sky satellite brilliance [10]. Marecal and Mahfouf [11] in 2000, exhibited that nudging of rain rate enhanced the dampness investigation and diminished the turn-up issue. As a result of deficient spread of rain measures and ground-based radars, satellite-recovered precipitation is one of the significant wellspring of precipitation perception. Treadon [8] in 1997, assimilated the satellite-recovered precipitation rate in the NCEP 3D-Var data assimilation framework. Lekhadiya and Jana [9] in 2018, shows the different physical paramererization options and it has ability to predict rainfall prediction using WRF model.

There are two ways to deal with NWP modeling system with different types of observation. The first one is the data-denial approach. It is the effect of different types of observation on forecast and analysis quality of NWP modeling system which is verified by performing two different side-by-side assimilation experiments. i.e. Control run;CNT and experiment run;EXP. In the CNT run all the observation assimilated and in EXP run only particular observation assimilated which gives best appropriate results and the effect of two experiment (CNT and EXP) are performed in different way [12, 13, 14, 15]. In the second approach, the adjoint of the NWP framework is utilized to evaluate the effect of specific types of observation [16, 17, 18]. Be that as it may, we utilized here data-denial Observing System Experiments (OSEs) with different sets of measurements which is computationally extremely costly on the grounds that requires an extensive number of analyses.

The objective of this study is to evaluate the impact of assimilation of INSAT-3D temperature and humidity profiles on WRF model forecast. An extreme rainfall event which occurred on 25^{th} July 2015 in north Madhya Pradesh and adjoining regions is taken as the case study.

2. Model description and assimilation methodology

2.1 WRF model

The model utilized as a part of this study is WRF version 3.7. The WRF model is a cutting edge mesoscale numerical weather prediction model intended to meet both research needs and operational forecasting. The subtle elements of the WRF model can be found on the site (http://www.wrf-model.org). WRF is a restricted region, compressible, and nonhydrostatic primitive equation model. It has different physical parameterization schemes [19]. There are two dynamic solvers in the WRF modeling system: the Advanced Research WRF (ARW) solver grew firstly at NCAR (National Center for Atmospheric Research), and the Nonhydrostatic Mesoscale Model (NMM) solver created at National Centers for Environmental Prediction (NCEP). Here we have utilised the ARW dynamic solver. We have used Arakawa C-grid staggering for horizontal grid and the completely compressible system of equations [20]. The territory following hydrostatic weight with vertical framework streching was utilised in vertical. The time split incorporation utilizes a third order Runge-Kutta scheme with little time ventures for acoustic and gravity wave modes. The parametrization schemes utilized as a part of this experiment comprised of WRF Single Moment (WSM) 6-class graupel conspire for microphysics, the New Kain-Fritsch [21] cumulus convection parameterization scheme and Yonsei University (YSU) planetary boundary layer scheme. The model domain (see Fig. 1) contained

330×320 framework focuses with 30 km spatial determination. The model had 36 vertical levels with the highest point level at 10 hPa.

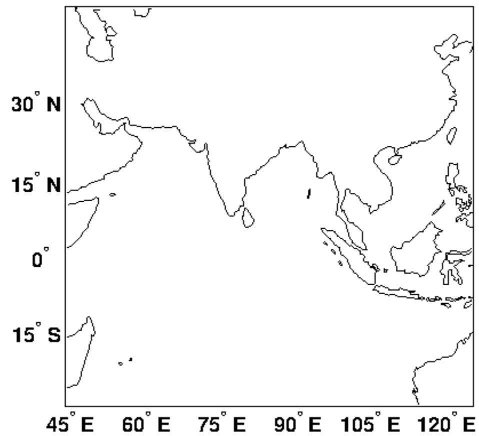

Figure 1: The model domain used in this study for WRF experiment

2.2 Assimilation methodology

The WRF three-dimensional variational (3D-Var) data assimilation framework is utilised as a part of this experiment. It is equipped with assimilating information from a wide range of observational stages got from conventional source as well as satellites. The WRF 3D-Var technique comprises of finding the most likely atmospheric state (i.e. analysis) by limiting a cost function ($J(x)$) given as

$$J(x)=J^b + J^o=\frac{1}{2}(x-x^b)^T B^{-1}(x-x^b)+\frac{1}{2}(H(x)-y^o)^T R^{-1}(H(x)-y^o). \tag{1}$$

The gradient of the cost function $J(x)$ with respect to x is given as

$$\nabla_x J(x) = B^{-1}(x-x^b) + H^T R^{-1}(H(x)-y^o). \tag{2}$$

As shown in Eq. (1), the cost function is characterised as the summed squared separation of the present state (x) to the background state (x^b) and to the perceptions (y^o) in which the agitators are weighted by the inverse of error covariance matrices. In Eq. (1), H is the (forward) perception administrator that maps model state to perception space. The covariance matrices of background error (B) and perception error (R) are expected Gaussian. These errors are additionally expected impartial and uncorrelated to each other. The setup of the WRF 3D-Var framework depends on an incremental plan turning in a multivariate incremental research inside the WRF model space. The incremental cost function minimization is accomplished in a preconditioned control variable space. The preconditioned manipulate variables applied as a part of this experiment are humidity, velocity potential, stream-function and unbalanced pressure.

Information of contrasts between 24 h and 12 h forecasts are utilised to evaluate background error covariances matrix by the National Meteorological Center (NMC) technique [22, 23]. Portrayal of the horizontal aspect of historical past blunders on a level plane homogeneous and isotropic recursive channels. The vertical component is hooked up through projection onto climatologically arrived on the midpoint of eigenvectors of vertical error evaluated by the NMC strategy [24]. Within WRF 3D-Var, all perception errors are thought to be uncorrelated in space and time. The corner to corner components of those covariance matrices are contrast for perceptions (in present case, temperature and humidity).

3. Data used

The temperature and humidity profiles are retrieved from INSAT-3D data at 43 pressure levels and the retrieved data is available at L2B product from Meteorological & Oceanographic Satellite Data Archival Centre (MOSDAC) (www.mosadc.gov.in). The spatial resolution of data is 10 km at nadir. The data is available hourly. The retrieved temperature and humidity profiles for 25^{th} July, 2015 are taken and data file is in the form of Hierarchical Data Format-5 (HDF5) which is dumped for the variable temperature, specific humidity, latitude, longitude and pressure levels in binary form. Combining all binary files to a single file and converting it into American Standard Code for Information Interchange (ASCII) format, which is readable for OBSPROC and subsequently for 3D-Var. Using them with GFS data file and processed through WRFDA. Model initial conditions and lateral boundary conditions are taken from NCEP analysis at every six hours with $0.5^o \times 0.5^o$ horizontal resolution and 26 vertical levels. The lateral boundary conditions must first be modified to reflect difference between background forecast and analysis. The model forecast verification is done with temperature and humidity profiles from ECMWF analysis. For rainfall forecast verification Global Satellite Mapping of Precipitation (GSMaP) data is used.

4. Case study

An extreme rainfall event which occurred on 25^{th} July, 2015 in North-Madhya Pradesh, central India is taken as the case study. The 24 h accumulated rainfall from GSMaP during 0000 UTC 25 - 0000 UTC 26 July 2015 is shown in Fig. 2. From Fig. 2, we can see that there was 160 mm rainfall in 24 h over North-Madhya Pradesh.

4.1 Experimental setup

Two experiments were performed for this study, the Control (CNT) and the Experiment (EXP). In the CNT run, only GFS analysis is taken as the model initial condition, while in EXP run, the INSAT-3D retrieved temperature and

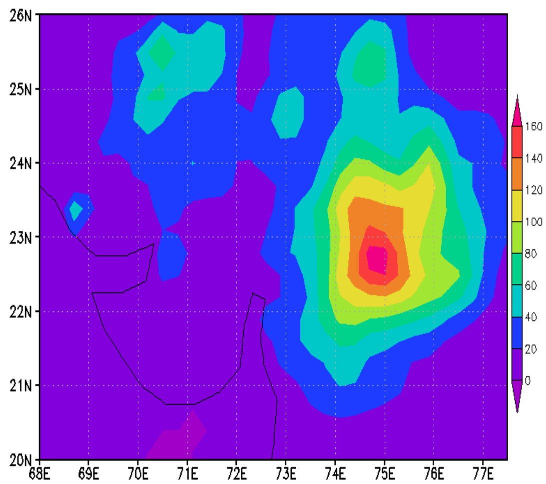

Figure 2: 24 h accumulated rainfall(mm) map from GSMaP during 0000 UTC 25 July 2015 - 0000 UTC 26 July 2015

humidity profiles are assimilated and GFS analysis are taken as model boundary conditions. The WRF model then run with the obtained analysis from CNT and EXP run to provide 24 h forecast.

WPS → Real → WRF → ARWPost → 24 h

Flowchart 1: Control Run

WPS → Real → WRFDA → WRF → ARWPost → 24 h

WRFDA ↓ Assimilation of INSAT-3D temperature and humidity

Flowchart 2: Experiment Run

5. Results and discussion

5.1 Impact on analysis

5.1.1 Overview of the fit to observations

The principal correlation that we made can be portrayed as an once-over to verify everything seems to be good or sanity check i.e. it is basic test to rapidly assess whether a claim or the after-effect of a figuring can be valid. The INSAT-3D analysed temperature and humidity data are plotted as a function of observed temperature and humidity respectively and compared with the first guess. In an effective assimilation, the investigation called as analysis departure (O-A) are smaller than the first guess departure (O-B); subsequently the analysis better matches the perceptions. The histogram plots of the first-guess and anlysis departures for temperature and humidity are shown in Fig. 3 and Fig. 4 respectively. The first-guess departures (O-B) for temperature has a Root Mean Square Deviation (RMSD) of about 0.4063 while the analysis depatures (O-A) has the RMSD of about 0.6831. The first-guess departure for humidity is found to have RMSD of about 0.2899 while analysis departures have the RMSD of about 9.2491e-04. The analysis bias and RMSD are altogether lower than their background counterparts. From Fig. 3 and Fig. 4, it is clear that the analysis is closer to the observations than the background.

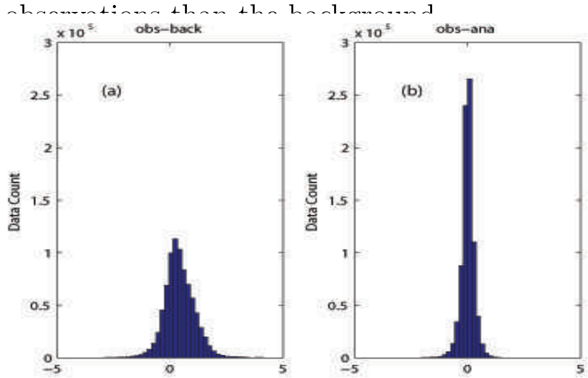

Figure 3: Histogram of the Temperature(K) (a) first guess departures (O-B) and (b) analysis departures (O-A)

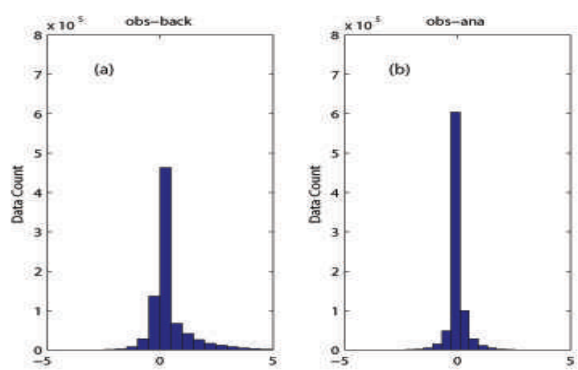

Figure 4: Histogram of the Humidity(gm/kg) (a) first guess departures (O-B) and (b) analysis departures (O-A)

5.1.2 Comparision with ECMWF analysis

The 24 h analysed specific humidity and temperature from both the experiments (CNT and EXP) are verified against the ECMWF analysis valid at 0000 UTC 25^{th} July, 2015. The vertical profiles of the domain averaged RMSD of the temperature and specific humidity with respect to the ECMWF analysis. Temperature and Humidity for both EXP and CNT runs are shown in Fig. 5(a) and Fig. 5(b) respectively. The RMSD values for the EXP runs for both temperature and humidity are less than the corresponding CNT values suggesting that the assimilation of the INSAT-3D temperature and humidity profiles have improved the analysis in EXP run.

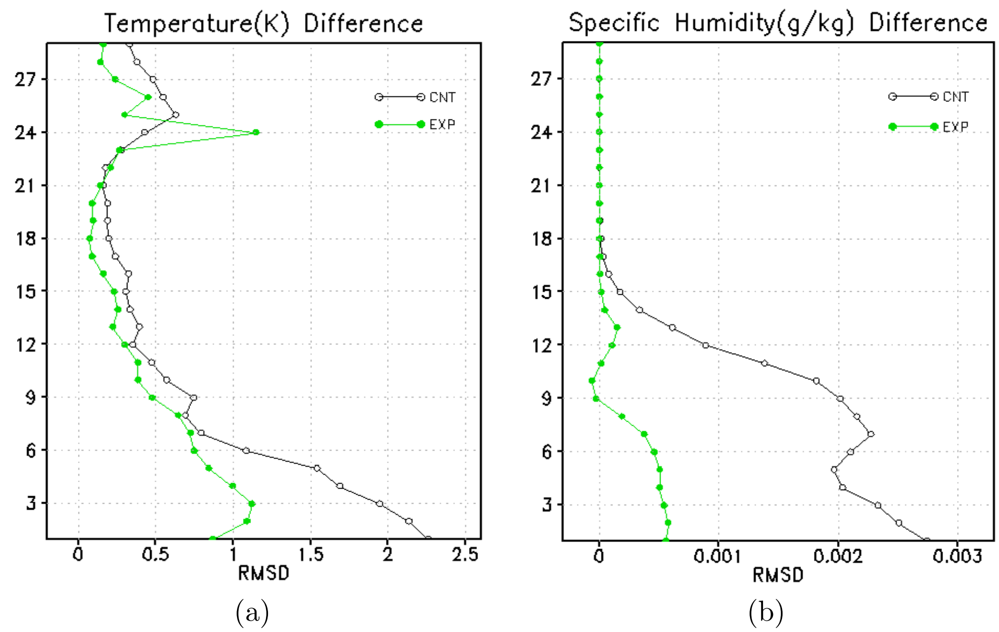

Figure 5: Domain averaged vertical profiles of the RMSD of (a) temperature and (b) humidity for CNT and EXP runs calculated with respect to the ECMWF analysis valid at 0000 UTC 25^{th} July 2015

5.2 Impact on forecast

5.2.1 Comparison with ECMWF analysis

The 24 h predicted temperature and specific humidity from both the experiments (CNT and EXP) are verified against the ECMWF analysis. The spatial distribution of the forecast improvement for the 24 h forecasted temperature and specific humidity on 26^{th} July 2015 are shown in Fig. 6 and Fig. 7 repectively. The assimilation of INSAT-3D temperature and specific humidity profiles show significant improvement in temperature and specific humidity throughout the domain. A few pockets of negative improvement are also observed. The vertical profiles of RMSD, for both CNT and EXP run, in 24 h forecast of specific humidity and temperature are shown in Fig. 8(a) and Fig. 8(b) respectively.

5.2.2 Rainfall Comparison with GSMaP rainfall

The 24 h precipitation forecast is verified against the observation from GSMaP. Here we examined the spatial distribution of 24 h accumulated rainfall. The

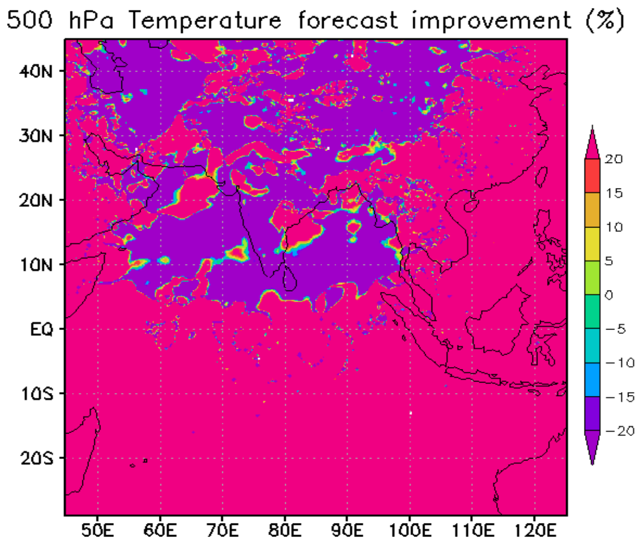

Figure 6: Spatial distribution of the temperature forecast improvement during 0000 UTC 26^{th} July 2015

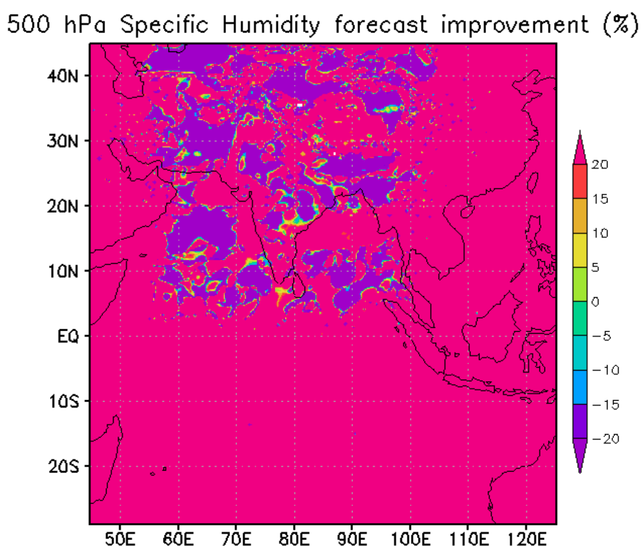

Figure 7: Spatial distribution of the specific humidity forecast improvement during 26^{th} July 2015

CNT minus GSMaP rainfall map is shown in Fig. 9(a), while EXP minus GSMaP rainfall map is shown in Fig. 9(b). The spatial distribution of the rainfall improvement parameter Fig. 9(c), clearly shows that the assimilation of temperature and specific humidity profiles from INSAT-3D data improved the accumulated rainfall prediction over whole Madhya Pradesh and its adjoin-

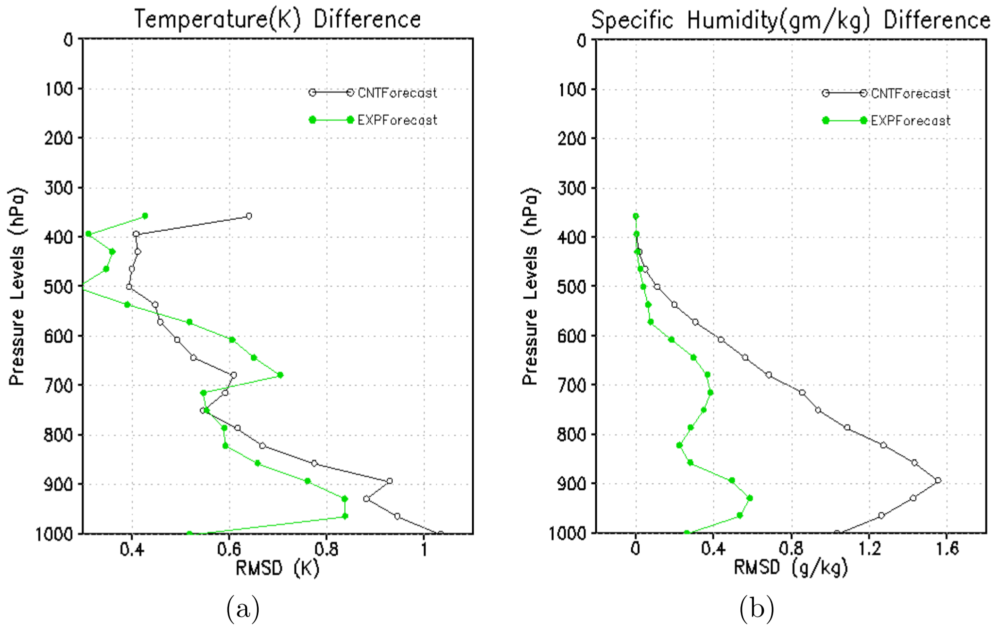

Figure 8: Domain averaged vertical profile of the RMSD of 24 h (a) Temperature and (b) Humidity forecast for CNTForcast and EXPForcast runs calculated with respect to the ECMWF analysis valid at 0000 UTC 26^{th} July 2015

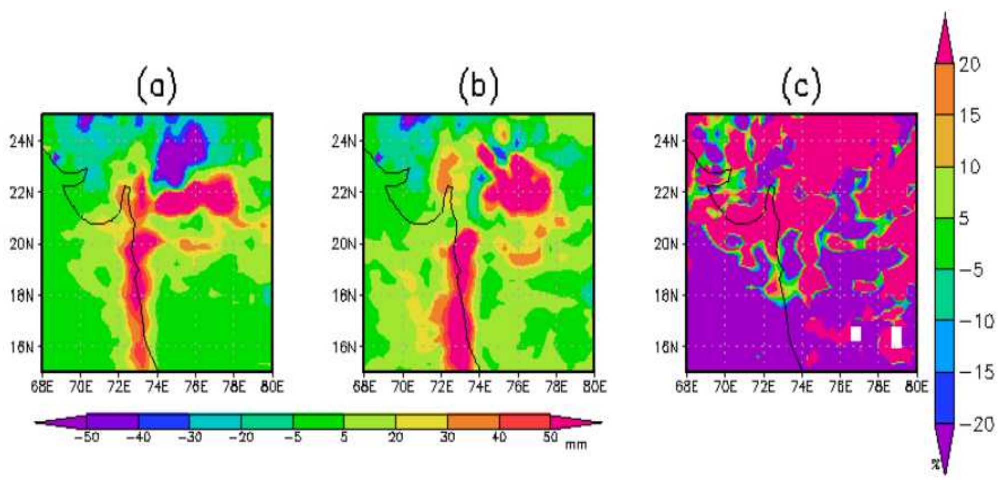

Figure 9: Spatial distribution of 24 h accumulated rainfall forecast improvement parameter in the form (a) CNT minus GSMaP rainfall map, (b) EXP minus GSMaP rainfall map and (c) rainfall improvement

ing regions. Overall, the plots suggests that accumulated rainfall prediction is improved on assimilation of thermodynamic profiles from INSAT-3D.

6. Conclusion

In this investigation, WRF model has been utilized to assimilate the precipitation information during 25 - 26 July, 2015 over North central part of India. Assimilation have been done to compare GFS analysis with ECMWF rainfall, GFS forecast with ECMWF rainfall as well as compared with GFS analysis and GSMaP rainfall. The results shows quite good improvement in temperature and specific humidity forecasts. It demonstrate that rainfall assimilation enhances the rainfall forecast. Thus, assimilation of rainfall observation in the NWP model can be seen as a positive advancement for enhancing the accuracy of numerical modeling for short-range weather forecast.

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A LOCAL MESHLESS RBF METHOD FOR SOLVING FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH OPTIMAL SHAPE PARAMETERS

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Abstract. This paper investigates the application of the meshless local radial basis functions collocation method (LRBFCM) for the numerical solution of fractional integro-differential equation and two-dimensional fractional Volterra integral equation. Unlike the traditional global RBF collocation method, dividing the collocation of the problem in the global domain into many local regions, and therefore, the ill-conditioning of the problem is reduced and becomes highly stable. Here, we use the multiquadric (MQ) radial basis function that includes a shape parameter, which plays an important role in the accuracy of method. Scaling of the shape parameter to make local RBF approximation insensitive is performed by particle swarm optimization (PSO) algorithm. Some test problems are studied and the numerical results shows the efficiency of the method.

Keywords: fractional calculus, local meshless methods, fractional integral-differential equations, collocation methods, optimal shape parameter.

1. Introduction

Meshless methods are very attractive and effective for solving boundary value problems, because they involve simple preprocessing, arbitrary node distribution and flexibility of placing nodes at arbitrary locations. Also, they are easily extendable to higher dimensional problems. These methods may use strong form [1, 2, 3] or weak form of governing equations [4, 5, 6, 7]. Also, these methods may treat the problem locally [8, 9, 10, 11] or globally [12]. Since the final matrix ob-

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tained in global methods are usually full and ill conditioned, local methods were proposed in the literature. Local multiquadric approximation (LMQ) method [13, 8, 14] and the finite collocation approach (FC) [9, 15] are among the popular local methods which use strong form equations. These local methods reduce the problem into many local sub-problems and finally assemble all these local equations into the final global matrix. Therefore, the final global matrix obtained by LMQ and FC methods are sparse. In this paper we use the multiquadric (MQ) radial basis function. Kansa [16] applied MQ functions for scattered data approximation in and presented a new approach to solve PDEs [17]. MQ was employed as a spatial approximation scheme for Hyperbolic, parabolic and the elliptic Poisson's equation. This function has a free parameter (c) called shape parameter. This parameter plays an important role for the accuracy of the method, and is achieved by various techniques. The particle swarm optimization (PSO) algorithm is applied to obtain the optimum value of this parameter. In this paper our aim is to suggest and apply the local RBF method based on multi-quadratics for the numerical solutions of the fractional integro-differential equation and fractional integral equations. These equations form an important part of applied mathematics, which links with many theoretical and practical fields. The concept of the fractional derivative was introduced in the middle of the 19th century by Riemann and Liouville. Many physical and biological models are formulated using fractional differentials. So, in recent years the number of publications about the fractional calculus has rapidly increased [4, 11, 18]. This paper is organized as follows: In Section 2, the basic defnitions in fractional calculus and fractional integral equation which is needed in the next sections are presented. In Section 3, the local RBF method for discretizing fractional integral and integro-differential equations is described. Section 4 is devoted to introducing particle swarm optimization algorithm for finding optimal shape parameters. Numerical results are given in Section 5. Our conclusions are summerized in Section 6.

2. Basic definitions

In this section, we give some basic deffnitions and properties of the fractional calculus theory which are used further in this paper [18, 19].

Let $f(x)$ be a function defined on (a, b), then we have the following definitions:

Definition 1. The Riemann-Liouville(R-L) fractional integration operator of order $\alpha \geq 0$ of a function f is defined as:

$$I^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, & \alpha > 0,\ x > 0, \\ f(x), & \alpha = 0, \end{cases}$$

where $\Gamma(.)$ is the gamma function and I is the fractional integral.

Definition 2. The Riemann - Liouville fractional derivative of $f(x)$ is:

$$^{R}D^{\alpha}f(x)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x\frac{f(t)}{(x-t)^{-m+1+\alpha}}dt, & x>0,\ m-1\le\alpha<m,\\ f^m(x), & \alpha=m,\end{cases}$$

where $m=\lceil\alpha\rceil$ is the smallest integer such that $m>\alpha$ and d^m/dx^m denotes the standard derivatives of integer order.

Definition 3. The Caputo fractional derivative of $f(x)$ is:

$$D^{\alpha}f(x)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_a^x\frac{f^{(m)}(t)}{(x-t)^{-m+\alpha+1}}dt, & x>0, m-1\le\alpha<m,\\ f^m(x), & \alpha=m.\end{cases}$$

The Caputo operator D^{α} advantages for fractional differential equations (FDEs) with initial conditions. The two definitions of Riemann-Liouville and Caputo are not equivalent and their relation is correlated by the following expression,

$$^{R}D^{\alpha}f(x)=D^{\alpha}f(x)+\sum_{k=0}^{m-1}f^{(k)}(a)\Phi_{k-\alpha+1}(x-a).$$

$$\Phi_{\alpha}(x)=\begin{cases}x^{\alpha-1}, & x>0,\\ 0, & x\le 0.\end{cases}$$

Definition 4. The left-sided mixed Riemann-Liouville integral of order $r=(r_1,r_2)$ for the function $u(x,y)$ is defined as

$$(I_{\theta}^{r}u)(x,y)=\frac{1}{\Gamma(r_1)}\frac{1}{\Gamma(r_2)}\int_a^x\int_a^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dsdt, \tag{1}$$

where $r\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1(J)$. So, we have the following:

(1) $(I_{\theta}^{\theta}u)(x,y)=u(x,y)$,

(2) $(I_{\theta}^{r}u)(x,y)=\int_a^x\int_a^y u(s,t)dsdt$ where $r=(1,1)$ *and for all* $(x,y)\in J$,

(3) $(I_{\theta}^{r}u)(x,0)=(I_{\theta}^{r}u)(0,y)=0$ for $x\in[0,a],\quad y\in[0,b]$,

(4) Let $\lambda,\ \omega\in(-1,\infty)$ then $(I_{\theta}^{r}x^{\lambda}y^{\omega})=\frac{\Gamma(\lambda+1)\Gamma(\omega+1)}{\Gamma(\lambda+r_1+1)\Gamma(\omega+r_2+1)}x^{\lambda+r_1}y^{\omega+r_2}$ for all $(x,y)\in J$.

For more information about the left-sided mixed Riemann-Liouville integral see also [20].

Two-dimensional fractional Volterra integral equation is difined as follows:

$$u(x,y)-\frac{1}{\Gamma(r_1)}\frac{1}{\Gamma(r_2)}\int_a^x\int_a^y(x-s)^{r_1-1}(y-t)^{r_2-1}K(x,y,s,t,u(s,t))dsdt$$
$$=g(x,y). \tag{2}$$

3. Discretization by LRBF-MQ

In this section, the local RBF method is used as a technique for approximation of boundary value problems and fractional integral equations . To illustrate the local RBF method, we consider the following boundary value problem:

$$\begin{cases} L(u(x)) = f(x), & x \in \Omega, \\ B(u(x)) = h(x), & x \in \partial\Omega, \end{cases} \tag{3}$$

where L and B denote the linear partial differential operator and boundary operation respectively. $\Omega \subset R^n$ is a bounded domain, and $\partial\Omega$ denotes its boundary. In the local RBF method, we consider a set of N scattered nodal points in the domain and on the boundary that is represented by $\Xi = \{x_k\}_{k=1}^{N}$. Then, to approximate the unknown solution u at an arbitrary point $x_k,\ k = 1, ..., N$, we consider a local region Ω_x around this point, called the domain of influence of point x_k, covering a number of n nodal points as shown in Fig.1. The solution

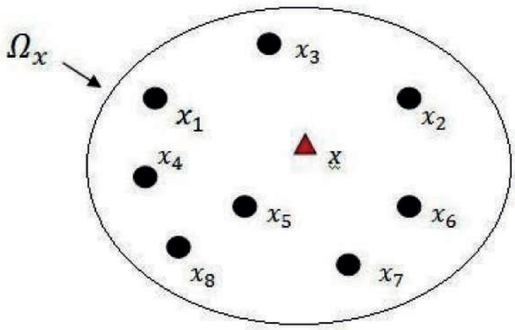

Figure 1. Influence domain Ω_x of a node x embracing 8 neighboring nodes.

$u(x)$ can be approximated by a localized formulation as follows:

$$\widetilde{u}(x) = \sum_{j=1}^{n} \lambda_j \phi_j(x), \tag{4}$$

where

$$\phi_j(x) = \sqrt{\|x - x_j\|^2 + c^2},$$

is the multiquadric(MQ), n is the number of nodal points fallen with in the influence domain Ω_x of x. The parameter $c > 0$ is known as the shape parameter, and describes the relative width of the RBFs around their centers. This parameter plays an important role for the accuracy of the method. In most articles, the authors choose this shape parameter by trial and error or some other techniques [21, 22]. Here the particle swarm optimization algorithm is applied to obtain the optimum value of this parameter which is explained in the next section.

To obtain the values of the coefficients λ_j, we can first evaluate Eq. (4) at all nodal points $x_i; i = 1, 2, ..., n$ in each influence domain. If all the collocation points are distinct, and $\phi(x)$ is a positive definite, it can be proved that the matrix $\Phi = (\phi(\|x_i^k - x_j^k\|_2))$ is non-singular. Hence, the unknown coefficients in Eq. (4) have the following matrix form:

$$\lambda^k = \Phi^{-1} u^k, \tag{5}$$

where $\lambda^k = (\lambda_1^k, \lambda_2^k, ..., \lambda_n^k)^T,\ u^k = (u(x_1^k), u(x_2^k), ..., u(x_n^k))^T$. Then the approximate solution $\widetilde{u}(x_k)$ can be revise according to the given nodal values $u(x_j^k)$ at influence domain x_k:

$$\widetilde{u}(x_k) = \Phi^k \lambda^k = \Phi^k \Phi^{-1} u^k = \Psi^k u^k, \tag{6}$$

where $\Phi^k = (\phi(\|x_k - x_j^k\|_2))$, and $\Psi^k = \Phi^k \Phi^{-1} = [\psi_1, \psi_1, ..., \psi_n]$. The functions $\psi_i,\ i = 1, 2, ..., n$ are called the shape functions for the local RBF interpolation. Fractional derivatives $D^\alpha \Psi$ can be computed as follows:

Using Eq (5),

$$D^\alpha(u(x)) = D^\alpha[\sum_{j=1}^{n} \lambda_j \phi_j(x)] = D^\alpha[\sum_{j=1}^{n} \Phi^{-1} \phi_j(x) \widetilde{u}] = [\Phi^{-1} D^\alpha(\Delta^k(x))]\widetilde{u}, \tag{7}$$

where $D^\alpha \Delta^k(x) = [D^\alpha \phi_1(x), D^\alpha \phi_2(x), ..., D^\alpha \phi_n(x)]$ and $\widetilde{u} = (\widetilde{u}(x_1), \widetilde{u}(x_2), ..., \widetilde{u}(x_n))$.

Finally, corresponding to each node, a local equation will be obtained and all these equations should be assembled in a global final system. When assembling the local equations in the final global system, the ith row of the global matrix is a vector with n non-zero elements $[0, ..., \vartheta(\psi_1), 0, ..., \vartheta(\psi_2), 0, ..., \vartheta(\psi_n), 0, ..., 0]$ in which $\vartheta(\psi_i)$ is equal either $L(\psi_i)$ or $B(\psi_i)$ depending on operator that acts on ψ_i. then Substituting these vectors into Eq. (3), yields

$$\begin{bmatrix} L\Psi \\ B\Psi \end{bmatrix} \widetilde{u} = [b],$$

which in general is the following system of equations:

$$Au = b.$$

Note that the number of columns, which ψ_k is located, is the global number of node k in all callocation node. In other words, this row is the extension of vector by patching zeros into entries associated with the nonselected $[\vartheta(\psi_1), \vartheta(\psi_2), ..., \vartheta(\psi_n)]$ nodes in the Ω_{x_i}.

By solving the above mentioned linear sparse system of equations, we get the approximate solutions $\widetilde{u}$ at all of the collocation points.

Similarly for two-dimensional fractional integral equation we have:

$$u(p) \simeq \sum_{\gamma=0}^{n} \lambda_\gamma \phi(\|p - p_\gamma\|) = \lambda^T \varphi(p), \tag{8}$$

then we have

$$\lambda = \varphi^{-1}u, \tag{9}$$

where $p = (x, y)$ and $p_\gamma = (x_\gamma, y_\gamma) \in R^2$. The dependence on the RBF expansion coefficients can be removed from Eq. (9) by the following:

$$u(p_k) \simeq \sum_{\gamma=0}^{n} \lambda_\gamma \phi(\|p_k - p_\gamma^s\|) = \Phi^k \lambda^k = \Phi^k \Phi^{-1} u^k = \Psi u^k, \tag{10}$$

substituting Eqs. (9) and (10) in Eq. (2) we have:

$$\begin{aligned} &\Psi u^k - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_a^x \int_a^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\cdot K(x, y, s, t, \Phi^{-1}\phi(s,t)\widetilde{u})dsdt = g(x, y). \end{aligned} \tag{11}$$

Substituting the given collocation points in the above equation and applying Legendre quadrature integration formula, we obtain

$$\begin{aligned} &\Psi\widetilde{u} - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=0}^{m}\sum_{l=0}^{m} w_k w_l (x_i - \xi_k)^{r_1-1}(y_j - \tau_l)^{r_2-1} \\ &\cdot K(x_i, y_j, \xi_k, \tau_j, \Phi^{-1}\phi(\xi_k, \tau_l)\widetilde{u}) = g(x_i, y_j). \end{aligned} \tag{12}$$

4. Choosing a Shape Parameter

In RBFs interpolation, different shape parameters correspond to different approximation results. In this section presented particle, swarm optimization algorithm (PSO) for optimizing shape parameters with respect to error in an global and local RBF interpolation is applied.

4.1 Particle swarm optimization algorithm (PSOA)

PSOA was firstly proposed by Eberhart and Kennedy (1995) based on the population (swarm) of particles [23]. Each particle is associated with velocity that indicates where the particle is traveling. The process is such that a group of particles in the particle swarm optimization algorithm are initially created randomly and by updating the generations, they try to find the optimal solution. In a bunch of N particles, the position of the ith particle in the search space is located under the influence of a n-dimensional spatial vector of Eq.(13).

$$X_i = (x_{i1}, x_{i2}, ..., x_{in})^T \in S, \tag{13}$$

The velocity vector of this particle is as Eq. (14).

$$V_i = (v_{i1}, v_{i2}, ..., v_{in})^T \in S, \tag{14}$$

The best position of the ith particle is represented by the Eq. (15).

$$P_{besti} = (P_{besti1}, P_{besti2}, ..., P_{bestin})^T \in S. \tag{15}$$

In each step, each particle is updated using the two best values. First situation is the best one that a particle can achieve so far. This position is known and stored as P_{BEST}. The best alternative used by the algorithm is the best situation ever achieved by the particle population. The other most appropriate value used by the algorithm, the best position ever has been achieved by the particle population. this position is shown by G_{BEST}. After finding the best values, the speed and location of each particle is updated using Eqs. (16) and (17).

$$V_{i,t+1} = \omega V_{i,t} + c_1 r_1 (P_{i,t}^{best} - X_{i,t}) + c_2 r_2 \tag{16}$$

$$X_{i,t+1} = X_{i,t} + V_{i,t+1}. \tag{17}$$

$V_{i,t}$ and $X_{i,t}$ are respectively the velocity vectors and the position of the particle i in the repetition t. ω is a stationary coefficient 1 and 2 are acceleration coefficients the implementation of optimization algorithm particles are usually considered 2. Also r_1 and r_2 are two nonlinear stochastic numbers between 0 and 1. The condition for stopping the PSO algorithm is usually considered to be such that, if the difference between two consecutive results is less than a certain value, the algorithm is stopped or a certain number of repetitions are considered for the algorithm [24].

Here, the error is considered to be the objective function, and we find the optimal shape parameter by finding the least error in the repetitions of the PSO algorithm.

LRBF implementation.

The step-wise procedure for the implementation of LRBF collocation method is as follows:

Step 1: Selection of scattered nodal points in the domain and on the boundary.

Step 2: Consider a local region around each collocate point, called the domain of influence including the point itself and $n-1$ other points.

Step 3: Applying local interpolation on each subdomain, Upon computing Φ^{-1}, the coefficient vector calculated by $\lambda^k = \Phi^{-1} u^k$.

Step 4: Approximated function, $u(x_k)$, (x_k is center subdomain) expressed in terms of the nodal values at each subdomain. That is $\widetilde{u}(x_k) = \Psi^k u^k$.

Step 5: Finally,corresponding to each node a local equation will be obtained and all these equations should be assembled in a final global system.

Step 6: Finding the optimal shape parameter with the pso algorithm.

Step 7: Solving linear sparse system of equations, we get the approximate solutions u at all of the collocation points.

5. Numerical experiment

Test problems that we consider in this section consist of fractional integro-differential equation with the nonlocal boundary conditions, Bagley-Torvik equation and two-dimensional Volterra integral equation of fractional order. Accuracies of the numerical results are measured by infinity norm error $\|e\|_\infty$ or root mean square (RMS) error defined by:

$$\delta_{er} = \|e\|_\infty = \max\{|u_N(x_i) - u^*(x_i)|\}, \qquad x_i \in X,$$

and

$$RMS = \left(\sum_{z_i \in Z} \frac{(u_N x_i - u^*(x_i))^2}{|X|} \right)^{1/2},$$

where u_N is the numerical solution, u^* is the exact solution and X is the number of testing nodes.

Remark. 1) Number of infuence domain points is selected based on the factors of accuracy and cost of calculations.

2) In PSO, population size is set to 10 and maximum number of iterations is selected 100.

3) The method is implemented in MATLAB and the numerical experiments are performed using a laptop with an Intel(R) Core(TM) i5-3230M, CPU 2.60GHz, and 4 GB RAM.

5.1 Example 1.

Consider the following fractional integro-differential equation [19].

$$ {}^{R}D^{\frac{5}{4}}u(x) = (cosx - sinx)u(x) + f(x) + \int_0^x sintu(t), \tag{18}$$

with the nonlocal conditions

$$u(0) + u(1) + (\frac{e+1}{e+2})u'(0) + \frac{1}{2}u'(1) - 8\int_0^1 tu(t) = 0,$$
$$2u(0) + 2u(1) + (\frac{e}{e+1})u'(0) - u'(1) = 0.$$

By choosing $f(x) = \frac{8}{3}\frac{x^{\frac{3}{4}}}{\Gamma(\frac{3}{4})} - 2\cos x - 2x\sin x + x^2\sin x + 2$, the exact solution of above problem is $u(x) = x^2$.

Numerical results versus the numbers of nodal points and stencil with optimal shape parameter are shown in Tabel 1. Fig. 2 presents the RMS error versus the number of the nodal points and stencil with optimal shape parameter. Fig. 3 shows the error curves for local and global RBF methods with 100 nodal points and optimal shape parameter. According to Tabel 1 and Figures 2 and 3, the local method is often just as accurate as the global RBF method.

Table 1: Numerical results with different number of nodal points by optimal neighborhood node and optimal shape parameter c and CPU time (seconds) for Ex.1.

		LRBF			GRBF		
N	n	δ_{er}	RMS	CPU(s)	δ_{er}	RMS	CPU(s)
20	7	1.33×10^{-4}	8.9799×10^{-4}	1.80	5.44×10^{-3}	2.5962×10^{-3}	4.17
30	7	5.45×10^{-4}	1.4264×10^{-4}	1.97	4.04×10^{-3}	1.7907×10^{-3}	5.67
50	11	3.38×10^{-4}	8.6699×10^{-5}	3.50	4.08×10^{-3}	1.7942×10^{-3}	12.12
80	16	1.36×10^{-4}	6.9852×10^{-5}	6.23	4.04×10^{-3}	2.0904×10^{-3}	26.13
100	17	5.85×10^{-5}	3.7125×10^{-5}	8.41	4.24×10^{-3}	2.7614×10^{-3}	39.30
150	21	8.18×10^{-5}	2.1590×10^{-5}	14.99	4.13×10^{-3}	2.2810×10^{-3}	85.90
200	25	5.45×10^{-5}	2.9491×10^{-5}	23.22	4.16×10^{-3}	1.7991×10^{-3}	149.35
250	32	3.84×10^{-4}	2.0969×10^{-4}	36.58	4.17×10^{-3}	4.0903×10^{-3}	239.13
300	35	2.92×10^{-4}	8.4034×10^{-5}	48.24	4.49×10^{-3}	4.2043×10^{-3}	339.24

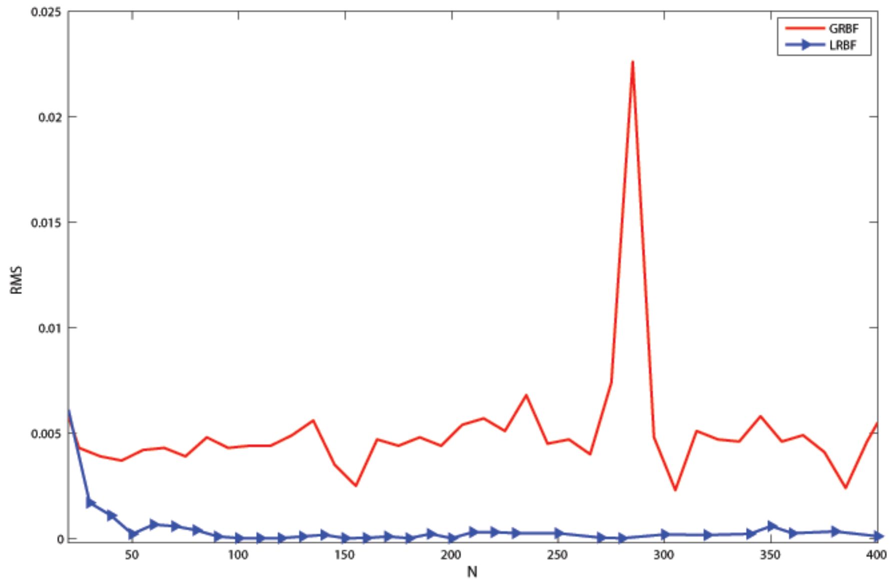

Fig. 2. RMS error versus N for Ex. 1.

5.2 Example 2.

Consider the following Bagley-Torvik equation [25]

$$u^{(2)}(x)+\theta D^{\alpha}u(x)+\sigma u(x) = f(x), \tag{19}$$

where

$$f(x) = (\lambda-1)(\lambda x-\lambda+2)x^{\lambda-3}+\theta\frac{(\lambda-1)!}{\Gamma(\lambda-\alpha)}(\frac{\lambda x}{\lambda-\alpha}-1)x^{\lambda-\alpha-1}+\sigma x^{\lambda-1}(x-1).$$

The exact solution for various values of $\theta = 0.5$, $\sigma = 1$ and $\lambda = 5$ and $\alpha = 0.3$ is $u(x) = x^{\lambda-1}(x-1)$.

Results for different values of the nodal points and the optimal stencil shown in the Table 2. Fig.4 presents the RMS error versus the number of the nodal points and Also, exact solution and approximate solution for various values α

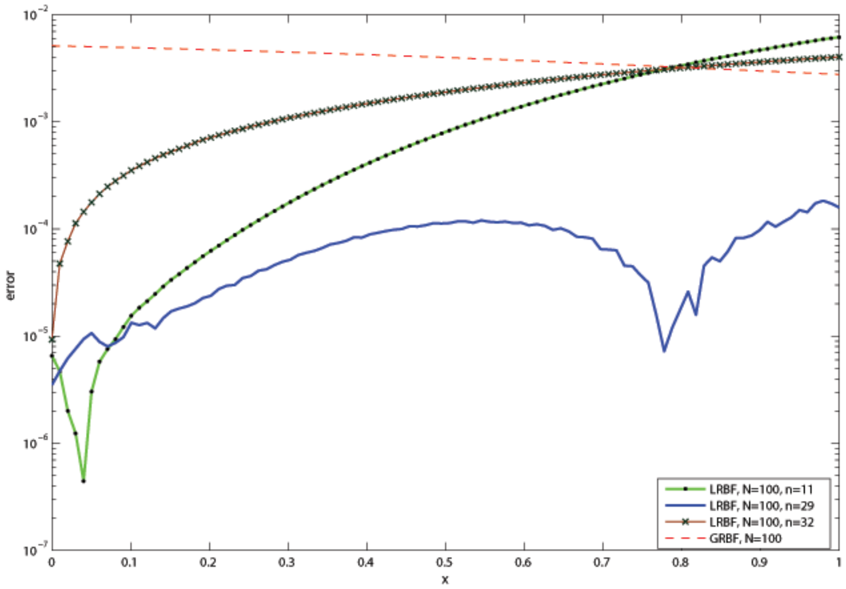

Fig. 3. Absolute errors for local and global RBF methods with optimal shape parameter for Ex.1.

Table 2: Numerical results with different number of nodal points and optimal shape parameter c for Ex.2.

N	n	LRBF δ_{er}	LRBF RMS	GRBF δ_{er}	GRBF RMS
30	9	4.9909×10^{-4}	2.0626×10^{-4}	9.8453×10^{-5}	9.8248×10^{-5}
50	12	8.9333×10^{-5}	1.2044×10^{-5}	4.4596×10^{-5}	4.4521×10^{-6}
80	16	6.2402×10^{-5}	2.5866×10^{-6}	1.3771×10^{-6}	3.3757×10^{-6}
100	19	5.5600×10^{-6}	6.2525×10^{-7}	2.4027×10^{-6}	2.4027×10^{-6}
150	39	5.7572×10^{-6}	8.9533×10^{-7}	1.2282×10^{-6}	1.1645×10^{-6}
200	35	6.3928×10^{-5}	5.7164×10^{-6}	1.1907×10^{-6}	1.0134×10^{-6}
250	26	1.2551×10^{-5}	2.0264×10^{-5}	2.2545×10^{-5}	1.5756×10^{-6}
300	33	8.0320×10^{-5}	3.6287×10^{-4}	2.1676×10^{-5}	1.4436×10^{-5}

presented in Fig. 5. Table 2 Shows that for the less number of collocation points, global RBFs method is better than local RBF method, but with increasing collocation points, the performance of the local RBFs method is better [25].

5.3 Example 3.

Consider the following Bagley-Torvik equation [26].

$$u^{(2)}(x)+\theta D^{\alpha}u(x) = -1-e^{(x-1)}, \tag{20}$$

In general, the exact solution of the problem is not known. However, for $\alpha = 1$, $\theta = -1$, the problem has exact solution is $u(x) = x(1 - e^{x-1})$.

Table 3 shows the numerical results at different the numbers of nodal points and stencil with optimal shape parameter. Approximate solutions with various α are presented in Fig. 6. RMS error and absolute error versus the number of the nodal points are shown in Fig. 7. Considering the results, accuracy of the

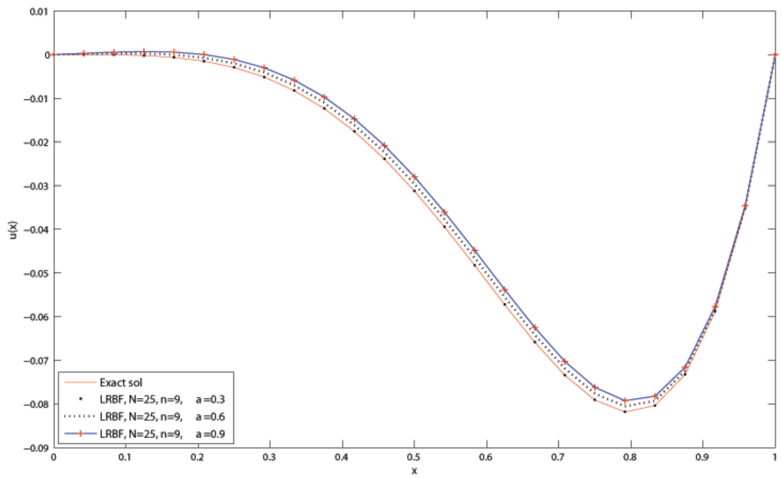

Fig. 4. Approximate solution obtained for Ex.2.

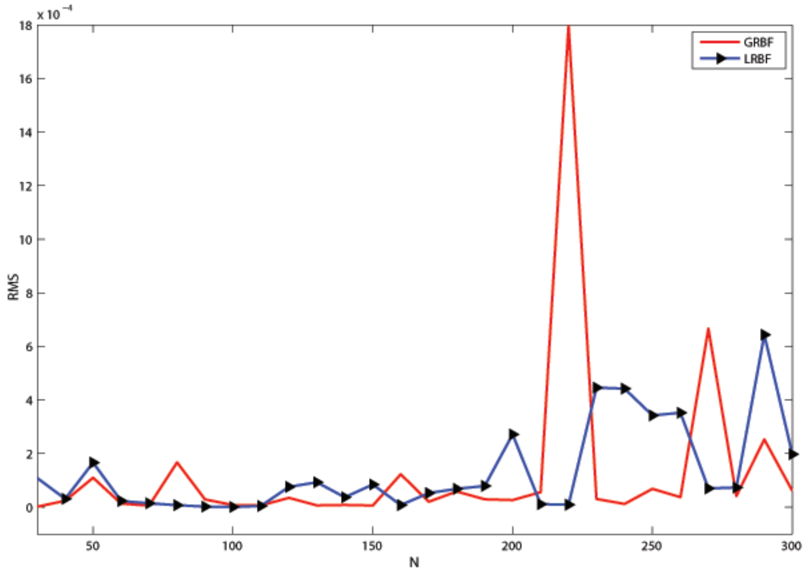

Fig. 5. RMS error versus N for Ex.2.

local method is more than the global method, and the local method is faster. Better result is obtained with local RBF collocation method than the method of [25].

Tabels 1 to 3 show that one of the advantages of using the optimal shape parameter in the local and global RBFs methods is that the approximate values obtained do not oscillate for the number of different nodal points.

Table 3: Numerical results with different number of nodal points and optimal shape parameter c for Ex.3.

N	n	LRBF δ_{er}	LRBF RMS	GRBF δ_{er}	GRBF RMS
20	6	3.7348×10^{-7}	4.4256×10^{-7}	4.9140×10^{-7}	1.9437×10^{-7}
30	6	3.0022×10^{-7}	1.7868×10^{-7}	7.5015×10^{-7}	2.2452×10^{-7}
50	6	1.5166×10^{-7}	6.9974×10^{-8}	1.9361×10^{-7}	8.5469×10^{-8}
80	11	1.3692×10^{-7}	2.5266×10^{-8}	3.1795×10^{-6}	1.0485×10^{-6}
100	20	1.1811×10^{-7}	1.9041×10^{-8}	8.5846×10^{-6}	3.1034×10^{-6}
150	21	3.0144×10^{-7}	2.5230×10^{-8}	9.0405×10^{-6}	2.2476×10^{-6}
200	21	4.5467×10^{-7}	4.7993×10^{-7}	3.0446×10^{-6}	1.9750×10^{-6}
250	21	4.6912×10^{-6}	1.2645×10^{-7}	1.0388×10^{-6}	1.6379×10^{-6}

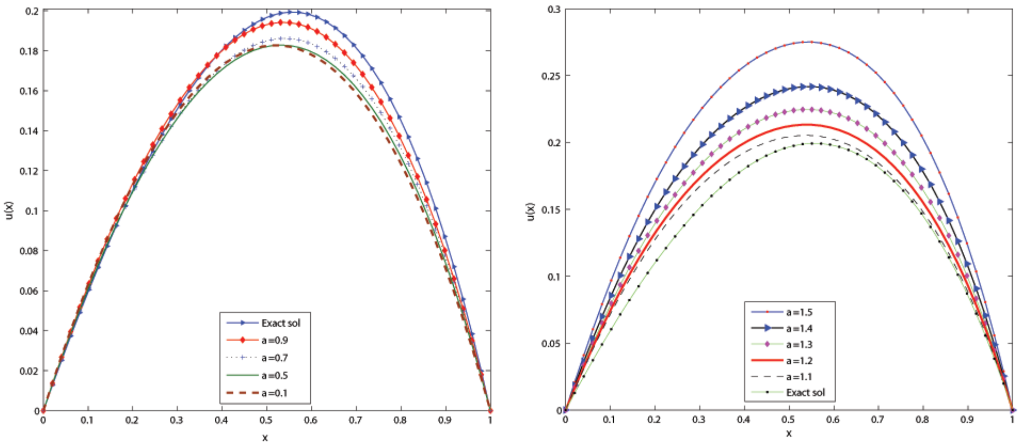

Fig. 6. Numerical solutions of Ex.3 for various α.

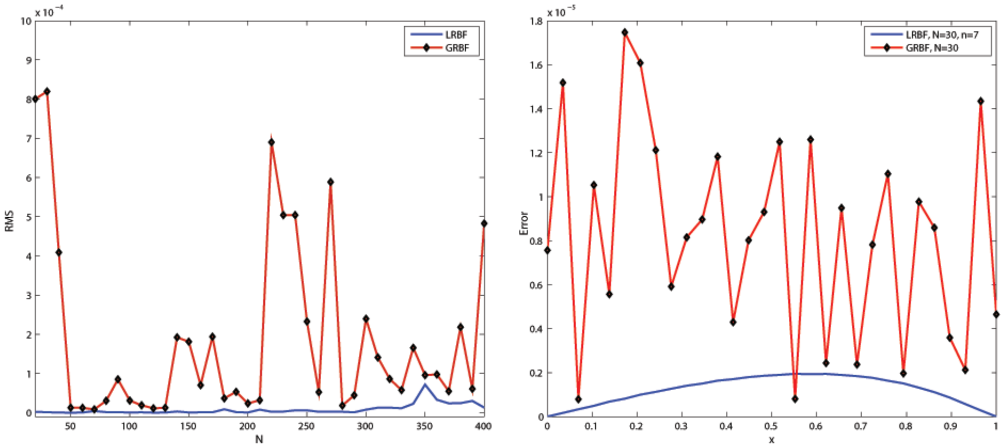

Fig. 7. RMS errors and abselot error local and global RBF methods for Ex.3.

5.4 Example 4.

Consider the following two-dimensional nonlinear fractional Volterra integral equation:

$$u(x,y) - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}\left[\int_0^x \int_0^y (x-s)^{\frac{1}{2}}(y-t)^{\frac{3}{2}}\sqrt{xyt}[u(s,t)]^2 dtds\right]$$

$$= \sqrt{y}\left(\frac{-1}{180}x^3 y^{\frac{7}{2}} + \sqrt{\frac{x}{3}}\right), \tag{21}$$

Table 4: Numerical results obtained with different number of nodal points for Ex.4.

	3 × 3 stencil			5 × 5 stencil		
N	δ_{ex}	RMS	CPU(s)	δ_{ex}	RMS	CPU(s)
6^2	1.042×10^{-2}	5.8280×10^{-3}	127.5	8.5×10^{-2}	3.5273×10^{-3}	343.2
7^2	2.872×10^{-2}	1.2000×10^{-3}	188.2	4.6×10^{-2}	2.9653×10^{-3}	495.5
8^2	1.107×10^{-2}	3.6457×10^{-4}	311.1	3.0×10^{-3}	3.6065×10^{-4}	663.9
9^2	1.480×10^{-3}	2.6248×10^{-4}	534.0	2.9×10^{-3}	9.5724×10^{-4}	1603.0
10^2	2.072×10^{-4}	8.6700×10^{-5}	827.9	5.1×10^{-3}	3.3647×10^{-4}	2042.3
11^2	3.421×10^{-4}	9.2053×10^{-5}	961.5	4.2×10^{-2}	3.2812×10^{-3}	2973.6

In which the exact solution is $u(x,y) = \frac{\sqrt{3xy}}{3}$. The results with optimal shape parameter are presented in Table 4. Fig. 8 show the cross section of the approximate solutions with $N = 10^2$, 3×3 stencils and fixed values of y. The results obtained in Test problems (4) with local RBF method are more accurate than the results obtained in [27].

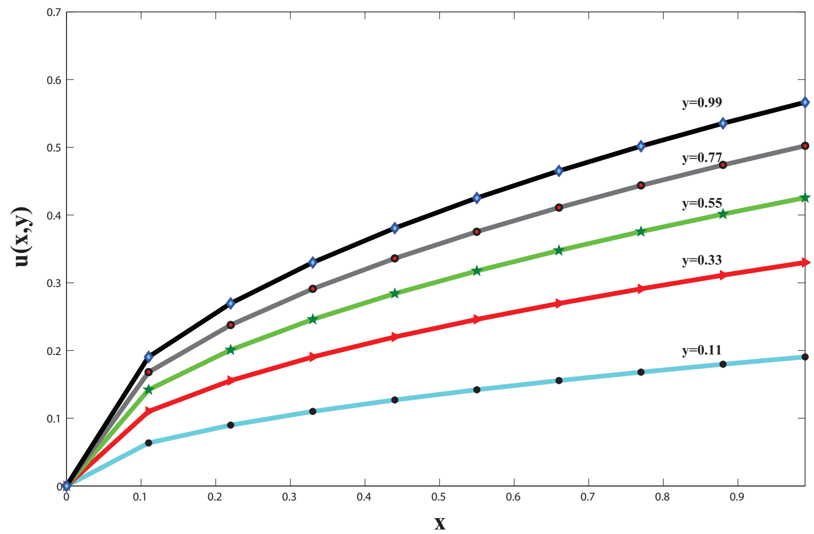

Fig. 8. Cross section of approximate solution obtained with $N = 10^2$ and 3×3 stencils for Ex. 4

5.5 Example 5.

Consider the two-dimensional fractional Volterra integral equation:

$$u(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\sqrt{xy}stu(s,t)dsdt = f(x,y),$$
$$f(x,y) = x^3(y^2-y) - \frac{1}{60}x^{\frac{11}{2}}y^{\frac{7}{2}}(3y-4).$$

We applied the presented method for various values of r_1 and r_2. For $r_1 = r_2 = 1$, the exact solution is given as $u(x,y) = x^3(y^2-y)$. Note that as r_1 and r_2 approach to 1, the numerical solution converges to the analytical solution $u(x,y) = x^3(y^2-y)$. Table 5 compares the absolute errors with 100 number of nodal points, 3×3 stencil between the local RBFs method and method [28].

Table 5: Numerical results obtained with 100 number of nodal points, with 3×3 and optimal shape parameter c for Ex.5.

	$r_1 = r_2 = 0.8$		$r_1 = 0.8,\ r_2 = 0.95$	
x=y	δ_{er}	Error [28]	δ_{er}	Error [28]
0.0	0.0	3.544×10^{-4}	0.0	3.068×10^{-3}
0.1	8.4684×10^{-7}	1.388×10^{-3}	1.2140×10^{-6}	1.240×10^{-3}
0.2	1.1030×10^{-7}	8.772×10^{-4}	4.5487×10^{-5}	1.166×10^{-3}
0.3	1.6701×10^{-6}	1.407×10^{-3}	4.0259×10^{-4}	1.863×10^{-3}
0.4	2.0687×10^{-5}	1.153×10^{-3}	9.4018×10^{-4}	5.133×10^{-3}
0.5	1.2684×10^{-5}	5.673×10^{-3}	1.2354×10^{-3}	4.848×10^{-3}
0.6	4.8970×10^{-4}	9.748×10^{-3}	2.1018×10^{-3}	8.742×10^{-3}
0.7	1.2566×10^{-4}	1.089×10^{-3}	2.9120×10^{-3}	9.716×10^{-3}
0.8	1.3564×10^{-3}	7.730×10^{-3}	3.3028×10^{-3}	6.411×10^{-3}
0.9	3.4802×10^{-3}	1.222×10^{-3}	4.2901×10^{-3}	2.162×10^{-4}

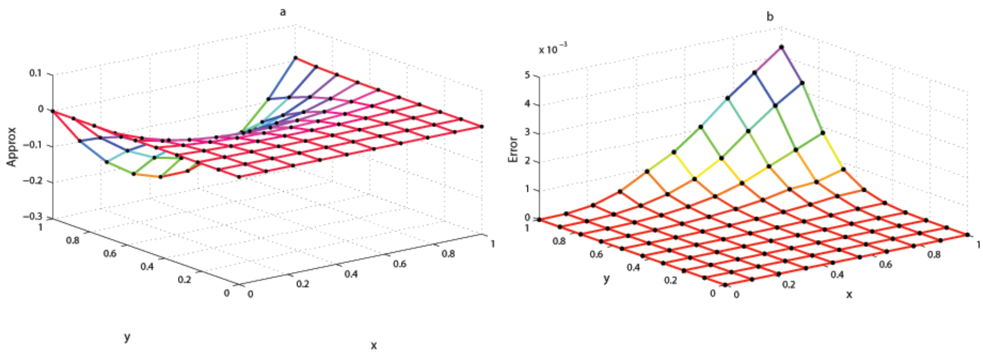

Fig. 9. Graphs of approximate solution (a) and absolute error (b) with $r_1 = 0.8$ and $r_2 = 0.95$ for Ex.5.

Graphs of approximate solution and absolute error with $r_1 = 0.8$, $r_2 = 0.95$, $N = 100$ and 3×3 stencil are given in Fig. 9. The results obtained in this paper, with local RBF method are more accurate than the results obtained in [28].

6. Conclusion

A meshless local RBF method was proposed to solve fractional integro-differential equation and two-dimensional fractional Volterra integral equation. In one-dimensional problems, by increasing the number of collocation points to find an optimal and more stable response, the number of nodes in local domain of influence increases. Numerical results showed that in the two dimensional fractional integral equations the local RBF method is much more efficient than the global RBF method, so, it can be concluded that local RBF method is more suitable for high dimensional problems.

The effectiveness of the method has the following reasons:

1) The use of the strong form equation and collocation approach made the method simpler than similar methods.

2) Using the locallization approach, the matrix operations required only the inversion of matrices of small size and the final global matrix became sparse.

So the method is suitable for large-scale and complicated problems.

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ON BI-Γ-IDEALS IN Γ-SEMIGROUPS WITH INVOLUTION

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Abstract. The concept of involution in semigroups was given by Nordahl et al [13]. In this paper, we introduce involution in Γ-semigroups. Also, we define bi-Γ-ideals in Γ-semigroups with involution and prove many interesting results characterizing Γ-semigroups with involution by using bi-Γ-ideals.
Keywords: Γ-semigroups, involution, quasi-Γ-ideal, bi-Γ-ideal.

1. Introduction and preliminaries

An involution semigroup S will mean a bijection $x \to x^\star$ of S onto itself, satisfying $(a^\star)^\star = a$, $(ab)^\star = b^\star a^\star$. If we consider involutions on various algebraic structures, it is generally needed that the defined involution is also an antiautomorphism of the underlying algebraic structures of period two. In that sense, involutions depict a fixed kind of internal symmetry of such systems. The natural example of an algebraic involution is the transposition of matrices in the algebra of matrices over a ring. Furthermore, an involution can be taken as a fundamental operation, and consequently, a part of the algebra on which it acts. For example, an involution semigroup is a triple $(S, \cdot, \star)$ such that $(S, \cdot)$ is a semigroup, while $\star$ is an involution on S such that $(xy)^\star = y^\star x^\star$ holds for all $x, y \in S$. In a similar fashion, if $(S, +, \cdot)$ is a semiring, then $(S, +, \cdot, \star)$ is

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called an involution semiring, provided that $\star$ is an involution of S satisfying the identities $(x+y)^\star = x^\star + y^\star, (xy)^\star = y^\star x^\star$. If it requires that semirings may be equipped with a zero 0, and/or an identity 1, then the involution satisfies $0^\star = 0$ and $1^\star = 1$. The motivation for investigation of involution semigroups arises from a wide range of involution rings, involution algebras, by leaving the additive structure. Γ-semigroups with involution is in a quite fascinating way strongly related to various classes of ideals like Γ-ideals, quasi-Γ-ideals and bi-Γ-ideals. Abbasi et al. [8] defined the involution in po-Γ-semigroups and studied many results on prime, semiprime and weakly prime ideals in involution po-Γ-semigroups. Furthermore, they characterized intra-regular involution po-Γ-semigroups. Various varieties of semigroups and algebras have unary operations imposed on them, including the classes of groups, inverse semigroups [6], cellular algebras [5], algebras [2], [6], [9], [15], primitive involution rings [7, 10] and regular $\star$-semigroups [13]. Scheiblich [4] constructed examples of bands for which two involutions lead to non-isomorphic regular $\star$-semigroups. For other results and examples, we refer [2],[3], [11], [12] and [14].
For Γ-semigroups, we refer [1]. In order to prove our main results, we introduce the following definitions and examples:

Definition 1.1. *Let A and Γ be any two nonempty sets. If there exists a mapping $A \times \Gamma \times A \to A$ such that $a\gamma b \in A \forall a, b \in A$ and $\gamma \in \Gamma$ and $A^\star \subseteq A$. Then A is called Γ-semigroup with involution.*

Example 1.1. Let $A = \left\{x : x = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in R\right\}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{gathered} * : A^* \to A \\ \text{s.t. } x^* \to x^T \end{gathered}$$

Then A is a Γ-semigroup with involution as $A \times \Gamma \times A \longrightarrow A$ and $A^T \subset A$.

Example 1.2. Let us consider $A = (0, 2]$ and $* : A \to A$ such that
$a^* \to 1/a \forall a \in A$.
Then it is not a semigroup. If we define $\Gamma = \{1/4n : n \in N\}$. Then A is a Γ-semigroup with involution $\star$.

Definition 1.2. *Let S be a Γ-semigroup with involution $\star$. A sub-Γ-semigroup B of a Γ-semigroup S with involution $\star$ is called a bi-Γ-ideal of S with involution if $B\Gamma S\Gamma B \subseteq B$ and $B^\star \subseteq B$.*

Example 1.3. Let $S = \left\{x : x = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in R\right\}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Then S is a Γ-semigroup with involution, where

$$\begin{gathered} * : A^* \to A \\ \text{s.t. } x^* \to x^T \end{gathered}$$

Let $A = \left\{x : x = \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} : \alpha \in R\right\}$. Then A is a bi-Γ-ideal of S.

Definition 1.3. *A non-empty subset Q of a Γ-semigroup S with involution $\star$ is called a quasi-Γ-ideal of S if*
$Q\Gamma S \cap S\Gamma Q \subseteq Q$ *and* $Q^\star \subseteq Q$.

2. Γ-semigroups with involution

In this section, we prove a number of important results concerning characterizations of Γ-semigroups with involution using bi-Γ-ideals.

Theorem 2.1. *Every quasi-Γ-ideal of a Γ-semigroup S with involution $\star$ is a bi-Γ-ideal of a Γ-semigroup S with involution $\star$.*

Proof. Let Q be a quasi-Γ-ideal of a Γ-semigroup S with involution $\star$. Then $Q\Gamma S \cap S\Gamma Q \subseteq Q$ and $Q^\star \subseteq Q$. Now, $Q\Gamma S\Gamma Q \subseteq Q\Gamma S\Gamma S \subseteq Q\Gamma S$.
Also, $Q\Gamma S\Gamma Q \subseteq S\Gamma S\Gamma Q \subseteq S\Gamma Q$. This implies that $Q\Gamma S\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q$. This further implies that $Q\Gamma S\Gamma Q \subseteq Q$ and $Q^\star \subseteq Q$. Hence, Q is a bi-Γ-ideal of a Γ-semigroup S with involution. □

Theorem 2.2. *Let S be a Γ-semigroup with order preserving involution $\star$. Then:*
(1) $(x\Gamma s\Gamma y)^\star = y^\star\Gamma S\Gamma x^\star$, *for any $x, y \in S$.*
(2) $(S\Gamma x\Gamma S)^\star = S\Gamma x^\star\Gamma S$, *for any $x \in S$.*

Proof. Let $t \in (x\Gamma s\Gamma y)^\star$. By definition, $t^\star \in x\Gamma s\Gamma y$, $t^\star \in x\beta s\gamma y$ for some $s \in S$ and $\beta,\gamma \in \Gamma$. This implies that
$t \in (x\beta s\gamma y)^\star \subseteq y^\star\beta s^\star\gamma x^\star \subseteq y^\star\Gamma S\Gamma x^\star$, because $\star$ is an order preserving involution. Thus $(x\Gamma s\Gamma y)^\star \subseteq y^\star\Gamma S\Gamma x^\star$.
On the other hand if $t \in y^\star\Gamma S\Gamma x^\star$, then for some $s \in S$, and $\beta, \gamma \in \Gamma$, we have $t \in y^\star\beta s^\star\gamma x^\star$. This implies that $t^\star \in x\beta s^\star \subseteq x\Gamma s\Gamma y$. As $t \in (x\Gamma s\Gamma y)^\star$. Therefore, $y^\star\Gamma S\Gamma x^\star \subseteq (x\Gamma s\Gamma y)^\star$.
Consequently, $(x\Gamma s\Gamma y)^\star = y^\star\Gamma S\Gamma x^\star$.
(2) The proof is similar to (1). □

Theorem 2.3. (1) *Suppose that A is a sub-Γ-semigroup of a Γ-semigroup S with involution $\star$, $s \in S$ and $(s\Gamma A\Gamma s) \cap A \neq \emptyset$, then $(s\Gamma A\Gamma s) \cap A$ is bi-Γ-ideal of A.*
(2) *Let S be a Γ-semigroup with involution and T a non-empty subset of S. Then, $T \cup T\Gamma S\Gamma T$ is the bi-Γ-ideal of S with involution $\star$.*

Proof.

$$\begin{aligned}
(1)(s\Gamma A\Gamma s \cap A)\Gamma A\Gamma(s\Gamma A\Gamma s \cap A) &\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A\Gamma A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [[(s\Gamma A\Gamma s\Gamma A)\Gamma(s\Gamma A\Gamma s)] \cap [A\Gamma(s\Gamma A\Gamma s)\Gamma A \cap A)]]\\
&\subseteq [(s\Gamma A\Gamma s) \cap (A\Gamma s\Gamma A\Gamma s)] \cap A\\
&\subseteq (s\Gamma A\Gamma s \cap A).
\end{aligned}$$

Hence $(s\Gamma A\Gamma s)\cap A$ is a bi-Γ-ideal of A.
Now

$$\begin{aligned}[(s\Gamma A\Gamma s\cap A)]^{\star} &\subseteq (s\Gamma A\Gamma s)^{\star}\cap A^{\star}\\ &\subseteq [(s\Gamma A)\Gamma s]^{\star}\cap A^{\star}\\ &\subseteq s^{\star}\Gamma(s\Gamma A)^{\star}\cap A^{\star}\\ &\subseteq s^{\star}\Gamma(A^{\star}\Gamma s^{\star})\cap A^{\star}\\ &\subseteq (s\Gamma A\Gamma s)\cap A.\end{aligned}$$

Hence $(s\Gamma A\Gamma s)\cap A$ is bi-Γ-ideal of A with involution.

(2) Let $B=T\cup T\Gamma S\Gamma T$. Then $T\subseteq B$. So

$$\begin{aligned}&B\Gamma S\Gamma B=(T\cup T\Gamma S\Gamma T)\Gamma S\Gamma(T\cup T\Gamma S\Gamma T)\\ &\subseteq[T(\Gamma S\Gamma)(T\cup T\Gamma S\Gamma T)]\cup[T\Gamma S\Gamma T(\Gamma S\Gamma)(T\cup T\Gamma S\Gamma T)]\\ &\subseteq[T(\Gamma S\Gamma)T\cup T(\Gamma S\Gamma)T\Gamma S\Gamma T]\cup[T\Gamma S\Gamma T(\Gamma S\Gamma)T\cup T\Gamma S\Gamma T(\Gamma S\Gamma)T\Gamma S\Gamma T]\\ &\subseteq[T\Gamma S\Gamma T\cup T\Gamma S\Gamma T]\cup[T\Gamma S\Gamma T\cup T\Gamma S\Gamma T]\\ &=T\Gamma S\Gamma T\cup T\cup T\Gamma S\Gamma T=B.\end{aligned}$$

Hence $B=T\cup T\Gamma S\Gamma T$ is bi-Γ-ideal. Consider:

$$\begin{aligned}(T\cup T\Gamma S\Gamma T)^{\star} &\subseteq T^{\star}\cup(T\Gamma S\Gamma T)^{\star}\\ &\subseteq T\cup[(T\Gamma S)\Gamma T]^{\star}\\ &\subseteq T\cup[T^{\star}\Gamma(T\Gamma S)^{\star}]\\ &\subseteq T\cup[T^{\star}\Gamma s^{\star}\Gamma T^{\star}]\\ &\subseteq T\cup T\Gamma S\Gamma T.\end{aligned}$$

Hence $T\cup T\Gamma S\Gamma T$ is a bi-Γ-ideal with involution. □

Theorem 2.4. *Let S be a Γ-semigroup with order preserving involution $\star$. Then the following statements hold:*
(1) If $\{A_i^{\star}: i\in I\}$ is a family of left(resp., right) Γ-ideals of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a left (resp.,right)Γ-ideals of S.
(2) If $\{A_i^{\star}: i\in I\}$ is a family of bi-Γ-ideals of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a bi-Γ-ideal of S.
(3) If $\{A_i^{\star}: i\in I\}$ is a family of quasi-Γ-ideal of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a quasi-Γ-ideal of S.

Proof. (1) Let $\{A_i^{\star}: i\in I\}$ be a family of left-Γ-ideals of S. Then $S\Gamma A_i^{\star}\subseteq A_i^{\star}$.
Consider:

$$\begin{aligned}S\Gamma\cap A_i^{\star} &\subseteq S\Gamma A_i^{\star}\\ &\subseteq A_i^{\star}\text{ for all i}\in\text{I}\\ &\subseteq \cap A_i^{\star}\end{aligned}$$

(2) Let $\{A_i^{\star}: i\in I\}$ be a family of bi-Γ-ideals of S. Then $A_i^{\star}\Gamma S\Gamma A_i^{\star}\subseteq A_i^{\star}$. Now consider:

$$\begin{aligned}\cap A_i^{\star}\Gamma S\Gamma\cap A_i^{\star} &\subseteq A_i^{\star}\Gamma S\Gamma A_i^{\star}\\ &\subseteq A_i^{\star}\text{ for all i}\in\text{I}\\ &\subseteq \cap A_i^{\star}.\end{aligned}$$

(3) Let $\{A_i^\star : i \in I\}$ be a family of quasi-Γ-ideals of S. Then $A_i^\star \Gamma S \cap S\Gamma A_i^\star \subseteq A_i^\star$. Consider $\cap A_i^\star \Gamma S \cap S\Gamma \cap A_i^\star \subseteq A_i^\star \Gamma S \cap S\Gamma A_i^\star \subseteq A_i^\star$ for all $\mathrm{i} \in \mathrm{I} \subseteq \cap \mathrm{A}_\mathrm{i}^\star$. □

Theorem 2.5. *Let S be a Γ-semigroup with order preserving involution $\star$. Then:*

(1) *$A^\star$ is a left (resp.,right) Γ-ideal for any right (resp.,left) Γ-ideal A of S.*
(2) *$B^\star$ is a bi-Γ-ideal for any bi-Γ-ideal B of S.*
(3) *$Q^\star$ is a quasi-Γ-ideal for any quasi-Γ-ideal Q of S.*

Proof. (1) Let A be right-Γ-ideal of S. Then $A\Gamma S \subseteq A$ and $S^\star = S$. Consider:

$$S\Gamma A^\star = S^\star \Gamma A^\star = (A\Gamma S)^\star \subseteq A^\star.$$

Thus $A^\star$ is a left-Γ-ideal of S.

(2) Let B be a bi-Γ-ideal of S. This implies $B\Gamma S\Gamma B \subseteq B$ and $S^\star = S$.

Now consider $B^\star \Gamma S \Gamma B^\star = B^\star \Gamma S^\star \Gamma B^\star = (B\Gamma S\Gamma B)^\star \subseteq B^\star$. Hence $B^\star$ is a bi-Γ-ideal of S.

(3) Let Q be quasi-Γ-ideal of S. Then $Q\Gamma S \cap S\Gamma Q \subseteq Q$ and $S^\star = S$. Now consider

$$\begin{aligned} Q^\star \Gamma S \cap S\Gamma Q^\star &\subseteq Q^\star \Gamma S^\star \cap S^\star \Gamma Q^\star \\ &\subseteq S^\star \Gamma Q^\star \cap Q^\star \Gamma S^\star \\ &\subseteq (Q\Gamma S)^\star \cap (S\Gamma Q)^\star \\ &\subseteq (Q\Gamma S \cap S\Gamma Q)^\star \\ &\subseteq Q^\star. \end{aligned}$$

Hence $Q^\star$ is a quasi-Γ-ideal of S. □

Theorem 2.6. *Let S be a Γ-semigroup with involution $\star$. If $A = A^\star \Gamma A^\star$, then $A^\star \cap B^\star = A\Gamma B$ for any Γ-ideals A and B of S.*

Proof. Let A and B be two Γ-ideals of S. By Theorem 2.5, $A^\star$ and $B^\star$ are Γ-ideals of S. Now we have $A\Gamma B \subseteq A\Gamma S \subseteq A = A^\star \Gamma A^\star \subseteq A^\star$. Similarly, $A\Gamma B \subseteq S\Gamma B \subseteq B = B^\star \Gamma B^\star \subseteq B^\star$. Thus $A\Gamma B \subseteq A^\star \cap B^\star$.
On the other hand, $A^\star \cap B^\star$ is a Γ-ideal of S. This implies that $A^\star \cap B^\star = (A^\star \cap B^\star)^\star \Gamma (A^\star \cap B^\star)^\star = (A \cap B)\Gamma(A \cap B) \subseteq A\Gamma B$. So $A^\star \cap B^\star = A\Gamma B$. □

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DYNAMICS OF TWO-GENE ANDRECUT-KAUFFMAN SYSTEM: CHAOS AND COMPLEXITY

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Abstract. Evolutionary dynamics of a two-gene model for chemical reactions, corresponding to gene expression and regulation, has been studied in detail. Bifurcation analysis has been carried out to understand behavior of steady state solutions leading to chaotic evolution for different cases. Numerical simulations have been performed and measurable quantities like Lyapunov exponents, topological entropies and correlation dimensions have been calculated for certain sets of parameter values. These measures explain complexity and chaotic nature of evolution.
Keywords: chaos, Lyapunov exponents, bifurcation, topological entropy.

1. Introduction

Mathematical equations dealing with natural and biological systems are nonlinear in nature and are mostly in complicated form. Nonlinearity can be defined by parameters involved in these systems. Behavior of such systems can be understood during evolution by varying parameters under different initial conditions. Computers have added much to the numerical study of this subject by producing many exciting and interesting results. A simple system evolves in simple ways but a complex or complicated system evolves in complicated ways and between simplicity and complexity there cannot be a common ground [1]. Complex systems have features like cascading failures, far from energetic equilibrium, often exhibit hysteresis, bistability, may be nested, network of multiplicity, emergent phenomena and some more properties. All these are related to the nonlinearity. A systematic evolutionary description and emergence of chaos can be obtained in the beginning chapters of the book edited by Hao-Bin-Lin [2]. Chaos and

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irregular phenomena may not require very complicated equations. During evolution, biological systems may display the properties like complexity and chaos. Complexity can be viewed via its systematic nonlinear properties and it is due to the interaction among multiple agents within the system [3, 4]. Chaotic systems display varied forms of attractors, depending on different sets of parameter values. Complexity and chaos observed in a system can be well understood by measuring elements like Lyapunov exponents (LCEs), topological entropies, correlation dimension etc. Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy signifies more complexity. Actually, it measures the exponential growth rate of the number of distinguishable orbits as time advances [5, 6, 7]. Since complexity and chaos appear mostly in nonlinear systems, it is necessary to find certain measure of the quantities causing these. Positive measure of LCEs signifies presence of chaos [8, 9, 10, 11]. Measure of topological entropy signifies the complexity [5, 11, 12, 13], and the correlation dimension provides the dimensionality of the attractor of the system [14, 16].

While dealing with natural systems, principles of nonlinear dynamics have been extensively used in diverse areas of sciences. In biochemical context nonlinear equations are obtained from chemical reactions appearing in a two-gene model [6, 15]. Here, chemical reactions are assumed to correspond to gene expression and regulation.

The studies performed in the present article deal with a two-gene Andrecut-Kauffman model [6]. In this two-dimensional discrete system, dynamical variables describe the evolution of the concentration levels of transcription factor proteins. To study the characteristics of complex nature of evolutionary phenomena, bifurcation diagrams have been drawn by varying a certain parameter. Then, some numerical investigations are carried forward to obtain LCEs, topological entropies and correlation dimensions for different sets of parameters of the system. Results obtained are shown through graphics. Finally, the complex nature of evolutions has been discussed on the basis of results obtained through this study.

2. Two-gene Andrecut-Kauffman system

In the present study, we consider a two-dimensional map proposed by Andrecut and Kauffmann [6, 7]. The map was used to investigate the dynamics of two-gene models for chemical reactions corresponding to gene expression and regulation. The discrete dynamical variables, denoted by x_n and y_n, describe the evolutions of the concentration levels of transcription factor proteins. The map is given by the following pair of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{a}{1+(1-b)x_n^t + by_n^t} + cx_n, \\ y_{n+1} &= \frac{a}{1+(1-b)y_n^t + bx_n^t} + dy_n, \end{aligned} \tag{1}$$

with parameters $a = 25$, $b = 0.1$, $c = d = 0.18$, and $t = 3$, one obtains four different fixed points with coordinates $(2.30409, 2.30409)$, $(-2.52688, 2.44162)$, $(2.44162, -2.52866)$, $(-2.39464, -2.39464)$, and all are unstable.

For $c \neq d$, and when $a = 25$, $b = 0.1$, $c = 0.18$, $d = 0.42$, and $t = 3$, again, four unstable fixed points exist as $(2.2832, 2.5413)$, $(-2.5458, 2.6566)$, $(2.4613, -2.7288)$, and $(-2.3744, -2.61705)$. Threrefore, for all of these cases, orbit with initial point taken nearby any of the fixed points may be unstable and may be chaotic as well.

We intend to investigate certain dynamic behavior of system (1) for cases when $c = d$ and when $c \neq d$ for evolutions showing irregularities due to presence of chaos and complexity.

3. Numerical simulations

Performing various numerical simulations, the dynamics of evolution have been investigated by obtaining bifurcation diagrams, calculating LCEs, topological entropy and correlation dimensions of the system for different cases. For the values of control parameters within the system the following ranges have been proposed: $a \in [0, 50]$, $c \in [-0.4, 0.4]$, $b = 0.1$, $d = 0.5$, $t = 3, 4, 5$.

Taking $c = d$, bifurcation diagrams are drawn along the directions x and y, by varying c for cases $t = 3, 4, 5$ and certain fixed values of other parameters as show in Fig. 1. Then, plots of attractors have been obtained for parameters $a = 25$, $b = 0.1$, $t = 3$ and (i) for regular case $c = d = 0.32$ and (ii) for chaotic case $c = d = 0.18$ and shown in Fig. 2. In each case when $t = 3, 4, 5$, bifurcations show period doubling leading to chaos and then to regularity. Also, bistability and folding nature of phenomena are appearing here.

3.1 Lyapunov Exponents and Topological Entropies

For chaotic evolution, when $a = 25$, $b = 0.1$, $t = 3$, $c = d = 0.18$, LCEs are obtained and their plots are shown in Fig. 3. Numerical investigations further proceeded for calculation of topological entropies. In Fig. 4, plots of topological entropies are presented for $t = 3, 4, 5$ and for different ranges of parameter c. Analysis of these plots, gives an impression that for the case $t = 3$, system shows enough complexity in the range $0.05 \leq c \leq 0.23$. For the case $t = 4$, the system shows high complexity in the range $0 \leq c \leq 0.22$ and in the case $t = 5$, high complexity appears in $0 \leq c \leq 0.44$.

In Fig. 6, plots of LCEs for chaotic evolution for different cases discussed above are shown in the upper row and plots of topological entropies are shown in the lower row for these cases. For all the plots, parameters $a = 25$ and $b = 0.1$ are common. Here, topological entropy plots are drawn for different ranges of parameter c.

3.2 Correlation Dimensions

Correlation dimension gives its measure of dimensionality. Chaotic evolutions in dynamical systems are characterized by a chaotic set, "strange attractor", which has fractal structure. Being one of the characteristic invariants of non-linear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. A statistical method can be used to determine correlation dimension. It is an efficient and practical method in comparison to other methods, like box counting etc. The procedure to obtain correlation dimension follows from some steps calculation [14, 17, 16].

Extending further the numerical study, the correlation dimensions of system (1) have been calculated for various chaotic cases discussed above. For this the method used is that of Martelli with Mathematica codes [16]. In briefly, the method can be described as follows:

Consider an orbit $O(\mathbf{x}_i) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots\}$ of a map $f : U \to U$, where U is an open bounded set in $\mathbb{R}^n$. To compute correlation dimension of $O(\mathbf{x}_i)$, for a given positive real number r, we form the correlation integral,

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H(r - \|\mathbf{x}_i - \mathbf{x}_j\|) \tag{2}$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is the unit-step function. The summation indicates the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. $C(r)$ measures the density of pair of distinct vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ that are close to r. The correlation dimension D_c of $O(\mathbf{x}_1)$ is then defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log(r)} \tag{3}$$

To obtain D_c, $\log C(r)$ is plotted against $\log(r)$, Fig. 8, and then we find a straight line fitted to this curve. The intercept of this straight line on y-axis provides the value of the correlation dimension D_c.

Computation of correlation dimension has been carried out for all the cases described in this article for different set of values of parameters as shown in Table 1.

4. Discussion

Two-gene Andrecut-Kauffmann system represented by map (1) has been studied carefully to understand chaotic phenomena during its evolution together with complexities present in the system. Investigation is made for cases $t = 3, 4, 5$ only but one can extend it for cases $t > 6$ also. Bifurcation plots in Fig. 1 and

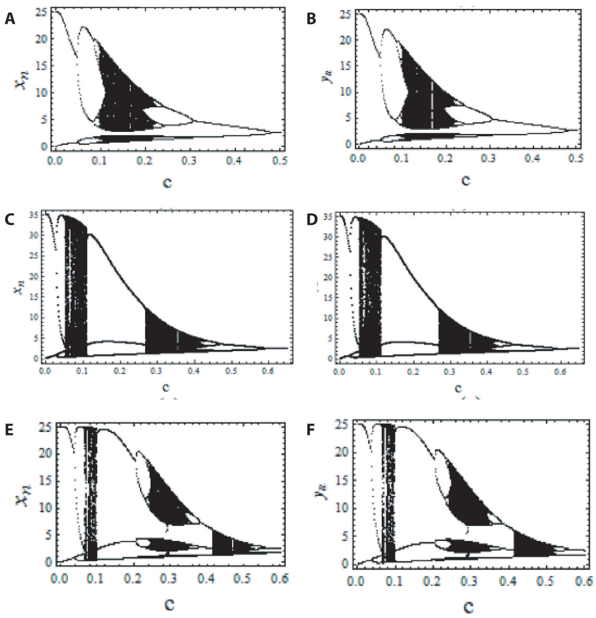
t	a	b	c	d	dimension
3	25	0.1	0.18	0.18	3.81869
4	25	0.1	0.18	0.18	3.05258
5	25	0.1	0.18	0.18	3.11754
3	25	0.1	0.28	0.12	3.16201
4	25	0.1	0.28	0.12	3.96724
5	25	0.1	0.28	0.12	4.05859
3	35	0.1	0.20	0.20	3.80410
4	35	0.1	0.20	0.20	3.41640
5	35	0.1	0.20	0.20	4.73368

Table 1: Table showing correlation dimension for different set of parameter values.

in Fig. 5, show the phenomena of period doubling and bistability in all these cases. Chaotic evolutions with periodic windows are clearly visible. Presence of complexity in the system can be observed by plots of topological entropies in Fig. 3, Fig. 4 and Fig. 6. Variations of topological entropies can be observed in 3D plots shown in Fig. 7. Numerical values of correlation dimensions, shown in Table 1, provide approximate dimensionality of chaotic attractors.

List of Figures

Figure 1: Three cases of bifurcations along x-axis (A, C, E) and y-axis (B, D, F) for map (1)



when $c = d$ are shown: (A) $t = 3, a = 25, b = 0.1$ and $0 \leq c \leq 0.5$; (B) $t = 3, a = 25, b = 0.1$ and $0 \leq c \leq 0.5$; (C) $t = 4, a = 35, b = 0.1$ and $0 \leq c \leq 0.65$; (D) $t = 4, a = 35, b = 0.1$ and $0 \leq c \leq 0.65$; (E) $t = 5, a = 25, b = 0.1$ and $0 \leq c \leq 0.5$; (F) $t = 5, a = 25, b = 0.1$ and $0 \leq c \leq 0.5$.

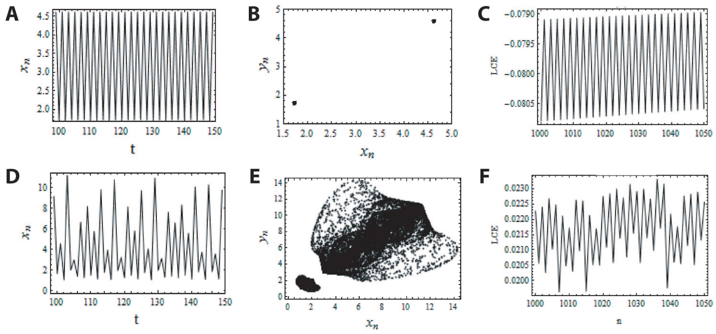


Figure 2: Time series, phase plane attractors and LCEs for regular (A, B, C) and chaotic (D, E, F) case of map (1). Parameter values are taken as: (A) time series for $a = 25, b = 0.1, t = 3, c = d = 0.32$; (B) phase plane attractors for $a = 25, b = 0.1, t = 3, c = d = 0.32$; (C) LCEs for $a = 25, b = 0.1, t = 3, c = d = 0.32$; (D) time series for $a = 25, b = 0.1, t = 3, c = d = 0.18$; (E) phase plane attractors for $a = 25, b = 0.1, t = 3, c = d = 0.18$; (F) LCEs for $a = 25, b = 0.1, t = 3, c = d = 0.18$.

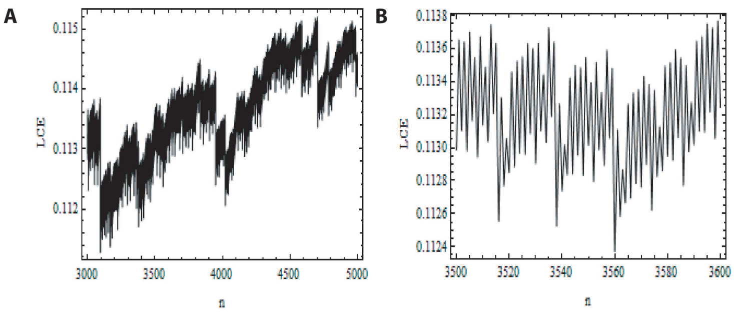


Figure 3: Plots of LCEs for chaotic evolution of map (1). Parameter values are $a = 25, b = 0.1, t = 3, c = d = 0.18$ while evolving from initial point $(2.1, 2.1)$.

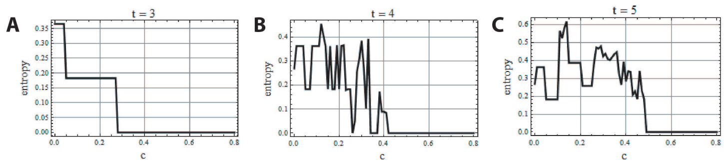


Figure 4: Plots of topological entropy for map (1) when $c = d$. The three different cases arise for the following values: (A) $t = 3, a = 25, b = 0.1$ and $0 \leq c \leq 0.5$; (B) $t = 4, a = 35, b = 0.1$ and $0 \leq c \leq 0.65$; (C) $t = 5, a = 25, b = 0.1$ and $0 \leq c \leq 0.8$.

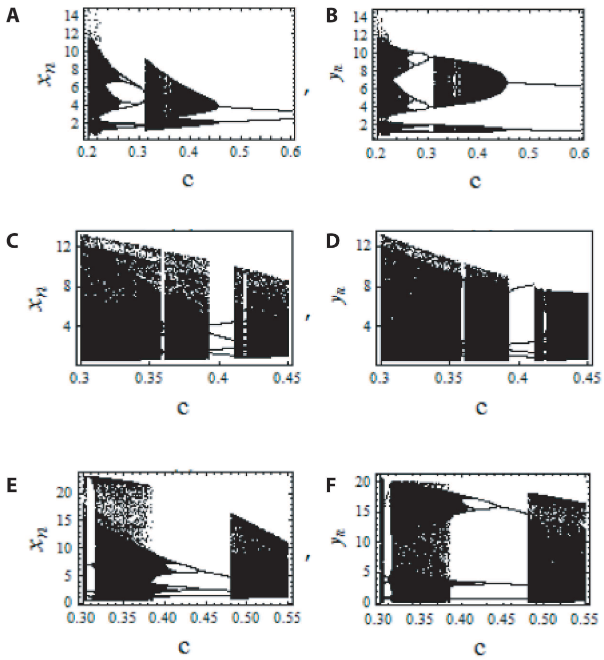


Figure 5: Bifurcation plots along x-axis (A, C, E) and y-axis (B, D, F) when $c \neq d$ for different ranges of parameter c: (A) $t = 3, a = 25, b = 0.1$ and $d = 0.2$; (B) $t = 3, a = 25, b = 0.1$ and $d = 0.2$; (C) $t = 4, a = 25, b = 0.1$ and $d = 0.3$; (D) $t = 4, a = 25, b = 0.1$ and $d = 0.3$; (E) $t = 5, a = 25, b = 0.1$ and $d = 0.2$; (F) $t = 5, a = 25, b = 0.1$ and $d = 0.2$.

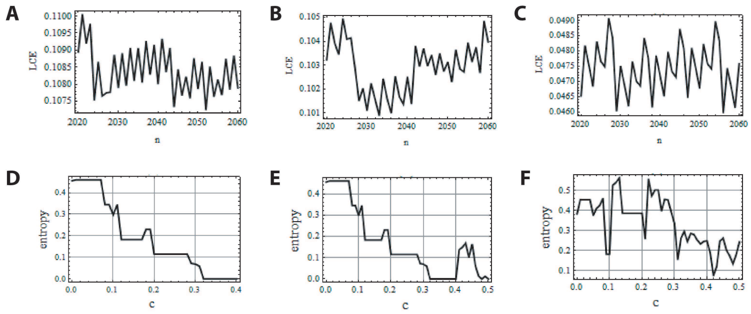


Figure 6: The plots for LCEs (A, B, C) and topological entropies (D, E, F) are shown here. Parameter values are taken as: (A) $t = 3, a = 25, b = 0.1, c = 0.2$ and $d = 0.15$; (B) $t = 4, a = 25, b = 0.1, c = 0.2$ and $d = 0.15$; (C) $t = 5, a = 25, b = 0.1, c = 0.28$ and $d = 0.12$; (D) $t = 3, a = 25, b = 0.1$ and $d = 0.15$; (E) $t = 4, a = 25, b = 0.1$ and $d = 0.15$; (F) $t = 5, a = 25, b = 0.1$ and $d = 0.15$.

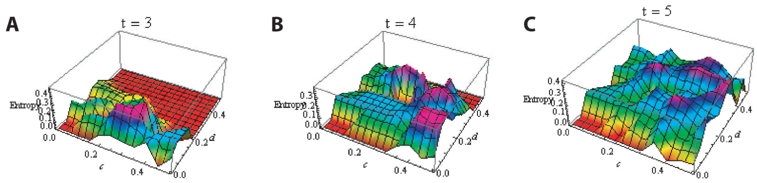
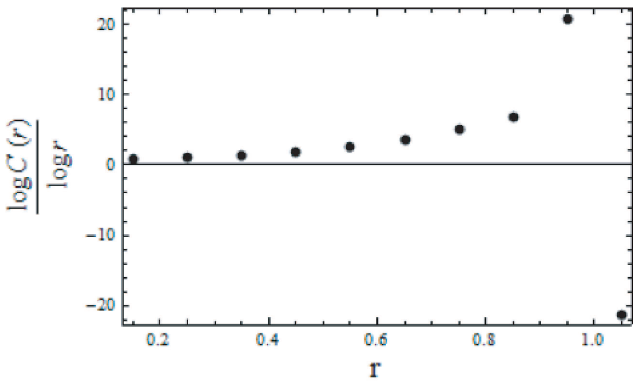


Figure 7: Three-dimensional plots for topological entropy variations. Parameter values are taken as: (A) $t = 3, a = 25, b = 0.1, 0 \leq c \leq 0.5$ and $0 \leq d \leq 0.5$; (B) $t = 4, a = 25, b = 0.1, 0 \leq c \leq 0.5$ and $0 \leq d \leq 0.5$; (C) $t = 5, a = 25, b = 0.1, 0 \leq c \leq 0.5$ and $0 \leq d \leq 0.5$. Figure 8: Plot of correlation integral curve for



the case $t = 3$. Parameter values are $a = 25, b = 0.1, c = 0.28, d = 0.12$.

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Z_3-CONNECTED GRAPHS WITH NEIGHBORHOOD UNIONS AND MINIMUM DEGREE CONDITION

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Abstract. Let G be a 2-edge-connected simple graph on $n \geq 15$ vertices, and let A denote an abelian group with the identity element 0. If a graph G^* is obtained by repeatedly contracting nontrivial A-connected subgraphs of G until no such a subgraph left, we say G can be A-reduced to G^*. In this paper, we prove that if for every $uv \notin E(G)$, $|N(u) \cup N(v)| + \delta(G) \geq n$, then G is not Z_3-connected if and only if G can be Z_3-reduced to one of $\{C_3, K_4, K_4^-, L\}$, where L is obtained from K_4 by adding a new vertex which is joined to two vertices of K_4. Our results extend the early theorem by Li et al. (Graphs and Combin., 29 (2013): 1891-1898).

Keywords: neighborhood unions, minimum degree, Z_3-connectivity, 3-flow.

1. Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notation not defined here are from [1].

For $S \subseteq V(G)$, let $N_S(v)$ denote the set of vertices in S that are adjacent to v in G and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For a vertex v, $N(v)$ is called the *neighborhood* of v. For two subsets $A, B \subseteq V(G)$, let $e_G(A, B)$ ($e(A, B)$ for short) denote the number of edges with one endpoint in A and the other endpoint in B. For simplicity, if H_1 and H_2 are two subgraphs of G, we write $e(H_1, H_2)$ instead of $e(V(H_1), V(H_2))$. A complete graph on n vertices is denoted by K_n, and K_n^- is the graph obtained from K_n by deleting one edge. A *k-cycle*, denoted by C_k, is a cycle of length k. For simplicity, we use δ to denote $\delta(G)$, the minimum degree of G.

∗. Corresponding auhtor

Let G be a graph, and let D be an orientation of G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail*$(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let $E^+(v)$ denote the set of edges with tail v and $E^-(v)$ the set of edges with head v. Let A denote an (additive) abelian group with the identity element 0 and let $A^* = A - \{0\}$. We define $F(G, A) = \{f | f : E(G) \to A\}$ and $F^*(G, A) = \{f | f : E(G) \to A^*\}$.

Given a function $f \in F(G, A)$, define $\partial f : V(G) \to A$ by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. The value $\partial f(v)$ is known as the *net flow out of* v *under* f.

For a graph G, a function $b : V(G) \to A$ is an A*-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $\mathbb{Z}(G, A)$. Given $b \in \mathbb{Z}(G, A)$, a function $f \in F^*(G, A)$ is an (A, b)*-nowhere-zero flow* if G has an orientation D such that $\partial f = b$. A graph G is A*-connected* if for every $b \in \mathbb{Z}(G, A)$, G admits an (A, b)-nowhere-zero flow. A *nowhere-zero* A*-flow* is an $(A, 0)$-nowhere-zero flow. More specifically, a *nowhere-zero* k*-flow* is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [18] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

An edge is *contracted* if it is deleted and its two ends are identified into a single vertex. Let H be a connected subgraph of G. Let G/H denote the graph obtained from G by contracting all edges of H and deleting all the loops. A graph G is A*-reduced* if it contains no nontrivial A-connected subgraph. We say that a graph G^* is an A*-reduction* of G if G^* is A-reduced and if G^* can be obtained from G by contracting all maximally A-connected subgraphs of G. It is known that the A-reduction of a graph is A-reduced and an A-reduction of a reduced graph is itself.

Integer flow problems were introduced by Tutte [17, 18]. Group connectivity was introduced by Jaeger *et al.* [7] as a generalization of nowhere-zero flows. The following conjecture is due to Jaeger *et al.*.

Conjecture 1.1. ([7]) *Every* 5*-edge-connected graph is* Z_3*-connected.*

Recently, Thomassen [16] confirmed the weak 3-flow conjecture, and Lovász *et al.* [13] proved that every 6-edge-connected graph is Z_3-connected. However, Conjecture 1.1 is still open.

On the other hand, degree conditions, local structure and forbidden subgraphs are used to investigate the existence of nowhere-zero 3-flows and Z_3-connectivity of graphs. One can find sufficient conditions for the existence of nowhere-zero 3-flows and Z_3-connectivity, and such conditions are related with ones for hamiltonian graphs. It is known that every graph which contains a hamiltonian cycle admits a nowhere-zero 4-flow and there are infinite graphs containing a hamiltonian cycle do not admit a nowhere-zero 3-flow [15]. For the literature, some results can be seen in [8, 14, 19, 20, 21].

In this paper, we still focus on the neighborhood unions condition, which was first introduced by Faudree *et al.* [6] as sufficient conditions for the existence of hamiltonian graphs. Faudree *et al.* [6] proved that if G is a 2-connected simple graph on $n \geq 3$ vertices such that $|N(u) \cup N(v)| \geq (2n-1)/3$ for each pair of nonadjacent vertices u and v, then G is hamiltonian. For this Faudree *et al.*'s result, the first author and X. Li proved that if $|N(u) \cup N(v)| \geq \lceil \frac{2n}{3} \rceil$ for any pair of nonadjacent vertices u and v, then G is Z_3-connected if and only if G cannot be Z_3-reduced one of four specified graphs $\{C_3, K_4, K_4^-, L\}$, where G is a 2-edge-connected graph. On the other hand, Faudree *et al.* [5] proved that if G is a graph on n vertices such that $|N(u) \cup N(v)| + \delta \geq n$ for each pair of nonadjacent vertices u and v, then G is hamiltonian, which improved the result of Faudree *et al.* [6]. Motivated by above observations, we present the following theorem in this paper.

Theorem 1.2. *Let G be a 2-edge-connected simple graph on $n \geq 15$ vertices. If $|N(u) \cup N(v)| + \delta \geq n$ for every $uv \notin E(G)$, then G is not Z_3-connected if and only if G can be Z_3-reduced to one of $\{C_3, K_4, K_4^-, L\}$, where L is obtained from K_4 by adding a new vertex which is joined to two vertices of K_4.*

2. Proof of the main result

For simplicity, define $\mathcal{F}$ to be the set of all 2-edge-connected simple graphs on $n \geq 15$ vertices such that $G \in \mathcal{F}$ if and only if $|N(u) \cup N(v)| + \delta \geq n$ for each $uv \notin E(G)$.

In order to prove Theorem 1.2, we need some lemmas. Some results [2, 3, 8, 9] on group connectivity are summarized as follows.

Lemma 2.1 ([2, 3, 8, 9])**.** *Let A be an abelian group. Then the following results are known:*

(1) *K_1 is A-connected.*

(2) *If $e \in E(G)$ and if G is A-connected, then G/e is A-connected.*

(3) *If H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.*

(4) *Each even wheel is Z_3-connected and each odd wheel is not.*

(5) *Let G be a simple graph and H a nontrivial subgraphs of G. If H is Z_3-connected, then $|V(H)| \geq 5$.*

(6) *Let H be a Z_3-connected subgraph of G. If $e(v, V(H)) \geq 2$ for $v \in V(G-H)$, then the subgraph induced by $V(H) \cup \{v\}$ is Z_3-connected.*

Let G be a graph and let u, v, w be three vertices of G with $uv, uw \in E(G)$. $G_{[uv,uw]}$ is defined to be the graph obtained from G by deleting two edges uv and uw and adding one edge vw. It is clear that $d_{G_{[uv,uw]}}(u) = d(u) - 2$.

Lemma 2.2 ([2, 9]). *Let A be an abelian group. Let G be a graph and let u, v, w be three vertices of G with $d(u) \geq 4$ and $uv, uw \in E(G)$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

Next we give two Theorems of Z_3-connectivity about degree conditions, which are important to prove our main Theorem.

Theorem 2.3 (Theorem 1.8 of [14]). *If G is a simple graph satisfying the Ore-condition with at least three vertices, then G is not Z_3-connected if and only if G is one of the 12 specified graphs shown in Fig. 1.*

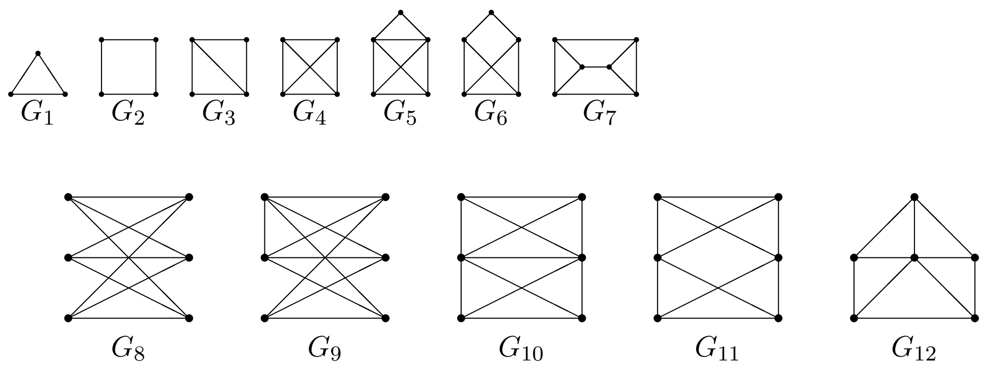

Fig. 1: 12 specified graphs for Theorem 2.3

Theorem 2.4 (Theorem 1.5 of [10]). *Let G be a 2-edge-connected simple graphs on $n \geq 14$ vertices. If $|N(u) \cup N(v)| \geq \lceil \frac{2n}{3} \rceil$ for every $uv \notin E(G)$, then G is not Z_3-connected if and only if G can be Z_3-reduced to one of $\{C_3, K_4, K_4^-, L\}$.*

Lemma 2.5 ([13]). *Every 6-edge-connected graph is Z_3-connected.*

Before proving Theorem 1.2, we summarize some characterizes of graphs in $\mathcal{F}$ with $\delta \geq \lfloor \frac{n}{3} \rfloor + 1$.

Lemma 2.6. *Suppose that $G \in \mathcal{F}$ with $\delta \geq \lfloor \frac{n}{3} \rfloor + 1$. If G contains a nontrivial Z_3-connected subgraph, then G is Z_3-connected.*

Proof. Assume that H is the maximum nontrivial Z_3-connected subgraph of G. If $H = G$, then we are done. Otherwise H is a proper subgraph of G. Let $G' = G/H$ and let v' denote the new vertex which H is contracted to. By the choice of H, each vertex of $V(G - H)$ has at most one neighbor in $V(H)$. It follows that G' is a simple graph. Since G is 2-edge-connected, G' is 2-edge-connected, and so $d_{G'}(v') \geq 2$.

We claim that $|V(H)| > \lfloor \frac{n}{3} \rfloor + 1$. Firstly, we prove it for $n \geq 21$. Suppose otherwise that $|V(H)| \leq \lfloor \frac{n}{3} \rfloor + 1$. By Lemma 2.1(5), $5 \leq |V(H)| \leq \lfloor \frac{n}{3} \rfloor + 1$. Assume $|V(H)| = t$. Thus H contains at most $t(t-1)/2$ edges. Since $\delta \geq \lfloor \frac{n}{3} \rfloor + 1$, $d_{G'}(v') \geq t(\lfloor \frac{n}{3} \rfloor + 1) - t(t-1) = t(\lfloor \frac{n}{3} \rfloor + 2) - t^2$. Define a real value function

$f(t) = t(\lfloor \frac{n}{3} \rfloor + 2) - t^2 - (n-t) = t(\lfloor \frac{n}{3} \rfloor + 3) - t^2 - n$, where $t \in [5, \lfloor \frac{n}{3} \rfloor + 1]$. When $t \in [5, \lfloor \frac{n}{3} \rfloor - 1]$, it is easy to verify that $f(t) > 0$. In this case, we get $d_{G'}(v') > n - t = |V(G' - v')|$. This contradicts that G' is a simple graph. This implies that $t = \lfloor \frac{n}{3} \rfloor$ or $\lfloor \frac{n}{3} \rfloor + 1$. We firstly assume that $t = \lfloor \frac{n}{3} \rfloor$. In this case, note that $f(\lfloor \frac{n}{3} \rfloor) = 3\lfloor \frac{n}{3} \rfloor - n$ and $d_{G'}(v') \geq 2\lfloor \frac{n}{3} \rfloor \geq 14$. Let u and v be two adjacent vertices of $N(v')$. By the choice of H and Lemma 2.1 (4), $|(N_{G'}(u) \cap N_{G'}(v)) \cap N(v')| \leq 1$. When $(N_{G'}(u) \cap N_{G'}(v)) \cap N(v') = \{w\}$, then $N_{G'}(u) \cup N_{G'}(v)$ has $2\lfloor \frac{n}{3} \rfloor - 4$ vertices in $N(v')$ other than w since $d_{G'}(u) + d_{G'}(v) \geq 2\lfloor \frac{n}{3} \rfloor + 2$. It is easy to see that $G'_{[uw,vw]}$ contains a 2-cycle. Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. By Lemma 2.2 and 2.1(3), G is Z_3-connected. When $(N_{G'}(u) \cap N_{G'}(v)) \cap N(v') = \emptyset$, we know that $|N_{G'}(u) \cup N_{G'}(v)| \geq 2\lfloor \frac{n}{3} \rfloor$. Let z be a neighbor of u in $N(v')$. It is easy to see that $G'_{[zu,zv']}$ contains a 2-cycle. Iteratively contracting 2-cycles generated in the processing leads eventually to a K_1, which is Z_3-connected. Therefore G is Z_3-connected by Lemmas 2.2 and 2.1(3). Now we assume that $|V(H)| = \lfloor \frac{n}{3} \rfloor + 1$. Clearly $G' - v' = G - H$. In this case, $d_{G'}(v') \geq (\lfloor \frac{n}{3} \rfloor + 1)(\lfloor \frac{n}{3} \rfloor + 2) - (\lfloor \frac{n}{3} \rfloor + 1)^2 = \lfloor \frac{n}{3} \rfloor + 1$. Hence $d_{G'}(x) + d_{G'}(y) \geq 2(\lfloor \frac{n}{3} \rfloor + 1) \geq n - (\lfloor \frac{n}{3} \rfloor + 1) + 1 \geq |G'|$ for each two nonadjacent vertices x and y in G' and $|V(G')| \geq d_{G'}(v) + 1 \geq \lfloor \frac{n}{3} \rfloor + 2 \geq 7$. By Lemmas 2.3 and 2.1 (3), G is Z_3-connected.

Now we claim that $|V(H)| \geq \lfloor \frac{n}{3} \rfloor + 2$ for $15 \leq n \leq 20$. Similarly, we get $5 \leq |V(H)| \leq \lfloor \frac{n}{3} \rfloor + 1$. For $15 \leq n \leq 17$, note that $\lfloor \frac{n}{3} \rfloor = 5$. In this case, the proof is similarly to the case $|V(H)| = \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1$ for $n \geq 21$. Therefore $|V(H)| \geq \lfloor \frac{n}{3} \rfloor + 2$ for $15 \leq n \leq 17$. For $18 \leq n \leq 20$, we firstly verify that $|V(H)| \neq 5$. Suppose otherwise that $|V(H)| = 5$. Since $\delta \geq 7$, $d(v') \geq 15$. If $n = 18$ or 19, then $d(v') \geq |G - H|$. It contradicts that G' is simple. If $n = 20$, then we get $N(v') = V(G) - V(H)$. Let $x, y \in N(v')$ be two adjacent vertices in G'. Consider the graph $G'_{[xv',xy]}$. It is easy to see that $G'_{[xv',xy]}$ contains at least five 2-cycles with one common vertex v'. Iteratively contracting 2-cycles generated in the processing leads eventually to the graph G''. Denote the new vertex by v''. If $G'' = K_1$, then G' is Z_3-connected by Lemmas 2.2 and 2.1 (3). We may assume that $G'' \neq K_1$. It is easy to verify that $d_{G''-v''}(v) \geq 6$ for $v \in V(G'') - \{x, v''\}$. This implies that $G'' - v''$ satisfies Ore-condition. Therefore, $G'' - v''$ is Z_3-connected by Theorem 2.3. By Lemmas 2.1 and 2.2, G' is Z_3-connected. Thus, G is Z_3-connected by Lemma 2.1. Then we get $|V(H)| \geq \lfloor \frac{n}{3} \rfloor = 6$ for $18 \leq n \leq 20$. When $|V(H)| = \lfloor \frac{n}{3} \rfloor$or $\lfloor \frac{n}{3} \rfloor + 1$ for $18 \leq n \leq 20$, the proof is similarly to the case $n \geq 21$. Therefore, $|V(H)| \geq \lfloor \frac{n}{3} \rfloor + 2$.

Thus, we may assume that $|V(H)| \geq \lfloor \frac{n}{3} \rfloor + 2$. Note that $|V(G' - v')| = n - |V(H)| \leq n - \lfloor \frac{n}{3} \rfloor - 2 = \lceil \frac{2n}{3} \rceil - 2$. Since $e(v, H) \leq 1$ for each $v \in V(G - H)$ and $n \geq 15$, $\delta(G' - v') \geq \lfloor \frac{n}{3} \rfloor \geq 5$. Hence $d_{G'-v'}(x) + d_{G'-v'}(y) \geq 2\delta(G' - v') \geq 2\lfloor \frac{n}{3} \rfloor \geq |V(G' - v')|$ for every two nonadjacent vertices x and y of $G' - v'$. Hence $G' - v'$ satisfies the Ore-condition. Since $\delta(G' - v') \geq 5$, $G' - v'$ is Z_3-connected by Lemma 2.3. For $e(v', V(G-H)) \geq 2$, G' is Z_3-connected by Lemma 2.1 (6). Therefore, G is Z_3-connected by Lemma 2.1 (3). $\square$

Lemma 2.7. *Let $G \in \mathcal{F}$ and $\delta \geq \lfloor \frac{n}{3} \rfloor + 1$. If G contains no Z_3-connected subgraph, then G is 6-edge-connected.*

Proof. Suppose that $E_0 = (X, Y)$ is minimum edge cut of the graph G such that $|X|$ is smallest. If $e(X, Y) \geq 6$, then we have done. Otherwise we assume that $2 \leq e(X, Y) \leq 5$. Now we claim that $G[X]$ contains a Z_3-connected subgraph. Note that $\lfloor \frac{n}{3} \rfloor + 1 \leq |X| \leq \frac{n}{2}$. Without loss of generality, we assume that $x_1, x_2, \ldots, x_l \in X$ are incident to the edge of E_0, where $1 \leq l \leq 5$. When $l = 1$, we consider the graph $H = G[X - \{x_1\}]$. Since $X - x_1$ is not adjacent to any vertex of Y, $\delta(H) \geq \lfloor \frac{n}{3} \rfloor \geq 5$. Thus, $d_H(x) + d_H(y) \geq 2\lfloor \frac{n}{3} \rfloor \geq |H|$ for nonadjacent two vertices x, y in H. By Theorem 2.3, H is Z_3-connected. When $l = 2$, we consider the graph $H = G[X - \{x_1, x_2\}]$. In this case $\delta(H) \geq \lfloor \frac{n}{3} \rfloor - 1 \geq 4$. Therefore, for nonadjacent two vertices x, y in H, $d_H(x) + d_H(y) \geq 2(\lfloor \frac{n}{3} \rfloor - 1) = 2\lfloor \frac{n}{3} \rfloor - 2 \geq |H|$. Thus H is Z_3-connected by Theorem 2.3. When $l = 3, 4, 5$, it is easy to verify that $G[X]$ satisfies the Ore-condition. Therefore, by Theroem 2.3, $G[X]$ is Z_3-connected. It contradicts that G contains no Z_3-connected subgraph. This complete the proof of the lemma. $\square$

Proof of Theorem 1.2 If $|N(u) \cup N(v)| \geq \lceil \frac{2n}{3} \rceil$ for every pair of nonadjacent vertices u and v of G, then G is Z_3-connected or can be Z_3-reduced to one of $\{C_3, K_4, K_4^-, L\}$ by Theorem 2.4. Therefore, in the following, we may assume that there are at least a pair of nonadjacent vertices u and v such that $|N(u) \cup N(v)| \leq \lceil \frac{2n}{3} \rceil - 1$.

Since G is 2-edge-connected, $\delta \geq 2$. When $2 \leq \delta \leq \lfloor \frac{n}{3} \rfloor$, $|N(u) \cup N(v)| \geq \lceil \frac{2n}{3} \rceil$ for each $uv \notin E(G)$. In this case we are done. Therefore, without loss of generality, we may assume that $\delta(G) \geq \lfloor \frac{n}{3} \rfloor + 1$. If G contains a nontrivial Z_3-connected graph, then G is Z_3-connected by Lemma 2.6. If G contains no nontrivial Z_3-connected graph, then, by Lemma 2.7, G is 6-edge-connected. Thus, by Lemma 2.5, G is Z_3-connected. This complete the proof of the theorem.

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t-PROPERTY OF METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. The prime goal of this article is to prove some fixed point results in partially ordered metric spaces that are not necessarily complete. This is achieved by introducing the concept of t-property. Moreover, we have established the existence of fixed points

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for variant contractive mappings in the framework of incomplete ordered metric spaces. Few examples have been given to illustrate the new concepts and results.
Keywords: t-property, ordered metric space, fixed point.

1. Introduction and preliminaries

Uniqueness of fixed points for contraction mappings in complete metric spaces was proved long ago in 1922 by Banach [10]. It was popular by the name of Banach Contraction Principle. This has played a pivotal role in the evolution of fixed point theory. This principle has been generalized in framework of different spaces, see [4, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28]. In 2004, Ran-Reuring [26], Nieto and Rodríguez-López [24] and many others have generalized Banach Contraction Principle in the setting of ordered metric spaces, see [2, 3, 5, 6, 8, 9, 25, 27, 29, 30]. In this article, t-property of partially ordered metric spaces has been introduced.Using this concept, we present some fixed point results for variant contraction mappings.

Definition 1.1 ([7])**.** *An ordered metric space $(X, d, \preceq)$ is said to be $\overline{O}$-complete if every increasing Cauchy sequence in X converges in X. In an ordered metric space, completeness implies $\overline{O}$-completeness, but the converse is not true in general.*

Now, we introduce some definitions.

Definition 1.2. *Let $(X, \preceq)$ be any ordered set and $x, y \in X$. Such x is called a strict upper bound of y, if $y \preceq x$ and $y \neq x$. We denote it by $y \prec x$.*

Definition 1.3. *Let $(X, d, \preceq)$ be any ordered metric space. X has the t-property if every strictly increasing Cauchy sequence $\{x_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $x_n \prec u$.*

We present the following examples illustrating Definition 1.3.

Example 1.1. Let $X = \mathbb{R}, \mathbb{Q}, (a, b], a, b \in \mathbb{R}$ be equipped with the natural ordering $\leq$ and the usual metric. Then X has t-property.

Example 1.2. Let $X = \{(x, y) : x, y \in \mathbb{Q}\}$. We define $\preceq$ in X by $(x_1, x_2) \preceq (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. Let d be the Euclidean metric on X. Then $(X, d, \preceq)$ has the t-property.

Example 1.3. Let $X = C[a, b]$ be equipped with the metric d defined as $d(f, g) = \int_a^b \mid f - g \mid dx$. Then (X, d) is not a complete metric space. Now, we define $\preceq$ in X as: $f \preceq g$ iff $f(x) \leq g(x)$, for each $x \in [a, b]$. Obviously, $(C[a, b], d, \preceq)$ has t-property.

In the following example, the increasing Cauchy sequence does not have any strict upper bound.

Example 1.4. Let us consider $X = \{(x, y, z) : x, y, z \in \mathbb{Q} \ with \ max\{x, y, z\} < \sqrt{2}\}$. Endow X with the Euclidean metric on $\mathbb{R}^3$. Define $\preceq$ in X by $(x_1, y_1, z_1) \preceq (x_2, y_2, z_2)$ if $x_1 \leq x_2$, $y_1 \leq y_2$ and $z_1 \leq z_2$. Consider $x_n = (q_n, q_n, q_n)$ in X such that $q_0 = 1$ and $\{q_n\}$ is strictly increasing in $\mathbb{Q}$. We have that $q_n < \sqrt{2}$ for all $n \geq 0$. Also, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, but it does not have any strict upper bound in X.

Remark 1.1. Mention that every totally ordered complete metric space has t-property provided that there exists a strictly increasing Cauchy sequence. But, every metric space having t-property is not complete. This fact is described In Example 1.1 (except the case $\mathbb{R}$), Example 1.2 and Example 1.3.

2. Main Result

In all our given results, the completeness of the metric space is omitted. To overcome this lack, we require that the space has the t-property. Our first fixed point result is

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered metric space satisfying the t-property. Let $f : X \to X$ be a self-mapping. Assume that f is monotonic non-decreasing. Further, if*
(1) *there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;*
(2) *for all $x, y \in X$ with $x \prec y$,*

$$d(y, f(y)) \leq \alpha d(x, f(x)), \tag{1}$$

where $\alpha \in (0, 1)$. Then f has at least one fixed point in X. Moreover, every strict upper bound of a fixed point is also a fixed point.

Proof. By assumption (1), we have $x_0 \preceq f(x_0)$. If $x_0 = f(x_0)$, the proof is completed. Otherwise, choose $x_1 = f(x_0)$ such that $x_0 \prec x_1$. By monotonicity of f, we have $f(x_0) \preceq f(x_1)$, that is, $x_1 \preceq f(x_1)$. If $x_1 = f(x_1)$, the proof is completed. Otherwise, choose $x_2 = f(x_1)$ such that $x_1 \prec x_2$. Again, by monotonicity of f, we have $f(x_1) \preceq f(x_2)$. Continuing in this process, we get a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{2}$$

As $x_0 \prec x_1$, by (1), we have

$$d(x_1, f(x_1)) \leq \alpha d(x_0, f(x_0)). \tag{3}$$

Again as $x_1 \prec x_2$, by (1), we have

$$d(x_2, f(x_2)) \leq \alpha d(x_1, f(x_1)). \tag{4}$$

Using (3) in (4), we get

$$d(x_2, f(x_2)) \leq \alpha^2 d(x_0, f(x_0)).$$

Continuing in this way, we get

$$d(x_n, f(x_n)) \leq \alpha^n d(x_0, f(x_0)). \tag{5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. For $n < m$, by using triangular inequality, (2) and (5), we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m), \\ &= d(x_n, f(x_n)) + d(x_{n+1}, f(x_{n+1})) + ... + d(x_{m-1}, f(x_{m-1})), \\ &\leq \alpha^n d(x_0, f(x_0)) + \alpha^{n+1} d(x_0, f(x_0)) + ... + \alpha^{m-1} d(x_0, f(x_0)), \\ &\leq (\alpha^n + \alpha^{n+1} + ... + \alpha^{m-n-1}) d(x_0, f(x_0)), \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_0, f(x_0)). \end{aligned}$$

This shows that $\{x_n\}$ is an increasing Cauchy sequence in X, which has the t-property, so there exists $u \in X$ such that $x_n \prec u$ for all n. Thus, from (1) and (5), we have

$$d(u, f(u)) \leq \alpha d(x_n, f(x_n)) \leq \alpha^{n+1} d(x_0, f(x_0)) \to 0, \quad as \quad n \to \infty.$$

Thus $f(u) = u$. Hence u is a fixed point of f. Now, let k be any strict upper bound of u in X, that is, $u \prec k$. By (1), we have

$$d(k, f(k)) \leq \alpha d(u, f(u)) = 0,$$

so $k = f(k)$, that is, k is also a fixed point of f in X. □

Example 2.1. Let $X = \{a_n : \ a_{n+1} = 3a_n + 1 \text{ for } n \geq 0 \text{ and } a_0 = -1\} \cup (-1, 0]$. Then $X = \{..., -41, -14, -5, -2, -1\} \cup (-1, 0]$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\leq$. Clearly, $(X, d, \preceq)$ has the t-property. Define $f : X \to X$ by

$$f(x) = \begin{cases} 3x + 1, & if x < -1, \\ x, & if x \geq -1. \end{cases}$$

Obviously, f is non-decreasing. Now, it remains to prove that f satisfies (1). Let $x, y \in X$ with $x < y$. If $y \geq -1$, then $f(y) = y$, so $d(y, f(y)) = 0$ and the proof is completed. Assume now that $x < y \leq -2$. Then $d(y, f(y)) = -(2y + 1)$ and $d(x, f(x)) = -(2x + 1)$. It should be noted that for $x, y \in X$ with $x < y \leq -2$, we have $y \geq \frac{5}{12}x$. Then

$$\begin{aligned} d(y, f(y)) &= -(2y + 1) \\ &\leq -\frac{5}{6}x - 1 \\ &= -\frac{1}{2}[\frac{5x + 6}{3}] \\ &\leq -\frac{1}{2}(2x + 1) \\ &= \alpha d(x, f(x)), \end{aligned}$$

where $\alpha = \frac{1}{2} \in (0,1)$. Hence all the conditions of Theorem 2.1 are satisfied. Therefore f has at least one fixed point in X. In fact, any element in the set $[0,1]$ is a fixed point of f.

On the other hand, neither Banach Contraction, nor Kannan-type and nor Chatterjea-type contraction holds. Indeed, by taking $x = -5$ and $y = -2$, we have

$$d(f(x), f(y)) > kd(x,y) \quad \text{for all } k \in (0,1),$$

$$d(d(x), f(y)) > k[d(x, f(x)) + d(y, f(y)] \quad \text{for all}\, k \in (0, \frac{1}{2}),$$

and

$$d(d(x), f(y)) > k[d(x, f(y)) + d(y, f(x)] \quad \text{for all } k \in (0, \frac{1}{2}).$$

Theorem 2.2. *Let $(X, d, \preceq)$ be an $\overline{O}$-complete ordered metric space. Let $f : X \to X$ be a self-mapping such that f is continuous and monotonic non-decreasing. Further if*
(1) *there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$.*
(2) *for all $x, y \in X$ with $x \prec y$, $x \neq f(x)$ and for any $\alpha \in (0, \frac{1}{2})$,*

$$d(y, f(y)) \leq \alpha[d(x,y) + d(f(x), f(y)]. \tag{6}$$

Then f has at least one fixed point in X.

Proof. As Theorem 2.1, we construct a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{7}$$

As $x_0 \prec x_1$, by using (6) and (7), we have

$$\begin{aligned} d(x_1, f(x_1)) &\leq \alpha[d(x_0, x_1) + d(f(x_0), f(x_1)] \\ &= \alpha d(x_0, f(x_0)) + \alpha d(x_1, f(x_1)). \end{aligned} \tag{8}$$

Then

$$d(x_1, f(x_1)) \leq \frac{\alpha}{1-\alpha} d(x_0, f(x_0)). \tag{9}$$

Again as $x_1 \prec x_2$, by using (6) and (7), we have

$$\begin{aligned} d(x_2, f(x_2)) &\leq \alpha[d(x_1, x_2) + d(f(x_1), f(x_2)], \\ &= \alpha d(x_1, f(x_1)) + \alpha d(x_2, f(x_2)). \end{aligned} \tag{10}$$

Then

$$d(x_2, f(x_2)) \leq \frac{\alpha}{1-\alpha} d(x_1, f(x_1)).$$

By using (9),

$$d(x_2, f(x_2)) \leq (\frac{\alpha}{1-\alpha})^2 d(x_0, f(x_0)).$$

Continuing this process, we get

$$d(x_n, f(x_n)) \leq (\frac{\alpha}{1-\alpha})^n d(x_0, f(x_0)). \tag{11}$$

As $0 < \alpha < \frac{1}{2}$, we get $0 < k = \frac{\alpha}{1-\alpha} < 1$. (11) becomes

$$d(x_n, f(x_n)) \leq k^n d(x_0, f(x_0)). \tag{12}$$

As Theorem 2.1, $\{x_n\}$ is an increasing Cauchy sequence in X. Since (X, d) is $\overline{O}$-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{13}$$

Since f is continuous,

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{14}$$

Taking $n \to \infty$ in (12) and making use of (13) and (14), we obtain $d(u, f(u)) = 0$. Hence u is a fixed point of f in X. □

Now, we are going to prove Theorem 2.2 when f is not continuous.

Definition 2.1 (SICU-property)**.** *An ordered metric space $(X, d, \preceq)$ is said to have SICU-property, if every strictly increasing convergent sequence has the limit as it's strict upper bound, i.e., if $\{x_n\}$ is strictly increasing convergent sequence with $x_n \to x \Rightarrow x_n \prec x$, for all n.*

Example 2.2. Let $X = \mathbb{R}^n$ endowed with Euclidean metric and $\preceq$ is defined as $(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_n)$, if $x_i \leq y_i$, for all $i = 1, 2, ..., n$. Then $\mathbb{R}^n$ has SICU-property.

Theorem 2.3. *In Theorem 2.2, if we leave the continuity of f, but assume that $(X, d, \preceq)$ has SICU-property, then f has at least one fixed point in X.*

Proof. Going through same lines of proof in Theorem 2.2, we get $x_n \to u$ such that $x_n \prec u$. Thus, by using triangular inequality and (6), we have

$$\begin{aligned} d(u, f(u)) &\leq \alpha[d(x_n, u) + d(f(x_n), f(u)], \\ &\leq \alpha[d(x_n, u) + d(f(x_n), x_n) + d(x_n, u) + d(u, f(u))]. \end{aligned}$$

Then

$$d(u, f(u)) \leq \frac{\alpha}{1-\alpha}[2d(x_n, u) + d(x_n, f(x_n))].$$

By using (12) and taking $n \to \infty$, we have $d(u, f(u)) \leq 0$. Thus $f(u) = u$. □

Example 2.3. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric. Consider $(x, y) \preceq (u, v)$ iff $x \le u$ and $y \le v$. Then $(X, d, \preceq)$ is $\overline{O}$-complete and has SICU-property.Take $A = \{1, 3, 5, 7, ...\}$ as the set of all positive odd numbers. Let $E \subset X$ defined by $E = \{(a, b) : a \in A \ \ and \ \ 0 < b < 1\}$. Clearly, for all $(a, b), (c, d) \in E$ such that $(a, b) \prec (c, d)$, we have

$$a + \frac{3}{2} \le c.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \begin{cases} (x, 1), & \text{if } (x, y) \in E \\ (x, y), & \text{if } (x, y) \in E^c. \end{cases}$$

It is easy to verify that f is non-decreasing. We show that f satisfies (6). For any $x = (x_1, y_1), y = (x_2, y_2) \in X$ with $x \prec y$ and $x \ne f(x)$, there exist only two cases:
(i) $x \in E$ and $y \in E^c$,
(ii) $x, y \in E$.

Case (i). $y \in E^c$, so $y = f(y)$, that is, $d(y, f(y)) = 0$. Hence (6) holds.

Case (ii) $x = (x_1, y_1), y = (x_2, y_2) \in E$. Then $0 < y_1, y_2 < 1$. Also, by definition of E, we have $x_2 - x_1 \ge \frac{3}{2}$. Also,

$$d(y, f(y)) = d((x_2, y_2), (x_2, 1)) = (1 - y_2) < 1,$$

and

$$d(x, y) = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ge (x_2 - x_1).$$

Similarly, $d(f(x), f(y)) = (x_2 - x_1)$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, we have

$$\alpha[d(x, y) + d(f(x), f(y)] \ge \frac{1}{3}2(x_2 - x_1) \ge \frac{2}{3}\frac{3}{2} = 1 > d(y, f(y)).$$

All the conditions of Theorem 2.3 are satisfied. Thus f has a fixed point in X. Any elemnt in E^c is a fixed point of f.

Example 2.4. Let $X = \{a_n : \ \ a_{n+1} = 5a_n + 1 \text{ for } n \ge 0 \text{ and } a_0 = -1\} \cup (-1, 0]$. Then $X = \{..., -94, -19, -4, -1\} \cup (-1, 0]$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\le$. Then $(X, d, \preceq)$ is an $\overline{O}$-complete ordered metric space and satisfies the SICU-property. Define $f : X \to X$ by

$$f(x) = \begin{cases} 5x + 1, & \text{if } x \le -1, \\ x, & \text{if } x > -1. \end{cases}$$

Then f is non-decreasing. We shall prove that f satisfies (6). Let $x, y \in X$ with $x < y$. If $y > -1$, then $d(y, f(y)) = 0$ and so (6) holds. Assume that $x < y \le -1$.

Then $d(y, f(y)) = -(4y+1)$, $d(x,y) = (y-x)$ and $d(f(x), f(y)) = 5(y-x)$. It should be noted that for $x, y \in X$ with $x < y \leq -1$, we have $y \geq \frac{x}{3}$ or $-2x \geq -6y$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, we get

$$\alpha[d(x,y) + d(f(x), f(y))] = 2y - 2x \geq 2y - 6y = -4y \geq -(4y+1) = d(y, f(y)).$$

Thus all the conditions of Theorem 2.3 are satisfied, and hence there exists a fixed point of f in X. Any $x \in [-1, 0]$ is a fixed point of f.

Example 2.5. Let $X = \{a_1, a_2, a_3, a_4\}$ be any ordered set where $\preceq$ is defined as: $a_i \preceq a_j$ iff $i \leq j$. If we define a metric $d : X \times X \to [0, \infty)$ by

$$d(a_i, a_i) = 0, \forall i = 1, 2, 3, 4.$$

$$d(a_i, a_{i+1}) = d(a_{i+1}, a_i) = 1, \text{ for } i = 1, 2, 3$$
$$d(a_i, a_{i+2}) = d(a_{i+2}, a_i) = 2, \text{ for } i = 1, 2$$
$$d(a_1, a_4) = d(a_4, a_1) = 3.$$

Note that $(X, d, \preceq)$ is a finite ordered metric space. We define $f : X \to X$ by $f(a_1) = a_2, f(a_2) = a_2, f(a_3) = a_4, f(a_4) = a_4$. If $a_i \preceq a_j$ for all $i \leq j$, then f is a monotonic non-decreasing mapping. Let $x, y \in X$ such that $x \prec y$ with $x \neq f(x)$. Take $x = a_1$ and $y = a_3$. Otherwise, $y = f(y)$, so $d(y, f(y)) = 0$ and the proof is completed in this case. For $x = a_1$ and $y = a_3$, We have $d(x,y) = d(a_1, a_3) = 2$, $d(f(x), f(y)) = d(f(a_1), f(a_3)) = d(a_2, a_4) = 2$ and $d(y, f(y)) = d(a_3, a_4) = 1$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, then

$$d(y, f(y)) \leq \alpha[d(x,y) + d(f(x), f(y))].$$

For $x \prec y = a_3, a_4$, we have $d(y, f(y)) = 0$. Thus all conditions of Theorem 2.2 are satisfied. The elements a_2 and a_4 are fixed points of f.

Here, neither Banach Contraction, nor Kannan-type contraction and nor Chatterjea-type contraction is satisfied. This can be proved by taking $x = a_1$ and $y = a_3$.

Now, let Φ be set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:
(i) ϕ is non-decreasing;
(ii) $\phi(t) < t, \quad \forall t > 0$;
(iii) $\lim_{r \to t^+} \phi(r) < t, \quad \forall t > 0$.

We state the following known lemma.

Lemma 2.1 ([1]). *Let $\phi \in \Phi$ and $\{u_n\}$ be a given sequence such that $u_n \to 0^+$ as $n \to \infty$. Then $\phi(u_n) \to 0^+$ as $n \to \infty$. Also $\phi(0) = 0$.*

Theorem 2.4. *Let $(X, d, \preceq)$ be any ordered metric space having the t-property and $f : X \to X$ be a monotonic non-decreasing self-mapping. Assume for all $x, y \in X$ with $x \prec y$, we have*

$$d(y, f(y)) \leq \phi(d(x, f(x))), \tag{15}$$

where $\phi \in \Phi$. Suppose that the series $\sum_{n\geq 1} \phi^n(t)$ converges for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has at least one fixed point in X. Moreover, every strict upper bound of fixed point of f is again a fixed point of f.

Proof. As Theorem 2.1 and without loss of generality, we construct a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{16}$$

We take $D_n = d(x_n, f(x_n))$. Since $x_n \neq f(x_n)$ $\forall n$, we have $D_n > 0$ for all n. As $x_n \prec x_{n+1}$ for all n, using (15), we get

$$D_{n+1} = d(x_{n+1}, f(x_{n+1})) \leq \phi(d(x_n, f(x_n))) = \phi(D_n) < D_n. \tag{17}$$

This shows that $\{D_n\}$ is a monotonic decreasing sequence in $\mathbb{R}^+$, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} D_n = r. \tag{18}$$

From (17), we have

$$\lim_{n\to\infty} \phi(D_n) = r. \tag{19}$$

Suppose that $r > 0$. By (19) and $\lim\limits_{r\to t^+} \phi(r) < t$ for $t > 0$, we get

$$r = \lim_{n\to\infty} \phi(D_n) = \lim_{D_n \to r^+} \phi(D_n) < r,$$

which is a contradiction, that is, $r = 0$, i.e.,

$$\lim_{n\to\infty} D_n = 0. \tag{20}$$

From (15)

$$d(x_1, f(x_1)) \leq \phi(d(x_0, f(x_0))).$$

Repeating this process n times, we get

$$D_n = d(x_n, f(x_n)) \leq \phi^n(d(x_0, f(x_0))), \text{ for all } n \geq 1.$$

Since $\sum_{n\geq 1} \phi^n(t)$ converges for all $t > 0$, we have that $\sum_{n\geq 1} D_n$ converges. We shall show that $\{x_n\}$ is a Cauchy sequence in X. As $\{x_n\}$ is strictly decreasing sequence, for $n, m \in \mathbb{N}$ with $n < m$, we have by using (15), (16) and (20)

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &= d(x_n, f(x_n)) + d(x_{n+1}, f(x_{n+1})) + ... + d(x_{m-1}, f(x_{m-1})) \\ &= D_n + D_{n+1} + ... + D_{m-1} \leq \sum_{k=n}^{\infty} D_k \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is a monotonic increasing Cauchy sequence in X, which has the t-property, so there exists $u \in X$ such that $x_n \prec u$ for all n. By using (15) and (19), we have

$$d(u, f(u)) \leq \phi(d(x_n, f(x_n))) = \phi(D_n) \to 0, \text{ as } n \to \infty.$$

This shows that u is a fixed point of f in X. Let $z \in X$ be any strict upper bound of u, *i.e.*, $u \prec z$. By using (15) and Lemma 2.1, we have

$$d(z, f(z)) \leq \phi(d(u, f(u))) = \phi(0) = 0.$$

Hence z is also a fixed point of f in X. □

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CHARACTERIZATIONS OF MV-ALGEBRAS IN TERMS OF CUBIC SETS

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Abstract. The operations of cubic sum, cubic product, cubic intersection, cubic union are given in MV-algebras, and the concepts of cubic MV-ideals and cubic prime MV-ideals in MV-algebras are introduced. Then some characterizations of cubic MV-ideals and cubic prime MV-ideals are obtained. The image set of cubic prime MV-ideals is proved to be a chain under the order relation $\preceq$ by discussing the properties of cubic prime ideals, and the cubic prime MV-ideal theory and extension theorem of MV-algebras are presented. Finally, the quotient structure of cubic MV-ideals is constructed by cubic cosets, and three isomorphism theorems concerning the quotient of cubic MV-ideals are presented by using the notion of invariant cubic sets.

Keywords: MV-algebra, cubic MV-ideal, cubic prime MV-ideal, quotient structure.

1. Introduction

Non-classical logic systems which lay logical foundation for dealing with uncertain information processing and fuzzy information in computer science, have become one of the most active research directions in artificial intelligence field. The study of logic algebraic systems not only promotes the development of non-classical mathematical logics, but also enriches the content and methods of algebras. MV-algebras were introduced by Chang [1] as the algebraic counterpart of Łukasiewicz infinite-valued calculus, and MV-algebras entered deeply in many areas of mathematics and logics. The notions of pseudo MV-algebras [2]

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and generalized MV-algebras [3] as two non-commutative but equivalent generalizations of MV-algebras have independently appeared, and they are used for algebraic foundations of non-commutative fuzzy logic.

Ideal theory is a very effectively tool to study logical algebras and the completeness of the corresponding nonclassical logics. On the one hand, ideals are closely related to congruence relations with which one can associate quotient algebras; on the other hand, the sets of provable formulas in the corresponding inference systems from the point of view of uncertain information can be described by fuzzy ideals of those algebraic semantics. A number of researches have motivated to develop nonclassical logics, and also to enrich the ideal theory of algebras [4, 5, 6]. In addition, based on the fuzzy set theory introduced by Zadeh, the related fuzzy structures (i.e., the fuzzification) of ideals in MV-algebras were further studied [7, 8]. Hedayati [9] extended the notions of fuzzy ideals to $(\in, \in \vee q)$-fuzzy (implicative) ideals in pseudo MV-algebras by using the concept of quasicoincidence of a fuzzy value with a fuzzy set. Using falling shadows theory, [10] proposed the concept of falling fuzzy (implicative) ideals which as a generalization of a $T_\wedge$-fuzzy (implicative) ideal in MV-algebras. Moreover, based on the concept of the soft set, [11] established the int-soft ideal theory in pseudo MV -algebras.

Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [12] introduced a new notion, called a cubic set, and investigated several properties, then they applied the cubic theory to BCK/BCI-algebras, and proposed cubic P-ideals and cubic α-ideals [13, 14]. Continue the Jun's work in [15], Khan et al. [16] introduced the concepts of cubic h-ideals, cubic h-bi-ideals and cubic h-quasi-ideals in hemirings, and provided some basic properties. Combining cubic sets and soft sets, [17] introduce the notions of cubic soft o-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras, and investigate related properties.

The paper aims to investigate ideals of MV-algebras based on the cubic theory. The concepts of cubic MV-ideals and cubic prime MV-ideals in MV-algebras are given, and some characterizations of them are present by the introduced cubic operations. Inspired by the fuzzy prime filter theorem in [18], the cubic prime MV-ideal theorem is provided in MV-algebras. A congruence relation on an MV-algebra is constructed via a cubic MV-ideal. Furthermore, a quotient structure of MV-algebras is constructed by cubic cosets, and some certain isomorphism theorems are proved by using the notion of invariant cubic sets.

2. Preliminaries

In this section, we will provide basic terminologies and notations of MV-algebras which are necessary for the understanding of subsequent results.

An algebra $(M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called an MV-algebra if it satisfies the following axioms: for any $x, y, z \in M$,

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Let $(M, \oplus, \neg, 0)$ be an MV-algebra, for any $x, y \in M$, we put $1 = \neg 0$, $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \ominus y = x \otimes \neg y$, $x \vee y = \neg(\neg x \oplus y) \oplus y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y) = (x \oplus \neg y) \otimes y$. In what follows, unless mentioned otherwise, $(M, \oplus, \neg, 0)$ is an MV-algebra and will often be referred to by its support set M.

Proposition 2.1 ([19, 20])**.** *Let $(M, \oplus, \neg, 0)$ be an MV-algebra. Then the following assertions are valid: for any $x, y, z, s, t \in M$,*

(1) *$x \leq y$ if and only if $\neg x \oplus y = 1$ if and only if $x \ominus y = 0$;*

(2) *$x \ominus y \leq z$ if and only if $x \leq y \oplus z$;*

(3) *$x \otimes \neg x = 0$, $x \oplus \neg x = 1$, $(x \ominus y) \wedge (y \ominus x) = 0$;*

(4) *$x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$;*

(5) *$x \ominus y = \neg y \ominus \neg x$, $x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$;*

(6) *$(x \oplus s) \ominus (y \oplus t) \leq (x \ominus y) \oplus (s \ominus t)$;*

(7) *if $x \leq y$, then $\neg y \leq \neg x$, $x \otimes z \leq y \otimes z$ and $x \oplus z \leq y \oplus z$;*

(8) *$(x \vee y) \ominus y = x \ominus y$, $x \ominus (x \wedge y) = x \ominus y$.*

Let $(M, \oplus, \neg, 0)$ be an MV-algebra and I a nonempty set of M. Then I is called an ideal of M if it satisfies: for any $x, y \in M$, (1) $x, y \in I$ implies $x \oplus y \in I$; (2) $x \leq y$ and $y \in I$ imply $x \in I$. An ideal I is proper iff $I \neq M$. We say that an ideal P is prime iff it is proper and satisfies for any $x, y \in M$, either $x \ominus y \in P$ or $y \ominus x \in P$.

Filters, the order duals of lattice ideals, have a variety of applications in logic and topology. Since MV-algebra M is a lattice, we can give the notion of lattice filters. A nonempty subset F of M is called a lattice filer if it satisfies: for any $x, y \in M$, (1) $x, y \in F$ implies $x \oplus y \in F$; (2) $x \leq y$ and $x \in F$ imply $y \in F$[21].

Let M_1 and M_2 be MV-algebras. A function $f : M_1 \to M_2$ is a homomorphism iff it satisfies the following conditions: for any $x, y \in M_1$, (1) $f(0) = 0$, (2) $f(x \oplus y) = f(x) \oplus f(y)$, (3) $f(\neg x) = \neg f(x)$.

Now we will recall the concept of interval-valued fuzzy sets. A closed subinterval $\tilde{a} = [a^-, a^+]$ of a closed unit interval $[0, 1]$ is called an interval number, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $D[0,1]$ the set of all interval numbers. We define the operations $\wedge$, $\vee$, $\geq$, $\leq$ and $=$ in case of two elements in $D[0,1]$. Consider two elements $\tilde{a}_1 = [a_1^-, a_1^+], \tilde{a}_2 = [a_2^-, a_2^+]$ in $D[0,1]$, then

(1) $\tilde{a}_1 \geq \tilde{a}_2$ if and only if $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$;

(2) $\tilde{a}_1 \leq \tilde{a}_2$ if and only if $a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$;

(3) $\tilde{a}_1 = \tilde{a}_2$ if and only if $a_1^- = a_2^-$ and $a_1^+ = a_2^+$;

(4) $\tilde{a}_1 \wedge \tilde{a}_2 = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]$;

(5) $\tilde{a}_1 \vee \tilde{a}_2 = [\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}]$;

(6) $\mathrm{rinf}_{i\in\Lambda}\tilde{a}_i = [\inf\limits_{i\in\Lambda} a_i^-, \inf\limits_{i\in\Lambda} a_i^+]$, where $\tilde{a}_i \in D[0,1]$, $i \in \Lambda$;

(7) $\mathrm{rsup}_{i\in\Lambda}\tilde{a}_i = [\sup\limits_{i\in\Lambda} a_i^-, \sup\limits_{i\in\Lambda} a_i^+]$, where $\tilde{a}_i \in D[0,1]$, $i \in \Lambda$;

other operations $>$ and $<$ can be defined analogously.

An interval-valued fuzzy set (briefly, IVF-set) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A = \{(x, [\mu_A^-(x), \mu_A^+(x)])|x \in X\},$$

where $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^- : X \to [0,1]$ and $\mu_A^+ : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\tilde{\mu}_A$, respectively.

3. Cubic MV-ideals of MV-algebras

In this section, we define some cubic operations on MV-algebras, then introduce a new notion called cubic MV-ideal of MV-algebras and study several properties of it.

Definition 3.1 ([12, 13]). *Let X be a nonempty set. A cubic set A in X as an object having the following form:*

$$A = \{(x, \tilde{\mu}_A(x), \lambda_A(x))|x \in X\},$$

which is briefly denoted by $A = (\tilde{\mu}_A, \lambda_A)$, where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ_A is a fuzzy set in X. In order to facilitate our subsequent discussion, for any $x \in X$, the number $A(x) = (\tilde{\mu}_A(x), \lambda_A(x))$ is called a cubic element, where the numbers $\tilde{\mu}_A(x)$ and $\lambda_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A, and $\mu_A^+(x) + \lambda_A(x) \leq 1$.

For two cubic elements $A(x)$ and $A(y)$ of the cubic set A, we give the following operations:

(1) $A(x) \preceq A(y)$ if and only if $\tilde{\mu}_A(x) \leq \tilde{\mu}_A(y)$, $\lambda_A(x) \geq \lambda_A(y)$;

(2) $A(x) \prec A(y)$ if and only if $\tilde{\mu}_A(x) < \tilde{\mu}_A(y)$, $\lambda_A(x) > \lambda_A(y)$;

(3) $A(x) \succeq A(y)$ if and only if $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$;

(4) $A(x) \succ A(y)$ if and only if $\tilde{\mu}_A(x) > \tilde{\mu}_A(y)$, $\lambda_A(x) < \lambda_A(y)$;

(5) $A(x) = A(y)$ if and only if $\tilde{\mu}_A(x) = \tilde{\mu}_A(y)$, $\lambda_A(x) = \lambda_A(y)$;

(6) $A(x) \veebar A(y) = (\tilde{\mu}_A(x) \vee \tilde{\mu}_A(y), \lambda_A(x) \wedge \lambda_A(y))$;

(7) $A(x) \barwedge A(y) = (\tilde{\mu}_A(x) \wedge \tilde{\mu}_A(y), \lambda_A(x) \vee \lambda_A(y))$.

If $A_i = (\tilde{\mu}_{A_i}, \lambda_{A_i})$ $(i \in \Lambda)$ are cubic elements, where Λ is an index set, then we define:

$$\overline{\sup}_{i\in\Lambda} A_i = \left(\mathrm{rsup}_{i\in\Lambda}\tilde{\mu}_{A_i}, \inf_{i\in\Lambda} \lambda_{A_i}\right).$$

Let X be a nonempty set and A a nonempty subset of X. The cubic characteristic function of A is defined as $\chi_A = \{(x, \tilde{\mu}_{\chi_A}(x), \lambda_{\chi_A}(x)) | x \in X\}$, where

$$\tilde{\mu}_{\chi_A}(x) = \begin{cases} [1,1], & x \in A, \\ [0,0], & \text{otherwise,} \end{cases} \quad \lambda_{\chi_A}(x) = \begin{cases} 0, & x \in A, \\ 1, & \text{otherwise.} \end{cases}$$

Let $A = (\tilde{\mu}_A, \lambda_A)$ and $B = (\tilde{\mu}_B, \lambda_B)$ be two cubic sets of X, we put $A \sqsubseteq B$ if and only if $A(x) \preceq B(x)$ for any $x \in X$; $A \sqsubset B$ if and only if $A(x) \prec B(x)$ for any $x \in X$.

In what follows, we introduce the operations $\circledast$ and $\circledcirc$ which provide interesting further characterizations of cubic MV-ideals in the subsequent discussions.

Definition 3.2. *Let $A = (\tilde{\mu}_A, \lambda_A)$ and $B = (\tilde{\mu}_B, \lambda_B)$ be two cubic sets of M. Then:*

(1) *the cubic sum $\circledast$ of A and B is defined as*

$$A \circledast B = \{(x, (A \circledast B)(x)) | x \in M\} := \{(x, (\tilde{\mu}_A + \tilde{\mu}_B)(x), (\lambda_A + \lambda_B)(x)) | x \in M\},$$

where $(A \circledast B)(x) = \overline{\sup}\{A(y) \barwedge A(z) | x = y \oplus z, y, z \in M\}$.

(2) *the cubic product $\circledast$ of A and B is defined as*

$$A \circledcirc B = \{(x, (A \circledcirc B)(x)) | x \in M\} := \{(x, (\tilde{\mu}_A \circ \tilde{\mu}_B)(x), (\lambda_A \circ \lambda_B)(x)) | x \in M\},$$

where $(A \circledcirc B)(x) = \overline{\sup}\{A(y) \barwedge A(z) | x = y \otimes z, y, z \in M\}$.

Inspired by [16], we can also give the intersection and union of two cubic sets as follows. Let A and B be cubic sets of an MV-algebra M. The intersection and union of A and B, denote by $A \sqcap B$ and $A \sqcup B$ respectively, are cubic sets:

$$A \sqcap B = \{(x, A(x) \barwedge B(x)) | x \in M\} := \{(x, (\tilde{\mu}_A \Cup \tilde{\mu}_B)(x), (\lambda_A \Cap \lambda_B)(x)) | x \in M\},$$
$$A \sqcup B = \{(x, A(x) \veebar B(x)) | x \in M\} := \{(x, (\tilde{\mu}_A \Cap \tilde{\mu}_B)(x), (\lambda_A \Cup \lambda_B)(x)) | x \in M\}.$$

Lemma 3.3. *For any cubic sets A, B, C, D and E in an MV-algebra M, then we have:*

(1) $A \circledast (B \sqcup C) = (A \circledast B) \sqcup (A \circledast C)$;

(2) $A \circledcirc (B \sqcup C) = (A \circledcirc B) \sqcup (A \circledcirc C)$;

(3) $A \circledast (B \sqcap C) = (A \circledast B) \sqcap (A \circledast C)$;

(4) $A \circledcirc (B \sqcap C) = (A \circledcirc B) \sqcap (A \circledcirc C)$;

(5) $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$;

(6) $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$;

(7) $A \circledast (B \circledast C) = (A \circledast B) \circledast C$;

(8) $A \circledcirc (B \circledcirc C) = (A \circledcirc B) \circledcirc C$;

(9) *if* $D \sqsubseteq A$, $E \sqsubseteq B$, *then* $D \circledast E \sqsubseteq A \circledast B$;

(10) *if* $D \sqsubseteq A$, $E \sqsubseteq B$, *then* $D \circledcirc E \sqsubseteq A \circledcirc B$.

Proof. (5), (6), (9) and (10) are straightforward. The proofs of (2), (3) and (4) are similar to that of (1), and the proof of (8) is similar to that of (7), therefore, we only give the proofs of (1) and (7).

(1) For any $x \in M$, assume that there exist $y, z \in X$ such that $x = y \oplus z$, then

$$\begin{aligned}
&(\tilde{\mu}_A + (\tilde{\mu}_B \Cup \tilde{\mu}_C))(x) = \operatorname{rsup}\{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B \Cup \tilde{\mu}_C)(z) | x = y \oplus z\}\\
&= \operatorname{rsup}\{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B(z) \vee \tilde{\mu}_C(z)) | x = y \oplus z\}\\
&= \operatorname{rsup}\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z)) \vee (\tilde{\mu}_A(y) \wedge \tilde{\mu}_C(z)) | x = y \oplus z\}\\
&= \operatorname{rsup}\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z)\} \vee \operatorname{rsup}\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_C(z) | x = y \oplus z\}\\
&= (\tilde{\mu}_A + \tilde{\mu}_B)(x) \vee (\tilde{\mu}_A + \tilde{\mu}_C)(x)\\
&= ((\tilde{\mu}_A + \tilde{\mu}_B) \Cup (\tilde{\mu}_A + \tilde{\mu}_C))(x),\\
&(\lambda_A + (\lambda_B \Cap \lambda_C))(x) = \inf\{\max\{\lambda_A(y), (\lambda_B \Cap \lambda_C)(z)\} | x = y \oplus z\}\\
&= \inf\{\max\{\lambda_A(y), \min\{\lambda_B(z), \lambda_C(z)\}\} | x = y \oplus z\}\\
&= \inf\{\min\{\max\{\lambda_A(y), \lambda_B(z)\}, \max\{\lambda_A(y), \lambda_C(z)\}\} | x = y \oplus z\}\\
&= \min\{\inf\{\max\{\lambda_A(y), \lambda_B(z)\} | x{=}y \oplus z\}, \inf\{\max\{\lambda_A(y), \lambda_C(z)\} | x = y \oplus z\}\}\\
&= \min\{(\lambda_A + \lambda_B)(x), (\lambda_A + \lambda_C)(x)\}\\
&= ((\lambda_A + \lambda_B) \Cap (\lambda_A + \lambda_C))(x).
\end{aligned}$$

Hence, $A \circledast (B \sqcup C) = (A \circledast B) \sqcup (A \circledast C)$

(7) For any $x \in M$, assume that there exist $y, z \in M$ such that $x = y \oplus z$, then

$$\begin{aligned}
&(\tilde{\mu}_A + (\tilde{\mu}_B + \tilde{\mu}_C))(x) = \operatorname{rsup}_{x=y\oplus z} \{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B + \tilde{\mu}_C)(z)\} \\
&= \operatorname{rsup}_{x=y\oplus z} \left\{\tilde{\mu}_A(y) \wedge \operatorname{rsup}_{z=z_1\oplus z_2} \{\tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}\right\} \\
&= \operatorname{rsup}_{x=y\oplus z} \left\{\operatorname{rsup}_{z=z_1\oplus z_2} \{\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}\right\} \\
&= \operatorname{rsup}_{x=y\oplus(z_1\oplus z_2)} \{\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}, \\
&((\tilde{\mu}_A + \tilde{\mu}_B) + \tilde{\mu}_C)(x) = \operatorname{rsup}_{x=y\oplus z} \{(\tilde{\mu}_A + \tilde{\mu}_B)(y) \wedge \tilde{\mu}_C(z)\} \\
&= \operatorname{rsup}_{x=y\oplus z} \left\{\operatorname{rsup}_{y=y_1\oplus y_2} \{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_1)\} \wedge \tilde{\mu}_C(z)\right\} \\
&= \operatorname{rsup}_{x=y\oplus z} \left\{\operatorname{rsup}_{y=y_1\oplus y_2} \{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_2) \wedge \tilde{\mu}_C(z)\}\right\} \\
&= \operatorname{rsup}_{x=(y_1\oplus y_2)\oplus z} \{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_2) \wedge \tilde{\mu}_C(z) \in M\},
\end{aligned}$$

since $x = y \oplus (z_1 \oplus z_2) = (y_1 \oplus y_2) \oplus z$, then $(\tilde{\mu}_A + (\tilde{\mu}_B + \tilde{\mu}_C))(x) = ((\tilde{\mu}_A + \tilde{\mu}_B) + \tilde{\mu}_C)(x)$.

$$\begin{aligned}
&(\lambda_A + (\lambda_B + \lambda_C))(x) = \inf_{x=y\oplus z} \{\max\{\lambda_A(y), (\lambda_B + \lambda_C)(z)\}\} \\
&= \inf_{x=y\oplus z} \left\{\max\left\{\lambda_A(y), \inf_{z=z_1\oplus z_2} \{\max\{\lambda_B(z_1), \lambda_C(z_2)\}\}\right\}\right\} \\
&= \inf_{x=y\oplus z} \left\{\inf_{z=z_1\oplus z_2} \{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\}\right\} \\
&= \inf_{x=y\oplus(z_1\oplus z_2)} \{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\}, \\
&((\lambda_A + \lambda_B) + \lambda_C)(x) \\
&= \inf_{x=y\oplus z} \{\max\{(\lambda_A + \lambda_B)(y), \lambda_C(z)\}\} \\
&= \inf_{x=y\oplus z} \left\{\max\left\{\inf_{y=y_1\oplus y_2} \{\max\{\lambda_A(y_1), \lambda_B(y_2)\}\}, \lambda_C(z)\right\}\right\} \\
&= \inf_{x=y\oplus z} \left\{\inf_{y=y_1\oplus y_2} \{\max\{\lambda_A(y_1), \lambda_B(y_2), \lambda_C(z)\}\}\right\} \\
&= \inf_{x=(y_1\oplus y_2)\oplus z} \{\max\{\lambda_A(y_1), \lambda_B(y_2), \lambda_C(z)\}\},
\end{aligned}$$

since $x = y \oplus (z_1 \oplus z_2) = (y_1 \oplus y_2) \oplus z$, hence

$$\begin{aligned}
&\inf_{x=y\oplus(z_1\oplus z_2)} \{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\} \\
&= \inf_{x=y\oplus(z_1\oplus z_2)} \{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\},
\end{aligned}$$

and so $(\lambda_A + (\lambda_B + \lambda_C))(x) = ((\lambda_A + \lambda_B) + \lambda_C)(x)$.

Therefore $A \circledast (B \circledast C) = (A \circledast B) \circledast C$. □

Lemma 3.4. *Let A and B be nonempty subsets of an MV-algebra M. Then the followings hold:*

(1) *$A \subseteq B$ if and only if $\chi_A \sqsubseteq \chi_B$;*

(2) *$\chi_A \sqcup \chi_B = \chi_{A\cup B}$, $\chi_A \sqcap \chi_B = \chi_{A\cap B}$;*

(3) *$\chi_A \circledast \chi_B = \chi_{A\oplus B}$, where $A \oplus B = \{x \oplus y | x \in A, y \in B\}$;*

(4) *$\chi_A \circledcirc \chi_B = \chi_{A\otimes B}$, where $A \otimes B = \{x \otimes y | x \in A, y \in B\}$.*

Proof. (1) and (2) is obviously, the proof of (4) is similar to that of (3), here we only need to prove (3). For any $x \in M$, we consider two cases.

(i) if $x \in A \oplus B$, then there exist $y_1 \in A$ and $z_1 \in B$ such that $x = y_1 \oplus z_1$. Then $\tilde{\mu}_{\chi_{A\oplus B}}(x) = [1,1]$. Since $(\tilde{\mu}_{\chi_A}+\tilde{\mu}_{\chi_B})(x) = \mathrm{rsup}\,\{\tilde{\mu}_{\chi_A}(y) \wedge \tilde{\mu}_{\chi_B}(z) | x = y \oplus z\}$ $\geq \tilde{\mu}_{\chi_A}(y_1) \wedge \tilde{\mu}_{\chi_B}(z_1) = [1,1]$, then $(\tilde{\mu}_{\chi_A} + \tilde{\mu}_{\chi_B})(x) = [1,1] = (\tilde{\mu}_{\chi_{A\oplus B}}(x)$, and so, $\tilde{\mu}_{\chi_{A\oplus B}}(x) = [1,1]$.

$(\lambda_{\chi_A}+\lambda_{\chi_B})(x) = \inf\limits_{x=y\oplus z} \{\max\{\lambda_{\chi_A}(y), \lambda_{\chi_B}(z)\}\} \leq \max\{\lambda_{\chi_A}(y_1), \lambda_{\chi_B}(z_1)\} = 0$, therefore $(\lambda_{\chi_A} + \lambda_{\chi_B})(x) = 0 = \lambda_{\chi_{A\oplus B}}(x)$.

(ii) if $x \notin A \oplus B$, then there exist $y_1 \in M\backslash A$ or $z_1 \in M\backslash A$ such that $x = y_1 \oplus z_1$, then $(\tilde{\mu}_{\chi_A} + \tilde{\mu}_{\chi_B})(x) = \mathrm{rsup}\,\{\tilde{\mu}_{\chi_A}(y) \wedge \tilde{\mu}_{\chi_B}(z) | x = y \oplus z\} = [0,0] = \tilde{\mu}_{\chi_{A\oplus B}}(x)$, and $(\lambda_{\chi_A}+\lambda_{\chi_B})(x) = \inf\limits_{x=y\oplus z} \{\max\{\lambda_{\chi_A}(y) \wedge \lambda_{\chi_B}(z)\}\} = 1 = \lambda_{\chi_{A\oplus B}}(x)$.

Therefore, we have $\chi_A \circledast \chi_B = \chi_{A\oplus B}$. □

Definition 3.5. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is called a cubic MV-ideal of M if it satisfies the following conditions: for any $x, y \in M$, if (1) $A(x) \barwedge A(y) \preceq A(x \oplus y)$; (2) $x \leq y$ implies $A(y) \preceq A(x)$.*

For better understanding the notion of cubic MV-ideals, we illustrate it by the following example.

Example 3.6. Let $M = \{0, a, b, 1\}$ be a set such that $0 < a < 1$ and $0 < b < 1$. The operations $\oplus$ and $\neg$ are defined as follows:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\neg$	0	a	b	1
	1	b	a	0

then $(M, \oplus, \neg, 0)$ is an MV-algebra. Define a cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in M as follows:

$$\tilde{\mu}_A(x) = \begin{cases} [0.8, 0.9], & x = 0, \\ [0.3, 0.4], & x = a, \\ [0.2, 0.5], & x = b, \\ [0.1, 0.3], & x = 1; \end{cases} \qquad \lambda_A(x) = \begin{cases} 0.1, & x = 0, \\ 0.5, & x = a, \\ 0.4, & x = b, \\ 0.6, & x = 1, \end{cases}$$

it is to check that A is a cubic MV-ideal of M.

(2) Let I be an ideal of MV-algebra M, $\Gamma_1 = (\tilde{\alpha}_1, \beta_1)$ and $\Gamma_2 = (\tilde{\alpha}_2, \beta_2)$ be cubic elements such that $(\tilde{\alpha}_1, \beta_1) \prec (\tilde{\alpha}_2, \beta_2)$. Define a function as follows:

$$I_{\Gamma_1}^{\Gamma_2}(x) = \begin{cases} \Gamma_2, & x \in I, \\ \Gamma_1, & x \notin I. \end{cases}$$

Routine calculation shows that $I_{\Gamma_1}^{\Gamma_2}$ is a cubic MV-ideal of M. Here the cubic set $I_{\Gamma_1}^{\Gamma_2}$ is called the generalized cubic characteristic function of I.

Proposition 3.7. *Let A be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if for any $x, y \in M$,*

(1) $A(x) \preceq A(0)$;

(2) $A(y) \barwedge A(x \ominus y) \preceq A(x)$.

Proof. The sufficiency is very clear, we now give the proof of the necessity. For any $x, y \in M$, if $x \leq y$, then $A(x) \succeq A(y) \barwedge A(x \ominus y) = A(y) \barwedge A(0) = A(y)$. Notice that $(x \oplus y) \ominus y \leq x$, we have $A((x \oplus y) \ominus y) \succeq A(x)$, and $A(x \oplus y) \succeq A(y) \barwedge A((x \oplus y) \ominus y) \succeq A(x) \barwedge A(y)$, therefore A is a cubic MV-ideal of M. □

Note that the concept of level sets in the fuzzy set theory, Khan et al. [16] give the notion of cubic level sets which serves as a bridge between of cubic sets and crisp sets.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of a nonempty set X, $r \in [0, 1]$ and $[s, t] \in D[0, 1]$ such that $r + t \leq 1$. The set

$$L(A; ([s, t], r)) = \{x \in X | \tilde{\mu}_A(x) \geq [s, t], \lambda_A(x) \leq r\}$$

is called a $([s, t], r)$-cubic level set of A. The proof of the next proposition is obviously, and will be omitted.

Proposition 3.8. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of M. Then the following statements are equivalent:*

(1) *A is a cubic MV-ideal;*

(2) *for any $r \in [0, 1]$, $[s, t] \in D[0, 1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; ([s, t], r))$ is an ideal of M.*

Analogues to the notion of cubic MV-ideals, we can present the notion of cubic lattice filters as follows.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is called a cubic lattice filter of M if it satisfies that for any $x, y \in M$, $A(x) \barwedge A(y) = A(x \wedge y)$. It is easy verify that if A is a cubic lattice filter of M, then for any $r \in [0, 1]$, $[s, t] \in D[0, 1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; ([s, t], r))$ is a lattice filter of M.

Theorem 3.9. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if $z \ominus x \le y$ implies $A(x) \barwedge A(y) \preceq A(z)$ for any $x, y, z \in M$.*

Proof. Assume that A is a cubic MV-ideal of M and there exist $x, y, z \in M$ such that $z \ominus x \le y$, then $A(z \ominus x) \succeq A(y)$. In view of Proposition 3.7, we have $A(z) \succeq A(x) \barwedge A(z \ominus x) \succeq A(x) \barwedge A(y)$.

Conversely, it follows immediately from $0 \ominus x = 0 \le x$ that $A(0) \succeq A(x) \barwedge A(x) = A(x)$. Notice that $x \ominus (x \ominus y) \le y$, we obtain that $A(x) \succeq A(y) \barwedge A(x \ominus y)$. Thus A is a cubic MV-ideal of M. □

Proposition 3.10. *Let A be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if for any $x, y \in M$,*

(1) $A(x) \barwedge A(y) \preceq A(x \oplus y)$;

(2) $A(y) \preceq A(x \otimes y)$.

Proof. The proof will be complete if we show that (2) is equivalent to the condition (2) of Definition 3.5. Assume that A is a cubic MV-ideal of M, since $x \otimes y \le y$ for any $x, y \in M$, then we get the condition (2) of Proposition 3.10.

Conversely, suppose that the condition (2) of Proposition 3.10 holds. For any $x, y \in M$, if $x \le y$, then $(\neg y \oplus x) \otimes y = x \wedge y = x$, and hence $A(x) = A((\neg y \oplus x) \otimes y) \succeq A(y)$, therefore (2) of Definition 3.5 is valid. □

Proposition 3.11. *Let A be a cubic MV-ideal of M. Then the following results hold: for any $x, y, z \in M$,*

(1) *if* $A(x \ominus y) = A(0)$*, then* $A(y) \preceq A(x)$;

(2) $A(x \vee y) = A(x) \barwedge A(y)$;

(3) $A(x \oplus y) = A(x) \barwedge A(y)$;

(4) $A(x) \barwedge A(\neg x) = A(0)$;

(5) $A(x \ominus y) \barwedge A(y \ominus z) \preceq A(x \ominus z)$.

Proof. (1) Since A is a cubic MV-ideal of M, then we have $A(y) \barwedge A(x \ominus y) = A(y) \preceq A(x)$ by Proposition 3.7.

(2) Using $(x \vee y) \le x \oplus y$ and together with Theorem 3.9, we obtain that $A(x) \barwedge A(y) \preceq A(x \vee y)$. As for the reverse inequality, from $x, y \le x \vee y$, we have $A(x \vee y) \preceq A(x)$ and $A(x \vee y) \preceq A(y)$, and so $A(x \vee y) \preceq A(x) \barwedge A(y)$. Hence (2) is valid.

(3) Since $(x \vee y) \le x \oplus y$, one more application of Definition 3.5 yields $A(x \oplus y) \preceq A(x \vee y) = A(x) \barwedge A(y)$. The reverse inequality follows from Proposition 3.10, which completes the proof of (3).

(4) is a consequence of (3).

(5) is immediately from Proposition 2.1 (5) and Theorem 3.9. □

Theorem 3.12. *Let A be a cubic set of M. Then A is a cubic MV-ideal of M if and only if the following conditions are valid:*

(1) $A \circledast A \sqsubseteq A$;

(2) $\chi_M \circledcirc A \sqsubseteq A$.

Proof. Assume that A is a cubic MV-ideal of M and x is an element of M. We will first show that (1) holds. Let x be expressed as $x = y \oplus z$ for some $y, z \in M$. Then we get that $\tilde{\mu}_A(x) = \tilde{\mu}_A(y \oplus z) \geq \tilde{\mu}_A(y) \wedge \tilde{\mu}_A(z)$, $\lambda_A(x) = \lambda_A(y \oplus z) \leq \max\{\lambda_A(y), \lambda_A(z)\}$. And so $\tilde{\mu}_A(x) \geq \mathrm{rsup}\,\{\tilde{\mu}_A(y) \wedge \tilde{\mu}_A(z) | x = y \oplus z\} = (\tilde{\mu}_A + \tilde{\mu}_A)(x)$, $\lambda_A(x) \leq \inf\{\max\{\lambda_A(y), \lambda_A(z)\} | x = y \oplus z\} = (\lambda_A + \lambda_A)(x)$, hence $A \circledast A \sqsubseteq A$.

For any $x \in M$, we have $(\tilde{\mu}_{\chi_M} \circ \tilde{\mu}_A)(x) = \mathrm{rsup}\{\tilde{\mu}_{\chi_M}(y) \wedge \tilde{\mu}_A(z) | x = y \otimes z\} = \mathrm{rsup}\{[1,1] \wedge \tilde{\mu}_A(z) | x = y \otimes z\} = \mathrm{rsup}\{\tilde{\mu}_A(z) | x = y \otimes z\} \leq \mathrm{rsup}\{\tilde{\mu}_A(y \otimes z) | x = y \otimes z\} = \tilde{\mu}_A(x)$, and $(\lambda_{\chi_M} \circ \lambda_A)(x) = \inf\{\max\{\lambda_{\chi_M}(y), \lambda_A(z)\} | x = y \otimes z\} = \inf\{\max\{0, \lambda_A(z)\} | x = y \otimes z\} = \inf\{\lambda_A(z) | x = y \otimes z\} \geq \inf\{\lambda_A(y \otimes z) | x = y \otimes z\} = \lambda_A(x)$, therefore $\chi_M \circledcirc A \sqsubseteq A$.

Conversely, suppose that $A \circledast A \sqsubseteq A$ and $\chi_M \circledcirc A \sqsubseteq A$. For any $x, y \in M$, we get that $\tilde{\mu}_A(x \oplus y) \geq (\tilde{\mu}_A + \tilde{\mu}_A)(x \oplus y) \geq \tilde{\mu}_A(x) \wedge \tilde{\mu}_A(y)$ and $\lambda_A(x \oplus y) \leq (\lambda_A + \lambda_A)(x \oplus y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, that is $A(x) \barwedge A(y) \preceq A(x \oplus y)$. Due to the fact that $\tilde{\mu}_A(x \otimes y) \geq (\tilde{\mu}_{\chi_M} \circ \tilde{\mu}_A)(x \otimes y) \geq \tilde{\mu}_{\chi_M}(x) \wedge \tilde{\mu}_A(y) = \tilde{\mu}_A(y)$ and $\lambda_A(x \otimes y) \leq (\lambda_{\chi_M} \circ \lambda_A)(x \otimes y) \leq \max\{\lambda_{\chi_M}(x), \lambda_A(y)\} = \lambda_A(y)$, that is, $A(y) \preceq A(x \otimes y)$, we get that A is a cubic MV-ideal of M by Proposition 3.10. □

Proposition 3.13. *Let A and B be cubic MV-ideals of M. Then the following results are valid:*

(1) *$A \sqcap B$ is a cubic MV-ideal of M;*

(2) *if $A \circledast B$ is an inverse isotone mapping, then $A \circledast B$ is a cubic MV-ideal of M.*

Proof. (1) It is obviously.

(2) For any $x, y, z \in M$, if $z \ominus x \leq y$, that is, $z \leq x \oplus y$, according to Proposition 3.11 (3), we have $(A \circledast B)(x) \barwedge (A \circledast B)(y) = \overline{\sup}\{A(x_1) \barwedge B(x_2) | x = x_1 \oplus x_2\} \barwedge \overline{\sup}\{A(y_1) \barwedge B(y_2) | y = y_1 \oplus y_2\} = \overline{\sup}\{A(x_1) \barwedge B(x_2) \barwedge A(y_1) \barwedge B(y_2) | x = x_1 \oplus x_2, y = y_1 \oplus y_2\} \preceq \overline{\sup}\{(A(x_1) \barwedge A(y_1)) \barwedge (B(x_2) \barwedge B(y_2)) | x \oplus y = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)\} = \overline{\sup}\{(A(x_1 \oplus y_1)) \barwedge (B(x_2 \oplus y_2)) | x \oplus y = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)\} = (A \circledast B)(x \oplus y) \preceq (A \circledast B)(z)$. Thus $A \circledast B$ is a cubic MV-ideal of M. □

Let X be a non-empty set and $(\tilde{\alpha}, \beta)$ a cubic element. For any $x \in X$, if $\mathcal{C}_{(\tilde{\alpha},\beta)}(x) = (\tilde{\alpha}, \beta)$, then $\mathcal{C}_{(\tilde{\alpha},\beta)}$ is called a constant cubic set of X. For the sake of convenience, a nonconstant cubic MV-ideal is called a proper cubic MV-ideal. We define the image set $Im(A)$ of the cubic set A of X as: $Im(A) = \{A(x) | x \in X\}$.

Definition 3.14. *Let A be a proper cubic MV-ideal of M. For any $x, y \in M$, if $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$, then A is called a cubic prime MV-ideal.*

In what follows we will show some characterizations of cubic prime MV-ideals.

Proposition 3.15. *Let A be a proper cubic MV-ideal of M. Then A is a cubic prime MV-ideal of M if and only if for any $r \in [0,1]$, $[s,t] \in D[0,1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; [s,t], r)$ is a prime ideal of M.*

Proposition 3.16. *Let A be a proper cubic MV-ideal of M. Then the following assertions are equivalent:*

(1) *A is a cubic prime MV-ideal of M;*

(2) *$A(x \wedge y) = A(0)$ implies $A(x) = A(0)$ or $A(y) = A(0)$ for any $x, y \in M$.*

Proof. Assume that A is a cubic prime MV-ideal of M. Let $A(x \wedge y) = A(0)$. Notice that $x \ominus (x \wedge y) = x \ominus y$ and A is a cubic MV-ideal of M, we have $A(x) \succeq A(x \wedge y) \barwedge A(x \ominus (x \wedge y)) = A(x \ominus (x \wedge y)) = A(x \ominus y)$ by Proposition 3.7, and so $A(0) \succeq A(x) \succeq A(x \ominus y)$. Similarly $A(0) \succeq A(y) \succeq A(y \ominus x)$. From A is a cubic prime MV-ideal, it follows that $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$, hence $A(x) = A(0)$ or $A(y) = A(0)$.

Conversely, suppose that (2) is valid. For any $x, y \in M$, $(x \ominus y) \wedge (y \ominus x) = 0$, then $A((x \ominus y) \wedge (y \ominus x)) = A(0)$. Then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ by hypothesis, therefore A is a cubic prime MV-ideal. □

Theorem 3.17. *Let A be a proper cubic MV-ideal of M. Then A is a cubic prime MV-ideal of M if and only if $Im(A)$ is a chain under the order relation $\preceq$, and $A(x \wedge y) = A(x) \veebar A(y)$ for any $x, y \in M$.*

Proof. Suppose that A is a cubic prime MV-ideal of M, then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ for any $x, y \in M$. If $A(x \ominus y) = A(0)$, consider that $x \ominus (x \ominus y) \leq x \wedge y$, we get that $A(x) \succeq A(x \ominus y) \barwedge A(x \wedge y) = A(x \wedge y)$ by Theorem 3.9. Combining with $A(x \wedge y) \succeq A(x)$ and $A(x \wedge y) \succeq A(y)$, we have $A(x \wedge y) = A(x)$ and $A(x) \succeq A(y)$. Similarly, if $A(y \ominus x) = A(0)$, we can prove that $A(x \wedge y) = A(y)$ and $A(y) \succeq A(x)$. Hence, $Im(A)$ is a chain under the order relation $\preceq$, and $A(x \wedge y) = A(x) \veebar A(y)$.

Conversely, due to the fact that $(x \ominus y) \wedge (y \ominus x) = 0$, we obtain that $A(0) = A((x \ominus y) \wedge (y \ominus x)) = A(x \ominus y) \veebar A(y \ominus x)$. Since $Im(A)$ is a chain, then $A(0) = A(x \ominus y)$ or $A(0) = A(y \ominus x)$, thus A is a cubic prime MV-ideal. □

Proposition 3.18. *Let A, B be cubic sets of M. If A is a cubic prime MV-ideal of M, and B is a proper cubic MV-ideal of M such that $A \sqsubseteq B$ and $A(0) = B(0)$, then B is a cubic prime MV-ideal of M.*

Proof. Since A is a cubic prime MV-ideal of M, then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ for any $x, y \in M$. If $A(x \ominus y) = A(0)$, from $A \sqsubseteq B$ and $A(0) = B(0)$ it follows that $B(x \ominus y) = B(0)$. Similarly, $B(y \ominus x) = B(0)$ if $A(y \ominus x) = A(0)$. Thus, B is a cubic prime MV-ideal. □

Proposition 3.19. *Let A and $\mathcal{C}_{(\tilde{\alpha},\beta)}$ be a cubic set and a constant cubic set of M, respectively. If A is a cubic prime MV-ideal of M and $(\tilde{\alpha},\beta) \prec A(0)$, then $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic prime MV-ideal of M.*

Proof. For any $x, y, z \in M$, if $z \ominus x \leq y$, then $A(z) \succeq A(x) \barwedge A(y)$ by Theorem 3.9, and so $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(z) = A(z) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z) \succeq (A(x) \barwedge A(y)) \vee \mathcal{C}_{(\tilde{\alpha},\beta)}(z) = (A(x) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z)) \barwedge (A(y) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z)) = (A(x) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(x)) \barwedge (A(y) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(y)) = (A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(x) \barwedge (A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(y)$. Therefore, $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic MV-ideal of M.

Nextly, we will show that $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is prime. In view that A is a cubic prime MV-ideal of M, and $(\tilde{\alpha},\beta) \prec A(0)$, we get that $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(0) = A(0) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(0) = A(0) \veebar (\tilde{\alpha},\beta) = A(0) \neq A(1)$, thus $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a proper cubic MV-ideal of M. Since $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(0) = A(0)$ and $A \sqsubseteq A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$, then $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic prime MV-ideal of M by Proposition 3.18 . □

Definition 3.20. *Let $(\tilde{\alpha},\beta)$ be a cubic element, and X be a set of some cubic elements. The cubic element $(\tilde{\alpha},\beta)$ satisfies the chain property on X if it is comparable with all elements of X.*

Lemma 3.21. [19] *Let I be an ideal of MV-algebra M and $F(\neq \emptyset)$ be a lattice filter of M with $I \cap F = \emptyset$. There is a prime ideal P of M such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Inspired by the fuzzy prime filter theorem in [18], we give the cubic prime ideal theorem in MV-algebras as follows.

Theorem 3.22. (*Cubic Prime Ideal Theory*) *Let A be a proper cubic MV-ideal of M with $A(0) \prec ([1,1],0)$. Suppose that there is a cubic lattice filter B of M such that $A \sqcap B \sqsubset \mathcal{C}_{(\tilde{\alpha},\beta)}$, and $(\tilde{\alpha},\beta)$ satisfies the chain property on $Im(A) \cup Im(B)$, then there is a cubic prime MV-ideal D such that $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.*

Proof. Since A is a proper cubic MV-ideal, then $L(A, A(0))$ is a proper ideal of M. Nextly, we consider the following three cases relative to the $(\tilde{\alpha},\beta)$-cubic level sets of A and B:

Case (1): $L(B; (\tilde{\alpha},\beta)) = \emptyset$. It follows that $B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$. Taking $F = \{0\}$, then there is a prime ideal P of M such that $L(A, A(0)) \subseteq P$ by Lemma 3.21. Here we put $D = P^{([1,1],0)}_{A(0)}$, that is, D is the generalized cubic characteristic function of the ideal P. And so D is a cubic prime MV-ideal of M, with $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.

Case (2): $L(A; (\tilde{\alpha},\beta)) = \emptyset$. We obtain that $A \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$ and $A(0) \prec (\tilde{\alpha},\beta)$. If P is the prime ideal given in case (1), and $D = P^{(\tilde{\alpha},\beta)}_{A(0)}$, then D is a cubic prime MV-ideal of M, with $A \sqsubseteq D \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.

Case (3): $L(A;(\tilde{\alpha},\beta)) \neq \emptyset$ and $L(B;(\tilde{\alpha},\beta)) \neq \emptyset$. Then $L(B;(\tilde{\alpha},\beta))$ is a lattice filter of M, and $(\tilde{\alpha},\beta) \preceq A(0)$. From $A \sqcap B \sqsubset \mathcal{C}_{(\tilde{\alpha},\beta)}$, we obtain that $L(A;(\tilde{\alpha},\beta)) \cap L(B;(\tilde{\alpha},\beta)) = \emptyset$, thus $L(A;(\tilde{\alpha},\beta)) \neq M$, and so $L(A;(\tilde{\alpha},\beta))$ is a proper ideal of M. By Lemma 3.21, it follows that there is a prime lattice ideal P of M such that $L(A;(\tilde{\alpha},\beta)) \subseteq P$ and $L(B;(\tilde{\alpha},\beta)) \cap P = \emptyset$. We put $D = P^{([1,1],0)}_{(\tilde{\alpha},\beta)}$, it is easy to see that D is a cubic prime MV-ideal of M, and we will show that $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$. For any $x \in M$, if $x \in P$, then $x \notin L(B;(\tilde{\alpha},\beta))$, and so $A(x) \preceq ([1,1],0) = D(x)$ and $B(x) \prec (\tilde{\alpha},\beta)$, hence $(D \sqcap B)(x) \prec (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$; if $x \notin P$, then $x \notin L(A;(\tilde{\alpha},\beta))$, thus $A(x) \prec (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$ and $(D \sqcap B)(x) \preceq (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$. Therefore, in any case we have $A(x) \preceq \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$ and $(D \sqcap B)(x) \preceq \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$. □

4. Quotient structures of MV-algebras based on cubic MV-ideals

In the section, we defined the quotient structure of cubic MV-ideals, then present three isomorphism theorems concerning the quotient of cubic MV-ideals.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic MV-ideal of an MV-algebra M and $x \in M$. The cubic set A^x is called the cubic coset of A which is defined as: for any $y \in M$,

$$A^x(y) = A(x \ominus y) \,\overline{\wedge}\, A(y \ominus x).$$

We denote M/A the set of all cubic cosets with respect to A.

Lemma 4.1. *Let A be a cubic MV-ideal of an MV-algebra M. Then the following assertions hold: for any $x, y, z, s, t \in M$,*

(1) $A^x = A^y$ *if and only if* $A(x \ominus y) = A(y \ominus x) = A(0)$*;*

(2) *the set* $A_* = \{x \in M | A(x) = A(0)\}$ *is an ideal of* M*;*

(3) $A^x(y) = A^y(x)$*;*

(4) $A^x(y) = A^{\neg x}(\neg y)$*;*

(5) $A^x(y) \,\overline{\wedge}\, A^y(z) \preceq A^x(z)$*;*

(6) $A^x(y) \,\overline{\wedge}\, A^s(t) \preceq A^{x \oplus s}(y \oplus t)$*.*

Proof. (1) Suppose that $A^x = A^y$, then $A^x(x) = A^y(x)$, it follows that $A(x \ominus x) = A(0) = A(y \ominus x) \,\overline{\wedge}\, A(x \ominus y)$. Thus $A(x \ominus y) = A(y \ominus x) = A(0)$.

Conversely, assume that $A(x \ominus y) = A(y \ominus x) = A(0)$. According to Proposition 3.11 (5), we get that $A(x \ominus z) \succeq A(x \ominus y) \,\overline{\wedge}\, A(y \ominus z) = A(y \ominus z)$ and $A(z \ominus x) \succeq A(z \ominus y) \,\overline{\wedge}\, A(y \ominus x) = A(z \ominus y)$. Hence $A^x(z) = A(x \ominus z) \,\overline{\wedge}\, A(z \ominus x) \succeq A(y \ominus z) \,\overline{\wedge}\, A(z \ominus y) = A^y(z)$. Similarly, $A^y(z) \succeq A^x(z)$, therefore $A^x = A^y$.

(2) and (3) are obviously.

(4) is immediately from Proposition 2.1 (5).

(5) Since $x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$ and $z \ominus x \leq (y \ominus x) \oplus (z \ominus y)$, then $A(x \ominus y) \,\overline{\wedge}\, A(y \ominus z) \preceq A(x \ominus z)$ and $A(y \ominus x) \,\overline{\wedge}\, A(z \ominus y) \preceq A(z \ominus x)$ by 3.11 (3), whence (5) follows form the monotonicity of $\overline{\wedge}$.

(6) The proof of (6) is similar to that of (5). □

As an immediate consequence of Lemma 4.1, we have

Proposition 4.2. *Let A be a cubic MV-ideal of M. A relation $\equiv_A$ on M is defined as follows: for any $x, y \in M$,*

$$x \equiv_A y \text{ if and only if } A(x \ominus y) = A(y \ominus x) = A(0),$$

Then $\equiv_A$ is a congruence relation on M.

Given $x \in M$, the equivalence class of x with respect to $\equiv_A$ will be denoted by $[x]_A$ and the quotient set $M/\equiv_A$. Since $\equiv_A$ is a congruence, defining the operations on the set $M/\equiv_A$ as: $\neg[x]_A = [\neg x]_A$ and $[x]_A \oplus [y]_A = [x \oplus y]_A$ for any $x, y \in M$. Then the system $(M/\equiv_A, \oplus, \neg, [0]_A)$ becomes an MV-algebra.

The next corollary is an easy consequence of Lemma 4.1 and Proposition 4.2.

Corollary 4.3. *Let A be a cubic MV-ideal of M. Then*

(1) *$A^x = A^y$ if and only if $x \equiv_A y$, for any $x, y \in M$;*

(2) *$A_* = [0]_A$.*

Let A be a cubic MV-ideal of an MV-algebra M. For any $A^x, A^y \in M/A$, we define $A^x \vee A^y = A^{x \vee y}$, $A^x \wedge A^y = A^{x \wedge y}$, $A^x \oplus A^y = A^{x \oplus y}$, $\neg A^x = A^{\neg x}$, and the order $\leq$ on M/A by $A^x \leq A^y$ if and only if $A^x \vee A^y = A^y$.

Lemma 4.4. *Let A be a cubic MV-ideal of M. Then $A^x \leq A^y$ if and only if $A(x \ominus y) = A(0)$.*

Proof. For any $x, y \in M$, $A^x \leq A^y$ if and only if $A^x \vee A^y = A^{x \vee y} = A^y$ if and only if $A(y \ominus (x \vee y)) = A((x \vee y) \ominus y) = A(0)$, that is, $A(x \ominus y) = A(0)$. □

Corollary 4.5. *Let A be a cubic MV-ideal of M. Then $A^x \leq A^y$ if and only if $A^x \wedge A^y = A^x$.*

Theorem 4.6. *Let A be a cubic MV-ideal of M. Then $(M/A, \oplus, \neg, A^0)$ is an MV-algebra, which is called a cubic quotient MV-algebra.*

Proof. We can claim that the operations on M/A are well-defined. In fact, if $A^x = A^y$ and $A^s = A^t$, according to Corollary 4.3, we have that $x \equiv_A y$ and $s \equiv_A t$, and so $x \oplus s \equiv_A y \oplus t$, it follows that $A^{x \oplus s} = A^{y \oplus t}$. Similarly, we can prove $A^{x \vee s} = A^{y \vee t}$ and $A^{x \wedge s} = A^{y \wedge t}$. Then we can easily check that M/A is an MV-algebra. □

Theorem 4.7. *Let A be a cubic MV-ideal of M. Then $M/A \cong M/\equiv_A$.*

Proof. Define a map $\varphi : M/A \to M/\equiv_A$ by $\varphi(A^x) = [x]_A$ for any $x \in M$. Assume that $A^x, A^y \in M/A$, then $A^x = A^y$ if and only if $x \equiv_A y$, which implies that φ is an one-to-one function. Obviously, φ is surjective. Moreover, $\varphi(A^x \oplus A^y) = \varphi(A^{x\oplus y}) = [x \oplus y]_A = [x]_A \oplus [y]_A = \varphi(A^x) \oplus \varphi(A^y)$, $\varphi(\neg A^x) = \varphi(A^{\neg x}) = [\neg x]_A = \neg[x]_A = \neg\varphi(A^x)$, thus φ is an isomorphism and the proof is complete. □

From the above theorem we immediately obtain:

Corollary 4.8. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A be a cubic MV-ideal of M with $ker f = A_*$. Then $M/A \cong f(M_1)$.*

Definition 4.9. *Let A be a cubic MV-ideal of M and B a cubic set of M. A cubic set B/A of the MV-algebra M/A is defined as follows: for any $A^x \in M/A$,*

$$(B/A)(A^x) = \overline{\sup}\{B(y) | A^x = A^y, y \in M\},$$

and B/A is called a quotient cubic set of M/A.

Proposition 4.10. *Let A, B be cubic MV-ideals of M. Then B/A is a cubic MV-ideal of M/A.*

Proof. For any $x_1, x_2 \in M$, $(B/A)(A^{x_1} \oplus A^{x_2}) = (B/A)(A^{x_1\oplus x_2}) = \overline{\sup}\{B(y_1 \oplus y_2) | A^{x_1\oplus x_2} = A^{y_1\oplus y_2}\} \geq \overline{\sup}\{B(y_1) \overline{\wedge} B(y_2) | A^{x_1} = A^{y_1}, A^{x_2} = A^{y_2}\} = B/A(A^{x_1}) \overline{\wedge} B/A(A^{x_2})$.

For any $x_1, x_2 \in M$ such that $A^{x_1} \leq A^{x_2}$, then $A^{x_1} = A^{x_1} \wedge A^{x_2}$, and $(B/A)(A^{x_1}) = \overline{\sup}\{B(y_1) | A^{x_1} = A^{y_1}\} = \overline{\sup}\{B(y_1) | A^{x_1} = A^{y_1}, A^{x_2} = A^{y_2}\} = \overline{\sup}\{B(y_1 \wedge y_2) | A^{x_1} = A^{y_1\wedge y_2}, A^{x_2} = A^{y_2}\} \geq \overline{\sup}\{B(y_2) | A^{x_2} = A^{y_2}\} = B/A(A^{x_2})$.

Thus B/A is a cubic MV-ideal of M/A. □

Definition 4.11. *Let f be a mapping from an MV-algebra M_1 into an MV-algebra M_2, and A, B be cubic sets of M_1 and M_2, respectively. Then*

(1) *the inverse image $f^{-1}(B)$ of B under f is defined as $f^{-1}(B)(x) = B(f(x))$, for any $x \in M_1$;*

(2) *the image $f(A)$ of A under f is defined as*

$$f(A)(y) = \begin{cases} \overline{\sup}\{A(x) | f(x) = y\}, & f^{-1}(y) \neq \emptyset, \\ ([0,0], 1), & \textit{otherwise}. \end{cases}$$

The following result can be easy proved together with Definition 3.5, and so we omit the proof.

Proposition 4.12. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A, B be cubic MV-ideals of M_1 and M_2, respectively. Then*

(1) *the inverse image $f^{-1}(B)$ is a cubic MV-ideal of M_1;*

(2) *the image $f(A)$ is a cubic MV-ideal of M_2.*

Proposition 4.13. *Let A, B be cubic MV-ideals of an MV-algebra M and $\varphi : M \to M/A$ a natural homomorphism, that is, $\varphi(x) = A^x$ for any $x \in M$. Then*

(1) *$\varphi(B) = B/A$ for any cubic set B of M;*

(2) *$\varphi^{-1}(B)/A = B$ for any cubic set B of M/A.*

Proof. (1) For any $A^x \in M/A$, we get that $\varphi(B)(A^x) = \overline{\sup}\{B^t|\varphi(t) = A^x, t \in M\} = \overline{\sup}\{B^t|A^t = A^x, t \in M\} = B/A(A^x)$. And therefore $\varphi(B) = B/A$.

(2) For any $A^x \in M/A$, we obtain that $(\varphi^{-1}(B)/A)(A^x) = \overline{\sup}\{\varphi^{-1}(B)(t)|A^t = A^x, t \in M\} = \overline{\sup}\{B(\varphi(t))|A^t = A^x, t \in M\} = \overline{\sup}\{B(A^t)|A^t = A^x, t \in M\} = B(A^x)$. Thus $\varphi^{-1}(B)/A = B$. □

For the purpose of investigating homomorphism theorems of MV-algebras based on cubic MV-ideals, we introduce the following notions.

Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A, B be cubic MV-ideals of M_1 and M_2, respectively. If $f(A) \sqsubseteq B$, we say that A is weakly homomorphic to B, and we write $A \sim B$. If $f(A) = B$, we say that A is homomorphic to B, and we write $A \approx B$. If f is bijective and $f(A) = B$, we say that A is isomorphic to B, and we write $A \cong B$.

As an immediate consequence of the above Proposition 4.13, we record here the following result.

Corollary 4.14. *Let A, B be cubic MV-ideals of M. Then $B \approx B/A$.*

Theorem 4.15. *Let A, B be cubic MV-ideals of an MV-algebra M_1, and $f : M_1 \to M_2$ be an epimorphism of MV-algebras such that $kerf = A_*$. Then $B/A \cong f(B)$.*

Proof. Define a map $\varphi : M_1/A \to M_2$ by $\varphi(A^x) = f(x)$ for any $x \in M_1$. Then for any $x_1, x_2 \in M_1$, we have that $A^{x_1} = A^{x_2}$ if and only if $x_1 \ominus x_2, x_2 \ominus x_1 \in A_* = kerf$ if and only if $f(x_1 \ominus x_2) = f(x_2 \ominus x_1) = f(0) = 0$ if and only if $f(x_1) = f(x_2)$. Therefore φ is an one-to-one function. It follows that φ is surjective due to the fact that f is a surjective function. For any $x_1, x_2, x \in M_1$, $\varphi(A^{x_1} \oplus A^{x_2}) = \varphi(A^{x_1 \oplus x_2}) = f(x_1 \oplus x_2) = f(x_1) \oplus f(x_2) = \varphi(A^{x_1}) \oplus \varphi(A^{x_2})$, $\varphi(\neg A^x) = \varphi(A^{\neg x}) = f(\neg x) = \neg f(x) = \neg\varphi(A^x)$, thus φ is a homomorphism.

Moreover, for any $y \in M_2$, $\varphi(B/A)(y) = \overline{\sup}\{(B/A)(A^x)|\varphi(A^x) = y\} = \overline{\sup}\{\overline{\sup}\{B(z)|A^z = A^x\}|f(x) = y\} = \overline{\sup}\{B(z)|A^z = A^x, f(x) = y\} = \overline{\sup}\{B(z)|f(z) = f(x), f(x) = y\} = f(B)(y)$, that is $\varphi(B/A) = f(B)$. And so $B/A \cong f(B)$. □

Definition 4.16. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A be a cubic MV-ideal of M_1. A is called an invariant cubic set with respect to f if $f(x_1) = f(x_2)$ implies $A(x_1) = A(x_2)$, for any $x_1, x_2 \in M_1$.*

Proposition 4.17. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras, and A be a cubic MV-ideal of M_1. Then A is invariant with respect to f if and only if $kerf \subseteq A_*$.*

Proof. Suppose that A is invariant with respect to f. For any $x \in kerf$, we have $f(x) = 0 = f(0)$, then $A(x) = A(0)$, hence $x \in A_*$, and so $kerf \subseteq A_*$.

Conversely, assume that $kerf \subseteq A_*$. For any $x_1, x_2 \in M_1$, if $f(x_1) = f(x_2)$, according to the proof of Theorem 4.15 we get that $x_1 \ominus x_2, x_2 \ominus x_1 \in kerf \subseteq A_*$, and so $A(x_1 \ominus x_2) = A(x_2 \ominus x_1) = A(0)$. Noting that A is a cubic MV-ideal of M_1, then $A(x_2) \preceq A(x_1)$ and $A(x_1) \preceq A(x_2)$ by Proposition 3.11 (1). Therefore $A(x_1) = A(x_2)$, and thus A is invariant with respect to f. □

Lemma 4.18. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A be a cubic MV-ideal of M_1 such that A is invariant with respect to f. Then for any $x_1, x_2 \in M_1$, $A^{x_1} = A^{x_2}$ if and only if $f(A)^{f(x_1)} = f(A)^{f(x_2)}$.*

Proof. Suppose that $A^{x_1} = A^{x_2}$, using Lemma 4.1 we obtain that $A(x_1 \ominus x_2) = A(x_2 \ominus x_1) = A(0)$. Since A is a cubic MV-ideal of M_1 and f is an epimorphism, it follows that $f(A)$ is a cubic MV-ideal of M_2 by Proposition 4.12. Then $f(A)(f(x_1) \ominus f(x_2)) = f(A)(f(x_1 \ominus x_2)) = f^{-1}(f(A))(x_1 \ominus x_2) = A(x_1 \ominus x_2) = A(0) = f(A)(0)$. Similarly, we can show that $f(A)(f(x_2) \ominus f(x_1)) = f(A)(0)$, thus $f(A)^{f(x_1)} = f(A)^{f(x_2)}$.

Conversely, let $f(A)^{f(x_1)} = f(A)^{f(x_2)}$. It follows that $f(A)(f(x_1) \ominus f(x_2)) = f(A)(f(x_1 \ominus x_2)) = f(A)(0) = A(0)$. And $A(x_1 \ominus x_2) = f^{-1}(f(A))(x_1 \ominus x_2) = f(A)(f(x_1 \ominus x_2)) = A(0)$, analogously, $A(x_2 \ominus x_1) = A(0)$. Hence $A^{x_1} = A^{x_2}$. □

Theorem 4.19. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A, B be cubic MV-ideals of M_1. Then $B/A \cong f(B)/f(A)$.*

Proof. Define $h : M_1/A \to M_2/f(A)$ by $h(A^x) = f(A)^{f(x)}$ for any $x \in M_1$, it follows from Lemma 4.18, we get that h is an one-to-one function. Note that f is a surjective function, hence h is surjective. For any $x_1, x_2, x \in M_1$,

$h(A^{x_1} \oplus A^{x_2}) = h(A^{x_1 \oplus x_2}) = f(A)^{f(x_1 \oplus x_2)} = f(A)^{f(x_1) \oplus f(x_2)} = f(A)^{f(x_1)} \oplus f(A)^{f(x_2)} = h(A^{x_1}) \oplus h(A^{x_2})$, $h(\neg A^x) = h(A^{\neg x}) = f(A)^{f(\neg x)} = f(A)^{\neg f(x)} = \neg h(A^x)$, thus h is a homomorphism.

Moreover, according to Definition 4.9, Definition 4.11 and Lemma 4.18, for any $x \in M_1$, we obtain that

$$\begin{aligned}
&h^{-1}(f(B)/f(A))(A^x) = (f(B)/f(A))(h(A^x)) = (f(B)/f(A))(f(A)^{f(x)}) \\
&= \overline{\sup}\{f(B)(y) | f(A)^{f(x)} = f(A)^y, y \in M_2\} \\
&= \overline{\sup}\{\overline{\sup}\{B(z) | f(z) = y, z \in M_1\} | f(A)^{f(x)} = f(A)^y, y \in M_2\} \\
&= \overline{\sup}\{B(z) | f(A)^{f(z)} = f(A)^x, z \in M_1\} \\
&= \overline{\sup}\{B(z) | A^z = A^x, z \in M_1\} = (B/A)(A^x),
\end{aligned}$$

that is, $h^{-1}(f(B)/f(A)) = B/A$. Due to the fact that h is isomorphic, we get that $h(B/A) = f(B)/f(A)$, hence $B/A \cong f(B)/f(A)$. □

Corollary 4.20. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A, B be cubic MV-ideals of M_2. Then $f^{-1}(B)/f^{-1}(A) \cong B/A$.*

Lemma 4.21. *Let A, B be cubic MV-ideals of M and $A \sqsubseteq B$. If $A(0) = B(0)$, then $B^x = B^y$ if and only if $(B/A)(A^x) = (B/A)(A^y)$, for any $x, y \in M$.*

Proof. For any $x, y \in M$, if $B^x = B^y$, then $B(x \ominus y) = B(y \ominus x) = B(0)$. And we have $(B/A)(A^{x\ominus y}) = \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} \succeq B(x \ominus y) = B(0) = (B/A)(A^0)$, thus $(B/A)(A^{x\ominus y}) = (B/A)(A^x \ominus A^y) = (B/A)(A^0)$, which means $(B/A)(A^y) \preceq (B/A)(A^x)$ by Proposition 3.11 (1). Similarly, we can prove $(B/A)(A^x) \preceq (B/A)(A^y)$, hence $(B/A)(A^x) = (B/A)(A^y)$.

Conversely, if $(B/A)(A^x) = (B/A)(A^y)$, it follows that $(B/A)(A^{x\ominus y}) = (B/A)(A^x \ominus A^y) = \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} = (B/A)(A^0) = B(0)$ by Proposition 3.11 (1). To prove $B^x = B^y$, we only need to show that $B(x \ominus y) = B(y \ominus x) = B(0)$. For any $z \in M$, if $A^z = A^{x\ominus y}$, then $A(z \ominus (x \ominus y)) = A(0)$ by Lemma 4.1 (1). Note that B is a cubic MV-ideal of M and $A \sqsubseteq B$, from Proposition 3.7 (2) we get that $B(x\ominus y) \succeq B(z) \barwedge B(z\ominus(x\ominus y)) = B(z) \barwedge A(z\ominus(x\ominus y)) = B(z) \barwedge A(0) = B(z) \barwedge B(0) = B(z)$. And so $B(x\ominus y) \succeq \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} = B(0)$, thus $B(x \ominus y) = B(0)$. Similarly, we can show $B(y \ominus x) = B(0)$, hence $B^x = B^y$. □

Theorem 4.22. *Let A, B, C be cubic MV-ideals of M and $A \sqsubseteq B$. If $A(0) = B(0)$, then $(C/A)/(B/A) \cong C/B$.*

Proof. It is easy to prove that $(C/A)/(B/A)$, C/B are cubic MV-ideals of MV-algebras $(M/A)/(B/A)$ and M/B, respectively.

Define $h : (M/A)/(B/A) \to M/B$ by $h((B/A)^{A^x}) = B^x$ for any $x \in M$, from Lemma 4.21 we get that h is an one-to-one function. Obviously, h is surjective. For any $x_1, x_2, x \in M$,

$$\begin{aligned}
&h((B/A)^{A^{x_1}} \oplus (B/A)^{A^{x_2}}) = h((B/A)^{A^{x_1} \oplus A^{x_2}}) \\
&= h((B/A)^{A^{x_1 \oplus x_2}}) = B^{x_1 \oplus x_2} \\
&= B^{x_1} \oplus B^{x_2} = h((B/A)^{A^{x_1}}) \oplus h((B/A)^{A^{x_2}}),
\end{aligned}$$

and $h(\neg(B/A)^{A^x})=h((B/A)^{\neg A^x})=h((B/A)^{A^{\neg x}})=B^{\neg x} = \neg B^x = \neg h((B/A)^{A^x})$, thus h is a homomorphism. Moreover, using Lemma 4.21 we have that: for any $x \in M$, $(C/A)/(B/A)((B/A)^{A^x}) = \overline{\sup}\{(C/A)(A^w)|(B/A)^{A^w} = (B/A)^{A^x}, w \in M\} = \overline{\sup}\{\overline{\sup}\{C(v)|A^v = A^w, v \in M\}|(B/A)^{A^w} = (B/A)^{A^x}, w \in M\} = \overline{\sup}\{C(v)|(B/A)^{A^v} = (B/A)^{A^x}, v \in M\} = \overline{\sup}\{C(v)|B^v = B^x, v \in M\} = (C/B)(B^x)=(C/B)(h((B/A)^{A^x}))=h^{-1}(C/B)((B/A)^{A^x})$, that is, $(C/A)/(B/A) = h^{-1}(C/B)$. Since h is isomorphic, we get that $(C/A)/(B/A) \cong C/B$. □

Theorem 4.23. *Let A, B be cubic MV-ideals of M and $A(0) = B(0)$. If $A \circledast B$ is an inverse isotone mapping, then $A/(A \sqcap B) \sim (A \circledast B)/B$ and $A/(A \sqcap B) \sim (A \circledast B)/A$.*

Proof. Since $A \circledast B$ is an inverse isotone mapping, it follows that $A \circledast B$ is a cubic MV-ideal by Proposition 3.13. It is easy to obtain that $A/(A \sqcap B)$, $(A \circledast B)/B$ are cubic MV-ideals of MV-algebras $M/(A \sqcap B)$ and M/B, respectively.

Define the map $h : M/(A \sqcap B) \to M/B$ by $h((A \sqcap B)^x) = B^x$ for any $x \in M$. For any $x_1, x_2 \in M$, if $(A \sqcap B)^{x_1} = (A \sqcap B)^{x_2}$, from Lemma 4.1 it follows that $(A \sqcap B)(x_1 \ominus x_2) = (A \sqcap B)(x_2 \ominus x_1) = (A \sqcap B)(0) = B(0)$, so we get that $B(x_1 \ominus x_2) = B(x_2 \ominus x_1) = B(0)$, thus $B^{x_1} = B^{x_2}$. hence h is an one-to-one function. Obviously, h is surjective.

Moreover, for any $x_1, x_2, x \in M$, we have $h((A \sqcap B)^{x_1} \oplus (A \sqcap B)^{x_2}) = h((A \sqcap B)^{x_1 \oplus x_2}) = B^{x_1 \oplus x_2} = B^{x_1} \oplus B^{x_2} = h((A \sqcap B)^{x_1}) \oplus h((A \sqcap B)^{x_2})$, and $h(\neg(A \sqcap B)^x) = h((A \sqcap B)^{\neg x}) = B^{\neg x} = \neg B^x = \neg h((A \sqcap B)^x)$, thus h is a homomorphism.

For any $x \in M$,

$$
\begin{aligned}
&((A \circledast B)/B)(B^x) = \overline{\sup}\{A \circledast B)(t)|B^t = B^x\} \\
&= \overline{\sup}\{A(t_1) \barwedge B(t_2)|t = t_1 \oplus t_2, B^t = B^x\} \succeq \overline{\sup}\{A(t) \barwedge B(0)|B^t = B^x\} \\
&= \overline{\sup}\{A(t)|B^t = B^x\}, \\
&h(A/(A \sqcap B))(B^x) = \overline{\sup}\{(A/(A \sqcap B))((A \sqcap B)^z)|h((A \sqcap B)^z) = B^x\} \\
&= \overline{\sup}\{(A/(A \sqcap B))((A \sqcap B)^z)|B^z{=}B^x\}{=}\overline{\sup}\{A(t)|(A \sqcap B)^t{=}(A \sqcap B)^z, B^z{=}B^x\} \\
&\preceq \overline{\sup}\{A(t)|B^t = B^z, B^z = B^x\} = \overline{\sup}\{A(t)|B^t = B^x\}.
\end{aligned}
$$

It follows that $h(A/(A \sqcap B))(B^x) \preceq ((A \circledast B)/B)(B^x)$, hence $h(A/(A \sqcap B)) \sqsubseteq ((A \circledast B)/B)$, and so $A/(A \sqcap B) \sim (A \circledast B)/B$. Similarly, we can prove $A/(A \sqcap B) \sim (A \circledast B)/A$. □

Definition 4.24. *The cubic MV-ideal A of an MV-algebra M has cubic sup-property if for any nonempty subset H of M, there exists $x_0 \in H$ such that $A(x_0) = \sup\{A(x)|x \in H\}$.*

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A NEW VERIFIABLE MULTI-SECRET SHARING SCHEME BASED ON ELLIPTIC CURVES AND PAIRINGS

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Abstract. In 2008, Liu, Huang, Luo and Dai, proposed a (t, n) multi-point sharing scheme by using self-pairings on the elliptic curves. The Liu's scheme is not verifiable, needs a secure channel and also there exists some restrictions in the number of secrets to be shared. In this paper we propose a new verifiable multi-secret sharing scheme which is based on that of Liu. In our scheme, there is no need to a secure channel and also there is no limit on the number of secrets. Furthermore, to identify the cheaters, the combiner can verify the secrets which have been sent by other participants during the reconstruction phase.

Keywords: secret sharing, cryptography, elliptic curves, bilinear maps.

1. Introduction

Secret sharing schemes are important tools used in many cryptographic protocols and security techniques. A secret sharing scheme consists of a dealer, who knows a secret, a set P any of whose elements is called a participant, and a family $\mathcal{A}$ of subsets of P, called an access structure. In such a scheme, the dealer distributes shares among the participants in such a way that any $A \in \mathcal{A}$ is able to recover the prescribed secret by pooling its members shares together, whereas any subset of P not lying in $\mathcal{A}$ knows nothing about the secret. If P is of cardinality n, and $\mathcal{A}$ consists of all subsets of P with at least t elements, then the scheme is referred to as a $(t, n)-$threshold secret sharing scheme. Secret sharing schemes can be used in many different domains such as secure data storage, secure multi-party computational, group key management and secure information communication. Secret sharing schemes for general access structures were proposed by Ito, Saito, and Nishizeki [17] in 1993, and more efficient schemes were introduced in, e.g., [1, 2, 5].

The first (t, n)-threshold secret sharing schemes were independently introduced by Shamir [24] and Blakley [3]. Shamir's scheme is based on the Lagrange interpolating polynomial, while Blakley's scheme is based on linear projective geometry.

A Multi-secret sharing (MSS) scheme, is a scheme in which several secrets are shared among participants and when any predetermined subset of them pool their information, they will be able to reconstruct all the secrets. The first MSS scheme was introduced by He and Dawson [15] in 1994, and was improved in e. g., [7, 6, 12, 13, 14, 16, 21]. MSS scheme can be used in many different domains, for example, launching intercontinental ballistic missiles, authenticating electronic transactions and opening a bank vault.

In 1994, Jackson et al. [18] classified the MSS scheme into the following two categories: the one-time-use schemes and the multi-use schemes. In a one-time-use scheme, the dealer updates the information distributed amongst the participants after reconstructing the secrets, while in a multi-use scheme, every participant only needs to keep one shadow and use it iteratively. Because distributing shadows to the participants is costly and difficult, the implementation of multi-use schemes is much better.

It should be noticed that the early secret sharing schemes were initiated on the assumption that both the dealer and participants are honest. However, it is very often in practice that a dishonest dealer distributes a fake shadow or a malicious participant provides the other ones with some fake shares. Hence, to remedy this pathology, the researchers were stimulated to work out the schemes which have the capability of being verified. Indeed, a so-called verifiable secret sharing (VSS) scheme is one in which all the participants are able to verify each other and, of course, the dealer.

The first VSS scheme was introduced in 1985 by chore et. al. [10]. Thenceforward, Harn [14] proposed a verifiable multi-secret sharing (VMSS) scheme in 1995. That immediately turned out to be of high computational costs. In fact, Harn's verification process needs any participant to solve a variety of equations. Chen's scheme [8], introduced in 1997, was one of the next attempts to improve Harn's scheme that, despite being of a partial success, was still a scheme with rather high computational costs. Finally, Shao and Cao [25] introduced a new efficient VMSS scheme based on YCH [29] and the hardness of discrete logarithm problem.

In 2008 Chen et al. [9] proposed a threshold secret sharing scheme based on bilinear maps. Chen's scheme was a single secret sharing scheme based on the idea of constructing a Vandermonde matrix to change the threshold. Chen's scheme was improved to a multi-secret sharing by Wang et al. [27]. They proposed a verifiable (t, n)-threshold multi-secret sharing scheme based on bilinear maps; it was subject to the restriction that the number of secrets should not exceed the threshold. Afterwards, Eslami et al. [11] modified Wang's scheme and proposed a new one which dispelled the aforementioned restriction.

In 2008 Liu et. al. [22] presented a $(t, n)-$threshold multi-secret sharing scheme by using self-pairing on an elliptic curve. It is worth pointing out that the privilege of the usage of elliptic curves stands on the fact that solving discrete logarithm problem over elliptic curves is usually difficult and far-reaching. In Liu's scheme the number of secrets, m, must be less than or equal to the threshold t. Moreover, his scheme is not a verifiable one and, at the same time, needs a secure channel.

In this paper we propose a verifiable (t, n)-threshold multi-secret sharing scheme based on elliptic curves and bilinear maps. The approach we take here is to try to exhibit a modified version of Liu's scheme which, at the same time, eliminates the need to restricting the number of secrets and does not require any secure channel. Moreover, our scheme benefits from the fact that the combiner (who can be one the participants) is also able to verify the shares pooled in the reconstruction phase.

The rest of this paper is organized as follows: In Section 2, we summarize the elementary notions of elliptic curves and bilinear maps. Review of Liu's scheme will be given in Section 3. Finally, Sections 4 and 5 are devoted to the presentation of our scheme besides analysing it and comparing to some of the schemes known in the literature.

2. Preliminaries

In this section, we will briefly provide the necessary background on elliptic curves and bilinear maps. The reader could consult a standard text book on the subject, e. g. [26, 28].

2.1 Elliptic curve cryptography

The elliptic curve cryptography (ECC) was suggested separately by Neal Koblitz [19] and Victor S. Miller [23] in 1985. It should be pointed out that merely the finite fields $\mathbb{F}_q$ where q is either a prime or $q = 2^n$ for some integer n, were firstly involved. Recently, elliptic curve cryptography has attained much attention as it has many advantages like a short key length and fast computation speed. In this subsection, we will give the definitions and some elementary properties of the elliptic curves. An elliptic curve E over the finite field $\mathbb{F}_q$ is defined by Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{2.1}$$

where $a_1, a_2, a_3, a_4, a_6 \in \mathbb{F}_q$ and its discriminant, $\Delta = -4a_2^3a_6 + a_2^2a_4^2 + 18a_2a_4a_6 - 4a_4^3 - 27a_6^2$, is supposed to be nonzero. If q is not dividable by 2 or 3, then by an appropriate change of variables, Eq. (2.1) can be reformulated as the short

Weierstrass form

$$y^2 = x^3 + Ax + B, \tag{2.2}$$

for $A, B \in \mathbb{F}_q$.

The points on an elliptic curve together with an extra point $\mathcal{O}$, which is called the point at infinity, form a finite abelian group with an addition law. Indeed, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points on E, then set $P + Q = (x_3, y_3)$, where

$$\begin{cases} x_3 = \lambda^2 - x_1 - x_2, \\ y_3 = \lambda(x_1 - x_3) - y_1 \end{cases}$$

and

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1}, & P \neq Q, \\ \dfrac{3x_1^2 + A}{2y_1}, & P = Q. \end{cases}$$

Discrete logarithm problem on elliptic curves

Throughout, we let q be some power of a prime and let the elliptic curve E be defined by Eq. (2.2). Also, let $E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q^2 | y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$. Assume next that $P \in E(\mathbb{F}_q)$ and $Q \in \langle P \rangle$, the subgroup generated by P. The discrete logarithm problem on E is to find the integer k satisfying $Q = kP$. In general, there is no polynomial time algorithm on $\log q$ to solve the discrete logarithm problem on $E(\mathbb{F}_q)$. This is why the cryptographic schemes which are based on elliptic curves have become valuable and interesting.

2.2 Bilinear maps

The definition of a bilinear map lies on the following two well-known problems in cryptography; namely CDHP and DDHP. To state these problems, let G be a cyclic additive group of order a prime number r with P as a generator. The so-called computational Diffie-Hellman problem (CDHP) involves computing abP for a given triple (P, aP, bP) with $a, b \in \mathbb{Z}_r^*$. Also the Decision Diffie-Hellman problem (DDHP) concerns the decision on whether $c = ab$ holds for a given quadruple (P, aP, bP, cP) with $a, b, c \in \mathbb{Z}_r^*$.

We say that a group G is a Gap Diffie-Hellman (GDH) group provided DDHP in G is easy to solve while simultaneously, CDHP in G is hard to treat; see [4] for more information.

Let G_1 be a cyclic additive group of order a prime r, and let G_2 be a cyclic multiplicative group of the same order. We assume that the DDHP in G_1 is easy while it is hard in G_2. Suppose moreover that CDHP in G_1 and DLP G_2 are hard to solve. A bilinear map is a map (or a pairing) $e : G_1 \times G_1 \to G_2$ with the following properties:

1. Bilinearity: $e(aP, bQ) = e(P, Q)^{ab}$ for all $P, Q \in G_1$ and $a, b \in \mathbb{Z}_r^*$.
2. Non-Degeneracy: there exist $P, Q \in G_1$ such that $e(P, Q) \neq 1$.

3. Computability: e is efficiently computable, i.e., there exists a polynomial time algorithm to compute $e(P, Q) \in G_2$, for all $P, Q \in G_1$.

Pairing-based cryptography stands over the idea of using the construction and properties of bilinear maps between two suitable groups. The strategy is to exploit such maps to reduce a problem in one of the groups to a problem in the second one in such a way that, compared to the original one, the resulting problem is usually much more convenient to treat.

It should be emphasized that the existence of G_1 and G_2 with these properties follows from Weil pairing [28, 11.2] and Tate pairing [28, 11.3]. Indeed, one can take for G_1 an elliptic curve and for G_2 the underlying field of G_1.

Let K be a field, $\overline{K}$ its algebraic closure and let $E = E(\overline{K})$ be an elliptic curve over $\overline{K}$. For $l \neq 0$, let $E[l]$ denote the subgroup of $l-$torsion points, which contains all the points P with $lP = \mathcal{O}$. If $char(K) = 0$ or $char(K) = p$ where l is not dividable by p, then $E[l]$ can be represented as a direct sum of two cyclic groups namely, $E[l] \cong Z_l \oplus Z_l$. If this is the case, let $\{G, H\}$ be a generating set for $E[l]$. Then any point in $E[l]$ can be represented as a linear combination of G and H. Consider two points P and Q in $E[l]$, so that we have $P = a_1G + b_1H$ and $Q = a_2G + b_2H$, for integers $a_1, a_2, b_1, b_2 \in [0, l-1]$. We can now define the following pairing map for any two arbitrary integers $\alpha, \beta \in [0, l-1]$:

$$\begin{gathered} e_{\alpha,\beta} : E[l] \times E[l] \to E[l] \\ e_{\alpha,\beta}(P, Q) = (a_1b_2 - a_2b_1)(\alpha G + \beta H) \end{gathered} \tag{2.3}$$

The trivial case when $\alpha = \beta = 0$ has been excluded. The pairing $e_{\alpha,\beta}$ is called self-pairing because it maps $E[l] \times E[l]$ to $E[l]$.

We notice that this is the particular pairing by means of which we will define our proposed scheme in Section 4.

Theorem 2.1 ([20],Prop.3.1)**.** *The pairing $e_{\alpha,\beta}$ has the following properties:*

1. *Identity: for all $P \in E[l]$, $e_{\alpha,\beta}(P, P) = \mathcal{O}$.*
2. *Bilinearity: for all $P, Q, R \in E[l]$, $e_{\alpha,\beta}(P+Q, R) = e_{\alpha,\beta}(P, R)+e_{\alpha,\beta}(Q, R)$ and $e_{\alpha,\beta}(P, Q + R) = e_{\alpha,\beta}(P, Q) + e_{\alpha,\beta}(P, R)$.*
3. *Anti-symmetry: for all $P \in E[l]$, $e_{\alpha,\beta}(P, Q) = -e_{\alpha,\beta}(Q, P)$.*
4. *Non-degeneracy: for all $P \in E[l]$, $e_{\alpha,\beta}(P, \mathcal{O}) = \mathcal{O}$. Moreover, if $e_{\alpha,\beta}(P, Q) = \mathcal{O}$ for all $P \in E[l]$, then $Q = \mathcal{O}$.*

3. A Review of Liu's scheme

In this section, we briefly introduce the scheme posed by Liu et. al. in [22] which is actually a scheme for sharing points on an elliptic curve. This scheme consists of the following phases: initialization, shadow distribution, point sharing and point reconstruction. Below, we provide overview of any of these phases.

Let D be a trusted dealer, and let $U_1, U_2, \dots, U_n$ be honest participants. The dealer wants to distribute secrets $M_1, M_2, \dots, M_m$ between the participants such that any group consisting of at least t participants can reconstruct all the secrets, but no group of less than t participants can do.

3.1 Initialization

In Liu's scheme, the dealer publishes public information on a public bulletin which can be accessed by every participant. The dealer uses the following steps to set up the parameters of the sharing scheme.

1. The dealer chooses an elliptic curve E over the finite field $\mathbb{F}_q$ where $q = p^r$, p being a large enough prime for which the DLP and ECDLP are simultaneously hard in $\mathbb{F}_q^*$ and $E(\mathbb{F}_q)$ respectively. The dealer then chooses a large prime l such that $E[l] \subseteq E(\mathbb{F}_{q^k})$ for some integer k.
2. The dealer D chooses a generating set $\{G, H\}$ of $E[l]$ and two integers $\alpha, \beta \in [1, l-1]$, which determine the pairing $e_{\alpha,\beta}$ as defined before.
3. Finally, the dealer publishes $\{E, q, l, k, \alpha G + \beta H\}$ in the public bulletin.

3.2 Shadow distribution

In this phase, the dealer D uses the following steps to distribute the shadows to the participants. As stated before, this should be done in such a way that any group consisting of at least t participants can reconstruct the shared points, but no group with less than t participants can.

1. D considers the matrix A of size $n \times t$ as

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{t-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n & n^2 & \dots & n^{t-1} \end{bmatrix}$$

2. The dealer randomly chooses t pairs of numbers $a'_i, b'_i \in [1, l-1]$ for $1 \leq i \leq t$.
3. D computes

$$\begin{aligned} (a_1, a_2, \dots, a_n)^T &= A \cdot (a'_1, a'_2, \dots, a'_t)^T, \\ (b_1, b_2, \dots, b_n)^T &= A \cdot (b'_1, b'_2, \dots, b'_t)^T. \end{aligned}$$

4. Finally, D sends (a_j, b_j) to user U_j through a secret channel for all $1 \leq j \leq n$.

3.3 Point sharing

After distributing the shadows, the dealer shares the points among all the participants through the following steps:

1. To share m different points $M_1, \ldots, M_m$, the dealer chooses $c_i, d_i \in [0, l-1]$ randomly and computes $Q_i = c_i G + d_i H$ for all $1 \leq i \leq m$.
2. The dealer computes $R_i = e_{\alpha,\beta}(Q_i, P'_t) + M_i$, for $1 \leq i \leq m$, where $P'_t = a'_t G + b'_t H$.
3. Finally, the dealer publishes $\{c_i, d_i, R_i\}$, for $1 \leq i \leq m$, in the public bulletin.

3.4 Point reconstruction

Without loss of generality, we may assume by a relabeling that the participants $U_1, U_2, \ldots, U_t$ want to reconstruct the secrets $M_1, M_2, \ldots, M_m$. Each participant computes the pseudo share from his secret share and the public information. The reconstructing procedure is as follows:

1. Each U_j downloads the pair of integers $\{c_i, d_i\}$ from the public bulletin board, where $1 \leq j \leq t$.
2. Each U_j computes $Q_{i,j} = e_{\alpha,\beta}(Q_i, P_j)$, where $P_j = a_j G + b_j H$ and $Q_i = c_i G + d_i H$, for $1 \leq i, j \leq t$.
3. Each U_j multicasts the pseudo shadow $Q_{i,j}$ to $U_1, \ldots, U_{j-1}, U_{j+1}, \ldots, U_t$, for $1 \leq j \leq t$.
4. Each participant U_i computes $T_i = \sum_{k=1}^{t} y_k Q_{i,k}$, where $y_k = (\prod_{j=1, j\neq k}^{t}(k-j))^{-1}$.
5. Each participant U_i downloads the point R_i from the public bulletin and recovers $M_i = R_i - T_i$.

4. Proposed scheme

Recall that the MSS scheme proposed by Liu was based on the elliptic curves cryptography, as pointed out earlier in the paper, is addressed to the fact that solving the discrete logarithm problem on the elliptic curves is really a challenging and far-reaching problem to treat and this provides the scheme with a rather high security in comparison to some of the other ones.

Nevertheless, from several points of view, it suffers from some deficiencies; namely it needs a secure channel, it is not verifiable, and also there exists some restrictions on the number of secrets to be shared. So it seems quite reasonable and natural to deal with new technique in order to make Liu's scheme into a more satisfactory one.

Here, we will make use of the elliptic curves and bilinear pairings to propose a new verifiable (t, n)-threshold MSS scheme. The procedure can be divided into four parts: Initialization phase, Point sharing phase, Point distribution phase, Secret reconstruction and Verification phase.

In our proposed scheme, which is based on that of Liu, we will drop the restriction on the number of secrets. Moreover, each participant will be able to identify cheaters in the reconstruction phase by using bilinear maps, and since any participant chooses his/her secret shadow by him/herself, there is no need to a secure channel and the dealer can never distribute a fake shadow.

Let $U_1, U_2, \ldots, U_n$ be the participants involved in the secret sharing process and let $K_1, K_2, \ldots, K_m$ be the secrets to be shared.

4.1 Initialization phase

In the initialization phase, the dealer D publishes some public information on the public bulletin which is accessible by every participant.

1. The Dealer D chooses an elliptic curve E over $\mathbb{F}_q$, $q = p^r$, where p is a large prime such that the DLP and ECDLP are hard respectively in $\mathbb{F}_q^*$ and $E(\mathbb{F}_q)$. The dealer then chooses $E[l]$, a torsion subgroup of a large prime order l. It is well-known that $E[l] \subseteq E(\mathbb{F}_{q^k})$, for some integer k.

2. D chooses a generating pair $\{G, H\} \subseteq E[l]$ and a pair of integers $\alpha, \beta \in [1, l-1]$. The dealer then forms the pairing (2.3) and then computes $W = \alpha G + \beta H$.

3. The dealer chooses a hash function $h : E[l] \to \mathbb{Z}_l^*$, and finally publishes $\{E, p, l, G, H, W, h\}$ on the public bulletin.

4.2 Point sharing phase

In this step, the dealer may use the following steps to distribute the shadows to the participants; this distribution is subject to the prescribed conditions, as pointed out before.

1. The dealer D maps the secrets $K_1, K_2, \ldots, K_m$ to a set of points $M_1, M_2, \ldots,$ M_m on the elliptic curve E.

2. D chooses private numbers $a_0, a_1, b_0, b_1 \in [1, l-1]$ and computes the points $Q_0 = a_0 G + b_0 H$ and $P_i = a_1^i G + b_1^i H$ for $i = 1, \ldots, m$.

3. Corresponding to the secret M_i, the dealer publishes $R_i = e_{\alpha,\beta}(Q_0, P_i) + M_i$ for $i = 1, \ldots, m$.

4.3 Point distribution phase

1. D chooses some $d \in \mathbb{Z}_l^*$ randomly and publishes $G' = dG$.

2. Each participant U_i selects the secret shadow s_i and publishes $G_i = s_i G$.
3. D considers the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (n-t+4) & (n-t+4)^2 & \dots & (n-t+4)^{n+3} \end{bmatrix}. \tag{4.1}$$

4. The dealer computes ds_iG, and constructs the column vector $X = [h(ds_1G), h(ds_2G), \dots, h(ds_nG), a_0, b_0, a_1, b_1]^T$.
5. D publishes $[I_1, I_2, \dots, I_{n-t+4}]$ where

$$A \times X = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (n-t+4) & (n-t+4)^2 & \dots & (n-t+4)^{n+3} \end{bmatrix} \begin{bmatrix} h(ds_1G) \\ \vdots \\ h(ds_nG) \\ a_0 \\ b_0 \\ a_1 \\ b_1 \end{bmatrix}$$

$$= \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{n-t+4} \end{bmatrix}. \tag{4.2}$$

4.4 Secret reconstruction and verification phase

It is clear that Eq. (4.2) is a system of $(n-t+4)$ linear equations with $(n+4)$ unknowns. To reconstruct all the secrets, we need to know the values of a_0, a_1, b_0 and b_1. Suppose that t-out-of-n participants $U_1, \dots, U_t$ intend to reconstruct all the secrets. To this goal, assume U_i computes ds_iG for $i = 1, \dots, t$. Firstly, the combiner ensures verifiability of the shares by using the bilinearity property of the pairing maps and checks if $e_{\alpha,\beta}(ds_iG, G) = e_{\alpha,\beta}(dG, s_iG)$. Then, the combiner uses the public hash function h to compute $h(ds_iG)$, for $i = 1, \dots, t$ and generates the i-th row of the unknown's matrix. Therefore, t unknowns of Eq. (4.2) are computed and the combiner may now solve a system of $(n-t+4)$ equations and $(n-t+4)$ unknowns to reconstruct a_0, a_1, b_0 and b_1. Finally, the secrets can be obtained by putting $M_i = R_i - e_{\alpha,\beta}(Q_0, P_i)$, for $i = 1, \dots, m$.

5. Security analysis phase

The security analysis of the proposed scheme goes ahead through the following lines.

Theorem 5.1. *Any t or more participants are able to reconstruct all the secrets.*

Proof. Without loss of generality, we suppose that $U_1, U_2, \ldots, U_t$ share their secret shadows ds_iG for $i = 1, \ldots, t$. Then, Eq. (4.2) converts to a system of $n-t+4$ equations and $n-t+4$ unknowns with the invertible coefficients matrix

$$A' = \begin{bmatrix} 1 & \ldots & 1 \\ 2^t & \ldots & 2^{n+3} \\ \vdots & & \vdots \\ (n-t+4)^t & \ldots & (n-t+4)^{n+3} \end{bmatrix}. \tag{5.1}$$

(Indeed, the determinant of A' might be calculated via $det(A') = 2^t \times \cdots \times (n-t+4)^t \times det(A'')$, for some Vandermonde matrix A''.) Hence, the participants obtain the secrets by computing the inverse matrix of A'. □

Theorem 5.2. *Any group of less than t participants cannot compute any of the secrets.*

Proof. Suppose, to the contrary, that this is the case. Then Eq. (4.2) reduces to a system of $n-t+4$ equations and more than $n-t+4$ unknowns which has certainly an infinite set of solutions. □

The following theorem ensures that using a secure channel in order to share the secrets is in fact not mandatory.

Theorem 5.3. *The proposed scheme does not require a secure channel.*

Proof. We must make sure that no participant's shadow s_i might be grasped from s_iG. In fact, if an attacker wants to compute s_i from s_iG, he/she must solve a discrete logarithm problem in the elliptic curve E, which is hard according to our assumptions. □

Theorem 5.4. *The dealers private key d cannot be obtained from the public information dG.*

Proof. The theorem follows from the argument provided in Theorem 5.3. □

Finally, Theorem 5.5 below illustrates the verifiability of the proposed scheme.

Theorem 5.5. *The shares provided by the participants in the reconstruction phase can be verified.*

Proof. Suppose that the participant U_i provides ds_iG. During the reconstruction phase, this share can be verified, because as mentioned before, given s_iG and dG, it is infeasible to compute ds_iG in E; this follows from the hardness of the Diffie-Hellman problem. Therefore, only the dealer and the participant U_i are able to compute this value. By using the bilinearity property of the pairing maps, the combiner can check whether $e_{\alpha,\beta}(ds_iG, G) = e_{\alpha,\beta}(dG, s_iG)$ holds. If the verification passes, he/she accepts ds_iG. □

Table 1 provides a comparison between the proposed scheme in this paper and some other one's that are based on the techniques of elliptic curves and pairings.

Table 1: Comparison of some schemes based on elliptic curve and pairing

Scheme	Chen [9]	Liu [22]	Wang [27]	Proposed
Multi-secret	No	Yes	Yes	Yes
Number of secrets	1	t	t	unrestricted
Public parameters	$2n-t+9$	$3m+5$	$2n+7$	$2n+m-t+11$
Verifiability	Yes	No	Yes	Yes
Cheater detection	Yes	No	No	Yes
Cheater identification	Yes	No	No	Yes
Need a secure channel	No	Yes	No	No

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TRANSLATION AND DENSITY OF A BIPOLAR-VALUED FUZZY SET IN UP-ALGEBRAS

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Abstract. We apply the notion of bipolar fuzzy translations of a bipolar-valued fuzzy set to UP-algebras. For any bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A, the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II are introduced, their basic properties are investigated and some useful examples are discussed. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. Moreover, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.
Keywords: UP-algebra, bipolar fuzzy translation, bipolar fuzzy UP-subalgebra, bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [3], K-algebras [1], KU-algebras [16], SU-algebras [12], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of fuzzy sets of a set was first considered by Zadeh [22] in 1965. The fuzzy set theories developed by Zadeh and others have found many appli-

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cations in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc. The notion of bipolar-valued fuzzy sets was first introduced by Lee [14] in 2000, is an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 0]$. After the introduction of the notion of bipolar-valued fuzzy sets by Lee [14], several researches were conducted on the generalizations of the notion of bipolar-valued fuzzy sets and application to many logical algebras such as: In 2008, Jun and Song [10] introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals in BCH-algebras. In 2009, Jun and Park [9] introduced the notions of bipolar fuzzy regularities, bipolar fuzzy regular subalgebras, bipolar fuzzy filters, and bipolar fuzzy closed quasi filters in BCH-algebras. In 2011, Lee and Jun [13] introduced the notion of bipolar fuzzy a-ideals of BCI-algebras. In 2012, Jun et al. [8] introduced the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras. In 2014, Muhiuddin [15] introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. In 2015, Senapati [20] introduced the notion of bipolar fuzzy BG-subalgebras in BG-algebras. In 2016, Sabarinathan et al. [17] introduced the notion of bipolar valued fuzzy ideals of BF-algebras. In 2017, Sabarinathan et al. [18] introduced the notion of bipolar valued fuzzy H-ideals of BF-algebras.

Moreover, bipolar-valued fuzzy sets were extended to bipolar fuzzy translations in many algebras such as: In 2009, Jun et al. [7] introduced the notions of bipolar fuzzy translations and bipolar fuzzy S-extensions of a bipolar fuzzy subalgebra in BCK/BCI-algebras. In 2012, Sardar et al. [19] introduced the notions of bipolar valued fuzzy translations and bipolar valued fuzzy S-extensions of a bipolar valued fuzzy subsemigroup (bi-ideal) in semigroups.

In this paper, we apply the notion of bipolar fuzzy translations of a bipolar-valued fuzzy set to UP-algebras. For any bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A, the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II are introduced, their basic properties are investigated and some useful examples are discussed. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. Moreover, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.

2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [4] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1 ([4])**.** Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.2 ([4])**.** *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.3 ([4])**.** A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [4] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.4 ([4, 21])**.** A subset S of A is called

(1) a *UP-filter* of A if

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y \in A, x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(2) a *UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(3) a *strongly UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [2] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Bipolar fuzzy (α, β)-translations in UP-algebras

Let X be the universe of discourse. A *bipolar-valued fuzzy set* [13] φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

We recall the definitions of bipolar fuzzy UP-subalgebras, bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals, and bipolar fuzzy strongly UP-ideals.

Definition 3.1 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$, and

(2) $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$.

Definition 3.2 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-filter* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$, and

(4) $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$.

Definition 3.3 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$, and

(4) $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$.

Definition 3.4 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy strongly UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$, and

(4) $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$.

Kawila et al. [11] proved the generalization that the notion of bipolar UP-subalgebras is a generalization of bipolar UP-filters, the notion of bipolar UP-filters is a generalization of bipolar UP-ideals, and the notion of bipolar UP-ideals is a generalization of bipolar strongly UP-ideals. Moreover, they also proved that a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is constant if and only if it is a bipolar fuzzy strongly UP-ideal of A.

3.1 Bipolar fuzzy (α, β)-translations of a bipolar fuzzy set of type I

Definition 3.5. The inclusion "$\subseteq$" is defined by setting, for any bipolar fuzzy sets $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ in A,

$$\varphi \subseteq \psi \Leftrightarrow \varphi^-(x) \geq \psi^-(x) \text{ and } \varphi^+(x) \leq \psi^+(x) \text{ for all } x \in A.$$

We say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy extension* of $\varphi = (A; \varphi^-, \varphi^+)$, and $\varphi = (A; \varphi^-, \varphi^+)$ is a *bipolar fuzzy intension* of $\psi = (A; \psi^-, \psi^+)$.

Definition 3.6. For any bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A, we denote

$$\begin{aligned} \bot &:= -1 - \inf\{\varphi^-(x) \mid x \in A\}, \\ \top &:= 1 - \sup\{\varphi^+(x) \mid x \in A\}. \end{aligned}$$

Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. By a *bipolar fuzzy* (α, β)*-translation of* $\varphi = (A; \varphi^-, \varphi^+)$ *of type I*, we mean a bipolar fuzzy set ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ where

$$\begin{aligned} {\varphi^-}_{(\alpha,\mathrm{T}_1)} &: A \to [-1, 0], x \mapsto \varphi^-(x) + \alpha, \\ {\varphi^+}_{(\beta,\mathrm{T}_1)} &: A \to [0, 1], x \mapsto \varphi^+(x) + \beta. \end{aligned}$$

Theorem 3.7. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-subalgebra of A, then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. For any $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and for all $x, y \in A$, we have

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot y) &= \varphi^-(x \cdot y) + \alpha \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y) &= \varphi^+(x \cdot y) + \beta \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\} + \beta \\ &= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.8. *If there exists $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-subalgebra of A for $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and for all $x, y \in A$, we have

$$\begin{aligned}\varphi^-(x \cdot y) + \alpha &= \varphi^-_{(\alpha,T_1)}(x \cdot y) \\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\} \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-(x), \varphi^-(y)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x \cdot y) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y) \\ &\geq \{\varphi^+_{(\beta,\mathrm{T}_1)}(x), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\} \\ &= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\ &= \min\{\varphi^+(x), \varphi^+(y)\} + \beta.\end{aligned}$$

Thus $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$ and $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.9. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-filter of A then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. For any $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_1)}(0) = \varphi^+(0) + \beta \geq \varphi^+(x) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(x).$$

Next, let $x, y \in A$. Then $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$ and $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_1)}(y) &= \varphi^-(y) + \alpha \\
&\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} + \alpha \\
&= \max\{\varphi^-(x \cdot y) + \alpha, \varphi^-(x)) + \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\}
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+_{(\beta,\mathrm{T}_1)}(y) &= \varphi^+(y) + \beta \\
&\geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} + \beta \\
&= \min\{\varphi^+(x \cdot y) + \beta, \varphi^+(x) + \beta\} \\
&= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y), \varphi^+_{(\beta,\mathrm{T}_1)}(x)\}.
\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-filter of A. $\square$

Theorem 3.10. *If there exists $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-filter of A for $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then

$$\varphi^-(0) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha$$

and

$$\varphi^+(0) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y \in A$. Then

$$\begin{aligned}\varphi^-(y) + \alpha &= \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x\cdot y), \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\}\\ &= \max\{\varphi^-(x\cdot y)+\alpha, \varphi^-(x)+\alpha\}\\ &= \max\{\varphi^-(x\cdot y), \varphi^-(x)\}+\alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^-(y) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(y)\\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x\cdot y), \varphi^+_{(\beta,\mathrm{T}_1)}(x)\}\\ &= \min\{\varphi^-(x\cdot y)+\beta, \varphi^-(x)+\beta\}\\ &= \min\{\varphi^-(x\cdot y), \varphi^-(x)\}+\beta.\end{aligned}$$

Thus $\varphi^-(y) \leq \max\{\varphi^-(x\cdot y), \varphi^-(x)\}$ and $\varphi^-(y) \geq \min\{\varphi^-(x\cdot y), \varphi^-(x)\}$. Hence, $\varphi^- = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.11. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-ideal of A, then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. For any $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_1)}(0) = \varphi^+(0) + \beta \geq \varphi^-(x) + \beta = \varphi^-_{(\beta,\mathrm{T}_1)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-(x\cdot z) \leq \max\{\varphi^-(x\cdot(y\cdot z)), \varphi^-(y)\}$ and $\varphi^+(x\cdot z) \geq \min\{\varphi^+(x\cdot(y\cdot z)), \varphi^+(y)\}$. Thus

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_1)}(x\cdot z) &= \varphi^-(x\cdot z)+\alpha\\ &\leq \max\{\varphi^-(x\cdot(y\cdot z)), \varphi^-(y)\}+\alpha\\ &= \max\{\varphi^-(x\cdot(y\cdot z))+\alpha, \varphi^-(y)+\alpha\}\\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x\cdot(y\cdot z)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_1)}(x\cdot z) &= \varphi^+(x\cdot z)+\beta\\ &\geq \min\{\varphi^+(x\cdot(y\cdot z)), \varphi^+(y)\}+\beta\\ &= \min\{\varphi^+(x\cdot(y\cdot z))+\beta, \varphi^+(y)+\beta\}\\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x\cdot(y\cdot z)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}.\end{aligned}$$

Hence, $\varphi^{T_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,T_1)}, \varphi^+_{(\beta,T_1)})$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.12. *If there exists $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{T_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,T_1)}, {\varphi^+}_{(\beta,T_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi^{T_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,T_1)}, \varphi^+_{(\beta,T_1)})$ is a bipolar fuzzy UP-ideal of A for $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and let $x \in A$. Then

$$\varphi^-(0) + \alpha = \varphi^-_{(\alpha,T_1)}(0) \leq \varphi^-_{(\alpha,T_1)}(x) = \varphi^-(x) + \alpha$$

and

$$\varphi^+(0) + \beta = \varphi^+_{(\beta,T_1)}(0) \geq \varphi^+_{(\beta,T_1)}(x) = \varphi^+(x) + \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x \cdot z) + \alpha &= \varphi^-_{(\alpha,T_1)}(x \cdot z) \\
&\leq \max\{\varphi^-_{(\alpha,T_1)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,T_1)}(y)\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} + \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x \cdot z) + \beta &= \varphi^+_{(\beta,T_1)}(x \cdot z) \\
&\geq \min\{\varphi^+_{(\beta,T_1)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,T_1)}(y)\} \\
&= \min\{\varphi^+(x \cdot (y \cdot z)) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} + \beta.
\end{aligned}$$

Thus $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$ and $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.13. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy strongly UP-ideal of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{T_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,T_1)}, {\varphi^+}_{(\beta,T_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. For any $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,T_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,T_1)}(x)$$

and

$$\varphi^+_{(\beta,T_1)}(0) = \varphi^+(0) + \beta \geq \varphi^+(x) + \beta = \varphi^+_{(\beta,T_1)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-(x) \leq \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. Thus

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_1)}(x) &= \varphi^-(x)+\alpha\\ &\leq \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}+\alpha\\ &= \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)))+\alpha, \varphi^-(y)+\alpha\}\\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}((z\cdot y)\cdot(z\cdot x)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_1)}(x) &= \varphi^+(x)+\beta\\ &\geq \min\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}+\beta\\ &= \min\{\varphi^+((z\cdot y)\cdot(z\cdot x))+\beta, \varphi^+(y)+\beta\}\\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}((z\cdot y)\cdot(z\cdot x)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy strongly UP-ideal of A. □

Theorem 3.14. *If there exists $(\alpha,\beta) \in [\bot, 0]\times[0,\top]$ such that the bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A, then $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy strongly UP-ideal of A for $(\alpha,\beta) \in [\bot, 0]\times[0,\top]$ and let $x \in A$. Then

$$\varphi^-(0)+\alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x)+\alpha$$

and

$$\varphi^+(0)+\beta = \varphi^+_{(\beta,\mathrm{T}_1)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x)+\beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}\varphi^-(x)+\alpha &= \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}((z\cdot y)\cdot(z\cdot x)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}\\ &= \max\{\varphi^-((z\cdot y)\cdot(z\cdot x))+\alpha, \varphi^-(y)+\alpha\}\\ &= \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}+\alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x)+\beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(x)\\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_1)}((z\cdot y)\cdot(z\cdot x)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}\\ &= \min\{\varphi^+((z\cdot y)\cdot(z\cdot x))+\beta, \varphi^+(y)+\beta\}\\ &= \min\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}+\beta.\end{aligned}$$

Thus $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. $\square$

Remark 3.15. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy set in A, then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, ${\varphi^-}_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha \leq \varphi^-(x)$ and ${\varphi^+}_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta \geq \varphi^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy extension of $\varphi = (A; \varphi^-, \varphi^+)$ for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$.

Lemma 3.16. *Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A. If $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \psi$ for all $(\alpha_1, \beta_1) \in [\perp, 0] \times [0, \top]$, then there exists $(\alpha_2, \beta_2) \in [\perp, 0] \times [0, \top]$ with $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, that is, $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$ such that $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \varphi^{\mathrm{T}_1}_{(\alpha_2,\beta_2)} \subseteq \psi$.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \psi$ for all $(\alpha_1, \beta_1) \in [\perp, 0] \times [0, \top]$. Then $\psi^-(x) \leq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)$ and $\psi^+(x) \geq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Put $\alpha_2 = \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\}$. Then

$$\begin{aligned}
\sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\} &= \sup\{\psi^-(x) - (\varphi^-(x) + \alpha_1)\} \\
&\geq \sup\{-1 - (\varphi^-(x) + \alpha_1)\} \\
&= -1 + \sup\{-\varphi^-(x) - \alpha_1\} \\
&= -1 + \sup\{-\varphi^-(x)\} - \alpha_1 \\
&= -1 - \inf\{\varphi^-(x)\} - \alpha_1 \\
&= \perp - \alpha_1,
\end{aligned}$$

so $\alpha_2 = \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\} \geq \alpha_1 + \perp - \alpha_1 = \perp$. Thus $\alpha_2 \in [\perp, 0]$ and $\alpha_2 \leq \alpha_1$, so $\varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) \leq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Now for all $x \in A$, we have

$$\begin{aligned}
\varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) &= \varphi^-(x) + \alpha_2 \\
&= \varphi^-(x) + \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\} \\
&\geq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) + \psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) \\
&= \psi^-(x).
\end{aligned}$$

Thus $\varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) \geq \varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) \geq \psi^-(x)$ for all $x \in A$. Put $\beta_2 = \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\}$. Then

$$\begin{aligned}
\inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} &= \inf\{\psi^+(x) - (\varphi^+(x) + \beta_1)\} \\
&\leq \inf\{1 - (\varphi^+(x) + \beta_1)\} \\
&= 1 + \inf\{-\varphi^+(x) - \beta_1\} \\
&= 1 + \inf\{-\varphi^+(x)\} - \beta_1 \\
&= -1 - \sup\{\varphi^+(x)\} - \beta_1 \\
&= \top - \beta_1,
\end{aligned}$$

so $\beta_2 = \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} \leq \beta_1 + \top - \beta_1 = \top$. Thus $\beta_2 \in [0, \top]$ and $\beta_2 \geq \beta_1$, so $\varphi^+_{(\beta_2,\mathrm{T}_1)}(x) \geq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Now for all $x \in A$, we have

$$\begin{aligned}
\varphi^+_{(\beta_2,\mathrm{T}_1)}(x) &= \varphi^+(x) + \beta_2 \\
&= \varphi^+(x) + \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} \\
&\leq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x) + \psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x) \\
&= \psi^+(x).
\end{aligned}$$

Thus $\varphi^+_{(\beta_1,\mathrm{T}_1)}(x) \leq \varphi^+_{(\beta_2,\mathrm{T}_1)}(x) \leq \psi^+(x)$ for all $x \in A$. Hence, $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \varphi^{\mathrm{T}_1}_{(\alpha_2,\beta_2)} \subseteq \psi$. □

Definition 3.17. Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A with $\varphi \subseteq \psi$. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, and we say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy UP-subalgebra* (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) *extension* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 3.18. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13) and Remark 3.15. □

Theorem 3.19. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α_1, β_1)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha_1,\beta_1)} = (A; {\varphi^-}_{(\alpha_1,\mathrm{T}_1)}, {\varphi^+}_{(\beta_1,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of the bipolar fuzzy (α_2, β_2)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha_2,\beta_2)} = (A; {\varphi^-}_{(\alpha_2,\mathrm{T}_1)}, {\varphi^+}_{(\beta_2,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ with $(\alpha_1, \beta_1) \geq (\alpha_2, \beta_2)$.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13). □

Theorem 3.20. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A. For every bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar* *fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension* $\psi = (A; \psi^-, \psi^+)$ *of the bipolar fuzzy* (α, β)*-translation* $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$ *there exists* $(k^-, k^+) \in [\perp, 0] \times [0, \top]$ *such that* $(k^-, k^+) \geq (\alpha, \beta)$*, that is,* $k^- \leq \alpha$ *and* $k^+ \geq \beta$*, and* $\psi = (A; \psi^-, \psi^+)$ *is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of bipolar fuzzy* (k^-, k^+)*-translation* $\varphi^{\mathrm{T}_1}_{(k^-,k^+)} = (A; \varphi^-_{(k^-,\mathrm{T}_1)}, \varphi^+_{(k^+,\mathrm{T}_1)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$*.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13) and Lemma 3.16. $\square$

3.2 Bipolar fuzzy (α, β)-translations of a bipolar fuzzy set of type II

Definition 3.21. For any bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A, we denote

$$\begin{aligned} \pm &:= \sup\{\varphi^-(x) \mid x \in A\}, \\ \mp &:= \inf\{\varphi^+(x) \mid x \in A\}. \end{aligned}$$

Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. By a *bipolar fuzzy* (α, β)*-translation of* $\varphi = (A; \varphi^-, \varphi^+)$ *of type II*, we mean a bipolar fuzzy set ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ where

$$\begin{aligned} {\varphi^-}_{(\alpha,\mathrm{T}_2)} &: A \to [-1, 0], x \mapsto \varphi^-(x) - \alpha, \\ {\varphi^+}_{(\beta,\mathrm{T}_2)} &: A \to [0, 1], x \mapsto \varphi^+(x) - \beta. \end{aligned}$$

Theorem 3.22. *If a bipolar fuzzy set* $\varphi = (A; \varphi^-, \varphi^+)$ *in* A *is a bipolar fuzzy UP-subalgebra of* A*, then for all* $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$*, a bipolar fuzzy* (α, β)*-translation* ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$ *is a bipolar fuzzy UP-subalgebra of* A*.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and for all $x, y \in A$, we have

$$\begin{aligned} \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y) &= \varphi^-(x \cdot y) - \alpha \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} - \alpha \\ &= \max\{\varphi^-(x) - \alpha, \varphi^-(y) - \alpha\} \\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y) &= \varphi^+(x \cdot y) - \beta \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\} - \beta \\ &= \min\{\varphi^+(x) - \beta, \varphi^+(y) - \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}. \end{aligned}$$

Hence, ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-subalgebra of A. $\square$

Theorem 3.23. *If there exists $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ such that the bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (\varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-subalgebra of A for $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$. Then for all $x, y \in A$, we have

$$\begin{aligned}
\varphi^-(x \cdot y) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \\
&= \max\{\varphi^-(x) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-(x), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x \cdot y) - \beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y) \\
&\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\} \\
&= \min\{\varphi^+(x) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+(x), \varphi^+(y)\} - \beta.
\end{aligned}$$

Thus $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$ and $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$. Hence, $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.24. *If a bipolar fuzzy set $\varphi = (A;\varphi^-,\varphi^+)$ in A is a bipolar fuzzy UP-filter of A, then for all $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$, a bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi = (A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-filter of A. For any $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0) = \varphi^-(0) - \alpha \leq \varphi^-(x) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0) = \varphi^+(0) - \beta \geq \varphi^+(x) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x, y \in A$. Then $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$ and $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_2)}(y) &= \varphi^-(y) - \alpha \\
&\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} - \alpha \\
&= \max\{\varphi^-(x \cdot y) - \alpha, \varphi^-(x) - \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_2)}(x)\}
\end{aligned}$$

and

$$\begin{aligned}\varphi^{+}_{(\beta,\mathrm{T}_2)}(y) &= \varphi^{+}(y) - \beta \\ &\geq \min\{\varphi^{+}(x\cdot y), \varphi^{+}(x)\} - \beta \\ &= \min\{\varphi^{+}(x\cdot y) - \beta, \varphi^{+}(x) - \beta\} \\ &= \min\{\varphi^{+}_{(\beta,\mathrm{T}_2)}(x\cdot y), \varphi^{+}_{(\beta,\mathrm{T}_2)}(x)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^{-}_{(\alpha,\mathrm{T}_2)}, \varphi^{+}_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.25. *If there exists $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ such that the bipolar fuzzy (α,β)-translation $\varphi^{\mathrm{T}_2}{}_{(\alpha,\beta)} = (A; \varphi^{-}{}_{(\alpha,\mathrm{T}_2)}, \varphi^{+}{}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^{-}, \varphi^{+})$ is a bipolar fuzzy UP-filter of A, then $\varphi = (A; \varphi^{-}, \varphi^{+})$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^{-}_{(\alpha,\mathrm{T}_2)}, \varphi^{+}_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-filter of A for $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ and let $x \in A$. Then

$$\varphi^{-}(0) - \alpha = \varphi^{-}_{(\alpha,\mathrm{T}_2)}(0) \leq \varphi^{-}_{(\alpha,\mathrm{T}_2)}(x) = \varphi^{-}(x) - \alpha$$

and

$$\varphi^{+}(0) - \beta = \varphi^{+}_{(\beta,\mathrm{T}_2)}(0) \geq \varphi^{+}_{(\beta,\mathrm{T}_2)}(x) = \varphi^{+}(x) - \beta.$$

Thus $\varphi^{-}(0) \leq \varphi^{-}(x)$ and $\varphi^{+}(0) \geq \varphi^{+}(x)$. Next, let $x, y \in A$. Then

$$\begin{aligned}\varphi^{-}(y) - \alpha &= \varphi^{-}_{(\alpha,\mathrm{T}_2)}(y) \\ &\leq \max\{\varphi^{-}_{(\alpha,\mathrm{T}_2)}(x\cdot y), \varphi^{-}_{(\alpha,\mathrm{T}_2)}(x)\} \\ &= \max\{\varphi^{-}(x\cdot y) - \alpha, \varphi^{-}(x) - \alpha\} \\ &= \max\{\varphi^{-}(x\cdot y), \varphi^{-}(x)\} - \alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^{+}(y) - \beta &= \varphi^{+}_{(\beta,\mathrm{T}_2)}(y) \\ &\geq \min\{\varphi^{+}_{(\beta,\mathrm{T}_2)}(x\cdot y), \varphi^{+}_{(\beta,\mathrm{T}_2)}(x)\} \\ &= \min\{\varphi^{+}(x\cdot y) - \beta, \varphi^{+}(x) - \beta\} \\ &= \min\{\varphi^{+}(x\cdot y), \varphi^{+}(x)\} - \beta.\end{aligned}$$

Thus $\varphi^{-}(y) \leq \max\{\varphi^{-}(x), \varphi^{-}(x\cdot y)\}$ and $\varphi^{+}(y) \geq \min\{\varphi^{+}(x), \varphi^{+}(x\cdot y)\}$. Hence, $\varphi = (A; \varphi^{-}, \varphi^{+})$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.26. *If a bipolar fuzzy set $\varphi = (A; \varphi^{-}, \varphi^{+})$ in A is a bipolar fuzzy UP-ideal of A, then for all $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$, a bipolar fuzzy (α,β)-translation $\varphi^{\mathrm{T}_2}{}_{(\alpha,\beta)} = (A; \varphi^{-}{}_{(\alpha,\mathrm{T}_2)}, \varphi^{+}{}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^{-}, \varphi^{+})$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0) = \varphi^-(0) - \alpha \leq \varphi^-(x) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0) = \varphi^+(0) - \beta \geq \varphi^+(x) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) \leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}$ and $\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot z) \geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) &= \varphi^-(x \cdot z) - \alpha \\
&\leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} - \alpha \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot z) &= \varphi^+(x \cdot z) - \beta \\
&\geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} - \beta \\
&= \min\{\varphi^+(x \cdot (y \cdot z)) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}.
\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.27. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-ideal of A for $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then

$$\varphi^-(0) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_2)}(x) = \varphi^-(x) - \alpha$$

and

$$\varphi^+(0) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_2)}(x) = \varphi^+(x) - \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x \cdot z) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x\cdot z)-\beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(x\cdot z)\\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x\cdot(y\cdot z)),\varphi^+_{(\beta,\mathrm{T}_2)}(y)\}\\ &= \min\{\varphi^+(x\cdot(y\cdot z))-\beta,\varphi^+(y)-\beta\}\\ &= \min\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}-\beta.\end{aligned}$$

Thus $\varphi^-(x\cdot z)\leq \max\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$ and $\varphi^+(x\cdot z)\geq \min\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Hence, $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-ideal of A. □

Theorem 3.28. *If a bipolar fuzzy set $\varphi=(A;\varphi^-,\varphi^+)$ in A is a bipolar fuzzy strongly UP-ideal of A, then for all $(\alpha,\beta)\in[\pm,0]\times[0,\mp]$, a bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)}=(A;{\varphi^-}_{(\alpha,\mathrm{T}_2)},{\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. For any $(\alpha,\beta)\in[\pm,0]\times[0,\mp]$ and let $x\in A$. Then $\varphi^-(0)\leq\varphi^-(x)$ and $\varphi^+(0)\geq\varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0)=\varphi^-(0)-\alpha\leq\varphi^-(x)-\alpha=\varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0)=\varphi^+(0)-\beta\geq\varphi^+(x)-\beta=\varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x,y,z\in A$. Then $\varphi^-(x)\leq\max\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$ and $\varphi^+(x)\geq\min\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}$. Thus

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_2)}(x) &= \varphi^-(x)-\alpha\\ &\leq \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}-\alpha\\ &= \max\{\varphi^-((z\cdot y)\cdot(z\cdot x))-\alpha,\varphi^-(y)-\alpha\}\\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}((z\cdot y)\cdot(z\cdot x)),\varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_2)}(x) &= \varphi^+(x)-\beta\\ &\geq \min\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}-\beta\\ &= \min\{\varphi^+((z\cdot y)\cdot(z\cdot x))-\beta,\varphi^+(y)-\beta\}\\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}((z\cdot y)\cdot(z\cdot x)),\varphi^+_{(\beta,\mathrm{T}_2)}(y)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)}=(A;\varphi^-_{(\alpha,\mathrm{T}_2)},\varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy strongly UP-ideal of A. □

Theorem 3.29. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{T_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,T_2)}, {\varphi^+}_{(\beta,T_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that ${\varphi^{T_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,T_2)}, {\varphi^+}_{(\beta,T_2)})$ is a bipolar fuzzy strongly UP-ideal of A for $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then

$$\varphi^-(0) - \alpha = \varphi^-_{(\alpha,T_2)}(0) \leq \varphi^-_{(\alpha,T_2)}(x) = \varphi^-(x) - \alpha$$

and

$$\varphi^+(0) - \beta = \varphi^+_{(\beta,T_2)}(0) \geq \varphi^+_{(\beta,T_2)}(x) = \varphi^+(x) - \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x) - \alpha &= \varphi^-_{(\alpha,T_2)}(x) \\
&\leq \max\{\varphi^-_{(\alpha,T_2)}((z \cdot y) \cdot (z \cdot x)), \varphi^-_{(\alpha,T_2)}(y)\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x) - \beta &= \varphi^+_{(\beta,T_2)}(x) \\
&\geq \min\{\varphi^+_{(\beta,T_2)}((z \cdot y) \cdot (z \cdot x)), \varphi^+_{(\beta,T_2)}(y)\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} - \beta.
\end{aligned}$$

Thus $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. □

Remark 3.30. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy set in A and $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, then ${\varphi^-}_{(\alpha,T_2)}(x) = \varphi^-(x) - \alpha \geq \varphi^-(x)$ and ${\varphi^+}_{(\beta,T_2)}(x) = \varphi^+(x) - \beta \leq \beta^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α, β)-translation ${\varphi^{T_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,T_2)}, {\varphi^+}_{(\beta,T_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy intension of $\varphi = (A; \varphi^-, \varphi^+)$ for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$.

Lemma 3.31. *Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A. If $\psi \subseteq \varphi^{T_2}_{(\alpha_1,\beta_1)}$ for $(\alpha_1, \beta_1) \in [\pm, 0] \times [0, \mp]$, then there exists $(\alpha_2, \beta_2) \in [\pm, 0] \times [0, \mp]$ with $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ such that $\psi \subseteq \varphi^{T_2}_{(\alpha_2,\beta_2)} \subseteq \varphi^{T_2}_{(\alpha_1,\beta_1)}$.*

Proof. Assume that $\psi \subseteq \varphi^{T_2}_{(\alpha_1,\beta_1)}$ for $(\alpha_1, \beta_1) \in [\pm, 0] \times [0, \mp]$. Then $\varphi^-_{(\alpha_1,T_1)}(x) \leq \psi^-(x)$ and $\varphi^+_{(\beta_1,T_1)}(x) \geq \psi^+(x)$ for all $x \in A$. Put $\alpha_2 = \alpha_1 + \sup\{\varphi^-_{(\alpha_1,T_2)}(x) -$

$\psi^-(x)\}$. Then

$$\begin{aligned}\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\} &\geq \sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\}\\ &= \sup\{\varphi^-(x)-\alpha_1\}\\ &= \sup\{\varphi^-(x)\}-\alpha_1\\ &= \pm-\alpha_1,\end{aligned}$$

so $\alpha_2=\alpha_1+\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\}\geq\alpha_1+\pm-\alpha_1=\pm$. Thus $\alpha_2\in[\pm,0]$ and $\alpha_2\leq\alpha_1$, so $\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x)\geq\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Now for all $x\in A$, we have

$$\begin{aligned}\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x) &= \varphi^-(x)-\alpha_2\\ &= \varphi^-(x)-(\alpha_1+\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\})\\ &= \varphi^-(x)-\alpha_1-\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\}\\ &= \varphi^-(x)-\alpha_1+\inf\{\psi^-(x)-\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\}\\ &\leq \varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)+\psi^-(x)-\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\\ &= \psi^-(x).\end{aligned}$$

Thus $\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\leq\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x)\leq\psi^-(x)$ for all $x\in A$. Put $\beta_2=\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}$. Then

$$\begin{aligned}\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\} &\leq \inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\}\\ &= \inf\{\varphi^+(x)-\beta_1\}\\ &= \inf\{\varphi^+(x)\}-\beta_1\\ &= \mp-\beta_1,\end{aligned}$$

so $\beta_2=\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}\leq\beta_1+\mp-\beta_1=\mp$. Thus $\beta_2\in[0,\mp]$ and $\beta_2\geq\beta_1$, so $\varphi^+_{(\beta_2,\mathrm{T}_1)}(x)\leq\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Now for all $x\in A$, we have

$$\begin{aligned}\varphi^+_{(\beta_2,\mathrm{T}_2)}(x) &= \varphi^+(x)-\beta_2\\ &= \varphi^+(x)-(\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\})\\ &= \varphi^+(x)-\beta_1-\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}\\ &= \varphi^+(x)-\beta_1+\sup\{\psi^+(x)-\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\}\\ &\geq \varphi^+_{(\beta_1,\mathrm{T}_2)}(x)+\psi^+(x)-\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\\ &= \psi^+(x).\end{aligned}$$

Thus $\psi^+(x)\leq\varphi^+_{(\beta_2,\mathrm{T}_2)}(x)\leq\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Hence, $\psi\subseteq\varphi^{\mathrm{T}_2}_{(\alpha_2,\beta_2)}\subseteq\varphi^{\mathrm{T}_2}_{(\alpha_1,\beta_1)}$. □

Definition 3.32. Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A with $\psi \subseteq \varphi$. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, and we say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy UP-subalgebra* (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) *intension* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 3.33. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28) and Remark 3.30. □

Theorem 3.34. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α_1, β_1)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha_1,\beta_1)} = (A; {\varphi^-}_{(\alpha_1,\mathrm{T}_2)}, {\varphi^+}_{(\beta_1,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of the bipolar fuzzy (α_2, β_2)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha_2,\beta_2)} = (A; {\varphi^-}_{(\alpha_2,\mathrm{T}_2)}, {\varphi^+}_{(\beta_2,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ with $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28). □

Theorem 3.35. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A. For every bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension $\psi = (A; \psi^-, \psi^+)$ of the bipolar fuzzy (α, β)-translation $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ there exists $(k^-, k^+) \in [\pm, 0] \times [0, \mp]$ such that $(k^-, k^+) \leq (\alpha, \beta)$, that is, $k^- \geq \alpha$ and $k^+ \leq \beta$, and $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of bipolar fuzzy (k^-, k^+)-translation $\varphi^{\mathrm{T}_2}_{(k^-,k^+)} = (A; \varphi^-_{(k^-,\mathrm{T}_2)}, \varphi^+_{(k^+,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28) and Lemma 3.31. □

3.3 Complement of a bipolar fuzzy set

In this part, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.

Definition 3.36. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. The bipolar fuzzy set $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ defined by: for all $x \in A$,

$$\begin{array}{rcl}\overline{\varphi^-}(x) & = & -1 - \varphi^-(x), \\ \overline{\varphi^+}(x) & = & 1 - \varphi^+(x),\end{array}$$

is called the *complement* of $\varphi = (A; \varphi^-, \varphi^+)$ in A.

Definition 3.37. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and for any $(t^-, t^+) \in [-1, 0] \times [0, 1]$. The sets

$$N_L(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \leq t^-\}$$

and

$$N_U(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \geq t^-\}$$

are called the *negative lower* t^-*-cut* and the *negative upper* t^-*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively. The sets

$$P_L(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \leq t^+\}$$

and

$$P_U(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \geq t^+\}$$

are called the *positive lower* t^+*-cut* and the *positive upper* t^+*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively.

Lemma 3.38. *Let $a, b, c \in \mathbb{R}$. Then the following statements hold:*

(1) $a - \min\{b, c\} = \max\{a - b, a - c\}$, *and*

(2) $a - \max\{b, c\} = \min\{a - b, a - c\}$.

Proof. (1) If $\min\{b, c\} = b$, then $c \geq b$. Thus $a - c \leq a - b$, so $\max\{a - b, a - c\} = a - b = a - \min\{b, c\}$. Similarly, if $\min\{b, c\} = c$, then

$$\max\{a - b, a - c\} = a - c = a - \min\{b, c\}.$$

(2) If $\max\{b, c\} = b$, then $b \geq c$. Thus $a - b \leq a - c$, so $\min\{a - b, a - c\} = a - b = a - \max\{b, c\}$. Similarly, if $\max\{b, c\} = c$, then

$$\min\{a - b, a - c\} = a - c = a - \max\{b, c\}.$$

$\square$

Theorem 3.39. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A if and only if for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-subalgebras of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$ be such that $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x, y \in N_U(\varphi; t^-)$. Then $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so t^- is a lower bound of $\{\varphi^-(x), \varphi^-(y)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A, we have $\overline{\varphi^-}(x \cdot y) \leq \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1 - \varphi^-(x \cdot y) \leq \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} = -1 - \min\{\varphi^-(x), \varphi^-(y)\}$. Thus $\varphi^-(x \cdot y) \geq \min\{\varphi^-(x), \varphi^-(y)\} \geq t^-$ and so $x \cdot y \in N_U(\varphi; t^-)$. Therefore, $N_U(\varphi; t^-)$ is a UP-subalgebra of A.

(ii) Let $x, y \in P_L(\varphi; t^+)$. Then $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x), \varphi^+(y)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A, we have $\overline{\varphi^+}(x \cdot y) \geq \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}$. By Lemma 3.38 (2), we have $1 - \varphi^+(x \cdot y) \geq \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} = 1 - \max\{\varphi^+(x), \varphi^+(y)\}$. Thus $\varphi^+(x \cdot y) \leq \max\{\varphi^+(x), \varphi^+(y)\} \leq t^+$ and so $x \cdot y \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a UP-subalgebra of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-subalgebras of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x, y \in A$. Then $\varphi^-(x), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. Thus $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so $x, y \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-subalgebra of A and so $x \cdot y \in N_U(\varphi; t^-)$. Thus $\varphi^-(x \cdot y) \geq t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned} \overline{\varphi^-}(x \cdot y) &= -1 - \varphi^-(x \cdot y) \\ &\leq -1 - \min\{\varphi^-(x), \varphi^-(y)\} \\ &= \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} \\ &= \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}. \end{aligned}$$

(ii) Let $x, y \in A$. Then $\varphi^+(x), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \max\{\varphi^+(x), \varphi^+(y)\}$. Thus $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $x, y \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-subalgebra of A and so $x \cdot y \in P_L(\varphi; t^+)$. Thus $\varphi^+(x \cdot y) \leq t^+ = \max\{\varphi^+(x), \varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned} \overline{\varphi^+}(x \cdot y) &= 1 - \varphi^+(x \cdot y) \\ &\geq 1 - \max\{\varphi^+(x), \varphi^+(y)\} \\ &= \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} \\ &= \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}. \end{aligned}$$

Hence, $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.40. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A if and only if for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-filters of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$ be such that $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $a \in N_U(\varphi; t^-)$. Then $\varphi^-(a) \geq t^-$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^-}(0) \leq \overline{\varphi^-}(a)$. Thus $-1 - \varphi^-(0) \leq -1 - \varphi^-(a)$, so $\varphi^-(0) \geq \varphi^-(a) \geq t^-$. Hence, $0 \in N_U(\varphi; t^-)$. Next, let $x, y \in A$ be such that $x \cdot y \in N_U(\varphi; t^-)$ and $x \in N_U(\varphi; t^-)$. Then $\varphi^-(x \cdot y) \geq t^-$ and $\varphi^-(x) \geq t^-$, so t^- is a lower bound of $\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^-}(y) \leq \max\{\overline{\varphi^-}(x \cdot y), \overline{\varphi^-}(x)\}$. By Lemma 3.38 (1), we have $-1-\varphi^-(y) \leq \max\{-1-\varphi^-(x\cdot y), -1-\varphi^-(x)\} = -1-\min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Thus $\varphi^-(y) \geq \min\{\varphi^-(x \cdot y), \varphi^-(x)\} \geq t^-$ and so $y \in N_U(\varphi; t^-)$. Therefore, $N_U(\varphi; t^-)$ is a UP-filter of A.

(ii) Let $b \in P_L(\varphi; t^+)$. Then $\varphi^+(b) \leq t^+$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is bipolar fuzzy UP-filter of A, we have $\overline{\varphi^+}(0) \geq \overline{\varphi^+}(b)$. Thus $1 - \varphi^+(0) \geq 1 - \varphi^+(b)$, so $\varphi^+(0) \leq \varphi^+(b) \leq t^+$. Hence, $0 \in P_L(\varphi; t^+)$. Next, let $x, y \in A$ be such that $x \cdot y \in P_L(\varphi; t^+)$ and $x \in P_L(\varphi; t^+)$. Then $\varphi^+(x \cdot y) \leq t^+$ and $\varphi^+(x) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^+}(y) \geq \min\{\overline{\varphi^+}(x \cdot y), \overline{\varphi^+}(x)\}$. By Lemma 3.38 (2), we have $1 - \varphi^+(y) \geq \min\{1 - \varphi^+(x \cdot y), 1 - \varphi^+(x)\} = 1 - \max\{\varphi^-(x \cdot y), \varphi^+(x)\}$. Thus $\varphi^+(y) \leq \max\{\varphi^+(x \cdot y), \varphi^+(x)\} \leq t^+$ and so $y \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a UP-filter of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-filters of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Thus $\varphi^-(x) \geq t^-$, so $x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-filter of A and so $0 \in N_U(\varphi; t^-)$. Thus $\varphi^-(0) \geq t^- = \varphi^-(x)$ and so $\overline{\varphi^-}(0) = -1 - \varphi^-(0) \leq -1 - \varphi^-(x) = \overline{\varphi^-}(x)$.

(ii) Let $x, y \in A$. Then $\varphi^-(x \cdot y), \varphi^-(x) \in [-1, 0]$. Choose $t^- = \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Thus $\varphi^-(x \cdot y) \geq t^-$ and $\varphi^-(x) \geq t^-$, so $x \cdot y, x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-filter of A and so $y \in N_U(\varphi; t^-)$. Thus $\varphi^-(y) \geq t^- = \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}
\overline{\varphi^-}(y) &= -1 - \varphi^-(y) \\
&\leq -1 - \min\{\varphi^-(x \cdot y), \varphi^-(x)\} \\
&= \max\{-1 - \varphi^-(x \cdot y), -1 - \varphi^-(x)\} \\
&= \max\{\overline{\varphi^-}(x \cdot y), \overline{\varphi^-}(x)\}.
\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-filter of A and so $0 \in P_L(\varphi; t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1 - \varphi^+(0) \geq 1 - \varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x, y \in A$. Then $\varphi^+(x \cdot y), \varphi^+(x) \in [0, 1]$. Choose $t^+ = \max\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus $\varphi^+(x \cdot y) \leq t^+$ and $\varphi^+(x) \leq t^+$, so $x \cdot y, x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-filter of A and so $y \in P_L(\varphi; t^+)$. Thus $\varphi^+(y) \leq t^+ = \max\{\varphi^+(x \cdot y), \varphi^+(x)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}
\overline{\varphi^+}(y) &= 1-\varphi^+(y)\\
&\geq 1-\max\{\varphi^+(x\cdot y),\varphi^+(x)\}\\
&= \min\{1-\varphi^+(x\cdot y),1-\varphi^+(x)\}\\
&= \min\{\overline{\varphi^+}(x\cdot y),\overline{\varphi^+}(x)\}.
\end{aligned}$$

Hence, $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.41. *Let $\varphi=(A;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A if and only if for all $(t^-,t^+)\in[-1,0]\times[0,1], N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A. Let $(t^-,t^+)\in[-1,0]\times[0,1]$ be such that $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $a\in N_U(\varphi;t^-)$. Then $\varphi^-(a)\geq t^-$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^-}(0)\leq\overline{\varphi^-}(a)$. Thus $-1-\varphi^-(0)\leq-1-\varphi^-(a)$, so $\varphi^-(0)\geq\varphi^-(a)\geq t^-$. Hence, $0\in N_U(\varphi;t^-)$.

(ii) Let $x,y,z\in A$ be such that $x\cdot(y\cdot z)\in N_U(\varphi;t^-)$ and $y\in N_U(\varphi;t^-)$. Then $\varphi^-(x\cdot(y\cdot z))\geq t^-$ and $\varphi^-(y)\geq t^-$, so t^- is a lower bound of $\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^-}(x\cdot z)\leq\max\{\overline{\varphi^-}(x\cdot(y\cdot z)),\overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1-\varphi^-(x\cdot z)\leq\max\{-1-\varphi^-(x\cdot(y\cdot z)),-1-\varphi^-(y)\}=-1-\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Thus $\varphi^-(x\cdot z)\geq\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}\geq t^-$ and so $x\cdot z\in N_U(\varphi;t^-)$. Therefore, $N_U(\varphi;t^-)$ is a UP-ideal of A.

(iii) Let $b\in P_L(\varphi;t^+)$. Then $\varphi^+(b)\leq t^+$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^+}(0)\geq\overline{\varphi^+}(b)$. Thus $1-\varphi^+(0)\geq1-\varphi^+(b)$, so $\varphi^+(0)\leq\varphi^+(b)\leq t^+$. Hence, $0\in P_L(\varphi;t^+)$.

(iv) Let $x,y,z\in A$ be such that $x\cdot(y\cdot z)\in P_L(\varphi;t^+)$ and $y\in P_L(\varphi;t^+)$. Then $\varphi^+(x\cdot(y\cdot z))\leq t^+$ and $\varphi^+(y)\leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^+}(x\cdot z)\geq\min\{\overline{\varphi^+}(x\cdot(y\cdot z)),\overline{\varphi^+}(y)\}$. By Lemma 3.38 (2), we have $1-\varphi^+(x\cdot z)\geq\min\{1-\varphi^+(x\cdot(y\cdot z)),1-\varphi^+(y)\}=1-\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Thus $\varphi^+(x\cdot z)\leq\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}\leq t^+$ and so $x\cdot z\in P_L(\varphi;t^+)$. Therefore, $P_L(\varphi;t^+)$ is a UP-ideal of A.

Conversely, assume that for all $(t^-,t^+)\in[-1,0]\times[0,1], N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $x\in A$. Then $\varphi^-(x)\in[-1,0]$. Choose $t^-=\varphi^-(x)$. Thus $\varphi^-(x)\geq t^-$, so $x\in N_U(\varphi;t^-)\neq\emptyset$. By assumption, we have $N_U(\varphi;t^-)$ is a UP-ideal of A and so $0\in N_U(\varphi;t^-)$. Thus $\varphi^-(0)\geq t^-=\varphi^-(x)$ and so $\overline{\varphi^-}(0)=-1-\varphi^-(0)\leq-1-\varphi^-(x)=\overline{\varphi^-}(x)$.

(ii) Let $x,y,z\in A$. Then $\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\in[-1,0]$. Choose $t^-=\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Thus $\varphi^-(x\cdot(y\cdot z))\geq t^-$ and $\varphi^-(y)\geq t^-$, so $x\cdot(y\cdot z),y\in N_U(\varphi;t^-)\neq\emptyset$. By assumption, we have $N_U(\varphi;t^-)$ is a UP-ideal of A and so $x\cdot z\in N_U(\varphi;t^-)$. Thus $\varphi^-(x\cdot z)\geq t^-=\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}\overline{\varphi^-}(x\cdot z) &= -1-\varphi^-(x\cdot z)\\ &\leq -1-\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}\\ &= \max\{-1-\varphi^-(x\cdot(y\cdot z)),-1-\varphi^-(y)\}\\ &= \max\{\overline{\varphi^-}(x\cdot(y\cdot z)),\overline{\varphi^-}(y)\}.\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0,1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi;t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi;t^+)$ is a UP-ideal of A and so $0 \in P_L(\varphi;t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1-\varphi^+(0) \geq 1-\varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x,y,z \in A$. Then $\varphi^+(x\cdot(y\cdot z)),\varphi^+(y) \in [0,1]$. Choose $t^+ = \max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Thus $\varphi^+(x\cdot(y\cdot z)) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $x\cdot(y\cdot z), y \in P_L(\varphi;t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi;t^+)$ is a UP-ideal of A and so $x\cdot z \in P_L(\varphi;t^+)$. Thus $\varphi^+(x\cdot z) \leq t^+ = \max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}\overline{\varphi^+}(x\cdot z) &= 1-\varphi^+(x\cdot z)\\ &\geq 1-\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}\\ &= \min\{1-\varphi^+(x\cdot(y\cdot z)),1-\varphi^+(y)\}\\ &= \min\{\overline{\varphi^+}(x\cdot(y\cdot z)),\overline{\varphi^+}(y)\}.\end{aligned}$$

Hence, $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A. □

Theorem 3.42. *Let $\varphi = (A;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A if and only if for all $(t^-,t^+) \in [-1,0]\times[0,1]$, $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are strongly UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A. Let $(t^-,t^+) \in [-1,0]\times[0,1]$ be such that $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $a \in N_U(\varphi;t^-)$. Then $\varphi^-(a) \geq t^-$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^-}(0) \leq \overline{\varphi^-}(a)$. Thus $-1-\varphi^-(0) \leq -1-\varphi^-(a)$, so $\varphi^-(0) \geq \varphi^-(a) \geq t^-$. Hence, $0 \in N_U(\varphi;t^-)$.

(ii) Let $x,y,z \in A$ be such that $(z\cdot y)\cdot(z\cdot x) \in N_U(\varphi;t^-)$ and $y \in N_U(\varphi;t^-)$. Then $\varphi^-((z\cdot y)\cdot(z\cdot x)) \geq t^-$ and $\varphi^-(y) \geq t^-$, so t^- is a lower bound of $\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^-}(x) \leq \max\{\overline{\varphi^-}((z\cdot y)\cdot(z\cdot x)),\overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1-\varphi^-(x) \leq \max\{-1-\varphi^-((z\cdot y)\cdot(z\cdot x)),-1-\varphi^-(y)\} = -1-\min\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$. Thus $\varphi^-(x) \geq \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\} \geq t^-$ and so $x \in N_U(\varphi;t^-)$. Therefore, $N_U(\varphi;t^-)$ is a strongly UP-ideal of A.

(iii) Let $b \in P_L(\varphi;t^+)$. Then $\varphi^+(b) \leq t^+$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^+}(0) \geq \overline{\varphi^+}(b)$. Thus $1-\varphi^+(0) \geq 1-\varphi^+(b)$, so $\varphi^+(0) \leq \varphi^+(b) \leq t^+$. Hence, $0 \in P_L(\varphi;t^+)$.

(iv) Let $x,y,z \in A$ be such that $(z\cdot y)\cdot(z\cdot x) \in P_L(\varphi;t^+)$ and $y \in P_L(\varphi;t^+)$. Then $\varphi^+((z\cdot y)\cdot(z\cdot x)) \leq t^+$ and $\varphi^+(y) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^+}(x) \geq \min\{\overline{\varphi^+}((z\cdot y)\cdot(z\cdot x)),\overline{\varphi^+}(y)\}$. By Lemma 3.38

(2), we have $1-\varphi^+(x) \geq \min\{1-\varphi^+((z\cdot y)\cdot(z\cdot x)), 1-\varphi^+(y)\} = 1-\max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. Thus $\varphi^+(x) \leq \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\} \leq t^+$ and so $x \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a strongly UP-ideal of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are strongly UP-ideals of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Thus $\varphi^-(x) \geq t^-$, so $x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a strongly UP-ideal of A and so $0 \in N_U(\varphi; t^-)$. Thus $\varphi^-(0) \geq t^- = \varphi^-(x)$ and so $\overline{\varphi^-}(0) = -1-\varphi^-(0) \leq -1-\varphi^-(x) = \overline{\varphi^-}(x)$.

(ii) Let $x, y, z \in A$. Then $\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}$. Thus $\varphi^-((z\cdot y)\cdot(z\cdot x)) \geq t^-$ and $\varphi^-(y) \geq t^-$, so $(z\cdot y)\cdot(z\cdot x), y \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a strongly UP-ideal of A and so $x \in N_U(\varphi; t^-)$. Thus $\varphi^-(x) \geq t^- = \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}
\overline{\varphi^-}(x) &= -1-\varphi^-(x) \\
&\leq -1-\min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\} \\
&= \max\{-1-\varphi^-((z\cdot y)\cdot(z\cdot x)), -1-\varphi^-(y)\} \\
&= \max\{\overline{\varphi^-}((z\cdot y)\cdot(z\cdot x)), \overline{\varphi^-}(y)\}.
\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a strongly UP-ideal of A and so $0 \in P_L(\varphi; t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1-\varphi^+(0) \geq 1-\varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x, y, z \in A$. Then $\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. Thus $\varphi^+((z\cdot y)\cdot(z\cdot x))) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $(z\cdot y)\cdot(z\cdot x), y \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a strongly UP-ideal of A and so $x \in P_L(\varphi; t^+)$. Thus $\varphi^+(x) \leq t^+ = \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}
\overline{\varphi^+}(x) &= 1-\varphi^+(x) \\
&\geq 1-\max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\} \\
&= \min\{1-\varphi^+((z\cdot y)\cdot(z\cdot x)), 1-\varphi^+(y)\} \\
&= \min\{\overline{\varphi^+}((z\cdot y)\cdot(z\cdot x)), \overline{\varphi^+}(y)\}.
\end{aligned}$$

Hence, $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A. □

4. Conclusions and future work

In the present paper, we have introduced the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II for a bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. We think this work would enhance the scope for further study in UP-algebras and related algebraic systems. It is our hope that this work would serve as a foundation for the further study in a new concept of UP-algebras.

In our future study of UP-algebras, may be the following topics should be considered:

- To get more results in bipolar fuzzy translations of a bipolar-valued fuzzy set in UP-algebras.
- To define bipolar-valued fuzzy sets with thresholds in UP-algebras.
- To define bipolar-valued fuzzy soft sets in UP-algebras.

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P-EXPANDABLE SPACES

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Abstract. We introduce the concept of P-expandable spaces as a variation of expandable spaces. A space (X, τ) is said to be P-expandable if every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$. We characterize P-expandable spaces and study their basic properties. We show that if a space (X, τ) is a quasi submaximal space, then (X, τ) is P-expandable if and only if it is expandable.

Keywords: preopen set, p-locally finite collection, expandable space, P-expandable space.

1. Introduction

By a space, we mean a topological space in which no separation axioms is assumed unless explicitly stated. Let (X, τ) be a space and A be a subset of X. The closure of A, the interior of A and the relative topology on A in (X, τ) will be denoted by $cl(A)$, $\text{int}(A)$ and τ_A, respectively. A is called a preopen subset of (X, τ) [3] if $A \subseteq \text{int}(cl(A))$. The complement of a preopen set is called a preclosed set. A is called semi-open [12] (resp. α-sets [13], regular closed) if $A \subseteq cl(\text{int}(A))$ (resp. $A \subseteq \text{int}(cl(\text{int}(A)))$, $A = cl(\text{int}(A))$). The family of all subsets of a space (X, τ) which are preopen (resp. preclosed, semi-open, regular closed) is denoted by $PO(X, \tau)$ (resp. $PC(X, \tau)$, $SO(X, \tau)$, $RC(X, \tau)$). It is known that the collection of all α-sets of (X, τ) forms a topology on X, denoted by τ^α, finer than τ and $PO(X, \tau) = PO(X, \tau^\alpha)$.

A space (X, τ) is called submaximal [11] if every dense subset of (X, τ) is open. It is known that (X, τ) is submaximal if and only if $\tau = PO(X, \tau)$. In [4], Al-Nashef introduced the notion of quasi-submaximal spaces where a space (X, τ) is quasi-submaximal if $cl(D) - D$ is nowhere dense subset for each dense subset D of (X, τ). This is equivalent to saying that $\text{int}(D)$ is dense for each dense subset D of (X, τ) [4].

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Mashhour et al. [2] used preopen sets to define P_1-paracompact and P_2-paracompact spaces. In 2007, Al-Zoubi and Al-Ghour [8] define P_3-paracompact space and the notion P-locally finite collections and study their properties. In this paper we introduce P-expandable spaces by using preopen sets and p-locally finiteness and study their topological properties. We deal with subspaces, sum, image and the inverse images of P-expandable.

Lemma 1.1. *Let A and B be subsets of a space (X, τ).*

i. *If $A \in PO(X, \tau)$ and $B \in SO(X, \tau)$, then $A \cap B \in PO(B, \tau_B)$ ([6]).*

ii. *If $A \in PO(B, \tau_B)$ and $B \in PO(X, \tau)$, then $A \in PO(X, \tau)$ ([6]).*

iii. *If $A \in PO(X, \tau)$ and $B \in \tau$, then $A \cap B \in PO(X, \tau)$ ([7]).*

Definition 1.2. A collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of a space (X, τ) is called locally finite (resp. p-locally finite [8]) if for each $x \in X$, there exists $W_x \in \tau$ (resp. $W_x \in PO(X, \tau)$) containing x and W_x intersects at most finitely many members of $\mathcal{F}$.

Corollary 1.3 ([8]). *Let (X, τ) be any space:*

i. *Every locally finite collection subset of X is p-locally finite collection subset of X.*

ii. *Every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of preopen subsets of a quasi- submaximal space X is locally finite.*

iii. *Every p-locally finite collection of open sets (α-sets, regular closed sets) is locally finite.*

Definition 1.4. A function $f : (X, \tau) \to (Y, \sigma)$ is called

i. Preirresoulte [5] if and only if $f^{-1}(A) \in PO(X, \tau)$ for each $A \in PO(Y, \sigma)$.

ii. Strongly preclosed [16] if $f(A) \in PC(Y, \sigma)$ for each $A \in PC(X, \tau)$.

iii. M-preopen [2] if $f(A) \in PO(Y, \sigma)$ for each $A \in PO(X, \tau)$.

iv. Countable perfect [10] if f is continuous, closed, surjective function such that $f^{-1}(y)$ is countable compact for each y in Y.

Lemma 1.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function.*

i. *If $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of (Y, σ), then $f^{-1}(\mathcal{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection in (X, τ) [10].*

ii. *Let f be a countable perfect function. If $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of (X, τ), then $f(\mathcal{F}) = \{f(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection in (Y, σ) ([1]).*

Recall that a space (X,τ) is called strongly compact relative to X [2] if every cover of A by preopen sets of X has a finite subcover.

Theorem 1.6 ([8]). *Let $f:(X,\tau)\to(Y,\sigma)$ be a function:*

i. If f is a preirresolute function and $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ is a p-locally finite collection in (Y,σ), then $f^{-1}(\mathcal{F})=\{f^{-1}(F_\alpha):\alpha\in\Delta\}$ is a p-locally finite collection in (X,τ).

ii. If f is a strongly preclosed function such that $f^{-1}(y)$ is strongly compact relative to (X,τ) for every $y\in Y$ and $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ is a p-locally finite collection of subsets of (X,τ), then $f(\mathcal{F})=\{f(F_\alpha):\alpha\in\Delta\}$ is a p-locally finite collection in (Y,σ).

Corollary 1.7. *Let (X,τ) be a space, then the following are equivalent:*

i. (X,τ) is expandable.

ii. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of open subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$ [8].

iii. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subset of X there exists a locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen subset of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

iv. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ collection of α-open subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

Proof. (i$\to$ii $\to$iii $\to$iv) These implication follow from definitions, Corollary 1.3 and the fact that $\tau\subseteq\tau_\alpha$.

(iv$\to$i) Let $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ be a locally finite collection of subsets of a space (X,τ). Then, by (iv), there exists a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of α-open sets subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$. Then, by Corollary 1.3, $\{\mathrm{int}(cl(\mathrm{int}(G_\alpha))):\alpha\in\Delta\}$ is a locally finite collection of open subset of X such that $F_\alpha\subseteq\mathrm{int}(cl(\mathrm{int}(G_\alpha)))$ for all $\alpha\in\Delta$. Hence (X,τ) is expandable. □

2. P-expandable spaces

Definition 2.1. A space (X,τ) is said to be P-expandable (resp. P_1-expandable, pre-expandable) if every locally finite (resp. p-locally finite, p-locally finite) collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite (resp. locally finite, p-locally finite) collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen (resp. open, preopen) subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

It is clear (from the fact that the closure of any locally finite collection is locally finite) that a space (X,τ) is P-expandable iff every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of closed subsets of X, there exists a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

The following diagram follows immediately from the definitions in which none of these implications is reversible.

$$\begin{array}{ccc}\text{Expandable} & \rightarrow & \text{P-expandable}\\ \uparrow & & \uparrow\\ \text{P}_1\text{-expandable} & \rightarrow & \text{pre-expandable}\end{array}$$

To show that none of these implications is reversible, In the above diagram, we consider the following examples.

Example 2.2. Let $X=N\cup N^-$ with the topology $\tau=\{U\subseteq X:N\subseteq U\}\cup\{\phi\}$ such that N is the set of all positive integers and N^- is the set of all negative integers. Then $PO(X,\tau)=\{A\subseteq X:A\cap N\neq\phi\}$.

(i) Note that (X,τ) is not expandable since the collection $\{\{x\}:x\in N^-\}$ is locally finite in (X,τ) and there exists no locally finite collection $\{U_x:x\in N^-\}$ of open sets in (X,τ) such that $x\in U_x$ for $x\in N^-$.

(ii) To see that (X,τ) is pre- expandable (hence P-expandable). Let $\mathcal{U}=\{U_\alpha:\alpha\in\Delta\}$ be a p-locally finite collection in (X,τ). Put $\Delta_1=\{\alpha\in\Delta:U_\alpha\cap\mathbb{N}\neq\phi\}$ and $\Delta_2=\{\alpha\in\Delta:U_\alpha\cap\mathbb{N}=\phi\}$. Now, for $\alpha\in\Delta_2$, choose $x_\alpha\in U_\alpha$ and put $U_\alpha^*=U_\alpha\cup\{-x_\alpha\}$. Put $\mathcal{U}^*=\{U_\alpha:\alpha\in\Delta_1\}\cup\{U_\alpha^*:\alpha\in\Delta_2\}$. Then, it is clear $\mathcal{U}^*$ is a collection of preopen sets in (X,τ) such that for all $\alpha\in\Delta$, there exists $H_\alpha\in\mathcal{U}^*$ such that $U_\alpha\subseteq H_\alpha$. Finally, we show that $\mathcal{U}^*$ is p-locally finite in (X,τ). Let $x\in X$. Then, there exists a preopen set P_x in (X,τ) such that $x\in P_x$ and a finite subset Δ_1' of Δ_1 and a finite subset Δ_2' of Δ_2 such that $P_x\cap U_\alpha=\phi$ for all $\alpha\in\Delta-(\Delta_1'\cup\Delta_2')$. Now, if $x\in\mathbb{N}^-$, put $P_x^*=(P_x-\{-x_\alpha:\alpha\in\Delta_2-\Delta_2'\})\cup\{-x\}$. Then P_x^* is a preopen set in (X,τ) such that $x\in P_x^*$ and P_x^* intersect at most finitely many members of $\mathcal{U}^*$. If $x\in\mathbb{N}$, put $P_x^*=(P_x-\{-x_\alpha:\alpha\in\Delta_2-\Delta_2'\})\cup\{x\}$. Then P_x^* is a preopen set in (X,τ) such that $x\in P_x^*$ and P_x^* intersect at most finitely many members of $\mathcal{U}^*$. Thus $\mathcal{U}^*$ is a p-locally finite collection of preopen sets in (X,τ) and so (X,τ) is pre-expandable.

Example 2.3. Let $X=\mathbb{R}$ with the topology $\tau=\{U:U\subseteq\mathbb{Q}\}\cup\{\mathbb{R}\}$. Note that $PO(X,\tau)=\{U:U\subseteq\mathbb{Q}\}\cup\{U:\mathbb{Q}\subseteq U\}$ and every locally finite collection is finite. Hence (X,τ) is expandable (and so P-expandable). On the other hand, (X,τ) is not pre-expandable since the collection $\{\{x\}:x\in\mathbb{R}-\mathbb{Q}\}$ is p-locally finite in X but there does not exists a p-locally finite collection of preopen set $\{G_x:x\in\mathbb{R}-\mathbb{Q}\}$ in (X,τ) such that $x\in G_x$ for each $x\in\mathbb{R}-\mathbb{Q}$. If $\mathcal{G}=\{G_x:x\in\mathbb{R}-\mathbb{Q}\}$ is p-locally finite collection of preopen sets, then $\{x\}\cup\mathbb{Q}\subseteq G_x$ for all $x\in\mathbb{R}-\mathbb{Q}$. Choose $x_\circ\in\mathbb{Q}$ and $p_\circ\in PO(X,\tau)$ such that $x_\circ\in p_\circ$. Then $p_\circ\cap G_x\neq\phi$ for all x.

Example 2.4. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. Note that $PO(X, \tau) = \tau_{dis}$ and so (X, τ) is pre-expandable. On the other hand, every locally finite is finite, therefore (X, τ) is expandable. To show that (X, τ) is not P_1-expandable, we consider the collection $\mathcal{U} = \{\{x\} : x \in \mathbb{Q}\}$. $\mathcal{U}$ is *p*-locally finite in X but there does not exists a locally finite collection of open set $\{G_x : x \in \mathbb{Q}\}$ in (X, τ) such that $x \in G_x$ for each $x \in \mathbb{Q}$. Note that, if $G_x \in \tau$ such that $\{x\} \subseteq G_x$ then either $G_x = \mathbb{Q}$ or $G_x = \mathbb{R}$ and so $\{G_x\}$ is not locally finite.

Note that Example 2.2 and Example 2.3 shows that expandable and pre-expandable spaces are independent notions.

Proposition 2.5. *Let (X, τ) be a space, then the following are equivalent:*

i. (X, τ) *is* P_1*-expandable.*

ii. *For every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.*

iii. *For every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.*

Proof. (i→ii→iii) These implication follow from the definition and Corollary 1.3..

(iii→i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a *p*-locally finite collection of subsets of X. Then, by (iii) there exists a locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Then, $\{\mathrm{int}(cl(G_\alpha)) : \alpha \in \Delta\}$ is a locally finite collection of open subset of X such that $U_\alpha \subseteq \mathrm{int}(cl(G_\alpha))$ for all $\alpha \in \Delta$. Hence (X, τ) is P_1-expandable. □

Proposition 2.6. *Let (X, τ) be any space:*

i. *If (X, τ) is a quasi-submaximal, then (X, τ) is expandable iff it is* P*-expandable.*

ii. *If (X, τ) is a submaximal, then (X, τ) is expandable iff it is pre-expandable.*

Proof. The easy proof is left to the reader. □

In Example 2.2 and Example 2.3 show that the conditions in Proposition 2.6 are essential. Recall that a space (X, τ) is called countable P-compact [15], if every countable preopen cover of (X, τ) has a finite subcover. It is clear that every *p*-locally finite collection of countably P-compact space is finite [14].

Proposition 2.7. *Let (X, τ) be a countably* P*-compact space. Then (X, τ) is expandable if and only if it is* P*-expandable.*

Proof. The necessity is clear and we need only prove the sufficiency. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite collection of X. Then there exists a P-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $U_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$, Since (X, τ) is countably P-compact then $\{\text{int}(cl(G_\alpha)) : \alpha \in \Delta\}$ is a locally finite collection of open subset of X such that $U_\alpha \subseteq \text{int}(cl(G_\alpha))$ for all $\alpha \in \Delta$. Hence (X, τ) is expandable. □

Recall that a space (X, τ) is called P_1-paracompact [2], (resp. P_2-paracompact [2], P_3-paracompact [8]) if every preopen (resp. preopen, open) cover of X has a locally finite open (resp. locally finite preopen, p-locally finite preopen) refinement.

Theorem 2.8. *Every* P_3*-paracompact space is* P*-expandable.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of closed subsets of X. Let Δ' be the collection of all finite subsets of Δ. For $\beta \in \Delta'$, let $V_\beta = X - \cup\{F_\alpha : \alpha \notin \beta\}$. Because $\mathcal{F}$ is the locally finite collection, V_β is open. Also, V_β meets only finitely many elements of $\mathcal{F}$. Let $\mathcal{V} = \{V_\beta : \beta \in \Delta'\}$. Then $\mathcal{V}$ is an open cover of X. Since X is P_3-paracompact, $\mathcal{V}$ has a p-locally finite preopen refinements, say $\mathcal{W} = \{W_\gamma : \gamma \in \Delta\}$. Set $U_\alpha = \cup\{W_\gamma \in \mathcal{W} : W_\gamma \cap F_\alpha \neq \phi\}$ for each $\alpha \in \Delta$. Because arbitrary unions of preopen sets are preopen set, U_α is preopen and $F_\alpha \subseteq U_\alpha$ for each $\alpha \in \Delta$. Now, we shall try to show that $\{U_\alpha : \alpha \in \Delta\}$ is p-locally finite. Since $\mathcal{W}$ is p-locally finite, for each $x \in X$, there exists a preopen set U_x in (X, τ) containing x and U_x intersects at most finitely many members of $\mathcal{W}$. Also, by the definition of U_α, we say that $U_x \cap U_\alpha \neq \phi$ if and only if $U_x \cap W_\gamma \neq \phi$ and $W_\gamma \cap F_\alpha \neq \phi$ for some $\gamma \in \Delta$. Since $\mathcal{W}$ is refinement of $\mathcal{V}$, there is number V_β of $\mathcal{V}$ containing W_γ for each number W_γ of $\mathcal{W}$. Then W_γ meets only finitely many F_α for each $\gamma \in \Delta$. Thus, $\{U_\alpha : \alpha \in \Delta\}$ is p-locally finite. □

Corollary 2.9. *Every* P_1*-paracopmact (reps.* P_2*-paracompact) space is* P*-expandable.*

The following example shows that the converse of the above corollary need not be true.

Example 2.10. Let ω_1 denote the first uncountable ordinal and let $X = [0, \omega_1)$ with the usual order topology. Then, from [9], X is countable compact but not paracompact since the collection $\{[0, \alpha) : \alpha < \omega_1\}$ is an open cover of X which has no open locally finite refinement. Hence X is P-expandable but neither P_1-paracompact nor P_2-paracompact.

Theorem 2.11. *Let* (X, τ) *be a space:*

i. If (X, τ^α) *is* P*-expandable, then* (X, τ) *is* P*-expandable.*

ii. If (X, τ) *is* P*-expandable submaximal space, then* (X, τ^α) *is* P*-expandable.*

Proof. This follows immediately from the definitions and the facts that $\tau \subseteq \tau^\alpha$ and $PO(X, \tau^\alpha) = PO(X, \tau)$. □

The converse of part (i) of Theorem 2.11 is not true in general as the following example shows.

Example 2.12. Let X be an infinite set and $q \in X$. Let $\tau = \{\phi, X, \{q\}\}$. Then (X, τ) is P-expandable. But (X, τ^α) is not P-expandable and not submaximal. Since $\tau^\alpha = PO(X, \tau) = PO(X, \tau^\alpha) = \{\phi\} \cup \{U \subseteq X : q \in U\}$. Now, the collection $\{\{x\} : x \in X - \{q\}\}$ is locally finite in (X, τ^α) and there is no p-locally finite collection$\{G_\alpha : x \in X - \{q\}\}$ of preopen subset in (X, τ^α) such that $x \in G_\alpha$ and $x \in X - \{q\}$.

Definition 2.13. A space (X, τ) is said to be $\omega -$P-expandable if every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta,\ |\Delta| \leq \omega\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$.

Theorem 2.14. *Let (X, τ) be a space. Then (X, τ) is $\omega -$P-expandable if and only if every countable open cover of X has a p-locally finite preopen refinement.*

Proof. Sufficiency is similar to the proof of Theorem 2.8.

To prove necessity, let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open cover of X. Put $A_i = \cup\{U_j : j \leq i\}$ for each $i \in \mathbb{N}$. Let $B_1 = A_1$ and $B_i = A_i - A_{i-1}$ such that $i = 2, 3, 4, ...$ Therefore $B_i \subseteq U_i$ for each $i \in \mathbb{N}$. For $x \in X$, let $i(x) = \min\{i \in \mathbb{N} : x \in U_i\}$. Then $x \in B_{i(x)}$. Put $\mathcal{A} = \{B_i : i \in \mathbb{N}\}$. Then, $\mathcal{A}$ is a refinement of $\mathcal{U}$ and $\mathcal{A}$ is locally finite since $U_i \cap B_i = \phi$ for $j > i$. Because X is $\omega -$ P-expandable, there exists a p-locally finite collection $\{G_i : i \in \mathbb{N}\}$ of preopen subsets of X such that $B_i \subseteq G_i$ for each $i \in \mathbb{N}$. Let $V_i = U_i \cap G_i$ for each $i \in \mathbb{N}$. By Lemma 1.1, V_i is preopen set in (X, τ) for each $i \in \mathbb{N}$. Let $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$. Since $\{G_i : i \in \mathbb{N}\}$ is p-locally finite, $\mathcal{V}$ is p-locally finite. Because $\mathcal{A}$ is a cover of X, there exists some $i \in \mathbb{N}$ such that $x \in B_i$ for each $x \in X$. Since $B_i \subseteq V_i$, $x \in V_i$. Thus, $\mathcal{V}$ is a p-locally finite preopen refinement of $\mathcal{U}$. □

3. Operations

In this section we study some basic operation on P-expandable spaces.

Definition 3.1. A subset A of a space (X, τ) is called an:

i. α P-expandable set in (X, τ) if every locally finite (in X) collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of A there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of (X, τ) such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

ii. β P-expandable set in (X, τ) if and only if (A, τ_A) is P-expandable.

Note that α P-expandable and β P-expandable sets are linearly independent. To see that we give the following examples.

Example 3.2. Let $X = \mathbb{R}$ with the topology $\tau = \{U : 1 \in U\} \cup \{\phi\}$. Note that the set of all $PO(X, \tau) = \tau$. Put $A = \mathbb{R} - \{1\}$. Then $A \notin PO(X, \tau)$ and $\tau_A = \tau_{dis}$. Therefore, A is β P-expandable but not α P-expandable.

Example 3.3. Let (X, τ) be as in Example 2.2 and let $A = \mathbb{N}^- \cup \{1\}$. Then $\tau_A = \{\{1\} \cup H : H \subseteq \mathbb{N}^-\} \cup \{\phi\}$ and $PO(A, \tau_A) = \tau_A$. To show that (A, τ_A) is not βP-expandable we consider the collection $\mathcal{U} = \{\{x\} : x \in \mathbb{N}^-\}$. Then $\mathcal{U}$ is a locally finite collection of subsets of A and note that if $\mathcal{U}$ is a locally finite (this equivalent p-locally finite) collection of open (preopen) subsets of (A, τ_A), then $\mathcal{U}$ is finite. Therefore, (A, τ_A) is not β P-expandable. On the other hand, A is α P-expandable. Indeed, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite (in X) collection of subsets of A. Then $1 \notin U_\alpha$ for every $\alpha \in \Delta$. As in Example part (2) we show that there exists a p-locally finite collection $\rho = \{P_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $U_\alpha \subseteq P_\alpha$ for all $\alpha \in \Delta$. Thus A is α P-expandable.

A subset A of a space (X, τ) is called pre-clopen if A is preopen and preclosed.

Theorem 3.4. *Let A and B be subsets of a space (X, τ) such that $A \subseteq B$.*

i. If B is pre-clopen in (X, τ) and A is α P-expandable in (B, τ_B) then A is α P-expandable in (X, τ).

ii. If B is semi-open in (X, τ) and A is α P-expandable in (X, τ), then A is α P-expandable in (B, τ_B).

Proof. i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of A. Then there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of (B, τ_B) such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Since B is pre-clopen subset in (X, τ), then , by Lemma 1.1, $\mathcal{G}$ is p-locally finite collection of preopen subsets of (X, τ). For, let $x \in X$. Then either $x \in B$ or $x \notin B$. If $x \in B$, then there exists a preopen set W in (B, τ_B) containing x such that W intersects at most finitely many members of $\mathcal{G}$. Since B is preopen in (X, τ) then W is preopen in (X, τ), by Lemma 1.1 and hence $\mathcal{G}$ is p-locally finite collection in (X, τ). However, if $x \notin B$, then $X - B$ is preopen set in (X, τ) containing x which intersects no member of $\mathcal{G}$. Hence $\mathcal{G}$ is a p-locally finite collection in (X, τ).

ii) Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of A. Then there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of (X, τ) such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Now consider $\mathcal{G}^* = \{G_\alpha \cap B : \alpha \in \Delta\}$, by Lemma 1.1, $\mathcal{G}^*$ is a p-locally finite collection of preopen subset of (B, τ_B) such that $F_\alpha \subseteq G_\alpha \cap B$ for all $\alpha \in \Delta$. Thus A is α P-expandable in (B, τ_B). □

Corollary 3.5. *Let A be a subset of a space (X, τ).*

i. If A is pre-clopen in (X, τ) and β P-expandable, then A is α P-expandable.

ii. If A is semi-open in (X,τ) and $\alpha\,$P-expandable, then A is $\beta\,$P-expandable.

Note that Example 3.2 shows that the assumption A is pre-clopen in Corollary 3.5 can not be replaced by the statement A is preclosed.

Lemma 3.6. *If A is a closed subset of a space (X,τ), then any locally finite collection of subsets of A is a locally finite collection in X.*

Proposition 3.7. *Let (X,τ) be a P-expandable space, then:*

i. Every regular closed subset of (X,τ) is $\beta\,$P-expandable.

ii. Every closed subsets of (X,τ) is $\alpha\,$P-expandable.

Proof. i) Let A be a regular closed subset of a P-expandable space (X,τ). Let $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ be a locally finite collection of subset of A. Since A is closed by Lemma 3.6, $\mathcal{F}$ is locally finite in (X,τ), so there exists a p-locally finite collection of preopen subset of (X,τ), say $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ such that $F_\alpha\subseteq G_\alpha$, for each $\alpha\in\Delta$. Consider $\mathcal{G}^*=\{G_\alpha\cap A:\alpha\in\Delta\}$. Then, by Lemma 1.1 and the fact that $RC(X,\tau)\subseteq SO(X,\tau)$, $\mathcal{G}^*$ is a p-locally finite collection of preopen subsets of A such that $F_\alpha\subseteq G_\alpha\cap A$ for each $\alpha\in\Delta$. Thus A is $\beta\,$P-expandable.

ii) It is follow from Lemma 3.6. □

Let $\{(X_\alpha,\tau_\alpha):\alpha\in\Delta\}$ be a collection of topological spaces such that $X_\alpha\cap X_\beta=\phi$ for each $\alpha\neq\beta$. Let $X=\bigcup\limits_{\alpha\in\Delta}X_\alpha$ be topologized by $\tau=\{G\subseteq X: G\cap X_\alpha\in\tau_\alpha$ for each $\alpha\in\Delta\}$. Then (X,τ) is called the sum of the spaces $\{(X_\alpha,\tau_\alpha):\alpha\in\Delta\}$ and we write $X=\bigoplus\limits_{\alpha\in\Delta}X_\alpha$.

Theorem 3.8. *The topological sum $\bigoplus\limits_{\alpha\in\Delta}X_\alpha$ is P-expandable if and only if (X_α,τ_α) is P-expandable, for each $\alpha\in\Delta$.*

Proof. Necessity follows from Proposition 3.7. To prove sufficiency, let $\mathcal{U}$ be a locally finite collection of $\bigoplus\limits_{\alpha\in\Delta}X_\alpha$. For each $\alpha\in\Delta$ the family $\mathcal{U}_\alpha=\{U\cap X_\alpha:U\in\mathcal{U}\}$ is a locally finite collection of the P-expandable space (X_α,τ_α). Therefore there exists a p-locally finite collection $\mathcal{G}_\alpha=\{G_{U_\alpha}:U\in\mathcal{U}\}$ of a preopen subsets of (X_α,τ_α) such that for all $\alpha\in\Delta$, $U\cap X_\alpha\subseteq G_{U_\alpha}$ for all $U\in\mathcal{U}$. Put $G_U=\bigcup\limits_{\alpha\in\Delta}G_{U_\alpha}$ and $\mathcal{G}^*=\{G_U:U\in\mathcal{U}\}$. We note that (i) G_U is preopen in X for each $U\in\mathcal{U}$(by Lemma 1.1) (ii) G^* is p-locally finite in X. Let $x\in X$. Then there exists $\alpha_\circ\in\Delta$ such that $x\in X_{\alpha_\circ}$. So there exists a preopen subset $W_{\alpha_\circ}$ of $X_{\alpha_\circ}$ such that $W_{\alpha_\circ}$ intersects at most finitely many member of $\mathcal{G}_{\alpha_\circ}$, say $G_{U_{1_{(\alpha_\circ)}}},G_{U_{2_{(\alpha_\circ)}}},...G_{U_{n_{(\alpha_\circ)}}}$. Note that $G_{U_\beta}\cap W_{\alpha_\circ}=\phi$ for each $U\in\mathcal{U}$ and so for every $U\in\mathcal{U}-\{U_1,\ ....,\ U_n\}$, $W_{\alpha_\circ}\cap G_U=\phi$. Thus $\mathcal{G}^*$ is p-locally finite in X such that for each $U\in\mathcal{U}$, $U=U\cap X=U\cap(\bigcup\limits_{\beta\in\Delta}X_\beta)\subseteq G_U$. □

Theorem 3.9. *Let $f:(X,\tau)\to(Y,\sigma)$ be a M-preopen and strongly preclosed surjective continuous function such that $f^{-1}(y)$ is strongly compact relative to (X,τ) for every $y\in Y$. If (X,τ) is* P*-expandable then (Y,σ) is* P*-expandable.*

Proof. Assume that (X,τ) is P-expandable and $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ is a locally finite collection of subsets of Y. Then $f^{-1}(\mathcal{F})=\{f^{-1}(F_\alpha):\alpha\in\Delta\}$ is a locally finite collection of subsets of the P-expandable (X,τ) and so there is a p-locally finite $\{G_\alpha:\alpha\in\Delta\}$ of preopen subsets of X such that $f^{-1}(F_\alpha)\subseteq G_\alpha$ for each $\alpha\in\Delta$. Since f is M-preopen and by Theorem 1.6, the collection $f(G_\alpha)$ is p-locally finite collection of preopen subsets of Y such that $F_\alpha\subseteq f(G_\alpha)$. □

Theorem 3.10. *Let $f:(X,\tau)\to(Y,\sigma)$ be a countably perfect preirresolute continuous function. If (Y,σ) is* P*-expandable, then so is (X,τ).*

Proof. Let $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ be a locally finite collection of subsets of X, by Lemma 1.5, $\{f(F_\alpha):\alpha\in\Delta\}$ is a locally finite collection in Y. Hence there is a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen subsets of Y such that $f(F_\alpha)\subseteq G_\alpha$ for each $\alpha\in\Delta$. Then, by Theorem 1.6, $F_\alpha\subseteq f^{-1}f(F_\alpha)\subseteq f^{-1}(G_\alpha)$ and $\{f^{-1}(G_\alpha):\alpha\in\Delta\}$ is a p-locally finite collection of preopen subsets of X. □

It clear that every continuous open function is preirresolute and M-preopen [8].

Corollary 3.11. *Let (X,τ) be compact and (Y,σ) be* P*-expandable. Then the product space $(X,\tau)\times(Y,\sigma)$ is P-expandable.*

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A COLOR IMAGE ENCRYPTION SCHEME WITH SYNCHRONOUS PERMUTATION-DIFFUSION STRUCTURE

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Abstract. Permutation and diffusion are two basic principles in designing an image encryption algorithm. Almost all image encryption methods are based on a scheme that separates permutation and diffusion, namely, asynchronous permutation and diffusion scheme (APDS). This paper analyses the flaws of APDS and cracks it with a chosen plaintext attack, and then proposes a synchronous permutation-diffusion scheme (SPDS). Experimental simulations and performance evaluations in key space, key sensibility, correlation coefficient, Shannon entropy, differential attack and data loss/noise attacks all show that the proposed scheme processes better performance compared with the APDS and some others, and can ensure a secure communication in practical applications.
Keywords: Image encryption, Synchronous permutation-diffusion, Chaotic map.

1. Introduction

With the increasing degree of interconnection, openness, and sharing of computer networks, the Internet has rapidly developed and is used for an extensive number of applications. Data, images, and multimedia information have become the largest online information flow. It is precisely because of the popularization of the Internet, as well as the simplicity, visualization, and information richness of image information, image information has become the most common information transmitted on the Internet. However, images often include important personal information, business secrets, or even information containing national

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secrets. Therefore, it is essential to encrypt the images securely for transmitting. However, the encryption technology used in image transmission is now facing the danger of being cracked with the increase of computing power. Therefore, new encryption algorithms are required for the security of image transmission.

Two basic methods used in image encryption are permutation and diffusion [1]. In permutation, each coordinate in the image is changed, and in diffusion, pixel values of the image are modified. A good image encryption scheme should present good permutation and diffusion effects which make the original image into a noisy one and can also resist all kinds of attacks. A permutation-only image encryption scheme based on chaos has been proposed in [2] and a diffusion-only scheme has been proposed in [3]. In [7, 5, 8, 9, 10, 11, 12, 13, 6, 15, 14], encryption schemes based on permutation and diffusion structures are proposed, where [7, 5, 8, 9, 10, 11, 12, 13, 6, 14] encrypts images using permutation first and is followed by diffusion, or diffusion is first, followed by permutation. Image encryption with permutation and diffusion performed separately has a defect that cryptanalysts can exploit to obtain one of the key streams of permutation and diffusion first, and can then obtain another by certain chosen plaintext attacks [5, 4]. Authors in [15] try to encrypt images using a synchronous permutation-diffusion scheme, but throughout their entire algorithm, the scheme is no different from permutation and diffusion done separately.

Chaos is a deterministic, random process in a nonlinear dynamic system that is neither periodic nor convergent, and has a very sensitive dependence on initial values. Given a discrete chaotic system and iterating it with two very close initial values, the output results are completely uncorrelated. Therefore, by using the extremely sensitive dependence of the chaotic system on the initial conditions, we can obtain a large number of uncorrelated, random, and deterministic chaotic sequences which have been widely used in image encryptions [7, 8, 9, 10, 11, 12, 13, 6, 15, 14]. Chaotic image encryption schemes [3, 6, 4] are usually based on a low-dimensional chaotic map which has the problems of a short code period and low accuracy, and therefore cannot guarantee the security of the scheme [14]. Subsequently, high-dimensional chaotic, or spatiotemporal chaotic encryption schemes, have been proposed to overcome the problems [7, 8, 9, 10, 11, 12, 13, 15, 14].

In this paper, we find the defects of APDS and propose a chosen plaintext attack to crack the APDS scheme and further propose a synchronous permutation-diffusion scheme to remedy the defects of APDS. The proposed scheme first determines the permutation sequence using a two-dimensional chaotic map, and then permutates the position of pixels one by one, and diffuses image pixel values related to the current permutation sequence, their former encrypted pixel values and a chaotic key stream; therefore, the diffusion is closely dependent on permutation and an attacker cannot crack the scheme by respectively extracting key-streams of permutation and diffusion using a chosen plaintext attack.

The rest of the paper is organized as follows: Section 2 gives a brief review of an asynchronous permutation and diffusion scheme proposed in [6] and analyses its defects to further attack the scheme successfully. After that, a new image encryption scheme of synchronous permutation-diffusion is proposed in Section 3. Experimental performances of the proposed scheme are presented in Section 4, followed by the conclusions of the paper in Section 5.

2. APDS and its defects

2.1 Overview the APDS scheme

Image encryption using an APDS separately processes plain images with permutation and diffusion. The simple image encryption steps of this structure are listed as follows [6].

1. Reshape the plain image $I_{M\times N\times 3}$ into 1D array $P = \{p_1, p_2, \cdots, p_{3MN}\}$.
2. Iterate a 1D chaotic map $(3MN + N_0)$ times and throw out the former N_0 elements to obtain a sequence X with the length of $3MN$.
3. Sort X in ascending order to obtain an index sequence PX.
4. Let $P' = P(PX)$ be the permutated image matrix.
5. Let $D = mod(floor(X \times 10^{14}), 256)$ be the diffusion matrix.
6. Let $C(i) = mod(P'(i) + D(i), 256) \oplus C(i-1), i = 1, 2, \cdots, 3MN$ be the encrypted image matrix.
7. Circularly shift the elements in C towards left by the amount of lp: $C' = circshift(C, lp)$.
8. Reshape C into I_c with size of $M \times N \times 3$, then the final encryption image is I_c.

2.2 Defects of the APDS

Some defects of the above encryption algorithm are presented in detail below.

1. Vulnerable to CPA: using CPA as a chosen plaintext attack, adversaries can access the encryption machinery and choose arbitrary plaintexts from its corresponding ciphertexts. The adversary aims to obtain some useful information which helps divulge the other plaintexts' encrypted information with the same encryption scheme and the same secret keys.

   We choose two $M \times N \times 3$ plain images $I_1 = 0$ and $I_2 = 0$, and set one of the elements in I_2 to 1. Due to the encryption machinery available, the corresponding cipher image of I_1 and I_2 are denoted as I_{c1} and I_{c2}. Fig. 1 shows each step of the encryption process for two simple plain images. Next, we detail the steps of cracking APDS.

(a) Analyse and obtain key lp: As can be seen in Fig. 1, P_1 and P_2 are both 1D arrays which correspond to I_1 and I_2 respectively. Due to P_1 being a full 0 array, the permutation step is inoperative, thus the encryption algorithm is analogous to $C_1(i) = D(i) \oplus C_1(i-1)$, and $C_1' = circshift(C_1, lp)$. P_2 only has one element that equals 1; all other values equal 0. After permutated, the element 1 is moved to position s_1. The former s_1 elements of C_1 and C_2 are the same. The last lp elements of C_1' and C_2' are the former lp elements of C_1 and C_2 respectively. Thus, if $s_1 >= lp$, then lp is the number of identical elements between the tail of C_1' and C_2'; otherwise, lp can not be determined, and we should reset the index of element 1 in P_2 to meet the condition.

(b) Determine key stream D: After obtaining lp, we circularly shift the elements in C_1' towards the right lp times to get C_1, then $D(i) = C_1(i) \oplus C_1(i-1)$.

(c) Obtain the index array PX: We reselect a plain image and denote its 1D array as $P_{i1}, P_{i2}, \cdots, P_{in} (n = 3MN/256)$. The first 256 pixel values of P_{i1}, the 257th–512th pixel values of P_{i2}, $\cdots$, the $256(n-1)$th–$256n$th pixel values of P_{in} are all in the array $[0, 1, \cdots, 255]$ and the rest of the pixel values are all 0. We encrypt the n plain images to obtain their corresponding cipher images $I_{c1}, I_{c2}, \cdots, I_{cn}$ and then compare the n cipher images with I_1 respectively to extract all of the elements in PX.

The simple process of the CPA cryptanalysis algorithm is clearly shown in Fig. 2. A total of $\frac{3MN}{256} + 2$ images are needed to completely break the APDS.

2. Key stream generated by a low-dimensional chaotic map: A low-dimensional chaotic sequence cannot ensure the security of encryption schemes due to its shorter period and lower accuracy compared with high-dimensional maps and therefore cannot ensure the security of encryption schemes.

3. Synchronous permutation-diffusion scheme

Similar to the defects of APDS listed above, this section proposes a synchronous permutation-diffusion encryption scheme. In this scheme, 2D Logistic-adjusted-Sine map as Eq. (1) is used to generate the chaotic key streams [16].

$$\begin{cases} x(i+1) = \sin(\pi\mu(y_i + 3)x_i(1 - x_i)), \\ y(i+1) = \sin(\pi\mu(x_{i+1} + 3)y_i(1 - y_i)), \end{cases} \tag{1}$$

where $\mu \in [0, 1]$ is map parameter. When $\mu \in [0.37, 0.38] \cup [0.4, 0.42] \cup [0.44, 0.93] \cup \{1\}$, the LASM shows chaotic behavior. We iterate the map $3MN + N_0$ times

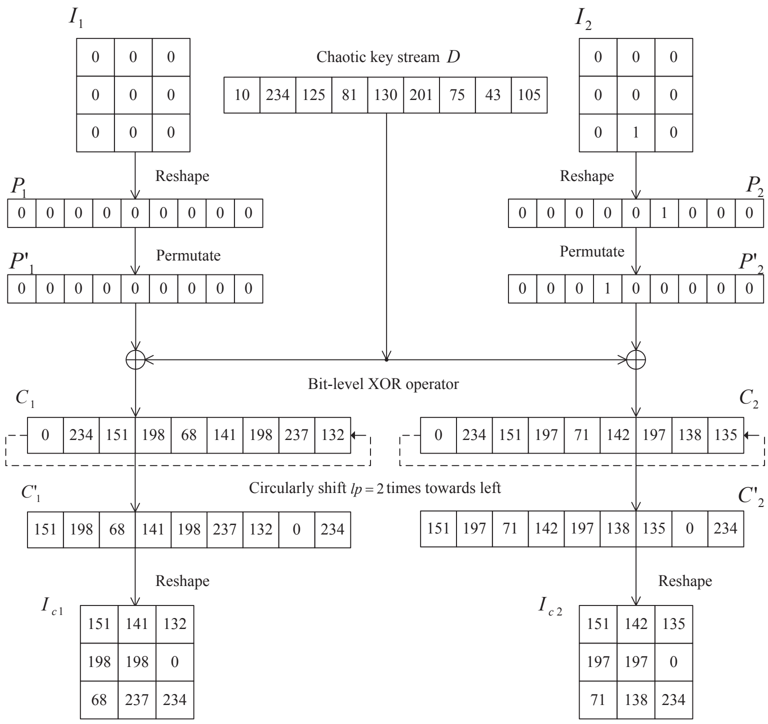


Figure 1: A simple demonstration of APDS image encryption

with proper initial values and parameters, and then discard the former N_0 data to obtain two chaotic sequences denoted as x, y. Then we obtain a permutation position matrix PX according to x and obtain a diffusion matrix D according to y. The detailed processes of the proposed encryption scheme are presented in the following subsection.

3.1 Encryption algorithm

1. Reshape the plain image $I_{M\times N\times 3}$ into 1D array $P = \{p_1, p_2, \cdots, p_{3MN}\}$.

2. Choose the secret keys μ, x_0, y_0, N_0 of LASM and iterate the chaotic map $(3MN + N_0)$ times and throw out the former N_0 elements to generate two chaotic sequences x, y with length $3MN$.

3. Sort x in ascending order to obtain an index sequence PX.

4. Permutate ith image pixel position with $P'(i) = P(PX(i))$.

5. Make $D(i) = mod(floor(y(i) \times 10^{14}), 256)$ the diffusion value.

6. Rotate $lp = mod(PX(i), 8)$ times.

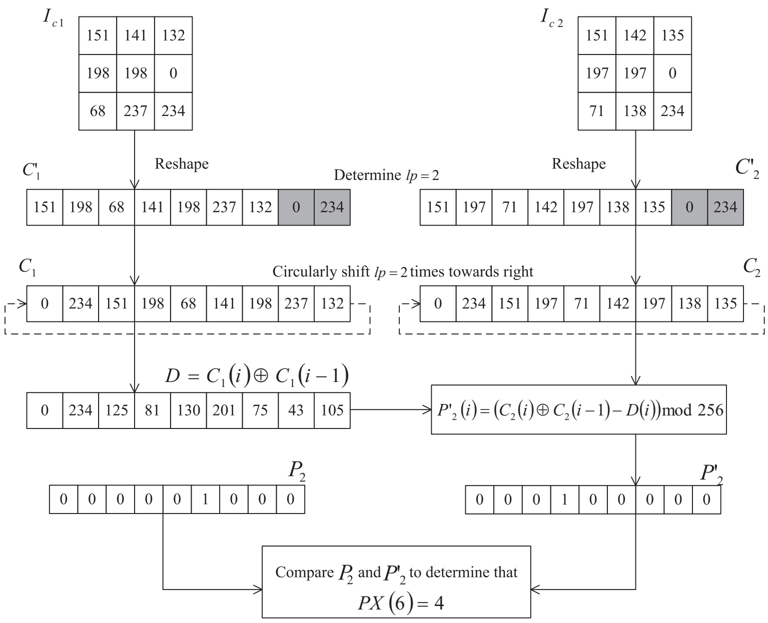


Figure 2: The process of cryptanalysis

7. Circularly shift the binary number that corresponds to the decimal number $C(i-1)$, towards the left by the amount of lp: $C'(i-1) = circshift(C(i-1), lp)$.

8. Encrypt the ith image pixel value by $C(i) = mod(P'(i) + D(i), 256) \oplus C(i-1)$.

9. Repeat steps 4 to 8 $3MN$ times to obtain all encrypted values in C.

10. Reshape C into I_c with a size of $M \times N \times 3$.

A flowchart of the encryption process is shown in the following Fig. 3. The decryption algorithm is an inverse process of the encryption algorithm.

The proposed algorithm encrypts plain images with permutation and diffusion associated encryption methods, thus, a cryptanalyst can break a cipher image only when he knows both PX and D. Therefore, the proposed scheme can effectively resist CPA.

4. Experimental results and discussion

The experimental simulations were run on desktop computer with an Intel(R) Core(TM) i5-3470 CPU 3.20GHz, 4GB RAM, and a 500GB hard drive. The operating system was Microsoft Windows 7 and the software run was Matlab

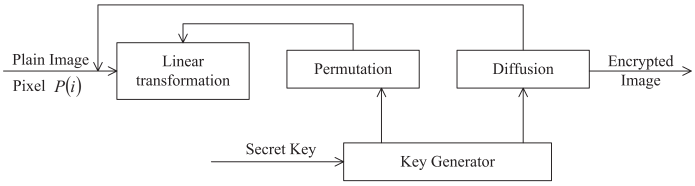
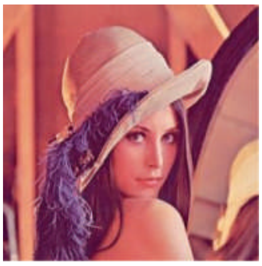
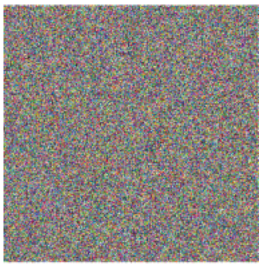


Figure 3: SPDS image encryption scheme

8.3.0.532 (R2014a). In order to evaluate the performance of the proposed encryption, the standard color image Lena.bmp is $256 \times 256 \times 3$ pixels, as shown in Fig. 4 (a), it was chosen as the plain image. The encrypted image and decrypted image of Lena are shown in Fig. 4 (b) and (c) respectively. Fig. 4 (d-f) are the histograms of the plain, ciphered, and deciphered image corresponding to Fig. 4 (a-c) respectively. Fig. 4 (e) shows an uniform distribution histogram, therefore, the cipher image encrypted by the proposed scheme cannot provide any useful information about the plain image.

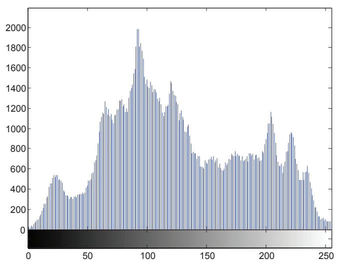


(a) The plain image

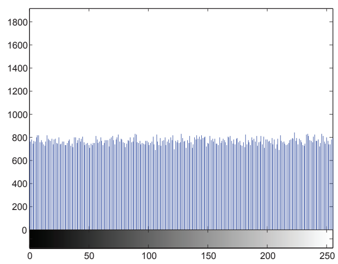


(b) The final encrypted image

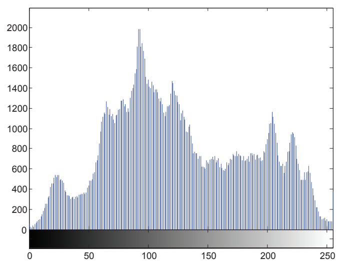
(c) The decrypted image



(d) Histogram of plain image



(e) Histogram of encrypted image



(f) Histogram of decrypted image

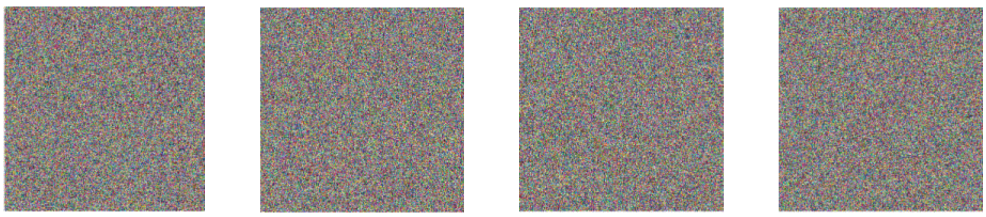
Figure 4: Plain image, encrypted image, decrypted image and their histogram.

4.1 Key space

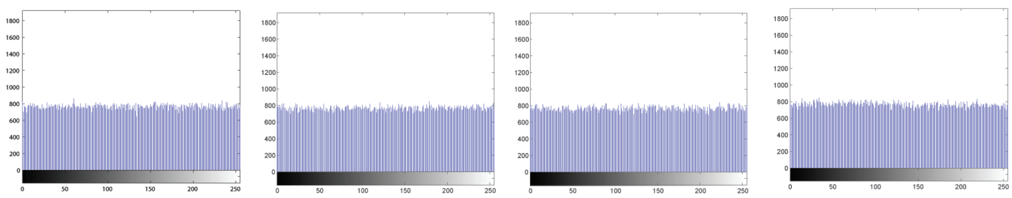
A larger space than 2^{100} is usually needed for an encryption scheme to resist a brute-force attack. Our proposed algorithm has four security keys: x_0, y_0, μ, N_0, where $x_0, y_0 \in (0,1]$ with accuracy 10^{16}, $\mu \in [0.37, 0.38] \cup [0.4, 0.42] \cup [0.44, 0.93] \cup \{1\}$ with accuracy 10^{16} and N_0 is a positive integer. Let $100 < N_0 < 10100$, then the space of N_0 is 10^4. The total key space can be calculated as $(10^{16})^3 \times 10^4 = 10^{52} \approx 2^{172}$, therefore, the proposed scheme can effectively resist a brute-force attack.

4.2 Key sensitivity

We encrypted the plain Lena image with secret keys $N_0 = 1000, x_0 = 0.3, y_0 = 0.4, \mu = 0.5$ to a cipher image, then we respectively changed one of the following four keys $N_0 = 1001, x_0 = 0.30001, y_0 = 0.40001, \mu = 0.50001$ and maintained the other three keys, then decrypted the ciphered Lena image with these changed keys. Fig. 5 (a-d) show the decrypted image with the wrong keys and their corresponding histogram are shown in Fig. 5 (e-h) respectively. From these figures, we know that a tiny change in any secret key can make the decrypted image noisy; because the decrypted image is noisy, we know the proposed scheme is extremely sensitive to a secret key.



(a) Decryption with wrong key N_0 (b) Decryption with wrong key x_0 (c) Decryption with wrong key y_0 (d) Decryption with wrong key μ



(e) Histogram of (a) (f) Histogram of (b) (g) Histogram of (c) (h) Histogram of (d)

Figure 5: Key sensitivity analysis

4.3 Correlation analysis

Due to the possibility that some images have strong correlation among adjacent pixels, cryptanalysts may be able to access some useful information. In our experiment, 2000 adjacent pixels of plain and cipher images were randomly selected from horizontal, vertical, and diagonal directions respectively, and their correlation can be calculated by Eq. (2).

$$r_{xy} = \frac{N^2 \cdot \operatorname{cov}(x,y)}{\sum_{i=1}^{N}(x_i - E_x)^2 \cdot \sum_{i=1}^{N}(y_i - E_y)^2}, \tag{2}$$

where $E_x = \frac{\sum_{i=1}^{N} x_i}{N}$, $\operatorname{cov}(x,y) = E((x-E_x)(y-E_y))$; x, y are the two neighboring pixels' sequences. Table 1 shows experimental data and compares the correlation of the proposed scheme with some other references. From the table we know that the plain image has high correlation coefficients, as close to 1, in all three channels, but the cipher image encrypted by the proposed scheme has correlation coefficients close to 0, which represents that the encrypted image is a random-like image. Thus, the proposed scheme dramatically randomizes adjacent image pixel values.

Table 1: Correlation of two adjacent pixels in the plain and encrypted Lena image with different encryption scheme.

Lena	Orientation	R component	G component	B Component
Plain-image	Horizontal	0.9336	0.9501	0.8989
	Vertical	0.9746	0.9693	0.9361
	Diagonal	0.9299	0.9288	0.8485
Ref. [3]	Horizontal	-0.0463	0.0435	0.0136
	Vertical	-0.0587	-0.0682	-0.0688
	Diagonal	-0.0200	-0.0052	0.0127
Ref. [4]	Horizontal	0.0005	0.0011	-0.0023
	Vertical	-0.0070	0.0001	0.0078
	Diagonal	0.0005	-0.0016	-0.0009
Ref. [6]	Horizontal	0.0038	0.0069	0.236
	Vertical	0.0026	0.0125	0.0054
	Diagonal	0.0017	0.0037	0.0296
Ref. [17]	Horizontal	0.0104	0.0095	-0.0215
	Vertical	-0.0029	0.0126	0.0135
	Diagonal	0.0123	-0.0116	-0.0304
Ref. [18]	Horizontal	0.0049	0.0054	0.0053
	Vertical	0.0031	0.0001	0.0022
	Diagonal	0.0007	0.0017	0.0007
Proposed	Horizontal	0.0001	0.0004	0.0023
	Vertical	0.0081	0.0101	0.0003
	Diagonal	0.0004	0.0016	0.0002

4.4 Shannon entropy

Shannon entropy [4] is defined to measure the randomness of the test image. The greater entropy corresponds to the more uniform image gray value distribution.

Shannon entropy of an 8-bit image is defined as follows:

$$H(m) = -\sum_{i=0}^{255} P(m_i) \log P(m_i), \tag{3}$$

where m_i is the ith gray value and $P(m_i)$ is the probability of symbol m_i in a test image. Entropy for an 8-bit true random image is 8, which shows that the pixel values in an image are completely random. Therefore, entropy of encrypted an image for a good encryption algorithm is close to 8, and the closer the entropy is to 8, the smaller the possibility of information disclosure for the encryption scheme.

Entropy of the different images and their cipher images, encrypted by different schemes, are shown in Table 2. From this table, we know that the proposed algorithm has an entropy that is closer to 8 compared to the other schemes, which illustrates that the proposed scheme can rarely leak any useful information.

Table 2: Shannon entropy of different image encrypted by the IECTM and the improved scheme.

Image	Plain image	Ref. [3]	Ref. [4]	Ref. [18]	Proposed
Lena	7.758377	7.990966	7.997287	7.9972	7.99903
Baboon	7.774815	7.991296	7.998973	7.9972	7.99907
Girl	6.904487	7.991575	7.998989	-	7.99902
Couple	6.300791	7.991349	7.999136	-	7.99907

4.5 Differential attack

The plain image's sensitivity to an encryption scheme is the measure of the influence of a tiny change in a plain image compared to the cipher image. Differential attacks aim to find the relationship between the two plain images and their cipher images. Therefore, a slight change in a sensitive plain image can result in a completely different cipher image, and cryptanalysts cannot gather any useful information. The degree of an image's sensitivity can be reflected by $NPCR$ and $UACI$ [4] which are shown in Eq. (4).

$$NPCR = \frac{\sum_{i,j} D(i,j)}{m \times n} \times 100\%, \tag{4}$$

$$UACI = \frac{1}{m \times n} \left(\sum_{i,j} \frac{|C(i,j) - C'(i,j)|}{255} \right) \times 100\%, \tag{5}$$

$$D_{i,j} = \begin{cases} 1, & \text{if } C(i,j) \neq C'(i,j), \\ 0, & \text{if } C(i,j) = C'(i,j), \end{cases} \tag{6}$$

where m, n are the size of the image; C and C' are two cipher images. An ideal $NPCR$ value is 1, which represents that a plain image has high sensitivity and therefore can resist differential attacks. The theoretical values of $UACI$ is 0.33 [4]. In our test, a randomly selected pixel from the plain image was changed, then we encrypted the changed and unchanged plain images to obtain two cipher images C and C', respectively. Comparison of $NPCR$ and $UACI$ scores between different encryption schemes are shown in Table 3 which illustrates that the proposed scheme has the largest $NPCR$ score and can therefore resist stronger differential attacks.

Table 3: $NPCR$ and $UACI$ of different encrypted image with one bit differ from the plain image.

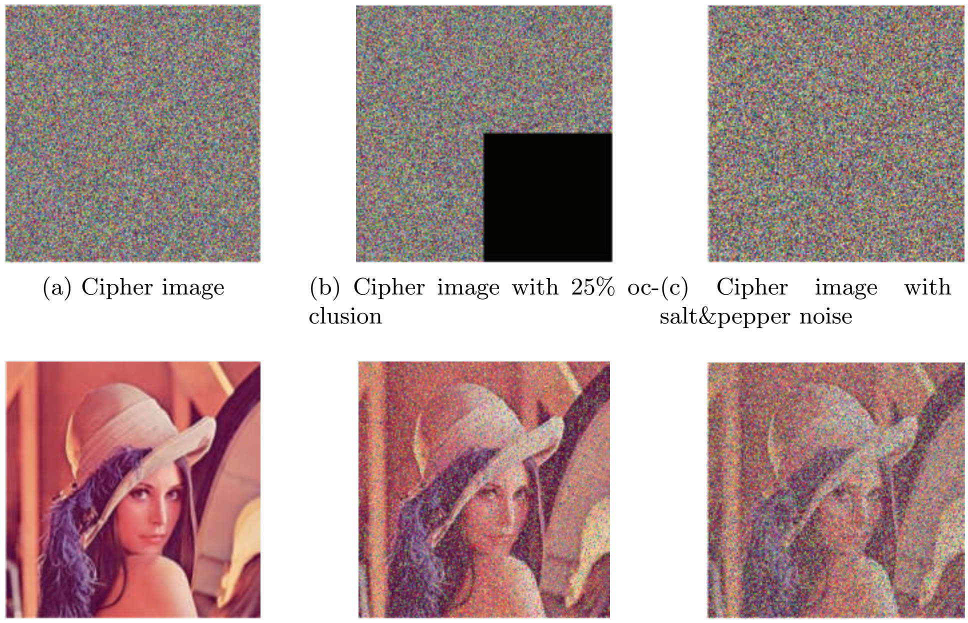
Encryption scheme		Lena	Baboon	Pepper	Barbarb
Ref. [6]	$NPCR$	0.9965	0.9955	0.9960	0.9940
	$UACI$	0.3348	0.3342	0.3340	0.3341
Ref. [3]	$NPCR$	5.0862×10^{-6}	5.0862×10^{-6}	5.0862×10^{-6}	5.0862×10^{-6}
	$UACI$	1.9946×10^{-8}	1.9946×10^{-8}	1.9946×10^{-8}	1.9946×10^{-8}
Ref. [17]	$NPCR$	0.9962	0.9943	0.9964	0.9960
	$UACI$	0.3377	0.3353	0.3353	0.3341
Ref. [18]	$NPCR$	0.9966	0.9965	0.9963	-
	$UACI$	0.3344	0.3350	0.3347	-
Proposed	$NPCR$	0.9967	0.9972	0.9970	0.9973
	$UACI$	0.3346	0.3350	0.3351	0.3355

4.6 Data loss and noise attack

When images are transferred or stored, cipher images, noise, and data loss are inevitable. A good encryption scheme should resist these influences well. In order to test the ability of the proposed scheme in resisting noise and data loss attacks, we decrypted two cipher images which both encrypted the Lena image of size 256×256 and one was cut as Fig. 6 (b) , another was influenced by 3% 'salt&pepper' noise as Fig. 6 (c), and their deciphered images are Fig. 6 (e,f) which both contain massive visual information of the original Lena image as Fig. 6 (d). PSNR (Peak Signal to Noise Ratio) expressed in Eq. (7) is employed to evaluate the ability of a scheme to restore an image. The larger the PSNR value, the better the image recovered. Table 4 compares $NPCR$ coefficients between [6] and the proposed scheme. Experimental results show that the proposed scheme has excellent performance against noise attacks.

$$\begin{cases} PSNR = 10 \times \lg \dfrac{255^2}{MSE}, \\ MSE = \dfrac{1}{W \times H} \sum_{i=0}^{H} \sum_{j=0}^{W} (P(i,j) - D(i,j))^2, \end{cases} \tag{7}$$

where M, N are the height and the weight of image respectively; $P(i,j)$ and $D(i,j)$ are the pixels from the plain and deciphered images respectively.



(a) Cipher image (b) Cipher image with 25% occlusion (c) Cipher image with 3% salt&pepper noise

(d) Decipher image of (a) (e) Decipher image of (b) (f) Decipher image of (c)

Figure 6: Robustness against data loss and noise attacks.

Table 4: Comparison the PSNR of the different scheme.

Encryption scheme	Ref. [6]	Proposed
Data loss	47.4199	77.0979
Noise attack	48.1770	70.0263

5. Conclusion

In this paper, we first introduced image encryption based on asynchronous permutation and diffusion, then presented its defects and proposed a synchronous permutation-diffusion encryption scheme to overcome the defects. Experiments and security comparisons between the synchronous and other schemes show that the proposed has better encryption effects and keeps all the merits of asynchronous ones.

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INTEGRAL EQUATION FOR THE NUMBER OF INTEGER POINTS IN A CIRCLE

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Abstract. The problem is to obtain the most accurate upper estimate for the absolute value of the difference between the number of integer points in a circle and its area (when the radius tends to infinity). In this paper we obtain an integral equation for the function expressing the dependence of the number of integer points in a circle on its radius. The kernel of the equation contains the Bessel functions of the first kind, and the equation itself is a kind of the Hankel transform.

Keywords: Gauss circle problem, integral equation, Hankel transform.

1. The problem and calculations

The Gauss circle problem is the problem of determining how many integer lattice points there are in a circle centered at the origin and with given radius. Let us consider the circle $K(R) : x^2 + y^2 \leq R$ and let $A(\sqrt{R})$ be the number of

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points with integer coordinates within this circle. As R increases, $A(\sqrt{R})$ is approximately equal to the area inside the circle πR . Let us define $\Delta(R) = |A(\sqrt{R}) - \pi R|$. The Gauss circle problem consists in estimating the upper bound for $\Delta(R)$ when $R \to \infty$ as much precisely as possible.

Similarly, the Dirichlet divisor problem consists in finding the number of integer points under the hyperbola $K_1(R) : xy \leq R,\ 0 < x \leq R,\ 0 < y \leq R$. The problems of Gauss and Dirichlet were investigated by many authors. Gauss himself found the estimation $O(R^{1/2})$ for$\Delta(R)$. Voronoy [1] obtained the result $O(R^{1/3} lnR)$ for the Dirichlet problem, while Serpinksij found the estimation $O(R^{1/3})$ in 1903 and Hua $O(R^{13/40})$ in 1942 [2] for the circle. Hardy and Littlewood [3] proved that it is impossible to get a better estimation than $O(R^{1/4} ln^2 R)$. Probably the most precise estimation for the circle up to now is $O(R^{131/208})$ obtained by Huxley in the early 2000s [4]. To give the readers more information about the problem, they can take a look to the references [5, 6]. Our aim is to derive an integral equation for the number of integer points in a circle.

Let $\delta_\epsilon(x) = 1 + \sum_{n=1}^{\infty} \frac{sin\, 2\pi n\epsilon}{2\pi n\epsilon}\, cos\, 2\pi nx$ be a periodic function with period 1, which is tending to the periodic delta-function of Dirac when $\epsilon \to 0$. Let us consider the integral

$$A_\epsilon(\sqrt{R}) = \int_K \int_{(R)} \delta_\epsilon(x)\delta_\epsilon(y)\, dx\, dy. \tag{1}$$

We can see that $\lim_{\epsilon\to 0} A_\epsilon(\sqrt{R}) = A(\sqrt{R})$. Let us calculate the integral in the right hand side of (1). The function $\delta_\epsilon(x)$ may be written as follows: $\delta_\epsilon(x) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$, where $c_n = \frac{sin\, 2\pi n\epsilon}{2\pi n\epsilon}$ for $n \neq 0$ and $c_n = 1$ for $n = 0$. Then

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \int_K \int_{(R)} \delta_\epsilon(x)\delta_\epsilon(y)\, dx\, dy \\ &= \int_0^{\sqrt{R}} r\, dr \int_0^{2\pi} \sum_{n,m=-\infty}^{+\infty} c_n c_m e^{2\pi i r\, (n\, cos\, \phi + m\, sin\, \phi)}\, d\phi. \end{aligned}$$

After changing the order of integration and summation and transforming the function $n\, cos\, \phi + m\, sin\, \phi$, the expression takes the form:

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r\, dr \int_0^{2\pi} e^{2\pi i r\, \sqrt{n^2+m^2} cos(\phi+\phi_0)}\, d\phi \\ &= \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r\, dr \int_0^{2\pi} e^{2\pi i r\, \sqrt{n^2+m^2} cos(\phi)}\, d\phi. \end{aligned}$$

Taking into account that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{ix\, cos\, \phi}\, d\phi$ is the Bessel function of the first kind and zero order, we get:

$$A_\epsilon(\sqrt{R}) = 2\pi \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r\, J_0(2\pi r\sqrt{n^2+m^2})\, dr. \tag{2}$$

Using the relation

$$xJ_0(x) = \frac{d}{dx}\left(xJ_1(x)\right), \tag{3}$$

where $J_1(x)$ is the Bessel function of the first kind, we get by integrating from (2):

$$A_\epsilon(\sqrt{R}) = \sqrt{R}\sum_{n,m=-\infty}^{+\infty} c_n c_m \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}}. \tag{4}$$

Since C_n tend to 1 while $\epsilon \to 0$ we get:

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \sqrt{R}\sum_{n,m=-\infty}^{+\infty} \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}} \\ &= \pi R + \sqrt{R}\sum_{n^2+m^2\neq 0} \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}}. \end{aligned}$$

The series for $A(\sqrt{R})$ does not converge absolutely, but it sums up to the number of integer points in a round area with radius $R^{1/2}$ (if R is an integer, the points on the circumference are counted with the coefficient 1/2). The expression $A(\sqrt{R})$ may be rewritten in the following way:

$$\begin{aligned} A(\sqrt{R}) &= \lim_{\epsilon\to 0}\sqrt{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_1(2\pi\sqrt{R}\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\delta_\epsilon(x)\,\delta_\epsilon(y)dxdy \\ &= \lim_{\epsilon\to 0}\sqrt{R}\int_0^{\infty} J_1(2\pi r\sqrt{R})\,dr\int_0^{2\pi}\delta_\epsilon(r\,cos\,\phi)\,\delta_\epsilon(r\,sin\,\phi)\,d\phi. \end{aligned}$$

Since (see above)

$$\int_0^{2\pi}\delta_\epsilon(r\,cos\,\phi)\,\delta_\epsilon(r\,sin\,\phi)\,d\phi = 2\pi\sum_{n,m=-\infty}^{+\infty} c_n c_m\,J_0(2\pi r\sqrt{n^2+m^2}),$$

we get:

$$A(\sqrt{R}) = \lim_{\epsilon\to 0} 2\pi\sqrt{R}\int_0^{+\infty} J_1(2\pi r\sqrt{R})\sum_{n,m=-\infty}^{+\infty} c_n c_m\,J_0(2\pi r\sqrt{n^2+m^2})\,dr.$$

Let us denote $\sqrt{R} = \rho$. Using the property (3) of Bessel functions and integrating by parts, we get:

$$A(\rho) = \lim_{\epsilon\to 0}\int_0^{+\infty}\left(\frac{1}{r}J_1(2\pi\rho r) - 2\pi\rho J_1'(2\pi\rho r)\right)\left(\sum_{n,m=-\infty}^{+\infty} c_n c_m\frac{J_1(2\pi r\sqrt{n^2+m^2})}{\sqrt{n^2+m^2}}\right)dr.$$

In view of (4), the following expression is derived:

$$A(\rho) = \lim_{\epsilon \to 0} \int_0^{+\infty} \left(\frac{1}{r^2} J_1(2\pi\rho r) - \frac{2\pi\rho}{r} J_1'(2\pi\rho r) \right) A_\epsilon(r)\, dr.$$

Now, replacing $J_1'(2\pi\rho r)$ according to formula (3) and taking the limit $\epsilon \to 0$ we finally obtain the integral equation:

$$A(\rho) = \int_0^{+\infty} A(r)\, K(\rho,\, r)\, dr, \tag{5}$$

where the core is

$$K(\rho,\, r) = \frac{2}{r^2} J_1(2\pi\rho r) \;-\; \frac{2\pi\rho}{r} J_0(2\pi\rho r).$$

2. Conclusion

Let us note that the integral transform $F(\rho) = \int_0^{+\infty} A(r)\, K(\rho, r)\, dr$ with the core $K(\rho, r) = r\, J_s(2\pi\rho r)$, $s \geq -1/2$ is known as the Hankel transform. Thus, for the function expressing the dependence of the number of integer points in a circle on its radius, the integral equation is obtained, which is a kind of the Hankel transform. It can be used for further investigations of the Gauss circle problem. Also of interest is a possible generalization of the methodology and applying it to other similar problems (for example, the Dirichlet divisor problem).

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A NEW CHARACTERIZATION OF SIMPLE JANKO-GROUPS

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Abstract. It is proved that simple Janko-groups J_1, J_2, J_3 and J_4 can be determined by their order and one irreducible character degree.
Keywords: Finite group, simple group, character degree, characterization.

1. Introduction and preliminary results

Let G be a finite group and $cd(G)$ the set of all complex irreducible character degrees of G. Characters of a finite group can give some important information of the group's structure.

In [1], Huppert posed the following conjecture: if H is a finite non-abelian simple group such that $cd(G) = cd(H)$, then $G \cong H \times A$, where A is an abelian group. And it was verified that the conjecture holds for many non-ableian simple

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groups. Also it was proved that a finite simple group can be uniquely determined by its character table in [2]. In this paper, we manage to characterize the finite simple groups by less character degrees. In fact, we shall prove the four simple Janko-groups, J_1, J_2, J_3 and J_4 can be unique determined by the group order and one irreducible character degree. The following theorems will be proved:

Theorem A. *Simple Janko-groups J_1, J_3 and J_4 can be uniquely determined by their orders and the largest irreducible character degrees.*

Theorem B. *Janko-group J_2 is uniquely determined by its order and the second largest irreducible character degree.*

For convenience we denote the largest irreducible character degree of G as $L(G)$, and the second largest irreducible character degree of G as $S(G)$. We use $A\ .\ B$ to denote any group having a normal subgroup isomorphic to, for which the corresponding quotient group isomorphic to B, and use Z_n to denote the cyclic group of order n. All further unexplained symbols and notations are standard and can be found, for instance, in [3] and [4].

In order to prove the above theorems, we need the following lemmas:

Lemma 1. *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.*

Proof. Let H be a normal subgroup of G, which is maximal such that G/H is non-solvable. Then if K/H is a chief factor, it is a direct product of isomorphic non-abelian simple groups, and also it is the unique minimal normal subgroup of G/H, and from this, the desired conclusion is immediate.

Lemma 2. *Suppose that a Sylow $p-$subgroup of a solvable group G is not normal. Then some prime power dividing $|G|$ and exceeding 1 is congruent to 1 mod p.*

Proof. Let P be a Sylow $p-$subgroup of G. If P is not normal, let N be its normalizer, and let $N \leq M$, where M is maximal in G. Then $|G : M|$ is a prime power and $|G : N| = |G : M| \cdot |M : N|$. By Sylow's theorem, $|G : N|$ and $|M : N|$ are congruent to 1 mod p, so $|G : M|$ is congruent to 1.

Lemma 3. *Let G be a finite solvable group and $|G| = 2^\alpha \cdot 3^\beta \cdot 5^2 \cdot 7^\gamma$, where $\alpha \leq 7$, $\beta \leq 3$, $\gamma \leq 1$. Then $O_5(G) \neq 1$.*

Proof. If $O_5(G) = 1$, then a Sylow $5-$subgroup P acts faithfully on the Fitting subgroup F of G for G is solvable. Consequently, P acts nontrivially on some Sylow p'-subgroup of F, which implies that there exists a power of 2, 3 or 7 dividing $|G|$ that is congruent to 1 mod 5. Since the only prime power q dividing $|G|$ that is congruent to 1 mod 5 is $q = 2^4$, it follows that P acts faithfully on the Sylow 2−subgroup of F, which means some 2−group of order dividing 2^7 having an automorphism group of order divisible by 25. But the order of $GL_7(2)$ cannot be divided by 25, this is a contradiction.

Lemma 4. *Let G be a non-solvable group. Suppose that G has a normal series $1 \trianglelefteq K \trianglelefteq G$ such that K is abelian and G/K is a non-abelian simple group. If $Aut(K)$ does not contain any simple section isomorphic to G/K. Then $K = Z(G)$.*

Proof. It follows straightforward from N/C theorem by observing.

Lemma 5. *Let $H \trianglelefteq G$ such that $G/H \cong A_5$ and let $\sigma \in Irr(H)$, $\varphi \in Irr(G)$ with $[\varphi_H, \sigma] \neq 0$. Suppose σ is invariant in G and $5 \mid \varphi(1)/\sigma(1)$. Then φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. Moreover, $\varphi(1) = 5 \cdot \sigma(1)$.*

Proof. Write $\sigma^G = e_i \varphi_i$, where $\varphi_i \in \mathrm{Irr}(G)$ and $\varphi_1 = \varphi$, e_i are positive integers. Since σ is invariant in G, we have $(\varphi_i)_H = e_i \sigma$. Therefore, $\varphi_i(1) = e_i \cdot \sigma(1)$ and

$$|G:H|\sigma(1) = \sigma^G(1) = \sum e_i \cdot \varphi_i(1) = \sum e_i^2 \cdot \sigma(1),$$

so that $\sum e_i^2 = |G:H| = 60$. By the assumption, $5 \mid e_1$. Thus the fact $e_1^2 \leq \sum e_i^2 = 60$ forces that $e_1 = 5$. If some $e_i = 1$, then σ is extendible to φ_i. Hence $\varphi_i \lambda$ for $\lambda \in \mathrm{Irr}(G/H)$ are all of the irreducible constituents of σ^G. Since $H/A \cong A_5$, we have that φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. If all of $e_i > 1$, it follows by $\sum e_i^2 = 60$ and $e_1 = 5$ that $e_i \neq 5$ for $i \geq 2$. Therefore φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. Moreover, $\varphi(1) = e_1 \cdot \sigma(1) = 5 \cdot \sigma(1)$.

2. Proof of Theorems

Proof of Theorem A. We write the proof in several cases.

Case 1. Let G be a group having the same order of J_1 and having an irreducible character χ with $\chi(1) = L(J_1)$, we prove that $G \cong J_1$.

By [3], $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $\chi(1) = L(J_1) = 11 \cdot 19$. If $O_{19}(G) \neq 1$, then $O_{19}(G)$ is abelian and thus $\xi(1) \mid |G/O_{19}(G)|$ for every $\xi \in \mathrm{Irr}(G)$. But $\chi(1) = 11 \cdot 19 \nmid |G/O_{19}(G)|$, a contradiction. Therefore $O_{19}(G) = 1$. Hence G is non-solvable by Lemma 2. Therefore by Lemma 1 G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, we have K/H is isomorphic to one of A_5, $L_2(7)$, $L_2(11)$ and J_1.

Suppose $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|G/K| = 1$ or 2, and get $|H| = 2^t \cdot 7 \cdot 11 \cdot 19$, where $t = 0$ or 1. Clearly H is solvable for 15 does not divides $|H|$, thus Sylow 19−subgroups of H is normal in H by Lemma 2, which of course is normal in G too, a contradiction. Similarly we can prove that $K/H \ncong L_2(7)$ and $L_2(11)$.

Now, we have $K/H \cong J_1$, which concludes $G \cong J_1$ eventually.

Case 2. Let G be a group having the same order of J_3 and having an irreducible character χ with $\chi(1) = L(J_3)$, we prove that $G \cong J_3$.

In this case, $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, and $\chi(1) = L(J_3) = 2 \cdot 3^4 \cdot 19$. Then $O_{19}(G) = 1$ by the similar reason as in Case 1. Further, G is non-solvable by Lemma 2 and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$ by Lemma 1. As $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, we have that K/H is isomorphic to one of A_5, A_6, $L_2(17), L_2(19), L_2(16), U_4(2)$ and J_3.

Suppose $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^\alpha \cdot 3^4 \cdot 17 \cdot 19$ where $\alpha = 4$ or 5. If H is solvable, then $O_{19}(H) \neq 1$ by Lemma 2, a contradiction to $O_{19}(G) = 1$. Now, we have H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that B/A is a direct product of isomorphic non-abelian simple groups and $|H/B| \mid |Out(B/A)|$. As $|H| = 2^\alpha \cdot 3^4 \cdot 17 \cdot 19$, one has that $B/A \cong L_2(17)$. Since $|Out(B/A)| = 2$, it follows that $|A| = 2^\beta \cdot 3^2 \cdot 19$ where $\beta = 1$ or 2. Clearly A is solvable, hence $O_{19}(A) \neq 1$ by Lemma 2. Consequently $O_{19}(G) \neq 1$, a contradiction.

If $K/H \cong A_6$ or $L_2(16)$, we get $O_{19}(G) \neq 1$ by the same arguments as before, a contradiction too.

Now, assume that $K/H \cong L_2(17)$. Since $|Out(L_2(17))| = 2$, we have $|H| = 2^\alpha \cdot 3^3 \cdot 5 \cdot 19$ where $\alpha = 2$ or 3. Let θ be an irreducible constituent of χ_H. Since $\chi(1)/\theta(1) \mid |G/H| = 2^\beta \cdot 3^2 \cdot 17$, where $\beta = 4$ or 5, we have that $3^2 \cdot 19 \mid \theta(1)$. Thus $\theta(1)^2 \geq 3^4 \cdot 19^2 > |H|$, a contradiction.

Suppose $K/H \cong L_2(19)$. Since $|Out(L_2(19))| = 2$, we have $|H| = 2^\alpha \cdot 3^3 \cdot 17$ where $\alpha = 4$ or 5. Let P be a Sylow $17-$subgroup of H. If H is solvable, then $P \trianglelefteq H$ by Lemma 2, and thus $P \trianglelefteq G$. Let φ be an irreducible constituent of χ_P. Since P is abelian, $\varphi(1) = 1$. Noticing $|Aut(P)| \cong Z_{16}$ and $t = |G : I_P(\varphi)|$ divides both of $|Aut(P)|$ and $\chi(1)$, we have that $t \leq 2$, which implies that $e = [\chi_P, \varphi] = 2 \cdot 3^4 \cdot 19$ or $3^4 \cdot 19$, and thus $[\chi_P, \chi_P] = e^2 \cdot t \geq 2 \cdot 3^8 \cdot 19^2 > |G : P|$, a contradiction. Hence H is non-solvable. Then H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong L_2(17)$. Since $|Mult(L_2(17))| = |Out(L_2(17))| = 2$, we have that H is isomorphic to one of $Z_3 \times L_2(17)$, $(Z_3 \times L_2(17))\ .\ Z_2$, $Z_3 \times SL_2(17)$ and $Z_6 \times L_2(17)$. Let θ be an irreducible constituent of χ_H. Since $\chi(1)/\theta(1) \mid |G/H|$, we have $3^2 \mid \theta(1)$. By the structure of H, we know that $\theta(1) = 3^2$ or $2 \cdot 3^2$ and H has at most 27 irreducible characters of degree $\theta(1)$, and thus $t = |G : I_G(\theta)| \leq 27$. Let U be a maximal subgroup of G containing $I_G(\theta)$. Then $1 \leq |G : U| \mid |G : I_G(\theta)|$. By checking maximal subgroups of $L_2(19)$ (see [3]), we have that $t = 1$, which implies that $e = [\chi_H, \theta] = 2 \cdot 3^2 \cdot 19$ or $3^2 \cdot 19$, and thus $[\chi_H, \chi_H] = e^2 \cdot t \geq 3^4 \cdot 19^2 > |G : H|$, a contradiction.

Therefore $K/H \cong J_3$, which concludes $G \cong J_3$ as we want.

Case 3. Let G be a group having the same order of J_4 and having an irreducible character χ with $\chi(1) = L(J_4)$, we prove that $G \cong J_4$.

Since the approach used in this case is the same as previous cases, we just write the idea of the proof. In this case, $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, and $\chi(1) = L(J_4) = 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37$. Since $29 \cdot 37 | \chi(1)$, we have that $O_{29}(G) = O_{37}(G) = 1$, and thus G is non-solvable by Lemma 2. Hence, by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, we have $K/H \cong A_5$, $L_3(2)$, A_6, $L_2(8)$, $L_2(11)$, A_7, $U_3(3)$, $L_2(23)$, M_{11}, $L_2(29)$, $L_2(31)$, A_8, $L_3(4)$, $L_2(32)$, M_{12}, M_{22}, $L_5(2)$, M_{23}, $U_3(11)$, M_{24} or J_4.

Except that K/H is not isomorphic to J_4, we can use the same approach to show that $O_{29}(G)$ or $O_{37}(G)$ is nontrivial by Lemma 2 and come to contradictions. Therefore we have that $K/H \cong J_4$, which concludes $G \cong J_4$.

Proof of Theorem B. By assumption and [3], we have that $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ and G has an irreducible character β with $\beta(1) = S(J_2) = 2^2 \cdot 3 \cdot 5^2$. If $O_5(G) \neq 1$, then $O_5(G)$ is abelian of order 5 or 5^2, and thus $\xi(1) \mid |G/O_5(G)|$ for every $\xi \in \mathrm{Irr}(G)$. But $\beta(1) = 2^2 \cdot 3 \cdot 5^2 \nmid |G/O_5(G)|$, a contradiction. Hence $O_5(G) = 1$, it follows that G is non-solvable by Lemma 3. In the following, we write the proof step by step.

Step 1. to prove that $O_7(G) = 1$.

Let $N = O_7(G) \neq 1$. Then $|N| = 7$. Since $G/C_G(N) \lesssim Aut(N)$, we have that $|G/C_G(N)| = 1, 2, 3$ or 6. Obviously, $O_7(G)$ is the Sylow 7-subgroup of G, so $C_G(N)$ has a normal subgroup M such that $C_G(N) \cong N \times M$. Notice that G is non-solvable, we have by Lemma 1 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq M \trianglelefteq C_G(N) \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|M/K| \mid |Out(K/H)|$. By $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, one has that $K/H \cong A_5,\ A_6,\ L_2(8),\ U_3(3)$ or $A_5 \times A_5$. We go on discussing the $G/C_G(N)$ case by case.

Case 1. to prove that $|G/C_G(N)| = 1$ is impossible.

Otherwise, $|M| = 2^7 \cdot 3^3 \cdot 5^2$. Let $\theta \in \mathrm{Irr}(M)$ such that $[\beta_M, \theta] \neq 0$. Then $\beta(1) = \theta(1) = 2^2 \cdot 3 \cdot 5^2$ and thus $\theta(1)^2 > |M|$, a contradiction.

Case 2. to prove that $|G/C_G(N)| = 2$ is impossible.

Otherwise, $|M| = 2^6 \cdot 3^3 \cdot 5^2$. Let $\theta \in \mathrm{Irr}(M)$ such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2 \cdot 3 \cdot 5^2$ or $2^2 \cdot 3 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |M|$, a contradiction. Hence $\theta(1) = 2 \cdot 3 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|H| = 2^\alpha \cdot 3^2 \cdot 5$, where $\alpha = 3$ or 4. We claim that H is non-solvable. Suppose that H is solvable. If $\alpha = 3$, then $O_5(H) \neq 1$ by Lemma 2, which implies that $O_5(G) \neq 1$, a contradiction. Therefore $\alpha = 4$. We assert that H has a normal series $1 \trianglelefteq R \trianglelefteq S \trianglelefteq H$ such that

$|R| = 2^4$, $|S/R| = 5$, R is elementary abelian and S/R acts fixed-point-freely on R. Since $O_5(H) \leq O_5(G)$ and $O_5(G)$=1, we have that $O_5(H) = 1$. Thus H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq C \trianglelefteq H$, such that $|B/A| = 2^4$, $|C/B| = 5$, B/A is elementary abelian and C/B acts fixed-point-freely on B/A.

Suppose first that $|A| = 3$. Because $H/C_H(A) \lesssim Aut(A)$, we see that $|C_H(A)| = 2^a \cdot 3^2 \cdot 5$, where $a = 3$ or 4. If $|C_H(A)| = 2^3 \cdot 3^2 \cdot 5$, then $O_5(H) \neq 1$ by Lemma 2, a contradiction. So $|C_H(A)| = 2^4 \cdot 3^2 \cdot 5$, that is $C_H(A) = H$. Let T be the Hall $\{2,5\}$-subgroup of C. Then T is characteristic in $C \trianglelefteq H$, and thus $T \trianglelefteq H$. Since $O_5(T) \leq O_5(G)$, we have $O_5(T) = 1$. It follows that $|O_2(T)| = 2^4$ and $O_2(T) \trianglelefteq H$. Therefore, $1 \trianglelefteq O_2(T) \trianglelefteq T \trianglelefteq H$ is the series as we want.

If $|A| = 3^2$, then by the same reason as above, $C_H(A) = H$, so $T \trianglelefteq H$, and thus $1 \trianglelefteq O_2(T) \trianglelefteq T \trianglelefteq H$ is the series as wanted.

If $|A| = 1$, then the normal series $1 \trianglelefteq B \trianglelefteq C \trianglelefteq H$ is what we want.

Since $GL(4, 2)$ has no subgroup of order $3^2 \cdot 5$, we have that $3 \mid |C_H(R)|$. It follows that $H \cong (S \times Z_3) \rtimes Z_3$ or $H \cong S \times E$, where $|E| = 9$. Let $\varphi \in Irr(H)$ such that $[\theta_H, \varphi] \neq 0$. Suppose that $H \cong (S \times Z_3) \rtimes Z_3$. Then H has exactly 45 linear characters and 3 irreducible characters of degree 15. Thus $\varphi(1) = 15$ and $t = |M : I_M(\varphi)| \leq 3$. Let U be a maximal subgroup of M containing $I_M(\varphi)$. Then $1 \leq |M : U| \mid |M : I_M(\varphi)|$. By checking maximal subgroups of A_5 (see [3]), we have that $t = 1$, which forces that $[\theta_H, \theta_H] = 2^2 \cdot 5^2 > |M : H|$, a contradiction. Suppose that $H \cong S \times E$, where $|E| = 9$. It is clear that S has exactly 3 non-linear irreducible characters of degree 5, and then H has exactly 45 linear characters and 27 irreducible characters of degree 5. Then $\varphi(1)$=5. Since both of S and E are the Hall subgroups of H, we have S and E are both normal in M. By checking the maximal subgroups of A_5, we know that the conjugate action of M/H on $\mathrm{Irr}_1(S)$ is trivial, and the orbit lengths under the conjugate action of M/E on $\mathrm{Irr}(E)$ is 1 or 6. Hence $t = |M : I_M(\varphi)| = 1$ or 6, which implies that $[\theta_H, \theta_H] \geq 2 \cdot 3 \cdot 5^2 > |M : H|$, a contradiction.

Now we have proved that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ or A_6 and $|H/B| \mid |Out(B/A)|$. Let $\varphi \in \mathrm{Irr}(H)$ and $\sigma \in \mathrm{Irr}(A)$ such that $e_1 = [\theta_H, \varphi] \neq 0$ and $e_2 = [\varphi_A, \sigma] \neq 0$.

Suppose that $B/A \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|A| = 2^b \cdot 3$, where $b \leq 2$. Suppose that $|A| = 2^2 \cdot 3$. By checking the order of G, we know that $K = M$ and $B = H$. If A is abelian, then $A = Z(H)$ by Lemma 4, and thus $A \trianglelefteq M$. By $\sigma(1) = 1$ and $|Aut(A)| \leq 6$, we have that $t = |M : I_M(\sigma)| \leq 6$, thus $[\theta_A, \theta_A] \geq 2 \cdot 3 \cdot 5^4 > |M : A|$, a contradiction. If A is non-abelian, then the first column of the character table of A is one of sequences $\{1, 1, 1, 3\}$ and $\{1, 1, 1, 1, 2, 2\}$. If $\sigma(1) = 1$, then $t_2 = |H : I_H(\sigma)| \leq 3$. By checking maximal subgroups of A_5, it is easy to get that $t_2 = 1$. Hence $e_2 = 5$, φ is the unique irreducible constituent of σ^H of degree 5 by Lemma 5. Therefore H has at most 4 irreducible characters of degree 5, which implies that $t_1 = |M : I_M(\varphi)| \leq 4$. By checking properties of maximal subgroups of A_5, it is easy to get that $t_1 = 1$, so that $[\theta_H, \theta_H] = 2^2 \cdot 3^2 \cdot 5^2 > |M : H|$, a contradiction. By the same discussion, we have that if $\sigma(1) \neq 1$, then $t_1 = 1$, and thus $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. Moreover we can get contradictions by the same arguments for the rest possibilities of $|A|$.

Now we consider the case that $B/A \cong A_6$. Since $|Out(A_6)| = 4$, we have that $|H/B|| \ 4$, which implies that $|A| = 1$ or 2. If $|A| = 2$, then $H \cong SL_2(9)$ or $H \cong Z_2 \times A_6$, which has at most 4 irreducible characters of degree 5 and at most 3 irreducible characters of degree 10. Then $\varphi(1) = 5$ or 10, and $|M : I_M(\varphi)| = 1$ by checking properties of maximal subgroups of A_6, which forces that $[\theta_H, \theta_H] \geq 3^2 \cdot 5^2 > |M : H|$, a contradiction. If $|A| = 1$, then $H \cong A_6$ or $H \cong A_6 \ . \ Z_2$. A contradiction appears through the same arguments.

While K/H is isomorphic to A_6 or $A_5 \times A_5$, we get contradiction by the same approach as above.

Case 3. To prove $|G/C_G(N)| = 3$ is impossible.

Otherwise, $|M| = 2^7 \cdot 3^2 \cdot 5^2$. Let $\theta \in$Irr(M) such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2^2 \cdot 3 \cdot 5^2$ or $2^2 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |M|$, a contradiction. Hence $\theta(1) = 2^2 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|K/H|| \ 2$, which implies that $|H| = 2^\alpha \cdot 3 \cdot 5$, where $\alpha = 4$ or 5. We claim that H is non-solvable. Suppose that H is solvable. Since $O_5(H) \leq O_5(G) = 1$, we have that $O_5(H) = 1$, which implies that $|O_2(H)| = 2^4$ or 2^5 and the elements of order 5 in H act nontrivially on $O_2(H)/O_2(H)'$. Furthermore $O_2(H)$ is elementary abelian while $|O_2(H)| = 2^4$, and $O_2(H)$ is extra special or elementary abelian while $|O_2(H)| = 2^5$. Since $O_2(H)$ *char* $H \trianglelefteq M$, we have $O_2(H) \trianglelefteq M$.

Suppose first that $|O_2(H)| = 2^5$. Then M has a normal series: $1 \trianglelefteq O_2(H) \trianglelefteq B \trianglelefteq H \trianglelefteq M$, such that $|B/O_2(H)| = 5$, $|H/B| = 3$ and $M/H \cong A_5$. By Lemma 4, $H/B = Z(M/B)$. Since $|Mult(A_5)| = 2$, one has that $M/B = H/B \times A_5$. Therefore M has a normal series $B \trianglelefteq H_1 \trianglelefteq M$ such that $H_1/B \cong A_5$ and $M/H_1 \cong H/B$. Let $\varphi \in$Irr(H_1) such that $[\theta_{H_1}, \varphi] \neq 0$. Then $\varphi(1) = 2^2 \cdot 5^2$ by Clifford Theorem, which means that $\varphi(1)^2 > |H_1|$, a contradiction. Now we have $|O_2(H)| = 2^4$. Let $D = O_2(H)$. Because $M/C_M(D) \leq Aut(D)$ and $5^2 \nmid |Aut(D)|$, we have that $5 \mid |C_M(D)|$. Recall that the elements of order 5 in H are not contained in $C_M(D)$, so $5^2 \nmid |C_M(D)|$. If $C_M(D)$ is solvable, then $O_5(C_M(D)/D) \neq 1$ by Lemma 2. Hence $O_5(C_M(D)) \neq 1$, so that $O_5(M) \neq 1$, a contradiction. Therefore $C_M(D)$ is not solvable. By Jordan-Hölder theorem, $C_M(D)/D$ has a section isomorphic to A_5. Meanwhile $|C_M(D)/D|$ equals one of $2^3 \cdot 3^2 \cdot 5$, $2^3 \cdot 3 \cdot 5$, $2^2 \cdot 3^2 \cdot 5$ and $2^2 \cdot 3 \cdot 5$. Let $\varphi \in$Irr$(C_M(D))$ such that $[\theta_{C_M(D)}, \varphi] \neq 0$. If $|C_M(D)/D| = 2^3 \cdot 3^2 \cdot 5$ or $2^3 \cdot 3 \cdot 5$, then $\varphi(1) = 2^2 \cdot 5$ by Clifford theorem. Let λ be an irreducible constituent of φ_D. Since λ is linear and invariant in $C_M(D)$, we have $[\varphi_D, \varphi_D] = 2^4 \cdot 5^2 > |C_M(D) : D|$, a contradiction. If $|C_M(D)/D| = 2^2 \cdot 3 \cdot 5$, then $\varphi(1) = 2^2 \cdot 5$ or $2 \cdot 5$, we still have $[\varphi_D, \varphi_D] \geq 2^2 \cdot 5^2 > |C_M(D) : D|$, a contradiction too. At last $|C_M(D)/D| = 2^2 \cdot 3^2 \cdot 5$ and $\varphi(1) = 2^2 \cdot 5$ or $2 \cdot 5$. Clearly $C_M(D)/D \cong Z_3 \times A_5$. Let $E/D = A_5$. Then φ_E is irreducible. By the same arguments as above, we get $[\varphi_D, \varphi_D] \geq 2^2 \cdot 5^2 > |E : D|$, a contradiction.

Now we have proved that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ and $|H/B| \mid |Out(B/A)|$. By $|Out(A_5)| = 2$, $|H/B||\ 2$, so $|A| = 2^b$, where $b = 1$, 2 or 3. Let $\varphi \in \mathrm{Irr}(H)$, $\sigma \in \mathrm{Irr}(A)$ such that $[\theta_H, \varphi] \neq 0$ and $[\varphi_A, \sigma] \neq 0$.

Suppose that $|A| = 2^3$. By checking the order of M, we know that $K = M$ and $B = H$. By Clifford theorem, $5 \mid \varphi(1)/\sigma(1)$ and $\sigma(1) = 1$ or 2. Assume first that $\sigma(1) = 1$. By Lemma 4 and 5, $A/A^{'} \leq Z(H/A^{'})$, and for each linear character λ of A, there is at most one irreducible constituent of λ^H with degree 5, which means that $\varphi(1) = 5$ and H has at most 8 irreducible characters of degree 5. Hence $|M : I_M(\varphi)| < 8$, which menas that $[\theta_H, \theta_H] \geq 2^4 \cdot 5 > |M : H|$, a contradiction. If $\sigma(1) = 2$, by the fact that A has exactly one irreducible character with degree 2, we have that σ is invariant in H. Hence by the same reasoning as above, we get $\varphi(1) = 10$, $|M : I_M(\varphi)| = 1$, and thus $[\theta_H, \theta_H] = 2^2 \cdot 5^2 > |M : H|$, a contradiction. For the rest possibilities of $|A|$, we can get contradictions by the same arguments as above.

By the same way, we get $K/H \ncong A_5 \times A_5$.

Now suppose $K/H \cong A_6$. Since $|Out(A_6)| = 4$, we have that $|K/H| \mid 4$, which implies that $|H| = 2^\alpha \cdot 5$, where $\alpha = 4$, 3 or 2. If $\alpha = 3$ or 2, then $O_5(H) \neq 1$ by Lemma 2, it follows that $O_5(M) \neq 1$, a contradiction. Now we have $\alpha = 4$. By $O_5(H) = 1$, it follows that $|O_2(H)| = 2^4$, meanwhile $O_2(H)$ is elementary abelian and $H/O_2(H)$ act fixed-point-freely on $O_2(H)$. Hence H has exactly 3 irreducible character of degree 5. Let φ be an irreducible constituent of θ_H such that $e = [\theta_H, \varphi] \neq 0$ and $t = |M : I_M(\varphi)|$. Then $\varphi(1)$=5 and $t = |M : I_M(\varphi)| \leq 3$ by Clifford theorem. Since $K/H \cong A_6$, we see that $t = 1$, which forces that $[\theta_H, \theta_H] = 2^4 \cdot 5^2 > |M : H|$, a contradiction.

Case 4. to prove that $|G/C_G(N)| = 6$ is impossible.

Otherwise, $|M| = 2^6 \cdot 3^2 \cdot 5^2$. Let $\theta \in$Irr(M) such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, $2^2 \cdot 5^2$, $2 \cdot 3 \cdot 5^2$ or $2 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$ or $2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |K|$, a contradiction. Hence $\theta(1) = 2^2 \cdot 5^2$ or $2 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|H| = 2^\alpha \cdot 3 \cdot 5$, where $\alpha = 3$ or 4. We claim that H is non-solvable. Otherwise if H is solvable. Then by $O_5(H) = 1$ we have $\alpha = 4$ by Lemma 2. By the same arguments as in Case 2, we can show that M has a normal series $1 \trianglelefteq O_2(H) \trianglelefteq B \trianglelefteq H_1 \trianglelefteq M$, such that $O_2(H)$ is elementary abelian with order 2^4, $|B/O_2(H)| = 5$, $H_1/B \cong A_5$, $|M/H_1| = 3$ and $B/O_2(H)$ act fixed-point-freely on $O_2(H)$. Let φ be an irreducible constituent of θ_{H_1} and σ an irreducible constituent of φ_B. By Clifford Theorem, $\varphi(1) = \theta(1)$ and $\sigma(1) = 5$. By the structure of B, it is to show that B has exactly 3 irreducible characters of degree 5, which implies that $t = |H_1 : I_{H_1}(\sigma)| \leq 3$. But $H_1/B \cong A_5$, we have that $t = 1$, so that $[\varphi_B, \varphi_B] \geq 2^2 \cdot 5^2 > |H_1 : B|$, a contradiction.

Now we have shown that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ and $|H/B| \mid 2$. Let φ be an irreducible constituent of θ_H and σ an irreducible constituent of φ_A.

Suppose that $|A| = 2^2$. By Lemma 4, $A = Z(H)$, and thus $A \trianglelefteq M$. Since $A = Z(H)$, $\sigma(1) = 1$ and $t_2 = |H : I_H(\sigma)| = 1$. By Lemma 5, φ is the unique irreducible constituent of σ^H with degree 5. Then $t_1 = |M : I_M(\varphi)| = 1$, thus $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. Now suppose that $|A| = 2$. Then H is isomorphic to one of $SL_2(5)\ .\ Z_2$, $Z_2 \times S_5$, $Z_2 \times A_5$ and $SL_2(5)$. Hence $\varphi(1) = 5$ and H has at most 4 irreducible characters with degree 5 respectively. But $K/H \cong A_5$, it is easy to show that $t = |M : I_M(\varphi)| = 1$. If $H \cong SL_2(5)\ .\ Z_2$ or $Z_2 \times S_5$, then $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. If $H \cong Z_2 \times A_5$ or $SL_2(5)$, let ψ be an irreducible constituent of θ_K. Again by Lemma 5 we have $\psi(1) = 5^2$ and $\theta_K = \psi$. But $\theta_K(1) = \theta(1) = 2^2 \cdot 5^2$ or $2 \cdot 5^2$, which does not equal to $\psi(1)$, a contradiction. If $|A| = 1$, then $M \cong S_6 \times S_6$, which has no irreducible character of degree $2^2 \cdot 5^2$ and $2 \cdot 5^2$, a contradiction.

By the same way as above, we can show $K/H \ncong A_5 \times A_5$.

Now suppose that $K/H \cong A_6$. By $|Out(A_6)| = 4$, we see that $|H| = 2^\alpha \cdot 5$, where $1 \leq \alpha \leq 3$. Clearly, H is solvable, and thus $O_5(H) \neq 1$ by Lemma 2. Hence $O_5(G) \neq 1$, a contradiction.

Step 2. to prove that $G \cong J_2$.

By G is non-solvable, we get by Lemma 1 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, we have $K/H \cong A_5,\ L_2(7),\ A_6,\ L_2(8),\ A_7,\ U_3(3),\ A_5 \times A_5$ or J_2.

Suppose that $K/H \cong A_5$. We claim that H is non-solvable. Assume that H is solvable. Because $O_5(H) \leq O_5(G) = 1$ and $O_7(H) \leq O_7(G) = 1$, we have that $O_5(H) = O_7(H) = 1$. Hence H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq C \trianglelefteq D \trianglelefteq H$, such that $|B/A| = 2^\alpha$, $|C/B| = 5$, $|D/C| = 7$ and B/A is elementary abelian and C/B acts fixed-point freely on B/A, where $\alpha = 4$ or 5. Since $GL(4, 2)$ and $GL(5, 2)$ have no subgroup of order $5 \cdot 7$, we have that $|C_{H/A}(B/A)|$ is divided by 5 or 7. If $5 \mid |C_{H/A}(B/A)|$, then $O_5(H/A) \neq 1$. Let $E/A = O_5(H/A)$. By $|A| = 2^{5-\alpha} \cdot 3^\gamma$, where $1 \leq \gamma \leq 2$, we have $O_5(E) \neq 1$ by Lemma 2. Noticing that $O_5(E)$ is characteristic in $E \trianglelefteq H$, one has that $O_5(E) \leq O_5(H) \leq O_5(G)$, a contradiction to $O_5(G) = 1$. By the same way, we can show $O_7(G) \neq 1$ while $7 \mid |C_{H/A}(B/A)|$.

Now H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$, A_6, A_7 or $L_2(7)$, and $|H/B| \mid |Out(B/A)|$. Since $O_5(G) = O_7(G) = 1$, it follows that $G \cong L_2(7) \times A_5 \times A_5$ or $B/A \cong A_i, (i = 5, 7)$ and A is solvable. For the first case, G has no irreducible character of degree $S(J_2)$, a contradiction. If $B/A \cong A_5$ and that A is solvable. Since $O_7(A) = 1$, A has a normal series $1 \trianglelefteq D \trianglelefteq E \trianglelefteq A$ such that $|D| = 2^3$, $|E/D| = 7$, $|A/E| = 3$ and D is elementary abelian and E/D acts nontrivially on D. Clearly that $|A/A^{'}| = 3$ or 21, and then the first column of the character table of A is one of the sequences $\{1, 1, 1, 3, 3, 7, 7, 7\}$ or $\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 7, 7, 7\}$ respectively. Let φ be an irreducible constituent of β_H and σ be an irreducible constituent of φ_A. Suppose first that $\sigma(1) = 1$. By

Lemma 4, $A/A' = Z(H/E)$, which implies that every linear character of A is invariant in H. By Lemma 5, it follows that $\varphi(1) = 5$, and for each $\lambda \in$Irr(A) there is at most one irreducible character of H with degree 5 lying over λ, this means that H has at most 21 irreducible character of degree 5. Hence $t = |G : I_G(\varphi)| \leq 20$, then $[\beta_H, \beta_H] \geq 2^2 \cdot 3^2 \cdot 5 > |G : H|$, a contradiction. Suppose that $\sigma(1) = 3$. Since A has exactly 2 irreducible characters of degree 3, we have that $t_1 = |H : I_H(\sigma)| \leq 2$. But $B/A \cong A_5$, by checking maximal subgroups of A_5, we have that $t_1 = 1$. Hence $\varphi(1) = 15$ and H has at most 2 irreducible characters of degree 15 by Lemma 5. Therefore, $|G : I_G(\varphi)| \leq 2$, which forces that $[\beta_H, \beta_H] = 2^4 \cdot 5^2 > |G : H|$, a contradiction.

If $B/A = A_7$, then $|H|$ divides $4|B/A|$. If $|G/K| = 2$, then $G/H = S_5$, $H \cong Z_2.A_7$, $Z_2 \times A_7$ or $A_7.Z_2$. If $G/K = 1$, then $G/H = A_5$, $H \cong A \times A_7$ ($|A| = 4$), $Z_2 \times Z_2.A_7$ or $Z_2.A_7.Z_2$. Let $\theta \in Irr(K)$ such that $[\beta_K,\ \theta] \neq 0$, and $\sigma \in Irr(H)$ such that $[\theta_H,\ \sigma] \neq 0$. Since $5|\sigma(1)$, by checking the character table of H, we get that $\sigma(1) = 10,\ 15$ or 20 and $t = |K : I_K(\sigma)| < 5$. But $K/H \cong A_5$, by checking maximal subgroup of A_5, we get $t = 1$, which force that $[\theta_H,\ \theta_H] > |K : H|$, a contradiction. Similarly, we can show that $K/H \neq A_7$.

Since $O_5(G) = O_7(G) = 1$, we have that K/H is not isomorphic to $L_2(8)$, A_6, $U_3(3)$ and $A_5 \times A_5$. And if K/H is isomorphic to $L_2(7)$ or A_6, then $G \cong L_2(7) \times A_5 \times A_5$, which has no irreducible character of degree $S(J_2)$, a contradiction.

Now we have proved that $K/H \cong J_2$, which concludes $G \cong J_2$. This ends the proof of Theorem B.

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AN ONC-CHARACTERIZATION OF A_{14} AND A_{15}

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Abstract. Let G be a finite group, $o_1(G)$ denote the largest element order of G, $n_1(G)$ the number of the elements of order $o_1(G)$. Assume that G totally has r elements of order $o_1(G)$, whose centralizers have distinct orders, say, they are $c_i(G)$, $i = 1,\ 2,\ \cdots,\ r$. The following quantity is called the 1st $ONC-$degree of G

$$ONC_1(G) = \{o_1(G);\ n_1(G);\ c_1(G),\ c_2(G),\ \cdots,\ c_r(G)\},$$

denoted as $ONC_1(G)$. It has been proved that K_3-simple groups, $L_2(q)$($q = 8$, 11, 13, 17, 19, 23, 29), Mathieu simple groups, Janko Groups and alternating groups $A_n(5 \leq n \leq 13)$ can be characterized by their 1st $ONC-$degrees, but unfortunately $L_2(q)(q = 16, 25)$ cannot be characterized by the 1st $ONC-$degree. Since the $ONC-$degree of an alternating group usually contains only 3 numbers, so it is interesting to study if an alternating group can be characterized by the 1st $ONC-$degree. We shall prove that A_{14} can be characterized by the 1st $ONC-$degree, but we can not prove A_{15} does by using our approaches. We shall prove if the prime graph of G is not connected and $ONC_1(G) = ONC_1(A_{15})$, then $G \cong A_{15}$.

Keywords: alternating group; $ONC-$Degree; ONC-characterization.

1. Introduction

Professor W. J. Shi put forward the famous conjecture in 1989:

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Shi's Conjecture. Let G be a finite group, M a finite simple group, then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of element orders in G (see [1]).

The conjecture is recorded as Problem 12.39 in Unsolved Problems in Group Theory (see [7]). Research on Shi's conjecture opened the era of quantitative characterization of finite simple groups since 1980's. In 2009, Shi's conjecture was completely proved. Afterwards, an interesting topic is trying to weaken conditions of Shi's conjecture since the set of element orders seems containing too many numbers. The last two authors defined the 1st $ONC-$ degree in [2]. Let G be a finite group, $o_1(G)$ denote the largest element order of G, $n_1(G)$ the number of the elements of order $o_1(G)$. Assume that G totally has r elements of order $o_1(G)$, whose centralizers have distinct orders, say, $c_i(G)$, $i = 1,\ 2,\ \cdots,\ r$. The following quantity is called the 1st $ONC-$degree of G

$$ONC_1(G) = \{o_1(G);\ n_1(G);\ c_1(G),\ c_2(G),\ \cdots,\ c_r(G)\},$$

denoted as $ONC_1(G)$. Notice $ONC_1(G)$ is not a set, but a series of numbers.

Because in many groups, orders of centralizers of elements having the largest order are the same, so the 1st $ONC-$ degree often contains only three numbers, for example, in alternating groups or symmetric groups. Hence the 1st $ONC-$ degree contains less numbers than Shi's conjecture in some cases. The reason why orders of centralizers of elements of largest orders are considered in the 1st $ONC-$ degree is that they almost determine the prime graph of a finite group. Hence, it is meaningful to study if a finite group, especially a finite simple group, can be characterized by the 1st $ONC-$degree.

Li-Guan He characterized some non-abelian simple groups by the 1st $ONC-$degree in his doctoral dissertation , such as K_3 simple groups , A_5, A_6, $L_2(8)$, $L_3(3)$ and $L_2(17)$ (see [2] and [4]). Apart from those, by comparing the second order or other special numbers, he also characterized some other sporadic simple groups in [2]. Later, Li-Guan He and Gui-Yun Chen continued to study the 1st $ONC-$degree. For example, it has been proved that Mathieu groups and Janko Groups can be characterized by the 1st $ONC-$degree in [2] and [3]. In [5], it is proved that $L_2(q)$ for $q = 11,\ 13,\ 19,\ 23,\ 29$ can be characterized by the 1st $ONC-$degree and $L_2(16)$ and $L_2(25)$ cannot, but a classification of finite groups G such that $ONC_1(G) = ONC_1(L_2(16))$ or $ONC_1(L_2(25))$ is given. For alternating groups, Li-Guan He proved that alternating groups A_n, $n \leq 13$, can be characterized by the 1st $ONC-$degree in [6]. In this paper, we continue to discuss the 1st $ONC-$degree characterization of alternating groups, and prove that A_{14} can be characterized by the 1st $ONC-$degree, A_{15} can be characterized by the 1st $ONC-$degree and its prime graph.

2. Prelimiaries

In this section we present some lemmas which are required in Section 3.

Lemma 2.1 ([10], Theorem A). *If G is a finite group whose prime graph has more than one component, then G has one of the following structures:*

(1) *G is a $Frobenius$ group or a $2-Frobenius$ group;*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 groups, K/H is a simple group, H is a nilpotent group, where $2 \in \pi_1$. And $|G/K|\big||Out(K/H)|$.*

Lemma 2.2. *Let G and H be two groups. Suppose that G acts on H co-primely, then for every prime $p\big||G|$, H has a $G-$invariant $p-$Sylow subgroup.*

Lemma 2.3. *Let G be a $p-$group of order p^n and K an elementary commutative group of order p^n, then $|Aut(K)|$ is divided by $|Aut(G)|$.*

Lemma 2.4. *Let G be a p group of order p^n, and G act on a q-group H of order q^α, where p and q are distinct primes. If $|G| \nmid \prod_{i=1}^{\alpha}(q^i-1)$, then $pq \in \pi_e(G \ltimes H)$.*

Proof. A group G acts group H, so $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$, then

$$|G/C_G(H)|\big||Aut(H)|\big|q^{\frac{\alpha(\alpha-1)}{2}} \cdot \prod_{i=1}^{\alpha}(q^i-1).$$

Since $(|p|,\ q^{\frac{\alpha(\alpha-1)}{2}})=1$ and $|G| \nmid \prod_{i=1}^{\alpha}(q^i-1)$, we have $|G| \nmid |Aut(H)|$. Therefore $C_G(H) \neq 1$, which concludes that $p\big||C_G(H)|$, $pq \in \pi_e(G \ltimes H)$. □

Lemma 2.5 ([10], Corollary). *If G is solvable with a non-connected prime graph. Then G is either a $Frobenius$ or a $2-Frobenius$ group. Moreover if G is a $2-Frobenius$ group, then G has exactly two components, and one of which consists of primes dividing the lower $Frobenius$ complement.*

Lemma 2.6. (1) *Let G be a $Frobenius$ group (not a $2-Frobenius$ group) with $Frobenius$ kernel H and $Frobenius$ complement K. then it has a non-connected prime graph, and the vertex sets of prime graph components of G are exactly $\{\pi(H),\ \pi(K)\}$.*

(2) *Let G be a $2-Frobenius$ group, then $G = ABC$, where A is normal in G, AB is a $Frobenius$ group with $Frobenius$ kernel A and $Frobenius$ complement B, BC is a $Frobenius$ group with $Frobenius$ kernel B and $Frobenius$ complement C, where B and C are cyclic groups. In addition, $|C|\big||Aut(B)|$. The vertex sets of prime graph components of G are exactly $\{\pi(A) \cup \pi(C),\ \pi(B)\}$.*

Proof. At first, we prove (2). By the definition of $2-Frobenius$ group, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are $Frobenius$ groups with $Frobenius$ kernels K/H and H respectively. Hence, let $K = HL$, where L is the $Frobenius$ complement of K. Based on the generalized $Frattini$ Argument, we have that $G = N_G(L)K = N_G(L)H$. Because $N_G(L) \cap H = N_H(L) = 1$, $N_G(L) \cong G/H$ is a $Frobenius$ group. (2) follows from Lemma 2.5.

Now we prove (1). Since H is a nilpotent group, vertexes in $\pi(H)$ belong to one component and K acts fixed-point-freely on H, so $\pi(H)$ must be an independent prime graph component. If K is unsolvable, owing to [8], we know that K has a normal subgroup K_0 which is isomorphic to $SL(2,\ 5)$, $|K:K_0|\le 2$, $K_0\cong Z\times SL(2,\ 5)$, $(|Z|,\ 30)=1$. Because the prime graph of $SL(2,\ 5)$ is connected, the prime graph of K must be connected. If K is solvable and has more than one prime graph component, by Lemma 2.5, K is a *Frobenius* or a $2-$*Frobenius* group, consequently G is a $2-$*Frobenius* group, G has the structure in (2). Now (1) follows. □

Theorem 2.7. *Let G be a finite group, $M=A_{14}$, then $G\cong M$ if and only if $ONC_1(G)=ONC_1(M)$.*

Proof. The necessity is obvious. It is enough to prove the sufficiency.

Because the largest element order of A_{14} is 45, the number of elements of order 45 in A_{14} is $\frac{P_{14}^{9}}{9}\cdot\frac{P_5^5}{5}=2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$, where P_m^n is the number of permutations of m letters taking from n letters, and all elements of order 45 in A_{14} are self-centralized, thus, $ONC_1(G)=ONC_1(A_{14})=\{45;\ 2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13;\ 45\}$. We may assume that elements of order 45 in G are partitioned into t conjugacy classes. Due to lengths of conjugacy classes of any two elements of order 45 in G are equal to $\frac{|G|}{45}$, then $t\cdot\frac{|G|}{45}=2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$. Hence $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$ and $|G|>2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$. Now $45\big||G|$ yields $3, 5\in\pi(G)$. If $2\notin\pi(G)$, then $|G|$ divides $3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$, so $|G|<2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$, a contradiction. Hence $2\in\pi(G)$. Noticing $2^{11}\cdot 3^5\cdot 5^2\cdot 11\cdot 13<2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$, we get $7\in\pi(G)$; By $2^{11}\cdot 3^5\cdot 5^2\cdot 7^2<2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$, we get that $\{11,\ 13\}\cap\pi(G)\ne\varnothing$.

Now we divide our proof into several steps.

(1) We prove that $\{11,\ 13\}\subset\pi(G)$.

It is enough to show both $13\in\pi(G)$, $11\notin\pi(G)$ and $13\notin\pi(G)$, $11\in\pi(G)$ are impossible.

(1.1) If $13\in\pi(G)$ and $11\notin\pi(G)$, then $|G|>2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$ yields $7^2\big||G|$. We assert that: G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that K/H is a non-abelian simple group and $\{7,\ 13\}\subset\pi(K/H)$.

In fact, considering chief series of G: $1=G_k\trianglelefteq G_{k-1}\trianglelefteq\cdots\trianglelefteq G_1\trianglelefteq G_0=G$, we may assume $\{7,\ 13\}\cap\pi(G_i)\ne\varnothing$ and $\{7,\ 13\}\cap\pi(G_{i+1})=\varnothing$. Let $H=G_{i+1}$ and $K=G_i$, then $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ is a normal series of G, K/H is a minimal normal subgroup of G/H, and further K/H is the direct product of isomorphic simple groups. Now we prove that $\{7,\ 13\}\subset\pi(K/H)$, hence K/H is a direct product of nonabelian simple groups.

Otherwise, either $7\in\pi(K/H)$, $13\notin\pi(K/H)$ or $7\notin\pi(K/H)$, $13\in\pi(K/H)$. If the former one holds, considering the action of an element of order 13 of G/H on K/H by conjugation, and noticing $13\nmid(7-1)(7^2-1)=2^5\cdot 3^2$, we conclude that $91\in\pi_e(G)$ by Lemma 2.2, 2.3 and 2.4. Hence $91>o_1(G)=45$, a contradiction. Therefore $13\in\pi(K/H)$ while $7\in\pi(K/H)$. Similarly, we can get a contradiction too while $7\notin\pi(K/H)$, $13\in\pi(K/H)$. Therefore $\{7,\ 13\}\subset\pi(K/H)$, and K/H is a direct product of non-abelian simple groups. Further by $13\big\|\,|G|$, we get that K/H is a nonabelian simple group, the assertion follows.

Now $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$. Checking simple groups of orders divided by 13 and dividing $2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 13$ in [9], one gets that K/H may be one of the following groups:

$$L_2(13)(2^2\cdot 3\cdot 7\cdot 13),\ L_2(27)(2^2\cdot 3^3\cdot 7\cdot 13),\ S_z(8)(2^6\cdot 5\cdot 7\cdot 13),$$
$$L_2(64)(2^6\cdot 3^2\cdot 5\cdot 7\cdot 13).$$

For above groups, it always holds that $7\big\||Aut(K/H)|$. Since $7^2\big||G|$ and $7\nmid|H|$, we have $7\in\pi(C_{G/H}(K/H))$ by $G/H/C_{G/H}(K/H)\le Aut(K/H)$, so $7\times 13=91\in\pi_e(G)$, which contradicts $o_1(G)=45$. Thus $11\in\pi(G)$ while $13\in\pi(G)$.

(1.2) If $11\in\pi(G)$, $13\notin\pi(G)$, by the same reasoning as above, we conclude that $\{7,\ 11\}\subset\pi(K/H)$, and K/H is a non-abelian simple group. Checking simple groups of orders divided by 11 and dividing $2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11$ in [9], we get that K/H is one of the following groups:

$$M_{22}(2^7\cdot 3^2\cdot 5\cdot 7\cdot 11),$$
$$A_{11}(2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 11),$$
$$A_{12}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11).$$

By the same reasoning as in (1.1), we come to $7\in\pi(C_{G/H}(K/H))$, hence $77\in\pi_e(G)$, again a contradiction to $o_1(G)=45$. Therefore $13\in\pi(G)$ while $11\in\pi(G)$, which concludes (1).

(2) We prove that $G\cong A_{14}$.

It follows by that $\pi(G)=\{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$, $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, where K/H is a non-abelian simple group and $\{11,\ 13\}\subset\pi(K/H)$. By [9], we get that K/H may be one of $A_{13}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13)$ and $A_{14}(2^{10}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13)$.

If $K/H=A_{13}$, then $7\big\||Aut(K/H)|$. If $7^2\big||G|$, it follows that $7\big||H|$ or $7\in\pi(C_{G/H}(K/H))$ by $G/H\big/C_{G/H}(K/H)\le Aut(K/H)$. While $7\big||H|$, H has a $7-Sylow$ subgroup of order 7, and $13\notin\pi(H)$. Consider the action of an element of order 13 of G on H by conjugation, we get $91\in\pi_e(G)$ by Lemma 2.2, which contradicts $o_1(G)=45$. Hence $7\in\pi(C_{G/H}(K/H))$, which means $91\in\pi_e(G)$, a contradiction. Thus $7\ \big\||G|$, so $|H|\big|4$ by comparing orders of G and A_{13}. Then $|Aut(H)|\big|6$, which implies that $|G/C_G(H)|\big|6$, hence $\{2,\ 3,\ 5,\ 7,\ 11,\ 13\}\subseteq\pi(C_G(H))$. However $H\le C_G(H)\trianglelefteq G$ yields $(C_G(H)/H)\cap K/H=1$ or K/H. Therefore $C_G(H)\cap K=H$ or K. Obviously, the former one is impossible. For the latter, it follows $H\le Z(K)$. But A_{13} has an element of order 35, thus K must have an element of order 70, a contradiction.

Now we have that $K/H=A_{14}$ and then $|H|\big|2$. If $|G|=2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$. If $|H|=1$, then $K=A_{14}$, then $G=S_{14}$ or $G=A_{14}\times C_2$, both of which has an element of order 90, a contradiction. Therefore $|H|=2$, let an element of order 45 of G act on H by conjugation, we conclude $90\in\pi_e(G)$, which contradicts $o_1(G) = 45$. Hence $|G| = 2^{10}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$. Therefore $H = 1$ and $G = K = A_{14}$. This concludes the theorem. □

Theorem 2.8. *Let G be a finite group with non-connected prime graph, $M = A_{15}$, then $G \cong M$ if and only if $ONC_1(G) = ONC_1(M)$.*

Proof. The necessity is obvious. It is enough to show the sufficiency.

Because the largest element order of A_{15} is $3\times 5\times 7$, the number of elements of order 105 in A_{15} is $\frac{P_{15}^3}{3}\cdot\frac{P_{12}^5}{5}\cdot\frac{P_7^7}{7} = 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$, and every element of the largest order is self-centralized, thus

$$ONC_1(G) = ONC_1(A_{15}) = \{105;\ 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13;\ 105\}.$$

Assume that elements of order 105 in G are divided into t conjugacy classes. Due to lengths of conjugacy classes of elements of order 105 in G are the same and equal to $\frac{|G|}{105}$, so $t\cdot\frac{|G|}{105} = 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$, which implies that $|G|\big|2^{11}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13$ and $|G| > 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$. Obviously $3,\ 5,\ 7\in\pi(G)$. If $2\notin\pi(G)$, then $|G|\le 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13 < 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$, a contradiction. Therefore $2\in\pi(G)$. If $(11\times 13,\ |G|) = 1$, then $2^{11}\cdot 3^6\cdot 5^3\cdot 7^2 < 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$, hence $\{11,\ 13\}\cap\pi(G)\ne\varnothing$. By $|G| > 2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$ and trivial comparing, we have that $2^8\big||G|$, $3^5\big||G|$ and $5^2\big||G|$.

Now we divide the proof into several steps.

(1) to prove G is neither a Frobenius group nor a 2-Frobenius group.

(1.1) If G is a *Frobenius* group with *Frobenius* kernel H and *Frobenius* complement K, then $G = HK$ and either $105\big||H|$ or $105\big||K|$ by Lemma 2.7. If $105\big||H|$, then $\pi(H) = \{3,\ 5,\ 7\}$. Otherwise, $o_1(G)\ge o_1(H) > 105$. As $\{11,\ 13\}\cap\pi(G)\ne\varnothing$, so $2^8\cdot 11$ or $2^8\times 13\big||K|$. Now consider the action of $2-Sylow$ subgroup of K on the $7-Sylow$ subgroup of H, one comes to that G has an element of order 14, consequently the prime graph of G is connected, a contradiction. If $105\big||K|$, then K has an element of order 105. If $11\big||H|$ or $13\big||H|$, we consider the action of an element of order 7 in K on the $11-Sylow$ subgroup of H, which concludes that there exists an element of order 77 or an element of order 91 in G, both contradict Lemma 2.6. So H is just a $2-$group and $2^8\big||H|\big|2^{11}$. Notice $5^2\big||K|$ and consider the action of a $5-Sylow$ subgroup of G on H by conjugation, we see by Lemma 2.4 that there exists an element of order 10 in G, a contradiction to Lemma 2.6.

(1.2) If G is a $2-Frobenius$ group, then $G = ABC$, where $A,\ B,\ C$ are as in Lemma 2.6. Since $\pi(B)$ is a vertex set of a prime graph component of G, so either $105\big||B|$ or $105\big||A||C|$. Hence if the former holds, then $5^2\times 3^5\times 7\big||G|$. But B is a cyclic group, so B has an element of order $\ge 5^2\cdot 3^5\cdot 7$, a contradiction. If the latter case holds, then $2^8\big||B|$ and at least one of $11\big||B|$ and $13\big||B|$ holds. Since B is a cyclic group, then $|B|\le 105$, so $2^8\nmid|B|$, $11\times 13\nmid|B|$. Hence $|B| = 11$ or 13, and then $|C|\big|10$ or 12 by Lemma 2.6. Thus $7\big||A|$. Note that the $7-$Sylow subgroup D of A is a normal subgroup of G, where $|D|\big|7^2$. We observe the action of B on D by conjugation and come to that there exists elements of order 77 or 91 in G by Lemma 2.4, which contradicts that AB is a *Frobenius* group. (1) follows.

Now by Lemma 2.1, we have the following:

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H ia a non-abelian simple group, and H is a nilpotent group, $\pi(K/H) \subset \{2,3,5,7,11,13\}$.

(3) It is impossible that $11 \in \pi(G)$, $13 \notin \pi(G)$ and $11 \big| |K/H|$.

Otherwise, $11 \big| |K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$. Checking simple groups of order divided by 11 and dividing $|K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$ in [9], we get that K/H may be one of the following groups: $L_2(11)(2^2 \cdot 3 \cdot 5 \cdot 11)$, $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $M_{22}(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $A_{11}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11)$, $HS(2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11)$, $A_{12}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11)$, $M^cL(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11)$.

(3.1) to prove that $5,\ 7 \notin \pi(H)$.

Note that $|G| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $11 \big| |K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, one has $|H| \big| 2^9 \cdot 3^5 \cdot 5^2 \cdot 7^2$.

If $7 \in \pi(H)$, let an element of order 11 of G act on H by conjugation, since $11 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we get $7 \times 11 \in \pi_e(G)$ by Lemma 2.4. If $2 \in \pi(H)$, then the $2-$Sylow subgroup of H is of order dividing 2^9. Since $11 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$, it follows $11 \times 2 \in \pi_e(G)$ by Lemma 2.4 and considering the action of an element of order 11 on 2-Sylow subgroup of H, this implies that G has a connected prime graph, a contradiction. Hence $2 \notin \pi(H)$, thus we can consider the co-prime action of subgroup of order 2^8 of G on H by conjugation. Since $2^8 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we come to $2 \times 7 \in \pi_e(G)$ by Lemma 2.2, 2.3 and 2.4, again G has a connected prime graph, a contradiction. Therefore $7 \notin \pi(H)$.

If $5 \in \pi(H)$, by $11 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, we get $5 \times 11 \in \pi_e(G)$. In this case, if $2 \in \pi(H)$, we can prove by the same approach that $11 \times 2 \in \pi_e(G)$, then G has a connected prime graph, a contradiction. So $2 \notin \pi(H)$. Consider the action of a subgroup of order 2^8 of G on H by conjugation and notice $2^8 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, we get $2 \times 5 \in \pi_e(G)$ by Lemma 2.4, this implies that G has a connected prime graph, a contradiction. This concludes (3.1).

(3.2) to prove that $K/H \neq L_2(11)$, M_{11}, M_{12} or $U_5(2)$.

If K/H is one of $L_2(11)$, M_{11}, M_{12} and $U_5(2)$, then $7 \notin \pi(K)$. Let an element of order 7 of G act on K by conjugation, because $7 \nmid (11-1) = 2 \times 5$, G has an element of order 77 by Lemma 2.4.

If $2 \in \pi(H)$, consider the action of an element of order 11 of G on the $2-$*Sylow* subgroup of H by conjugation, by $11 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$ and Lemma 2.4, we get $2 \times 11 \in \pi_e(G)$. Furthermore, the prime graph of G is connected, a contradiction. Hence $\{2,\ 5,\ 7\} \notin \pi(H)$ by (3.1). Therefore H is a $3-$group.

If $K/H = L_2(11)(2^2 \cdot 3 \cdot 5 \cdot 11)$, then $2^2 \big\| |K|$. Let an element of order 7 act on K by conjugation, we get $7 \times 2 \in \pi_e(G)$, which means that the prime graph of G is connected, a contradiction. Hence $K/H \neq L_2(11)$.

If K/H is one of $M_{11}(2^4\cdot 3^2\cdot 5\cdot 11)$, $M_{12}(2^6\cdot 3^3\cdot 5\cdot 11)$ and $U_5(2)(2^{10}\cdot 3^5\cdot 5\cdot 11)$, then $|H|\big|3^4$. Assume $H\neq 1$, considering the action of an element of order 77 on H by conjugation, we get by $(77,\ \prod_{i=1}^{4}(3^i-1))=(77,\ 2^9\cdot 5\cdot 13)=1$ and Lemma 2.4 that G has an element of order 77×3, which contradicts $o_1(G)=105$. So $H=1$ and K is one of $M_{11},\ M_{12}$ and $U_5(2)$. Because $G/C_G(K)\leq Aut(K)$. Notice that $|Out(K)|=1$ or 2, we come to $|Aut(K)||2^{11}\cdot 3^4\cdot 5\cdot 11$. However $5^2\big||G|$, hence $5\in\pi(C_G(K))$, this implies that $5\times 2,\ 5\times 11\in\pi_e(G)$, which implies that G has a connected prime graph, a contradiction.

(3.3) to prove $K/H\neq A_{11},\ HS,\ A_{12}$ or M^cL.

If K/H is one of $A_{11}(2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 11)$, $HS(2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11)$, $A_{12}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11)$ and $M^cL(2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11)$. It follows by (3.1) that $5,\ 7\notin\pi(H)$. Hence $|H|\big|2^4\cdot 3^4$. If $2\big||H|$, noticing $5^2\nmid\prod_{i=1}^{4}(2^i-1)=3^2\cdot 5\cdot 7,\ 11\nmid\prod_{i=1}^{4}(2^i-1)=3^2\cdot 5\cdot 7$, considering the actions of subgroups of order 5^2 and 11 in G on H by conjugation respectively, we conclude that $5\times 2,\quad 11\times 2\in\pi_e(G)$. Hence the prime graph of G is connected, a contradiction. This also means that if $H\neq 1$ then $|H|\big|3^4$. Because $11\nmid\prod_{i=1}^{4}(3^i-1)=2^9\cdot 5\cdot 13$, considering action of element of order 11 on H by conjugation, we can get $3\times 11\in\pi_e(G)$ by Lemma 2.4. Since $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$, and $|Aut(H)|\big|3^6\cdot\Pi_{i=1}^{4}(3^i-1)=2^9\cdot 3^6\cdot 5\cdot 13$, we have that $7\big||C_G(H)|$. If $2\in\pi(C_G(H))$, then $2\times 3\in\pi_e(G)$, the prime graph of G is connected, a contradiction. Therefore $2\notin\pi(C_G(H))$, any subgroup of order 2^8 in G acts on $C_G(H)$ co-primely by conjugation. Since $7\big||C_G(H)|$, there exists a $G-$invariant $7-Sylow$ subgroup of $C_G(H)$. But $2^8\nmid\prod_{i=1}^{2}(7^i-1)=2^5\cdot 3^2$, so Lemma 2.4 implies $2\times 7\in\pi_e(G)$, which concludes the prime graph of G is connected, a contradiction. Therefore $H=1$, and K is one of $A_{11},\ HS,\ A_{12}$ and M^cL. Moreover $G/C_G(K)\leq Aut(K)$. If $C_G(K)\neq 1$, G has a connected prime graph, a contradiction. Hence $C_G(K)=1$, $G\leq Aut(K)$. But $|Out(K)|=2$ for these groups, hence $|Aut(K)|\big|2^{10}\cdot 3^6\cdot 5^3\cdot 7\cdot 11$, which contradicts $|G|>2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$.

(4) It is impossible that $11\in\pi(G)$, $13\notin\pi(G)$ or $11\nmid|K/H|$.

Otherwise, by [9] and step (2), K/H is one of the following groups:
$A_5(2^2\cdot 3\cdot 5)$, $L_3(2)(2^3\cdot 3\cdot 7)$,$A_6(2^3\cdot 3^2\cdot 5)$, $L_2(8)(2^3\cdot 3^2\cdot 7)$, $A_7(2^3\cdot 3^2\cdot 5\cdot 7)$, $U_3(3)(2^5\cdot 3^3\cdot 7)$, $A_8(2^6\cdot 3^2\cdot 5\cdot 7)$, $U_4(2)(2^6\cdot 3^4\cdot 5)$, $L_2(49)(2^4\cdot 3\cdot 5^2\cdot 7^2)$, $U_3(5)(2^4\cdot 3^2\cdot 5^3\cdot 7)$, $A_9(2^6\cdot 3^4\cdot 5\cdot 7)$, $J_2(2^7\cdot 3^3\cdot 5^2\cdot 7)$, $L_3(4)(2^6\cdot 3^2\cdot 5\cdot 7)$, $S_6(2)(2^9\cdot 3^4\cdot 5\cdot 7)$, $A_{10}(2^7\cdot 3^4\cdot 5^2\cdot 7)$, $U_4(3)(2^7\cdot 3^6\cdot 5\cdot 7)$.

For the above groups, 11 does not divide the order of the outer automorphism group of any group above. Hence $11\in\pi(H)$, hence $11\big||H|\big|2^9\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11$. By Lemma 2.1, H is nilpotent. Considering the action of a subgroup of order 2^2 and a subgroup of order 3^2 of G on the $11-Sylow$ subgroup of H by conjugation, we come to $11\times 2,\ 11\times 3\in\pi_e(G)$, which implies that the prime graph of G is connected, a contradiction. Step (4) follows.

(5) It is impossible that $13\in\pi(G)$ or $11\notin\pi(G)$.

Here we mention the proved fact that $5^2\big||G|$, $3^4\big||G|$ and $2^8\big||G|$.

If $13 \notin \pi(K/H)$, then $\pi(K/H)$ does not contain 11 and 13, which is a case as Step (4), a contradiction.

If $13 \in \pi(K/H)$, then K/H is one of the following groups by [9]:
$L_2(13)(2^2 \cdot 3 \cdot 7 \cdot 13)$, $L_3(3)(2^4 \cdot 3^3 \cdot 13)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(27)(2^2 \cdot 3^3 \cdot 7 \cdot 13)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(64)(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9)(2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$.

For all simple groups above, $|H|\big|2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2$ always follows.

(5.1) to prove $5,\ 7 \notin \pi(H)$**.**

If $7 \in \pi(H)$, let an element of order 13 in G act on the $7-Sylow$ subgroup of H by conjugation, then $7 \times 13 \in \pi_e(G)$ by $13 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$ and Lemma 2.4. In this case, if $2 \in \pi(H)$, we consider action of an element of order 13 in G on the $2-Sylow$ subgroup of H by conjugation, by $13 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$ and Lemma 2.4, we conclude that $13 \times 2 \in \pi_e(G)$, thus the prime graph of G is connected, a contradiction. So $2 \notin \pi(H)$. Considering the action of a subgroup of order 2^8 in G on the $7-Sylow$ subgroup of H by conjugation, and noticing $2^8 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we get by Lemma 2.4 that $2 \times 7 \in \pi_e(G)$. Hence G has a connected prime graph, a contradiction. Therefore $7 \notin \pi(H)$.

If $5 \in \pi(H)$, by $13 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$ and Lemma 2.4, we have $5 \times 13 \in \pi_e(G)$. Now if $2 \in \pi(H)$, then $2 \times 13 \in \pi_e(G)$, the prime graph of G is connected, a contradiction. So $2 \notin \pi(H)$. Now let a subgroup of order 2^8 in G act on the $5-Sylow$ subgroup of H by conjugation, since $2^8 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, it follows $2 \times 5 \in \pi_e(G)$ by Lemma 2.4, so the prime graph of G is connected, a contradiction.

(5.2) to prove $K/H \neq L_2(13),\ L_3(3),\ L_2(27)$ **or** $G_2(3)$**.**

If $K/H{=}L_2(13)(|Out(K/H)| = 2)$, $L_3(3)(|Out(K/H)| = 2)$,

$L_2(27)(|Out(K/H)| = 6)$ or $G_2(3)$ $(|Out(K/H)| = 2)$, then $5 \notin \pi(Aut(K/H))$. By $G/H/C_{G/H}(K/H) \leqslant Aut(K/H)$, we get $5 \in \pi(C_{G/H}(K/H)$ by (5.1). Consequently, G has a connected prime graph, a contradiction.

(5.3) to prove $K/H \neq L_2(25),\ U_3(4),\ L_4(3)$ **or** ${}^2F_4(2)$**.**

Otherwise, if K/H is one of above groups, then $|Out(K/H)|\big|4$, $7 \notin \pi(K/H)$. Thus $7 \notin \pi(Aut(K/H))$, then $7 \in \pi(C_{G/H}(K/H)$ by (5.1), which implies that the prime graph of G is connected, a contradiction.

(5.4) to prove that K/H **cannot be any one of rest groups listed in beginning of the proof.**

If K/H is one of rest groups, then $|H|\big|2^5 \cdot 3^6$. If $2 \in \pi(H)$, by $13 \nmid \prod_{i=1}^{5}(2^i - 1) = 3^2 \cdot 5 \cdot 7 \cdot 31$, $5^2 \nmid \prod_{i=1}^{5}(2^i - 1) = 3^2 \cdot 5 \cdot 7 \cdot 31$ and Lemma 2.4, we come to $13 \times 2 \in \pi_e(G)$, $5 \times 2 \in \pi_e(G)$, which implies the prime graph of G connected, a contradiction.

If $H \neq 1$, then H is a $3-$ group. For possible choice of K/H, it always follows that $|Out(K/H)| = 2$ or 2^2, or 3, hence $|Aut(K/H)|\big|2^9 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$. But $5^2\big||G|$ yields $5 \in \pi(C_{G/H}(K/H))$, then $5 \times 2 \in \pi_e(G)$, $5 \times 13 \in \pi_e(G)$, so G has only one prime graph component, a contradiction.

(6) If $\{11,\ 13\} \subset \pi(G)$**, then** $G \cong A_{15}$.

It follows from (3)-(5) that $\pi(G) = \{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$.

Now we assert $\{11,\ 13\} \subset \pi(K/H)$. Otherwise, it follows by (4) and (5) that $11 \in \pi(K/H)$ and $13 \notin \pi(K/H)$, hence either $13 \big| |H|$ or $13 \big| |G/K|$. If the former holds, we consider the conjugate action of an element of order 11 of G on the $13-Sylow$ subgroup of H and get $11 \times 13 = 143 \in \pi_e(G)$, a contradiction. Hence $13 \in \pi(G/K)$. Since $(13,\ |K/H|) = 1$, we consider the action of an element of order 13 in G/H on K/H by conjugation and come to some $11-Sylow$ subgroup of K/H is fixed. So $11 \times 13 = 143 \in \pi_e(G)$, a contradiction.

Hence $\pi(K/H) = \{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$. Comparing the orders of K/H and G, we have that $K/H = A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$, $A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$ or $A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

For these groups, $|H| \big| 2^2 \cdot 3 \cdot 5 \cdot 7$. We assert that $7 \notin \pi(H)$. Otherwise if $7 \in \pi(H)$, since $13 \nmid (7-1) = 6$, $11 \nmid (7-1) = 6$, viewing the actions of element of order 11 and 13 of G on the $7-Sylow$ subgroup of H by conjugation, we come to $13 \times 7,\ 11 \times 7 \in \pi_e(G)$ by Lemma 2.4. If $2 \in \pi(H)$, then $11 \times 2 \in \pi_e(G)$, and G has only one prime graph component, a contradiction. Hence $2 \notin \pi(H)$. Now consider the action of subgroup of order 2^2 of G on the $7-Sylow$ subgroup of H and noticing that $2^2 \nmid (7-1) = 6$, we get by Lemma 2.4 that $7 \times 2 \in \pi_e(G)$, which implies that the prime graph of G is connected, a contradiction.

Suppose $5 \in \pi(H)$, considering actions of elements of order 11 and 13 of G on the $5-Sylow$ subgroup of H by conjugation respectively, we get $13 \times 5,\ 11 \times 5 \in \pi_e(G)$. Noticing that 5 is adjacent to $2,\ 3,\ 5,\ 7$ in A_{13}, A_{14} and A_{15}, we come to that the prime graph is connected, a contradiction. Hence $5 \notin \pi(H)$. Similarly, we can prove that $2 \notin \pi(H)$ and $3 \notin \pi(H)$. Hence $H = 1$.

Now $K = A_{13},\ A_{14}$ or A_{15}. By the prime graph of G is connected, it follows that $C_G(K) = 1$. Thus $K \leq G \leq Aut(K) = S_{13},\ S_{14}$ or S_{15}. at last by $o_1(G) = 105$, we come to $G = K = A_{15}$. □

Remark 2.9. Because the 1st $ONC-$degree of an alternating group usually contains only three numbers, so such kind of characterization involved not many numbers for alternating groups. But it is for this reason, counterexamples may appear in the alternating groups. Since we cannot show A_{15} can be characterized by the 1st $ONC-$degree, it is worth to study whether A_{15} is a counterexample. Surely if we add the condition that the prime graph of G is not connected, then we can prove $G \cong M$ for many alternating groups M while $ONC_1(G) = ONC_1(M)$. In fact, we checked for $M = A_n(n = 17,\ 18,\ 19,\ 20)$, if G is a finite group with non-connected prime graph, then $G \cong M$ if and only if $ONC_1(G) = ONC_1(M)$. Because of the length of the paper, we will not describe the results here.

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WEAKLY (I, J)-CONTINUOUS MULTIFUNCTIONS AND CONTRA (I, J)-CONTINUOUS MULTIFUNCTIONS

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Abstract. The purpose of the present paper is to introduce, study and characterize upper and lower weakly (I, J)-continuous multifunctions and contra (I, J)-continuous multifunctions. Also, we investigate its relation with another class of continuous multifunctions.

Keywords: weakly (I, J)-continuous multifunctions, I-open set, I-closed set, contra (I, J)-continuous multifunctions, (I, J)-continuous multifunctions.

1. Introduction

It is well known today, that the notion of multifunction playing a very important role in general topology, upper and lower continuity have been extensively studied on multifunctions $F : (X, \tau) \to (Y, \sigma)$. Currently using the notion of ideal topological space, different types of upper and lower continuity in a multifunction $F : (X, \tau, I) \to (Y, \sigma)$ have been studied and characterized [2], [8], [9], [15],

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[18]. The concept of ideal topological spaces has been introduced and studied by Kuratowski [12] and the local function of a subset A of a topological space (X,τ) was introduced by Vaidyanathaswamy [17] as follows: given a topological space (X,τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called the local function of A with respect to τ and I, is defined as follows: for $A \subseteq X$, $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I$ for every $U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(,)$ for a topology $\tau^*(\tau, I)$ called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by A^*. In 1990, Jankovic and Hamlett[10], introduced the notion of I-open set in a topological space (X,τ) with an ideal I on X. In 1992, Abd El-Monsef et al.[1] further investigated I-open sets and I-continuous functions. In 2007, Akdag [2], introduced the concept of I-continuous multifunctions in a topological space with and ideal on it. In 2007, Al-Omari and Noorani [3], introduced the notions of contra-I-continuous and almost I-continuous functions. Given a multifunction $F : (X,\tau) \to (Y,\sigma)$, and two ideals I, J associate, now with the topological spaces (X,τ,I) and (Y,σ,J), consider the multifunction $F : (X,\tau,I) \to (Y,\sigma,J)$. We want to study some type of upper and lower continuity of F as doing Rosas et al. [14]. In this paper, we introduce and study two new classes of multifunctions called a weakly (I,J)-continuous multifunctions and contra (I,J)-continuous multifunctions in topological spaces. Investigate its relation with another classes of continuous multifunctions. Also its relation when the ideal $J = \{\emptyset\}$.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if I is and ideal on X, (X,τ,I) mean an ideal topological space. For a subset A of (X,τ), $Cl(A)$ and $\mathrm{int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular open [16] (resp. semiopen [11], preopen[13], semi preopen [4]) if $A = \mathrm{int}(Cl(A))$ (resp.$A \subseteq Cl(\mathrm{int}(A)), A \subseteq \mathrm{int}(Cl(A)), A \subseteq Cl(\mathrm{int}(Cl(A)))$). The complement of a regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. A subset S of (X,τ,I) is an I-open[10], if $S \subseteq \mathrm{int}(S^*)$. The complement of an I-open set is called I-closed set. The I-closure and the I-interior, can be defined in the same way as $Cl(A)$ and $\mathrm{int}(A)$, respectively, will be denoted by $I\,Cl(A)$ and $I\mathrm{int}(A)$, respectively. The family of all I-open (resp. I-closed, regular open, regular closed, semiopen, semi closed, preopen, semi-preclosed) subsets of a (X,τ,I), denoted by $IO(X)$(resp. $IC(X), RO(X), RC(X), SO(X), SC(X), PO(X), SPO(X), SPC(X)$). We set $IO(X,x) = \{A : A \in IO(X)$ and $x \in A\}$. It is well known that in an ideal topological space (X,τ,I), $X^* \subseteq X$ but if the ideal is codense, that is $\tau \cap I = \emptyset$, then $X^* = X$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 2.1 ([7]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper semi continuous at a point $x \in X$ if for each open set V of Y with $x \in F^+(V)$, there exists an open set U containing x such that $F(U) \subseteq V$.*
2. *lower semi continuous at a point $x \in X$ if for each open set V of Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(a) \cap V \neq \emptyset$ for all $a \in U$.*

Definition 2.2 ([15]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper weakly continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an open set U containing x such that $U \subseteq F^+(Cl(V))$.*
2. *lower weakly continuous if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap Cl(V) \neq \emptyset$ for every $u \in U$.*
3. *weakly continuous if it is both upper weakly continuous and lower weakly continuous.*

Definition 2.3 ([2]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an I-open set U containing x such that $U \subseteq F^+(V)$.*
2. *lower I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists an I-open set U containing x such that $U \subseteq F^-(V)$.*
3. *I-continuous if it is both upper and lower I-continuous.*

Definition 2.4 ([5]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper weakly I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an I-open set U containing x such that $U \subseteq F^+(Cl(V))$.*
2. *lower weakly I-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists an I-open set U containing x such that $U \subseteq F^-(Cl(V))$*
3. *weakly I-continuous if it is both upper weakly I-continuous and lower I-weakly continuous.*

3. Weakly (I, J)-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper weakly (I, J)-continuous at a point $x \in X$ if for each J-open set V such that $x \in F^+(V)$, there exists an I-open set U containing x such that $U \subseteq F^+(J\,Cl(V))$*

2. *lower weakly (I, J)-continuous at a point $x \in X$ if for each J-open set V of Y such that $x \in F^-(V)$, there exists an I-open set U of X containing x such that $U \subseteq F^-(J\,Cl(V))$.*

3. *upper (resp. lower) weakly (I, J)-continuous on X if it has this property at every point of X.*

Example 3.2. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{a\}$, $F(b) = \{c\}$ and $F(c) = \{b\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

In consequence, F is upper(resp. lower) weakly (I, J)-continuous on X.

Example 3.3. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and two ideals $I = J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{a\}$, $F(b) = \{c\}$ and $F(c) = \{b\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, Y, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. In consequence, F is not upper (resp. lower) weakly (I, J)-continuous.

Recall that if (X, τ, I) is an ideal topological space and I is the empty ideal, then for each $A \subseteq X$, $A^* = cl(A)$, that is to said, every I-open set is a pre-open set, in consequence, if $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is upper weakly $(I, \{\emptyset\})$-continuous, then F is upper weakly I-continuous.

Example 3.4. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a, c\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{b\}$, $F(b) = \{c\}$ and $F(c) = \{a\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

$F : (X, \tau, I) \to (Y, \sigma)$ is upper weakly I-continuous but $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is not upper weakly $(I, \{\emptyset\})$-continuous.

Now consider (X, τ, I) and (Y, σ, J) two ideals topological spaces. If $J \neq \{\emptyset\}$, then the concepts of upper weakly (I, J)-continuous and upper weakly I-continuous are independent, as we can see in the following examples.

Example 3.5. In the Example 3.4, the multifunction F is upper weakly (I, J)-continuous on X but is not upper weakly I-continuous on X.

Example 3.6. In the Example 3.3, the multifunction F is upper weakly I-continuous on X but is not upper weakly (I, J)-continuous on X.

Remark 3.7. It is easy to see that if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$ and F is upper (lower) weakly I-continuous, then F is upper (lower) weakly (I, J)-continuous. Even more, if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \nsubseteq \sigma$, we can find upper (resp. lower) weakly (I, J)-continuous on X that are not upper (lower) weakly I-continuous.

The following theorem characterize the upper weakly (I, J) continuous multifunctions.

Theorem 3.8. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is upper weakly (I, J)-continuous.*
2. *$F^+(V) \subseteq I\,int(F^+(J\,Cl(V)))$ for any J-open set V of Y.*
3. *$I\,Cl(F^-(J\,int(B))) \subset F^-(B)$ for any every J-closed subset B of Y.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be any J- open set of Y. From (1), there exists an I-open set U_x containing x such that $U_x \subset F^+(J\,Cl(V))$. It follows that $x \in I\mathrm{int}(F^+(J\,Cl(V)))$, in consequence, $F^+(V) \subseteq I\mathrm{int}(F^+(J\,Cl(V)))$ for any J-open set V of Y.

(2)⇒(1): Let V any J-open subset of Y such that $x \in F^+(V)$. By (2), $x \in F^+(V) \subseteq I\mathrm{int}(F^+(J\,Cl(V))) \subseteq F^+(J\,Cl(V))$. Choose $U = I\mathrm{int}(F^+(J\,Cl(V)))$. U is an I-open subset of X, containing x. It follows that F is upper weakly (I, J)-continuous.

(2)⇒(3): Let B be any J- closed set of Y.

Then by (2), $F^+(Y\backslash B) = X\backslash F^-(B) \subseteq I\mathrm{int}(F^+(J\,Cl(Y\backslash B)))$
$= I\mathrm{int}(F^+(J\,Cl(Y\backslash I\mathrm{int}(B)))) = X\backslash I\,Cl(F^-(J\mathrm{int}(B)))$.
Thus, $I\,Cl(F^-(J\mathrm{int}(B))) \subset F^-(B)$.

(3)⇒(2): Let V be any J- open set of Y. Then by (3), $I\,Cl(F^-(J\mathrm{int}(Y\backslash V))) \subset F^-(Y\backslash V) = X\backslash F^+(V)$.

It follows that $I\,Cl(X\backslash F^+(I\,Cl(V)) = I\,Cl(F^-(Y\backslash I\,Cl(V)))$
$= I\,Cl(F^-(J\mathrm{int}(Y\backslash V))) \subset X\backslash F^+(V)$, and then $X\backslash I\mathrm{int}(F^+(I\,Cl(V)))$
$\subseteq X\backslash F^+(V)$. Therefore the result follows. □

Theorem 3.9. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is lower weakly (I, J)-continuous.*
2. *$F^-(V) \subseteq I\,int(F^-(J\,Cl(V)))$ for any J-open set V of Y.*

3. *$I\,Cl(F^+(J\,int(B))) \subset F^+(B)$ for any every J-closed subset B of Y.*

Proof. The proof is similar to that of Theorem 3.8. □

Definition 3.10 ([14])**.** *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper (I, J)-continuous at a point $x \in X$ if for each J-open set V containing $F(x)$, there exists an I-open set U containing x such that $F(U) \subset V$.*

2. *lower (I, J)-continuous at a point $x \in X$ if for each J-open set V of Y meeting $F(x)$, there exists an I-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) (I, J)-continuous on X if it has this property at every point of X.*

Example 3.11. The multifunction defined in Example 3.2 is upper weakly (I, J)-continuous on X but is not upper (I, J)-continuous on X.

Remark 3.12. Every upper (resp. lower) (I, J)-continuous multifunction on X is upper (resp. lower) weakly (I, J)-continuous multifunction on X, but the converse is not necessarily true, as we can see in the following example.

Example 3.13. Let $X = \mathbb{R}$ the set of real numbers with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $Y = \mathbb{R}$ with the topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = \{\emptyset\}$=J. Define $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(x) = \mathbb{Q}$ if $x \in \mathbb{Q}$ and $F(x) = \mathbb{R} \setminus \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Recall that in this case the I-open sets are the preopen sets. f is upper (resp. lower) weakly (I, J)-continuous on X, but is not upper (resp. lower) (I, J)-continuous on X..

Theorem 3.14 ([14])**.** *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is upper (I, J)-continuous.*

2. *$F^+(V)$ is I-open for each J-open set V of Y.*

3. *$F^-(K)$ is I-closed for every J-closed subset K of Y.*

4. *$I\,Cl(F^-(B)) \subset F^-(J\,Cl(B))$ for every subset B of Y.*

5. *For each point $x \in X$ and each J-open set V containing $F(x)$, $F^+(V)$ is an I-open containing x.*

There exist any additional condition in order to say that every upper (resp. lower) (I, J)-continuous if upper (resp. lower) weakly (I, J)-continuous.

Theorem 3.15. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction such that $F(x)$ is a J-open subset of Y for each $x \in X$. Then F is lower (I, J)-continuous if and only if lower weakly (I, J)-continuous.*

Proof. Let $x \in X$ and V any J-open subset of Y such that $x \in F^{-}(V)$. Then there exists an I-open subset U of X containing x such that $U \subset F^{-}(J\, Cl(V)$. It follows that $F(u) \cap J\, Cl(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is a J-open subset of Y for each $u \in U$, it follows that $F(u) \cap V \neq \emptyset$ and then F is lower (I, J)-continuous. The converse is clear because every (I, J)-continuous multifunction is weakly (I, J)-continuous. □

Theorem 3.16. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction such that $F(x)$ is a J-open subset of Y for each $x \in X$. Then F is upper (I, J)-continuous if and only if upper weakly (I, J)-continuous.*

Proof. The proof is similar to the above Theorem. □

4. Contra (I, J)-continuous multifunctions

Definition 4.1. *A multifunction $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper contra (I, J)-continuous if for each $x \in X$ and each J-closed set V such that $x \in F^{+}(V)$, there exists an I-open set U containing x such that $F(U) \subset V$.*

2. *lower contra (I, J)-continuous if for each $x \in X$ and each J-closed set V of Y such that $x \in F^{-}(V)$, there exists an I-open set U of X containing x such that $U \subseteq F^{-}(V)$.*

3. *Contra (I, J)-continuous if it is upper contra (I, J)-continuous and lower contra (I, J)-continuous.*

Example 4.2. Let $X = \mathbb{R}$ the set of real numbers with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $Y = \mathbb{R}$ with the topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = \{\emptyset\}$=J. Define $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(x) = \mathbb{Q}$ if $x \in \mathbb{Q}$ and $F(x) = \mathbb{R} \setminus \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Recall that in this case the I-open sets are the preopen sets. It is easy to see that F is upper (resp. lower) contra (I, J)-continuous.

Example 4.3. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{b\}$, $F(b) = \{a\}$ and $F(c) = \{c\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

The set of all J-closed is $\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, Y\}$.

In consequence, f is upper(resp. lower) (I, J)-continuous on X but is not upper (resp. lower) contra (I, J)-continuous.

Example 4.4. The multifunction F defined in Example 4.2 is upper (resp. lower) contra (I, J)-continuous but is not upper (resp. lower) (I, J)-continuous on X and the multifunction F defined in Example 4.3 is upper (resp. lower)

(I, J)-continuous but is not upper (resp. lower) contra (I, J)-continuous. In consequence both concepts are independent of each other.

Theorem 4.5. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is upper contra (I, J)-continuous.*
2. *$F^+(V)$ is I-open for each J-closed set V of Y.*
3. *$F^-(K)$ is I-closed for every J-open subset K of Y.*

Proof. (1)⇔(2): Let $x \in F^+(V)$ and V be any J-closed set of Y. From (1), there exists an I-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since any union of I-open sets is I-open, $F^+(V)$ is I-open in (X, τ). The converse is similar.

(2)⇔(3): Let K be any J- open set of Y. Then $Y \backslash K$ is a J-closed set of Y by (2), $F^+(Y \backslash K) = X \backslash F^-(K)$ is an I-open set. Then it is obtained that $F^-(K)$ is an I-closed set. The converse is similar. □

Theorem 4.6. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *F is lower contra (I, J)-continuous.*
2. *$F^-(V)$ is I-open for each J-closed set V of Y.*
3. *$F^+(K)$ is I-closed for every J-open subset K of Y.*
4. *For each $x \in X$ and each J-closed set K of Y such that $F(x) \cap K \neq \emptyset$, there exists an I-open set U containing x such that $F(y) \cap K \neq \emptyset$ for each $y \in U$.*

Proof. The proof is similar to the proof of Theorem 4.5. □

Remark 4.7. It is easy to see that if $J = \{\emptyset\}$ and $F : (X, \tau, I) \to (Y, \sigma, J)$ is upper (resp. lower) contra (I, J)-continuous then F is upper (resp. lower) contra I-continuous.

The following example shows the existence of upper (resp. lower) contra I-continuous that is not upper (resp. lower) contra $(I, \{\emptyset\})$-continuous.

Example 4.8. Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a, c\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{c\}$, $F(b) = \{b\}$ and $F(c) = \{a\}$. It is easy to see that:

The set of all I-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all J-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$.

The set of all J-closed is $\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, Y\}$.

Observe that $F : (X, \tau, I) \to (Y, \sigma)$ is upper contra I-continuous but $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is not upper contra $(I, \{\emptyset\})$-continuous.

Remark 4.9. It is easy to see that if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$. If F is upper (lower) contra I-continuous, then F is upper (lower) (I, J)-continuous. Even more, if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \nsubseteq \sigma$, we can find upper (resp. lower) contra (I, J)-continuous on X that are not upper (lower) contra I-continuous.

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A STUDY OF SECOND ORDER IMPULSIVE NEUTRAL DIFFERENTIAL EVOLUTION CONTROL SYSTEMS WITH AN INFINITE DELAY

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Abstract. In this article, we study sufficient conditions for the controllability of second-order impulsive partial neutral differential evolution systems with infinite delay in Banach spaces by using the theory of cosine families of bounded linear operators and fixed point theorem.

Keywords: Nonlinear ordinary differential operators, Impulsive optimal control problems; evolution equations.

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1. Introduction

The study of impulsive functional differential equations is related to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are executed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place instantaneously in the form of impulses. The theory of impulsive systems provides a common frame work for mathematical modeling of many real world phenomena. Moreover, these impulsive phenomena can also be found in fields such as information science, electronics, fed-batch culture in fermentative production, robotics and telecommunications (see [1, 5, 13, 15, 17, 19] and references therein).

In recent years, the study of impulsive control systems has received increasing interest. Due to its importance several authors have investigated the controllability of impulsive systems (see [2, 6, 9, 18]).

Motivated by the effort of the after mentioned papers [2, 10], the primary inspiration driving this manuscript is mainly concerned with the study of controllability of second order impulsive partial neutral system of the form

$$\frac{d}{d\grave{u}}\Big[\vartheta'(\grave{u}) - \theta_1(\grave{u}, \vartheta_{\grave{u}})\Big] = A(\grave{u})\vartheta(\grave{u}) + Bu(\grave{u}) + \theta_2(\grave{u}, \vartheta_{\grave{u}}),$$

$$\grave{u} \in J = [0, a],\ \grave{u} \neq \grave{u}_k,\ k = 1, 2, ..., n, \tag{1}$$

$$\vartheta_0 = \phi \in \mathcal{B}, \quad \vartheta'(0) = \xi \in X, \tag{2}$$

$$\Delta\vartheta(\grave{u}_k) = I_k(\vartheta_{\grave{u}_k}), \quad k = 1, 2, ..., m, \tag{3}$$

$$\Delta\vartheta'(\grave{u}_k) = J_k(\vartheta_{\grave{u}_k}), \quad k = 1, 2, ..., m, \tag{4}$$

where $\phi \in \mathcal{B}$ and $\zeta \in X$. The control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ as a bounded linear operator; For $\grave{u} \in J, x_{\grave{u}}$ represents the function $\vartheta_{\grave{u}} : (-\infty, 0] \to X$ defined by $\vartheta_{\grave{u}}(\theta) = \vartheta(\grave{u} + \theta), -\infty < \theta \leq 0$ which belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\theta_1, \theta_2 : J \times \mathcal{B} \to X$, $I_k : \mathcal{B} \to X, J_k : \mathcal{B} \to X$ are appropriate functions and will be specified later. $0 < \grave{u}_1 < \ldots < \grave{u}_n < a$ are fixed numbers and the symbol $\Delta\xi(\grave{u})$ represents the jump of a function ξ at $\grave{u}$, which is defined by $\Delta\xi(\grave{u}) = \xi(\grave{u}^+) - \xi(\grave{u}^-)$. Throughout the text we will assume that $A(\cdot)$ generates an evolution operator $S(\grave{u}, s)$.

2. Preliminaries

This section we review some basic concepts, notation, and properties required to find our main results. Nowadays there has been an increasing interest in studying the theoretical non-autonomous second order initial value problem

$$\vartheta''(\grave{u}) = A(\grave{u})\vartheta(\grave{u}) + f(\grave{u}), \qquad 0 \leq s, \grave{u} \leq a, \tag{5}$$

$$\vartheta(s) = v, \quad \vartheta'(s) = w, \tag{6}$$

where $A(\grave{u}) : D(A(\grave{u})) \subseteq \vartheta \to X,\ \grave{u} \in J = [0,a]$ is a closed densely defined operator and $f : J \to X$ is an suitable function. Equations of this form have been considered in several papers. We refer the reader to [14, 16] and the references therein. In the majority of works, the existence of results to the problem (5)-(6) is related to the existence of an evolution operator $S(\grave{u}, s)$ for the homogeneous equation,

$$\vartheta''(\grave{u}) = A(\grave{u})\vartheta(\grave{u}), \quad 0 \leq s,\ \grave{u} \leq a. \tag{7}$$

Let as assume that the domain of $A(\grave{u})$ is a subspace of D dense in X and not dependent of $\grave{u}$, and for each $\vartheta \in D$ the function $\grave{u} \longmapsto A(\grave{u})\vartheta$ is continuous. The fundamental solution for the second-order evolution equation (7), has been developed by Kozak [12], and we will use the following concept of evolution operator.

Definition 2.1. *A family S of a bounded linear operator $S(\grave{u}, s) : J \times J \to L(\vartheta)$ is called an evolution operator for (7), if the following conditions are satisfied:*

($\mathbb{Z}$1) *For each $\vartheta \in X$,the mappings $(\grave{u}, s) \in [0,a] \times [0,a] \to S(\grave{u}, s)\vartheta \in X$ of class C^1 and*

- (i) *For each $\grave{u} \in [0,a]$, $S(\grave{u}, \grave{u}) = 0$,*
- (ii) *For all $\grave{u}, s \in [0,a]$, and for each $\vartheta \in X$,*

$$\frac{\partial}{\partial \grave{u}} S(\grave{u}, s)\vartheta|_{\grave{u}=s} = \vartheta, \qquad \frac{\partial}{\partial s} S(\grave{u}, s)\vartheta|_{\grave{u}=s} = -\vartheta$$

($\mathbb{Z}$2) *For all $\grave{u}, s \in [0,a]$ if $\vartheta \in D(A)$, then $S(\grave{u}, s)\vartheta \in D(A)$,the mappings $(\grave{u}, s) \in [0,a] \times [0,a] \to S(\grave{u}, s)\vartheta \in \vartheta$ is of class C^2 and*

- (i) $\frac{\partial^2}{\partial \grave{u}^2} S(\grave{u}, s)\vartheta = A(\grave{u})S(\grave{u}, s)\vartheta$,
- (ii) $\frac{\partial^2}{\partial s^2} S(\grave{u}, s)\vartheta = S(\grave{u}, s)A(s)\vartheta$,
- (iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial \grave{u}} S(\grave{u}, s)\vartheta|_{\grave{u}=s} = 0$.

($\mathbb{Z}$3) *For all $\grave{u}, s \in [0,a]$ if $\vartheta \in D(A)$,then $\frac{\partial}{\partial s} S(\grave{u}, s)\vartheta \in D(A)$, there exists $\frac{\partial^2}{\partial \grave{u}^2}\frac{\partial}{\partial s} S(\grave{u}, s)\vartheta$, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial \grave{u}} S(\grave{u}, s)\vartheta$ and*

- (i) $\frac{\partial^2}{\partial \grave{u}^2}\frac{\partial}{\partial s} S(\grave{u}, s)\vartheta = A(\grave{u})\frac{\partial}{\partial s} S(\grave{u}, s)\vartheta$,
- (ii) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial \grave{u}} S(\grave{u}, s)\vartheta = \frac{\partial}{\partial \grave{u}} S(\grave{u}, s)A(s)\vartheta$ *and the mapping* $[0,a] \times [0,a] \ni (\grave{u}, s) \to A(\grave{u})\frac{\partial}{\partial s} S(\grave{u}, s)\vartheta$ *is continuous.*

Throughout this problem we assume that there exists an evolution operator $S(\grave{u}, s)$ associated to the operator $A(\grave{u})$. To abbreviate the text, we introduce the operator $C(\grave{u}, s) = -\frac{\partial S(\grave{u}, s)}{\partial s}$. In addition, we set N and $\widetilde{N}$ for positive constants such that $\sup_{0<s,\grave{u}<a} \|S(\grave{u}, s)\| \leq N$ and $\sup_{0<s,\grave{u}<a} \|C(\grave{u}, s)\| \leq \widetilde{N}$. In addition, we denote by $N1$ is a positive constant such that,

$$\|S(\grave{u} + h, s) - S(\grave{u}, s)\| \leq N_1 |h|, \tag{8}$$

for all $s, \grave{u}, \grave{u} + h \in [0, a]$. Assuming that $f : J \to X$ is an integrable function, the mild solution $\vartheta : [0, a] \to X$ of the problem (5)-(6) is given by,

$$\vartheta(\grave{u}) = C(\grave{u}, s)v + S(\grave{u}, s)w + \int_s^{\grave{u}} S(\grave{u}, \tau)f(\tau)d\tau. \tag{9}$$

In the literature a number of methods have been discussed to establish the existence of the evolution operator $S(.,.)$. In particular, a very studied situation is that $A(\grave{u})$ is that perturbation of an operator A that generates a cosine operator function. In this reason, below we briefly analysis some essential properties of the theory of cosine functions. Let $A : D(A) \subseteq \vartheta \to X$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(\grave{u}))_{\grave{u}\in R}$ on Banach space ϑ. We denote by $(S(\grave{u}))_{\grave{u}\in R}$ the sine function associated with $(C(\grave{u}))_{\grave{u}\in R}$ which is defined by $S(\grave{u})\vartheta = \int_0^{\grave{u}} C(s)\vartheta ds$, for $\vartheta \in X$ and $\grave{u} \in R$. We refer them to [3, 21] for the necessary concepts about cosine functions. After that we only mention a few results and notations about this matter needed to establish our results. It is immediate that

$$C(\grave{u})\vartheta - \vartheta = A \int_0^{\grave{u}} S(s)\vartheta ds, \tag{10}$$

for all $\vartheta \in X$. The notation $D(A)$ stands for the domain of the operator A endowed with the graph norm $\|\vartheta\|_A = \|\vartheta\| + \|A\vartheta\|, \vartheta \in D(A)$. Moreover, in this work, E is the space formed by the vectors $\vartheta \in \vartheta$ for which $C(.)\vartheta$ is of class C^1 on R. It was proved by Kisynski [11] that E endowed with the norm $\|\vartheta\|_E = \|\vartheta\| + \sup_{0\leq t\leq 1} \|AS(t)\vartheta\|, \vartheta \in E$, is a Banach space. The operator-valued function

$$\mathcal{H}(\grave{u}) = \begin{bmatrix} C(\grave{u}) & S(\grave{u}) \\ AS(\grave{u}) & C(\grave{u}), \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times \vartheta$ generated by the operator $\mathcal{H}(\grave{u}) = \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $S(\grave{u}) : X \to E$ is a bounded linear map such that the operator valued maps $S(.)$ is strongly continuous and $AS(\grave{u}) : E \to X$ is a bounded linear operator such that $A(S)\vartheta \to 0$ as $\grave{u} \to 0$, for each $\vartheta \in E$. Furthermore, if $\vartheta : [0, \infty) \to X$ is a locally integrable function, then the function $y(\grave{u}) = \int_0^{\grave{u}} S(\grave{u} - s)\vartheta(s)ds$ defines an E-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem,

$$\vartheta''(\grave{u}) = A\vartheta(\grave{u}) + h(\grave{u}), \quad 0 \leq \grave{u} \leq a \tag{11}$$

$$\vartheta(s) = \upsilon, \vartheta'(s) = \omega, \tag{12}$$

where $h : J \to X$ is an integrable function, has been discussed in [22]. Similarly the existence of solutions of semilinear second order abstract Cauchy problems has been treated in [23]. We only mention here that the function $\vartheta(\cdot)$ given by

$$\vartheta(\grave{u}) = C(\grave{u}-s)\upsilon + S(\grave{u}-s)\omega + \int_s^{\grave{u}} S(\grave{u}-\tau)h(\tau)d\tau, \quad 0 \leq \grave{u} \leq a, \tag{13}$$

is called a mild solution of (7)-(8) and that when $\upsilon \in E, \vartheta(.)$ is continuously differentiable and

$$\vartheta'(\grave{u}) = AS(\grave{u}-s)\upsilon + C(\grave{u}-s)\omega + \int_s^{\grave{u}} C(\grave{u}-\tau)h(\tau)d\tau \quad 0 \leq \grave{u} \leq a.$$

In addition ,if $\upsilon \in D(A), \omega \in E$ and f is a continuously differentiable function, then the function $\vartheta(\cdot)$ is a solution of the initial value problem (11)-(12).

Assume now that $A(\grave{u}) = A + \widetilde{B}(\grave{u})$ where $\widetilde{B}(\cdot) : R \to \mathcal{L}(E, \vartheta)$ is a map such that the function $\grave{u} \mapsto \widetilde{B}(\grave{u})\vartheta$ is continuously differentiable in ϑ for each $\vartheta \in E$. It has been established by serizawa [20] that for each $(\upsilon, \omega) \in D(A) \times E$ the non-autonomous abstract Cauchy problem

$$\vartheta''(\grave{u}) = (A + \widetilde{B}(\grave{u}))\vartheta(\grave{u}), \grave{u} \in R, \tag{14}$$

$$\vartheta(0) = \upsilon, \quad \vartheta'(0) = \omega, \tag{15}$$

has a unique solution $\vartheta(\cdot)$ such that the function $\grave{u} \mapsto \vartheta(\grave{u})$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that Eq.(14), with the initial condition (12) has a unique solution $\vartheta(\cdot, s)$ such that the function $\grave{u} \mapsto \vartheta(\grave{u}, s)$ is continuously differentiable in E. It follows from (13) that

$$\vartheta(\grave{u}, s) = C(\grave{u}-s)\upsilon + S(\grave{u}-s)\omega + \int_s^{\grave{u}} S(\grave{u}-\tau)\widetilde{B}(\tau)\vartheta(\tau, s)d\tau.$$

In particular ,for $\upsilon = 0$ we have

$$\vartheta(\grave{u}, s) = S(\grave{u}-s)\omega + \int_s^{\grave{u}} S(\grave{u}-\tau)\widetilde{B}(\tau)\vartheta(\tau, s)d\tau. \tag{16}$$

Consequently,

$$\|\vartheta(\grave{u}, s)\|_1 \leq \|S(\grave{u}-s)\|_{\mathcal{L}(\vartheta,E)} \|\omega\| + \int_s^{\grave{u}} \|S(\grave{u}, \tau)\|_{\mathcal{L}(\vartheta,E)} \left\|\widetilde{B}(\tau)\right\|_{\mathcal{L}(\vartheta,E)} \|\vartheta(\tau, s\|_1 \, d\tau$$

and, applying the Gronwall - Bellman lemma we infer that

$$\|\vartheta(\grave{u}, s)\|_1 \leq \widetilde{M} \|\omega\|, \quad s, \grave{u} \in J. \tag{17}$$

We define the operator $S(\grave{u}, s)\omega = \vartheta(\grave{u}, s)$. It follows from the previous estimate that $S(\grave{u}, s)$ is a bounded linear map on E. Since E is dense in X, we can extend $S(\grave{u}, s)$ to X. We keep the notation $S(\grave{u}, s)$ for this extension. It is well known that, exception the case $dim(X) < \infty$, the cosine function $C(\grave{u})$ cannot be compact for all $\grave{u} \in \mathbb{R}$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(\grave{u})$ is very often a compact operator for all $\grave{u} \in \mathbb{R}$. This motivates the result [[7], Theorem 1.2].

We now consider some notations and definitions concerning impulsive differential equations. A function $\vartheta : [\sigma, \tau] \to X$ is said to be a normalized piece wise continuous function on $[\sigma, \tau\ ]$ if ϑ is piece wise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau], X)$ the space of normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all normalized piece wise continuous functions $\vartheta : [0, a] \to X$ such that $\vartheta(\cdot)$ is continuous at $\grave{u} \neq \grave{u}_k, \vartheta(\grave{u}_{k^-}) = \vartheta(\grave{u}_k)$ and $\vartheta(\grave{u}_k^+)$ exists, for $k = 1, 2, \ldots, m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm $\|\vartheta\|_{\mathcal{PC}} = \sup_{s \in J} \|\vartheta(s)\|$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

In what follows, we put $\grave{u}_0 = 0, \grave{u}_{n+1} = a$ and, for $\vartheta \in \mathcal{PC}$, we denote by $\tilde{\vartheta}_k$, for $k = 0, 1, \ldots, m$, the function $\tilde{\vartheta}_k \in C([\grave{u}_k, \grave{u}_{k+1}]; \vartheta)$ given by $\tilde{\vartheta}_k(\grave{u}) = \vartheta(\grave{u})$ for $\grave{u} \in (\grave{u}_k, \grave{u}_{k+1})$ and $\tilde{\vartheta}_k(\grave{u}_k) = \lim_{\grave{u} \to \grave{u}^+} \vartheta(\grave{u})$. Moreover, for a set $E \subseteq \mathcal{PC}$, we denote by $\tilde{E}_k$, for $k = 0, 1, \ldots, m$, the set $\tilde{E}_k = \tilde{\vartheta}_k : \vartheta \in E$.

Lemma 2.1. *A set $E \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if each $\widetilde{E_k}$,$k = 0, 1, \ldots, m$, is relatively compact in $C([\grave{u}_k, \grave{u}_{k+1}]; \vartheta)$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$, similar to the one used in [8] and suitably modify to treat retarded impulsive differential equations. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty, 0]$ into ϑ endowed with a seminorm denoted $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms are hold:

(A) If $\vartheta : (-\infty, \mu + b] \to \vartheta, b > 0$, is such that $\vartheta_\mu \in \mathcal{B}$ and $\vartheta|_{[\mu, \mu+b]} \in \mathcal{PC}([\mu, \mu + b], X)$ then, for every $\grave{u} \in [\mu, \mu + b)$, the following conditions are hold: and

   (i) $\vartheta_{\grave{u}}$ is $\in \mathcal{B}$,

   (ii) $\|\vartheta(\grave{u})\| \leq H \|\vartheta_{\grave{u}}\|_{\mathcal{B}}$,

   (iii) $\|\vartheta_{\grave{u}}\|_{\mathcal{B}} \leq K(\grave{u} - \mu) \sup\{\|\vartheta(s)\| : \mu \leq s \leq \grave{u}\} + M(\grave{u} - \mu)\|\vartheta_\mu\|_{\mathcal{B}}$,

   where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $\vartheta(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Remark 2.1. In impulsive functional differential systems, the map $[\mu, \mu + b] \to \mathcal{B}$, $\grave{u} \to \vartheta_t$, is in general discontinuous. For this reason, this property has been omitted from our description of the phase space $\mathcal{B}$.

Now we include that some of our proofs are based on the following well-known result [[4], Theroem 6.5.4].

Lemma 2.2 (Leray-Schauder Alternative)**.** *Let D be a closed convex subsets of a normed lined space ϑ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{\vartheta \in D : \vartheta = \lambda F(\vartheta),\ for\ some 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

The terminology and notations are generally used in functional analysis. In particular, for Banach spaces $(Z, \|\cdot\|), (W, \|\cdot\|_w)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. By $\sigma(A)$ (respectively ,$\rho(A)$) we denote the spectrum (respectively ,the resolvent set)of a linear operator A. Moreover$B_r(\vartheta, Z)$ denotes the closed ball with center at ϑ and radius $r > 0$ in the space Z.

Remark 2.2. In what follows the notation $g(a)$ stands for the space

$$\theta_1(a) = \{y : (-\infty, a] \to \vartheta : y|_J \in \mathcal{PC}, y_0 = 0\}.$$

endowed with the sup norm. In addition, we denote by $\widetilde{\phi} : (-\infty, a] \to \vartheta$ the function defined by $\widetilde{\phi}_0 = \phi$ and $\widetilde{\phi}(\grave{u}) = C(\grave{u}, 0)\phi(0) + S(\grave{u}, 0)\zeta$, for $\grave{u} \geq 0$.

Definition 2.2. *A function $\vartheta : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem* (1)-(4), *if $\vartheta_0 = \phi \in \mathcal{B}, \vartheta|_J \in \mathcal{PC}$, the impulsive conditions $\Delta\vartheta(\grave{u}_k) = I_k(\vartheta_{\grave{u}_k})$, $\Delta\vartheta'(\grave{u}_k) = J_k(\vartheta_{\grave{u}_k})$, $k = 1, 2 \ldots, m$, are satisfied and the following integral equation*

$$\begin{aligned}\vartheta(\grave{u}) &= C(\grave{u}, 0)\phi(0) + S(\grave{u}, 0)\Big(\vartheta - \theta_1(0, \phi, 0)\Big) + \int_0^{\grave{u}} C(\grave{u}, s)\theta_1(s, \vartheta_s)ds \\ &+ \int_0^{\grave{u}} S(\grave{u}, s)\Big[Bu(s) + \theta_2(s, \vartheta_s)\Big]ds \\ &+ \sum_{0<\grave{u}_k<\grave{u}} C(\grave{u}, \grave{u}_k)I_k(\vartheta_{\grave{u}_k}) + \sum_{0<\grave{u}_k<\grave{u}} S(\grave{u}, \grave{u}_k)J_k(\vartheta_{\grave{u}_k}), \quad 0 < \grave{u} < a.\end{aligned}$$

is verified.

3. Controllability result

To establish our result, we introduce the following assumptions on system (1)-(4):

(H1) The function $\theta_i : J \times B \to X$ is continuous and there exist constants $L_{\theta_i} > 0$, $\widetilde{L_{\theta_i}} > 0$ for $\psi_1, \psi_2 \in \mathcal{B}$, such that

$$\|\theta_i(\grave{u}, \psi_1) - \theta_i(\grave{u}, \psi_2)\| \leq L_{\theta_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}$$

and $\widetilde{L}_{\theta_i} = sup_{\grave{u} \in J} \|\theta_i(\grave{u}, 0)\|, \quad i = 1, 2.$ we have $\|\theta_i(t, \psi)\| \leq L_{\theta_i}\|\psi\|_{\mathcal{B}} + \widetilde{L}_{\theta_i}, \quad i = 1, 2.$

(H2) B is a continuous operator from U to X and the linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^a S(a,s)Bu(s)ds,$$

has a bounded invertible operator W^{-1} which takes values in $L^2(J,U)/kerW$ and there exist positive constant M such that $\left\|BW^{-1}\right\| \leq M_1$.

(H3) The impulsive functions satisfy the following conditions:

(i) The maps $I_k : \mathcal{B} \to X, k = 1,2,\ldots,m$ is continuous and there exist constants $L_1 > 0, \widetilde{L}_1 > 0$ for $\psi_1, \psi_2 \in \mathcal{B}$ such that

$$\|I_k(\psi_1) - I_k(\psi_2)\| \leq L_I \|\psi_1 - \psi_2\|$$

and $\widetilde{L}_I = \|I_k(0)\|$.

(ii) The maps $J_k : \mathcal{B} \to X, k = 1,2,\ldots,m$ is continuous and there exists constants $L_J > 0, \widetilde{L}_J > 0$ for $\psi_1, \psi_2 \in \mathcal{B}$ such that

$$\|J_k(\psi_1) - J_k(\psi_2)\| \leq L_J \|\psi_1 - \psi_2\|$$

and $\widetilde{L}_J = \|J_k(0)\|$.

(H4) Let $aN\Big[L_{\theta_1}(K_a r + c_1) + \widetilde{L}_{\theta_1}\Big] + aNA_0 + \sum_{k=1}^m \Big(\widetilde{N}L_I + NL_J\Big)\Big[K_a r + \|\widetilde{\phi}_{\grave{u}_k}\|\Big] + \sum_{k=1}^m \Big(\widetilde{N}\widetilde{L}_I + N\widetilde{L}_J\Big) \leq r$, for some $r > 0$.

(H5) Let $\mu = k_a(1 + aNM_1)[a\widetilde{N}L_g + aNL_{\theta_1} + \sum_{k=1}^m (\widetilde{N}L_I + NL_J) < 1$ be such that $0 \leq \mu < 1$.

Definition 3.3. *The system* (14-17) *is said to be controllable on the interval J, if for every $\vartheta_0 = \phi \in \mathcal{B}, \vartheta'(0) = \zeta$ and $z_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $\vartheta(\cdot)$ of* (1)-(4) *satisfies $\vartheta(a) = z_1$.*

The following results is an immediate application of the contraction principle of Banach. To simplify the text, we denote $K_a = \sup_{0 \leq \grave{u} \leq a} K(\grave{u})$.

Since $\|\widetilde{\phi}_k\|_{\mathcal{B}} = \widetilde{N}\|\phi(0)\| + N\|\zeta\| + M\|\phi\|_{\mathcal{B}} < \infty$, $0 \leq t \leq a$, we denote, $c_1 = \sup_{0 \leq \grave{u} \leq a} \|\phi_{\grave{u}}\|_{\mathcal{B}}$ and $\|y_{\grave{u}} + \widetilde{\phi}_{\grave{u}}\| \leq K_a\|y_{\grave{u}}\| + \|\widetilde{\phi}\| \leq K_a r + c_1 = \rho$.

Theorem 3.1. *If the hypothesis $(H1)$-$(H5)$ are satisfied, then the impulsive second order system* (1)-(4) *is controllable on J.*

Proof. Using the assumption $(H2)$, we define the control function

$$u(\grave{u}) = W^{-1}[z_1 - C(a,0)\phi(0) - S(a,0)[\zeta - \theta_1(0,\phi)] + \int_0^a C(a,s)\theta_1(s,\vartheta_s)ds$$
$$- \int_0^a S(a,s)\theta_2(s,\vartheta_s)ds - \sum_{k=1}^m C(a,\grave{u}_k)I_k(\vartheta_{\grave{u}_k}) - \sum_{k=1}^m S(a,\grave{u}_k)J_K(\vartheta_{\grave{u}_k})](\grave{u}).$$

Let $B_r = \{\vartheta \in X, \|\vartheta\| \leq r\}$ for some $r > 0$. now we show that when using this control the operator Γ on the space $\theta_2(a)$ defined by$(\Gamma y)_0 = 0$ and

$$
\begin{aligned}
(\Gamma y)(\grave{u}) = {} & S(\grave{u}, 0)[\zeta - \theta_1(0, \phi)] - \int_0^{\grave{u}} C(\grave{u}, s)\theta_1(s, y_s \\
& + \widetilde{\phi}_s)ds + \int_0^{\grave{u}} S(\grave{u}, s)\theta_2(s, y_s + \widetilde{\phi}_s)ds \\
& + \int_0^{\grave{u}} S(\grave{u}, \eta)BW^{-1}\Big[z_1 - C(a, 0)\phi(0) - S(a, 0)[\zeta - \theta_2(0, \phi)] \\
& + \int_0^{a} C(a, s)\theta_1(s, y_s + \widetilde{\phi}_s)ds \\
& - \int_0^{\grave{u}} S(a, s)\theta_2(s, y_s + \widetilde{\phi}_s)ds - \sum_{k=1}^{m} C(a, \grave{u}_k)I_k(y_{\grave{u}_k} + \widetilde{\phi}_{\grave{u}_k}) \\
& - \sum_{k=1}^{m} S(a, \grave{u}_k)J_k(y_{\grave{u}_k} + \widetilde{\phi}_{\grave{u}_k})\Big](\eta)d\eta \\
& + \sum_{0<\grave{u}_k} <_{\grave{u}} C(\grave{u}, \grave{u}_k)I_k(y_{\grave{u}_k} + \widetilde{\phi}_{\grave{u}_k}) + \sum_{0<\grave{u}_k<\grave{u}} S(\grave{u}, \grave{u}_k)J_k(y_{\grave{u}_k} + \widetilde{\phi}_{\grave{u}_k}), \grave{u} \in J,
\end{aligned} \tag{18}
$$

has a fixed point $\vartheta(\cdot)$. This fixed point is then a mild solution of the system (1)-(4). Clearly $(\Gamma\vartheta)(a) = z_1$ which means that the control u steers the system from the initial state ϕ to z_1 in time a, provided we can obtain a fixed point of the operator Γ which implies that the system is controllable. From the assumptions, it is easy to see that Γ is well defined and continuous. For convenience let us take,

$$
\begin{aligned}
\|Bu(s)\| \leq {} & M_1[\|z_1\| + \widetilde{N}\|\phi(0)\| + N[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_{\theta_2}}] + \widetilde{N}a[L_{\theta_2}(K_a r + c_1)\widetilde{L_{\theta_2}}] \\
& + aN[L_{\theta_1}(K_a r + c_1) + \widetilde{L}_{\theta_1}] + \widetilde{N}\sum_{k=1}^{m}[L_I(K_a r + \|\widetilde{\phi}_{\grave{u}_k}\|) + \widetilde{L}_I] \\
& + N\sum_{k=1}^{m}[l_{\theta_1}(K_a r + \|\widetilde{\phi}\|) + \widetilde{L}_J] = A_0.
\end{aligned}
$$

First we show that Γ maps $B_r(0, \theta_2(a))$ into $B_r(0, \theta_2(a))$. To this end, from the definition of the operator Γ in (18) and our hypotheses, we obtain

$$
\begin{aligned}
\|(\Gamma y)(\grave{u})\| \leq {} & N[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_g}] + \widetilde{N}a[L_{\theta_2}(K_a r + c_1)\widetilde{L_{\theta_2}}] \\
& + aN[L_{\theta_1}(k_a r + c_1) + \widetilde{L}_{\theta_1}] + aNA_0 + \sum_{k=1}^{m}(\widetilde{N}\widetilde{L}_I + N\widetilde{L}_J) \\
& + \sum_{k=1}^{M}(\widetilde{N}L_J)[K_a r + \|\widetilde{\phi}_{\grave{u}_k}\|] \leq r.
\end{aligned}
$$

for $y \in \theta_2(a)$ and $\grave{u} \in J$. Hence $\|\Gamma y\|_a \leq r$. Therefore, Γ maps $B_r(0, \theta_2(a))$ into itself. Now for $y, z \in B_r(0, \theta_2(a))$, we have

$$\|(\Gamma y)(\grave{u}) - (\Gamma z)(\grave{u})\| \leq K_a(1 + aNM_1)\Big[a(\widetilde{N}L_{\theta_2} + aNL_{\theta_1} + \sum +k = 1^m((\widetilde{N}L_I + NL_J)\Big]\|y - z\| \leq \mu\|y - z\|_a.$$

Which implies that Γ is a contraction on $B_r(0, \theta_2(a))$. Hence by the Banach fixed point theorem, Γ has a unique fixed point y in $\theta_2(a)$. Defining $\vartheta(\grave{u}) = y(\grave{u}) + \widetilde{\phi}(\grave{u}), -\infty < \grave{u} \leq a$, we obtain that $\vartheta(\cdot)$ is a mild solution of the problem (1)-(4) and the proof is complete.
We use the below condition instead of (H1) to avoid the Lipschitz continuity of f used in Theorem 3.1.

(A1) The function $\theta_1 : J \times \mathcal{B} \to X$ satisfies the following conditions:

- (i) For each $\grave{u} \in J$, the function $\theta_1(\grave{u}, \cdot) : \mathcal{B} \to X$ is continuous and the function $\grave{u} \to \theta_1(\grave{u}, \vartheta_{\grave{u}})$ is strongly measurable.
- (ii) There exist an integrable function $p : J \to [0, \infty)$ and a continuous non-decreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that

$$\|\theta_1(\grave{u}, \psi)\| \leq p(\grave{u})\Omega(\|\psi\|_B, (\grave{u}, \psi) \in J \times \mathcal{B}.$$

Also, we have the following condition.

(A2)

$$\Big[\|z_1\| + N\|\phi(0)\| + \widetilde{N}[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_{\theta_2}}] + Na(L_{\theta_2}\rho + \widetilde{L_{\theta_2}}) + N\int_0^a p(s)\Omega(\rho)ds + \widetilde{N}\sum_{k=1}^m(\alpha_k^1(\rho) + \alpha_k^2) + N\sum_{k=1}^m(\beta_k^1(\rho) + \beta_k^2)\Big] = M^*.$$

Theorem 3.2. *Assume that θ_1 verify condition $(A1)$ and condition $(A2)$, $(H2)$ are satisfied. Also, the following condition hold:*

- (a) *For every $\grave{u}$ $\in J$ and every $r > 0$, the set $U(r, \grave{u}) = \{S(\grave{u}, s)\theta_1(s, \psi) : s \in [0, a], \psi \in B_r(0, \mathcal{B})\}$ is relatively compact in X.*
- (b) *The maps $I_k, J_k : \mathcal{B} \to X$ are completely continuous and there exist positive constants $\alpha_k^i, \beta_k^i, i = 1, 2, k = 1, 2, \ldots, m$, such that $\|I_k(\psi)\| \leq \alpha_k^1 \|\psi\|_{\mathcal{B}} + \alpha_k^2$ and $\|J_k(\psi)\| \leq \beta_k^1 \|\psi\|_{\mathcal{B}} + \beta_k^2$, for all $\psi \in \mathcal{B}$.*
- (c) *The constant $\mu = \tilde{N}aK_aL_g + K_a\sum_{k=1}^m(\tilde{N}\alpha_k^1 + N\beta_k^1) < 1$ and $\int_c^\infty \frac{ds}{\Omega(s)} > \frac{K_\alpha N}{1-\mu}\int_0^a p(s)ds$ where $c = \frac{1}{1-\mu}[N[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_{\theta_2}}] + \tilde{N}aK_a\widetilde{L_{\theta_2}} + aNM_1M^* + c_1 + K_a\sum_{k=1}^m(\tilde{N}\alpha_k^2 + N\beta_k^2)]$. Then the (1)-(4) is controllable on J.*

Proof. we define the map Γ on the space $\theta_2(a)$ as in eq (18). To prove the controllability of the problem (1)-(4), we must show that the operator Γ has a fixed point. This fixed point is then a mild solution of the system (1)-(4). From the assumptions, it is easy to see that Γ is well defined and continuous.

In order to apply Lemma 2.2, we need to obtain a priori bound for the solutions of the integral equation $y = \lambda\Gamma(y)$, $\lambda \in (0,1)$. To this end, let y^λ be a solution of $\lambda\Gamma(y) = y$, $\lambda \in (0,1)$. Using the notation $\nu^\lambda = \sup_{0\le s\le \grave{u}} \|y_s^\lambda + \tilde{\phi}_s\|_{\mathcal{B}} \le K_a\|y^\lambda\|_s + \|\tilde{\phi}_s\|_{\mathcal{B},a} \le K_a r + c_1 = \rho$, we observe that

$$\|y^\lambda(\grave{u})\| \le N[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_{\theta_2}}] + \tilde{N}aK_a(L_{\theta_2}v^\lambda(s) + \widetilde{L_{\theta_2}}) + N\int_0^a p(s)\Omega(v^\lambda(s))ds$$
$$+ aNM_1[M^*] + \sum_{0<\grave{u}_k<t}(\widetilde{N}\alpha_k^1 + N\beta_k^1)v^\lambda(\grave{u}_k) + \sum_{0<\grave{u}_k<t}(\widetilde{N}\alpha_k^2 + N\beta_k^2).$$

Hence follows that

$$v^\lambda(\grave{u}) \le N[\|\zeta\| + L_{\theta_2}\|\phi\| + \widetilde{L_{\theta_2}}] + \tilde{N}aK_a\widetilde{L_{\theta_2}} + K_aN\int_0^t p(s)\Omega(v^\lambda(s))ds$$
$$+ aNM_1M^* + \sup_{0\le s\le t}\|\phi_s\|_{\mathcal{B}} + K_a\sum_{k=1}^m(\widetilde{N}\alpha_k^2 + N\beta_k^2) + \mu v^\lambda(t)$$

which yields

$$v^\lambda(\grave{u}) \le c + \frac{K_aN}{1-\mu}\int_0^{\grave{u}} p(s)\Omega(v^\lambda(s))ds.$$

Denoting by $\omega'_\lambda(t)$ the right-hand side of the previous inequality, we see that

$$\omega'_\lambda(\grave{u}) \le \frac{KaN}{1-\mu}[p(\grave{u})\Omega(\omega_\lambda(\grave{u}))],$$

and subsequently, upon integrating over $[0,\grave{u}]$, we obtain

$$\int_c^{\omega_\lambda(\grave{u})}\frac{ds}{\Omega(s)} \le \frac{K_aN}{1-\mu}\int_0^{\grave{u}} p(s)ds \le \frac{K_aN}{1-\mu}\int_0^a p(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}.$$

This estimate permits us to conclude that the set of functions $\{\omega_\lambda : \lambda \in (0,1)\}$ is bounded and, in turn, that $\{y^\lambda : \lambda \in (0,1)\}$ is bounded in $\theta_2(a)$. Next we show that Γ is completely continuous. To clarify this proof, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 y(\grave{u}) = \int_0^{\grave{u}} S(\grave{u},s)[\theta_2(s, y_s + \tilde{\phi}_s) + Bu(s)]ds,$$
$$\Gamma_2 y(\grave{u}) = S(\grave{u},0)[\zeta - \theta_1(0,\phi)] - \int_0^{\grave{u}} C(\grave{u},s)\theta_1(s, y_s + \widetilde{\phi}_s)ds$$
$$+ \sum_{0<\grave{u}_k<\grave{u}} C(\grave{u},\grave{u}_k)I_k(y_{\grave{u}_k} + \tilde{\phi}_{\grave{u}_k}) + \sum_{0<\grave{u}_k<\grave{u}} S(\grave{u},\grave{u}_k)J_k(y_{\grave{u}_k} + \tilde{\phi}_{\grave{u}_k}), \grave{u} \in J.$$

Using the hypotheses, condition (b) and Lemma 2.1, we obtain that Γ_1 is continuous and that Γ_2 is completely continuous. In order to use the Ascoli-Arzela theorem we prove that Γ_1 takes bounded sets into relatively compact ones. As above, $B_r = B_r(0, \theta_2(a))$ and $\|y_{\grave{u}} + \tilde{\phi}_{\grave{u}}\|_{\mathcal{B}} \leq K_a r + c_1 = \rho \;\; for \;\; \grave{u} \in J$. And also, $\|(Bu)(s)\| \leq B_0$.

From the mean value theorem , we see that

$$\Gamma_1 y(\grave{u}) \in \overline{\grave{u} co \{S(\grave{u}, s)\theta_1(s, \psi) : s \in [0, a], \|\psi\|_{\mathbb{B}} \leq \rho\}}$$

which implies that the set $\{\Gamma_1 y(\grave{u}) : y \in B_r(0, \theta_2(a))\}$ is relatively compact for each $\grave{u} \in J$. Moreover, from

$$\begin{aligned}\Gamma_1 y(\grave{u}+h) - \Gamma_1 y(\grave{u}) &= \int_0^{\grave{u}} [S(\grave{u}+h, s) - S(\grave{u}, s)][\theta_2(s, y_s + \tilde{\phi}) + (Bu)(s)]ds \\ &+ \int_{\grave{u}}^{\grave{u}+h} S(\grave{u}+h, s)[\theta_2(s, y_s + \tilde{\phi}_s) + (Bu)(s)]ds\end{aligned}$$

and using that $S(., s)$ verifies a Lipschitz condition, we obtain that

$$\|\Gamma_1 y(\grave{u}+h) - \Gamma_1 y(\grave{u})\| \leq |h|\, N_1 \int_0^a [p(s)\Omega(\rho) + B_0]ds + N \int_{\grave{u}}^{\grave{u}+h} [p(s)\Omega(\rho) + B_0]ds$$

which implies that $\|\Gamma_1 y(t+h) - \Gamma_1 y(t)\| \to 0$ as $h \to 0$uniformly for y $\in B_r(0, g(a))$. From this we infer that $\Gamma_1 y(\grave{u}) : y \in B_r(0, \theta_2(a)$ is relatively compact in $G(a)$ and consequently that Γ_1 is completely continuous. This completes the proof of the assertion that the map Γ is completely continuous.

By an application of Lemma 2.1, we conclude that there exists a fixed point y of Γ. It is clear that the function $\vartheta = y + \tilde{\phi}$ is a mild solution of the system (1)-(4). This completes the proof.

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θ-CLOSURE AND $T_{2\frac{1}{2}}$ SPACES VIA IDEALS

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Abstract. We introduce θ-closure of a set with respect to an ideal using the local closure function and obtain some properties.We also introduce θ-convergence of a filter and $T_{2\frac{1}{2}}$ spaces with respect to an ideal and by using these concepts and other separation axioms obtain the sufficient conditions for a set to be θ-closed with respect to an ideal and also obtain some characterizations of local closure function. Finally, the sufficient conditions for the equivalence of θ-closure with respect to an ideal and closure in $*$-topology are given.

Keywords: $\mathcal{I}_\theta$-closed, T_2 mod $\mathcal{I}$, $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, $\mathcal{I}$-compact, $\mathcal{I}$-regular, almost-$\mathcal{I}$-regular, $\mathcal{I}$-QHC, $\mathcal{I}_\theta$ convergence, ideal.

1. Introduction

In [15], Veličko introduced strong form of closed sets called θ-closed sets and in [7], Janković utilized these sets to obtain new characterizations of separation axioms. On the other hand in [1], Al-Omari and Noiri defined the local closure function stronger than the local function with respect to ideal topological space and obtained various properties of it. The concept of ideals has arisen due to Kuratowski [9] to study various topological properties. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X such that it is closed downwards (i.e. every subset of member of $\mathcal{I}$ is in $\mathcal{I}$) and closed under finite union. This concept was further studied by Vaidyanathaswamy who obtained a new topology $\tau^*(\mathcal{I}, \tau)$ called the $*$-topology which is generally finer than the original topology having the corresponding Kuratowski closure operator $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [13], where $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open nhd. U of x in $X\}$ called a local function[9] of A with respect to $\mathcal{I}$ and τ and $\beta = \{V - A : V \in \tau$ and $A \in \mathcal{I}\}$ is a basis for the $*$-topology τ^*. We will write A^* for $A^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ or τ^* for $\tau^*(\mathcal{I}, \tau)$.

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For a topological space (X,τ) and any subset A of X, a point $x \in X$ is said to be in the θ-closure of A if for every open nhd. U of x in X, $\overline{U} \cap A \neq \emptyset$ denoted by $cl_\theta(A)$[15] and in θ-interior of A if there exists a nhd. U of x such that $U \subset \overline{U} \subset A$ denoted by $int_\theta(A)$. The subset A is said to be θ-closed (θ-open) if $cl_\theta(A) = A$ ($int_\theta(A) = A$). Also the collection of all θ-open sets forms a topology denoted by τ_θ which is generally weeker than the original topology. And for an ideal topological space $(X,\tau,\mathcal{I})$ for any subset A of X, the local closure function $\Gamma(A)(\mathcal{I},\tau)$ with respect to $\mathcal{I}$ and τ is given as $\Gamma(A)(\mathcal{I},\tau) = \{x \in X : \overline{U} \cap A \notin \mathcal{I}$ for every τ-nhd. U of x in X $\}$ [1] and the operator $\Psi_\Gamma(A) = X - \Gamma(X - A) = \{x \in X :$ there exist τ-nhd. U of x such that $\overline{U} - A \in \mathcal{I}$ $\}$. Also note that the collection $\sigma = \{A \subset X : A \subset \Psi_\Gamma(A)\}$ forms a topology for X [1]. Further a topological space (X,τ) is said to be S_2 [2] if for any two distinct points x, y of X, whenever one of them has an open set not containing the other, then there exist disjoint open subsets containing them and a space is said to be Urysohn space or $T_{2\frac{1}{2}}$ space [5] if for every distinct points x, y of X there exist open subsets U, V containing x, y respectively such that $cl(U) \cap cl(V) = \emptyset$. Also an ideal space $(X,\tau,\mathcal{I})$ is said to be T_2 mod $\mathcal{I}$ [11] if for any two distinct points x, y of X, there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. An ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-compact [10] if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \bigcup\{G_\alpha : \alpha \in \Delta_0\} \in \mathcal{I}$ and is said to be $\mathcal{I}$-regular [4] if for any closed subset F of X and any point $x \in X$ whenever $x \notin F$, there exist disjoint open subsets U, V such that $x \in U$ and $F - V \in \mathcal{I}$. Also we have the following:

Definition 1.1 ([6])**.** Let $(X,\tau,\mathcal{I})$ be an ideal space then $\mathcal{I}$ is said to be codense if $\tau \cap \mathcal{I} = \emptyset$.

Example 1.1. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{b, c\}\}$. Then the following are codense ideals:

(a) $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$

(b) $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

(c) $\mathcal{I} = \{\emptyset, \{a\}\}$

(d) $\mathcal{I} = \{\emptyset, \{b\}\}$

(e) $\mathcal{I} = \{\emptyset, \{c\}\}$

Definition 1.2 ([15])**.** Let (X,τ) be any topological space and $\mathcal{F}$ be any filter on X. Then for any point $a \in X$, $\mathcal{F}$ *is said to be* θ*-convergent to* a denoted by $\mathcal{F} \to_\theta a$ if for every open set U containing a, $\overline{U} \in \mathcal{F}$.

Lemma 1.1 ([1])**.** *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then for any subset* A *of* X *the following holds:*

(*a*) $A^* \subset \Gamma(A)(\mathcal{I},\tau) \subset cl_\theta(A)$.

(b) $\Gamma(A)(\mathcal{I},\tau) = cl(\Gamma(A)(\mathcal{I},\tau))$.

Lemma 1.2 ([6]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. then the following are equivalent :*

(a) *$\mathcal{I}$ is codense.*

(b) *$X = X^*$.*

(c) *For every $U \in \tau$, $U \subseteq U^*$.*

Theorem 1.1 ([11]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if for each $x \in X$ and open set U containing x , there is an open set V containing x such that $\overline{V} - U \in \mathcal{I}$.*

Notation. Throughout this paper (X,τ) will denote topological space and for an ideal $\mathcal{I}$ on X, $(X,\tau,\mathcal{I})$ is called an ideal space. When there is no chance of confusion, by a open subset (open nhd.) of X, we will mean open set in the topological space (X,τ). For a subset A of X, $cl(A)$ or $\overline{A}$ and $int(A)$ will denote the closure of A, interior of A in (X,τ), respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of A, interior of A in (X,τ^*), respectively, and $X - A = A^C$ will denote the complement of A in X.

In section 2 of this paper, firstly for any subset of an ideal topological space $(X,\tau,\mathcal{I})$, we define θ-closure of a set using the local closure function and obtain its relationship with other θ-closure (closure) of a set. We prove that unlike in the case of local function, the local closure function of any subset A of X with respect to τ and τ^* need not be same (Example 2.2 below), but in case of codense ideal the local closure function of any subset A of X with respect to τ and τ^* coincide (Theorem 2.1 below). Further in Section 3, we introduce θ-convergence of a filter and $T_{2\frac{1}{2}}$ spaces with respect to an ideal and obtain various properties. We also obtain the characterization of local closure function in terms of $\mathcal{I}_\theta$ convergence of a filter (Theorem 3.2 below) and the characterization of $T_{2\frac{1}{2}}$ space with respect to an ideal in terms of $\mathcal{I}_\theta$ convergence of a filter and local closure function (Theorems 3.5 and 3.6 below). Finally, the sufficient conditions for the equivalence of $\mathcal{I}_\theta$-closure and closure in $*$-topology (Theorems 2.4 and 2.5 below), for the $\mathcal{I}_\theta$-closedness of a set (Theorems 2.3 and 3.8 below) and for the $\mathcal{I}_\theta$-closedness of $\mathcal{I}_\theta$-closure of a set (Theorem 2.6 below) are obtained. Examples are given throughout the paper to give counterexamples and illustrations.

2. Results

We begin by defining the θ-closure of a set with respect to an ideal (briefly $\mathcal{I}_\theta$-closure) for any subset A of X in an ideal topological space $(X,\tau,\mathcal{I})$.

Definition 2.1. Let $(X,\tau,\mathcal{I})$ be an ideal space. For any subset A of X, θ-closure of A with respect to an ideal $\mathcal{I}$ is given by $cl_{\mathcal{I}_\theta}(A) = A \cup \Gamma(A)(\mathcal{I},\tau)$. The subset A is said to be $\mathcal{I}_\theta$-closed if $cl_{\mathcal{I}_\theta}(A) = A$.

Remark 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. For any subset A of X, define $Int_{\mathcal{I}_\theta}(A) = X - cl_{\mathcal{I}_\theta}(X - A)$ so that $Int_{\mathcal{I}_\theta}(A) = \{x \in A : \overline{U} - A \in \mathcal{I}$ for some τ-nhd. U of x in X $\}$. The subset A is said to be $\mathcal{I}_\theta$-open if $Int_{\mathcal{I}_\theta}(A) = A$. It can be easily checked that the collection of $\mathcal{I}_\theta$-open sets forms a topology. In our further results we denote it by $\tau_{\mathcal{I}_\theta}$. Also note that in view of Lemma 1.1(a), we have $\tau_\theta \subset \tau_{\mathcal{I}_\theta} \subset \tau^*$.*

Even though using Lemma 1.1(b), it follows that $cl_{\mathcal{I}_\theta}(A)$ is closed subset of X, but the following Example 2.1, shows that it need not be $\mathcal{I}_\theta$ closed.

Example 2.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $cl_{\mathcal{I}_\theta}(\{b\}) = \{b, c\}$ and $cl_{\mathcal{I}_\theta}(\{b, c\}) = \{a, b, c\}$. Hence $\{b\}$ is not $\mathcal{I}_\theta$ closed.

In [6], Janković and Hamlett proved that for any subset A of X in an ideal space $(X, \tau, \mathcal{I})$, $A^*(\mathcal{I}, \tau) = A^*(\mathcal{I}, \tau^*(\mathcal{I}))$. So the natural question arises is the result true for the local closure function. The following Theorem 2.1 shows that for codense ideals the result also holds for the local closure function. Before this we prove the following Lemma:

Lemma 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $cl^*(G) - I \subset cl^*(G - I)$ for any open subset G of X and $I \in \mathcal{I}$.*

Proof. Let G be any open subset of X and $I \in \mathcal{I}$. Let $x \in X$ be such that $x \in cl^*(G) - I$ and H be any τ^*-nhd. of x, then $H - I$ is also τ^*-nhd. of x (since every $I \in \mathcal{I}$ is τ^*-closed and $x \notin I$). Therefore, $x \in cl^*(G)$ implies that $(H - I) \cap G \neq \emptyset$ and so $H \cap (G - I) \neq \emptyset$. Hence $x \in cl^*(G - I)$. □

Theorem 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. Then $\Gamma(A)(\mathcal{I}, \tau) = \Gamma(A)(\mathcal{I}, \tau^*)$ for any subset A of X.*

Proof. Let A be any subset of X. Then it can be seen easily that $\Gamma(A)(\mathcal{I}, \tau^*) \subset \Gamma(A)(\mathcal{I}, \tau)$ since $\tau \subset \tau^*$ and $\mathcal{I}$ is closed downwards. Conversely, let $x \in X$ such that $x \notin \Gamma(A)(\mathcal{I}, \tau^*)$, so there exist τ^* nhd. G of x such that $cl^*(G) \cap A \in \mathcal{I}$. Since $\beta = \{V - A : V \in \tau$ and $A \in \mathcal{I}\}$ is a basis for the $*$-topology τ^*, so there exist basic open set $U - I$ such that $x \in U - I \subset G$ and so $cl^*(U - I) \cap A \in \mathcal{I}$. Now using Lemma 2.1, it follows that $cl^*(U) \cap A \in \mathcal{I}$. Further $\mathcal{I}$ is codense, so $cl^*(U) = cl(U)$ for every open subset U of X. Hence $x \notin \Gamma(A)(\mathcal{I}, \tau)$. □

The following Example 2.2 shows that if the ideal is not codense then Theorem 2.1 need not be true.

Example 2.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \wp(X)$. Then $\Gamma\{c\}(\mathcal{I}, \tau^*) = \{c\}$ and $\Gamma\{c\}(\mathcal{I}, \tau) = \{a, b, c\}$.

Theorem 2.2. *The topology $\tau^*(\mathcal{I}, \tau_\theta)$, the local function of which is given by $A^*(\mathcal{I}, \tau_\theta) = \{x \in X : U \cap A \notin \mathcal{I}$ for every nhd. U of x in $(X, \tau_\theta)\}$ is generally coarser than $\tau_{\mathcal{I}_\theta}$.*

Proof. For this we will prove $cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)(\mathcal{I}, \tau_\theta)$. Let $x \in X$ be any element such that $x \notin cl^*(A)(\mathcal{I}, \tau_\theta)$. So there exist τ_θ open subset U of x such that $U \cap A \in \mathcal{I}$. Therefore, U is θ-open implies that there exist open subset V of x such that $cl(V) \subset U$ and so $cl(V) \cap A \in \mathcal{I}$. Hence $x \notin cl_{\mathcal{I}_\theta}(A)$. □

Corollary 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then for any subset A of X, $cl^*(A)(\mathcal{I}, \tau) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)(\mathcal{I}, \tau_\theta) \subset cl_\theta(A)$. So every τ_θ-open set is $(\tau_\theta)^*$-open, $(\tau_\theta)^*$-open set is $\mathcal{I}_\theta$-open and hence τ^*-open.*

Here a natural question arise given any ideal space $(X, \tau, \mathcal{I})$, is there can be any relationship between $*$-topology of θ-open sets and θ-open sets with respect to τ^*. The following Example 2.3 shows that there is no relationship between the topological spaces $\tau^*(\mathcal{I}, \tau_\theta)$ and $(\tau^*)_\theta$, where $(\tau^*)_\theta$ means θ-open sets with respect to τ^*.

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. So $\tau^* = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $\tau_\theta = \{\emptyset, X\}$ and so $\tau^*(\mathcal{I}, \tau_\theta) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. But $(\tau^*)_\theta = \{\emptyset, \{b\}, \{a, c\}, X\}$.

In an ideal space $(X, \tau, \mathcal{I})$ even though $cl^*(A)(\mathcal{I}, \tau) \subset cl_{\mathcal{I}_\theta}(A)$ i.e. every $\mathcal{I}_\theta$-open set is τ^*-open but the following example shows that there is no relationship between $\mathcal{I}_\theta$-open and τ-open subset of X.

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then it can be easily checked that $\tau_{\mathcal{I}_\theta} = \{\emptyset, \{b, c\}, X\}$.

In [6], Janković and Hamlett proved that $A^*(\mathcal{I}_f, \tau) = A^d$ if and only if (X, τ) is T_1 (where $\mathcal{I}_f$ denotes the ideal of finite subsets of X and A^d denotes the derived set of A in (X, τ)). Therefore, we have the immediate results in the following:

Note 1. Let $(X, \tau, \mathcal{I}_f)$ be an ideal space and (X, τ) be T_1. Then $cl(A) = cl^*(A) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(\mathcal{I}, \tau_\theta)(A) \subset cl_\theta(A)$ for any subset A of X.

Further in [6], it is shown that for an ideal space $(X, \tau, \mathcal{I}_{cd})$, where $\mathcal{I}_{cd} = \{A \subset X : A^d = \emptyset\}$, $A^d \subset A^*$. So we have

Note 2. Let $(X, \tau, \mathcal{I}_{cd})$ be an ideal space. Then $cl(A) = cl^*(A) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(\mathcal{I}, \tau_\theta)(A) \subset cl_\theta(A)$ for any subset A of X.

Therefore, the above Notes 1 and 2 give the relationship between closed sets and $\mathcal{I}_\theta$ closed sets for particular ideal of finite sets and ideal of closed and discrete sets.

Further we will give characterizations of $\mathcal{I}_\theta$-closed sets using separation axioms.

It is well known that every compact set in T_2 space is closed and in [7], Janković proved the stronger result that a space is T_2 if and only if every compact set is θ-closed. On the other hand in [11], Sivaraj and Renukadevi proved that an $\mathcal{I}$-compact set in T_2 mod $\mathcal{I}$ space is τ^*-closed. Therefore, analogously the following Theorem 2.3 shows the stronger result that every $\mathcal{I}$-compact set in T_2 space is $\mathcal{I}_\theta$ closed.

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and (X, τ) be T_2. Then for any $\mathcal{I}$-compact subset A of X, A is $\mathcal{I}_\theta$ closed.*

Proof. We have to prove that $cl_{\mathcal{I}_\theta}(A) \subset A$. Let $x \in X$ be any element such that $x \notin A$. Since X is T_2, so for all $y \in A$, there exist disjoint open subsets U_y, V_y containing x and y respectively. This implies that $A \subset \bigcup_{y\in A} V_y$. Now A is $\mathcal{I}$-compact, so there exist finite subset of A such that $A - \bigcup_{i=1}^{n} V_{y_i} \in \mathcal{I}$. Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$, then $U \cap V = \emptyset$ and so $\overline{U} \cap V = \emptyset$. Therefore, $(\overline{U} \cap V) \bigcup (A - V) \in \mathcal{I}$ and so $\overline{U} \cap A \in \mathcal{I}$. Hence $x \notin cl_{\mathcal{I}_\theta}(A)$. □

The following Example shows that the result need not be true if we replace T_2 space by T_2 mod $\mathcal{I}$ space.

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then X is T_2 mod $\mathcal{I}$ space but not T_2. Also X is finite, so every subset of X is $\mathcal{I}$-compact. But $cl_{\mathcal{I}_\theta}(\{c\}) = \{a, b, c\}$. Hence $\{c\}$ is not $\mathcal{I}_\theta$ closed.

The following Example 2.6 shows that converse of Theorem 2.3 need not be true.

Example 2.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then it can be easily checked that every subset of X is $\mathcal{I}_\theta$ closed. But X is not T_2.

The following Theorem 2.4 gives the sufficient condition for the equivalence of $*$-closure and $\mathcal{I}_\theta$-closure.

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be S_2 ideal space, then $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for every $\mathcal{I}$-compact subset A of X.*

Proof. Let A be any $\mathcal{I}$-compact subset of X. Since $cl^*(A) \subset cl_{\mathcal{I}_\theta}(A)$, we only need to prove $cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)$. Also if $A \in \mathcal{I}$, then trivially $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$. Therefore, consider the case when $A \notin \mathcal{I}$. Let $y \notin cl^*(A)$, so there exist open set U containing y such that $U \cap A \in \mathcal{I}$ and so $U \cap A = I$ for some $I \in \mathcal{I}$. This implies that $U \cap (A - I) = \emptyset$. Now X is S_2, so for all $z \in A - I$, y has an open set U not containing z, so there exist open sets V_z and W_z such that $z \in V_z, y \in W_z$ and $V_z \cap W_z = \emptyset$. Therefore, $A \subset U \cup \bigcup_{z \in A-I} V_z$, but A is $\mathcal{I}$-compact implies that there exist finite subset of $A - I$ such that $A - \bigcup_{i=1}^{n} V_{z_i} \in \mathcal{I}$ (without loss of generality, we can remove the open subset U, since $U \cap A \in \mathcal{I}$ implies that $(A - \bigcup_{i=1}^{n} V_{z_i}) \cap U \in \mathcal{I}$). Let $G = \bigcup_{i=1}^{n} V_{z_i}$ and $H = \bigcap_{i=1}^{n} W_{z_i}$. Now for all

$i = 1, 2, ...n$, $V_{z_i} \cap W_{z_i} = \emptyset$ implies that $G \cap H = \emptyset$ and so $\overline{H} \cap G = \emptyset$. Since $A - G \in \mathcal{I}$, so $\overline{H} \cap A \in \mathcal{I}$. Therefore, H is the required open set containing y such that $\overline{H} \cap A \in \mathcal{I}$ and so $y \notin cl_{\mathcal{I}_\theta}(A)$. Hence $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for every $\mathcal{I}$-compact A of X. □

Even though we have seen in Example 2.4 that there is no relationship between closed and $\mathcal{I}_\theta$-closed sets. The following Corollary 2.2 gives the sufficient condition for a closed set to be $\mathcal{I}_\theta$-closed.

Corollary 2.2. *Let $(X, \tau, \mathcal{I})$ be S_2 ideal space, then $cl_{\mathcal{I}_\theta}(A) \subset cl(A)$ for every $\mathcal{I}$-compact subset A of X.*

Proof. Proof follows from Theorem 2.4. □

The following Theorem 2.5 shows that for $\mathcal{I}$-regular spaces the concept of $*$-closure and $\mathcal{I}_\theta$-closure coincide.

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if $\Gamma(A)(\mathcal{I}, \tau) \subset A^*$ for any subset A of X. Hence in particular, X is $\mathcal{I}$-regular if and only if $cl_{\mathcal{I}_\theta}(A) = cl^*(A)$ for any subset A of X.*

Proof. Firstly, let X be $\mathcal{I}$-regular space and $x \in X$ be any element such that $x \notin A^*$. So there exist open subset U of x in X such that $U \cap A \in \mathcal{I}$. Since X is $\mathcal{I}$-regular, so by Theorem 1.1 there exist open subset G of X such that $x \in G \in \overline{G}$ and $\overline{G} - U \in \mathcal{I}$. Therefore, $(\overline{G} - U) \bigcup (U \cap A) \in \mathcal{I}$ and so $\overline{G} \cap A \in \mathcal{I}$. Hence $x \notin \Gamma(A)(\mathcal{I}, \tau)$. Conversely, let F be any closed set and $a \in X$ such that $a \notin F$. Since F is closed and hence τ^*-closed, so $\Gamma(F)(\mathcal{I}, \tau) \subset F^* \subset F$ implies that $a \notin \Gamma(F)(\mathcal{I}, \tau)$. Therefore, there exist open subset G of a in X such that $\overline{G} \cap F \in \mathcal{I}$. Hence G and $(\overline{G})^C$ are the required disjoint open subsets of X such that $a \in G$ and $F - (\overline{G})^C \in \mathcal{I}$ and so X is $\mathcal{I}$-regular. □

Corollary 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and X be $\mathcal{I}$-regular space, then $cl_{\mathcal{I}_\theta}(A)$ is $\mathcal{I}_\theta$ closed for every subset A of X.*

Proof. Proof follows from Theorem 2.5, since X is $\mathcal{I}$-regular implies $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for any subset A of X. □

Corollary 2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if $cl_{\mathcal{I}_\theta}(A) \subset cl(A)$ for any subset A of X.*

Proof. Proof follows from Theorem 2.5 and the fact that $cl^*(A) \subset cl(A)$ for any subset A of X. □

Lemma 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $cl_{\mathcal{I}_\theta}(A) = \bigcap_{G \in \tau} \{cl_{\mathcal{I}_\theta}(G) : A \cap G^C \in \mathcal{I}\}$ for any subset A of X.*

Proof. Let $\bigcap_{G\in\tau}\{cl_{\mathcal{I}_\theta}(G) : A\cap G^C \in \mathcal{I}\} = T$. Firstly, let $x \notin T$, then there exist open G such that $A\cap G^C \in \mathcal{I}$ and $x \notin cl_{\mathcal{I}_\theta}(G)$. Therefore, there exist open subset V containing x such that $\overline{V}\cap G \in \mathcal{I}$ and so $(A\cap G^C)\cup(\overline{V}\cap G) \in \mathcal{I}$. Hence $\overline{V}\cap A \in \mathcal{I}$ implies that $x \notin cl_{\mathcal{I}_\theta}(A)$. Conversely, let $x \notin cl_{\mathcal{I}_\theta}(A)$, then there exist open V containing x such that $\overline{V}\cap A \in \mathcal{I}$ and so $(\overline{V}^C)^C \cap A \in \mathcal{I}$. Therefore, $\overline{V}\cap\overline{V}^C = \emptyset$ implies that $x \notin cl_{\mathcal{I}_\theta}(\overline{V}^C)$ and so $x \notin T$. Hence $cl_{\mathcal{I}_\theta}(A) = \bigcap_{G\in\tau}\{cl_{\mathcal{I}_\theta}(G) : A\cap G^C \in \mathcal{I}\}$. □

Now we will prove the stronger result than Corollary 2.3 that $\mathcal{I}_\theta$-closure of any subset A of X is $\mathcal{I}_\theta$ closed even for almost-$\mathcal{I}$-regular spaces where

Definition 2.2. An ideal space $(X,\tau,\mathcal{I})$ is said to be almost-$\mathcal{I}$-regular if for any point x and a regular closed set F not containing x there exist disjoint open subsets G, H such that $x \in G$, $F - H \in \mathcal{I}$.

It can be seen easily that every $\mathcal{I}$-regular space is almost-$\mathcal{I}$-regular but the converse need not be true, since every regular closed set is closed and closed set need not be regular closed.

Theorem 2.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space and X be almost-$\mathcal{I}$-regular space, then $cl_{\mathcal{I}_\theta}(A)$ is $\mathcal{I}_\theta$ closed for any subset A of X.*

Proof. Let A be any subset of X. We only need to prove that $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A)) \subset cl_{\mathcal{I}_\theta}(A)$. Firstly, we will prove that $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U)) \subset cl_{\mathcal{I}_\theta}(U)$ for any open subset U of X. Let $x \notin cl_{\mathcal{I}_\theta}(U)$. It can be easily checked that $cl_{\mathcal{I}_\theta}(U)$ is regular closed set, so there exist disjoint open sets G, H such that $x \in G$ and $cl_{\mathcal{I}_\theta}(U) - H \in \mathcal{I}$, since X is almost-$\mathcal{I}$-regular space. Now $G\cap H = \emptyset$ and so $\overline{G}\cap H = \emptyset$. Therefore, $(cl_{\mathcal{I}_\theta}(U)\cap H^C)\bigcup(\overline{G}\cap H) \in \mathcal{I}$ and so $\overline{G}\cap(cl_{\mathcal{I}_\theta}(U)) \in \mathcal{I}$. This implies that $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U))$ and so $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U)) \subset cl_{\mathcal{I}_\theta}(U)$ for any open subset U of X. Now let $x \notin cl_{\mathcal{I}_\theta}(A)$, then by Lemma 2.2 there exist open subset G of X such that $A\cap G^C \in \mathcal{I}$ and $x \notin cl_{\mathcal{I}_\theta}(G)$ and so $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(G))$. Also from Lemma 2.3, it follows that $cl_{\mathcal{I}_\theta}(A) \subset cl_{\mathcal{I}_\theta}(G)$ and so $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A)) \subset cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(G))$. Hence $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A))$. □

3. θ-convergence and $T_{2\frac{1}{2}}$ spaces with respect to an ideal

Now we will discuss $\mathcal{I}_\theta$-convergence of a filter.

Definition 3.1. Let $(X,\tau,\mathcal{I})$ be any ideal space and the filter $\mathcal{F}$ on X with $\mathcal{F}\cap\mathcal{I} = \emptyset$. Let $a \in X$ be any element then $\mathcal{F}$ is said to be $\mathcal{I}_\theta$-convergent to a, denoted by $\mathcal{F} \to_{\mathcal{I}_\theta} a$ if for every open subset U containing a, there exists $F \in \mathcal{F}$ such that $F - \overline{U} \in \mathcal{I}$. We denote the collection of all such points by $\mathcal{I}_\theta$-lim $\mathcal{F}$.

It can be seen easily that $\mathcal{F} \to_\theta a$ implies $\mathcal{F} \to_{\mathcal{I}_\theta} a$, but the converse need not be true as can be seen from the example below:

Example 3.1. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $\mathcal{F} = \{X\}$. Then $\mathcal{F} \to_{\mathcal{I}_\theta} a$, but $\mathcal{F}$ does not θ-converge to a, since $cl\{a\} = \{a, c\} \notin \mathcal{F}$.

Remark 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I} = \emptyset$, then $\mathcal{F} \to_\theta a$ if and only if $\mathcal{F} \to_{\mathcal{I}_\theta} a$.

Even though we have seen above $\mathcal{F} \to_{\mathcal{I}_\theta} a$ does not imply $\mathcal{F} \to_\theta a$, but the following result shows that in case of ultrafilter both concepts coincide.

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{F}$ be an ultrafilter with $\mathcal{F} \cap \mathcal{I} = \emptyset$, then $\mathcal{F} \to_{\mathcal{I}_\theta} a$ if and only if $\mathcal{F} \to_\theta a$.*

Proof. Let G be open subset of X containing a. Then there exists $F \in \mathcal{F}$ such that $F - \overline{G} \in \mathcal{I}$ since $\mathcal{F} \to_{\mathcal{I}_\theta} a$. Therefore, $F - \overline{G} = I$ for some $I \in \mathcal{I}$ and so $F \subset \overline{G} \cup I$. Thus $\overline{G} \cup I \in \mathcal{F}$. Further, $\mathcal{F}$ is ultrafilter implies that $\overline{G} \in \mathcal{F}$ as $I \notin \mathcal{F}$. Hence $\mathcal{F} \to_\theta a$. □

For an ideal space $(X, \tau, \mathcal{I})$ and any subset A of X, the following theorem gives various characterizations for a point to be in the local closure function of A in terms of θ-convergence ($\mathcal{I}_\theta$-convergence) of a filter.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be any subset of X. Then the following conditions are equivalent:*

(*a*) *$x \in \Gamma(A)(\mathcal{I}, \tau)$.*

(*b*) *there exists a filter $\mathcal{F}$ containing A with $\mathcal{F} \cap \mathcal{I} = \emptyset$ such that $\mathcal{F} \to_\theta x$.*

(*c*) *there exists a filter $\mathcal{F}$ containing A with $\mathcal{F} \cap \mathcal{I} = \emptyset$ such that $\mathcal{F} \to_{\mathcal{I}_\theta} x$.*

Proof. (a)⇒(b): Let $x \in \Gamma(A)(\mathcal{I}, \tau)$, so for every open subset G containing x, $\overline{G} \cap A \notin \mathcal{I}$. Consider the filter $\mathcal{F}$ generated by the filterbase $\mathcal{F}(\mathcal{B}) = \{\overline{G} \cap A : G$ is open subset of X containing x $\}$. Therefore, $\overline{G} \cap A \subset \overline{G}$ for every open subset G containing x and $\overline{G} \cap A \subset A$ implies that $\overline{G} \in \mathcal{F}$ and $A \in \mathcal{F}$. Hence $\mathcal{F}$ is the required filter containing A such that $\mathcal{F} \to_\theta x$.

(b)⇒(c): is obvious.

(c)⇒(a): Let G be open subset of X containing x. Then by (c), there exists $F \in \mathcal{F}$ such that $F - \overline{G} \in \mathcal{I}$ and so $(A \cap F) - \overline{G} \in \mathcal{I}$. On contrary, let $\overline{G} \cap A \in \mathcal{I}$ then $\overline{G} \cap A \cap F \in \mathcal{I}$ and so $((A \cap F) - \overline{G}) \cup ((A \cap F) \cap \overline{G}) \in \mathcal{I}$. This implies that $A \cap F \in \mathcal{I}$, which contradicts that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Hence $\overline{G} \cap A \notin \mathcal{I}$ and so $x \in \Gamma(A)(\mathcal{I}, \tau)$. □

Further, we introduce $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ spaces and obtain its various properties and characterizations.

Definition 3.2. An ideal space $(X, \tau, \mathcal{I})$ is said to be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ if for any two distinct points x, y of X, there exist open sets U and V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

Since $\emptyset \in \mathcal{I}$, it can be easily seen that every $T_{2\frac{1}{2}}$ space is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, but the following Example 3.6 shows that the converse need not be true.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ but not $T_{2\frac{1}{2}}$.

Theorem 3.3. *If an ideal space $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{I} \subset \mathcal{J}$ then $(X, \tau, \mathcal{J})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{J}$.*

Proof. Proof is obvious and hence is omitted. □

The following Example 3.3 shows that if (X, τ^*) is $T_{2\frac{1}{2}}$, then X need not be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \wp(X)$ and hence (X, τ^*) is obviously $T_{2\frac{1}{2}}$, but X is not $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. Since $\overline{\{a\}} \cap \overline{\{b\}} = \{a, c\} \cap \{b, c\} = \{c\} \notin \mathcal{I}$.

Note: It can be seen easily that if (X, τ) is $T_{2\frac{1}{2}}$, then (X, τ^*) is also $T_{2\frac{1}{2}}$. But the above Example 3.3 shows that the converse need not be true.

Even though we have seen that if (X, τ^*) is $T_{2\frac{1}{2}}$, then X need not be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. but the following Theorem 3.4 shows that for codense ideals (X, τ^*) is $T_{2\frac{1}{2}}$ implies X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense and (X, τ^*) be $T_{2\frac{1}{2}}$ then X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.*

Proof. Let $x, y \in X$ be any two distinct elements then (X, τ^*) is $T_{2\frac{1}{2}}$ implies there exists basic open subsets $G - I, H - J$ where G, H are open in X and $I, J \in \mathcal{I}$ such that $x \in G - I, y \in H - J$ and $cl^*(G - I) \cap cl^*(H - J) = \emptyset$ and so by Lemma 2.1, $[cl^*(G) - I] \cap [cl^*(H) - J] = \emptyset$. This implies that $(cl^*(G) \cap cl^*(H)) - (I \cup J) = \emptyset$. Therefore, $cl^*(G) \cap cl^*(H) \subset (I \cup J) \in \mathcal{I}$. Also $\mathcal{I}$ is codense implies that $cl^*(G) = cl(G)$ for every open subset G of X. Hence $cl(G) \cap cl(H) \in \mathcal{I}$ and so X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

The following Example 3.4 shows that if X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, then (X, τ^*) need not be $T_{2\frac{1}{2}}$, even if $\mathcal{I}$ is codense. Hence in particular, (X, τ) is not $T_{2\frac{1}{2}}$.

Example 3.4 ([12]). Consider the space $X = \Re^2$ with an additional point 0^* with double origin topology τ is given as follows: Neighbourhoods of points other than 0 and 0^* are the usual open sets of $\Re^2 - \{0\}$ and for the basis of neighbourhoods of 0 and 0^*, take $V_n(0) = \{(x, y) \in \Re^2 : x^2 + y^2 < \frac{1}{n^2}, y > 0\} \cup \{0\}$ and $V_n(0^*) = \{(x, y) \in \Re^2 : x^2 + y^2 < \frac{1}{n^2}, y < 0\} \cup \{0^*\}$, where $n \in \mathcal{N}$ and $\mathcal{I} = \wp(\Re) \equiv$ collection of all subsets of real numbers. Then X is not $T_{2\frac{1}{2}}$, since $\{0\}$ and $\{0^*\}$ do not have disjoint closed nhds. as any two nhds. of $\{0\}$ and $\{0^*\}$ contain a segment of the x-axis in the intersection of their closures. But $\mathcal{I} = \wp(\Re)$ implies that X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. Further it can be easily seen that nhds. of $\{0\}$ and $\{0^*\}$ are same in the given topology τ and its $*$ topology τ^* (since τ^* has basis $\beta = \{V - A : V \in \tau \text{ and } A \in \mathcal{I}\}$ and any two nhds. of $\{0\}$ and $\{0^*\}$ do not contain a segment of the x-axis). Also $\mathcal{I}$ is codense implies $cl^*(U) = cl(U)$ for every open subset U of X. Hence (X, τ^*) is not $T_{2\frac{1}{2}}$.

Theorem 3.5. *An ideal space $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ if and only if for all convergent filter $\mathcal{F}$ with $\mathcal{F} \cap \mathcal{I} = \emptyset$, $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique.*

Proof. Firstly, let X be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{F}$ be any convergent filter with $\mathcal{F} \cap \mathcal{I} = \emptyset$ Let $x \neq y$ be any two elements of X such that $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$. Therefore, X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets U, V of X containing x, y respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$. Further $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$ implies that there exists $F_1, F_2 \in \mathcal{F}$ such that $F_1 - \overline{U}, F_2 - \overline{V} \in \mathcal{I}$ and so $(F_1 \cap F_2) - (\overline{U} \cap \overline{V}) \in \mathcal{I}$. Thus $((F_1 \cap F_2) - (\overline{U} \cap \overline{V})) \cup (\overline{U} \cap \overline{V}) \in \mathcal{I}$ and so $F_1 \cap F_2 \in \mathcal{I}$, which contradicts that $\mathcal{F}$ does not contain the members of $\mathcal{I}$. Hence $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique.

Conversely, let $x \neq y$ be any two elements of X and there does not exist any open subsets U, V containing x, y respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$. Therefore, it can be easily checked that $\mathcal{F}$ is the filter generated by the filterbase $\mathcal{F}(\mathcal{B}) = \{\overline{U} \cap \overline{V} : U, V$ are open subsets of X containing x, y respectively $\}$. Further for every open subsets U, V containing x, y respectively, $\overline{U} \cap \overline{V} \subset \overline{U}$ and $\overline{U} \cap \overline{V} \subset \overline{V}$ implies that $\overline{U} \in \mathcal{F}$ and $\overline{V} \in \mathcal{F}$. This implies that $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$, contradicting the fact that $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique. Hence X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent:*

(*a*) *$(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.*

(*b*) *If $x \in X$, then for each $y \neq x$, $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$ for some open subset G containing x.*

(*c*) *$\bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing x $\} = \emptyset$ or $\{x\}$ for all $x \in X$.*

Proof. (a)$\Rightarrow$(b): Let $x \in X$ and $y \neq x$ be any element. Then (a) implies that there exists open subsets G and H such that $x \in G, y \in H$ and $\overline{G} \cap \overline{H} \in \mathcal{I}$. Therefore, $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$.

(b)$\Rightarrow$(c): Let $x \in X$ and $y \neq x$ be any element. Then by (b), $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$ for some open subset G containing x and so $y \notin \bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing x $\}$. Hence (c) holds.

(c)$\Rightarrow$(a): Let x, y be two distinct elements of X.

Then by (c), $y \notin \bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing x $\}$ and so there exist open subset G containing x such that $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$. This implies that there exist open subset H containing y such that $\overline{H} \cap \overline{G} \in \mathcal{I}$. Hence, $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

In [3], Gupta and Noiri introduced QHC spaces with respect to an ideal written $\mathcal{I}$-QHC(where An ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-QHC if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \bigcup\{cl(G_\alpha) : \alpha \in \Delta_0\} \in \mathcal{I}$). We now characterize $\mathcal{I}$-QHC spaces in terms of $\mathcal{I}_\theta$ closure of a set and also prove that every $\mathcal{I}$-QHC set in $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ space is $\mathcal{I}_\theta$ closed.

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent:*

(a) $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-QHC.

(b) for every filter $\mathcal{F}$ with $\mathcal{F} \cap \mathcal{I} = \emptyset$, $\bigcap_{F \in \mathcal{F}} \Gamma(F)(\mathcal{I}, \tau) \neq \emptyset$.

Proof. (a)$\Rightarrow$(b): Let $\mathcal{F}$ be any filter with $\mathcal{F} \cap \mathcal{I} = \emptyset$ such that $\bigcap_{F \in \mathcal{F}} \Gamma(F)(\mathcal{I}, \tau) = \emptyset$. Therefore, for all $x \in X$ there exist open set G_x containing x and $F_x \in \mathcal{F}$ such that $\overline{G_x} \cap F_x \in \mathcal{I}$. Now $X = \bigcup_{x \in X} G_x$, so (a) implies that there exist finite subset of X such that $X - \bigcup_{i=1}^n \overline{G_{x_i}} \in \mathcal{I}$. Let $G = \bigcup_{i=1}^n \overline{G_{x_i}}$ and $F = \bigcap_{i=1}^n F_{x_i}$, then $G \cap F \in \mathcal{I}$. Therefore, $(G \cap F) \cup (X - G) \in \mathcal{I}$ and so $F \in \mathcal{I}$ contradicting the fact that $\mathcal{F} \cap \mathcal{I} = \emptyset$ (since finite intersection of members of $\mathcal{F}$ is also in $\mathcal{F}$). Hence $\bigcap_{F \in \mathcal{F}} \Gamma(F)(\mathcal{I}, \tau) \neq \emptyset$.

(b)$\Rightarrow$(a): Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of X such that there does not exist any finite subset Δ_0 of Δ such that $X - \bigcup_{\alpha \in \Delta_0} \overline{G_\alpha} \in \mathcal{I}$. Therefore, for every finite subset Δ_0 of Δ, $\bigcap_{\alpha \in \Delta_0} (\overline{G_\alpha})^C \notin \mathcal{I}$. Let $\mathcal{F}(\mathcal{B}) = \{\bigcap_{\alpha \in \Delta_0} (\overline{G_\alpha})^C : \Delta_0$ is finite $\}$. Then it can be easily checked that $\mathcal{F}(\mathcal{B})$ is a filterbase not containing the members of $\mathcal{I}$. Now consider the filter $\mathcal{F}$ generated by $\mathcal{F}(\mathcal{B})$ and let $x \in X = \bigcup_\alpha G_\alpha$ and so $x \in G_\alpha$ for some $\alpha \in \Delta$. Therefore, $\overline{G_\alpha} \cap (\overline{G_\alpha})^C = \emptyset$ implies that $x \notin \bigcap_{F \in \mathcal{F}} \Gamma(F)(\mathcal{I}, \tau)$ (since $(\overline{G_\alpha})^C \in \mathcal{F}$) contradicting (b). Hence $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-QHC. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space and X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ then every $\mathcal{I}$-QHC set is $\mathcal{I}_\theta$ closed.*

Proof. Let K be an $\mathcal{I}$-QHC subset of X. We have to prove that $cl_{\mathcal{I}_\theta}(K) \subset K$. Let $x \in X$ such that $x \notin K$, then for all $y \in K$, $y \neq x$ and so X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets G_y and H_y containing x, y respectively such that $\overline{G_y} \cap \overline{H_y} \in \mathcal{I}$. Now $K \subset \bigcup_{y \in K} H_y$ but K is $\mathcal{I}$-QHC implies that there exist finite subset of K such that $K - \bigcup_{i=1}^n \overline{H_{y_i}} \in \mathcal{I}$. Let $G = \bigcap_{i=1}^n G_{y_i}$ and $H = \bigcup_{i=1}^n \overline{H_{y_i}}$ then $\overline{G} \subset \bigcap_{i=1}^n \overline{G_{y_i}}$ implies that $\overline{G} \cap H \in \mathcal{I}$ and so $(\overline{G} \cap H) \cup (K - H) \in \mathcal{I}$. Therefore, $\overline{G} \cap K \in \mathcal{I}$ and so $x \notin cl_{\mathcal{I}_\theta}(K)$ since G is open subset of x. Hence K is $\mathcal{I}_\theta$ closed. □

Corollary 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space and X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ then every $\mathcal{I}$-compact set is $\mathcal{I}_\theta$ closed.*

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ON SOME APPLICATIONS OF ALGEBRAIC HYPERSTRUCTURES FOR THE MANAGEMENT OF TEACHING AND RELATIONSHIPS IN SCHOOLS

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Abstract. In terms of school context, the effectiveness of a teaching process frequently depends to a large extent on the relationship system, which had been formed within a classroom. This paper is going to present how algebraic hyperstructures can contribute significantly to understanding the system of relationships within a classroom. Furthermore, it becomes possible to assess the impact of interventions targeted at improving the system of relationships and thus to establish undisturbed and fundamental participation of students in the learning process.

Keywords: algebraic hyperstructures, interpersonal relations.

1. Introduction

In recent decades, research on learning efficiency has attracted and still has been attracting the interest of world experts. Various experience has shown that at schools, in particular at elementary and primary level, both teacher training and pedagogical skills cannot reach required and acceptable results unless there is collaborative, friendly and positive relationship, e.g. [2, 3, 11, 25, 22].

Studying existing relationships among students is an inevitable prerequisite for planning interventions focused on reaching adequate required teaching/learning efficiency. [24, 35, 37]. Let S is a set of students of a particular scholastic classroom K. A scientific study covering the relationship among students within a classroom has lead to the determining a final set of relationships R.

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In the past, social relationships within school environment were described by many authors (e.g. Moreno, [30, 29] Sciarra, [36]) using a set of binary sharp relationships. The most efficient teaching methods in terms of the system of mutual relationships within a classroom have also been studied in several recent papers by Delli Rocili and Maturo [7, 9, 10] and also Hoskova-Mayerova [12, 13, 14, 15].

Some of our findings concluded (Hoskova-Mayerova & Maturo, [17, 18, 19, 20] that deeper and more profound knowledge on school environment relationship can be obtained through fuzzy relationships since they consider semantic uncertainty and degree of relationship intensity. See also [8, 25, 26]. This paper continues, deepens and expands the ideas presented in our previous several works, which demonstrate different perspectives. We also present that there is another tool for representing and evaluating uncertainty; these are the algebraic hyperstructures, more flexible than the common operations, because they are multi-results, which represent the possible outcomes of an agreement between individuals. The section dealing with results covers some algebraic hyperstructures associated with existing relationships within a school context.

2. Algebraic hyperstructures associated to the Moreno approach

The theory of hyperstructures originates from the work of Marty [23] both the ideas and definitions published here have been developing particularly in the last 40 years. The hypergroup has been the most studied hyperstructure; it is a concept that generalizes the concept of a group. In the book "Prolegomena hypergroup theory" (Corsini [4]), basic results in terms of hypergroup theory are presented up to 1992. The book supplement comprises the entire bibliography. The results overview until 2003 was published in 2006 by Corsini & Leoreanu, see [5]). A very detailed bibliography with respect to hyperstructure study is available on the website: `www.aha.eled.duth.gr/Thesaurus1.1.htm.` Further review can be found in the work done by Hoskova & Chvalina and published at proceedings of the conference Algebraic Hyperstructures and Applications–AHA 2005, see [16].

Perhaps, the most important impetus for the study of algebraic hyperstructures came from the basic material "Join Geometries" by Prenowitz and Jantosciak published in 1979; [34] in addition to providing an original and general approach to the study of geometry, this work introduces the interdisciplinary view of geometry and algebra: there is shown how to trace back the study of the Euclidean geometry of a specific commutative hypergroup that satisfies a particular axiom called *incidence property.* Various other geometries, such as "Projective geometry", can also be transferred to commutative hypergroups satisfying the incidence property.

The idea of studying hypergroup applications to solve problems of uncertainty and decision-making problems in architecture and social sciences was born after a series of conferences held at the Faculty of Architecture in Italian

Pescara organized by Giuseppe Tallini in 1993. The concept was being expanded, researched and developed at various AHA (Algebraic Hyperstructures and Applications) conferences as well as in domestic seminars and conferences organized by Prof. Piergiulio Corsini in the period 1994-2014; e.g., in December 1994 and October 1995, two conferences on "Hyperstructures and their applications in the field of cryptography, geometry and uncertainty" were organized by Corsini, Eugeni and Maturo in Chieti and Pescara.

Let us recall some fundamental definitions on algebraic hyperstructures. For more information, see the Vougiouklis book [40] and the papers (Jafarpour and Cristea [21], Chvalina and Hoskova [6], Massouros and Massouros [27, 28], Nikolaidou P. & Vougiouklis T. [31, 41] Novák [32, 33]; Vougiouklis [38]; Vougiouklis et al. [39, 42, 43]) and many others.

Definition 2.1. *Let S be a nonempty set. A function $\alpha : S \times S \to P(S)$, where $P(S)$ is the family of subsets of S, is said to be:*

- *a hyperoperation on S, if $\forall x, y \in S, \alpha(x, y) \neq \emptyset$;*
- *a partial hyperoperation on S, if $\exists x, y \in S, \alpha(x, y) = \emptyset$;*
- *commutative on S, if $\forall x, y \in S, \alpha(x, y) = \alpha(y, x)$;*
- *closed, if $\forall x, y \in S, \alpha(x, y) \supseteq \{x, y\}$;*
- *open, if $\forall x, y \in S, (x \neq y) \Rightarrow \alpha(x, y) \cap \{x, y\} = \emptyset$;*
- *idempotent, if $\forall x \in S, \alpha(x, x) = \{x\}$;*
- *reproductive, if $\forall x, y \in S, \exists u, v \in S : y \in \alpha(x, u) \cap \alpha(v, x)$.*

The pair (S, α), with S hyperoperation (resp. partial hyperoperation) on S is said to be a *hypergroupoid* (resp. *partial hypergroupoid*). Usually the set α(x, y) is written x α y and is called the hyperproduct of x by y (with respect to the hyperoperation α). If H and K are subsets of S then the set hyperproduct H α K is the union of the hyperproducts x α y with x∈H, y∈K.
A hypergroupoid (S, α) is said to be:

- a *quasi-hypergroup*, if the reproductive property is valid, i.e., $\forall x \in S$, $x\alpha S = S = S\alpha x$;
- a *semi-hypergroup*, if the following *associative property* is valid: $\forall x, y, z \in S$, $(x\alpha y)\alpha z = x\alpha(y\alpha z)$
- a *hypergroup*, if it is a quasi-hypergroup and a semi-hypergroup.

3. New results

Let ρ be a Moreno binary relation on the set S of students in a school class. Let us introduce the following definition:

Definition 3.1. *Let $\otimes$ be a binary operation, i. e. an operation in $\{0,1\}$, and ρ a reflexive relation on S. We define:*

- *active hyperoperation (eventually partial hyperoperation) associated with $(\otimes, \rho)$ the function $\otimes^a_\rho\colon (x,y) \in S \times S \to x \otimes^a_\rho y = \{z \in S\colon (x\,\rho\,z) \otimes (y\,\rho\,z) = 1\}$;*
- *passive hyperoperation (eventually partial hyperoperation) associated with $(\otimes, \rho)$ the function $\otimes^p_\rho\colon (x,y) \in S \times S \to x \otimes^p_\rho y = \{z \in S : (z\,\rho\,x) \otimes (z\,\rho\,y) = 1\}$;*
- *circular hyperoperation (eventually partial hyperoperation) associated with $(\otimes, \rho)$ the function $\otimes^c_\rho\colon (x,y) \in S \times S \to x \otimes^c_\rho y = \{z \in S : (x\,\rho\,z) \otimes (z\,\rho\,y) = 1\}$;*
- *inverse circular hyperoperation (eventually partial hyperoperation) associated with $(\otimes, \rho)$ the function $\otimes^i_\rho\colon (x,y) \in S \times S \to x \otimes^i_\rho y = \{z \in S : (z\,\rho\,x) \otimes (y\,\rho\,z) = 1\}$.*

Let us consider, to set the ideas, the active hyperoperation. As ρ is reflexive, the possible cases that can arise are those shown in the following Table 1.

ρ	x	y	u	v	w	t
x	1	0,1	0	1	0	1
y	0,1	1	0	0	1	1

Table 1:

3.1 Particular cases of active hyperoperations

Let $\otimes$ be the union $\cup$. Then $(x\,\rho\,z) \otimes (y\,\rho\,z) = \max\{(x\,\rho\,z),\ (y\,\rho\,z)\}$ and from Table 1 we can see that $(x\,\rho\,z) \otimes (y\,\rho\,z) = 1$ for $z \in \{x, y, v, w, t\}$. In particular, $\forall x, y \in S, x \cup^a_\rho y \supseteq \{x, y\}$. Then $(S, \cup^a_\rho)$ is a closed quasi-hypergroup. As $\cup$ is commutative, the associate active hyperoperation is commutative.

Let $\otimes$ be the intersection $\cap$. Then $(x\,\rho\,z) \otimes (y\,\rho\,z) = \min\{(x\,\rho\,z),\ (y\,\rho\,z)\}$ and from Table 1 we can see that $(x\,\rho\,z) \otimes (y\,\rho\,z) = 1$ for $z = t$.

The function $\cup^a_\rho$ is a commutative partial hyperoperations and is an hyperoperations if and only if the following condition holds: $\forall x, y \in S$, $\exists t \in S : x\,\rho\,t$ and $y\,\rho\,t$. The student t can be defined as "*a passive mediator*" between x and y. So each pair of students must have at least one passive mediator.

Let $\otimes$ be the implication $\to$. Then $(x\,\rho\,z) \otimes (y\,\rho\,z) = \max\{1-(x\,\rho\,z),\ (y\,\rho\,z)\}$ and from Table 1 we can see that $(x\,\rho\,z) \otimes (y\,\rho\,z) = 1$ for $z \in \{y, u, w, t\}$. In particular, $\forall x, y \in S, x \to^a_\rho y \supseteq \{y\}$. Then $(S, \cup^a_\rho)$ is a hypergroupoid.

Similar considerations can be made for the passive hyperoperation (resp. partial hyperoperation) associated with $(\otimes,\ \rho)$. It is sufficient to observe that it is reduced to active hyperoperation (resp. partial hyperoperation) associated with $(\otimes,\ \rho^{-1})$.

3.2 Particular case of circular hyperoperations

The circular hyperoperation (resp. partial hyperoperation) associated with $(\otimes, \rho)$ leads to the consideration of the paths of length 2 of the digraph (S, ρ), and to consider the binary operation on the arcs of each paths. As the inverse circular hyperoperation (resp. partial hyperoperation) associated with $(\otimes,\ \rho)$ reduces to the circular hyperoperation (resp. partial hyperoperation) associated with $(\otimes, \rho^{-1})$, it is sufficient to study the properties of the function $\otimes_p^c$.

Let $\otimes$ be the union $\cup$. Then $(x\rho z) \otimes (z\,\rho\,y) = \max\{(x\,\rho\,z),\ (z\,\rho\,y)\}$. As ρ and the inverse ρ^{-1} are reflexive, $\forall x, y \in S, x \cup_p^c y \supseteq \{x, y\}$. Then $(S, \cup_p^c)$ is a closed quasi-hypergroup.

Let $\otimes$ be the intersection $\cap$. Then $(x\,\rho\,z) \otimes (z\,\rho\,y) =\ min\{(x\,\rho\,z),\ (z\,\rho\,y)\}$. As the reflexivity of ρ, $(x\,\rho\,z) \otimes (z\,\rho\,y) = 1$ if $x, \rho\,y$ or there is a path of length 2 of consecutive vertices x, z, y with $z \neq x$ and $z \neq y$.

The function $\otimes_p^c$ is a hyperoperation if and only if the following condition hold: $\forall x, y \in S$, $(x(-\rho)y) \Rightarrow z \in S\colon x\,\rho\,z$ and $z\,\rho\,y$. The student z can be defined as"an intermediate mediator" between x and y. So each pair of students must have at least one intermediate mediator.

Let $\otimes$ be the implication $\rightarrow$. Then $(x\,\rho\,z) \otimes (z\,\rho\,t) = \max\{1-(x\,\rho\,z),\ (z\,\rho\,y)\}$. As ρ and the inverse ρ^{-1} are reflexive, $\forall x, y \in S$, $x \rightarrow_\rho^c y \supseteq \{y\}$. Then $(S, \rightarrow_\rho^a)$ is a hypergroupoid.

4. Conclusions and perspective of research

From Sections 1 we can see that the significant algebraic hyperstructures that can be associated with the relations system of a school class are very numerous. In some of our papers (see e.g., Hoskova-Mayerova and Maturo, [17, 18, 19]) many other types of hyperstructures, from points of view different from those considered in this work, have also been examined.

In the context of Moreno's binary approach it is also possible to directly construct algebraic hyperstructures in S by making interviews to the ordered pairs (x, y) of students, making the first element of the pair, x, assume, the role of indicating classmates that he considers suitable for an activity (at least 2) and the second element of the pair, y, the possibility to choose which elements indicated by x are accepted (at least 1).

If n is the number of elements of the class, to implement this procedure it is necessary to propose $2n$ meetings for interviews (as for a football league with n teams). This way to obtain hyperstructures associated with the class is therefore very significant but rather time consuming.

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ON THE NUMBER OF CYCLIC SUBGROUPS IN FINITE GROUPS

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Abstract. It is proved that a finite group G has $|G|-3$ cyclic subgroups if and only if $G \cong D_{10}$ or Q_8.
Keywords: finite groups, cyclic subgroups, 2-groups.

1. Introduction

Let G be a finite group and $C(G)$ be the poset of cyclic subgroups of G. Sometimes $C(G)$ can decide the structure of G. For example, G is an elementary abelian 2-group if and only if $|C(G)| = |G|$. Tărnăuceanu [3, 4] classified the groups G such that $|G| - |C(G)| = 1$ or 2.

In this paper, we shall continue this study by describing the finite groups G such that

$$|C(G)| = |G| - 3.$$

We prove that there are just two such groups: D_{10} and Q_8.

For any finite group G, denote by $\pi_e(G)$ the set of all element orders of G, and denote by $\pi(G)$ the set of all prime divisors of $|G|$. For convenience, let $\pi_c(G) = \pi_e(G) - (\pi(G) \cup \{1\})$. For any $i \in \pi_e(G)$, denote by $C_i(G)$ the set of all cyclic subgroups of order i in G, and denote $c_i(G) = |C_i(G)|$.

2. The main result

Throughout this section, let $c_i = c_i(G)$ for each $i \in \pi_e(G)$.

Lemma 2.1. *Let* $|G| = p_1^{a_1} \cdots p_r^{a_r}$ *with* $p_1 < \cdots < p_r$*. If* $r \geq 3$*, then* $|G| - c(G) > p_r$.

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Proof. For any finite group G, we know that

$$|G| = \sum_{k\in\pi_e(G)} c_k(\phi(k)),$$
$$|C(G)| = \sum_{k\in\pi_e(G)} c_k,$$

where ϕ is the Euler function. Hence

$$|G| - |C(G)| = \sum_{k\in\pi_e(G)} c_k(\phi(k)-1). \tag{1}$$

Let G be a group such that $|G| - |C(G)| \leq p_r$. By (1), we see that

$$\sum_{k\in\pi_e(G)} c_k(\phi(k)-1) \leq p_r.$$

By Cauchy theorem, $c_{p_i} \geq 1$ for all $i \leq r$. Hence we get that

$$c_{p_1}(p_1-2) + c_{p_2}(p_2-2) + \cdots + c_{p_r}(p_r-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq p_r.$$

Since $r \geq 3$, we get that $p_r \geq 5$. Thus $c_{p_r} = 1$ and

$$c_{p_1}(p_1-2) + c_{p_2}(p_2-2) + \cdots + c_{p_{r-1}}(p_{r-1}-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 2.$$

So we get $r = 3$, $p_2 = 3$ and $p_1 = 2$. It follows that

$$c_3 + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 2.$$

If $c_3 = 2$, then $\pi_c(G) = \emptyset$. Let X_1 and X_2 be the two cyclic subgroups of order 3. Consider the action of G on $\{X_1, X_2\}$, we see that X_1 is normalized by a Sylow p_r-subgroup, which implies $3p_r \in \phi_c(G)$, a contradiction. Hence $c_3 = 1$. Similarly, we get that $3p_r \in \pi_c(G)$. But $c_{3p_r}(\phi(3p_r)-1) \geq 2(p_r-1)-1 \geq 2(5-1)-1 = 7$, a contradiction. □

Lemma 2.2. *Suppose that $|G| = p^a q^b$, where p, q are primes such that $p < q$. Then $|G| - c(G) > q$ if $G \ncong D_{2q}$, C_6, D_{12}, C_6 or S_3, and*

$$|D_{2q}| - |C(D_{2q})| = q-2,$$
$$|C_6| - |C(C_6)| = 2,$$
$$|D_{12}| - |C(D_{12})| = 2,$$
$$|C_6| - |C(C_6)| = 2,$$
$$|S_3| - |C(S_3)| = 1.$$

Proof. Let G be a group such that $|G| - c(G) \leq q$. By (1), we have

$$c_p(p-2) + c_q(q-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq q. \tag{2}$$

i) $q \geq 5$. Then $c_q = 1$ and

$$c_p(p-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 2.$$

If $p \neq 2$, then $\pi_c(G) = \emptyset$, and $c_p(p-2) \leq 2$. It follows that $p = 3$ and $c_3 \leq 2$. But we can find an element of order $3q$, a contradiction. It follows that $p = 2$, and we see that $\pi_e(G) = \{1, 2, q\}$ or $\{1, 2, 2^2, q\}$. Thus G has only one Sylow q-subgroup Q which is isomorphic to C_q.

If $\pi_e(G) = \{1, 2, 4, q\}$, then $c_4 \leq 2$. Thus Q normalizes a cyclic subgroup of order 4. This implies that $4q \in \pi_c(G)$, a contradiction. Hence $\pi_e(G) = \{1, 2, q\}$. If $a \geq 2$, by considering the conjugate action of a Sylow 2-group on Q, we can find an element of order $2q$, a contradiction. Hence $|G| = 2q$, and $G = \langle u, v | u^q = 1, v^2 = 1, u^v = u^{-1}\rangle \cong D_{2q}$. In this case, $|G| - |C(G)| = q - 2$.

ii) $q = 3$ and $p = 2$. Now (2) becomes that

$$c_3 + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 3.$$

It follows that $c_3 \leq 3$ and $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$.

If $\pi_e(G) = \{1, 2, 3, 4, 6\}$, then $c_3 = c_4 = c_6 = 1$. And we get $12 \in \pi_e(G)$, a contradiction. If $\pi_e(G) = \{1, 2, 3, 4\}$, then $c_4 \leq 2$. We can get that $12 \in \pi_e(G)$ if $c_4 = 1$. Hence $c_4 = 2$. Therefore, a Sylow 3-group will normalizes a cyclic subgroup of order 4, and we get that $12 \in \pi_e(G)$, a contradiction.

If $\pi_e(G) = \{1, 2, 3, 6\}$, then $c_3 + c_6 \leq 3$. Then $c_3 = 1$ or $c_6 = 1$. Thus we can get a normal cyclic subgroup $X = \langle x\rangle$ of order 3. Thus $|G : C_G(x)| \leq 2$. If $a \geq 3$, then $C_G(x)$ will contain a subgroup $L \cong C_2 \times C_2 \times C_3$. Since $c_6(L) = 3$, we get a contradiction. Hence $a \leq 2$. From $c_3 \leq 2$, we get that $b = 1$. Hence $|G| \leq 12$, and $G \cong C_6$ or D_{12}.

Now we need to consider the case that $\pi_e(G) = \{1, 2, 3\}$. Thus $c_3 \leq 3$, and there are at most 6 nontrivial 3-element. It follows that a Sylow 3-subgroup is isomorphic to C_3. By Sylow theorem, $c_3 = 1$. Thus the Sylow 3-subgroup Q is normal in G. Since $6 \notin \pi_e(G)$, $a \leq 1$, and $|G| = 6$. Since $|C_6| - |C(C_6)| = 2$ and $|S_3| - |C(S_3)| = 1$, we get that $G \cong C_6$ or S_3 in this case. □

Lemma 2.3. *Let $|G| = 2^a$. If $|G| - |C(G)| = 2^a - 3$, then $G \cong Q_8$.*

Proof. Since $\phi(8) = 4$, from (1), $exp(G) \leq 4$. If $exp(G) = 2$, then $c(G) = |G|$, a contradiction. Hence $exp(G) = 4$, and $c_4 = 3$. We find a normal cyclic subgroup $X = \langle x\rangle$ of order 4. Let $C = C_G(X)$. Then $|C| = 2^a$ or 2^{a-1}.

We claim that C/X is an elementary 2-group. Otherwise, there exists an element $g \in C$ such that gX is an element of order 4 in C/X. Hence $|g| = 4$. Let $D = \langle g, x\rangle$. Then D is abelian. Since $|gX| = 4$, we get that $\langle g\rangle \cap X = 1$, and $D \cong C_4 \times C_4$. But $c_4(C_4 \times C_4) > 3$, a contradicton.

Hence the Frattini subgroup $\Phi(C) \leq X$, and C is a 2-group with cyclic Frattini subgroup. Suppose that C is non-abelian. By [1, Theorem 4.4], if $|\Phi(C)| > 2$, then there exists an element of order $2|\Phi(C) \geq 8$, a contradiction. Hence $\Phi(C) = C' \cong C_2$. By [1, Lemma 4.2], $C = EZ(C)$ and $|E \cap Z(C)| = 2$, where E is an extra-special 2-groups. By [2, Theorem 3.13.8], $E = A_1 * \cdots * A_m$, the central product of A_i, where $A_i \cong D_8$ or Q_8. Note that $c_4(Q_8) = 3$ and $c_4(D_8) = 1$ and $c_4(D_8 * D_8) > 3$. We get that $E = D_8$ for $X \not\leq E$. Let $y \in E$ with $|y| = 4$. Since $c_4 = 3$, we see that $Z(C) \cong C_4$ or $C_4 \times C_2$. If $Z(C) \cong C_4 \times C_2$, then $c_4(\langle y, Z(C)\rangle) > 3$, a contradiction. Hence $Z(C) = X \cong C_4$. But now $c_4(C) > 3$, a contradiction. So we see that C is abelian.

Since $c_4 = 3$, we get that $C \cong C_4$ or $C_4 \times C_2$. Suppose that $C \cong C_4 \times C_2$. Since $c_4(C) = 2$ and $c_4 = 3$, there exists $u \in G - C$ such that $|u| = 2$. Let $C = \langle x\rangle \times \langle w\rangle$, where $|w| = 2$. Thus $x^u = x^{\pm 1}$, and $w^u = w$ or wx^2, and we get 4 groups, but none of the them satisfy $c_4 = 3$. Hence $C \cong C_4$, and $G \cong Q_8$. □

Now by Lemma 2.1, 2.2 and 2.3, we get our main result.

Theorem 2.4. *If $|C(G)| = |G| - 3$, then $G \cong D_{10}$ or Q_8.*

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MULTI-FUZZY GROUP INDUCED BY MULTISETS

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Abstract. Multisets can be used to represent real life problems where repetition(s) of elements is necessary. Such cases occur in database query, chemical structures and computer programming but to mention a few. In this paper, some properties of algebraic sum of multisets $\uplus$ and some previous results on selection are mentioned. This work also introduces a new way to construct fuzzy sets and fuzzy groups structure on multiset.

Keywords: multisets, functions on multiset, selection operation, multi-fuzzy set, submultiset.

1. Introduction

Many real life problems can be represented by multisets. Such cases occur in database query, chemical structures and computer programming but to mention a few.

The term multiset can be traced back to 1888, when Dedekind in [5] stated that the element of the set may belongs to more than one. Multisets are also considered in and replaced with various notions such as bag, fireset (**f**initely **r**epeated **e**lement **s**et), heap, bunches, etc.

In the recent time, Nazmul *et al* [6] has put algebraic structure on multisets of a set in order to be able to consider multigroup and other related algebraic properties as in the classical group. His work was extended by Shinoj *et al* [8].

In this paper, we present some results on the algebraic structure of multi-fuzzy sets.

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2. Preliminaries

In what follows, we shall use X to denote a non-empty set.

Definition 2.1 ([6]). *A* multiset *M drawn from a set X is denoted by the count function $C_M: X \longrightarrow N$ defined by $C_M(x) = n \in N$, the multiplicity or number of occurrence of x in M, where N is the set of non-negative integers.*

Definition 2.2 ([6]). *Let multisets A and B be drawn from X. A is said to be a* submultiset *of B and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x)\, \forall x, y \in X$.*

The root set *or* support *of a multiset M, which is denoted by M^*, is the set which contains the distinct elements in the multiset. Hence, M^* is the set of $x \in M$ such that $C_M(x) > 0$.*

Let us recall some notions concerning multisets. See e.g. [2] for more details.

A multiset M is called a *regular multiset* if $C_M(x) = C_M(y)\ \forall x, y \in M$. The count function of the intersection of two multisets A and B both drawn from X is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and that of their union is denoted $C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\}$.

Multisets A and B are said to be equal if and only if $C_A(x) = C_B(x)$. Denote by $[X]^\alpha$, all the multisets whose elements have the multiplicity not more than α and $MS(X)$ the *set of all multisets over* X. An empty multiset ϕ is such that $C_\phi(x) = 0,\ \forall x \in X$. *Cardinality* of a multiset M is denoted by $|M| = \sum C_M(x),\ \forall x \in M$. The *peak element* $x \in M$ is such that $C_M(x) \geq C_M(y),\ \forall y \in M$.

Definition 2.3 ([10]). *Let us consider $A \in MS(X)$.*

i. The insertion *of x into A results into a multiset denoted by $\mathcal{C} = x \uplus A$ which has the count function*

$$C_{\mathcal{C}}(y) = \begin{cases} C_A(y), & y \neq x \\ C_A(x) + 1, & y = x. \end{cases}$$

ii. The removal *of x from A results into a multiset denoted by $\mathcal{D} = A \ominus x$ which has the count function*

$$C_{\mathcal{D}}(y) = \begin{cases} \max\{C_A(y) - 1, 0\}, & y = x \\ C_A(y), & y \neq x. \end{cases}$$

Let us consider $A, B \in MS(X)$.

iii. The insertion *of A into B or of B into A results into a multiset which has the count function denoted by $\mathcal{C}(x) = C_A(x) + C_B(x)$.*

iv. The removal *of B from A results into a multiset which has the count function denoted by $C_{\mathcal{D}}(x) = \max\{C_A(x) - C_B(x), 0\}$.*

It is clear that the set of all multisets over X that is $MS(X)$ is commutative and associative with respect to the sum $\uplus$.

The removal operation is neither commutative nor associative. Apart from, it is also possible to make some kind of selection in multisets using the following operations.

Definition 2.4 ([10])**.** *Consider $A \in MS(X)$ and $B \subseteq X$.*

i. The multiset $\mathcal{E} = A \otimes B$ is such that $\mathcal{E}$ only contains elements of A which also occur in B. The count function of $\mathcal{E}$ is denoted by

$$C_{\mathcal{E}}(x) = \begin{cases} C_A(x), & x \in B \\ 0, & x \notin B. \end{cases}$$

ii. The multiset $\mathcal{F} = A \bigodot B$ is such that $\mathcal{F}$ only contains elements of A which do not occur in B. The count function of $\mathcal{F}$ is denoted by

$$C_{\mathcal{F}}(x) = \begin{cases} C_A(x), & x \notin B \\ 0, & x \in B. \end{cases}$$

Operations such as "$\otimes$" or "$\odot$" are called selection operations*.*

Definition 2.5 ([6])**.** *Let X be a group and $e \in X$ its identity. Then, $\forall x, y \in X$, a multiset M drawn from X is called a* multigroup *if*

i. $C_M(xy) \geq C_M(x) \wedge C_M(y)$,

ii. $C_M(x^{-1}) \geq C_M(x)$.

The immediate consequence is that $C_M(e) \geq C_M(x)$. The set $MG(X)$ is called the *set of all multigroups over X*. The next definition can be found e.g. in [2].

Definition 2.6. *Let $A \in MS(X)$, where X is a group.*

i. $A_n = \{x : C_A(x) \geq n\}$;

*ii. We denote a multiset containing only one element x with multiplicity. It is called n as $[n]_x$–*a simple multiset*;*

iii. The complement of the multiset *$M \in [X]^\alpha$ denoted by M' is such that $C_{M'}(x) = \alpha - C_M(x)$;*

iv. $nA = \{x^n,\ \forall x \in A$ n is the multiplicity of each element that appears in $A\}$.

Remark 2.1. For a multigroup A over a group X, A_n is a group, indeed the subgroup of X [6].

Proposition 2.1 ([6], p. 645). *Let $A, B \in MS(X)$ and $m, n \in I\!N$.*

i. *If $A \subseteq B$, then $A_n \subseteq B_n$;*

ii. *If $m \leq n$, then $A_m \supseteq A_n$;*

iii. $(A \cap B)_n = A_n \cap B_n$;

iv. $(A \cup B)_n = A_n \cup B_n$;

v. *$A = B$ if and only if $A_n = B_n$, $\forall n \in I\!N$.*

Definition 2.7 ([6]). *Let X and Y be two nonempty sets such that $f\colon X \longrightarrow Y$ is a mapping. Consider the multisets $M \in [X]^\alpha$ and $N \in [Y]^\alpha$. Then,*

i. *the image of M under f denoted $f(M)$ has the count function*

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x), & \textit{if } f^{-1}(y) \neq \emptyset \\ 0, & \textit{otherwise}; \end{cases}$$

ii. *the inverse image of N under f denoted $f^{-1}(N)$ has the count function $C_{f^{-1}(N)}(x) = C_N[f(x)]$.*

The following propositions were proved in [6]. But we shall later show that the items (iv), (v) and (vii) are not true and that the Proposition 2.2 needs to be restated.

Proposition 2.2 ([6]). *Let X, Y and Z be three nonempty sets such that $f\colon X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are mappings. If $M_i \in [X]^\alpha$, $N_i \in [Y]^\alpha$, $i \in I$ then*

i. $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;

ii. $f(\cup_{i\in I} M_i) = \cup_{i\in I} f(M_i)$;

iii. $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;

iv. $f^{-1}(\cup_{i\in I} M_i) = \cup_{i\in I} f^{-1}(M_i)$;

v. $f^{-1}(\cap_{i\in I} M_i) = \cap_{i\in I} f^{-1}(M_i)$;

vi. $f(M_i) \subseteq N_j \Rightarrow M_j \subseteq f^{-1}(N_j)$;

vii. *$g[f(M_i)] = [gf](M_i)$ and $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.*

Definition 2.8 ([11]). *A fuzzy set A of a non-empty set X is a class of objects in X with the associated (or characteristic) membership function $\mu : X \longrightarrow [0, 1]$ which assigns to every $x \in X$ a real value between 0 and 1.*

The value of $\mu(x)$ restricted to A is actually the degree of membership of x in A. If $\mu(x) = 0$, it represents complete non-membership while $\mu(x) = 1$ represents complete membership. But since μ characterizes the fuzzy set A, we can simply refer to μ as fuzzy subset.

Definition 2.9 ([7]). *Let μ be a fuzzy subset of G. Then, μ is called a fuzzy subgroup of G if $\forall x, y \in G$*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x) = \mu(x^{-1})$

Proposition 2.3 ([7]). *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every $x, y \in G$.*

In paper [5] defined a kind of multiset (Dedekind's multiset) denoted by M_f as follows: if there are n elements in a domain X (of a function f mapping X to Y) which are mapped to an element $y \in Y$, then y has frequency n so that it is an n-fold element of Y.

But, on multisets (mentioned as Definition 2.7(i)) fails for Dedekind's multisets. So let us redefine Nazmul *et al*'s definition of function as follows:

Definition 2.10 ([2]). *Let X and Y be two non-empty sets and $f\colon X \longrightarrow Y$ a mapping such that $M \in [X]^\alpha$. Then, $C_{f(M)}(y) = \sum_{f^{-1}(y) \neq \emptyset f(x)=y} C_M(x)$.*

For more details concerning this problem see [2].

3. Induced fuzzy group

In the Introduction we have recalled that $[X]^\alpha$ is the collection of all multisets drawn from X so that if $A \in [X]^\alpha$, then $C_A(x) \leq \alpha \ \forall x \in A$. Now, let $[[X]]^\alpha$ be a subset of $[X]^\alpha$ such that if $B \in [[X]]^\alpha$, $C_B(x) = \alpha \ \forall x \in B$, meaning that B is a regular multiset in which each element has the multiplicity α.

Proposition 3.1. *Let A be a regular multiset over a group X. Then A is a multigroup if and only if A^* is a group (i.e. a subgroup of X).*

Proof. Assume that A^* is a group. Then for any $x, y \in A^*, xy^{-1} \in A^*$. Then, $C_A(xy^{-1}) > 0$. Since A is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y^{-1}) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$.

Conversely, assume that A is a multigroup. Since $A_1 = A^*$, by Remark 2.1, the proof is complete. □

Proposition 3.2. *Let A be a multigroup over a group X such that $A \in [X]^w$. Then A' is a regular multigroup over X if and only if A is a regular.*

Proof. Assume that A is a multigroup over X and regular. Then $\forall\ x, y \in A, C_A(x) = C_A(y)$. Since A is a multigroup, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y) = C_A(x) = C_A(y)$. Also, since $C_A(x) \neq 0 \neq C_A(y)$, then $C_A(xy^{-1}) \neq 0$. Hence, we have $xy^{-1} \in A$. But, A is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$ and $w - C_A(xy^{-1}) \geq w - C_A(x) \wedge w - C_A(y)$. Thus, $C_{A'}(xy^{-1}) \geq C_{A'}(x) \wedge C_{A'}(y)$.

Conversely, assume that A' is a regular multigroup over X. Since $C_{A'}(xy^{-1}) = C_{A'}(x) = C_{A'}(y)$, then we can have both $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$ and $C_A(x) = C_A(y) \forall x, y \in A$. □

There are some exceptions to what happens in classical algebra of set in multiset. If $A \in [X]^w$ is regular, it is possible to have the following:

(i) $A = A'$;

(ii) $A \cap A' \neq \emptyset$;

(iii) $A \subseteq A'$ or $A' \subseteq A$;

(iv) Also, $A \biguplus A' = B \in [[X]]^w$.

Example 3.1. Let $X = Z_6, A = \{0, 0, 2, 2, 4, 4\}$ and $B = \{0, 0, 0, 2, 2, 2, 4, 4, 4\}$. Then the complement $A' = \{0, 0, 0, 0, 2, 2, 2, 2, 4, 4, 4, 4\}$ and $B' = \{0, 0, 0, 2, 2, 2, 4, 4, 4\} = B$.

Definition 3.1. *Let $\bar{B} \in [X]^\alpha$ and $B \in [[X]]^\alpha$ such that $\bar{B} \subseteq B$. Then the degree to which the multiplicity of $b \in \bar{B}$ is near to the multiplicity of $b \in B$ or how near $\bar{B}$ is to B is defined by $\gamma = \frac{C_{\bar{B}}(b)}{\alpha}$.*

Remark 3.1. Then, any such $\bar{B}$ and $C_{\bar{B}}$ induce a *multi-fuzzy group* structure with membership function $\mu_{\frac{B}{\bar{B}}}(x) = \gamma$.

For simplicity, we shall use $\mu_{\mathcal{B}}$ in place of $\mu_{\frac{B}{\bar{B}}}$ and $\mu_{\bar{\mathcal{B}}}$ in place of $\mu_{\frac{\bar{B}}{\bar{\bar{B}}}}$. This multiset has a structure similar to that defined by Syropoulos [9] in that every $x \in X$ which has a multiplicity in $\bar{B}$ is such that it has only one membership degree and one multiplicity. The support of $\mu_{\mathcal{B}}$ yields a fuzzy subset of X.

It should be noted that if X is a group with identity e and $[X]^\alpha$ is the collection of all multigroups drawn from X such that for any $\bar{A} \in [X]^\alpha$ and $\forall x \in \bar{A}, C_{\bar{A}}(x) \leq \alpha$. Let $A \in [[X]]^\alpha$ be the collection of all regular multigroups drawn from X. Then, any such multigroup $\bar{A}$ from $[X]^\alpha$ induces a *multi-fuzzy group* with membership degree $\mu_{\mathcal{A}}$. It can be said that $\mu_{\mathcal{A}}$ is *fuzzy relative to* A. The support of $\mu_{\mathcal{A}}$ is a fuzzy subgroup of X. All the operations on fuzzy set such as intersection, union, inclusion and complement can be seen to hold for $\mu_{\mathcal{A}}$.

Example 3.2. Let

$$X = S_3 = \{e, (12) = a, (13) = b, (23) = c, (123) = d, (132) = f\}$$

be the group with identity. Let $A \in [X]^4$ be $A = \{e, e, e, e, d, d, d, d, f, f, f, f\}$, $\bar{A} = \{e, e, e, d, d, f, f\}$ and $\bar{\bar{A}} = \{e, d, d, f\}$. Observe the following:

(i) $\mu_{\mathcal{A}} = \{(e, 0.75), (e, 0.75), (e, 0.75), (d, 0.5), (d, 0.5), (f, 0.5), (f, 0.5)\}$ is a multi-fuzzy group.

(ii) $\mu_{\bar{\mathcal{A}}} = \{(e, 0.25), (d, 0.5), (d, 0.5), (f, 0.25)\}$ is a multi-fuzzy set.

(iii) $\bar{A}$ is a multigroup and $\mu_{\mathcal{A}}$ is a multi-fuzzy group. The support of $\mu_{\mathcal{A}}$ is the fuzzy subgroup $\{(e, 0.75), (d, 0.5), (f, 0.5)\}$ of X.

(iv) Considering $\mu_{\mathcal{A}}$ and $\mu_{\bar{A}}$ as multi-fuzzy sets,

$$\mu_{\mathcal{A}} \cap \mu_{\bar{A}} = \{(e, 0.25), (d, 0.5), (d, 0.5), (f, 0.25)\}$$

and

$$\mu_{\mathcal{A}} \cup \mu_{\bar{A}} = \{(e, 0.75), (e, 0.75), (e, 0.75), (d, 0.5), (d, 0.5), (f, 0.5), (f, 0.5)\}.$$

Proposition 3.3. *Let X be a group, $\bar{A} \in [X]^\alpha$ be a multigroup over X and $\mu_{\mathcal{A}}$ the multi-fuzzy group induced by $\bar{A}$ and $C_{\bar{A}}$. Then, $\forall x \in X$ the following hold:*

(*i*) $\mu_{\mathcal{A}}(e) \geq \mu_{\mathcal{A}}(x)$,

(*ii*) $\mu_{\mathcal{A}}(x^{-1}) = \mu_{\mathcal{A}}(x)$.

Proof. (i) Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(e) \geq C_{\bar{A}}(x)$. This implies that $\frac{C_{\bar{A}}(e)}{\alpha} \geq \frac{C_{\bar{A}}(x)}{\alpha}$. Thus, $\mu_{\mathcal{A}}(e) \geq \mu_{\mathcal{A}}(x)$.

(ii) The proof is similar. □

Proposition 3.4. *Let X be a group, $A \in [[X]]^\alpha$ and $\bar{A} \in [A]^\alpha$. Then, $\mu_{\mathcal{A}}$ has a fuzzy group structure.*

Proof. Let $x, y^{-1} \in \bar{A}$. Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(y) = C_{\bar{A}}(y^{-1}) = \alpha$. But, $\mu_{\mathcal{A}}(x) = \frac{C_{\bar{A}}(x)}{\alpha}$ and $\mu_{\mathcal{A}}(y) = \mu_{\mathcal{A}}(y^{-1}) = \frac{C_{\bar{A}}(y)}{\alpha}$. Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(xy^{-1}) \geq \min\{C_{\bar{A}}(x), C_{\bar{A}}(y^{-1})\}$. Since α is a natural number, we can have that $\frac{C_{\bar{A}}(xy^{-1})}{\alpha} \geq \min\{\frac{C_{\bar{A}}(x)}{\alpha}, \frac{C_{\bar{A}}(y^{-1})}{\alpha}\}$. Hence,

$$\mu_{\mathcal{A}}(xy^{-1}) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y^{-1})\}. \quad \square$$

The intersection of this multi-fuzzy group is also a multi-fuzzy group but the union is not necessarily a multi-fuzzy group. These properties are established by the following examples.

Example 3.3. Let $X = \{e, a, b, ab\}$ the Klein's 4-group. Consider the multigroups $A \in [X]^3$ such that $A = \{e, e, e, a, a, a, b, b, b, ab, ab, ab\}$. Let $\bar{A} = \{e, e, e, a, a\}$ and $\bar{\bar{A}} = \{e, e, b, b\}$. $\mu_{\mathcal{A}} = \{(e, 1), (e, 1), (e, 1), (a, \frac{2}{3}), (a, \frac{2}{3})\}$ and $\mu_{\bar{A}} = \{(e, \frac{2}{3}), (e, \frac{2}{3}), (b, \frac{2}{3}), (b, \frac{2}{3})\}$. The intersection of $\mu_{\mathcal{A}}$ and $\mu_{\bar{A}}$ is $\{(e, \frac{2}{3}), (e, \frac{2}{3})\}$, which is a trivial multi-fuzzy group. Their union is

$$\{(e, 1), (e, 1), (e, 1), (a, \frac{2}{3}), (a, \frac{2}{3}), (b, \frac{2}{3}), (b, \frac{2}{3})\},$$

which is obviously not a fuzzy group, since

$$0 = \mu_{\mathcal{A} \cup \bar{A}}(ab) \ngeq \min\{\mu_{\mathcal{A} \cup \bar{A}}(a), \mu_{\mathcal{A} \cup \bar{A}}(b)\} = \frac{2}{3}.$$

Conclusion

It is known that the theory of multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set just once. Given a regular multiset $A \in [X]$, where X is a group, we can fuzzify, relative to A, any multiset B which is a submultiset of A. Another interesting approach to multisets can be found e.g. in [1, 3, 4].

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APPLICATION OF REPRODUCING KERNEL ALGORITHM FOR SINGULAR BVPs INVOLVING FREDHOLM-VOLTERRA OPERATOR

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Abstract. This paper proposes asymptotically efficient algorithm for treating classes of singular boundary value problems involving Fredholm and Volterra operators associated with three-point boundary conditions. The algorithm methodology is proposed based on the novel reproducing kernel Hilbert space (RKHS) method, which is used directly without employing linearization and perturbation. The orthonormal system is generated in a favorable Hilbert space on a compact dense interval to expand the solution in Fourier series formula with accurately computable components. Numerical examples of singular multipoint BVPs are performed to support the theoretical statements that acquired by interrupting the n-term of the exact solutions. Besides, the results obtained indicate that the RK procedure is effective and competitive with a great capability in scientific and engineering applications.

Keywords: singular boundary value problems, Fredholm and Volterra operators, reproducing kernel Hilbert space method.

1 Introduction

Singular boundary value problems associated with three-point boundary conditions have been investigated in a wide area of mathematics, physics and engineering including dynamics, nuclear, chemical reaction, atomic structures and so on [1, 2, 3, 4]. Indeed, the scientific issues in this area often occur to be nonlinear with a finite set of singularity, which they are very difficult to be handled analytically through classical way. In this situation, solutions needed could not be accurately determined or fail to be convergent due to singularity. So, it has to be solved using advanced numerical and computational methods. Unfortunately, these methods are very limited unless we can resort to linearization and discretization of the variables to deal with them. Thus, it appears to be very important to develop an efficient numerical method for handling such problems. Anyhow, some numerical approaches for solving second-order, singular three point boundary value problems (BVPs) are available in literature [3, 4, 5, 6, 7]. But, there is a few research papers about second-order, singular three-point BVPs restricted by Fredholm or Volterra operators.

The purpose of this study is to investigate and implement a computational iterative technique, the reproducing kernel Hilbert space method (RKHS), in finding approximate solutions for a certain class of singular BVPs. More specifically, we consider second order three-point singular BVP in the following differential operator form:

$$u''(t)+\frac{a(t)}{p(t)}u'(t)+\frac{b(t)}{q(t)}u(t)=f(t,u(t),Tu(t)),0\leq t\leq 1, \tag{1}$$

associated with three-point boundary conditions

$$u(0)=0,\ u(1)=\alpha u(\eta),\ 0<\eta<1,\ \alpha>0,\ \alpha\eta<1, \tag{2}$$

where $a(t),b(t)\in C^2(0,1)$,

$$Tu(t)=\lambda_1\int_0^1 h_1(t,s)G_1(u(s))\,ds+\lambda_2\int_0^t h_2(t,s)G_2(u(s))\,ds,\lambda_i,i=1,2,$$

are positive parameters, $h_i(t,s),i=1,2,$ are arbitrary analytical kernel functions over the square $0<s<t<1$, $G_1(v_1),G_2(v_2)$ are linear or nonlinear continuous terms in $\Pi_1[0,1]$ as $v_i=v_i(t)\in\Pi_3[0,1]$, $0\leq t\leq 1$, $-\infty<v_i<\infty$, $i=1,2$, $f(t,u,Tu)\in\Pi_1[0,1]$ are sufficiently regular given functions such that singular BVPs (1) and (2) satisfies the existence and uniqueness of the solutions, and $u\in\Pi_3[0,1]$ is an unknown function to be determined. Here, the real-valued functions $p(t)$ and $q(t)$ are continuous and may be equal to zero at $\{t_i\}_{i=1}^m\in[0,1]$; that is, the equation may be singular at $t=t_i,i=1,2,...,m$.

The reproducing-kernel is a numerical as well as analytical algorithm for treating a wide variety of ODEs and PDEs associated to different kinds of order derivatives degree, which usually provides the solutions in terms of rapidly convergent series with components that can be elegantly computed [8, 9, 10, 11, 12]. The RKHS algorithm has been successfully applied to various areas in numerical analysis, computational mathematics, image processing, machine learning, quantum mechanics, and finance [13, 14, 15, 16] Moreover, in the recent years, a lot of research work has been devoted to utilize the RKHS method as a superb framework to find numerical approximate solutions to diverse matters [17, 18, 19, 20, 21, 22, 23]. On the other hand as well, the numerical solvability of different categories of BVPs can be found in [24, 25, 26, 27, 28, 29, 30].

The objective of this article is to highlight the importance of singular BVPs in sobolev spaces for specific applications. The structure of this article is organized as follows: In the next section, necessary details and preliminaries about the reproducing-kernel theory are briefly given. In section 3, theoretical and analytical basis with representation of solutions are introduced in the Hilbert space $\Pi_3[0,1]$. Convergence analysis of the method are presented in section 4 Numerical outcomes are investigated in section 5. A final section provides brief conclusions.

2 Background and preliminaries

In this section, we present the concept of essential materials about the RKHS method for constructing smooth reproducing-kernel functions which will used to produce a set of orthonormal basis functions, as well as, derive the solutions in terms of Fourier series coefficients in Sobolev spaces. To do that, multiply both sides of Eq. (1) by $p(t)\,q(t)$ to get

$$\widetilde{P}(t)\,u''(t)+\widetilde{Q}(t)\,u'(t)+\widetilde{R}(t)\,u(t)=\widetilde{F}(t,u(t),Tu(t)), \tag{3}$$

where $\widetilde{P}(t)=p(t)\,q(t)$, $\widetilde{Q}(t)=a(t)\,q(t)$, $\widetilde{R}(t)=b(t)\,p(t)$, and $\widetilde{F}(t,u,Tu)=p(t)\,q(t)\,f(t,u,Tu)$.

If we defined a function $v(t)=u(t)-\gamma t$, $\gamma=\alpha u(\eta)$, then Eq. (3) can be reduced to the following form:

$$\widetilde{P}(t)\,v''(t)+\widetilde{Q}(t)\,v'(t)+\widetilde{R}(t)\,v(t)=\widetilde{G}(t,v(t),Tv(t)), \tag{4}$$

with the homogeneous boundary conditions

$$v(0)=0,\ v(1)=0, \tag{5}$$

where $\widetilde{G}(t,v(t),Tv(t))=\widetilde{F}(t,u(t)+\gamma t,T(u(t)+\gamma t))-\gamma(t\widetilde{R}(t)+\widetilde{Q}(t))$. Obviously, it suffices for us to solve BVPs (4) and (5).

Definition 1 ([11])**.** *Let $\mathcal{H}$ be a Hilbert space of function $\mathcal{F}:\Omega\to\mathcal{H}$ on a set Ω. A function $\Gamma:\Omega\times\Omega\to\mathbb{R}$ is a reproducing kernel of $\mathcal{H}$ if the following conditions are satisfied: Firstly, $\Gamma(\cdot,s)\in\mathcal{H}$ for each $s\in\Omega$. Secondly, $\langle\mathcal{F}(\cdot),\Gamma(\cdot,s)\rangle=\mathcal{F}(s)$ for each $\mathcal{F}\in\mathcal{H}$ and each $s\in\Omega$.*

Definition 2 ([8])**.** *The space $\Pi_1[0,1]$ is defined as $\Pi_1[0,1]=\{v=v(t):v$ is one-variable absolutely continuous real-valued function on $[0,1]$ and $v'\in L^2[0,1]\}$. Whilst, the inner product and the norm of $\Pi_1[0,1]$ are given, respectively, by*

$$\langle v_1(t),v_2(t)\rangle_{\Pi_1}=v_1(0)\,v_2(0)+\int_0^1 v_1'(s)v_2'(s)ds, \tag{6}$$

and $||v_1(t)||^2_{\Pi_1}=\langle v_1(t),v_1(t)\rangle_{\Pi_1}$, where $v_1,v_2\in\Pi_1[0,1]$.

Theorem 3 ([8])**.** *The Hilbert space $\Pi_1[0,1]$ is a complete reproducing kernel with the reproducing kernel function*

$$\widehat{G}_s(t)=\begin{cases}1+t, & t\le s,\\ 1+s, & t>s.\end{cases} \tag{7}$$

Now, we construct the reproducing kernel space $\Pi_3[0,1]$ in which every function satisfies the boundary conditions $v(0)=0$ and $v(1)=0$.

Definition 4. *The space* $\Pi_3[0,1]$ *is defined as* $\Pi_3[0,1] = \{v = v(t) : v, v', v''$ *are one-variable absolutely continuous real-valued functions on* $[0,1]$, $v''' \in L^2[0,1]$, *and* $v(0) = 0, v(1) = 0\}$. *Whilst, the inner product and the norm of* $\Pi_3[0,1]$ *are given, respectively, by*

$$(8)\qquad \langle v_1(t), v_2(t)\rangle_{\Pi_3} = \sum_{i=0}^{1} v_1^{(i)}(0)\, v_2^{(i)}(0) + v_1(1)v_2(1) + \int_0^1 v_1'''(w)v_2'''(w)dw,$$

and $||v_1(t)||^2_{\Pi_3} = \langle v_1(t), v_1(t)\rangle_{\Pi_3}$, *where* $v_1, v_2 \in \Pi_3\,[0,1]$.

Theorem 5. *The Hilbert space* $\Pi_3\,[0,1]$ *is a complete reproducing kernel with reproducing kernel function*

$$(9)\quad G_s\,(t) = \begin{cases} \frac{1}{120}(1-t)^3 s^3 \left[6s^2t^2 + 3st(s-5t) + (10t^2 - 5st + s^2)\right], & t \le s, \\ \frac{1}{120}(1-s)^3 t^3 \left[6s^2t^2 + 3st(t-5s) + (10s^2 - 5st + t^2)\right], & t > s. \end{cases}$$

Proof. According to [8], $\Pi_3\,[0,1]$ is a complete reproducing kernel Hilbert space, that is, for each fixed $s \in [0,1]$ and any $v(t) \in \Pi_3\,[0,1]$, there exists a function $G_s\,(t) \in \Pi_3\,[0,1]$ such that $\langle v(s), G_t\,(s)\rangle_{\Pi_3} = v(t)$, $t \in [0,1]$ and the expression form of $G_s\,(t)$ can be denoted as $G_s\,(t) = \sum_{i=1}^{6} a_i(s)t^{i-1}$, if $t \le s$, and $G_s\,(t) = \sum_{i=1}^{6} b_i(s)t^{i-1}$, if $t > s$, where the coefficients $a_i(s)$ and $b_i(s)$, $i = 1, 2, ..., 6$, could be obtained by solving the following generalized differential equations using Maple 13 software package:

$$(10)\qquad \begin{array}{l} G_s\,(0) = 0, \partial_t G_s\,(0) + \partial_t^4 G_s\,(0) = 0, \partial_t^3 G_s\,(0) = 0, \\ G_s\,(1) = 1, \partial_t^i G_s\,(1) = 0, i = 3, 4, \\ \partial_t^6 G_s\,(t) = -\delta\,(s-t)\,,\ \delta \text{ dirac-delta function}, \\ \partial_t^i G_s\,(s-0) = \partial_t^i G_s\,(s+0)\,, i = 0, 1, ..., 4, \\ \partial_t^5 G_s\,(s+0) - \partial_t^5 G_s\,(s-0) = -1. \end{array}$$

The proof is complete. □

3 Theoretical and analytical basis

In order to illustrate the RKHS methodology to proposed model, we consider that $\mathcal{D} : \Pi_3\,[0,1] \to \Pi_1\,[0,1]$ is an invertible bounded linear operator such that $\mathcal{D}v(t) := \widetilde{P}\,(t)\,v''\,(t) + \widetilde{Q}\,(t)\,v'\,(t) + \widetilde{R}\,(t)\,v\,(t)$, and $\mathcal{D}^*$ is the adjoint operator of $\mathcal{D}$. Then, Eqs. (4) and (5) can be equivalently converted into the form:

$$(11)\qquad \begin{cases} \mathcal{D}v(t) = \widetilde{G}(t, v(t), Tv(t)), \\ v(0) = 0, v(1) = 0. \end{cases}$$

Let $\varphi_i\,(t) = G_{t_i}\,(t)$ and $\psi_i\,(t) = \mathcal{D}^*\varphi_i\,(t)$, where $\{t_i\}_{i=1}^{\infty}$ is a countable dense subset in $[0,1]$. From the reproducing-kernel property, it holds $\langle v\,(t)\,, \varphi_i\,(t)\rangle_{\Pi_1} = v(t_i)$.

Theorem 6. *Suppose that* $\{t_i\}_{i=1}^{\infty}$ *is dense in the interval* $[0,1]$*, then the sequence* $\{\psi_i(t)\}_{i=1}^{\infty}$ *is a complete function system in* $\Pi_3[0,1]$ *with* $\psi_i(t) = .(\widetilde{P}(s)\partial_s^2 + \widetilde{Q}(s)\partial_s + \widetilde{R}(s))[G_t(s)]|_{s=t_i}$.

Proof. In this proof, the subscript s by the operator $\mathcal{D}$, denoted by $\mathcal{D}_s$, indicates that the operator $\mathcal{D}$ applies to the function of s. However, it is clear that $\psi_i(t) = \mathcal{D}^*\varphi_i(t) = \langle \mathcal{D}^*\varphi_i(t), G_t(s)\rangle_{\Pi_3} = \langle \varphi_i(t), \mathcal{D}_s G_t(s)\rangle_{\Pi_1} = .\mathcal{D}_s G_t(s)|_{s=t_i} = .(\widetilde{P}(s)\partial_s^2 + \widetilde{Q}(s)\partial_s + \widetilde{R}(s))[G_t(s)]|_{s=t_i}$.

Since $\{t_i\}_{i=1}^{\infty}$ is dense in the interval $[0,1]$. For each $v(t)$ in $\Pi_3\,[0,1]$, if $\langle v\,(t)\,, \psi_i\,(t)\rangle_{\Pi_3} = \langle \mathcal{D}v\,(t)\,, \varphi_i\,(t)\rangle_{\Pi_1} = \mathcal{D}v\,(t_i) = 0$, $(i = 1, 2, ...)$, then from the density of $\{t_i\}_{i=1}^{\infty}$ and continuity of $v\,(t)$, we have $v\,(t) = 0$. □

The RKHS solution will be obtained by calculating a truncated series based on the orthonormal functions $\left\{\overline{\psi}_i\,(t)\right\}_{i=1}^{\infty}$ of the space $\Pi_3\,[0,1]$, which is constructed from $\{\psi_i\,(t)\}_{i=1}^{\infty}$ by using the Gram-Schmidt process such that

$$\bar{\psi}_i\,(t) = \sum_{k=1}^{i} \mu_{ik}\psi_k\,(t)\,,\ (\mu_{ik} > 0, i = 1, 2, ...), \tag{12}$$

where μ_{ik} are orthogonal coefficients that can be determined as in section 4.

Theorem 7. *Suppose that* $\{t_i\}_{i=1}^{\infty}$ *is dense in the interval* $[0,1]$. *If* $v(t) \in \Pi_3\,[0,1]$ *is a unique solution of Eq.* (11)*, then*

$$v\,(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \mu_{ik}\left[\widetilde{G}\,(t_k, v\,(t_k)\,, Tv(t_k))\right]\bar{\psi}_i\,(t)\,. \tag{13}$$

Proof. Let $v(t)$ be the solution of Eq. (11) that can be expanded in Fourier series. Since $\langle v\,(t)\,, \varphi_i\,(t)\rangle_{\Pi_1} = v\,(t_i)$ and $\sum_{i=1}^{\infty}\left\langle v\,(t)\,, \bar{\psi}_i\,(t)\right\rangle_{\Pi_3}\bar{\psi}_i\,(t)$ is convergent series for each $v\,(t) \in \Pi_3\,[0,1]$, then we have

$$\begin{aligned}
v\,(t) &= \sum_{i=1}^{\infty}\left\langle v\,(t)\,, \bar{\psi}_i\,(t)\right\rangle_{\Pi_3}\bar{\psi}_i\,(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\,\langle v\,(t)\,, \psi_k\,(t)\rangle_{\Pi_3}\bar{\psi}_i\,(t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\,\langle v\,(t)\,, \mathcal{D}^*\varphi_k\,(t)\rangle_{\Pi_3}\bar{\psi}_i\,(t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\,\langle \mathcal{D}v\,(t)\,, \varphi_k\,(t)\rangle_{\Pi_1}\bar{\psi}_i\,(t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\left\langle \widetilde{G}\,(t, v\,(t)\,, Tv(t))\,, \varphi_k\,(t)\right\rangle_{\Pi_1}\bar{\psi}_i\,(t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\left[\widetilde{G}\,(t_k, v\,(t_k)\,, Tv(t_k))\right]\bar{\psi}_i\,(t)\,.
\end{aligned}$$

Therefore, the form of Eq. (13) is the exact solution of Eq. (11). The proof is complete. □

Since $\Pi_3\,[0,1]$ is a Hilbert space, the series

$$\sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}\left\langle \mathcal{D}v\left(t\right),\varphi_k\left(t\right)\right\rangle_{\Pi_1}\bar{\psi}_i\left(t\right)<\infty.$$

Hence, the truncated series

$$v_n\left(t\right)=\sum_{i=1}^{n}\sum_{k=1}^{i}\mu_{ik}\left[\widetilde{G}\left(t_k,v\left(t_k\right),Tv(t_k)\right)\right]\bar{\psi}_i\left(t\right) \tag{14}$$

is convergent in the sense of $\|\cdot\|_{\Pi_3[0,1]}$ and the numerical solution of Eq. (11) can be directly calculated by Eq. (14).

To analyze the most comprehensive computations, we mention that the solution depends on the internal structure of $\widetilde{G}$. That is, if $\widetilde{G}$ is linear, then the exact and the numerical solutions can be directly obtained by using Theorem 7. Whilst, if $\widetilde{G}$ is nonlinear, then the exact and the numerical solutions can be obtained iteratively by using the following process:

$$v\left(t\right)=\sum_{i=1}^{\infty}\Lambda_i\bar{\psi}_i\left(t\right),\ \Lambda_i=\sum_{k=1}^{i}\mu_{ik}\widetilde{G}\left(t_k,v\left(t_k\right),Tv(t_k)\right). \tag{15}$$

Let $t_1=0$, then $v(t_1)=0$. Set the initial function $v_0(t_1)=v(t_1)$, then $\widetilde{G}\left(t_1,v\left(t_1\right),Tv(t_1)\right)$ is known. Define the n-term numerical solution of $v\left(t\right)$ as follows:

$$v_n\left(t\right)=\sum_{i=1}^{n}\Lambda_i\bar{\psi}_i\left(t\right), \tag{16}$$

where $\Lambda_i=\sum_{k=1}^{i}\mu_{ik}\widetilde{G}\left(t_k,v_{n-1}\left(t_k\right),Tv_{n-1}(t_k)\right).$

In the iterative process of Eq. (16), we can guarantee that the numerical solution v_n satisfies the constraints conditions of Eq. (11).

4 Convergence of the RKHS method

In this section, we show that the n-term approximate solution $v_n(t)$ converges uniformly to exact solution $v(t)$ as $n\to\infty$ in the Hilbert space $\Pi_3\,[0,1]$.

Theorem 8. *If $\|v_{n-1}-v\|_{\Pi_3}arrow0$, $t_n arrows$ $(n arrow\infty)$, $\|v_n\|$ is bounded, and $\widetilde{G}(t,w_1,w_2)$ is continuous in $[0,1]$ with respect to t,w_i, $i=1,2$, then $\widetilde{G}(t_n,v_{n-1}(t_n),Tv_{n-1}(t_n))arrow\widetilde{G}(s,v(s),Tv(s))$ as $n arrow\infty$.*

Proof. By the reproducing property of $G_s\left(t\right)$, we have $v\left(t\right)=\left\langle v\left(\cdot\right),G_t\left(\cdot\right)\right\rangle_{\Pi_3}$ and $v'\left(t\right)=\left\langle v\left(\cdot\right),\partial_tG_t\left(\cdot\right)\right\rangle_{\Pi_3}$. From the continuity of $G_s\left(t\right)$ and and the Schwarz inequality, we get that $\left|v'\left(t\right)\right|=\left|\left\langle v\left(s\right),\partial_tG_t\left(s\right)\right\rangle_{\Pi_3}\right|\le\left\|v\left(s\right)\right\|_{\Pi_3}\left\|\partial_tG_t\left(s\right)\right\|_{\Pi_3}\le M_1\left\|v\right\|_{\Pi_3}$, $\left|v_{n-1}\left(s\right)-v\left(s\right)\right|=\left|\left\langle v_{n-1}\left(s\right)-v\left(s\right),G_t\left(s\right)\right\rangle_{\Pi_3}\right|$

$\leq \|v_{n-1}(s) - v(s)\|_{\Pi_3} \|G_t(s)\|_{\Pi_3} \leq M_2 \|v_{n-1} - v\|_{\Pi_3}$. From the last description as well, it follows that

$$\begin{aligned} |v_{n-1}(t_n) - v(s)| &= |v_{n-1}(t_n) - v_{n-1}(s) + v_{n-1}(s) - v(s)| \\ &\leq |v_{n-1}(t_n) - v_{n-1}(s)| + |v_{n-1}(s) - v(s)| \\ &\leq \left|(v_{n-1})'(\xi)\right| |t_n - s| + |v_{n-1}(s) - v(s)| \\ &\leq M_1 \|v\|_{\Pi_3} |t_n - s| + M_2 \|v_{n-1} - v\|_{\Pi_3}, \end{aligned}$$

where ξ lies between t_n and s. From the assumptions $\|v_{n-1} - v\|_{\Pi_3} \to 0$, $t_n \to s$ and the boundedness of $\|v_n\|$ as $n \to \infty$, it implies that $v_n(t_n) \to v(s)$ as $n \to \infty$. By the continuation of $\widetilde{G}$, we get the result directly. □

Lemma 9. *Suppose that* $\{t_i\}_{i=1}^{\infty}$ *is dense in the interval* $[0,1]$ *and* $v(t)$ *is the solution of Eq. (15), then* $\mathcal{D}v(t_k) = \mathcal{D}v_n(t_k)$ *as* $n \to \infty$.

Proof. Let P_n be the projective operator such that $P_n{:}\Pi_3[0,1] {\to} \{\sum_{m=1}^{n} c_m\psi_m(t), c_m \in \mathbb{R}\}$.

Then, we have $\mathcal{D}v_n(t_k) = \langle v_n(\xi), \mathcal{D}_{t_k}G_{t_k}(\xi)\rangle_{\Pi_3} = \langle v_n(\xi), \psi_k(\xi)\rangle_{\Pi_3} = \langle P_n v(\xi), \psi_k(\xi)\rangle_{\Pi_3} = \langle v(\xi), P_n\psi_k(\xi)\rangle_{\Pi_3} = \langle v(\xi), \psi_k(\xi)\rangle_{\Pi_3} = \langle v(\xi), \mathcal{D}_{t_k}G_{t_k}(\xi)\rangle_{\Pi_3} = \mathcal{D}_{t_k}\langle v(\xi), G_{t_k}(\xi)\rangle_{\Pi_3} = \mathcal{D}v(t_k)$. □

Theorem 10. *Suppose that* $||v_n||_{\Pi_3}$ *is bounded in Eq.* (16) $\{t_i\}_{i=1}^{\infty}$ *is dense in the interval* $[0,1]$, *and Eq.* (11) *has a unique solution. Then, the* n*-term approximate solution* $v_n(t)$ *converges to the exact solution* $v(t)$ *of Eq.* (11) *and the exact solution is expressed as* $v(t) = \sum_{i=1}^{\infty} \Lambda_i \bar{\psi}_i(t)$, *where* Λ_i *is given in the iterative formula* (16).

Proof. From the iterative formula (16), we have $v_n(t) = v_{n-1}(t) + \Lambda_n \bar{\psi}_n(t)$. Hence, $\|v_n(t)\|_{\Pi_3}^2 = \|v_{n-1}(t)\|_{\Pi_3}^2 + \Lambda_n^2$, which implies that $\|v_n(t)\|_{\Pi_3}^2 = \|v_0(t)\|_{\Pi_3}^2 + \sum_{i=1}^{n} \Lambda_i^2$. Since $\|v_n(t)\|_{\Pi_3}$ is bounded, monotone increasing and convergent as soon as $n \to \infty$, then there exists a positive constant β such that $\sum_{i=1}^{n} \Lambda_i^2 = \beta$. For $m > n$, it follows that

$$\begin{aligned} \|v_m(t) - v_n(t)\|_{\Pi_3}^2 &= \|v_m - v_{m-1} + v_{m-1} - \dots + v_{n+1} - v_n\|_{\Pi_3}^2 \\ &= \|v_m - v_{m-1}\|_{\Pi_3}^2 + \|v_{m-1} - v_{m-2}\|_{\Pi_3}^2 + \dots + \|v_{n+1} - v_n\|_{\Pi_3}^2 \\ &= \sum_{l=n+1}^{m} \Lambda_l^2 \to 0, (m, n \to \infty), \end{aligned}$$

in view of $(v_m - v_{m-1}) \perp (v_{m-1} - v_{m-2}) \perp \dots \perp (v_{n+1} - v_n)$. Considering the completeness of $\Pi_3[0,1]$, there exist $v(t) \in \Pi_3[0,1]$ such that $v_n(t) \to v(t)$ as soon as $n \to \infty$.

Again, from the iterative formula (16), we have $v(t) = \lim_{n\to\infty} v_n(t) = \sum_{i=1}^{\infty} \Lambda_i \bar{\psi}_i(t)$.

Thus, $\mathcal{D}v(t) = \sum_{i=1}^{\infty} \Lambda_i \mathcal{D}\bar{\psi}_i(t)$ and $\mathcal{D}v(t_k) = \sum_{i=1}^{\infty} \Lambda_i \left\langle \mathcal{D}\bar{\psi}_i(t), \varphi_k(t)\right\rangle_{\Pi_1} = \sum_{i=1}^{\infty} \Lambda_i \left\langle \bar{\psi}_i(t), \mathcal{D}^*\varphi_k(t)\right\rangle_{\Pi_3} = \sum_{i=1}^{\infty} \Lambda_i \left\langle \bar{\psi}_i(t), \psi_k(t)\right\rangle_{\Pi_3}$.

Consequently, $\sum_{k=1}^{i}\mu_{ik}\mathcal{D}v\left(t_k\right) = \sum_{k=1}^{i}\mu_{ik}\left(\sum_{i=1}^{\infty}\Lambda_i\left\langle\bar{\psi}_i\left(t\right),\psi_k\left(t\right)\right\rangle_{\Pi_3}\right) = \sum_{i=1}^{\infty}\Lambda_i\left\langle\bar{\psi}_i\left(t\right),\sum_{k=1}^{i}\mu_{ik}\psi_k\left(t\right)\right\rangle_{\Pi_3} = \sum_{i=1}^{\infty}\Lambda_i\left\langle\bar{\psi}_i\left(t\right),\bar{\psi}_k\left(t\right)\right\rangle_{\Pi_3} = \Lambda_k$. In view of Eq. (11), we have $\mathcal{D}v\left(t_k\right) = \widetilde{G}\left(t_k, v\left(t_k\right), Tv(t_k)\right)$. For the conduct of proceedings in the proof, since $\{t_i\}_{i=1}^{\infty}$ is dense in the interval $[0,1]$, there exists a subsequence $\left\{t_{n_j}\right\}_{j=1}^{\infty}$ such that $t_{n_j} \to s$ as $j \to \infty$. Observing that $\mathcal{D}v_{n_j}\left(t_{n_j}\right) = \widetilde{G}\left(t_{n_j}, v_{n_j-1}\left(t_{n_j}\right), Tv_{n_j-1}(t_{n_j})\right)$, let $j \to \infty$, by the continuity of $\widetilde{G}$ and Theorem 8, we have $\mathcal{D}v\left(s\right) = \widetilde{G}\left(s, v\left(s\right), Tv(s)\right)$. Also, since $\bar{\psi}_i\left(t\right) \in \Pi_3\left[0,1\right]$, then $v\left(t\right)$ satisfies Eq. (11) and the constraints conditions. So, the proof of the theorem is complete. □

For the error behavior, if $\varepsilon_n = |v(t) - v_n(t)|$, where $v\left(t\right)$ and $v_n\left(t\right)$ are given in Eqs. (15) and (16), respectively. Then, one can write $\left\|\varepsilon_n\right\|_{\Pi_3}^2 = \left\|\sum_{i=n+1}^{\infty}\Lambda_i\bar{\psi}_i\right\|_{\Pi_3}^2 = \sum_{i=n+1}^{\infty}\left(\Lambda_i\right)^2$ and $\left\|\varepsilon_{n-1}\right\|_{\Pi_3}^2 = \left\|\sum_{i=n}^{\infty}\Lambda_i\bar{\psi}_i\right\|_{\Pi_3}^2 = \sum_{i=n}^{\infty}\left(\Lambda_i\right)^2$. Clearly, $\{\varepsilon_n\}_{n=1}^{\infty}$ is decreasing in the sense of $\left\|\cdot\right\|_{\Pi_3}$. Since $\sum_{i=1}^{\infty}\Lambda_i\bar{\psi}_i\left(t\right)$ is convergent series, then $\left\|\varepsilon_n\right\|_{\Pi_3} \to 0$ as $n \to \infty$.

5 Applications and numerical algorithm

To test the accuracy, simplicity and effectiveness of the RKHS algorithm, certain numerical examples with exact solutions are given. The results reveal that the method is highly accurate, rapidly converge, and convenient to handle various physical and engineering problems. Based on the algorithm, we pick $t_i = ih$, $i = 1, 2, ..., 6$, with step-size $h = 0.16$ and take $n = 26$, where n is the number of terms of the Fourier series of the unknown function $u(t)$. The numerical computations are performed using Maple 13 software package. To allocating more, we present the following algorithm to summarize the procedure in finding the approximate solutions.

Algorithm 11. *To approximate the solution $u_n\left(t\right)$ of $u\left(t\right)$ for Eqs.* (1) *and* (2)*, do the following steps.*

Step 1: Fixed t in $[0,1]$ and set $s \in [0,1]$;

If $s \le t$, let $G_s\left(t\right) = \sum_{i=1}^{6} a_i(s)t^{i-1}$;

else let $G_s\left(t\right) = \sum_{i=1}^{6} b_i(s)t^{i-1}$;

Step 2: Choose n collocation points and do the following subroutine:

Set $t_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$;

Set $\psi_i(t) = \mathcal{D}_s G_s\left(t\right)\big|_{s=t_i}$;

Step 3: Obtain the orthogonalization coefficients $\mu_{i\rho}$ as follows

Let $c_{ik} = \left\langle\psi_i\left(t\right), \overline{\psi}_k\left(t\right)\right\rangle_{\Pi_3}$, and do the following subroutine:

For $i = 1$, set $\mu_{11} = \|\psi_1\|_{\Pi_3}^{-1}$;

For $i = 2, ..., n$, set $\mu_{ii} = \left(\|\psi_i\|_{\Pi_3}^2 - \sum_{k=1}^{i-1} c_{ik}^2\right)^{-0.5}$;

else (for $\rho < i$), set $\mu_{i\rho} = -\left(\sum_{k=\rho}^{i-1} c_{ik}\mu_{k\rho}\right)\left(\|\psi_i\|_{\Pi_3}^2 - \sum_{k=1}^{i-1} c_{ik}^2\right)^{-0.5}$;

Step 3: For $i = 1, 2, ..., n$, set

$\overline{\psi}_i(t) = \sum_{k=1}^{i} \mu_{ik}\psi_k(t)$;

Step 4: Set $t_1 = 0$, and choose an initial approximation $v_0(t_1) = v(t_1)$;

For $i = 1$, set $\Lambda_1 = \mu_{11}\widetilde{G}(0, v_0(0), Tv_0(0))$ and $v_1(t) = \Lambda_1\overline{\psi}_1$;

For $i = 2, 3, ..., n$, set $\Lambda_i = \sum_{k=1}^{i} \mu_{ik}\widetilde{G}(t_k, v_{n-1}(t_k), Tv_{n-1}(t_k))$;

Set $v_n(t) = \sum_{i=1}^{n} \Lambda_i\bar{\psi}_i(t)$;

Step 5: Set $u_n(t) = v_n(t) + \gamma t$;

Solve a linear system $u_n(0) = 0$ and $u_n(1) - \alpha u_n(\eta) = 0$ to obtain γ.

Outcome: the numerical solution $u_n(t)$.

Step 6: Stop.

By applying Algorithm 1 throughout the numerical computations, we present some tabulate data, numerical comparison, and graphical results that discussed quantitatively at some selected grid points on $[0, 1]$.

Example 12. Consider the singular linear differential equation.$u''(t)+\frac{1}{\sin(t)}u'(t)-\frac{1}{t(t-1)}u(t)-Tu(t) = f(t)$, . with three-point boundary conditions$u(0) = 0$, $u(1)-4u\left(\frac{1}{9}\right) = 0$, where $Tu(t) = \int_0^1 t^2 su(s)ds + \int_0^t (t+1)su(s)ds$, $0 < s < t < 1$ and $f(t)$ satisfies the existence and uniqueness of $u(t)$. Here, the singularities at the two endpoints $\{0, 1\}$. However, the exact solution is $u(t) = t(t-1)\left(t - \frac{1}{9}\right)\cos(t)$.

The results of numerical analysis are approximate as much as is required within a logical error ratio that will be stored in a fixed number of digits. Using the RKHS algorithm, the comparison between exact and numerical solutions of Example 12 together with absolute and relative errors in $\Pi_3[0, 1]$ are shown in Table 1. This is an indication of stability of the presented method.

Table 1. Numerical solutions and errors for Example 12.

t	Exact solution	Numerical solution	Absolute error	Relative error
0.16	-0.0064867414033	-0.0064883615395	1.62014×10^{-6}	2.49761×10^{-4}
0.32	-0.0431467576347	-0.0431461185735	6.39061×10^{-7}	1.48113×10^{-5}
0.48	-0.0816697618499	-0.0816661092539	3.65260×10^{-6}	4.47240×10^{-5}
0.64	-0.0977401806727	-0.0977386264592	1.55421×10^{-6}	1.59015×10^{-5}
0.80	-0.0767925617414	-0.0767937406674	1.17893×10^{-6}	1.53521×10^{-5}
0.96	-0.0186952221593	-0.0186969923499	1.77019×10^{-6}	9.46868×10^{-5}

Example 13. Consider the singular nonlinear differential equation

$$u''(t) - \frac{1}{t^2(1-t)^2}u'(t) + \frac{1}{\sinh(t)}u(t) - u^2(t) - \sinh^{-1}(u(t)) - Tu(t) = f(t),$$

with three-point boundary conditions$u(0) = 0$, $u(1) - u\left(\frac{1}{2}\right) = 0$, where $Tu(t) = \int_0^1 tsu^3(s)\,ds + \int_0^t (t-s)u^2(s)\,ds$, $0 < s < t < 1$ and $f(t)$ satisfies the existence and uniqueness of $u(t)$. Here, the singularities at the two endpoints $\{0,1\}$. However, the analytical solution is $u(t) = \left(t - \frac{1}{2}\right)^2(t-1)^2\sinh(t)$.

Using the RKHS algorithm, the comparison between exact and numerical solutions of Example 13 together with absolute and relative errors in $\Pi_3[0,1]$ are shown in Table 2.

Table 2. Numerical solutions and errors for Example 13.

t	Exact solution	Numerical solution	Absolute error	Relative error
0.16	0.0131065322359	0.0131062996971	2.32539×10^{-7}	1.77422×10^{-5}
0.32	0.0048764035285	0.0048762184821	1.85046×10^{-7}	3.79473×10^{-5}
0.48	0.0000539334978	0.0000538664049	6.70929×10^{-8}	1.24399×10^{-3}
0.64	0.0017389788733	0.0017390244908	4.56175×10^{-8}	2.62324×10^{-5}
0.80	0.0031971815359	0.0031972745002	9.29643×10^{-8}	2.90770×10^{-5}
0.96	0.0003772918713	0.0003773895245	9.76532×10^{-8}	2.58827×10^{-4}

Example 14. Consider the singular nonlinear differential equation

$$u''(t) + \frac{1}{t(t-1)\left(t-\frac{1}{3}\right)}u'(t) + \frac{1}{(e^t - 1)}u(t) - \cosh(u(t)) - Tu(t) = f(t),$$

with three-point boundary conditions$u(0) = 0$, $u(1) - 2u\left(\frac{1}{3}\right) = 0$, where $Tu(t) = \int_0^1 ts^2(8s-11)\,e^{-s\left(2s-\frac{1}{3}\right)}e^{u(s)}ds + \int_0^t \cosh(t)\,u^4(s)\,ds$, $0 < s < t < 1$ and $f(t)$ satisfies the existence and uniqueness of $u(t)$. Here, the singularities at multi-points $\{0, \frac{1}{3}, 1\}$ of the interval $[0,1]$. However, the analytical solution is $u(t) = t\left(t - \frac{1}{3}\right)\left(2t^2 - 3t + 1\right)$.

Using the RKHS algorithm, the comparison between exact and numerical solutions of Example 14 together with absolute and relative errors in $\Pi_3[0,1]$ are shown in Table 3.

Table 3. Numerical solutions and errors for Example 14.

t	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.0158412800000	−0.0158410408525	2.39147×10^{-7}	1.50965×10^{-5}
0.32	−0.0010444799999	−0.0010442321864	2.47814×10^{-7}	2.37260×10^{-4}
0.48	0.0014643200000	0.0014645926865	2.72686×10^{-7}	1.86221×10^{-4}
0.64	−0.0197836799999	−0.0197832643578	4.15642×10^{-7}	2.10093×10^{-5}
0.80	−0.0448000000000	−0.0447993897465	6.10253×10^{-7}	1.36217×10^{-5}
0.96	−0.0221388799999	−0.0221381979890	6.82011×10^{-7}	3.08060×10^{-5}

Example 15. Consider the singular nonlinear differential equation

$$u''(t) + \frac{1}{\ln(t+1)}u'(t) - \frac{1}{t\left(t-\frac{1}{4}\right)(t-1)}u(t) - u(t)\,e^{u(t)} - Tu(t) = f(x),$$

with three-point boundary conditions$u(0) = 0$, $u(1) - 3u\left(\frac{1}{4}\right) = 0$, where $Tu(t) = \int_0^1 (t-s)^2 e^{u(s)} ds + \int_0^t e^{s+t} e^{u(s)} ds$, $0 < s < t < 1$ and $f(t)$ satisfies the existence and uniqueness of $u(t)$. Here, the singularities at multi-points $\{0, \frac{1}{4}, 1\}$ of the interval $[0,1]$. However, the the analytical solution is $u(t) = \ln\left(t^2(1-t)\left(t-\frac{1}{4}\right)+1\right)$.

Using the RKHS algorithm, the comparison between exact and numerical solutions of Example 15 together with absolute and relative errors in $\Pi_3[0,1]$ are shown in Table 4. It is clear from the tables that the numerical solutions are in close agreement with the exact solutions for all examples, while the accuracy is in advanced by using only few term of the RKHS iterations. The approximation values and the absolute errors of $u^{(i)}(t), i = 0,1,2$, at various t in $[0,1]$ of Example 15 are graphically plotted in Figures 1, 2, and 3, respectively. Here, the numerical values for the exact solution approach smoothly to the t-axis by satisfying their boundary conditions. Indeed, decreasing the step-size increases the accuracy of the results while increasing the time required to simulate the problem.

Table 4. Numerical solutions and errors for Example 15.

t	Exact solution	Numerical solution	Absolute error	Relative error
0.16	-0.0019372352291	-0.0019373982303	1.63001×10^{-7}	8.41412×10^{-5}
0.32	0.0048623993527	0.0048624655648	6.62121×10^{-8}	1.36172×10^{-5}
0.48	0.0271830114104	0.0271831781099	1.66700×10^{-7}	6.13249×10^{-6}
0.64	0.0559150456224	0.0559153019544	2.56332×10^{-7}	4.58431×10^{-6}
0.80	0.0680324103918	0.0680327200535	3.09662×10^{-7}	4.55168×10^{-6}
0.96	0.0258367772964	0.0258368978659	1.20569×10^{-7}	4.66658×10^{-6}

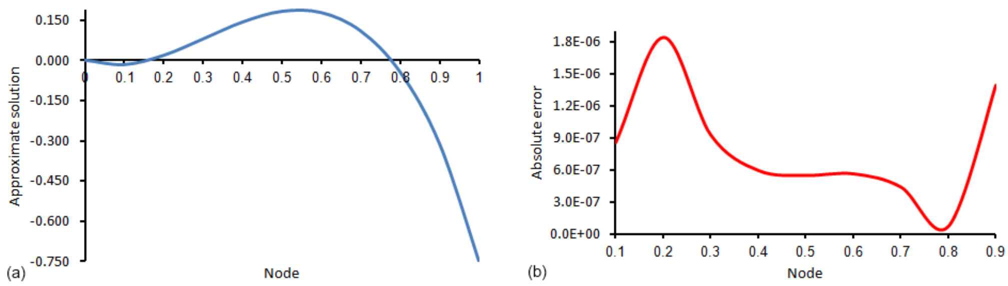

Figure 1: Graphical results of $u(x)$ of Example 15: (a) approximate solution (b) absolute error.

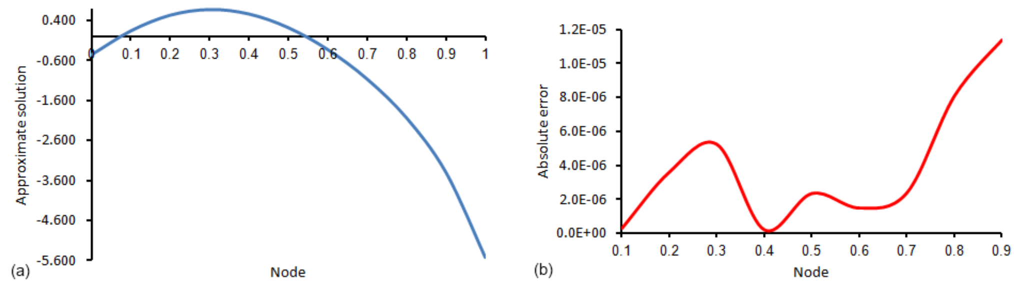

Figure 2: Graphical results of $u'(x)$ of Example 15: (a) approximate solution (b) absolute error.

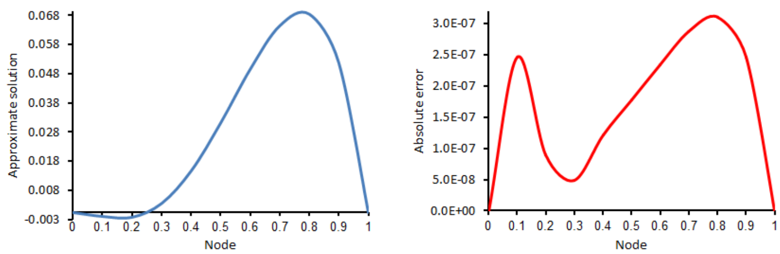

Figure 3: Graphical results of $u''(t)$ of Example 15: (a) solution solution, (b) absolute error.

6 Concluding remarks

This article is presented in the RKHS algorithm as a novel solver for a class of singular BVPs restricted by Fredholm-Volterra operators. This algorithm and its conjugate operator are employed to construct the complete orthonormal basis in the reproducing kernel space $\Pi_3[0,1]$. As well, it is applied in a direct way without using linearization, perturbation, or any restrictive assumptions. we can conclude that the RKHS algorithm is very powerful and efficient tool in finding analytical-numerical solutions for a wide class of such models arising in sciences and engineering. Besides, for numerical experiments, higher accuracy can be achieved in computing further RKHS iterations. Thus, the RKHS algorithm is capable of reducing the volume of the computational work and complexity while still maintaining the high accuracy of the numerical results.

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NEARLY SEMI -2-ABSORBING SUBMODULES AND RELATED CONCEPTS

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Abstract. In this article, R is commutative ring with identity and Y is a left unitary R-module. A proper submodule L of Y is called nearly semiprime submodule if whenever $r^n y \in L$, where $r \in R$ and $y \in Y$, $n \in Z^+$, implies that $ry \in L + J(Y)$, where $J(Y)$ is the Jacobson radical of Y. This concept in courage us to introduce the concept nearly semi-2-absorbing submodule as a generalization of nearly semiprime submodule, where a proper submodule L of Y is called nearly semi-2-absorbing submodule of Y if whenever $a^2 y \in L$, where $a \in R$, $y \in Y$, implies that either $ay \in L + J(Y)$ or $a^2 \in [L : Y]$. Many basic properties, and characterization of this concept are introduce. On the other hand the relation of this concept with other classes of modules are studied.
Keywords: semiprime submodule, semi 2-absorbing submodule, good ring.

1. Introduction

The notion of prime submodule was introduce by [6] . where a proper submodule N of an R-module Y is called prime submodule, if whenever $ry \in N$, $r \in R$, $y \in Y$, implies that either $y \in N$ or $r \in [N : Y]$ [6], where $[N : Y] = \{r \in R : rY \subseteq N\}$. Semiprime submodule as a generalization of prime submodule was introduced by [1], where a proper submodule N of Y is called semiprime if whenever $r^n y \in N$, $r \in R$, $y \in Y$, $n \in Z^+$ implies that $ry \in N$ [1]. This concept generalized in [7] to nearly semi prime sub module, where a proper sub module N of Y is called nearly semiprime if whenever $r^n y \in N$, $r \in R$, $y \in Y$, $n \in Z^+$ implies that $ry \in N + J(Y)$. Also, semiprime submodule generalized to semi -2- absorbing submodule in [3] ,where a proper submodule N of Y is called semi -2-absorbing sub module of Y if whenever $a^2 y \in N$, where $a \in R$, $y \in Y$, implies that either $ay \in N$ *or* $a^2 \in [N : Y]$. This led us to introduce the

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concept of nearly semi -2 absorbing sub module as a generalization of nearly semiprime submodule, where a proper submodule N of Y is called nearly semi-2-absorbing if whenever $a^2y \in N$, where $a \in R,\ y \in Y$, implies that either $y \in N + J\,(Y)$ *or* $a^2 \in [N + J\,(Y) : Y]$. We give many properties, characterizations and relationship between nearly semi-2-absorbing and other concepts.

2. Nearly semi -2-absorbing sub modules

We investigate in this section, the concept of nearly semi -2-absorbing sub module as a generalization of nearly semi prime sub module.

Definition 2.1. A proper submodule L of an R-module Y is called nearly Semi -2- absorbing, if whenever $a^2y \in L$, where $a \in R,\ y \in Y$, implies that either $ay \in L + J\,(Y)$ or $a^2 \in [L + J\,(Y) : Y]$.

A proper ideal J of a ring R is called nearly semi-2-absorbing ideal if J is nearly semi-2-absorbing sub module of an R-module R.

Proposition 2.2. *If L is a nearly semiprime submodule of an R-module Y, then L is nearly semi -2-absorbing sub module of Y.*

Proof. Assume that $a^2y \in L,\ a \in R\ ,\ y \in Y\ ,\ with\ a^2 \notin [L + J\,(Y) : Y]$, since L is a nearly semi prime , it follows that $ay \in L + J(Y)$. Thus L is nearly semi-2-absorbing sub module of Y.

The converse of Proposition 2.2 is not true in general, so the following example explain that.

Example 2.3. Consider the submodule $L = 4Z$ of the Z-module Z , L is nearly semi-2-absorbing but not nearly semi prime because if $2^2 \times 1 \in 4Z$, where $2 \in Z$, $1 \in Z$, implies that $2 \times 1 \notin 4Z + J\,(Z)$, but $2^2 \in [4Z + J\,(Z) : Z] = 4Z$. On the other hand if $2^2 \times 1 \in 4Z$, implies that $2 \times 1 \notin 4Z + J(Z)$ hence $4Z$ is not nearly semiprime Z-sub module.

Lemma 2.4 ([1]). *A proper submodule L of an R-module Y is semiprime if and only if whenever $a^2y \in L$, where $r \in R,\ y \in Y$, then $ry \in L$.*

Remark 2.5. Every semiprime submodule of an R-module Y is nearly semi -2-absorbing, but the converse is not true in general.

Proof. Since every semiprime submodule is nearly semi prime[7], hence the proof is follows by Proposition 2.3.

For the converse ,consider the following example:

Consider the sub module $(\overline{0})$ of the Z-module Z_4 . $(\overline{0})$ is nearly semi-2-absorbing but not semiprime submodule of Z_4 because if $2^2 \times 1 \in \left(\overline{0}\right)$ then $2 \times 1 \notin (\overline{0})$ hence $(\overline{0})$ is not semiprime, but $2^2 \in \left[\left(\overline{0}\right) + J\,(Z_4) : Z_4\right] = \left[\left(\overline{0}\right) + \left\{\overline{0}, \overline{2}\right\} : Z_4\right] = 2Z$, hence $(\overline{0})$ is nearly semi-2-absorbing in Z_4.

Proposition 2.6. *Let K be a submodule of an R-module Y, with $K + J(Y)$ is nearly semi-2-absorbing , then K is nearly semi-2-absorbing.*

Proof. Assume that $r^2y \in K$,*where* $r \in R$, $y \in Y$, then $r^2y \in K + J(y)$. But $K + J(Y)$ is nearly semi-2-absorbing then either $ry \in K + ?(Y) + J(Y) = K + J(Y)$ or $r^2 \in [K + J(Y) + J(Y) : Y]$ =$[K + J(Y) : Y]$. Hence K is nearly semi-2-absorbing in Y.

Lemma 2.7 ([5]). *Let N and K be two sub modules of an R-module Y, then*

1. *If $N \subset K$, then $[N : M] \subseteq [K : M]$.*
2. *If $N \subset K$,then $[N : M] \subseteq [N : K]$.*

Proposition 2.8. *Let E and F be two submodules of an R-module Y with $E \subsetneq F$. If E is a nearly semi-2-absorbing submodule of Y and $J(Y) \subseteq J(F)$, then E is nearly semi-2-absorbing in F.*

Proof. Assume that $r^2y \in E$, with $r \in R$, $y \in F$, since E is nearly semi-2-absorbing sub module of Y, then $ry \in E + J(Y)$ or $r^2 \in [E + J(Y) : Y]$. But $J(Y) \subseteq J(F)$, it follows that $ry \in E + J(F)$ or $r^2 \in [E + J(F) : Y] \subseteq [E + J(F) : F]$ by Lemma 2.7. Hence E is nearly semi-2-absorbing in F.

Remark 2.9. Every prime submodule of an R-module Y is nearly -2-absorbing submodule of Y ,while the converse is not true.

Proof. Since prime submodule is semiprime [1], hence the proof follows by Remark 2.5. For the converse consider the following example, let $Y = Z, R = Z$. $L = 9Z$ a submodule of $Y, 9Z$ is nearly semi-2-absorbing of Z since $3^2 \times 1 \in 9Z$ it follows that $3^2 \in [9Z + J(Z) : Z] = 9Z$. But $9Z$ is not prime submodule of Z, since $3 \times 3 \in 9Z$ but $3 \notin 9Z$ and $3 \notin [9Z : Z] = 9Z$.

Recall that a proper submodule L of an R-module is 2-absorbing if whenever $aby \in L$, with $a, b \in R$, $y \in Y$ implies that either $ay \in L$ or $by \in L$ or $ab \in [L : Y]$ [2].

Proposition 2.10. *Every 2-absorbing submodule of an R-module Y is nearly semi-2-absorbing submodule of Y.*

Proof. Let L be 2-absorbing submodule of Y, and $r^2y \in L$, where $r \in R$, $y \in Y$. Since L is 2-absorbing it follows that either $ry \in L$ or $r^2 \in [L : Y]$. That is either $ry \in L + J(Y)$ or $r^2 \in [L : Y] \leq [L + J(Y) : Y]$ by Lemma 2.7. Thus L is nearly semi-2-absorbing.

Proposition 2.11. *Let Y be an R-module, and K be a proper submodule of Y with $J(Y) \subseteq K$. Then K is semi-2-absorbing iff K is nearly semi-2-absorbing.*

Proof. $\Rightarrow$ Assume that $r^2y \in K$, where $r \in R$, $y \in Y$. Then either $ry \in K \subseteq K + J(Y)$ or $r^2 \in [K\ :Y] \subseteq [K + J(Y) : Y]$. Hence either $ry \in K + J(Y)$ or $r^2 \in [K + J(Y) : Y]$. Thus K is nearly semi-2-absorbing.

$\Leftarrow$ Assume that $a^2y \in K$ with $a \in R$, $y \in Y$. Then either $ay \in K + J(Y)$ or $a^2 \in [K + J(Y) : Y]$. But $J(Y) \subseteq K$, it follows that $K + J(Y) = K$. Hence either $ay \in K$ or $a^2 \in [K\ :Y]$. Thus K is semi-2-absorbing.

Remark 2.12. The intersection of two nearly semi-2-absorbing submodules of an R-module Y need not to be nearly semi-2-absorbing submodules. For example let $Y = Z$ and $R = Z, L = 2Z, K = 9Z$ are nearly semi-2-absorbing submodules of Y, but $2Z \cap 9Z = 18Z$ is not nearly semi-2-absorbing submodule of Y, since $3^2 \times 2 \in 18Z$, but $3 \times 2 \notin 18Z + J(Z)$ and $3^2 \notin [18Z + J(Z) : Z]$.

Proposition 2.13. *Let L be a nearly semi-2-absorbing submodule of an R-module Y, and K is a proper submodule of Y with $K \nsubseteq L$ and $J(K) = J(Y)$, then $L \cap K$ is nearly semi-2-absorbing submodule in K.*

Proof. Since $K \nsubseteq L$, then $L \cap K < K$. Let $r^2y \in\ K \cap L$, where $r \in R$, $y \in K$. Since L is nearly semi-2-absorbing and $r^2y \in\ L$, it follows that either $ry \in L + J(Y)$ or $r^2 \in [L + J(Y) : Y]$, but $J(K) = J(Y)$ then $ry \in L + J(K)$ or $r^2 \in [L + J(K) : Y]$. Since $y \in K$, then $ry \in K$, hence $ry \in (L + J(K)) \cap K$, it follows that $ry \in (L \cap K) + J(K)$, or $r^2 \in [L + J(K) : Y]$, implies that $r^2y \in L + J(Y)$ for all $y \in Y$. Since $ry \in K$, then $r^2y \in\ K$. Hence $r^2y \in (L + J(K)) \cap K$, implies that $r^2y \in (L \cap K) + J(K)$, hence $r^2 \in [(L \cap K) + J(K) : Y] \subseteq [(L \cap K) + J(K) : K]$. That is $r^2 \in [(L \cap K) + J(K) : K]$. Hence $L \cap K$ is nearly semi-2-absorbing in K.

Recall that a ring R is good ring if $J(R)Y = J(Y)$ where Y is an R-module [4].

Remark 2.14 ([4])**.** If R is good ring, then $J(Y) \cap N = J(N)$, where Y is an R-module, N submodule of Y.

Lemma 2.15 ([4], Lemma 2.3.15)**.** *Let Y be an R-module , and L, F and E are submodule of Y with $F \subsetneq E$. Then $(L+F) \cap E = (L \cap E)+F = (L \cap E) + (F \cap E)$.*

Proposition 2.16. *Let Y be an R-module over good ring, and L is nearly semi-2-absorbing submodule of Y, and K be a proper submodule of Y with $K \nsubseteq L$ and $J(Y) \leq K$, then $(L \cap K)$ is nearly semi-2-absorbing in K.*

Proof. Since $K \nsubseteq L$, then $(L \cap K) \subset K$. Assume that $r^2y \in (L \cap K)$, where $r \in R$, $y \in K$. Since L is a nearly semi-2-absorbing, and $r^2y \in L$, then either $ry \in L + J(Y)$ or $r^2 \in [L + J(Y) : Y]$. Since $y \in K$ then $ry \in K$ and $r^2y \in K$. Hence $ry \in (L + J(Y)) \cap K$ or $r^2y \in (L + J(Y)) \cap K$ for all $y \in Y$. Thus by Lemma 2.15, $ry \in (L \cap K) + J(Y) \cap K$ or $r^2y \in (L \cap K) + (J(Y) \cap K)$. Since R is good ring, then $J(Y) \cap K = J(K)$. That is $ry \in (L \cap K) + J(K)$ or $r^2y \in (L \cap K) + J(K)$. Hence $ry \in (L \cap K) + J(K)$ or

$r^2 \in [(L \cap K) + J(K) : Y] \leq [(L \cap K) + J(K) : K]$ by Lemma 2.7, (2), implies that $r^2 \in [(L \cap K) + J(K) : K]$. Thus $L \cap K$ is nearly semi-2-absorbing in K.

Proposition 2.17. *Let L and K be nearly semi-2-absorbing proper submodules of an R-module Y, with $K \nsubseteq L$ and either $J(Y) \subseteq L$ or $J(Y) \subseteq K$, then $L \cap K$ is semi-2-absorbing of Y.*

Proof. Since $K \nsubseteq L$, then $L \cap K \subsetneq L \subsetneq Y$, it follows that $L \cap K \subsetneq Y$. Assume that $r^2 y \in L \cap K$, $r \in R$, $y \in Y$. Then $r^2 y \in K$ and $r^2 y \in L$, but both K and L are nearly semi-2-absorbing in Y, then either $ry \in K + J(Y)$ or $r^2 \in [K + J(Y) : Y]$ and either $ry \in L + J(Y)$ or $r^2 \in [L + J(Y) : Y]$. It follows that either $ry \in (K + J(Y)) \cap (L + J(Y))$ or $r^2 \in [K + J(Y) : Y] \cap [L + J(Y) : Y]$. If $J(Y) \subseteq L$, then $ry \in (K + J(Y)) \cap L$, it follows that by Lemma 2.15, $ry \in (L \cap K) + J(Y)$. If $J(Y) \subseteq K$, then it follows that by Lemma 2.15, $ry \in (L + J(Y)) \cap K$, then $ry \in (L \cap K) + J(Y)$. Also $r^2 \in [K + J(Y) : Y] \cap [L + J(Y) : Y] \subseteq [(L \cap K) + J(Y) : Y]$, implies that $r^2 \in [L \cap K + J(Y) : Y]$. That is either $ry \in (L \cap K) + J(Y)$ or $r^2 \in [L \cap K + J(Y) : Y]$. Hence $L \cap K$ is nearly semi-2-absorbing submodule of Y.

Proposition 2.18. *Let Y be an R-module over a good ring R, and L, K be submodules of Y, with $L \subsetneq K$ and $J(Y) \subseteq K$. If L is a nearly semi-2-absorbing submodule of Y, then L is a nearly semi-2-absorbing submodule of K.*

Proof. Assume that $r^2 y \in L$, $r \in R$, $y \in K$, since L is a nearly semi-2-absorbing submodule of Y, and $y \in K \subseteq Y$, then either $ry \in L + J(Y)$ or $r^2 \in [L + J(Y) : Y]$. But $y \in K$, then $ry \in K$. Hence either $ry \in (L + J(Y)) \cap K$ or $r^2 y \in (L + J(Y)) \cap K$ for each y in K. Hence either $ry \in (L \cap K) + J(Y) \cap K$ or $r^2 y \in (L \cap K) + (J(Y) \cap K)$. Since R is a good ring, and $L \subsetneq K$, it follows that either $ry \in L + J(K)$ or $r^2 y \in L + J(K)$. Hence either $ry \in L + J(K)$ or $r^2 \in [L + J(K) : K]$. Thus L is a nearly semi-2-absorbing in K.

Proposition 2.19. *Let E be a submodule of an R-module Y. Then $E + J(Y)$ is a nearly semi-2-absorbing submodule of Y if and only if $\left[E + J(Y) : r^2 y\right] = [E + J(Y) : ry]$ for each $y \in Y$ or $r^2 \in [E + J(Y) : Y]$.*

Proof. $\Rightarrow$Assume that $r^2 \notin [E + J(Y) : Y]$. To prove that $\left[E + J(Y) : r^2 y\right] = [E + J(Y) : ry]$. It is clear that $[E + J(Y) : ry] \subseteq \left[E + J(Y) : r^2 y\right]$. Now, let $a \in \left[E + J(Y) : r^2 y\right]$, then $r^2 a y \in E + J(Y)$ since $E + J(Y)$ is nearly semi-2-absorbing in Y and $r^2 \notin [E + J(Y) : Y]$ so $a \in [E + J(Y) : ry]$. Thus $\left[E + J(Y) : r^2 y\right] = [E + J(Y) : ry]$.

$\Leftarrow$ Let $r^2 y \in E + J(Y)$, by hypothesis $[E + J(Y) : r^2 y] = [E + J(Y) : ry]$ or $r^2 \in [E + J(Y) : Y]$. If $[E + J(Y) : r^2 y] = [E + J(Y) : ry]$, then $[E + J(Y) : r^2 y] = R$ because $r^2 y \in E + J(Y)$. Implies that $[N + J(Y) : ry] = R$ and hance $ry \in E + J(Y)$. Thus either $ry \in E + J(Y)$ or $r^2 \in [E + J(Y) : Y]$. Hence $E + J(Y)$ is nearly semi 2-absorbing in Y.

Proposition 2.20. *Let Y be an R-module, and E be a submodule of Y. Then E is a nearly semi 2-absorbing in Y if and only if $r^2F \subseteq E$, implies that $rF \subseteq E + J(Y)$ or $a^2 \in [E + J(Y) : Y]$.*

Proof. $\Rightarrow$ Assume that $r^2F \subseteq E$, and suppose that there exist $y \in F$ such that $ry \notin E + J(Y)$. Since $r^2F \subseteq E$, so $r^2m \in E$ for each $m \in F$. But E is a nearly semi 2-absorbing in Y, and $ry \notin E + J(Y)$. Hence $r^2 \in [E + J(Y) : Y]$.

$\Leftarrow$ It is clear.

Proposition 2.21. *Let E be a submodule of an R-module Y such that $J(Y) \subseteq E$. If E is nearly semi 2-absorbing of Y, then$[E : Y]$ is a semi 2-absorbing ideal in R.*

Proof. Assume that $a^2b \in [E : Y]$, $a, b \in R$, implies that $a^2by \in E$ for each $y \in Y$. But E is a nearly semi 2-absorbing in Y, then either $aby \in E + J(Y)$ or $a^2 \in [E + J(Y) : Y]$, But $J(Y) \subseteq E$, it follows that $aby \in E$ or $a^2 \in [E : Y]$. That is $ab \in [E : Y]$ or $a^2 \in [E : Y]$.

$\Leftarrow$ Assume that $a^2y \in E$, $a \in R$, $y \in Y$, with the converse Proposition 2.21 hold under the class of cyclic modules.

Proposition 2.22. *Let E be a proper submodule of cyclic module Y. If $[E : Y]$ is semi 2-absorbing ideal of R, then E is a nearly semi 2-absorbing submodule of Y.*

Proof. Assume that $[E : Y]$ is semi 2-absorbing ideal of R then by [3], we have E is a semi 2-absorbing submodule of Y. Hence by Proposition 2.11, we get E is nearly semi 2-absorbing submodule of Y.

Corollary 2.23. *Let E be a proper submodule of cyclic R-module Y with $J(Y) \subseteq E$. Then E is a nearly semi 2-absorbing submodule of Y if and only if $[E : Y]$ is a semi 2-absorbing ideal of R.*

Proposition 2.24. *Let $\varphi : Y \longrightarrow Y$ be an R-epimorphism with $Ker\ \varphi \subseteq E$ whene E is a proper submodule of Y. Then*

1. *If E is a nearly semi 2-absorbing in Y, then $\varphi(E)$ is a nearly semi 2-absorbing submodule in $Y'e$.*
2. *If E^1 is a nearly semi 2-absorbing submodule in $Y'e$ and $Ker\varphi$ is small submodule in Y, then$\varphi^{-1}(E'e)$ is a nearly semi 2-absorbing submodule in Y.*

Proof.

1. $\varphi(E)$ is a proper submodule of $Y'e$, if not, that is $\varphi(E) = Y'e$, then for each $y \in Y$,$\varphi(y) \in Y'e = \varphi(E)$, implies that $\varphi(y) = \varphi(\eta)$ for some $n \in E$, hence $\varphi(y - n) = 0$, then $y - n \in Ker\varphi \subseteq E$, implies that $y \in E$,

hence E=Y contradiction (since $E \subsetneq Y$) Now, assume that $r^2 y'e \in \varphi(E)$, $r \in R, y'e \in Y'e$, since φ is onto, then there exist $y \in Y$ such that $\varphi(y) = y'e$ hence $r^2\varphi(y) \in \varphi(E)$, that is $\varphi(r^2 y) \in \varphi(E)$ then there exist $e \in E$ such that $\varphi(r^2 y) = \varphi(e)$, hence $\varphi(e - r^2 y) = 0$, implies that $e - r^2 y \in Ker\varphi \subseteq E$, hence $r^2 y \in E$, but E is a nearly semi 2-absorbing in Y, then either $ry \in E + J(Y)$ or $r^2 \in [E + J(Y) : Y]$. That is $ry \in E + J(Y)$ or $r^2 Y \in E + J(Y)$. It follows $r\varphi(y) \in \varphi(E) + \varphi(J(Y))$ or $r^2\varphi(y) \subseteq \varphi(E) + \varphi(J(Y))$. Hence $ry'e \in \varphi(E) + \varphi(J(Y'e))$ or $r^2 y'e \subseteq \varphi(E) + (J(Y'e))$. Hence either $ry'e \in \varphi(E) + J(Y')$ or $r^2 \in [\varphi(E) + J(Y'e) : Y'e]$. Therefore $\varphi(E)$ is a nearly semi 2-absorbing submodule in$Y'e$.

2. It is clear that $\varphi(E'e) \subsetneq Y$.Now, assume that $r^2 y \in \varphi^{-1}(E'e)$, $r \in R$, $y \in Y$, then $r^2\varphi(y) \in E'e$, implies that $r^2 y'e \in E'e$, $y'e \in Y'e$, since $E'e$ is a nearly semi 2-absorbing in $Y'e$, then either $ry'e \in E'e + J(Y'e)$ or $r^2 \in [E'e + J(Y'e) : Y'e]$. That is either $\varphi(ry) \in E + J(Y'e)$ or $r^2 y'e \subseteq E'e + (J(Y'e))$. Thus either $\varphi(ry) \in E + J(Y'e)$or $\varphi(r^2 y) \subseteq E'e + J(Y'e)$. It follows that either $ry \in \varphi^{-1}(E'e) + \varphi^{-1}(J(Y'e))$ or $r^2 y \subseteq \varphi^{-1}(E'e) + \varphi^{-1}(J(Y'e))$. Hence either $ry \in \varphi^{-1}(E'e) + J(Y)$ or $r^2 \in [\varphi^{-1}(E'e) + J(Y) : Y]$. Therefore $\varphi^{-1}(E'e)$ is a nearly semi 2-absorbing in Y.

Proposition 2.25. *Let E be a proper submodule of an R-module Y such that E is a nearly semi-2-absorbing submodule of Y, then $S^{-1}E$ is a nearly semi-2-absorbing submodule of $S^{-1}R$-module $S^{-1}Y$.*

Proof. Assume that$(\overline{a})^2\overline{y} \in S^{-1}E$ where $\overline{a} = \frac{a}{s_1} \in S^{-1}R$ and $\overline{y} = \frac{y}{s_2} \in S^{-1}Y$, $a \in R$, $y \in Y$, s_1, $s_2 \in S$. Hence $\frac{a^2 y}{{s_1}^2 s_2} \in S^{-1}E$, implies that $\frac{a^2 y}{t} \in S^{-1}E$, t $={s_1}^2 s_2$, then there exists $t_1 \in S$ such that $a^2 t_1 y \in E$. Since E is a nearly semi-2-absorbing in Y, then either $at_1 y \in$E+$J(Y)$ or $a^2 t_1 \in$ [E+$J(Y) : Y$], it follows that either $\frac{a^2 t_1 y}{{s_1}^2 t_1 s_2} \in S^{-1}$ [E+$J(Y)$] or $\frac{a^2 t_1}{{s_1}^2 t_1} \in S^{-1}$ [E+$J(Y) : Y$]. Hence either $(\overline{a})^2\overline{y} \in S^{-1}E + S^{-1}(J(Y))$ or $(\overline{a})^2 \in [S^{-1}(E) + S^{-1}(J(Y)) : S^{-1}Y]$, implies that either $(\overline{a})^2\overline{y} \in S^{-1}E + J(S^{-1}Y)$ or $(\overline{a})^2 \in [S^{-1}(E) + J(S^{-1}Y) : S^{-1}Y]$. Therefore $S^{-1}E$ is a nearly semi-2-absorbing submodule of $S^{-1}Y$.

Proposition 2.26. *Let $Y = Y_1 \oplus Y_2$ be an R-module, where Y_1, Y_2 are R-modules, and let E and F be a proper submodules of Y_1 and Y_2 respectively, then*

1. *E is a nearly semi-2-absorbing submodule in Y_1 if and only if $E \oplus Y_2$ is a nearly semi-2-absorbing in Y.*

2. *F is a nearly semi-2-absorbing in Y_2 if and only if $Y_1 \oplus F$ is a nearly semi-2-absorbing in Y.*

Proof. It is easy, we omitted.

Proposition 2.27. *If E and F are nearly semi-2-absorbing submodules in Y_1 and Y_2, respectively such that*$[E + J(Y_1) : Y_1] = [F + J(Y_2) : Y_2]$. *Then* $K = E \oplus F$ *is a nearly semi-2-absorbing submodule in R-module* $Y = Y_1 \oplus Y_2$, *where* Y_1, Y_2 *are R-modules.*

Proof. Assume that $r^2(y_1, y_2) \in E \oplus F$, $a \in R$, $(y_1, y_2) \in Y$, $y_1 \in Y_1$, $y_2 \in Y_2$, implies that $r^2 y_1 \in E$ and $r^2 y_2 \in F$. Since E and F are nearly semi-2-absorbing, then either $ry_1 \in E + J(Y_1)$ or $r^2 \in [E + J(Y_1) : Y_1]$ and either $ry_2 \in F + J(Y_2)$ or $r^2 \in [F + J(Y_2) : Y_2] = [E + J(Y_1) : Y_1]$ so $ry_1 \in E + J(Y_1)$ and $ry_2 \in F + J(Y_2)$ or $r^2 \in [E + J(Y_1) : Y_1]$. Thus $r(y_1, y_2) \in (E + J(Y_1)) \int (F + J(Y_2))$ or $r^2 \in [E \oplus F + J(Y_1 + Y_2) : Y_1 + Y_2]$. It follows that either $r(y_1, y_2) \in E \oplus F + J(Y_1 + Y_2)$ or $r^2 \in [E \oplus F + J(Y_1 + Y_2) : Y_1 + Y_2]$.

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RING FORMS

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Abstract. Determinant on a commutative ring of characteristics p can be extended by a linear mapping to provide a trilinear alternating form. We show some basic properties of such forms.

If the underlying ring is a chain-ring, we compute dimensions of radicals of all vectors and thus prove nonequivalence of forms arising from chain-rings with different sizes of ideals. Moreover, in the case $p = 2$ we show that all three nondegenerate forms on dimension 6 are ring forms.

Keywords: trilinear alternating form, commutative ring, chain ring.

1. Introduction

Let $f : V^3 \to F$ be a trilinear form on a vector space V over a field F, $\dim V = n < \infty$. The form f is called *alternating* if $f(u, v, w) = 0$ whenever two of the input vectors are equal. Two forms f and g on V are *equivalent* if there exists an automorphism of V satisfying $f(u, v, w) = g(\phi(u), \phi(v), \phi(w))$ for all $u, v, w \in V$. Classification of classes of this equivalence seems to be a very difficult problem (unlike in the bilinear case) even for small dimensions of V and not much has been done in this respect. This classification was done for the case $n \leq 7$ in [1] for a large family of fields including all finite fields and Gurevitch [2], D. Djokovic [3] and L. Noui [4] solved the case $n = 8$ for $F = \mathbf{C}$, $F = \mathbf{R}$ and F algebraically closed field of arbitrary characteristic, respectively. The case of dimension 8 over $\mathbf{GF}(2)$ is solved in [7].

Trilinear alternating forms over the two-element field appear as important invariants of doubly even binary codes and thus the accent is put on the case of characteristic 2.

In this paper we study trilinear alternating forms that arise as extensions of determinant over a ring R with a linear mapping $l : R \to \mathbf{GF}(p)$. This construction is a generalization of so called trace-derived forms introduced in [8]. We show when such a form is nondegenerate and when it is decomposable. If the ring R is a chain-ring then there exists a mapping l yielding a nondegenerate form. Two chain-rings with different sizes (or number) of ideals give rise to two nonequivalent forms.

There are only three four-element rings of characteristic 2 and we show that they provide exactly the three nondegenerate forms on dimension 6 over $\mathbf{GF}(2)$.

2. Definitions

A trilinear alternating form f satisfies the equality:

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = \mathrm{sgn}(\sigma) f(v_1, v_2, v_3),$$

for every permutation $\sigma \in S_3$. Since this paper deals mainly with forms over the two-element field, this equation often collapses into symmetry.

We shall denote the bilinear form $f(v, -, -)$ by $f[v]$ and similarly $f[v_1, v_2]$ shall denote the linear form $f(v_1, v_2, -)$.

An automorphism ϕ of V is said to be an *automorphism of the form* f if

$$f(v_1, v_2, v_3) = f(\phi(v_1), \phi(v_2), \phi(v_3)) \text{ for all } v_1, v_2, v_3 \in V.$$

The group of automorphisms of f will be denoted by $\mathrm{Aut}(f)$.

Let f be a trilinear form on V. The set

$$\{x \in V; f[x] = 0\}$$

is called the *radical* of f and will be denoted by $\mathrm{Rad} f$. If $\mathrm{Rad} f$ is trivial (contains only the zero vector), then f is called *nondegenerate*.

Fix $v \in V$ and define the *radical* $\mathrm{Rad}_f(v)$ *of* v as:

$$\mathrm{Rad}_f(v) = \{u \in V; f[v, u] = 0\}.$$

If it is clear which form is meant, we shall omit the index f. The radical of any vector v is clearly a subspace of V. The *rank* of $v \in V$ is the codimension of $\mathrm{Rad}(v)$ in V

$$r(v) = n - \mathrm{dim}\mathrm{Rad}(v).$$

To capture the information about ranks of vectors of forms (over finite fields) we shall use an invariant introduced in [7], called the *radical polynomial*

$$P(f) = \sum_{v \in V} x^{r(v)} y^{n-r(v)}.$$

$P(f)$ is a homogenous polynomial of degree n and if written in the form

$$P(f) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i} \tag{1}$$

then every α_i is a nonnegative integer and $\sum_{i=0}^{n-1} \alpha_i = q^n$. Since for every $u \in V$ we have $u \in \mathrm{Rad}(u)$, the rank $r(u)$ of any vector u is less than n and the sum in (1) can run only to $n-1$. Moreover, by Proposition 2.1 we get α_i is equal to zero whenever i odd.

Suppose that there is a fixed trilinear alternating form f on a vector space V. We say that nonzero vectors $u, v \in V$ are orthogonal, denoted by $u \perp v$, if $u \in \mathrm{Rad}(v)$. This relation is clearly reflexive and symmetric (the form is alternating) but is not necessarily transitive.

We shall use standard notation for forms: Let V be an n-dimensional vector space over a field F and fix a basis $B = \{b_1, \dots, b_n\}$ of V. Denote by $B^* = \{b_1^*, \dots, b_n^*\}$ its dual basis (defined as usual by $b_i^*(b_j) = \delta_{ij}$). Given B and B^* as above, a k-linear alternating form f can be expressed as

$$f_B = \sum_{1 \le i_1 < \dots < i_k \le n} f_{i_1 \dots i_k} b_{i_1}^* \wedge \dots \wedge b_{i_k}^*,$$

where the index B indicates the dependence of the presentation upon the chosen basis. Denote by Δ_f the set

$$\Delta_f = \{(i_1, \dots, i_k) \mid 1 \le i_1 < \dots < i_k \le n, f_{i_1 \dots i_k} \neq 0\}.$$

We shall shorten this notation to $f_B = \sum_{\Delta_f} f_{i_1 \dots i_k} \underline{i_1 \dots i_k}$.

If V is a vector space over the two element field $F = \mathbf{GF}(2)$ and $k = 3$, then to give a form f means to point out triples $\{i, j, k\}$ satisfying $f(b_i, b_j, b_k) = 1$, i.e., to give the set Δ.

In what follows we shall often use the well known characterization of bilinear alternating forms:

Proposition 2.1. *Let f be a bilinear alternating form on a vector space V of dimension n. Then there exists a basis $B = \{b_1, \dots, b_n\}$ and $k \le n$ such that*

$$f_B = \underline{12} + \underline{34} + \dots + \underline{(k-1)k}.$$

For two subspaces V_1 and V_2 of V we write $V_1 \perp V_2$ if $v_1 \perp v_2$ for any $v_1 \in V_1$ and $v_2 \in V_2$. We say that a nondegenerate form f on V is *decomposable* if $V = W_1 \oplus \dots \oplus W_m$, $m \ge 2$, and $W_i \perp W_j$ whenever $i \neq j$. Given an (orthogonal) decomposition of f, let π_i denote the projection of V onto W_i and f_i the restriction of f to W_i. Then we can express the form f as

$$f(u, v, w) = \sum_i f_i(\pi_i(u), \pi_i(v), \pi_i(w)),$$

and we shall write $f = \bigoplus f_i$. The difference between the bilinear case and k-linear case, $k \ge 3$, is that the finest decomposition of a nondegenerate multilinear form is unique, see [6]. On the other hand, there are many indecomposable trilinear forms if the dimension of V is at least six.

3. Ring forms

In this paper we are going to study trilinear forms arising primarily as determinants over a commutative ring R. Since some of the results need the ring to have the identity element, we consider only such rings. Moreover, the additive group $(R, +, -, 0)$ of the ring is assumed to be elementary abelian of prime exponent p. In some cases we shall have $p = 2$. The construction can be even more general:

Lemma 3.1. *Let p be a prime and R be a commutative ring with 1 satisfying $p \cdot r = 0$ for all $r \in R$. Let M be an R-module and let $f : M^3 \to R$ be a triadditive mapping. Let $l : R \to \mathbf{GF}(p)$ be a $\mathbf{GF}(p)$-linear mapping. Then $f_l = l \circ f : M^3 \to GF(p)$ is a trilinear form over $GF(p)$.*

Proof. Straightforward. □

A special case, which we are going to study is when $M = R^3$ and f is the determinant. Throughout this paper we shall call such forms *ring forms* and shall denote them as $d(l)$. Since the determinant is alternating, any ring form is a trilinear alternating form.

Note that in this paper we study only trilinear forms, but the construction could be clearly generalized to k-linear forms.

Let $E_i(r) : R \to R^3$ denote the mapping sending r to the triple (r_1, r_2, r_3), where $r_i = r$ and $r_j = 0$, $j \neq i$.

A degenerate form f can be factored by its radical $\mathrm{Rad} f = \{u, f[u] \equiv 0\}$ to obtain a nondegenerate form on a lower dimension. Thus we first determine a condition under which a ring form is nondegenerate.

Lemma 3.2. *Let $f = d(l)$ be a ring form. Then f is nondegenerate iff*

$$rR^2 = rR \not\subseteq \mathrm{Ker} l \text{ for every } 0 \neq r \in R. \tag{2}$$

Proof. If $rR \subseteq \mathrm{Ker} l$ for some $r \in R$ then clearly $u = (r, 0, 0)$ is in the radical of f. On the other hand, if $u = (r_1, r_2, r_3)$ is in the radical of f, then $f(u, E_2(1), E_3(s)) = l(r_1 \cdot s) = 0$ for every $s \in R$. □

Second step in the classification of trilinear forms is a decomposition to pairwise orthogonal subspaces. Propositions 3.3 and 3.6 show the connection between the decomposability of the ring form and the ring itself.

Proposition 3.3. *Let R_i be a ring of exponent p and let $d(l_i)$ be a nondegenerate ring form on $V_i = R_i^3$, $i \in \{1, 2\}$. Then there exists a linear mapping $l : R_1 \times R_2 \to \mathbf{GF}(p)$ such that $d(l)$ is a nondegenerate form on $R_1 \times R_2$ and $d(l) = d(l_1) \oplus d(l_2)$.*

Proof. Denote by W_i the kernel of l_i. By Lemma 3.2 we have $r_i R_i \not\subseteq W_i$ for every nonzero $r_i \in R_i$, $i \in \{1, 2\}$. Choose $x_i \in R_i$ such that $R_i = W_i \vee \langle x_i \rangle$ as a vector space, $i \in \{1, 2\}$. Put $W = W_1 \vee W_2 \vee \langle x_1 + x_2 \rangle$, which is clearly a hyperplane in $R_1 \times R_2$. Consider any nonzero $r = (r_1, r_2) \in R$. Without loss of generality we assume $r_1 \neq 0 \in R_1$. By assumption there exists an element $s_1 \in R_1$ such that $r_1 s_1 \notin W_1$. We prove that $(r_1, r_2) \cdot (s_1, 0) \notin W$. Suppose the contrary. Then

$$(r_1, r_2) \cdot (s_1, 0) = (r_1 s_1, 0) = a(w_1, 0) + b(0, w_2) + c(x_1, x_2), \tag{3}$$

$w_i \in W_i$ and $a, b, c \in \mathbf{GF}(p)$. From the second coordinate we get $0 = bw_2 + cx_2$ which implies $b = c = 0$ by the choice of x_2. Thus the equation (3) collapses to $r_1 s_1 = aw_1$, a contradiction. Finally, the mapping l can be any nonzero linear mapping with kernel W. □

Lemma 3.4. *Let $f = d(l)$ be a nondegenerate ring form on $V = R^3$. Then vectors $u = (r_1, r_2, r_3)$ and $v = (s_1, s_2, s_3)$ are orthogonal iff for every $i \neq j$ we have $r_i s_j - r_j s_i = 0$.*

Proof. Set $k = 6 - i - j$ and consider a vector $w = E_k(r)$. By the orthogonality of u and v we have $f(u, v, w) = l((r_i s_j - r_j s_i)r) = 0$ for any $r \in R$. Since f is assumed to be nondegenerate, using Lemma 3.2 yields the result. □

Lemma 3.5. *Let $R = R_1 \times R_2$ be a decomposable commutative ring. Then I is an ideal of R iff it is a direct sum $I_1 \times I_2$, I_i ideal of R_i, $i \in \{1, 2\}$. Moreover, I is principal iff both I_1 and I_2 are principal.*

Proposition 3.6. *Let $f = d(l)$ be a nondegenerate ring form on $V = R^3$. If f is decomposable then $R = R_1 \times R_2$ is a decomposable ring and $f = d(l_1) \oplus d(l_2)$, where l_i is a restriction of l to R_i, $i \in \{1, 2\}$.*

Proof. Let $V = V_1 \times V_2$ be an orthogonal decomposition of f. Then we can write the vector $E_1(1) = (1, 0, 0)$ as $(1, 0, 0) = (r_1, s_1, t_1) + (r_2, s_2, t_2)$, $u_i = (r_i, s_i, t_i) \in V_i$. We get $r_2 = 1 - r_1$, $s_2 = -s_1$ and $t_2 = -t_1$. Since the vectors u_1 and u_2 are orthogonal, we get by Lemma 3.4 $-r_1 s_1 - (1 - r_1)s_1 = 0$ and thus s_1 is equal to 0. By symmetry we get $t_1 = 0$. Denote r_1 by just r. Similarly we get the decomposition of $E_2(1)$ as $(0, 1, 0) = (0, s, 0) + (0, 1 - s, 0)$, $v_1 = (0, s, 0) \in V_1$ and $v_2 = (0, 1 - s, 0) \in V_2$. Using again Lemma 3.4 for pairs u_1, v_2 and u_2, v_1 we obtain equations

$$r(1 - s) = 0 \text{ and } s(1 - r) = 0, \tag{4}$$

respectively, and combining them we get $r = s$. Moreover, the equations (4) imply that r (and thus $1 - r$) is an idempotent and $R = rR \oplus (1 - r)R$.

It remains to prove that the restrictions of l to R_i satisfy the condition of Lemma 3.2. Since $\operatorname{Ker} l$ does not contain any principal ideal of R, we must have $\operatorname{Ker} l \cap R_1 \neq R_i$, otherwise $\operatorname{Ker} l$ would contain the ideal $R_1 \times 0$. Thus l_i is a nontrivial linear mapping and by Lemma 3.5 does not contain any principal ideal of R_i. □

Example. There are three nonisomorphic rings of order 4 satisfying the condition $r + r = 0$, namely $\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{GF}(2)[x]/(x^2)$ and $\mathbf{GF}(4)$.

The ring $\mathbf{Z}_2 \times \mathbf{Z}_2$ is decomposable and thus by Proposition 3.6 yields a decomposable form whenever l satisfies the condition of Lemma 3.2. The kernel of l is by Proposition 3.3 equal to $\mathrm{Ker}\, l = \{0, 1\}$ and we get the form $f_3 = \underline{123} + \underline{456}$ (the numbers of forms in this example correspond to the numbers used in [7]). The radical polynomial is compatible with the direct sum of forms (see [7]). Thus the radical polynomial of this form is a product of two radical polynomials of simple determinant. Radical polynomials for the other two forms can be computed using Proposition 3.9, see Table 1.

Now, consider the ring $\mathbf{GF}(2)[x]/(x^2)$. Setting $b_1 = E_1(1)$, $b_2 = E_3(x)$, $b_3 = E_2(1)$, $b_4 = E_1(x)$, $b_5 = E_3(1)$, $b_6 = E_2(x)$ and $\mathrm{Ker}\, l = \{0, 1\}$ we get exactly the form $f_4 = \underline{123} + \underline{345} + \underline{156}$.

For the underlying ring equal to $\mathbf{GF}(4)$ with elements $0, 1, \alpha, \alpha + 1$ setting $b_1 = E_1(1)$, $b_2 = E_2(1)$, $b_3 = E_3(1)$, $b_4 = E_1(\alpha + 1)$, $b_5 = E_2(\alpha)$, $b_6 = E_3(\alpha)$ and $\mathrm{Ker}\, l = \{0, \alpha\}$ yields the form $f_{10} = \underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156}$.

We see that all three nondegenerate forms on dimension 6 over $\mathbf{GF}(2)$ are ring forms.

R	f	$P(f)$
$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\underline{123} + \underline{456}$	$(y^3 + 7x^2y)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^2)$	$\underline{123} + \underline{345} + \underline{156}$	$y^6 + 7x^2y^4 + 56x^4y^2$
$\mathbf{GF}(4)$	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156}$	$y^6 + 63x^4y^2$

Table 1: Nondegenerate (ring) forms on dimension 6

Example. Rings of order 8 provide forms on dimension 9 over $\mathbf{GF}(2)$. There are six commutative rings with identity of this order. Three decomposable $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$, $\mathbf{GF}(4) \times \mathbf{Z}_2$ and three indecomposable $\mathbf{GF}(2)[x]/(x^3)$, $\mathbf{GF}(8)$ and $\mathbf{GF}(2)[x, y]/(x^2, y^2, xy)$. The ring $\mathbf{GF}(2)[x, y]/(x^2, y^2, xy)$ contains (two-element) principal ideals (x), (y) and $(x + y)$. Any hyperplane contains at least one of these ideals and thus this ring by Lemma 3.2 does not yield a nondegenerate form.

The decomposable forms are just direct sums of forms on dimension 6 with the determinant on b_7, b_8 and b_9, see Table 2, and thus we get the corresponding radical polynomials given in Table 3. Moreover, these three forms are the only (nondegenerate) decomposable forms on dimension 9, because there is no nondegenerate form on dimension 4 and thus the decomposition of the dimension must be $6 - 3$.

The rings $\mathbf{GF}(2)[x]/(x^3)$ and $\mathbf{GF}(8)$ are both chain rings and thus their radical polynomials can be computed using Proposition 3.9. The form arising from the field $\mathbf{GF}(8)$ has a transitive group of automorphisms (see [8]) and by

R	f
$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$\underline{123} + \underline{456} + \underline{789}$
$\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$	$\underline{123} + \underline{345} + \underline{156} + \underline{789}$
$\mathbf{GF}(4) \times \mathbf{Z}_2$	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156} + \underline{789}$
$\mathbf{GF}(2)[x]/(x^3)$	$\underline{149} + \underline{158} + \underline{167} + \underline{248} + \underline{257} + \underline{347}$
$\mathbf{GF}(8)$	$\underline{147} + \underline{148} + \underline{149} + \underline{157} + \underline{158} + \underline{167} + \underline{169} + \underline{247} +$ $+ \underline{248} + \underline{257} + \underline{259} + \underline{268} + \underline{347} + \underline{349} + \underline{358} + \underline{367}$

Table 2: Nondegenerate ring forms on dimension 9

(not yet published) classification of forms on dimension 9 over $\mathbf{GF}(2)$ we know that there is only one such form on this dimension.

R	$P(f)$
$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$(y^3 + 7x^2y)(y^3 + 7x^2y)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$	$(y^6 + 7x^2y^4 + 56x^4y^2)(y^3 + 7x^2y)$
$\mathbf{GF}(4) \times \mathbf{Z}_2$	$(y^6 + 63x^4y^2)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^3)$	$y^9 + 7x^2y^7 + 56x^4y^5 + 448x^6y^3$
$\mathbf{GF}(8)$	$y^9 + 511x^6y^3$

Table 3: Radical polynomials of ring forms on dimension 9

Now, we shall study forms arising from (commutative) chain rings - i.e., rings whose ideals form a chain. Every ideal is then principal. Moreover, any hyperplane not containing the minimal ideal satisfies the condition of Lemma 3.2, thus providing a nondegenerate form.

Let S be a subset of a ring R. The set $\{x \in R, xs = 0, \forall s \in S\}$ will be denoted by $\mathrm{Ann}(S)$ and is an ideal of R. We shall write $\mathrm{Ann}(s)$ instead of $\mathrm{Ann}(\{s\})$.

Lemma 3.7. *Let R be a commutative ring of exponent p and $r \in R$. Then* $\dim(rR) + \dim(\mathrm{Ann}(r)) = \dim R$.

Proof. The mapping $x \mapsto rx$ is a p-linear mapping from R to R, rR is its image and $\mathrm{Ann}(r)$ is its kernel. □

Lemma 3.8. *Let $d(l)$ be a nondegenerate chain ring form on $V = R^3$ and let $u = (r_1, r_2, r_3)$ be a vector in V. Then the size of* $\mathrm{Rad}(u)$ *is equal to* $|\, R \,| \cdot |\, \mathrm{Ann}(r_1, r_2, r_3)\, |^2$.

Proof. Since we compute only the size of the radical, we can assume without loss of generality that $r_3R \subseteq r_2R \subseteq r_1R$. Thus there are elements $x, y \in R$

such that $r_2 = r_1 x$ and $r_3 = r_1 y$. Consider a vector $v = (s_1, s_2, s_3)$. Then $d(l)(u, v, -)$ is equal to

$$l\left(\begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ - & - & - \end{vmatrix}\right) = l\left(\begin{vmatrix} r_1 & 0 & 0 \\ s_1 & s_2 + s_1 x & s_3 + s_1 y \\ - & - & - \end{vmatrix}\right).$$

By Lemma 3.4 vectors u and v are orthogonal if and only if $r_1(s_2 + s_1 x) = 0$ and $r_1(s_3 + s_1 y) = 0$, in other words if both $s_2 + s_1 x$, $s_3 + s_1 y$ belong to $\mathrm{Ann}(r_1)$. Since x and y are fixed, s_1 can be arbitrary and there are $|\mathrm{Ann}(r_1)|$ suitable vectors s_2 to satisfy $s_2 + s_1 x \in \mathrm{Ann}(r_1)$. Similarly for s_3. □

Proposition 3.9. *Let R be a finite chain ring of exponent p and $d(l)$ a non-degenerate form on n-dimensional vector space R^3. Let $0 = I_0 \subset I_1 \subset \ldots \subset I_m = R$ be the chain of all ideals of R and $0 = n_0 < n_1 < \ldots < n_m = n/3$ their respective dimensions as vector spaces over $\mathbf{GF}(p)$. Then the form $d(l)$ has only vectors of rank $2n_k$, $k \in \{0, \ldots, m\}$, and the number of such vectors is $p^{3n_k} - p^{3n_{k-1}}$, $k \in \{1, \ldots, m\}$.*

Proof. Consider a vector $u = (r_1, r_2, r_3)$ and let I_k be the ideal generated by $\{r_1, r_2, r_3\}$. The rank of u is $r(u) = n - \mathrm{Rad}(u)$, which is by Lemma 3.8 equal to $n - (n/3 + 2\dim(\mathrm{Ann}(r_1, r_2, r_3)))$. Moreover, by Lemma 3.7 we get $r(u) = n - (n/3 + 2(n/3 - \dim(I_k))) = 2n_k$. The number of nonzero vectors $u = (r_1, r_2, r_3)$ such that I_k is generated by $\{r_1, r_2, r_3\}$ is equal to $|I_k|^3 - |I_{k-1}|^3 = p^{3n_k} - p^{3n_{k-1}}$. □

The numbers $p^{3n_k} - p^{3n_{k-1}}$ together with 1 are the coefficients of the radical polynomial $\sum_{i=0}^{n-1} \alpha_i x^i y^{n-i}$ of the chain ring form. Thus every α_i, $i \in \{1, \ldots, n-1\}$, is divisible both by $p^{3n_{k-1}}$ and $p^3 - 1$, see Tables 1 and 3.

Proposition 3.10. *Let R_1 and R_2 be nonisomorphic chain rings of characteristics p. Then the corresponding ring forms are nonequivalent.*

Proof. It is well known, see for instance [5], that every chain ring R of characteristics p is isomorphic to $\mathbf{GF}(p^k)[x]/(x^t)$. Its size is p^{kt} and the size of the maximal ideal xR is $p^{k(t-1)}$. Nonisomorpic rings must have distinct numbers k (and t) and thus by Proposition 3.9 the numbers $p^{3kt} - p^{3k(t-1)}$ of vectors of rank $\frac{2n}{3}$ of the corresponding ring forms are distinct, too. □

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DERIVED NUMBERS OF ONE VARIABLE MONOTONIC FUNCTIONS

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Abstract. The Fermat, Roll and Lagrange theorems are generalized into the class of nondifferentiable functions, the necessary and sufficient conditions for monotonicity one variable functions are given.

Keywords: derived number, periodic solution, almost periodic solution, nonsmooth analysis, Dini-Hölder derived number.

1. Introduction

In this paper, a method of periodic and almost periodic ordinary differential equations development is considered. It is based on the ideas of functional analysis. I.P. Natanson briefly outlined the theory of derived numbers [1]. Developing this theory, several theorems of mathematical analysis are proved. Implementation of this theory let reducing the restrictions on smoothness degree of the right-hand sides of the equations considered, which made it possible to extend the scope of the results obtained [2-11]. In many problems of classical and celestial mechanics, robotics and mechatronics, there are processes which the time dependence is not periodic in [12-21]. In this connection, the interest in derived theory implementation to the study of periodic and almost periodic solutions of differential equations and differential equations with almost periodic coefficients has arisen [22-26].

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2. Basic definitions

Let f be a function defined on an open interval (a, b), taking values in the set of real numbers R, i.e. $f : (a, b) \to R$, $a, b \in R$, $a < b$. Consider an arbitrary point x_0 in (a, b).

Let a number λ be a derived number of function f at x_0 if there exists a sequence $\{x_k\}$, such that $x_k \to x_0$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = \lambda.$$

The fact that λ is the derived number of function f at x_0 is represented as $\lambda = \lambda[f](x_0)$.

The set of all derived numbers of function f at x_0 is denoted by $\Lambda[f](x_0)$

If in the definition of a derived number it is required the sequence $\{x_k\}$ to satisfy one more additional condition, which means that for all k the inequality $x_k - x_0 > 0$ is fulfilled, then such derived number is determined as the right derived number and denoted by $\lambda^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such derived number is determined as the left derived number of function f at x_0 and denoted by $\lambda^-[f](x_0)$

Let the set of right derived number of function f at x_0 be denoted by $\Lambda^+[f](x_0)$, and the set of left derived number be denoted by $\Lambda^-[f](x_0)$.

It is clear that $\sup_{\lambda\in\Lambda^+[f](x_0)} \lambda$ determines $D^+f(x_0)$ that is the right upper derived number of a Dini function at a point x_0. Similarly, the remaining three derived number of Dini function at a point x_0 can be introduced.

Suppose

$$\lambda^\alpha = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha}.$$

In this relation consider α such that for any $\varepsilon > 0$ the equalities $\lambda^{\alpha-\varepsilon} = 0$ and $\lambda^{\alpha+\varepsilon} = \infty$ are realised. If the function f is defined in some neighborhood of the point x_0, then such α obviously exists. The magnitude can depend only on the choice of convergence to x_0 of the subsequence $\{x_k\}$.

Let the number λ be called the derived number of a Hölder function at x_0 if there exist $\alpha \leq 0$ and a sequence $\{x_k\}$ converging to x_0, such that

$$\lambda = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha},$$

and for any $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha-\varepsilon}} = 0,$$

and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha+\varepsilon}} = \infty.$$

Let the number α appearing in the definition of the Hölder derived number be called the exponent of this derived number.

The fact that λ is a Hölder derived number of function f at x_0 can be represented as following:

$$\lambda = \lambda_H[f](x_0).$$

The set of Hölder derived number of function f at x_0 is denoted by $\Lambda_H[f](x_0)$.

If in the definition of the Hölder derived number it is required that $x_k - x_0 > 0$ for all k, then such a derived number is determined as the right Hölder derived number and denoted by $\lambda_H^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such a derived number is determined as the left Hölder derived number and denoted by $\lambda_H^-[f](x_0)$.

Let the set of all right Hölder derived numbers of function f at x_0 be denoted by $\Lambda_H^+[f](x_0)$, and the set of all left Hölder derived numbers at the same point be denoted by $\Lambda_H^-[f](x_0)$.

Let α^+ denote the minimal of the exponents of the derived numbers being into $\Lambda_H^+[f](x_0)$, and $\Lambda_H^{\alpha^+}[f](x_0)$ denote a set of derived numbers belonging to the set $\Lambda_H^+[f](x_0)$ and having the exponent α^+. Similarly, for a set $\Lambda_H^-[f](x_0)$, a number α^- and a set $\Lambda_H^{\alpha^-}[f](x_0)$ are introduced.

Let the number

$$\lambda = \sup_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right upper derivative of Dini-Hölder function f at x_0 and denoted by $DH^+[f](x_0)$.

Let the number

$$\lambda = \inf_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right lower derivative of Dini-Hölder function f at x_0.

Analogously, the notions of the left upper and left lower Dini-Hölder derivatives of function f at x_0 are introduced. These derivatives are denoted by $DH^-[f](x_0)$ and $DH_-[f](x_0)$, respectively. Let DH^*f denote any of the four Dini-Hölder derivatives of the function f.

3. Monotonic functions

Let the function f be called monotone if it follows from $x < y$ that $f(x) \leq f(y)$ or $f(x) \geq f(y)$. In the first case, the function f is called increasing, and in the second case decreasing. On the other hand, if from $x < y$ follows that $f(x) < f(y)$ or $f(x) > f(y)$, then f is called strictly monotonic. In oder to emphasize what kind of monotony is under consideration, then we stay that the function f strictly increases if $f(x) < f(y)$ follows from $x < y$, and strictly decreases if $f(x) \leq f(y)$ follows from $x < y$.

Theorem 1. *In order for the continuous function f to be strictly monotonic, it is necessary and sufficient that all its Dini derivatives be constant-sign and* *there is no interval at which at least one of the Dini derivatives of the function f is equal to zero.*

Proof. *Necessity.* Let f be continuous, and for definiteness assume that it is strictly increasing. Consider an arbitrary point x_0 and a sequence $\{x_k\}$ converging to it. Without loss of generality, a sequence $\{x_k\}$ can be chosen such that there exists a limit $\lim_{k\to\infty} \frac{f(x_k)-f(x_0)}{x_k-x_0}$.

By assumption, the function f strictly increases, and therefore for all k at once

$$\frac{f(x_k)-f(x_0)}{x_k-x_0} > 0.$$

Passing to the limit as $k \to \infty$ in this inequality, it follows that

$$\lim_{k\to\infty} \frac{f(x_k)-f(x_0)}{x_k-x_0} \geq 0.$$

This implies that all the Dini derivatives of the function f are constant-sign due to arbitrariness of the point x_0 and the sequence $\{x_k\}$. The case when f is strictly decreasing is treated similarly.

Let us show now that if f is strictly monotone and continuous, then the second condition of the theorem is also satisfied. Suppose, on the contrary, there exists an interval $[a, b]$ and at each point of it at least one of the Dini derivatives of the function f has a zero value. By continuity and monotonicity the function f transforms the interval $[a, b]$ to some interval $[\alpha, \beta]$ with measure $m[\alpha, \beta] = \beta - \alpha > 0$.

As follows from [4], if for some $p \geq 0$ the strictly monotonic function f at each point of the set $E \subset [a, b]$ has at least one Dini derivative such that

$$|D^*[f](x)| \leq p,$$

then

$$m^* f(E) \leq p \cdot m^* E,$$

where m^* denotes the outer measure of the corresponding set.

Applying this result to the considered case, the following is realised:
But this contradicts the inequality $|\beta - \alpha| > 0$ obtained above. Therefore the interval $[a, b]$ at each point of which at least one of the Dini derivatives of the function f vanishes can not exist.

Sufficiency. Let the conditions of the theorem be satisfied. Let us show that in this case the function f is strictly monotone. Suppose on the contrary that f is not monotonic. Then it is not monotonically increasing. Therefore, there exist two points x_1 and x_2, $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Consider a straight line $l(x) = qx + r$ through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Note that q is necessarily less than zero since $f(x_1) > f(x_2)$.

Consider all situations which arise at such construction. Suppose first that there exists a decreasing sequence $\{x_k^{'}\}$ converging to x_1 and such that $f(x_k^{'}) \leq l(x_k^{'})$ for all k. Then, taking into account that $f(x_1) = l(x_1)$

$$\frac{f(x_k^{'}) - f(x_1)}{x_k^{'} - x_1} \leq \frac{l(x_k^{'}) - l(x_1)}{x_k^{'} - x_1} = q < 0.$$

Passing to the limit $k \to \infty$ in this inequality, it can be assumed without loss of generality to exist. Then it can be concluded that the function f at the point x_1 has a negative derivative number that does not exceed q.

Suppose that such sequence $\{x_k^{'}\}$ does not exist, but there exist an increasing sequence $\{x_k^{''}\}$ converging to x_2 and such that $f(x_k^{''}) \geq l(x_k^{''})$ for all k. Repeating the arguments given for the sequence $\{x_k^{'}\}$also for the sequence $\{x_k^{'}\}$, it follows that the function f at the point x_2 has a derivative number not exceeding q.

Finally, if neither sequence $\{x_k^{'}\}$ nor sequence $\{x_k^{''}\}$ exists, then this means that $f(x) > l(x)$ for all points $x > x_1$ from a sufficiently small neighborhood of x_1, and $f(x) < l(x)$ for all points $x < x_2$ from a sufficiently small neighborhood of x_1. Then, in view of continuity of the function f, there exist a point $x_0 \in (x_1, x_2)$ and a decreasing sequence $\{x_k^*\}$ convergent to x_0, such that $f(x_0) = l(x_0)$ and $f(x_k^*) \leq l(x_k^*)$ for all k. But this situation completely coincides with the situation with the sequence $\{x_k^{'}\}$ considered. Therefore, the same conclusion made for the point x_1 in the presence of a sequence $\{x_k^{'}\}$ is also valid for a point x_0.

Thus, in any of the possible situations, there exists at least one point in which the function f has a negative derivative number, and therefore there exists at least one point at which at least one of the Dini derivatives of the function f is negative.

Further, the function f that is monotonic is also not decreasing. Then there are two points y_1 and y_2, $y_1 < y_2$ such that $f(y_1) < f(y_2)$. Repeating for the points y_1 and y_2 the arguments given above, it follows that there exists at least one point at which at least one of the Dini derivatives of the function f is positive.

So, if the function f is not monotonic, then it necessarily has Dini derivatives of different signs, which contradicts the requirements of the theorem. Therefore, if the conditions of the theorem are satisfied, then the function f is monotone.

Let f be a monotonic function, but not strictly monotone. Then there are two points z_1 and z_2 such that $f(z_1) = f(z_2)$. Since f is monotonic, then $f(x) = f(z_1)$ for all $x \in [z_1, z_2]$. In this case, $D^*[f](x) = 0$ for all $x \in (z_1, z_2)$, which contradicts the assumption of the theorem about the absence of intervals of this type.

Thus, if the conditions of the theorem are satisfied, then the function f is necessarily strictly monotone.

From the presented proof and the fact that if any of the Dini derivatives of the function f continuous on $[a, b]$ is constant on (a, b), then the same condition is true for the other three Dini derivatives, it follows that Theorem 7 can be reformulated as follows:

In order for the function f continuous on $[a, b]$ to be strictly monotonic, it is necessary and sufficient that one of the Dini derivatives of this function be constant-sign on (a, b) and that there exists no interval on which the function f has a derivative equal to zero.

Theorem 2. *If at each point of interval $[a, b]$ the continuous function f has a positive right derivative, then f strictly increases on $[a, b]$.*

Proof. Suppose f is a function not increasing on $[a, b]$. Then, by continuity it reaches its local maximum at some point $x_0 \in [a, b)$. But as shown in the proof of Theorem 6 [27], the right derivative of the function f does not exceed zero at this point. This contradiction shows that f is an increasing function.

If f increases but not strictly, then, as shown in the proof of Theorem 7, there exists an interval at each point of which the right derivative of the function f is equal zero. From this fact it follows a contradiction again.

Thus, if the right derivative of the function f is positive, then f is necessarily strictly increasing.

Theorem 3. *Let f be defined on $[a, b]$. If $D_+ f \geq 0$ and f does not have jumps down, then it increases on $[a, b]$.*

Proof. Suppose f satisfies the conditions of the theorem. Let us construct a function ϕ by setting that

$$\phi(x) = f(x) + \varepsilon x, \qquad \varepsilon > 0.$$

It is clear that ϕ is defined on $[a, b]$ and also has no jumps down.

Assume $\phi(a) > \phi(b)$. Let us construct a straight line $y = \frac{1}{2}[\phi(a) + \phi(b)]$. Since ϕ has no jumps down, then there exist points on (a, b) that are roots of the equation $\phi(x) = y$. Let c denote an exact lower bound of a set of roots for this equation. $\phi(c) = y$ since if $\phi(c) > y$, then obviously the function ϕ has a right derived number equal to $-\infty$ at the point c. But this is impossible, since the definition of the function ϕ implies that all its right-derived numbers are nonnegative. If $\phi(c) < y$, then by the assumption that the function f has no downward jumps there must exist a point $c^{'} < c$ such that $\phi(c^{'}) = y$ which contradicts the choice of the point c.

Consider now the interval $[a, c]$. It's clear that $\phi(a) > \phi(c)$. Let us repeat for the interval $[a, c]$ the construction made for the interval $[a, b]$. As a result, the minimum point c_1 is obtained on (a, c) for which $\phi(c_1) = \frac{1}{2}[\phi(a) + \phi(c)]$.

Continuing this process, a decreasing sequence $c > c_1 > c_2 > \ldots$ is constructed. This sequence is bounded below by a number a and, consequently, has a limit. Let this limit be denoted by c_0. Likewise for the point c, it is shown that $\phi(c_0) = \phi(a)$.

From the construction given above it follows that for any n

$$\frac{\phi(c_n) - \phi(c_0)}{c_n - c_0} < 0.$$

This inequality implies that the function ϕ has a nonpositive lower right Dini derivative at the point c_0.

But as follows from the definition of the function ϕ, all its right derivatives in $[a, b]$ are not less then ε. This contradiction shows that the inequality $\phi(a) > \phi(b)$ is impossible. Hence, $\phi(a) \leq \phi(b)$ or $f(a) + \varepsilon a \leq f(b) + \varepsilon b$. Since ε is arbitrary, then passing to the limit in the last inequality as $\varepsilon \to 0$ it follows that $f(a) \leq f(b)$, which proves the theorem, since there could be any interval $[x, y] \subset [a, b]$ taken instead of $[a, b]$.

Note that if $D_+f \geq 0$, then the condition that function f has no downward jump is satisfied, in particular, if D_-f is bounded below.

As an example, consider the function f with a derivative defined on an interval $I = [0, 1]$ as follows:

$$f'(x) = 1, x \in E_1 0, x \in E_2, 1.2$$

where E_1 is a set of measure zero dense everywhere on I, and I.

According to Theorems 1 [27] and 1, by virtue of the density of E_1, the function f is continuous and strictly increasing in I. But then there is a chain of inequalities [4]:

$$m^*f(I) \leq m^*f(E_1) + m^*f(E_2) \leq 1 \cdot m^*E_1 + 0 \cdot m^*E_2 = 0,$$

where m^*E denotes the outer measure of the set E. It obviously follows from the inequality obtained that an equality $mf(I) = 0$ holds, which contradicts the conclusion that the function f is strictly increasing on I, since if f is continuous and strictly increasing, then it transforms the interval $[0, 1]$ into some interval $[\alpha, \beta]$ with measure $m[\alpha, \beta] = \beta - \alpha > 0$.

Thus, these arguments lead to contradiction, since on the one hand, on the basis of Theorems 1 [27] and 1 the function f is continuous and strictly increasing on I, and on the other hand, according to the theory of monotone functions it follows from equality $mf(I) = 0$ that $f(0) = f\ (1)$, or in other words that f is constant on I. This contradiction shows that there is no function f. In particular, there is no function f which the Dirichlet function would play the role of the derivative for.

Let the function f be called almost continuous on the interval $[a, b]$ if there exists a continuous function g on $[a, b]$ such that $mE(f \neq g) = 0$, put this another way, if the changing the values of the function f on a set of measure zero implies that it can be made continuous.

Theorem 4. *If all the Dini derivatives of the function f are almost continuous and bounded on the interval $[a, b]$ functions, then f is continuously differentiable on $[a, b]$.*

Proof. If the conditions of the theorem are satisfied, then, the function $\Lambda[f](x)$ is obviously almost continuous. Let $\Lambda[f](x)$ be represented in the form

$$\Lambda[f](x) = g_1(x) + g_2(x) + g_3(x),$$

where g_1 is a function continuous on $[a, b]$, such that

$$\begin{aligned} &mE(\Lambda[f] \neq g) = 0, \\ &g_3(x) = \min(0, \Lambda[f](x)), \\ &g_2(x) = \Lambda[f](x) - g_1(x) - g_3(x). \end{aligned}$$

Let the function f can be represented as:

$$f = f_1 + f_2 + f_3,$$

where

$$\frac{df_1}{dx} = g_1, \qquad \frac{df_3}{dx} = g_3, \qquad \Lambda[f_2] = g_2.$$

It is easy to verify that the functions g_2 and g_3 are almost everywhere equal to zero, where g_3 is continuous, and g_2 does not take negative values. Thus, for each of the functions g_2 and g_3 the arguments given in the example above are applicable. Repeating these arguments, it follows that f_2 and f_3 are constants on $[a, b]$, which implies that

$$\frac{df}{dx} = \frac{df_1}{dx} = g_1,$$

so, f is continuously differentiable.

To complete the proof, it remains only to show that representation of the function f as a sum of three functions is possible. The validity of such a representation follows from the fact that the function $\bar{f} = f - f_1$ is continuous as a difference between two continuous functions, and the function $\Lambda[\bar{f}](x) = \Lambda[f](x) - g_1(x)$ are almost everywhere equal to zero.

4. Mean-value theorems

Thereafter it is assumed that all right-derived numbers of the function f are equal to each other, i.e. that the function f has a right derivative. Let us prove theorems that are analogous to the Rolle and Lagrange theorems for a considered class of functions.

Theorem 5. *If the function f is defined and continuous on the interval $[a, b]$, has a continuous right derivative (a, b) and has equal values at the ends of the interval, then there exists a point $x_0 \in (a, b)$ at which $f'^{+}(x_0) = 0$.*

Proof. Since the function f is continuous, then at some interior point x_0 of the interval $[a, b]$ it reaches its extremum. As stated in the theorem, f'^{+} is continuous in a neighborhood of point x_0, which implies that the right derivative vanishes at the point x_0 by virtue of Theorem 6 [27].

Theorem 6. *Suppose the function f is defined and continuous on the interval $[a,b]$ and has a right derivative in (a,b). Then there exist points $x_1 \in (a,b)$ and $x_2 \in (a,b)$ for which the following inequalities hold*

$$f'^{+}(x_1) \leq \frac{f(b)-f(a)}{b-a} \leq f'^{+}(x_2).$$

If f has a continuous right derivative in (a,b), then there exists a point $x_0 \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'^{+}(x_0).$$

Proof. Suppose that at each point of the open interval (a,b) the function f has a right derivative, and, contrary to the assertion of the theorem, the point $x_2 \in (a,b)$ is such that

$$\frac{f(b)-f(a)}{b-a} \leq f'^{+}(x_2),$$

does not exist. Then for any $x \in (a,b)$

$$\frac{f(b)-f(a)}{b-a} - f'^{+}(x) = \alpha(x) > 0.$$

Let us construct the following function

$$\phi(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a) - f(x).$$

The function ϕ is continuous, since it is a linear combination of continuous functions. It is easy to verify that $f'^{+}(x) = \alpha(x)$ for all $x \in (a,b)$. By assumption, $\alpha(x) > 0$ for all $x \in (a,b)$. Therefore, the function ϕ is strictly increasing by virtue of Theorem 8. But if ϕ is continuous and strictly increasing, then inequality $\phi(a) < \phi(b)$ must necessarily be fulfilled, while direct substitution shows that $\phi(a) = \phi(b) = 0$. This contradiction proves the existence of the point $x_2 \in (a,b)$ considered in the theorem.

Similarly, by contradiction, the existence of the point $x_1 \in (a,b)$ is proved, where

$$f'^{+}(x_1) \leq \frac{f(b)-f(a)}{b-a}.$$

Combining these two inequalities, the desired result is obtained.

If in addition it is known that the function f'^{+} is continuous on (a,b), then the function ϕ constructed above obviously satisfies all the requirements of Theorem 11, and consequently, by virtue of this theorem there must exist a point x_0 in (a,b) at which ϕ'^{+} vanishes. But

$$\phi'^{+}(x_0) = \frac{f(b)-f(a)}{b-a} - f'^{+}(x_0),$$

which the required equality is obtained from.

Conclusion. The method of derived numbers to study periodic and almost periodic solutions of ordinary differential equations is developed. Necessary and sufficient conditions for the monotonicity of one variable functions are presented.

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DERIVED NUMBERS OF ONE VARIABLE CONVEX FUNCTIONS

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Abstract. The Fermat, Roll and Lagrange theorems are generalized into the class of nondifferentiable functions, the necessary and sufficient conditions for convexity of one variable functions are given.

Keywords: derived number, periodic solution, almost periodic solution, nonsmooth analysis, Dini-Hölder derived number.

1. Introduction

In this paper, a method of periodic and almost periodic ordinary differential equations development is considered. It is based on the ideas of functional analysis. I.P. Natanson briefly outlined the theory of derived numbers [1]. Developing this theory, several theorems of mathematical analysis are proved. Implementation of this theory let reducing the restrictions on smoothness degree of the right-hand sides of the equations considered, which made it possible to extend the scope of the results obtained [2-11]. In many problems of classical and celestial mechanics, robotics and mechatronics, there are processes which the time dependence is not periodic in [12-21]. In this connection, the interest in derived theory implementation to the study of periodic and almost periodic solutions of differential equations and differential equations with almost periodic coefficients has arisen [22-26].

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2. Basic definitions

Let f be a function defined on an open interval (a, b), taking values in the set of real numbers R, i.e. $f : (a, b) \to R$, $a, b \in R$, $a < b$. Consider an arbitrary point x_0 in (a, b).

Let a number λ be a derived number of function f at x_0 if there exists a sequence $\{x_k\}$, such that $x_k \to x_0$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = \lambda.$$

The fact that λ is the derived number of function f at x_0 is represented as $\lambda = \lambda[f](x_0)$.

The set of all derived numbers of function f at x_0 is denoted by $\Lambda[f](x_0)$

If in the definition of a derived number it is required the sequence $\{x_k\}$ to satisfy one more additional condition, which means that for all k the inequality $x_k - x_0 > 0$ is fulfilled, then such derived number is determined as the right derived number and denoted by $\lambda^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such derived number is determined as the left derived number of function f at x_0 and denoted by $\lambda^-[f](x_0)$

Let the set of right derived number of function f at x_0 be denoted by $\Lambda^+[f](x_0)$, and the set of left derived number be denoted by $\Lambda^-[f](x_0)$.

It is clear that $\sup_{\lambda\in\Lambda^+[f](x_0)} \lambda$ determines $D^+f(x_0)$ that is the right upper derived number of a Dini function at a point x_0. Similarly, the remaining three derived number of Dini function at a point x_0 can be introduced.

Suppose

$$\lambda^\alpha = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha}.$$

In this relation consider α such that for any $\varepsilon > 0$ the equalities $\lambda^{\alpha-\varepsilon} = 0$ and $\lambda^{\alpha+\varepsilon} = \infty$ are realised. If the function f is defined in some neighborhood of the point x_0, then such α obviously exists. The magnitude can depend only on the choice of convergence to x_0 of the subsequence $\{x_k\}$.

Let the number λ be called the derived number of a Hölder function at x_0 if there exist $\alpha \le 0$ and a sequence $\{x_k\}$ converging to x_0, such that

$$\lambda = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha},$$

and for any $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha-\varepsilon}} = 0,$$

and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha+\varepsilon}} = \infty.$$

Let the number α appearing in the definition of the Hölder derived number be called the exponent of this derived number.

The fact that λ is a Hölder derived number of function f at x_0 can be represented as following:

$$\lambda = \lambda_H[f](x_0).$$

The set of Hölder derived number of function f at x_0 is denoted by $\Lambda_H[f](x_0)$.

If in the definition of the Hölder derived number it is required that $x_k - x_0 > 0$ for all k, then such a derived number is determined as the right Hölder derived number and denoted by $\lambda_H^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such a derived number is determined as the left Hölder derived number and denoted by $\lambda_H^-[f](x_0)$.

Let the set of all right Hölder derived numbers of function f at x_0 be denoted by $\Lambda_H^+[f](x_0)$, and the set of all left Hölder derived numbers at the same point be denoted by $\Lambda_H^-[f](x_0)$.

Let α^+ denote the minimal of the exponents of the derived numbers being into $\Lambda_H^+[f](x_0)$, and $\Lambda_H^{\alpha^+}[f](x_0)$ denote a set of derived numbers belonging to the set $\Lambda_H^+[f](x_0)$ and having the exponent α^+. Similarly, for a set $\Lambda_H^-[f](x_0)$, a number α^- and a set $\Lambda_H^{\alpha^-}[f](x_0)$ are introduced.

Let the number

$$\lambda = \sup_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right upper derivative of Dini-Hölder function f at x_0 and denoted by $DH^+[f](x_0)$.

Let the number

$$\lambda = \inf_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right lower derivative of Dini-Hölder function f at x_0.

Analogously, the notions of the left upper and left lower Dini-Hölder derivatives of function f at x_0 are introduced. These derivatives are denoted by $DH^-[f](x_0)$ and $DH_-[f](x_0)$, respectively. Let DH^*f denote any of the four Dini-Hölder derivatives of the function f.

Theorem 1. *For the function f to be continuous from the right at x_0, it is necessary and sufficient that either the two right Dini-Hölder derivatives $DH^+[f](x_0)$ and $DH_+[f](x_0)$ to be equal to zero or the exponent α^+ from the definition of Dini-Hölder derivative is greater than zero.*

Proof. *Necessity.* Let the function f be continuous from the right at x_0. Consider the right Dini-Hölder derivatives at this point, and let at least one of them, for example $DH^+[f](x_0)$, be non-zero. This means that there exist a sequence $\{x_k\}$ converging to x_0 and a number α^+, such that

$$DH^+[f](x_0) = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha^+}}.$$

If it turned out that in this expression $\alpha^+ = 0$, then the function f would obviously discontinue at the point x_0 which the assertion of this part of the theorem follows from.

Sufficiency. Let the conditions of the theorem be satisfied, and let the function f have discontinuity at a point x_0 contrary to our assertion. Then there exist $\varepsilon > 0$ and a sequence $\{x_k\}$ converging to x_0, such that inequality

$$\lim_{k\to\infty} |f(x_k) - f(x_0)| \geq \varepsilon.$$

It follows from this relation that the derived Hölder number realized on this sequence has an exponent equal to zero, and that at least one of the right Dini-Hölder derivatives at this point is nonzero. This contradiction proves the continuity on the right of the function f at the point x_0, and also the validity of the theorem assertion.

Similarly, the continuity of the function f at the point x_0 on the left is proved.

Obviously, in order for the function f to be continuous at the point x_0, it is necessary and sufficient that it be continuous both on the right and on the left.

Comparing the definition of the Dini-Holder derivative with the definition of the derived number of the function f or the derivative of a function f, it is obvious that Theorem 1 implies the validity of the following assertion.

The function f is continuous at a point x_0 if one of the following conditions is satisfied:

1. The set $\Lambda[f](x_0)$ is bounded;
2. Each of the Dini derivatives $D^*f(x_0)$ is bounded;
3. The derivative $f^{'}(x_0)$ is bounded.

Theorem 2. *Let the function f be defined in a neighborhood of the point x_0, and the function g be defined in a neighborhood of a point $f(x_0)$. Then, if sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$ are bounded, then every derived Hölder number of a function $h = g \circ f$ at the point x_0 can be represented in the form*

$$\lambda_H^{\gamma}[h](x_0) = \lambda_H^{\alpha}[g](f(x_0)) \cdot {\lambda_H^{\beta}}^{\alpha}[f](x_0),$$

where the exponent γ is equal to the product of exponents α and β, that is $\gamma = \alpha\beta$, $\lambda_H^{\alpha}[g](f(x_0))$ and $\lambda_H^{\beta}[f](x_0)$ are some derived Hölder numbers from sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$, respectively.

Proof. Since the function f is defined in a neighborhood of the point x_0 and g in a neighborhood of the point $f(x_0)$, a function h is also defined in some neighborhood of the point x_0. Let $\{x_k\}$ be a sequence from the range of definition of the function h, such that a certain Hölder number $\lambda_H^{\gamma}[h](x_0)$ with exponent γ is realized on it. Without loss of generality, it can assumed that the derived Hölder number $\lambda_H^{\beta}[f](x_0)$ of the function f with exponent β is realized on the sequence $\{x_k\}$, and the derived Hölder number $\lambda_H^\alpha[g](f(x_0))$ with exponent α of the function g is realized on the sequence $\{f(x_k)\}$. Since $h = g \circ f$, then

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^{\alpha\beta}} = \lim_{k\to\infty} \Big(\frac{g(f(x_k)) - g(f(x_0))}{(f(x_k) - f(x_0))^\alpha} \cdot \frac{(f(x_k) - f(x_0))^\alpha}{(x_k - x_0)^{\alpha\beta}}\Big) . \tag{1}$$

It can be shown that the derived Hölder number of the function h with exponent $\gamma = \alpha\beta$ is realized on the sequence $\{x_k\}$.

Consider an arbitrary positive number $\delta < \gamma$ and positive numbers $\alpha_0 \le \alpha$ and $\beta_0 \le \beta$, such that $\alpha_0\beta_0 = \delta$. Substituting α_0 and β_0 into (1) instead of α and β, respectively, and taking into account the definition of the derived Hölder number, it follows that for any

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\delta} = 0.$$

Consider now $\varepsilon > \gamma$, $\alpha_0 \ge \alpha$and $\beta_0 \ge \beta$, such that $\alpha_0\beta_0 = \varepsilon$ and repeating the arguments given above, it follows that for any $\varepsilon > \gamma$

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\varepsilon} = \infty.$$

The last two equalities imply that the exponent of the derived Hölder number of the function h realizable on a sequence $\{x_k\}$ is equal to γ.

Now, let $\{x_k\}$ be a sequence such that $f(x_k) \ne f(x_0)$ for $x_k \ne x_0$. Without loss of generality, it can assumed that the derived Hölder numbers $\lambda_1 = \lambda_H^\alpha[g](f(x_0)) \in \Lambda_H[g](f(x_0))$ and $\lambda_2 = \lambda_H^\beta[f](x_0) \in \Lambda_H[f](x_0)$ are realized on the sequence $\{x_k\}$. It follows from the boundedness of the sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$ that λ_1 and λ_2 are finite numbers. Then, considering [3] the following equality is realised:

$$\begin{aligned}\lambda_H^\gamma[h](x_0) = \lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\gamma} &= \lim_{k\to\infty} \frac{g(f(x_k)) - g(f(x_0))}{(f(x_k) - f(x_0))^\alpha} \\ \cdot \left(\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\beta}\right)^\alpha &= \lambda_1 \cdot \lambda_2^\alpha,\end{aligned}$$

which is assirted by the theorem.

Finally, let $\{x_k\}$ be a sequence converging to x_0 such that $f(x_k) = f(x_0)$ for $x_k \ne x_0$. In this case for the function f the derived number equal to zero is realised on the sequence $\{x_k\}$. Then, taking into account the boundedness of the set $\Lambda_H[g](f(x_0))$, it follows that for any $\lambda_1 \in \Lambda_H[g](f(x_0))$ an equality $\lambda_1 \cdot 0 = 0$ is true. But since for the function h $\{x_k\}$ the derived number equal to zero is realized on a sequence, then in the considered case it can be assumed that the theorem remains valid.

Theorem 3. *Suppose that for some $\delta > 0$ function f continuous at the point x_0 maps one-to-one interval $(x_0 - \delta, x_0 + \delta)$ into interval $(y_0 - \varepsilon, y_0 + \varepsilon)$, where $y_0 = f(x_0)$. Then*

$$\Lambda_H[f^{-1}](y_0) = \left(\Lambda_H[f](x_0)\right)^{-1},$$

where $\{\Lambda_H[f](x_0)\}^{-1}$ *is obtained from the set* $\Lambda_H[f](x_0)$ *by substituting each element* $\lambda_H^\alpha \in \Lambda_H[f](x_0)$ *by an element* $\mu_H^\beta = \frac{1}{\lambda_H^\alpha[f](x_0)^\beta}$*, where* $\beta = 1/\alpha$*.*

Proof. Let $\{y_k\}$ be a sequence converging to $\{y_k\}$ on which a certain derived Hölder number $\mu_H^\beta[f^{-1}](y_0)$ of a function f^{-1} at a point y_0 is realized, and let $\{x_k\}$ be a sequence corresponding to $\{y_k\}$ given by equalities $x_k = f^{-1}(y_k)$. Note that from the continuity at the point x_0 and the one-to-one mapping of f it follows that the sequence $\{x_k\}$ converges to x_0 as $k \to \infty$ and that if $y_k \neq y_0$, then $x_k \neq x_0$. Then

$$\begin{aligned}\mu_H^\beta[f^{-1}](y_0) &= \lim_{k\to\infty} \frac{f^{-1}(y_k) - f^{-1}(y_0)}{(y_k - y_0)^\beta} = \lim_{k\to\infty} \frac{x_k - x_0}{(f(x_k) - f(x_0))^\beta} \\ &= \left(\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{1/\beta}}\right)^{-\beta} = \frac{1}{\lambda_H^\alpha[f](x_0)^\beta} \in (\Lambda_H[f](x_0))^{-1},\end{aligned}$$

where $\alpha = 1/\beta$.

By virtue of the arbitrariness of the sequence $\{y_k\}$ choice, it follows from this relation that $\Lambda_H[f^{-1}](y_0) \subset (\Lambda_H[f](x_0))^{-1}$. It can be shown in a similar way that the reverse inclusion can also be realised. Combining these two results, the validity of the theorem is clear.

Theorems 2 and 3 are not only a generalization of the well-known analysis theorems on differentiation of a composite function and derivative of an inverse function, but also show how the degree of a composite function smoothness depends on the smoothness of the functions included in it.

3. Extremum of function

It is known that the derivative of a function allows to make a conclusion about the behavior of the function itself. Similar conclusions can be made based on the values of the derived numbers of function. In this section, considering the question of extremal values of a function, some generalizations of Fermat's theorem are presented.

Theorem 4. *Suppose for some* $\delta > 0$ *there is a function* f *defined on* $(x_0-\delta, x_0+\delta)$ *taking an extremal value at* x_0*. Then at this point the following inequalities are true:*

$$DH^- f(x_0) \leq 0 \leq DH_+ f(x_0),$$

if x_0 *is a local minimum point of the function* f*, and*

$$DH_+ f(x_0) \leq 0 \leq DH^- f(x_0),$$

if x_0 *is a local maximum point of the function* f*.*

Proof. Let x_0 be a local minimum point. Then in some neighborhood of the point x_0 $f(x) \geq f(x_0)$ for all $x < x_0$. Let $\{x_k\}$ be a sequence on which the left upper Dini-Holder derivative with exponent α^- is realised:

$$DH^- f(x_0) = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{-|x_k - x_0|^{\alpha^-}}.$$

Since for all k $\frac{f(x_k)-f(x_0)}{-|x_k-x_0|^{\alpha^-}} \leq 0$, then $DH^- f(x_0) \leq 0$.

It can be shown in a similar way that if x_0 is a local minimum point of the function f, then at this point

$$0 \leq DH_+ f(x_0).$$

The case when x_0 is a local maximum point of the function f can be shown in the same way.

It is clear that such a statement is also valid for the Dini derivatives.

Analysis of Theorem 4 proof allows to formulate a theorem giving sufficient conditions that x_0 is an extremum point for the function f, in other words a theorem being a kind of inverse to Theorem 4.

Theorem 5. *Suppose that for some $\delta > 0$ on $(x_0 - \delta, x_0 + \delta)$ function f is defined. Then, if*

$$D^- f(x_0) < 0 < D_+ f(x_0),$$

then x_0 is a local minimum point of the function f, and if

$$D_- f(x_0) > 0 > D^+ f(x_0),$$

then x_0 is a local maximum point of the function f.

Proof. Suppose x_0 is not an extremum of the function f. Then it follows from the definition of an extremum that there are two sequences $\{x_k\}$ and $\{y_k\}$ convergent to x_0 such that $f(x_k) > f(x_0)$ for all k, and $f(y_k) < f(x_0)$. Besides, without loss of generality, it can be assumed that on each of these sequences some derived number of the function f at the point x_0 is realised. Two cases are possible here: either these sequences are monotonically increasing or decreasing, or one of them is monotonically increasing and the other one is decreasing.

Consider the first case. Assume both sequences increase. Then immediately for all k

$$\frac{f(x_k) - f(x_0)}{x_k - x_0} < 0$$

and

$$\frac{f(y_k) - f(x_0)}{y_k - x_0} > 0.$$

From the estimates obtained and the assumption that the derived numbers of the function f are realizable on $\{x_k\}$and $\{y_k\}$, it follows that

$$\lambda_1 = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} \leq 0,$$

$$\lambda_2 = \lim_{k\to\infty} \frac{f(y_k) - f(x_0)}{y_k - x_0} \geq 0.$$

Since the sequences $\{x_k\}$ and $\{y_k\}$ are increasing by assumption, then $\lambda_1 \in \Lambda_-[f](x_0)$and $\lambda_2 \in \Lambda_-[f](x_0)$. Given that

$$D_- f(x_0) \leq \lambda_1 \leq 0 \leq \lambda_2 \leq D^- f(x_0),$$

it follows that in the considered case $0 \in [D_- f(x_0), D^- f(x_0)]$. Therefore none of the conditions of the theorem can be satisfied.

Consider now the second case Suppose the sequence $\{x_k\}$to increase and the sequence $\{y_k\}$ to decrease. Then for all k

$$\frac{f(x_k) - f(x_0)}{x_k - x_0} < 0$$

and

$$\frac{f(y_k) - f(x_0)}{y_k - x_0} < 0.$$

Repeating the arguments given in the first case analisis, it follows that $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$. Since by assumption the sequence $\{x_k\}$ is increasing and the sequence $\{y_k\}$ is decreasing, then $D_- f(x_0) \leq \lambda_1 \leq 0$ and $D_+ f(x_0) \leq \lambda_2 \leq 0$. Thus, in the second case the point 0 does not divide the intervals $[D_- f(x_0), D^- f(x_0)]$ and $[D_+ f(x_0), D^+ f(x_0)]$, i.e. none of the conditions of the theorem is satisfied.

So, if x_0 is not an extremum point of the function f, then either point 0 is the derived number of the function f at the point x_0, or both intervals $[D_- f(x_0), D^- f(x_0)]$ and $[D_+ f(x_0), D^+ f(x_0)]$ are on one side of the point 0. If the conditions of the theorem are satisfied, then neither of these two possibilities is realized at the point x_0, and, therefore, the point x_0 is the extremum point of the function f.

The points at which the function f can take extreme values can be selected based on the behavior of any one-sided, for example, right-sided derivative. But in this case it is no longer sufficient to know the value of this derivative only at one point in order to relate this point to a set at which function f can take extreme values or not. More precise representation on this phenomena is given in the following theorem.

Theorem 6. *Suppose for some $\delta > 0$ on $(x_0 - \delta, x_0 + \delta)$ there exists a continuous function f that reaches its extremal value at the point x_0. If in some neighborhood of the point x_0 the function f has a continuous right derivative f'^{+}, then it is necessary hat$f'^{+}(x_0) = 0$.*

Proof. Let the function f take a maximum value at the point x_0 to be definite. Then for all $x > x_0$

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Transferring to the limit for $x \to x_0 + 0$ in this inequality which exists by virtue of the assumption that there exists the right derivative at the point x_0, it follows that $f'^{+}(x_0) \leq 0$.

Suppose $f'^{+}(x_0) < 0$. By the continuity of the right derivative of the function f in a neighborhood of the point x_0, there exists $\delta' > 0$, such that an inequality $f'^{+}(x) < 0$ holds for all $x \in [x_0 - \delta', x_0 + \delta']$.

Let us take an arbitrary point x_1 in $[x_0 - \delta', x_0)$. Since the function f is continuous, it reaches its minimum at some point x_2 on $[x_1, x_0]$.

Let us show that $x_2 \neq x_0$. Indeed, otherwise for all $x \in [x_1, x_2]$ equality $f(x) = f(x_0)$ must be realised by the fact that at the point x_0 the function f reaches its maximum. But such a conclusion is incompatible with the assumption that $f'^{+}(x) < 0$ for all $x \in [x_1, x_2]$.

So, now it is prooved that $x_2 \in (x_1, x_0)$. Then there exists a monotonically decreasing sequence $\{y_k\}$ converging to x_2, such that for all k

$$\frac{f(y_k) - f(x_2)}{y_k - x_2} \geq 0.$$

But by assumption that $f'^{+}(x_2) < 0$ and for all $x > x_2$ and sufficiently close to x_2 the following is true

$$\frac{f(x) - f(x_2)}{x - x_2} < 0.$$

Th arguments given above imply that a sequence $\{y_k\}$ with the properties listed above does not exist.

Thus, the assumption that the inequality $f'^{+}(x_0) < 0$ is satisfied leads to a contradiction and$f'^{+}(x_0) = 0$.

4. A theorem on a convex function

The function f is called convex if from condition $x = \alpha x_1 + (1-\alpha)x_2$, $\alpha \in [0,1]$, the validity of inequality follows $f(x) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$. If for all $\alpha \in (0,1)$ there is a strict inequality, i.e. $f(x) < \alpha f(x_1) + (1-\alpha) f(x_2)$, then the function f is called strictly convex.

The main purpose of this section is to prove a theorem giving necessary and sufficient conditions for the function f to be convex. But before proceeding with the proof of this theorem, let us first prove an auxiliary statement on the reconstruction of a function from the values of its right derivative.

Theorem 7. *Let a continuous function f have a right derivative $f'^{+}(x)$ at each point $x \in [a,b]$. If f'^{+} is bounded on $[a,b]$, then it is integrable on $[a,b]$ and for any $x \in [a,b]$*

$$f(x) = f(a) + \int_a^x f'^{+}(y)\, dy.$$

Proof. Let us construct a function g by setting that

$$g(x) = f(x), x \in [a,b] f(b) + (x-b) f'^{+}(b), x > b.$$

It is obvious that the function g is continuous and has a finite right derivative on $[a, b+1]$. Let us introduce the following function for $x \in [a, b]$ and $n = 1, 2$

$$\phi_n(x) = n[g(x + \frac{1}{n}) - g(x)].$$

At each point $x \in [a, b]$

$$\lim_{n\to\infty} \phi_n(x) = g^{'+}(x) = f^{'+}(x),$$

and since each of the continuous functions ϕ_n is measurable, then $f^{'+}$ is also measurable, which implies the integrability of this function due to the condition of boundedness. Further, by Theorem 6 [27]

$$g^{'+}(x + \frac{\theta'}{n}) \leq \phi_n(x) = [g(x + \frac{1}{n}) - g(x)] \leq g^{'+}(x + \frac{\theta^{''}}{n}), \quad \theta^{'}, \theta'' \in (0, 1),$$

so that all of the functions ϕ_n are bounded by one number and, by the Lebesgue theorem on the passage to the limit under the integral sign

$$\int_a^b f^{'+}(x)\,dx = \int_a^b g^{'+}(x)\,dx = \lim_{n\to\infty} \int_a^b \phi_n(x)\,dx.$$

But

$$\begin{aligned}\int_a^b \phi_n(x)\,dx &= n\int_a^b g(x + \frac{1}{n})\,dx - n\int_a^b g(x)\,dx \\ &= n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} g(x)\,dx - n\int_a^b g(x)\,dx \\ &= n\int_b^{b+\frac{1}{n}} g(x)\,dx - n\int_a^{a+\frac{1}{n}} g(x)\,dx.\end{aligned}$$

Applying the mean-value theorem to each of the last two integrals, the following is obtained:

$$\int_a^b \phi_n(x)dx = g(b + \frac{\theta_n^{'}}{n}) - g(a + \frac{\theta_n^{''}}{n}), \quad \theta_n^{'}, \theta_n^{''} \in (0, 1),$$

which on the basis of the continuity of the function g implies the following

$$\lim_{n\to\infty} \int_a^b \phi_n(x)dx = \int_a^b g^{'+}(x)dx = \int_a^b f^{'+}(x)dx = g(b) - g(a) = f(b) - f(a).$$

Replacing b by an arbitrary $x \in [a, b]$, the equality required in the condition of the theorem is obtained.

In conclusion, note that substitution of variable in the integral and the application of the mean-value theorem in the proof of the theorem are admissible, since in both cases the continuous function are under integral.

Theorem 8. *In order for a function f bounded on (a,b) to be strictly convex, it is necessary and sufficient that it be continuous and have a strictly increasing right derivative $f^{'+}$ bounded at each point.*

Proof. *Necessity.* If the function f is convex and bounded on (a,b), then it is continuous on any interval $[p,q] \subset (a,b)$, and hence it is continuous on (a,b), too. Further, at each point of the open interval (a,b) the continuous convex function has a bounded right derivative. Let us show that this derivative is strictly increasing function, if f is strictly convex.

Consider two arbitrary points x and $y > x$. Assume $\alpha = \frac{1}{2}(y-x)$ and let z denote a point $x+\alpha = y-\alpha$. For the right derivative of a convex function at each point $x_0 \in (a,b)$ the following the estimates are realised:

$$\frac{f(x_0)-f(x_0-\beta)}{\beta} \leq f^{'+}(x_0) \leq \frac{f(x_0+\beta)-f(x_0)}{\beta},$$

where $\beta > 0$ so little that $[x_0-\beta, x_0+\beta] \subset (a,b)$.
Applying these estimates to the function f at the points x and y for $\beta = \alpha$, the following two inequalities are obtained:

$$f^{'+}(x) \leq \alpha^{-1}(f(z)-f(x)) = u,$$
$$f^{'+}(y) \geq \alpha^{-1}(f(y)-f(z)) = v.$$

By assumption, f it is strictly convex, and by virtue of this fact the folowing inequality is valid:

$$u - v = \alpha^{-1}(2f(z) - f(x) - f(y)) < 0,$$

i.e. this proves the inequality $u < v$. But, as noted above, $f^{'+}(x) \leq u$ and $v \leq f^{'+}(y)$, which implies that $f^{'+}(x) \leq u < v \leq f^{'+}(y)$, and therefore $f^{'+}(x) < f^{'+}(y)$, which proves a strict increase of the function $f$due to the arbitrariness of the points x and y.

Sufficiency. Suppose f is continuous on (a,b) and at each of its points has a bounded right derivative, which is strictly a increasing function (a,b). First of all, note that $f^{'+}$ is bounded on each interval $[p,q] \subset (a,b)$. Indeed, consider an arbitrary point x_1 on (a,p), and an arbitrary point x_2 on (q,b). Then, by the monotonicity of the function $f^{'+}$, for any $x \in [p,q]$ the following estimation is true:

$$f^{'+}(x_1) < f^{'+}(x) < f^{'+}(x_2).$$

The note that the function $f^{'+}$ takes finite values at the points x_1 and x_2 proves the validity of the assertion.

Thus, it is shown that all the conditions of Theorem 13 are satisfied on an arbitrary interval $[p,q] \subset (a,b)$, and therefore for any $x \in [p,q]$ the followng representation holds:

$$f(x) = f(p) + \int_p^x f^{'+}(y)\,dy.$$

By condition, the function f'^{+} is strictly increasing, and hence the function f is strictly convex on $[p, q]$. Since $[p, q]$ is an arbitrary interval belonging to (a, b), it is strictly convex on (a, b).

Remark. The boundedness of the function f on (a, b) is used only to prove its continuity. Thus, if it is known in advance that the function f is continuous on (a, b), then the requirement of its boundedness on this interval can be omitted.

Conclusion. The method of derived numbers to study periodic and almost periodic solutions of ordinary differential equations is developed. Necessary and sufficient conditions for the convexity of one variable functions are presented.

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NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH DEFORMED (α,β)-METRIC

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Abstract. The purpose of present paper is to determine the Finsler spaces due to deformation of special (α,β)-metrics. Consequently, we determine the non-holonomic frames for Finsler space with help of Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$,one form metric $\beta = b_i(x)y^i$ and some special Finsler (α,β)- metric.
Keywords: (α,β)-metric, Riemannian metric, One form metric, Nonholonomic Finsler frame.

1. Introduction

In 1982, P.R. Holland [5, 6] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field . Further in 1995, R.G. Beil [2] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered that the fundamental tensor field might be the deformation of four different special Finsler spaces from the (α, β)- metrics. First we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as:

I. $L(\alpha, \beta) = ((\alpha + \beta) + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$ i.e. product of (Randers + Kropina metric) and Riemannian metric.

II. $L(\alpha, \beta) = ((\alpha + \beta) + \frac{\alpha^2}{\beta})\beta = \alpha^2 + \alpha\beta + \beta^2$, i.e. product of (Randers + Kropina metric) and one form metric.

III. $L(\alpha, \beta) = (\alpha + \beta + \frac{\beta^2}{\alpha})\alpha = \alpha^2 + \alpha\beta + \beta^2$, i.e. product of first approximate Matsumoto Metric and Riemannian metric.

IV. $L(\alpha, \beta) = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \beta^2 + \alpha\beta + \frac{\beta^3}{\alpha}$, i.e. product of first approximate Matsumoto Metric and one form metric.

Further we obtain a corresponding frame for each of these four Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [4], Tripathi [12, 13] and Narasimhamurthy [10].

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces (α, β)-metrics [8]. The first Finsler spaces with (α, β)-metrics were introduced by physicist G.Randers in 1940,are called Randers spaces [4].Recently, R.G. Beil suggested to consider a more general case, the class of Lagrange spaces with (α, β)-metric,which was discussed in [2]. A unified formalism which uses a nonholonomic frame on a space time,a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland [5, 6].If we do not ask for the function L to be homogeneous of order two with respect to the (α, β) variables,then we have Lagrange space with (α, β)-metric. Next we look for some different Finsler space with (α, β)-metrics.

Definition 2.1. *Let U be an open set of TM and $V_i : u \in U \longmapsto V_i(u) \in V_uTM, i \in \{1, 2.....n\}$ be a vertical frame over U. If $V_i(u) = V_i^j(u)\frac{\partial}{\partial y^j}|_u$,then $V_i^j(u)$ are the entries of invertible matrix for all $u \in U$.Denote by $\check{V}_k^j(u)$ the inverse of this matrix. This means that : $V_j^i\check{V}_k^j = \delta_k^i$, $\check{V}_j^i V_k^j = \delta_k^i$. We call V_j^i a nonholonomic Finsler Frame.*

Definition 2.2. *A Finsler space $F^n = \{M, F(x, y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}, \tag{2.1}$$

where $\alpha^2(x, y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M, F). Taking into account the homogeneity of a and F we have the following formulae:

$$\begin{aligned}
&p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial\alpha}{\partial y^j}; \quad P_i = a_{ij}p^j = \frac{\partial\alpha}{\partial y^i};\\
&l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^j}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = p_i + b_i\\
&l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha};\\
&b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}
\end{aligned} \tag{2.2}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j. \tag{2.3}$$

Theorem 2.1 ([4])**.** *For a Finsler space (M, F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j), \tag{2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i}), j \in 1, 2, 3, \dots, n$ is a non holonomic frame.

Theorem 2.2 ([7])**.** *With respect to frame the holonomic components of the Finsler metric tensor $a_{\alpha\beta}$ is the Randers metric g_{ij}, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)-metric $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ we have the Finsler invariants [9].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α, β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [9].

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j).$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho^2_{-1})b_i b_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the theorem (7.9.1) in [4], given by,

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II^{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3 ([4]). *Let $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) *and* (2.12) *respectively.*

3. Nonholonomic frames for Finsler geometry with (α, β)-metric

In this section we consider four cases of nonholonomic Finlser frames with special (α, β)-metrics, such a I^{st} Finsler frame product of (Randers metric + Kropina metric) and Riemannian metric ; II^{nd} Finsler frame product of (Randers metric + Kropina metric) and 1-form metric;III^{rd} Finsler frame product of approxomate Matsumoto metric and Riemannian metric ;IV^{th} Finsler frame product of approxomate Matsumoto metric and 1-form metric.

3.1 Nonholonomic frame for $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$

In the first case, for a Finsler space with the fundamental function $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$ the Finsler invariants (2.6) are given by

$$\begin{aligned}
&\rho_1 = 1 + \frac{\beta}{2\alpha} + \frac{3\alpha}{2\beta}, \rho_0 = \frac{\alpha^2\beta - \alpha^3}{2\beta^2},\\
&\rho_{-1} = \frac{\beta^2 - 3\alpha^2}{2\alpha\beta^2}, \quad \rho_{-2} = \frac{3\alpha^2 - \beta^2}{2\alpha^3\beta},\\
&B^2 = \frac{(\beta^2 - 3\alpha)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^4\beta^4}.
\end{aligned} \tag{3.1}$$

Using (3.1) in (2.10) we have,

$$X_j^i = \sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}}\delta_j^i - \frac{(\beta^2-3\alpha^2)^2}{4\alpha^2\beta^6}[\sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}} \sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}}+\frac{2\alpha^3\beta^3}{(3\alpha^2-\beta)^2}](b^i-\frac{\beta}{\alpha^2}y^i)(b_j-\frac{\beta}{\alpha^2}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1+\frac{2\beta^3C^2}{\alpha\beta^3-\alpha^3\beta-3\alpha^3+\alpha\beta^2}}\}b^ib_j, \tag{3.3}$$

where $C^2 = (1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta})b^2 - \frac{(3\alpha^2-\beta^2)}{2\alpha^3\beta^3}(\alpha^2b^2-\beta^2)^2$.

Theorem 3.1. *Let $L = (\alpha+\beta+\frac{\alpha^2}{\beta})\alpha = \alpha^2+\alpha\beta+\frac{\alpha^3}{\beta}$ be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V_j^i = X_k^iY_j^k$$

is nonholonomic Finsler Frame with X_k^i and Y_j^k are given by (3.2) *and* (3.3) *respectively.*

3.2 Nonholonomic frame for $L = (\alpha+\beta+\frac{\alpha^2}{\beta})\beta = (\alpha+\beta+\frac{\beta^2}{\alpha})\alpha = \alpha^2+\alpha\beta+\beta^2$

In the second and third case, for a Finsler space with the fundamental function $L = (\alpha^2+\alpha\beta+\beta^2)$ are the same , the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= 1+\frac{\beta}{2\alpha}, \quad \rho_0 = 1, \\ \rho_{-1} &= \frac{1}{2\alpha}, \quad \rho_{-2} = \frac{-\beta}{2\alpha}, \\ B^2 &= \frac{(\alpha^2b^2-\beta^2)}{4\alpha^4}. \end{aligned} \tag{3.4}$$

Using (3.4) in (2.10) we have,

$$X_j^i = \sqrt{1+\frac{\beta}{2\alpha}}\delta_j^i - \frac{1}{4\alpha^2\beta}[\sqrt{1+\frac{\beta}{2\alpha}} + \sqrt{1+\frac{\beta}{2\alpha}}-2\alpha^3\beta](b^i-\frac{\beta}{\alpha^2}y^i)(b_j-\frac{\beta}{\alpha^2}y_j). \tag{3.5}$$

Again using (3.4) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1+\frac{2\alpha\beta C^2}{1+2\alpha\beta}}\}b^ib_j, \tag{3.6}$$

where $C^2 = (1+\frac{\beta}{2\alpha})b^2 - \frac{1}{2\alpha^3\beta}(\alpha^2b^2-\beta^2)^2$.

Theorem 3.2. *Let* $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\beta = (\alpha + \beta + \frac{\beta^2}{\alpha})\alpha = \alpha^2 + \alpha\beta + \beta^2$ *be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is nonholonomic Finsler Frame with X^i_k *and* Y^k_j *are given by* (3.5) *and* (3.6) *respectively.*

3.3 Nonholonomic frame for $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$

In the fourth case, for a Finsler space with the fundamental function $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\alpha^2\beta - \beta^3}{2\alpha^3}, \quad \rho_0 = \frac{\alpha + 3\beta}{\alpha}, \\ \rho_{-1} &= \frac{\alpha^2 - 3\beta^2}{2\alpha^3}, \quad \rho_{-2} = \frac{3\beta^3 - \alpha\beta^2}{2\alpha^5}, \\ B^2 &= \frac{(\alpha^2 - 3\beta^2)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^8}. \end{aligned} \tag{3.7}$$

Using (3.7) in (2.10) we have,

$$\begin{aligned} X^i_j &= \sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3}}\delta^i_j - \frac{(\alpha^2 - 3\beta^2)^2}{4\alpha^6\beta^2}[\sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3}} \\ &+ \sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3} + \frac{2\alpha^5\beta}{3\beta^2 - \alpha^2}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \end{aligned} \tag{3.8}$$

Again using (3.7) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\alpha\beta C^2}{\alpha^2 + 3\beta^2 + 2\alpha\beta}}\}b^i b_j, \tag{3.9}$$

where $C^2 = (\frac{\alpha^2\beta - \beta^3}{2\alpha^3})b^2 + \frac{(3\beta^2 - \alpha^2)}{\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.3. *Let* $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$ *be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is nonholonomic Finsler Frame with X^i_k *and* Y^k_j *are given by* (3.8) *and* (3.9) *respectively.*

4. Conclusion

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru [1, 2], have determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [10]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α,β)-metric, it appears a natural question: Does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α, β)-metrics.

In this work, we consider the special Finsler (α, β) metrics,first approximate Matsomoto metric, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many (α,β)-metrics, in future work we can determine the frames for them also.

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A NOTE ON GROUPS WHOSE ALL NON-LINEAR IRREDUCIBLE CHARACTERS ARE DEFECT ZERO

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Abstract. Let G be a finite group, χ a non-linear irreducible character of G. $\mathrm{dz}(\chi)$ denotes the set of all prime divisors p of $|G|$ such that χ is a $p-$defect zero character, $\mathrm{dz}(G)$ denotes the union of $\mathrm{dz}(\chi)$ for all non-linear irreducible characters χ of G, i.e., $\chi \in \mathrm{Irr}(G|G')$. A finite group G such that $\bigcap_{\chi\in\mathrm{Irr}(G|G')} \mathrm{dz}(\chi) \neq \emptyset$ was studied in 1996. Finite groups G satisfying $\mathrm{dz}(\chi) = \pi(\chi(1))$ for all $\chi \in \mathrm{Irr}(G|G')$ were classified in 2007. We are motivated to study more general case, i.e., a finite group G satisfying $\mathrm{dz}(\chi) \neq \emptyset$ for every $\chi \in \mathrm{Irr}(G|G')$. At first, we study a solvable group G and come to a necessary and sufficient condition. Secondly, for a non-solvable group, we prove that K_3-simple groups can be uniquely determined by $\mathrm{dz}(G)$ and the order of G.
Keywords: finite group, irreducible character, p-defect zero, structure of a group.

1. Introduction

All groups considered are finite, all characters considered are afforded by ordinary representations. Let n be a positive integer, define $\pi(n)$ to be the set of prime divisors of n. For a set of some primes π, we define n_π to be the positive divisor of n such that $n = n_\pi \times k$, where $\pi(n_\pi) \subseteq \pi$ and $\pi(k) \cap \pi = \emptyset$, especially while $\pi = \{p\}$, we write n_p instead of n_π. Let G be a finite group and N a normal subgroup of G, $\pi(G)$ denotes the set of prime divisors of $|G|$, $\mathrm{Irr}(G)$ the set of irreducible characters of G, and $\mathrm{Irr}(G|N) = \{\chi \big| \chi \in \mathrm{Irr}(G) \text{ and } N \nsubseteq Ker(\chi)\}$, $\mathrm{Irr}(G/N) = \{\chi \big| \chi \in \mathrm{Irr}(G) \text{ and } N \subseteq Ker(\chi)\}$. For a prime $p \in \pi(G)$, an irreducible character $\chi \in \mathrm{Irr}(G)$ is called p-defect zero if $\chi(1)_p = |G|_p$. Set

$$\mathrm{dz}(\chi) = \{\ p \in \pi(G) \mid \chi \text{ is } p\text{-defect zero }\} \text{ and } \mathrm{dz}(G) = \bigcup_{\chi\in\mathrm{Irr}(G|G')} \mathrm{dz}(\chi).$$

All other notations are referred to [2] and [4].

In 1996, Ren studied finite groups G such that $\bigcap_{\chi\in\mathrm{Irr}(G|G')} \mathrm{dz}(\chi) \neq \emptyset$ in [7]. In 2007, Liang, Qian and Shi classified finite groups G satisfying $\mathrm{dz}(\chi) = \pi(\chi(1))$ for all $\chi \in \mathrm{Irr}(G|G')$ in [5].

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We are motivated to study more general case, that is, a finite group G satisfying $\mathrm{dz}(\chi) \neq \emptyset$ for every $\chi \in \mathrm{Irr}(G|G')$. We first focus on a finite solvable group, and come to the following theorem in Section 2:

Theorem A. *Let G be a non-abelian solvable group. Then $\mathrm{dz}(\chi) \neq \emptyset$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if there exists a series of normal Hall-subgroups*

$$G = M_0 > M_1 > \cdots > M_r > 1$$

with $r \geqslant 1$, such that following statements hold.

(*a*) *For $0 \leq i \leq r-1$, M_i are non-abelian groups;*

(*b*) *For $1 \leqslant i \leqslant r$, M_{i-1}/M_i is a cyclic Sylow subgroup of G;*

(*c*) *For $1 \leqslant i \leqslant r$ and every $\lambda \in \mathrm{Irr}(M_i/M_i')$, it follows $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$ or M_{i-1}. Moreover for each non-linear irreducible character χ of M_r with $\mathrm{dz}(\lambda) = \emptyset$, if there exists, it follows $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$.*

By Theorem A, we have the following two corollaries.

Corollary A. *Let G be a non-abelian solvable group. Then $\mathrm{dz}(\chi) = \pi(\chi(1))$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if there exists a series of normal Hall subgroups*

$$G = M_0 > M_1 > \cdots > M_r > 1$$

with $r \geqslant 1$, such that following statements hold.

(*a*) *For $0 \leq i \leq r-1$, M_i are non-abelian groups, but M_r is an abelian group;*

(*b*) *For $1 \leqslant i \leqslant r$, M_{i-1}/M_i is a cyclic Sylow subgroup of G;*

(*c*) *For $1 \leqslant i \leqslant r$ and every $\lambda \in \mathrm{Irr}(M_i)$, it follows $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$ or M_{i-1}.*

Corollary B. *Let G be a group and π be a nonempty proper subset of $\pi(G)$. Suppose $\pi \subseteq \mathrm{dz}(\chi)$ for every $\chi \in \mathrm{Irr}(G|G')$. Then $\mathrm{cd}(G) = \{1,\ f\}$ if and only if G has an abelian normal subgroup N of index f such that*

(*a*) *G/N is cyclic;*

(*b*) *for every $\lambda \in \mathrm{Irr}(N)$, $\mathrm{I}_G(\lambda) = G$ or N.*

For a non-solvable group, we cannot get a beautiful result as Theorem A. It is worth to mention a result about simple group, which is in [3, Corollary 2], Granville and Ono proved that every finite simple group M satisfies $\mathrm{dz}(M) = \pi(M)$ with the following exceptions:

(a) M has no character of 2-defect zero if it is isomorphic to M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM, or A_n, where $n \neq 2m^2+m$ nor $2m^2+m+2$ for any integer m;

(b) M has no character of 3-defect zero if it is isomorphic to Suz, Co_3, or A_n with $3n+1 = m^2r$, where r is square free and divisible by some prime $q \equiv 2 \mod 3$.

Above result means that most simple groups M satisfies $\mathrm{dz}(M) = \pi(M)$. But if a finite group G satisfying $\mathrm{dz}(G) = \mathrm{dz}(M)$, can we get $G \cong M$? Surely we cannot. For example, let r be a positive integer such that $(r,\ |M|) = 1$ and C_r a cyclic group of order r, then $\mathrm{dz}(M) = \mathrm{dz}(M \times C_r)$, but $M \ncong M \times C_r$. So it is a meaningful topic to study 'adding what kind of condition to $\mathrm{dz}(G) = \mathrm{dz}(M)$, we can get $G \cong M$'.

Here we try to study K_3-simple groups. A finite group M is called a K_n-group if the order of M has exactly n distinct prime divisors. We shall prove in Section 3 that a K_3-simple group M can be uniquely determined by $\mathrm{dz}(M)$ and $|M|$, that is the following theorem.

Theorem B. *Let G be a finite group, M a K_3-simple group. Then $G \cong M$ if and only if*

(a) $|G| = |M|$,

(b) $\mathrm{dz}(G) = \mathrm{dz}(M)$.

2. Proof of Theorem A

At first we introduce a result in [6].

Lemma 2.1 ([6], Theorem A)**.** *Let G be a group, $1 < N \lhd G$, P a Sylow p-subgroup of G, where $p \in \pi(G)$. Then $p \notin V(G|N)$ if and only if N is a p'-group and P acts frobeniusly on N, where $V(G|N) = \bigcup_{\chi \in \mathrm{Irr}(G|N)} \pi\left(\frac{|G|}{|\ker\chi|\chi(1)}\right)$.*

Lemma 2.2. *Let G be a group, K a normal Hall subgroup of G. If G/K is solvable and $\xi \in \mathrm{Irr}(K)$ is invariant in G. Then ξ has a unique extension $\chi \in \mathrm{Irr}(G)$ with $o(\chi) = o(\xi)$.*

Proof. The lemma follows straight forward from [4, Corollary 6.28]. □

Lemma 2.3. *Let G be a group, π a nonempty proper subset of $\pi(G)$. Suppose $\pi \subseteq \mathrm{dz}(\chi)$ for every $\chi \in \mathrm{Irr}(G|G')$. Then the following statements hold.*

(a) For every $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, P acts frobeniusly on G'. In particular, G is solvable;

(b) $G = H \ltimes K$, where H is a cyclic Hall π-subgroup of G, K is a normal Hall π'-subgroup of G;

(c) For every $\xi \in \mathrm{Irr}(K)$, $\mathrm{I}_G(\xi) = G$ or K. Especially while $\mathrm{I}_G(\xi) = G$, ξ is linear.

Proof. By formula

$$|G| = |G/G'| + \sum_{\chi \in \mathrm{Irr}(G|G')} \chi(1)^2,$$

we have $|G|_\pi \big| |G/G'|$. Hence $1 < G' \lneqq G$. For every $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, it follows by Lemma 2.1 that P acts frobeniusly on G'. Then G' is a nilpotent π'-group. Thus G is solvable.

Since G/G' is abelian, we have $G/G' = HG'/G' \times K/G'$, where H is a Hall π-subgroup of G, K a normal Hall π'-subgroup of G. Note that $HG'/G' \cong H$ is abelian. Then H is cyclic by argument in preceding paragraph.

Let $\xi \in \mathrm{Irr}(K)$. If some constituents of ξ^G are non-linear, say, one is ϕ, then $\phi(1)/\xi(1) \big| |H|$. Since $\pi(H) = \pi$, we have $|H| \big| \phi(1)$ by the hypothesis. Note that $\xi(1)$ is a π'-number. We get $\phi(1) = |H|\xi(1) = \xi^G(1)$. Then $\xi^G = \phi$, i.e., $\mathrm{I}_G(\xi) = K$. If all irreducible constituents of ξ^G are linear, then ξ is extendible to G. Therefore $\mathrm{I}_G(\xi) = G$ and $\xi(1) = 1$. □

Remark 2.1. The statements (a) and (b) have been given in [7, Theorem 1]. However, the approach of proof in [7] is different from ours.

Conversely, assume that G is solvable and satisfies statements (b) and (c) of Lemma 2.3. Let $\chi \in \mathrm{Irr}(G|G')$ and ξ be an irreducible constituent of χ_K. If $\mathrm{I}_G(\xi) = K$, then $\chi = \xi^G$ and $\chi(1) = |H|\xi(1)$. So $\pi \subseteq \mathrm{dz}(\chi)$.

If $\mathrm{I}_G(\xi) = G$, then ξ is linear by (c). By Lemma 2.2, there exists $\phi \in \mathrm{Irr}(G)$ such that $\phi_K = \xi$. Hence characters of form like $\beta\phi$ for $\beta \in \mathrm{Irr}(G/K)$ are all irreducible constituents of ξ^G by [4, Corollary 6.17]. Therefore all irreducible constituents of ξ^G are linear, so χ is linear, a contradiction. Thus we come to the following lemma.

Lemma 2.4. *Let G be a group and π a nonempty proper subset of $\pi(G)$. Then $\pi \subseteq \mathrm{dz}(\chi)$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if G is solvable and satisfies*

(*a*) *$G = H \ltimes K$ where H is a cyclic Hall π-subgroup of G, K is a normal Hall π'-subgroup of G;*

(*b*) *for every $\xi \in \mathrm{Irr}(K)$, $\mathrm{I}_G(\xi) = G$ or K. Especially while $\mathrm{I}_G(\xi) = G$, ξ is linear.*

In order to write the proof of Theorem A to be readable, we make following hypothesis for brevity.

Hypothesis 2.1. *Let X be a non-abelian solvable group. Suppose $\mathrm{dz}(\chi) \neq \emptyset$ holds for every $\chi \in \mathrm{Irr}(X|X')$.*

The Proof of Theorem A. Firstly, we show the necessity. Now G satisfies Hypothesis 2.1. Let K be a maximal normal subgroup of G such that G/K is non-abelian. By [4, Lemma 12.3], it follows that $|\mathrm{cd}(G/K)| = 2$. Let χ be a non-linear irreducible character of G/K, then $\mathrm{cd}(G/K) = \{1,\ \chi(1)\}$, and there exists a prime $p \in \mathrm{dz}(\chi)$. By setting $\pi = \{p\}$ in Lemma 2.4, we see that the Sylow p-subgroup of G/K is cyclic and G/K has a normal Hall p'-subgroup M/K. Since K is a p'-group, we conclude that M is a normal Hall p'-subgroup of G. By Schur-Zassenhaus Theorem the Sylow p-subgroup of G is isomorphic to G/M and is cyclic.

We denote above M as M_1, p as p_1. If M_1 does not satisfy Hypothesis 2.1, we stop and let $r = 1$. If M_1 satisfies Hypothesis 2.1, by the same argument in the preceding paragraph, we get that M_1 has a cyclic Sylow p_2-subgroup and has a normal p_2-complement M_2. If M_2 does not satisfy Hypothesis 2.1, we stop and let r=2. Repeating this process, we obtain a series of normal Hall subgroups

$$G = M_0 > M_1 > \cdots > M_r > 1$$

such that $M_1, M_2, \cdots, M_{r-1}$ satisfy Hypothesis 2.1 and M_r does not. This means the statements (a) and (b) are proved.

Let $|M_{i-1}/M_i| = p_i^{\alpha_i}$, where $p_i^{\alpha_i} \| |G|$, $1 \leqslant i \leqslant r$. For every $\lambda \in \mathrm{Irr}(M_i/M_i')$, if all irreducible constituents of $\lambda^{M_{i-1}}$ are linear, then λ is extendible to M_{i-1}, moreover $\mathrm{I}_{M_{i-1}}(\lambda) = M_{i-1}$. Otherwise there exists $\phi \in \mathrm{Irr}(M_{i-1}|M_{i-1}')$ to be an irreducible constituent of $\lambda^{M_{i-1}}$. Then $\phi(1)/\lambda(1)\big||M_{i-1}/M_i|$, i.e., $\phi(1)\big|p_i^{\alpha_i}$. Since M_{i-1} satisfies Hypothesis 2.1, it follows $\mathrm{dz}(\phi) \neq \emptyset$. So $\phi(1) = p_i^{\alpha_i} = p_i^{\alpha_i}\lambda(1)$, and consequently $\phi = \lambda^{M_{i-1}}$. This implies $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$. We have proved the first part of (c).

Now assume M_r is non-abelian and $\xi \in \mathrm{Irr}(M_r|M_r')$ with $\mathrm{dz}(\xi) = \emptyset$. Let $\varphi \in \mathrm{Irr}(M_{r-1})$ be an irreducible constituent of $\xi^{M_{r-1}}$. Then $\varphi(1)/\xi(1)\big|p_r^{\alpha_r}$. Since M_{r-1} satisfies Hypothesis 2.1 and $\xi(1)$ is a p_r'-number and $\mathrm{dz}(\xi) = \emptyset$, it follows that φ has to be p_r-defect zero, and $\varphi(1) = p_r^{\alpha_r}\xi(1)$, further $\varphi = \xi^{M_{r-1}}$ and $\mathrm{I}_{M_{r-1}}(\xi) = M_r$. The second part of (c) is proved.

Now we prove the sufficiency. Assume $\chi \in \mathrm{Irr}(M_{r-1}|M_{r-1}')$. Let λ be an irreducible constituent of χ_{M_r}.

If λ is linear, then $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ or M_{r-1} by (c). If $\mathrm{I}_{M_{r-1}}(\lambda) = M_{r-1}$, λ is extendible to M_{r-1} by Lemma 2.2. Hence all irreducible constituents of $\lambda^{M_{r-1}}$ are linear by [4, Corollary 6.17] and (b). This contradicts that χ is a non-linear irreducible constituent of $\lambda^{M_{r-1}}$. Hence $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ and $\chi(1) = p_r^{\alpha_r}\lambda(1) = p_r^{\alpha_r}$. So $p_r \in \mathrm{dz}(\chi)$.

If λ is non-linear. Then M_r must be non-ableian. If $\mathrm{dz}(\lambda) \neq \emptyset$, then $\mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi) \neq \emptyset$. Otherwise $\mathrm{dz}(\lambda) = \emptyset$, and it follows by (c) that $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$, which implies that $\chi(1) = p_r^{\alpha_r}\lambda(1)$, $p_r \in \mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi)$. Hence it always follows that $\mathrm{dz}(\chi) \neq \emptyset$. Therefore M_{r-1} satisfies Hypothesis 2.1.

If $r = 1$, then sufficiency follows. Now assume $r > 1$. For any $\chi \in \mathrm{Irr}(M_{r-2}|M_{r-2}')$, let λ be an irreducible constituent of $\chi_{M_{r-1}}$. If λ is linear, then $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$ or M_{r-2} by (c). By the same arguments as in preceding paragraph, we have $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$, and $p_{r-1} \in \mathrm{dz}(\chi)$. If λ is non-linear. Since M_{r-1} satisfies Hypothesis 2.1 , we have $\mathrm{dz}(\lambda) \neq \emptyset$. By $\mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi)$, we have $\mathrm{dz}(\chi) \neq \emptyset$. So anyhow $\mathrm{dz}(\chi) \neq \emptyset$ always follows. Thus M_{r-2} satisfies Hypothesis 2.1. Repeating above process, we get at last that G satisfies Hypothesis 2.1. □

Remark 2.2. By the proof of Theorem A, one can see to that M_i satisfies Hypothesis 2.1, $0 \leqslant i \leqslant r-1$. But M_r does not.

In order to write the proof readable, we make a another hypothesis.

Hypothesis 2.2. Let X be a finite non-abelian solvable group. Suppose $\mathrm{dz}(\chi) = \pi(\chi(1))$ for every $\chi \in \mathrm{Irr}(X|X')$.

The Proof of Corollary A. Firstly, by assumption G satisfies Hypothesis 2.2. By Theorem A, in order to prove statement (a), it is enough to prove M_r is abelian.

By Remark 2.2, we take r as large as possible, and M_r does not satisfy Hypothesis 2.1. By [5, Lemma 2.2], every non-abelian normal Hall subgroup of G satisfies Hypothesis 2.2. Therefore M_r is abelian. The statement (a) follows.

The statement (b) follows trivially from Theorem A.

Now we prove statement (c). For $1 \leqslant i \leqslant r$, let $\lambda \in \mathrm{Irr}(M_i)$ and χ an irreducible constituent of $\lambda^{M_{i-1}}$. Let $|M_{i-1}/M_i| = p_i^{\alpha_i}$, $1 \leqslant i \leqslant r$, where $p_i^{\alpha_i} || \, |G|$. If $\mathrm{dz}(\chi) = \mathrm{dz}(\lambda)$, then $\chi(1) = \lambda(1)$. It follows that χ is an extension of λ, so $\mathrm{I}_{M_{i-1}}(\lambda) = M_{i-1}$. If $\mathrm{dz}(\chi) \neq \mathrm{dz}(\lambda)$. Then $\mathrm{dz}(\chi) \setminus \mathrm{dz}(\lambda) = \{p_i\}$, so $\chi(1) = p_i^{\alpha_i}\lambda(1) = \lambda^{M_{i-1}}(1)$. Hence $\chi = \lambda^{M_{i-1}}$ and $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$.

Conversely, suppose $\chi \in \mathrm{Irr}(M_{r-1}|M'_{r-1})$ and λ is an irreducible constituent of χ_{M_r}. By statement (c), we see that $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ or M_{r-1}. Assume $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$. Then $\chi = \lambda^{M_{r-1}}$ and $\chi(1) = p_r^{\alpha_r} \cdot \lambda(1)$. Notice that λ is linear. Then $\mathrm{dz}(\chi) = \{p_r\} = \pi(\chi(1))$. Now assume $\mathrm{I}_{M_{r-1}}(\lambda) = M_{r-1}$. By Lemma 2.2, λ is extendible to M_{r-1}. By (b) and [4, Corollary 6.17], we have $\chi(1) = \lambda(1) = 1$. This contradicts $\chi \in \mathrm{Irr}(M_{r-1}|M'_{r-1})$. Therefore M_{r-1} satisfies Hypothesis 2.2.

If $r = 1$, then proof is finished. Now assume $r > 1$.

For any $\chi \in \mathrm{Irr}(M_{r-2}|M'_{r-2})$, let λ be an irreducible constituent of $\chi_{M_{r-1}}$. By (c), $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$ or M_{r-2}. Firstly, assume $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$. Then $\chi = \lambda^{M_{r-2}}$ and $\chi(1) = p_{r-1}^{\alpha_{r-1}} \cdot \lambda(1)$. If λ is linear, then $\mathrm{dz}(\chi) = \pi(\chi(1))$. Otherwise λ is non-linear. Since M_{r-1} satisfies Hypothesis 2.2, we have $\mathrm{dz}(\lambda) = \pi(\lambda(1))$. It follows that $\mathrm{dz}(\chi) = \pi(\lambda(1)) \cup \{p_r\} = \pi(\chi(1))$. Secondly, assume $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-2}$. By the same argument as in the preceding paragraph, we get $\chi(1) = \lambda(1)$. It follows that λ is non-linear, which implies $\mathrm{dz}(\lambda) = \pi(\lambda(1))$ since M_{r-1} satisfies Hypothesis 2.2. Therefore it always follows $\mathrm{dz}(\chi) = \pi(\chi(1))$, so M_{r-2} also satisfies Hypothesis 2.2. Repeating this process, we get that G satisfies Hypothesis 2.2. □

The Proof of Corollary B. Suppose $\mathrm{cd}(G) = \{1, f\}$. Let K be a maximal normal subgroup such that G/K is non-abelian. By Lemma 2.4, G is solvable. Thus G/K satisfies the assumption of Lemma 12.3 in [4]. By the proof of Theorem 12.5 in [4], G has an abelian normal subgroup N of index f such that G/N is cyclic.

Assume $\lambda \in \mathrm{Irr}(N)$ and ϕ is a non-linear irreducible constituent of λ^G. Notice that $\lambda^G(1) = |G : N| = f = \phi(1)$, we have that λ^G is irreducible. Hence $\mathrm{I}_G(\lambda) = N$.

Conversely, let $\chi \in \mathrm{Irr}(G|G')$ and λ be an irreducible constituent of χ_N. If $\mathrm{I}_G(\lambda) = N$, then $\lambda^G = \chi$ and $\chi(1) = |G : N| = f$.

Assume $\mathrm{I}_G(\lambda) = G$. We assert that λ^G has no linear irreducible constituent. Otherwise, let ϕ be a linear irreducible constituent of λ^G, then $\phi_N = \lambda$. By [4, Corollary 6.17], all irreducible constituents of λ^G are linear, a contradiction to χ is a non-linear irreducible constituent of λ^G.

Let $\mathrm{Irr}(\lambda^G) = \{\chi_1 = \chi,\ \chi_2,\ \cdots,\ \chi_t\}$, where $\chi_i \in \mathrm{Irr}(G|G')$, $i = 1,\ 2,\ \cdots,\ t$. By Frobenius reciprocity, we have

$$[\chi_i, \lambda^G] = [(\chi_i)_N, \lambda] = \chi_i(1), \quad i = 1, 2, \cdots, t.$$

This implies $\lambda^G = \sum_{i=1}^t \chi_i(1)\chi_i$, so $\lambda^G(1) = \sum_{i=1}^t \chi_i(1)^2$. Thus $\sum_{i=1}^t \chi_i(1)^2 = f$.

For $p \in \pi$, let $p^r \big|\big| |G|$. By the assumption, we have $p^r | \chi_i(1)$, so $p^{2r} | f$. But this contradicts $f \big| |G|$ and $p^r \big|\big| |G|$. The proof is finished. □

3. Proof of Theorem B

Lemma 3.1. *Let G be a group. If $\mathrm{dz}(G) = \pi(G)$, then G is a non-solvable group. Moreover, the minimal subnormal subgroup of G is a non-abelian simple group.*

Proof. We assert that every nontrivial normal subgroup N of G is non-abelian. Otherwise, let N be an abelian non-trivial subgroup of G, then $\chi(1) \big| |G : N|$ for every $\chi \in \mathrm{Irr}(G)$. But by assumption, for every $p \in \pi(N)$, there exists $\xi \in \mathrm{Irr}(G)$ such that $\xi(1)_p = |G|_p$, a contradiction.

By the fact that every finite solvable group contains an abelian minimal normal subgroup, we get that G is non-solvable.

By Clifford Theorem, it is easy to see that the condition $\mathrm{dz}(G) = \pi(G)$ is inherited by a normal subgroup, thus a minimal subnormal subgroup of G must be a non-abelian simple group. □

Since the group of order $p^a q^b$ is solvable, a finite group G satisfying $\mathrm{dz}(G) = \pi(G)$ has at least three prime factors by Lemma 3.1. That is to say, when we investigate a finite group G satisfying $\mathrm{dz}(G) = \pi(G)$, we must start from investigating a finite group with $|\pi(G)| = 3$, i.e., a K_3-simple group, they are A_5, A_6, $L_2(7)(= L_3(2))$, $L_2(17)$, $L_3(3)$, $U_4(2)(= S_4(3))$, $L_2(8)(= R(3)')$ and $U_3(3)$. Now we set up following lemma.

Lemma 3.2. *Let G be a group and $|G| = p^a q^b r$, where p, q, r are distinct primes with $p < q < r$. Suppose that G satisfies $\mathrm{dz}(G) = \pi(G)$. Then G is one of K_3-simple groups, $Aut(L_2(8))$, $Aut(L_3(3))$, $Aut(U_3(3))$ and subgroups of $Aut(A_6)$.*

Proof. Since every minimal normal subgroup N of G is a direct product of isomorphic non-abelian simple group by Lemma 3.1, we have $pqr \big| |N|$. But $r \big|\big| |G|$, so N is a non-abelian simple group and is a unique minimal normal subgroup of G. Consequently $C_G(N) = 1$, thus $N \trianglelefteq G \lesssim \mathrm{Aut}(N)$, and N is a K_3-simple group. For $N = A_5,\ A_6,\ L_2(7),\ L_2(17),\ L_3(3),\ U_4(2),\ L_2(8)$ and $U_3(3)$, checking character tables of group G such that $N \trianglelefteq G \lesssim \mathrm{Aut}(N)$ in [1], we found that $\mathrm{dz}(G) = \pi(G)$ holds only if $G = N$ is a K_3-group, $Aut(L_2(8))$, $Aut(L_3(3))$, $Aut(U_3(3))$ or a subgroup of $Aut(A_6)$. This completes the proof. □

The Proof of Theorem B. It is enough to prove the sufficiency. For any K_3-simple group M, it always follows $\mathrm{dz}(M) = \pi(M)$. By assumption $|G| = |M|$, one has that $\pi(G) = \pi(M)$, and $\mathrm{dz}(G) = \mathrm{dz}(M) = \pi(M) = \pi(G)$. Hence G satisfies conditions of Lemma 3.2, G is a K_3-simple group. The sufficiency follows from $|G| = |M|$. Thus Theorem B follows. $\square$

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ON TWO NEW APPROACHES IN MODULAR SPACES

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Abstract. In this paper, we prove a Reich-type fixed point theorem in modular spaces. Also, we introduce the concept of h-convex modular spaces and we get the related Banach-type theorem. Our results generalize several ones in the existing literature. Moreover, some examples are given supporting theoretical approaches.

Keywords: fixed point, metric modular, h-convex modular.

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1. Introduction and preliminaries

Two very important kinds of the notions in mathematical analysis are well known: modular function spaces and modular metric spaces. For details on modular function spaces, readers can see [22], while for modular metric spaces see [9].

Both the kinds of modular concept are in fact generalizations of the standard metric spaces, was introduced by Nakano [27] and was intensively developed by several authors. Example 2.1 presented by Abdon and Khamsi [1] is an important motivation for developing the theory of modular metric spaces. Also, see the introduction section in [3].

Otherwise, modular function spaces, that is, modular metric spaces theory has many applications for example in physical interpretations of the modular, the electrorheological fluids, economy, engineering, further in applications to integral operators, approximation and fixed point results. For more details, see ([1]-[7], [9]-[12], [14]-[28]).

Now, we will begin with a brief recollection of basic notions and the facts in modular (metric) spaces. Let X be a nonempty set and $w : (0,\infty) \times X \times X \to [0,\infty]$ be a function. For simplicity, it is denoted by

$$w_\lambda(x,y) = w(\lambda,x,y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 ([9]). *Let X be a nonempty set. Assume that the map $\omega : (0,\infty) \times X \times X \to [0,\infty]$ satisfies the following conditions for all $x, y, z \in X$:*

(1) $\omega_\lambda(x,y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$*;*

(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ *for all* $\lambda > 0$*;*

(3) $\omega_{\lambda+\mu}(x,z) \leq \omega_\lambda(x,y) + \omega_\mu(y,z)$ *for all* $\lambda, \mu > 0$*.*

In this case, ω is a said a (metric) modular on X.

A (metric) modular w is said to be strict if it has the following property: given $x, y \in X$ with $x \neq y$, we have $\omega(\lambda, x, y) > 0$ for all $\lambda > 0$.

A (metric) modular w on X is said to be convex [10] if, instead of (3), it satisfies the stronger inequality:

$$\omega_{\lambda+\mu}(x,z) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,y) + \frac{\mu}{\mu+\lambda}\omega_\mu(y,z) \tag{1.1}$$

for all $\lambda, \mu > 0$.

Let (X,d) be a metric space such that X has at least two elements. Then we obtain the following examples of a metric modular w.

Example 1.1 ([3])**.** Let

$$\omega(\lambda,x,y)=w_\lambda(x,y)=d(x,y),$$

for all $\lambda>0$. This modular is not convex. Indeed, putting $z=y$ and $\mu=\lambda$ in (1.1), the result follows.

Example 1.2 ([3])**.** Let

$$\omega(\lambda,x,y)=w_\lambda(x,y)=\frac{d(x,y)}{\lambda},$$

for all $\lambda>0$. This modular is convex.

Example 1.3 ([3])**.** Let

$$w_\lambda(x,y)=\begin{cases}\infty \text{ if } \lambda<d(x,y),\\ 0 \text{ if } \lambda\geq d(x,y).\end{cases}$$

This modular w is also convex.

Example 1.4. Let $X=\mathbb{R}$ be endowed with the mapping $w_\lambda:(0,\infty)\times X\times X$ defined as

$$w_\lambda(x,y)=\begin{cases}\frac{|x|+|y|}{\lambda}, \text{ if } x\neq y\\ 0 \text{ if } x=y\end{cases}$$

for all $x,y\in X$ and $\lambda>0$. It is clear that this mapping is a modular.

Definition 1.2 ([9, 11])**.** *Given a modular ω on X, the two sets*

$$X_\omega\equiv X_\omega(x_0)=\{x\in X:\omega_\lambda(x,x_0)\to 0 \text{ as } \lambda\to\infty\}$$

and

$$X_\omega^*\equiv X_\omega^*(x_0)=\{x\in X:\exists\lambda=\lambda(x)>0 \text{ such that } \omega_\lambda(x,x_0)<\infty\}$$

are said to be modular spaces (around x_0).

It is not hard to see that $X_w\equiv X_w(x_0)=\{x_0\}$ in Examples 1.1 and 1.2, while $X_w^*\equiv X_w^*(x_0)=X$ in both cases.

Definition 1.3 ([10])**.** *Let ω be a modular on X. A sequence $\{x_n\}$ in X is said to be ω-convergent (or modular convergent) to an element $x\in X$ if there exists a number $\lambda>0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty}\omega_\lambda(x_n,x)=0$. A sequence $\{x_n\}$ in X is said to be ω-Cauchy if there exists $\lambda>0$, possibly depending on the sequence, such that $\omega_\lambda(x_m,x_n)\to 0$ as $m,n\to\infty$. X is said to be $\omega-$complete if every $\omega-$Cauchy sequence is $\omega-$convergent.*

Definition 1.4 ([10]). *Let w be a modular on a set X and let X_ω^* be a modular set. A mapping $T : X_\omega^* \to X_\omega^*$ is said to be modular contractive (or an $\omega-$contraction) if there exist numbers $k \in (0,1)$ and $\lambda_0 > 0$ such that*

$$\omega_{k\lambda}(Tx, Ty) \leq \omega_\lambda(x, y) \tag{1.2}$$

for all $0 < \lambda \leq \lambda_0$ and for all $x, y \in X_\omega^*$.

Definition 1.5 ([10]). *Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T : X_w^* \to X_w^*$ is said to be strongly modular contractive (or a strongly ω-contraction) if there exist $k \in (0,1)$ and $\lambda_0 > 0$ such that*

$$w_{k\lambda}(Tx, Ty) \leq k w_\lambda(x, y) \tag{1.3}$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

The following proposition is the key in enough proofs in the context of modular metric spaces.

Proposition 1.1 ([3], Proposition 2.4). *Let w be a modular on the set X.*

(a) *For every $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing;*

(b) *Let w be a convex modular. For $x, y \in X$, if $w_\lambda(x, y)$ is finite for at least one value of λ, then $w_\lambda(x, y) \to 0$ as $\lambda \to \infty$ and $w_\lambda(x, y) \to \infty$ as $\lambda \to 0^+$;*

(c) *If w is a convex modular, then the function $v_\lambda(x, y) = \frac{w_\lambda(x,y)}{\lambda}$ is a modular on X.*

2. Reich-type theorem in modular spaces

Definition 2.1. *Let w be a modular on a set X and let X_ω^* be a modular set. A mapping $T : X_\omega^* \to X_\omega^*$ is said to be a Reich $\omega-$contraction if there exist $a, b, c \in (0,1)$ with $a + b + c < 1$ and $\lambda_0 > 0$ such that*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{a}}(x, y) + \omega_{\frac{\lambda}{b}}(x, Tx) + \omega_{\frac{\lambda}{c}}(y, Ty), \tag{2.1}$$

for all $0 < \lambda \leq \lambda_0$ and for all $x, y \in X_\omega^*$.

The following theorem is an analog of the fixed point theorem by Reich [29] in the framework of modular spaces.

Theorem 2.1. *Let ω be a strict convex modular on X such that the modular space X_ω^* is ω-complete and let $T : X_\omega^* \to X_\omega^*$ be a Reich ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X_\omega^*$ such that $\omega_\lambda(x, Tx) < \infty$. Then T has a fixed point x^* in X_ω^*. If the modular w assumes only finite values on X_ω^*, then the condition $\omega_\lambda(x, Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X_\omega^*$, the sequence of iterates $T^n x_0$ is modular convergent to x^*.*

Proof. Let $x_0 \in X^*_\omega$ and $x_n = T^n x_0, n = 0, 1, 2, \ldots$ Putting $x = x_n$ and $y = x_{n-1}$ in inequality (2.1), we obtain

$$\omega_\lambda(x_{n+1}, x_n) \leq \omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n). \tag{2.2}$$

Next, we have

$$\begin{aligned} \omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) &= \omega_{\lambda + \lambda\frac{1-a}{a}}(x_n, x_{n-1}) \\ &\leq \frac{\lambda}{\frac{\lambda}{a}}\omega_\lambda(x_n, x_{n-1}) + \frac{\lambda\frac{1-a}{a}}{\frac{\lambda}{a}}\omega_{\lambda\frac{1-a}{a}}(x_{n-1}, x_{n-1}) \\ &= a\omega_\lambda(x_n, x_{n-1}). \end{aligned}$$

So,

$$\omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) \leq a\omega_\lambda(x_n, x_{n-1}). \tag{2.3}$$

Similarly,

$$\omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) \leq b\omega_\lambda(x_n, x_{n+1}) \tag{2.4}$$

and

$$\omega_{\frac{\lambda}{c}}(x_n, x_{n-1}) \leq c\omega_\lambda(x_n, x_{n-1}). \tag{2.5}$$

Using (2.2), (2.3), (2.4) and (2.5), we obtain that

$$\omega_\lambda(x_{n+1}, x_n) \leq \frac{a+c}{1-b}\omega_\lambda(x_n, x_{n-1}). \tag{2.6}$$

From (2.6), by induction we have

$$\omega_\lambda(x_{n+1}, x_n) \leq \left(\frac{a+c}{1-b}\right)^n \omega_\lambda(x_1, x_0). \tag{2.7}$$

If we put $= x_{m-1}$ and $y = x_{n-1}$ in the inequality (2.1), we get that

$$\omega_\lambda(x_m, x_n) \leq \omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_{m-1}, x_m) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n). \tag{2.8}$$

On the other hand, we have

$$\begin{aligned} \omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) &= \omega_{\frac{\lambda(1-a)}{2a} + \lambda + \frac{\lambda(1-a)}{2a}}(x_{m-1}, x_{n-1}) \\ &\leq \frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_{m-1}, x_m) + a\omega_\lambda(x_m, x_n) \\ &+ \frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_n, x_{n-1}). \end{aligned}$$

Using (2.8), we get

$$\begin{aligned}(1-a)\omega_\lambda(x_m,x_n) &\le \omega_{\frac{\lambda}{b}}(x_{m-1},x_m)+\frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_{m-1},x_m)\\ &+ \omega_{\frac{\lambda}{c}}(x_{n-1},x_n)+\frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_n,x_{n-1}).\end{aligned}$$

Now, by (2.7), we conclude that $\{x_n\}$ is a Cauchy sequence. Since ω is strict, the modular limit X^* of the sequence $\{x_n\}$ is determined uniquely. Let us show that x^* is a fixed point of T. We have the following

$$\begin{aligned}\omega_{\frac{\lambda}{c}}(x^*,Tx^*) &= \omega_{\frac{\lambda(1-c)}{c}+\lambda}(x^*,Tx^*)\\ &\le (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c\omega_\lambda(x_{n+1},Tx^*)\\ &= (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c\omega_\lambda(Tx_n,Tx^*)\\ &\le (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c[\omega_{\frac{\lambda}{a}}(x_n,x^*)\\ &+ \omega_{\frac{\lambda}{b}}(x_n,x_{n+1})+\omega_{\frac{\lambda}{c}}(x^*,Tx^*)].\end{aligned}$$

If we let $n\to\infty$ in the last inequality, we get

$$\omega_{\frac{\lambda}{c}}(x^*,Tx^*)\le c\omega_{\frac{\lambda}{c}}(x^*,Tx^*),$$

so, $Tx^*=x^*$.

Uniqueness. Let x^* and y^* be two fixed points. Then we have

$$\begin{aligned}\omega_\lambda(x^*,y^*) &= \omega_\lambda(Tx^*,Ty^*)\\ &\le \omega_{\frac{\lambda}{a}}(x^*,y^*)+\omega_{\frac{\lambda}{b}}(x^*,Tx^*)\\ &+ \omega_{\frac{\lambda}{c}}(y^*,Ty^*)\\ &\le a\omega_\lambda(x^*,y^*),\end{aligned}$$

so, $x^*=y^*$. □

Corollary 2.1. *If in the condition* (2.1)*, b and c tend to 0^+, then we obtain the Banach contraction principle in the context of strict convex modular metric spaces.*

Corollary 2.2. *If in the condition* (2.1)*, a tends to 0^+, then we obtain the Kannan-type contraction result in the context of strict convex modular metric spaces.*

Now, we give an example supporting Theorem 2.1.

Example 2.1. Let $X_w=[0,1]$ be endowed with the modular metric

$$w_\lambda(x,y)=\begin{cases}\frac{x+y}{\lambda}, & \text{if } x\neq y\\ 0, & \text{if } x=y\end{cases}$$

for all $x, y \in X_w$ and $\lambda > 0$. Clearly, (X_w, w_λ) is a $w-$complete modular metric space. Define the self-mapping $T : X_w \to X_w$ by $Tx = \frac{x}{2}$ and put $a = b = c = \frac{1}{4}$. It is easy to check that all the conditions of Theorem 2.1 hold and T has a fixed point, which is $x = 0$.

Definition 2.2. *Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T : X_w^* \to X_w^*$ is said to be a Hardy-Rogers ([28], (18)) $\omega-$ contraction if there exist $a, b, c, d, e \in (0, 1)$ with $a + b + c + d + e < 1$ and $\lambda_0 > 0$ such that*
(2.9)
$$w_\lambda (Tx, Ty) \le w_{\frac{\lambda}{a}} (x, y) + w_{\frac{\lambda}{b}} (x, Tx) + w_{\frac{\lambda}{c}} (y, Ty) + w_{\frac{\lambda}{d}} (x, Ty) + w_{\frac{\lambda}{e}} (y, Tx),$$

for all $0 < \lambda \le \lambda_0$ and all $x, y \in X_w^*$.

Now, we present the following open question (Hardy-Rogers $\omega-$ contraction): Prove or disprove the following statement:

• Let w be a strict convex modular on X such that the modular space X_w^* is w-complete and let $T : X_w^* \to X_w^*$ be a Hardy-Rogers ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X_w^*$ such that $w_\lambda (x, Tx) < \infty$. Then T has a fixed point x^* in X_w^*. If the modular w assumes only finite values on X_w^*, then the condition $w_\lambda (x, Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X_w^*$, the sequence of iterates $T^n x_0$ is modular convergent to x^*.

3. Contractions on h-convex modular spaces

We introduce the concept of $C-$type functions.

Definition 3.1. *A function $h : (0, 1) \to [0, \infty)$ is said to be $C-$type if it has the following properties:*

(C_1) *there exists $C_h > 0$ such that $h(x)h(y) \le C_h h(xy)$ for all $x, y \in (0, 1)$;*

(C_2) $\lim_{t \to 0+} h(t) = 0$.

Definition 3.2. *Let X be a nonempty set and h be a $C-$type function. A map $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a h-convex modular if for all $x, y, z \in X$, the following conditions are satisfied:*

(1_h) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;

(2_h) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$;

(3_h) $\omega_{\lambda+\mu}(x, z) \le h\left(\frac{\lambda}{\lambda+\mu}\right) \omega_\lambda(x, y) + h\left(\frac{\mu}{\mu+\lambda}\right) \omega_\mu(y, z)$ *for all* $\lambda, \mu > 0$.

A triplet (X, ω, h) is called a h-convex modular space.

Note that, if $h(x) = x$, then we obtain a convex modular. A convex modular is also a h-convex modular, but the converse is not true in general.

Example 3.1. 1. Let $h_1 : (0,1) \to [0,\infty)$ be a function defined by $h_1(x) = x^s$ with $s \in (0,1]$. Then h_1 is a $C-$type with $C_{h_1} = 1$ and we obtain

$$\omega_{\lambda+\mu}(x,z) \leq \left(\frac{\lambda}{\lambda+\mu}\right)^s \omega_\lambda(x,y) + \left(\frac{\mu}{\mu+\lambda}\right)^s \omega_\mu(y,z)$$

for all $\lambda, \mu > 0$.

2. Similarly, let $h_2 : (0,1) \to [0,\infty)$ be a function defined by $h_2(x) = sx$ with $s \geq 1$. Then h_2 is a $C-$type with $C_{h_2} = s$ and we obtain

$$\omega_{\lambda+\mu}(x,z) \leq s\left[\frac{\lambda}{\lambda+\mu}\omega_\lambda(x,y) + \frac{\mu}{\mu+\lambda}\omega_\mu(y,z)\right]$$

for all $\lambda, \mu > 0$.

Lemma 3.1. *Let ω be an $h-$convex modular. If for $x, y \in X$, $\omega_\lambda(x,y)$ is finite for at least one value of λ, then $\omega_\alpha(x,y) \to 0$ as $\alpha \to \infty$.*

Proof. If we set $z = y$ in condition (3_h), then

$$\omega_{\lambda+\mu}(x,y) \leq h\left(\frac{\lambda}{\lambda+\mu}\right)\omega_\lambda(x,y).$$

Now, from condition (C_2), it follows the assertion (by taking $\alpha = \lambda + \mu$ and $\mu \to \infty$). □

The following theorem is the analogue of Banach contraction principle in h-convex modular spaces.

Theorem 3.1. *Let ω be a strict h-convex modular on X such that the modular space X^*_ω is ω-complete and let $T : X^*_\omega \to X^*_\omega$ be a ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x, Tx) < \infty$. Then T has a fixed point x^* in X^*_ω. If the modular w assumes only finite values on X^*_ω, then the condition $\omega_\lambda(x, Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X^*_\omega$ the sequence of iterates $T^n x_0$ is modular convergent to x^*.*

Proof. From the condition $\lim\limits_{t\to 0+} h(t) = 0$, we obtain that there exists $n_0 \in \mathbb{N}$ such that

$$h(k^{n_0}) < \min\left\{1, \frac{1}{C_h}\right\}. \tag{3.1}$$

Further, let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 + \lambda_2 + k^{n_0}\lambda = \lambda$. Since ω is $h-$convex modular, we obtain that

$$\begin{aligned}
\omega_{\lambda_1+k^{n_0}\lambda+\lambda_2}(x,y) &\leq h(\frac{\lambda_1+k^{n_0}\lambda}{\lambda})\omega_{\lambda_1+k^{n_0}\lambda}w(x,T^{n_0}y)\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)\\
&\leq h(\frac{\lambda_1+k^{n_0}\lambda}{\lambda})[h(\frac{\lambda_1}{\lambda_1+k^{n_0}\lambda})\omega_{\lambda_1}(x,T^{n_0}x)\\
&+ h(\frac{k^{n_0}\lambda}{\lambda_1+k^{n_0}\lambda})\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y)]\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)\\
&\leq C_h[h(\frac{\lambda_1}{\lambda}t)\omega_{\lambda_1}(x,T^{n_0}x)+h(k^{n_0})\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y)]\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y).
\end{aligned}$$

The condition (1.2) yields that

$$\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y)\leq \omega_\lambda(x,y),$$

so we obtain

$$\omega_\lambda(x,y)\leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{\lambda_1}(x,T^{n_0}x)+h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)}{1-C_h h(k^{n_0})}. \tag{3.2}$$

For $x_0 \in X^*_\omega$, let $x_n = T^n x_0$. By (3.2), we obtain

$$\begin{aligned}
\omega_\lambda(x_m,x_n) &\leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{\lambda_1}(x_m,x_{m+n_0})+h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(x_{n+n_0},x_n)}{1-C_h h(k^{n_0})}\\
&\leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{k^{-m}\lambda_1}(x_0,x_{n_0})+h(\frac{\lambda_2}{\lambda})\omega_{k^{-n}\lambda_2}(x_{n_0},x_0)}{1-C_h h(k^{n_0})}\\
&\to 0 \text{ as } m,n\to\infty,
\end{aligned}$$

that is, $\{x_n\}$ is Cauchy. Let $\lim_{n\to\infty} x_n = x^*$. Then we have

$$\begin{aligned}
\omega_{(k+1)\lambda}(Tx^*,x^*) &\leq h\left(\frac{k\lambda}{(k+1)\lambda}\right)\omega_{k\lambda}(Tx^*,x_n)+h\left(\frac{\lambda}{(k+1)\lambda}\right)\omega_\lambda(x_n,x^*)\\
&\leq h\left(\frac{k}{k+1}\right)\omega_\lambda(x^*,x_{n-1})+h\left(\frac{1}{k+1}\right)\omega_\lambda(x_n,x^*)\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

By the strictness of ω, $Tx^* = x^*$. Suppose $x^*, y^* \in X^*_\omega$ are such that $Tx^* = x^*$ and $Ty^* = y^*$. Then we have

$$\begin{aligned}
\omega_\lambda(x^*,y^*) &\leq h(k^{n_0})\omega_{k^{n_0}\lambda}(x^*,y^*)+h(1-k^{n_0})\omega_{(1-k^{n_0})\lambda}(y^*,y^*)\\
&= h(k^{n_0})\omega_{k^{n_0}\lambda}(x^*,y^*)\\
&= h(k^{n_0})\omega_{k^{n_0}\lambda}(T^{n_0}x^*,T^{n_0}y^*)\\
&\leq h(k^{n_0})\omega_\lambda(x^*,y^*).
\end{aligned}$$

Since $\omega_\lambda(x^*, y^*)$ is finite, by inequality (3.1), we obtain $x^* = y^*$. □

Remark 3.1. For some details on so-called h-convexity, see [31].

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ACCRUAL BASED AND REAL EARNING MANAGEMENT ASSOCIATION WITH DIVIDENDS POLICY "THE CASE OF JORDAN"

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Abstract. This study aims to examine the association between earning management and dividend policy. Industrial companies listed in Amman Stock Exchange (ASE) are used as a sample for the years from 2010 to 2016. The dividend policy (DP) measured by dividends (DPO) payout ratio while earnings management (EM) measured by discretionary accruals (DAER) and real earning management (REM). Variables such as Firm size (SIZE), return on equity (ROE), financial leverage (LIV) and market to book value (MB) were also utilized as control variables. Ordinary least square was used to estimate the model built for the study and modified Cross Sectional models of this paper adopts the two modified Models of Kothari et al. (2005) and Roychowdhury model (2006). Our results show that dividend policy of a firm has a positive significant association with earnings management for. On the other hand the association between dividends policy and real earning management it has not been supported statistically.

Keywords: earning management, real earning management, return on equity, firm size, financial leverage, dividend policy, market to book value.

1. Introduction

Firm's financial reports are aimed to provide an accurate, true and faithful picture about the firms' activities and financial situation. The accurateness of such reports are approved when it is prepared based on historical date but when it is dealt with future uncertainties it become complex and questionable. Therefore, accounting uncertainties such as provisions and accruals may be seen a questionable act by different users of financial reports. The issuance of International Accounting Standard (IAS, 37) worked as a mechanism to prohibit manager's discretion from preparing accounting information for their benefit. Even nowadays, this problem still existed between managers and external information users about the asymmetric information; management always seeks to present accounting results using the most favorable methods by making a great one-off provision in years with high levels of underlying earnings was generated. Such provisions are called big-bath provisions, which were in turn made available to shield expenditure in future years when the earnings are not so good. That is, provisions are being used for earnings smoothened and the stakeholders are made to believe by relying on the financial statements produced, that the firm is performing well (Monsuru and Adetunji, 2014)

After several worldwide scandals such as Enron, WorldCom earning management practices gained more interest by external users of financial reports; auditors, analysts and tax authorities gave more attention to unusual results from earning management practices. This issue also promoted researchers and scholars to devote some of their efforts to bring more insight to the problem in terms of legitimacy and the side effects for reflecting earning management numbers in financial statements. Given the importance of historical dividend policy to firms dividends can has been used as a predictor of earnings whereas earnings can also be used as a predictor of dividends. Thus both of these concepts are interrelated as both determine each other's value. This mutual relationship helps to understand why managers of a firm are more interested in maximizing the firm's earnings. Earnings are considered the most important item that signals how much firm is involved in value adding services. With this much increased emphasis over importance of earnings for a firm, it is no wonder if company management takes vital interest in the manner their earnings are reported. An increase in earnings depicts the increase in overall value of a company and vice versa (Lev, 1989). Particularly to conceal the losses of a company, earnings are managed to show beneficial situation (Hayn, 1995). This presents the idea of earnings management that is use of accounting choices to amend reported earnings for the sake of managers' benefits.

Corporate managers in their daily course of action make a number of crucial decisions related to their entities financial status. From such decisions, Dividend payout decisions which is regarded as one of the most crucial financial decisions to the entity (Baker and Powell, 1999). The dividend payout ratio is regarded as a symbol for good financial health of entity; hence, managers work hardily to improve this ratio in the eyes of external users. According to efficient market theory value of firm cannot be enhanced by increasing the dividends payout ratio due to the fact that perfect efficient market unobtainable (Miller and Modigliani, 1961). Although prior literature provided evidence in favor for the use of earning management to effect the dividends payout ratio and thus increase up the market value (e.g. Clientele theory, bird-in-hand theory, agency theory, and signaling theory); many arguments were raised that such act might have been managed and cooked to suite dividends distributions. Therefore, this study is promoted by the need to bring more insight to this issue by investigating the role of earning management in effecting the dividends policy in emerged country Jordan as example. In Jordan, several authors have investigated the relationship between corporate governance and earnings management (e.g. Fayoumi et al., 2010; Abbadi et al., 2016), but the relationship between earnings management and dividend policy has not been ascertained. As a contribution to the existing literature, this present study intends to examine the association between earnings management on dividend policy using quoted financial data for industrial companies in Jordan.

The rest of the paper is sectioned as follows; literature review hypotheses development, methodology, discussion of results and discussion and the last section for conclusion.

2. Literature review and development of hypotheses

It is will know from literatures earnings management occurs when managers use judgment in financial reporting to alter financial reports and mislead some stakeholders about the underlying economic performance of the company. (Akers et al, 2007) indicates that earnings management is attempts by management to influence or manipulate reported earnings by using specific accounting methods, recognizing, deferring or accelerating expense or revenue transactions, or influencing short-term earnings, to achieve stable and predictable financial results.

According to the agency theory high level of information asymmetry leads to several agency conflicts e.g. moral hazard between managers (agent) and owners (principle). Information asymmetry is attributed to the lack of available public information about the firm when it was a private company (Aharony et al., 1993). Monsuru and Adetunji (2014) argued that asymmetric information opens the door for managers to manipulate the accounting information for their own benefits and achieve private gains by use their discretion to mislead stakeholders about the economic performance of the company or to influence contractual outcomes. Likewise, Bukit and Iskandar (2009) in their outcomes also found that misrepresentation and lake of disclosure in financial information are normal results for the behavior of earnings management through using bonus plans, implicit contract, need for external financing, political and regulatory process and earnings decreases or losses. Earnings management is regarded as one of the major indicators of earnings quality, since earnings management can be utilized to report more relevant information about earnings numbers that reflects true performance of the institutions (Dechow et al, 2010).

Many of previous literature on earning management suggests that Firms confirmed the use of some earnings management strategies, i.e., accrual-based and real earnings management, in order to manage their earnings (e.g. Cohen and Zarowin, 2010; Dechow et al., 2010; Badertscher, 2011; Kothari et al., 2012). Accrual-based earnings management is known as; choosing accounting policies from a set of accounting policies in order to hit earnings objectives whether upward or downward. Other strategies for earning management such as real earnings management is undertaken by managers by changing the time or restructuring of operations to deviate from normal business practices, like manipulating sales, reducing discretionary expenditures and overproducing inventory in order to decrease the costs of goods sold, (Roychowdhury, 2006). Literally real earnings management concept is regarded to be more expensive than that of accrual-based earnings management (Graham et al., 2005; Kim and Sohn, 2013). Darrough and Rangan (2005) and Mizik and Jacobson (2008) argued that greatest effect of accrual-based and real earnings management would be attained through a coordinated use of both tools.

Accrual-based or real earnings management is used by companies to manage their earnings (Badertscher, 2011). These strategies are basically constructed on selecting the accounting policies, estimates or changing accounting methods within the generally accepted proper accounting principles to achieve earnings desired level (Dechow and Skinner, 2000). Although of real earning management complexity it is a preferred option for managers since it is hard to detected, and applied with high secrecy (Manowan and Lin, 2013). On the other hand accruals earnings management is generally more constraint to specific times and period's while real earnings management can be applied throughout the year, (Zang, 2012).

A great deal of the extant literature has examined how earning management can affect dividend policy but the direction of this relationship still ambiguous and mixed. Monsuru and Adetunji (2014) when investigated the effect of earnings management on dividend policy in Nigeria, concluded that if managers increase discretionary accruals of company dividend percentage will not increase; the same conclusion was found by shah (2010) who showed that earnings management measured by discretionary accruals has no impact on dividend payout policy for Pakistani Listed Companies and Chinese listed Companies. Several other studies also found the same results and denied the existence of such relationship (Haider et al, 2012; Aurangzeb and Dilawe, 2012).

Despite of previous literature that denies the relationship between earning management and dividends policy, other studies results proved this relationship. Im et al, (2015) when examined discretionary accruals and real earnings management, found a pressure on firms to maintain consistency in the payout of dividends, this turn managers to become dependable on earnings management as a means to increase the income from which dividends are paid especially when facing decreased earnings or losses. In the same direction Chansarn and Chansarn (2016) when investigated the influence of earnings management on dividend policy in Thailand revealed that earnings management is positively correlated to dividends. Similarly, for a sample of German firms, Savov (2006) showed that firms having high investments tend to report more discretionary accrual in their earnings.

Several studies suggested that managers are likely conduct in earning management as an attempt to increase earnings when they realize that their firms are not likely to meet the earnings level required for dividend payouts thus manipulate earnings upward to maintain dividend levels (Morghri and Galogah, 2013; Chansarn and Chansarn, 2016). Some researchers argued that, high dividend payout ratios lead to more informative earnings and there is positive impact for dividend policy on informativeness of reported earnings in countries with relatively poor information environment (Farooq, 2018). The explanation of this is increased sensitivity of earnings management to permanent earnings and tendency to increase dividend payment to attract investors and thus leading to an increase in stock prices. Greater earnings numbers and stock prices do not necessarily reflect the true performance and earnings of companies because earnings management reflects manager's intent to present the best financial picture of the company outcomes through discretionary accruals (Baatour et al, 2017).

Although prior studies have found evidence that Jordanian public firms utilize earnings management to meet several targets (Abed et al., 2012; Hamdan et al., 2013; Abu Jebril and Al.Thuneibat, 2016; Alzoubi, 2016; Alqatamin et al., 2017; Ibrahim and Al Awawdeh, 2017, Alhadab, 2018; Alkurdi et al., 2017), no studies to date have examined the relationship between dividends policy and earning management in Jordan. Based on our prior review and discussion of related letreture we assume the following two hypotheses for our study.

H1: There Is An Association Between Dividends Policy And Earning Management Engagement Using Discretionary Accruals.

H2: There Is An Association Between Dividends Policy And Earning Management Engagement Using Real Earning Management.

3. Methodology

3.1 Method and sample

Investigating the study hypothesis was based on applying multiple regression analysis using Ordinary Least square (OLS) in order to examine the relationship between dividend policy and earnings management. The population used in this study comprises the listed industrial companies in Amman stock exchange. The industrial sector in Jordan is very crucial sector to our Jordanian economy, and regarded as the biggest sector in Jordanian economy; and a source of employment and economic growth. The study sample consisted of all industrial companies that reported distributions of dividends from year 2010 to 2016. The sample consisted of 57 company; companies that do not show dividends were excluded. The final total sample consisted of 392 observations.

3.2 Model specification

Traditionally the distribution of dividends by firms is regarded as a strong signal about the firm's future cash flows. A considerable number of prior literature studies reported several important factors that are capable to influence firm dividends decisions; examples of such factors are: previous cash flows, investment returns, general earnings, the level of liquidity; the expected future returns, previously distributed dividend, inflation, interest rates, legal factors, future growth, ownership structure and the size of firms (Brigham, 1995; Foong et al., 2007; Uwuigbe et al., 2012). In order to capture the association between dividends policy and earnings management we followed the two popular real earning models of Kothari et al. (2005) and Roychowdhury (2006). The first model of Kothari et al. (2005) suggests that firm's earnings management behavior is a function of discretionary accruals; and according to this model discretionally accruals (DISACC) can be estimated by the following equation:

$$\begin{aligned} DISACC_{it} = \left(\frac{TA_{it}}{A_{it-1}}\right) - \beta_0\left(\frac{1}{A_{it-1}}\right) + \beta_1\left(\frac{\Delta RET_{it} - \Delta REC_{it}}{A_{it-1}}\right) \\ + \beta_2\left(\frac{PPE_{it}}{A_{it-1}}\right) + \beta_3 lagged(ROA_{it-1}) + E. \end{aligned} \tag{1}$$

Where: TA_{it} = the total accruals in year t of the firm, measured by the difference between income before extraordinary items and discontinued operations and cash flows from operations (Collins and Hribrar, 2002); PPE_{it} = the net value of property, plant, and equipment at the end of year t-1 of the firm. A_{it-1} = Total assets in year $t1$ RET_{it} = Sale revenue in year t REC_{it} = Account receivable in year t PPE_{it} = properties, plant and equipment in year t; lagged ROA_{it} = Return on assets in year t.

Our methodology also employs Roychowdhury (2006) model, our second model this model suggests that earning management is a function of three elements: abnormal levels of cash flow from operations (CFO), abnormal levels of production costs (PROD) and abnormal levels of discretionary expenses ($DISCexp$). Given sales levels, firms that conduct in real earnings management essentially exhibit one or more of the following signs: abnormal low cash flows from operations, and/or abnormal high production costs, and/or abnormal low discretionary expenses. Based on this argument the estimation of firm's real earnings management behavior is captured through equations (2) to (4) as follows:

$$\begin{aligned} \left(\frac{\Delta CFO_{it}}{ASSETS_{it-1}}\right) = \gamma_1\left(\frac{1}{ASSETS_{it-1}}\right) + \gamma_2\left(\frac{\Delta SALES_{it}}{ASSETS_{it-1}}\right) \\ + \gamma_3\left(\frac{\Delta SALES_{it-1}}{ASSETS_{it-1}}\right) + E \end{aligned} \tag{2}$$

Where: CFO_{it} = net cash from the operations of firm i in year t; $ASSETS_{it-1}$ = total assets at the end of year $t-1$ of the firm; $SALES_{it}$ = net sales in year t of the firm; $\Delta SALES_{it}$ = change in net sales from year $t-1$ to t of the firm. Abnormal CFO was measured by the estimated residual from Equation (2). Since price discounts and more lenient credit terms will decrease cash flows for the period. The low negative residuals imply unusual low levels of cash flows from operations suggesting more sales manipulation to manage reported upward earnings (Roychowdhury, 2006; Cohen and Zarowin, 2010; Dechow et al., 1995).

$$\begin{aligned} (\frac{PROD_{it}}{ASSETS_{it-1}}) = \gamma_1\left(\frac{1}{ASSETS_{it-1}}\right) + \gamma_2\left(\frac{\Delta SALES_{it}}{ASSETS_{it-1}}\right) \\ + \gamma_3\left(\frac{\Delta SALES_{it-1}}{ASSETS_{it-1}}\right) + E. \end{aligned} \tag{3}$$

Where: $PROD_{it}$ = costs of goods sold of firm i in year t. The abnormal production cost ($PROD$) is the difference between actual and normal levels of production costs; this measure is calculated using the estimated coefficients from Equation (3). Overproduction will result in positive residuals in equation (3). High positive values of $PROD$ imply a real activity for manipulation through overproduction, which results in a reduction of cost of goods sold (Roychowdhury, 2006; Cohen and Zarowin, 2010; e.g. Dechow et al., 1995).

$$\left(\frac{DISC_{\ exp\ it}}{ASSETS_{it-1}}\right) = \gamma_1 \left(\frac{1}{ASSETS_{it-1}}\right) + \gamma_2 \left(\frac{\Delta SALES_{it-1}}{ASSETS_{it-1}}\right) + E. \tag{4}$$

Where: $DISC_{\ exp\ it}$ = the discretionary expenses, and calculated as the sum of selling, general, administrative expenses (S&GA) and (research & development) expenses. S&GA are those expenses do not directly attribute to production activities rather they related to selling, general and administrative functions and also includes advertising expenses and R&D expenses which consist of all direct and indirect costs that are related to the creation and development of new processes, techniques, applications and products with commercial possibilities. The abnormal expenses could be generated by cutting the discretionary expenses such as advertising, research and development and administrative (SG&A) expenses. The abnormal levels of discretionary expenses are ($DISCexp$) as a measure is estimated as the residual from Equation (4). Low negative residuals indicate that firms cut amounts of discretionary expenses to increase reported earnings. For interpretation purposes higher residuals imply high levels of real activities of manipulation (Roychowdhury, 2006; Cohen and Zarowin, 2010; e.g. Dechow et al., 1995).

3.3 Research model

Based on previous discussion we constructed the following two models to investigate the association between firm's dividends payout ratio and firm's earnings management:

$$DAEM_{it} = \alpha_0 + \alpha_1 DPO + \alpha_2 SIZE + \alpha_3 MB + \alpha_4 ROE + \alpha_5 LEV + E, \tag{5}$$

$$REM_{it} = \alpha_0 + \alpha_1 DPO + \alpha_2 SIZE + \alpha_3 MB + \alpha_4 ROE + \alpha_5 LEV + E. \tag{6}$$

The main variables that are included in the models are as follows: we used the discretionary accruals of earning management ($DAEM$) as the dependent variable for the first model as appears in Equation (5); this dependent variable is to show the relationship between earnings management and the dividend policy (DPO). In model 2 equation (6) we substitutes discretionary accruals with real earning management (REM); many of previous research regarded both of real and accrual based earnings management as substitutes in managing the earnings behavior (Cohen and Zarowin, 2010; Zang, 2012; Chang et al., 2015). As also appears in the two models dividends policy was twice measured by the payout ratio (DPO). And in order to capture the real effect of dividends policy on earning management and isolate any other factors effects' the following control variable were added to the model; first, the study controls the firm size; prior literature revealed mixed results for the effect of $SIZE$ effect on discretionary accruals (Gu et al., 2005 and Aini et al., 2006). On one hand, Gu et al., 2005, found negative association between size and discretionary accruals. While on the other hand, Aini et al., 2006, claimed that the larger the firm size, the more likely it could select income decreasing policy to avoid political costs. The second control variable was market to book value (MB); this variable represents growth factor. Generally, growth firms are more likely to manage their performance; some results found a strong association between market to book value and earning management (Cohen and Zarowin, 2010; Chaney et al., 2011).

The third control variable was profitability of firm (ROE); previous literature suggests that low profitable firms have considerable incentives to conduct earning management; dividend payments will impact the net shareholder equity on the balance sheet and will therefore influence the ROE figure. When a business pays dividends, its retained earnings will decline. Since retained earnings is added to the paid-in capital to calculate the total shareholder equity, dividend payments will reduce the total shareholder equity on the balance sheet. A reduction in shareholder equity translates to a smaller denominator in the ROE equation. In other words, the analyst divides the net income figure by a smaller number, which results in a larger ROE. In sum, dividends reduce shareholder equity and boost ROE (Kothari et al., 2005; Jiraporn et al., 2007). And finally; we control the leverage (LEV); this factor represents the firm financial stability and risk, some studies found positive relationship between leverage and EM, in contrast to other studies that showed negative relationship between leverage and EM (Sweeney, 1994; Dichev and Skinner, 2002; Gu et al., 2005; Rashidah and Haneem, 2006).

3.4 Variable measurement

Table 1 shows the Variable measurements for the employed variable in the study models:

4. Results and discussion

4.1 Descriptive statistics

Table (2) show the results of descriptive statistics for the variables regarding 392 observations of 57 industrial companies listed on ASE during the period (2010-2016). The results in table 2 show that the dividends payout ratio varied from -0.714 to 1.255 with an average of 0.571 which indicates that profitable companies tend to report at least 50 percent distributions of their earnings; this policy of showing high percentages of dividends aimed to boost the market

Table 1: Variable measurement

Variable	Type of variable	Abbreviation	Description
DA. earning management	dependent	DAEM	Discretionary accruals computed using the Kothari et al. (2005)
Real earning management	dependent	REM	Real earning management using Roychowdhury model (2006)
Dividends policy	dependent	DPO	Payout ratio = Dividends per share/ earning per share
Firm size	control	SIZE	Log of total assets
Return on equity	control	ROE	Net income/ shareholders' equity
leverage	control	LIV	Total debt/ total assets
Market to book		MB	market capitalization/ common equity

value of company and share prices. The same table also show that discretionary accruals ranges from 0.000 to 0.491 with an average of 0.137; this proves that engage in earning management behavior, either by raising profit levels to denote higher profitability or, reducing profit levels to avoid taxes and distributions. The results of real earning management ranged from -0.529 to 0.277 with an average of 0.093; similarly this rate also indicates management manipulation in earnings through cash flow or sales or expenses. The values of both real and accrual earnings management were greater than the mean values this implies the existence of earning management behavior. The size effect of these companies varies from 11.150 to 25.916 with an average of 14.331, on average these assets are financed through debt rate of 0.000 to 0.438. Finally, return on assets varied from -41.255 to 28.019 with an average of 3.976, which implies how management is utilizing company's assets to make profits.

Table 2: Descriptive statistics for the study variable

Variables	Minimum	Mean	Maximum	Std. deviation
DAEM	0	0.137	0.491	0.161
REM	-0.529	0.093	0.277	0.319
DPO	-0.714	0.571	1.255	2.593
SIZE	11.15	14.331	25.916	0.832
ROE	-41.255	3.976	28.019	5.61
LIV	0	0.295	0.438	0.188
MB	0	0.627	2.791	3.477
* N= 392				

4.2 Empirical results

4.2.1 Correlation results

Table 3 depicts the correlation matrix for the coefficients of the study variables. As presented in table (3) there is a positive association between dividends policy and both of accrual and real earnings management. The results show that this association is significant at a level 5 percent with accrual earning management and at 10 percent level with real earning management. This result implies and provides preliminary evidence that Jordanian industrial companies manage upwards their reported earnings. Table 3 also reveals positive correlation between financial leverage (LEV), this indicate that companies with a higher debt ratio engage in a higher level of earnings management. The size effect was found negatively associated with earning management; this result suggests that large companies avoid earning management behavior. Market to book value variable was found negatively associated with earning management; this also suggests that high market value companies do not apply earning management management practices. Finally the profitability showed mixed results with both of accrual and real earning management.

Table 3: The correlation results for the study variables

Variables	DAEM	REM	DPO	SIZE	ROE	LIV	MB
DAEM	1						
REM	0.621**	1					
DPO	0.352**	0.247*	1				
SIZE	-0.318**	-0.293**	0.159*	1			
ROE	0.337**	-0.264*	0.215*	0.141*	1		
LIV	0.364**	0.311**	0.128*	0.112*	0.163*	1	
MB	-0.351**	-0.259*	0.266*	0.098	0.196*	0.170*	1
***** sig 1%, ** sig 5%, * sig 10%**							

4.2.2 Regression results

Table 4 and 5 show the results of the regression analyses for the hypothesized association between accrual and real earnings management with dividends policy. Table 4 represents the results of discretionary accruals are estimated using the modified version of Jones (1991) model as suggested by Kothari et al. (2005); and table 5 show the results of real earning measures estimated using Roychowdhury (2006) model. ***, ** and *, represent significance at the one percent, five percent, and ten percent levels, respectively; all variables are defined earlier in variables measurement.

In table 4 the three models are the results for the association between earnings management measured by DAEM and dividends policy measured by payout ratio (DPO) while controlling the other variables (SIZE), (ROE), (LIV) and

(MB). If we review the three models it appears that the association between (DAEM) and (DPO) found positive and statically significant; this suggests that the effect of a firm's dividend policy on the firm's earnings management during the sample period to verify for our first hypothesis that asserts that there is an association between payout policy behavior drives earnings management behavior with discretionary accruals. From Column (1) to Column (3) in Table 3, a positive coefficient is observed on DPO (i > Coef: 0.083, t-value: 2.714, ii > Coef: 0.113, t-value: 3.017, iii > Coef: 0.097, t-value: 1.933).

Table 4: Regression results for dependent variable discretionary accruals (DAEM)

variable	Model 1		Model 2		Model 3	
	B	T- value	B	T-value	B	T-value
Constant	0.261	6.337**	0.186	2.580**	0.144	2.299**
DPO	0.083	2.714**	0.113	3.017**	0.097	1.933**
SIZE	-0.052	-1.481*	-0.039	-2.105**	-0.031	-1.866**
MB	-0.049	-1.277*	-0.043	-1.250*	-0.039	-1.199
ROE	0.063	1.511*	0.049	1.361*	0.017	0.979
LIV	0.071		0.055		0.03	
F-value	17.109		13.922		11.677	
Adj-R2	0.351		0.319		0.299	
N	392		392		392	
*** sig 1%, ** sig 5%, sig 10%						

Similarly, in table 5 the three models provide evidence for the positive association between earnings management measured by REM and dividends policy measured by payout ratio (DPO). The positive coefficient of DPO shows a positive association with REM (real earnings management); (i > Coef: 0.009, t-value: 0.551, ii > Coef: 0.007, t-value: 0.480, iii > Coef: 0.008, t-value: 0.611). Unfortunately, although these results show an association between a firm's dividend policy and real earnings management behavior but they are not significance at a level of 5 percent. Generally, the Overall results of DPO in tables 4 and 5 provided evidence that firm's dividend policy is likely to increase earnings management. This result supports our hypotheses for the existence of the relationship between dividend policy and both of accrual and real earnings management; this assures that Jordanian companies engage in earnings management, especially by using discretionary accruals to create retained earnings from which managers can pay dividends to shareholders.

Regarding the other variables, included as control variables, tables 4 and 5 showed that; the Size of company (SIZE) was negatively significant associated with both accrual and real earning management; this result assures the avoidance of large companies in earning management behavior, This is may possibly attributed to their benefits from economies of scale compared with

Table 5: Regression results for dependent variable real earning management (REM)

variable	Model 1		Model 2		Model 3	
	B	T- value	B	T-value	B	T-value
Constant	0.264	6.893**	0.173	2.105**	0.161	2.080**
DPO	0.009	0.551*	0.007	0.480*	0.008	0.611*
SIZE	-0.047	-1.399**	-0.044	-2.388**	-0.029	-1.155**
MB	-0.023	-1.019*	-0.022	-1.614*	-0.027	-2.366**
ROE	0.059	1.488*	0.071	2.233**	0.041	1.377*
LIV	0.051		0.034		0.021	
F-value	14.333		9.725		8.166	
Adj-R2	0.383		0.316		0.309	
N	392		392		392	
*** sig 1%, ** sig 5%, sig 10%						

small companies that tend to manipulate earning to cover their high marginal cost (Alhadab, 2018). we also found that leverage (LEV) is not been supported statistically in all models but proved to be positively associated with earning management this results assures that high financial leverage companies have no incentives to manage their income. Likewise, profitability (ROE) results were not significantly affecting earning management but positively related with the two types of earning management that means high profitable companies are rarely engaged in earnings management. The last control variable market to book value (MB); the results revealed that this variable is negatively associated with earning management in all models; however, it has not been supported statistically in all models.

In summary, the results reported in table 4& 5 provide evidence to literature that earing management in Jordanian industrial companies is associated with dividends policy using accrual earnings management; this evidence is consistent with the first main hypothesis of this study. Based on the above mentioned results, our first hypothesis is accepted and the second hypothesis is rejected.

5. Conclusion

This study was an attempt to contribute to accounting literature by investigating the association between dividend policy and both of accrual and real earnings management in emerged country context. This study examines whether Jordanian industrial companies engage in earning management both real and accrual-based in order to discrete earnings during the period that took place between 2010 and 2016. The findings of this study provide supportive evidence to the literature that dividends policy is associated with accrual-based earning management; This supports our assertion that managers are likely to conduct earnings management and practice opportunistic behavior to increase retained earnings from which dividends are paid out, as the pressure on the firm is to maintain consistency in the payout of dividends, even when facing decreased earnings or losses. Regarding the association between dividend policy and real earning management the study results did not provide conclusive evidence for such relationship. Regarding control variables the results found the Size of company (SIZE) was negatively significant associated with both accrual and real earning management; leverage (LEV) has not proved to be positively associated with earning management; profitability (ROE) results were not also not found significantly affecting earning management but positively related with the two types of earning management; market to book value (MB) was found negatively associated with earning management.

This study contributes to the knowledge through several aspects. First, it provides new evidence on the use earnings management in Jordanian context. Hence, our findings can be generalized to for developing countries that have stock exchanges with similar characteristics to more developed countries. Second, this study provides more insight for earnings management behavior association with dividends policy. In particular, the findings revealed that accrual earnings management is applied by Jordanian companies to manipulate financial results. Third, higher levels of public monitoring and governance and increasing constraints over accounting discretion regarding the different forms of earnings management, positively or negatively affect for the efficient allocation of resources. Finally, this study suggests a new avenue for future research in Jordan to investigate earning management with other financial information factors.

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C^*-ALGEBRA-VALUED M-METRIC SPACES AND SOME RELATED FIXED POINT RESULTS

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Abstract. In this paper, the concept of C^*-algebra-valued M-metric spaces is initiated, generalizing the M-metric spaces introduced by Asadi *et al.* [8]. Some fixed point theorems are also established via C_*-class functions in such spaces. Moreover, some illustrative examples are given. The obtained results generalize and improve some fixed point results in the literature.

Keywords: fixed point, C^*-algebra-valued M-metric space, C_*-class function.

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1. Introduction

The fixed point theorem, generally known as the Banach contraction mapping principle [11], appeared in explicit form in Banach thesis in 1922. Fixed point theory is also very famous due to its variety of applications in numerous areas such as engineering, computer sciences, economics, etc. The contractive type conditions play an important role in the fixed point theory. Many researchers have extended and generalized Banach contraction principle because it is the heart of this theory.

In 1994, Matthews [20] introduced the notion of a partial metric space and proved the contraction principle of Banach in this new framework. Next, many fixed point theorems in partial metric spaces have been given by several mathematicians. Recently, Haghi *et al.* published [17] a paper which stated that we should be careful on partial metric fixed point results along with giving some results. They showed that some fixed point results in partial metric spaces can be obtained from the corresponding results in metric spaces. Going in same direction, see [7, 26].

In 2014, Asadi *et al.* [8] extended the partial metric space to a M-metric space, and proved some of the main theorems by generalizing contractions to get existence of (common) fixed points. For more information on M-metric spaces, see also [1, 9, 10, 22, 23, 27].

Consider the operator equation

$$X - \sum_{n=1}^{\infty} L_n^* X L_n = Q, \tag{1.1}$$

where $\{L_1, L_2, \cdots, L_n\}$ is a subset of the set of linear bounded operators on an Hilbert space H, $X \in L(H)$ and $Q \in L(H)_+$ is a positive linear bounded operator on the Hilbert space H. Then we convert the operator equation to the mapping $F : L(H) \to L(H)$ which is defined by

$$F(X) = \sum_{n=1}^{\infty} L_n^* X L_n + Q. \tag{1.2}$$

Observe that the range of mapping F is not a real number, but it is a linear bounded operator on the Hilbert space H. Therefore, the Banach contraction principle cannot be applied with this problem. Afterward, does such mapping have a fixed point which is equivalent to the solution of operator equation?

Recently in 2014, Ma et al. in [19] introduced C^*-algebra-valued metric spaces as a new concept which is more general than metric spaces, replacing the set of real numbers by C^*-algebras, and established a fixed point theorem for self-maps involving contractive or expansive conditions on such spaces, analogous to the Banach contraction principle. As applications, the existence and uniqueness results for an integral type equation and operator type equation were given and were able to solve the above problem if the elements $L_1, L_2, \cdots, L_n \in L(H)$ satisfy $\sum_{n=1}^{\infty} \|L_n\|^2 < 1$.

In 2014, Ansari [2] introduced the concept of C-class functions covering a large class of contractive conditions. For more details, see also [4, 5, 6, 12, 16, 21].

This paper is organized as follows: In section 2, we give the required information, notions and definitions about M-metric spaces and C^*-algebras. In section 3, we introduce the concept of C^*-algebra-valued M-metric spaces. Some properties and examples of such spaces are given and several essential lemmas are proved. Finally in section 4, our main results are established and by applying C_*-class functions, some fixed point results are proved in C^*-algebra-valued M-metric spaces.

2. Preliminaries

To begin with, we give some basic definitions, notations and theorems which will be used later.

Definition 2.1 ([8]). *Let X be a non empty set. A function $m : X \times X \to \mathbb{R}_+$ is called a M-metric if the following conditions are satisfied:*

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$;

(m2) $m_{xy} \leq m(x,y)$ *where* $m_{xy} := \min\{m(x,x), m(y,y)\}$;

(m3) $m(x,y) = m(y,x)$;

(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Then the pair (X,m) is called a M-metric space.

We begin with the basic concept of C^*-algebras. A real or a complex linear space $\mathbb{A}$ is an algebra if the vector multiplication is defined for every pair of elements of $\mathbb{A}$ satisfying two conditions such that $\mathbb{A}$ is a ring with respect to vector addition and vector multiplication and for every scalar α and every pair of elements $x, y \in \mathbb{A}$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$. A norm $\|.\|$ on $\mathbb{A}$ is said to be sub-multiplicative if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathbb{A}$. In this case $(\mathbb{A}, \|.\|)$ is called normed algebra. A complete normed algebra is called Banach algebra. An involution on the algebra $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. $(\mathbb{A}, *)$ is called an $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is an $*$-algebra $\mathbb{A}$ with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. There are many examples of C^*-algebras, such as the set of complex numbers, the set of all bounded linear operators on a Hilbert space H, $L(H)$, and the set of $n \times n$-matrices, $M_n(\mathbb{C})$. If a normed algebra $\mathbb{A}$ admits a unit I, $Ia = aI = a$ for all $a \in \mathbb{A}$ and $\|I\| = 1$, we say that $\mathbb{A}$ is a unital normed algebra. A complete unital normed algebra $\mathbb{A}$ is called a unital Banach algebra.

For properties on C^*-algebras, we refer to [13, 14, 24] and the references therein. A positive element of $\mathbb{A}$ is an element $a \in \mathbb{A}$ such that $a^* = a$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathbb{A}$. That is,

$$b \succeq a \text{ if and only if } b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}$ is positive, then we write $a \succeq \theta$, where θ is the zero element of $\mathbb{A}$ ($\theta = 0_{\mathbb{A}}$). Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$, we mean a unital C^*-algebra with identity element I. Further, $\mathbb{A}_+ = \{a \in \mathbb{A} : a \succeq \theta\}$ and $(a^*a)^{\frac{1}{2}} = |a|$.

Lemma 2.2 ([15])**.** *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) *For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \|x\| \leq 1$;*

(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$;*

(3) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$;*

(4) *By $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$ if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'$ is an invertible operator, then*

$$(I-a)^{-1}b \succeq (I-a)^{-1}c.$$

Notice that in a C^*-algebra, if $\theta \preceq a, b$, one cannot conclude that $\theta \preceq ab$. For example, consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, then $ab = \begin{bmatrix} -1 & 2 \\ -4 & 8 \end{bmatrix}$. Clearly $a, b \in \mathbb{M}_2(\mathbb{C})_+$, while ab is not.

Definition 2.3 ([19])**.** *Let X be a non empty set. A function $d : X \times X \to \mathbb{A}$ is called a C^*-algebra-valued metric on X if the following conditions are satisfied:*

($c1$) *$\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \Leftrightarrow x = y$;*

($c2$) *$d(x,y) = d(y,x)$ for all $x, y \in X$;*

($c3$) *$d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.*

Then the pair $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

Definition 2.4 ([19])**.** *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space, $x \in X$ and $\{x_n\}$ is a sequence in $(X, \mathbb{A}, d)$. Then*

(1) *$\{x_n\}$ converges to x with respect to $\mathbb{A}$, if for any $\epsilon > 0$, there is a positive integer N such that $\|d(x_n, x)\| \leq \epsilon$ for all $n \geq N$. We denote it by $\lim_{n\to\infty} x_n = x$;*

(2) $\{x_n\}$ *is Cauchy with respect to* $\mathbb{A}$*, if for any* $\epsilon > 0$ *there is a positive integer* N *such that* $\|d(x_n, x_m)\| \leq \epsilon$ *for all* $n, m \geq N$*;*

(3) $(X, \mathbb{A}, d)$ *is complete if every Cauchy sequence with respect to* $\mathbb{A}$ *in* X *converges to a point in* X*.*

In 2017, Ansari *et al.* [3] introduced the concept of complex C-class functions as follows:

Definition 2.5. *Let* $S = \{z \in \mathbb{C} : 0 \precsim z\}$*, then a continuous function* $F : S^2 \to \mathbb{C}$ *is called a complex* C*-class function if for any* $s, t \in S$*, the following conditions hold:*
(1) $F(s,t) \precsim s$*;*
(2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$*.*

The same letter $\mathcal{C}$ will denote the class of all complex C*-class* functions. For some examples of these functions, see [3].

3. C^*-algebra-valued M-metric spaces

In this section, let $\mathbb{A}$ be a unital C^*-algebra with unit I. We introduce the concept of C^*-algebra-valued M-metric spaces, which is more general than M-metric spaces.

Define a partial ordering $\succeq$ on the elements of $\mathbb{A}$ as

$$B \succeq A \iff B - A \in \mathbb{A}_+ \iff B - A \succeq \theta.$$

$A, B \in \mathbb{A}$ are comparable if and only if

$$A \succeq B \quad \text{or} \quad B \succeq A.$$

So if all elements of a set $\mathbb{D} \subseteq \mathbb{A}$ are comparable pairwise, then we can define "min" and "max" for $\mathbb{D}$ as follows:

$$\begin{aligned} &\max\{A_i : A_i \in \mathbb{D}, i = 1, 2, \cdots, n\} = A_k \\ &\iff A_k \succeq A_i, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_k - A_i \in \mathbb{A}_+, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_k - A_i \succeq \theta, \quad \forall i = 1, 2, \cdots, n, \end{aligned}$$

and

$$\begin{aligned} &\min\{A_i : A_i \in \mathbb{D}, i = 1, 2, \cdots, n\} = A_k \\ &\iff A_i \succeq A_k, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_i - A_k \in \mathbb{A}_+, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_i - A_k \succeq \theta, \quad \forall i = 1, 2, \cdots, n, \end{aligned}$$

for some $k \in \{1, 2, 3, \cdots, n\}$.

Definition 3.1. *Let X be a non empty set. A function $m : X \times X \to \mathbb{A}$ is called a C^*-algebra-valued M-metric if the following conditions are satisfied:*

(*cm*1) $\theta \preceq m(x,y)$ *for all* $x, y \in X$ *and* $m(x,x) = m(y,y) = m(x,y) \iff x = y$,

(*cm*2) $m(x,x)$ *and* $m(y,y)$ *be comparable for all* $x, y \in X$;

(*cm*3) $m_{xy} \preceq m(x,y)$ *for all* $x, y \in X$, *where* $m_{xy} = \min\{m(x,x), m(y,y)\}$;

(*cm*4) $m(x,y) = m(y,x)$ *for all* $x, y \in X$;

(*cm*5) $(m(x,y) - m_{xy}) \preceq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$ *for all* $x, y, z \in X$.

Then the pair $(X, \mathbb{A}, m)$ is called a C^*-algebra-valued M-metric space.

Remark 3.2. Note that if we take $\mathbb{A} = \mathbb{R}$, then the new notion of C^*-algebra-valued M-metric space becomes equivalent to Definition 2.1 of the real M-metric space.

Let $(X, \mathbb{A}, m)$ be a C^*-algebra-valued M-metric space. Define M_{xy} by

$$M_{xy} = \max\{m(x,x), m(y,y)\}.$$

Remark 3.3. For every $x, y, z \in X$, we have

1. $\theta \preceq M_{xy} + m_{xy} = m(x,x) + m(y,y)$;
2. $\theta \preceq M_{xy} - m_{xy} = (m(x,x) - m(y,y)) \vee (m(y,y) - m(x,x))$;
3. $M_{xy} - m_{xy} \preceq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

It is clear that each C^*-algebra-valued M-metric p on X generates a T_0 topology τ_m on X. Let $\theta \prec \varepsilon \in \mathbb{A}$. The set

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon \succ \theta\},$$

where

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) \prec m_{xy} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon \succ \theta$, forms the base of τ_m.

Definition 3.4. *Let $(X, \mathbb{A}, m)$ be a C^*-algebra-valued M-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

1. *$\{x_n\}$ converges to x with respect to $\mathbb{A}$, whenever for every $\epsilon > 0$ there is a natural number N such that $\|m(x_n,x) - m_{x_n x}\| \le \epsilon$ for all $n \ge N$. We denote this by*

$$\lim_{n\to\infty}(m(x_n,x) - m_{x_n x}) = \theta; \qquad (3.1)$$

2. *$\{x_n\}$ is a m-Cauchy sequence with respect to $\mathbb{A}$, whenever for every $\epsilon > 0$ there is a natural number N such that $\|m(x_n, x_m) - 2m_{x_n x_m} + M_{x_n x_m}\| \leq \epsilon$, for all $m, n \geq N$;*

3. *$(X, \mathbb{A}, m)$ is complete if every m-Cauchy sequence with respect to $\mathbb{A}$, converges to a point X such that*

$$\lim_{n\to\infty}(m(x_n, x) - 2m_{x_n x} + M_{x_n x}) = \theta.$$

The next example states that m^s and m^w are C^*-algebra-valued metrics.

Example 3.5. Let m be a C^*-algebra-valued M-metric. Put

1. $m^w(x, y) = m(x, y) - 2m_{xy} + M_{xy}$;
2. $m^s(x, y) = m(x, y) - m_{xy}$ if $x \neq y$ and $m^s(x, y) = \theta$ if $x = y$.

Then m^w and m^s are C^*-algebra-valued metrics.

Proof. We have $m^w(x, y) \succeq \theta$ and if $m^w(x, y) = \theta$, then

$$m(x, y) = 2m_{xy} - M_{xy}. \tag{3.2}$$

From (3.2) and the fact that $m_{xy} \preceq m(x, y)$, we get $m_{xy} = M_{xy} = m(x, x) = m(y, y)$, so by (3.2), we obtain $m(x, y) = m(x, x) = m(y, y)$. Therefore, $x = y$. For the triangle inequality, it is enough that we consider $(cm5)$ together with Remark 3.3. Similarly, we can show that m^s is a C^*-algebra-valued metric. □

Remark 3.6. For every $x, y \in X$, we have

1. $m(x, y) - M_{xy} \preceq m^w(x, y) \preceq m(x, y) + M_{xy}$;
2. $(m(x, y) - M_{xy}) \preceq m^s(x, y) \preceq m(x, y)$.

Lemma 3.7. *Let $(X, \mathbb{A}, m)$ be a C^*-algebra-valued M-metric space. Then*

1. *$\{x_n\}$ is a m-Cauchy sequence in $(X, \mathbb{A}, m)$ if and only if it is Cauchy in the C^*-algebra-valued metric space $(X, \mathbb{A}, m^w)$;*
2. *A C^*-algebra-valued M-metric space $(X, \mathbb{A}, m)$ is complete if and only if the C^*-algebra-valued metric space $(X, \mathbb{A}, m^w)$ is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n, x) = \theta \iff (\lim_{n\to\infty}(m(x_n, x) - m_{x_n x}) = \theta \quad \& \quad \lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = \theta).$$

Proof. It suffices to use Definition 2.3, Definition 3.4 and Example 3.5. □

Likewise, above lemma also holds for m^s.

Lemma 3.8. *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a C^*-algebra-valued M-metric space $(X, \mathbb{A}, m)$. Then*

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n y_n}) = m(x, y) - m_{xy}.$$

Proof. It suffices to write that

$$(m(x_n, y_n) - m_{x_n y_n}) - (m(x, y) - m_{xy}) \preceq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n}),$$

and

$$(m(x, y) - m_{xy}) - (m(x_n, y_n) - m_{x_n y_n}) \preceq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n}).$$

□

From Lemma 3.8, we can deduce the following.

Lemma 3.9. *Assume that $x_n \to x$ as $n \to \infty$ in a C^*-algebra-valued M-metric space $(X, \mathbb{A}, m)$. Then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n y}) = m(x, y) - m_{xy},$$

for all $y \in X$.

Similarly, we may state

Lemma 3.10. *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in a C^*-algebra-valued M-metric space $(X, \mathbb{A}, m)$. Then $m(x, y) = m_{xy}$. Further, if $m(x, x) = m(y, y)$, then $x = y$.*

Proof. By Lemma 3.8, we have $\theta = \lim_{n\to\infty}(m(x_n, x_n) - m_{x_n x_n}) = m(x, y) - m_{xy}$. □

We present the following examples.

Example 3.11. Let $X = [0, \infty)$ and $\mathbb{A} = M_n(\mathbb{C})$. An element $A = (a_{ij})_{n\times n} \in \mathbb{A} = M_n(\mathbb{C})$ is a positive element (written as $A \succeq \theta$) means that

$$a_{ij} \succeq 0 \quad \forall i, j \in \{1, 2, \cdots, n\},$$

where θ is the zero matrix in $M_n(\mathbb{C})$. We define a partial ordering $\preceq$ on $\mathbb{A}$ as follows

$$A \preceq B \text{ iff } \theta \preceq B - A.$$

It is clear that $\preceq$ is a partial order relation. Define

$$m(x, y) = \begin{bmatrix} \frac{x+y}{2} + i\frac{x+y}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{x+y}{2} + i\frac{x+y}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{x+y}{2} + i\frac{x+y}{2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{x+y}{2} + i\frac{x+y}{2} \end{bmatrix}, \tag{3.3}$$

where $x, y \in X$. A norm $\|.\|$ on $\mathbb{A}$ is defined by

$$\|A\| = \max_{i,j} |a_{ij}|^2,$$

where $A = (a_{ij})_{n\times n} \in \mathbb{A}$. The involution is given by $A^* = (\overline{A})^T$, the conjugate transpose of matrix A:

$$A^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

It is easy to verify that m is a C^*-algebra valued M-metric and $(X, M_n(\mathbb{C}), m)$ is a complete C^*-algebra valued M-metric space of $\mathbb{C}$.

Example 3.12. Let $X = \{1, 2, 3\}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$m(1,2) = m(2,1) = m(1,1) = \left[\begin{array}{cc} 8 & 0 \\ 0 & 8 \end{array}\right]$$

$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = \left[\begin{array}{cc} 7 & 0 \\ 0 & 7 \end{array}\right]$$

$$m(2,2) = \left[\begin{array}{cc} 9 & 0 \\ 0 & 9 \end{array}\right] \quad m(3,3) = \left[\begin{array}{cc} 5 & 0 \\ 0 & 5 \end{array}\right],$$

so m is a C^*-algebra-valued M-metric. Consider $D(x,y) = m(x,y) - m_{xy}$. We have $m(1,2) = m_{12} = \left[\begin{array}{cc} 8 & 0 \\ 0 & 8 \end{array}\right]$. Also, $D(1,2) = \theta = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$. Since $1 \neq 2$, this means that D is not a C^*-algebra-valued metric.

Example 3.13. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. Take $\phi : \mathbb{A}_+ \to \{A \in \mathbb{A} : \phi(\theta) \preceq A\}$ a one to one, nondecreasing or strictly increasing mapping where $\phi(\theta) \succeq \theta$ is defined, such that

$$\phi(x+y) \preceq \phi(x) + \phi(y) - \phi(\theta), \quad \forall x, y \succeq \mathbb{A}_+.$$

Then m defined by $m(x,y) = \phi(d(x,y))$, is a complex valued M-metric.

Proof. (cm1), (cm2), (cm3) and (cm4) are clear. For (cm5), we have

$$\begin{aligned} \phi(d(x,y)) &\preceq \phi(d(x,z) + d(z,y)) \\ &\preceq \phi(d(x,z)) + \phi(d(z,y)) - \phi(\theta), \end{aligned}$$

then

$$(\phi(d(x,y)) - \phi(\theta)) \preceq (\phi(d(x,z)) - \phi(\theta)) + (\phi(d(z,y)) - \phi(\theta)).$$

This means that $(m(x,y) - m_{xy}) \preceq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$. $\square$

Example 3.14. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. Then $m(x,y) = ad(x,y) + bI$ where $a, b > 0$, is a C^*-algebra-valued M-metric. It suffices to take $\phi(t) = at + bI$, for all $t \in \mathbb{A}_+$.

4. Main results

In this section, first we introduce the concept of a C_*-class function. The main idea consists in using the set of elements of a unital C^*-algebra instead of the set of complex numbers.

Definition 4.1 (C_*-class function)**.** *Suppose that $\mathbb{A}$ is a unital C^*-algebra, then a continuous function $F : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ is called a C_*-class function if for any $A, B \in \mathbb{A}_+$, the following conditions hold*

(1) *$F(A,B) \preceq A$;*

(2) *$F(A,B) = A$ implies that either $A = \theta$ or $B = \theta$.*

The letter $\mathcal{C}_*$ will denote the class of all C_*-class functions.

Remark 4.2. The class $\mathcal{C}_*$ includes the set of complex C-class functions introduced in [3]. It is sufficient to take $\mathbb{A} = \mathbb{C}$ in Definition 4.1.

The following examples show that the class $\mathcal{C}_*$ is nonempty.

Example 4.3. Let $\mathbb{A} = M_2(\mathbb{R})$, of all 2×2 matrices with the usual operation of addition, scalar multiplication, and matrix multiplication. Define a norm on $\mathbb{A}$ by $\|A\| = \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$. Consider $* : \mathbb{A} \to \mathbb{A}$, given by $A^* = A$, for all $A \in \mathbb{A}$. It defines a convolution on $\mathbb{A}$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

we denote $A \preceq B$ if and only if $(a_{ij} - b_{ij}) \leq 0$, for all $i, j = 1, 2$.

(1) Define $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ by

$$F_*(A,B) = A - B, \text{ i.e.,}$$

$$F_*\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}$$

for all $a_{i,j}, b_{i,j} \in \mathbb{R}_+$ and $i, j \in \{1, 2\}$. Then F_* is a C_*-class function.

(2) Define $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ by

$$F_*\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) = m \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

for all $a_{i,j}, b_{i,j} \in \mathbb{R}_+$, $(i, j \in \{1, 2\})$, where $m \in (0, 1)$. Then F_* is a C_*-class function.

Example 4.4. Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. Denote $B(H)$ the set of bounded linear operators on the Hilbert space H. Clearly, $B(H)$ is a C^*-algebra with the usual operator norm.
Define $F_* : B(H)_+ \times B(H)_+ \to B(H)$ by

$$F_*(U, V) = U - \varphi(U),$$

where $\varphi : B(H)_+ \to B(H)_+$ is a continuous function such that $\varphi(U) = \theta$ if and only if $U = \theta$ $(\theta = 0_{B(H)})$. Then F_* is a C_*-class function.

Let Ψ be the set of all continuous functions $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and non-decreasing;

(ψ_2) $\psi(T) = \theta$ if and only if $T = \theta$.

Our essential main result is

Theorem 4.5. *Let $(X, \mathbb{A}, m)$ be a C^*-algebra-valued M-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \preceq F_*\Big(\psi(m(x, y)), \phi(m(x, y))\Big) \quad \text{for all} \quad x, y \in X, \tag{4.1}$$

where $\psi, \phi \in \Psi$ and $F_* \in \mathcal{C}_*$. Then T has a unique fixed point.

Proof. Fix $x_0 \in X$. Define $x_n = T^n x_0$ for every $n = 1, 2, 3, \cdots$. We shall prove that

$$m(x_n, x_{n+1}) \to \theta \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned} \psi(m(x_n, x_{n+1})) &= \psi(m(Tx_{n-1}, Tx_n)) \\ &\preceq F_*\Big(\psi(m(x_{n-1}, x_n)), \phi(m(x_{n-1}, x_n))\Big) \\ &\preceq \psi(m(x_{n-1}, x_n)). \end{aligned} \tag{4.2}$$

So we get

$$\psi(m(x_n, x_{n+1})) \preceq \psi(m(x_{n-1}, x_n)).$$

ψ is nondecreasing, so the sequence $\{m(x_n, x_{n+1})\}$ is monotone decreasing in $\mathbb{A}_+$ and hence there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$m(x_n, x_{n+1}) \to t \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (4.2), by definition of F_* and continuity of ψ and ϕ, we obtain $\psi(t) \preceq F_*(\psi(t), \phi(t)) \preceq \psi(t)$. Thus $F_*(\psi(t), \phi(t)) = \psi(t)$, so $\psi(t) = \theta$ or $\phi(t) = \theta$, hence $t = \theta$. That is

$$m(x_n, x_{n+1}) \to \theta \quad \text{as} \quad n \to \infty. \tag{4.3}$$

Now, we want to show that $\{x_n\}$ is a m-Cauchy sequence in $(X, \mathbb{A}, m)$. By Lemma 3.7, it suffices to prove that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, m^w)$. We obtained $\lim_{n\to\infty} m(x_n, x_{n+1}) = \theta$. Having in mind that $\theta \preceq m_{x_n x_{n+1}} \preceq m(x_n, x_{n+1})$, so

$$\lim_{n\to\infty} m_{x_n x_{n+1}} = \theta. \tag{4.4}$$

Also, $m_{x_n x_{n+1}} = \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\}$. In view of the above, one can write

$$\lim_{n\to\infty} m(x_n, x_n) = \theta.$$

Recall that $m_{x_n x_m} = \min\{m(x_n, x_n), m(x_m, x_m)\}$ and $M_{x_n x_m} = \max\{m(x_n, x_n), m(x_m, x_m)\}$. We deduce that

$$\lim_{n,m\to\infty} m_{x_n x_m} = \lim_{n,m\to\infty} M_{x_n x_m} = \theta. \tag{4.5}$$

Assume that $\{x_n\}$ is not Cauchy in $(X, \mathbb{A}, m^w)$. Then there exist $\epsilon > 0$ and subsequences $\{x_{l_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > l_k > k$ such that $\|m^w(x_{l_k}, x_{n_k})\| > \epsilon$. Now, corresponding to l_k, we can choose n_k such that it is the smallest integer with $n_k > l_k$ and satisfying above inequality. Hence $\|m^w(x_{l_k}, x_{n_k-1})\| \leq \epsilon$. So, we have

$$\begin{aligned} \epsilon < \|m^w(x_{l_k}, x_{n_k})\| &\leq \|m^w(x_{l_k}, x_{n_k-1})\| + \|m^w(x_{n_k-1}, x_{n_k})\| \\ &\leq \epsilon + \|m^w(x_{n_k-1}, x_{n_k})\|. \end{aligned} \tag{4.6}$$

We know that

$$m^w(x_{n_k-1}, x_{n_k}) = m(x_{n_k-1}, x_{n_k}) - 2m_{x_{n_k-1}x_{n_k}} + M_{x_{n_k-1}x_{n_k}}. \tag{4.7}$$

Clearly, by (4.3) and (4.5),

$$\lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{n_k})\| = 0. \tag{4.8}$$

Using (4.8) in (4.6), we have

$$\lim_{k\to\infty} \|m^w(x_{l_k}, x_{n_k})\| = \epsilon. \tag{4.9}$$

Again,

$$\begin{aligned} \|m^w(x_{n_k}, x_{l_k})\| \leq &\|m^w(x_{n_k}, x_{n_k-1})\| + \|m^w(x_{n_k-1}, x_{l_k-1})\| \\ &+ \|m^w(x_{l_k-1}, x_{l_k})\|, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|m^w(x_{n_k-1}, x_{l_k-1})\| &\leq \|m^w(x_{n_k-1}, x_{n_k})\| + \|m^w(x_{n_k}, x_{l_k})\| \\ &\quad + \|m^w(x_{l_k}, x_{l_k-1})\|. \end{aligned} \tag{4.11}$$

Letting $k \to \infty$ in (4.10) and (4.11) and using (4.8) and (4.9), we have

$$\lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{l_k-1})\| = \epsilon.$$

Thus

$$\begin{aligned}\lim_{k\to\infty} \|m(x_{n_k-1}, x_{l_k-1})\| &= \lim_{k\to\infty} \|(m(x_{n_k-1}, x_{l_k-1}) - 2m_{x_{n_k-1}x_{l_k-1}} + M_{x_{n_k-1}x_{l_k-1}})\| \\ &= \lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{l_k-1})\| = \epsilon.\end{aligned}$$

Since $m(x_{n_k-1}, x_{l_k-1}), m(x_{n_k}, x_{l_k}) \in \mathbb{A}_+$ and

$$\lim_{k\to\infty} \|m(x_{n_k-1}, x_{l_k-1})\| = \lim_{k\to\infty} \|m(x_{n_k}, x_{l_k})\| = \epsilon,$$

so there exists $a \in \mathbb{A}_+$ with $\|a\| = \epsilon$ such that

$$\lim_{k\to\infty} m(x_{n_k-1}, x_{l_k-1}) = \lim_{k\to\infty} m(x_{n_k}, x_{l_k}) = a.$$

Now, by (4.1), we have

$$\begin{aligned}\psi(a) &= \lim_{k\to\infty} \psi(m(x_{n_k}, x_{l_k})) \\ &\preceq \lim_{k\to\infty} F_*\Big(\psi(m(x_{n_k-1}, x_{l_k-1})), \phi(m(x_{n_k-1}, x_{l_k-1}))\Big).\end{aligned}$$

Therefore,

$$\psi(a) \preceq F_*\Big(\psi(a), \phi(a)\Big) \preceq \psi(a).$$

Hence $\psi(a) = \theta$ or $\phi(a) = \theta$, so $a = \theta$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in the complete C^*-algebra-valued metric space $(X, \mathbb{A}, m^w)$, and so $\{x_n\}$ is m-Cauchy in the complete C^*-algebra-valued M-metric space $(X, \mathbb{A}, m)$. Hence there exists some $v \in X$ such that

$$\lim_{n\to\infty} (m(x_n, v) - m_{x_n v}) = \theta.$$

Due to (4.4), we have $\lim_{n\to\infty} m_{x_n v} = \theta$, hence $\lim_{n\to\infty} m(x_n, v) = \theta$. By Remark 3.3, $m(v, v) = \theta$. Now, we want to show that v is the fixed point of T. By (4.1), we have $\theta \preceq \psi(m(Tv, Tv)) \preceq F_*(\psi(m(v, v)), \phi(m(v, v))) = F_*(\psi(\theta), \phi(\theta)) = \theta$. Thus $\psi(m(Tv, Tv)) = \theta \Rightarrow m(Tv, Tv) = \theta$. On the other hand

$$\psi(m(x_n, Tv)) \preceq F_*\Big(\psi(m(x_{n-1}, v)), \phi(m(x_{n-1}, v))\Big).$$

Then letting $n \to \infty$, making use of Lemma 3.9 and continuity of functions F_*, ψ and ϕ, we obtain that $m(v, Tv) = \theta$. Hence we have

$$m(v, v) = m(Tv, Tv) = m(v, Tv) = \theta, \tag{4.12}$$

so by (cm2), we have $Tv = v$. Now, let $u, v \in X$ be two fixed points of T . From (4.1),

$$\psi(m(v, v)) = \psi(m(Tv, Tv)) \preceq F_*\Big(\psi(m(v, v)), \phi(m(v, v))\Big) \preceq \psi(m(v, v)),$$

so $\psi(m(v, v)) = \theta$ or $\phi(m(v, v)) = \theta$. Thus $m(v, v) = \theta$. Similarly, we obtain $m(u, u) = \theta$. Again, by (4.1), we have

$$\psi(m(v, u)) = \psi(m(Tv, Tu)) \preceq F_*\Big(\psi(m(v, u)), \phi(m(v, u))\Big) \preceq \psi(m(v, u)).$$

Hence $\psi(m(v, u)) = \theta$ or $\phi(m(v, u)) = \theta$, so $m(v, u) = \theta$. we obtained that $m(v, v) = m(u, u) = m(v, u) = \theta$. By (cm2), $u = v$. □

If in Theorem 4.5, we take $F_*(s, t) = s - t$, where $s, t \in \mathbb{A}_+$, then we get the following.

Corollary 4.6. *Let $(X, \mathbb{A}, m)$ be a complete C^*-algebra-valued M-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \preceq \psi(m(x, y)) - \phi(m(x, y)) \quad \text{for all} \quad x, y \in X, \tag{4.13}$$

where $\psi, \phi \in \Psi$. Then T has a unique fixed point.

Remark 4.7. If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$, then we obtain Theorem 3.1 of [23].

Remark 4.8. If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$ and $\phi(t) = (1 - k)\psi(t)$ with $0 < k < 1$, then we obtain the M-metric generalization of the result in [18].

Remark 4.9. If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$ and $\psi(t) = t$, then we obtain the M-metric generalization for the weakly contractive fixed point theorem in [25].

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FLUCTUATIONS OF STOCK PRICE AND REVENUE AFTER THE EARLY ADOPTION OF IFRS 15, "REVENUE FROM CONTRACTS WITH CUSTOMERS"

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Abstract. The adoption of International Financial Reporting Standards (IFRS) around the world is gaining popularity and Jordan not being an exception. This adoption brings about improvement in accountability and quality of accounting Information through a uniform set of standards for financial reporting. The objective of this paper is to examine whether the early adoption of IFRS 15 that supersede the International Accounting Standards (IAS) 18 concerning revenue recognition has effected Jordanian companies revenue levels and the value of stock price or not. The paper measures revenue and stock prices pre and post the IFRS 15 implementation. Two hypotheses was developed and tested at five (5) percent level of significance. The findings revealed that there is significance quantitative difference between the arithmetic means for both of revenue and stock price pre and post the standard application. Further, the results of the study provided conclusive evidence that IFRS 15 have impacted on accountability and quality of information that reported in financial statement for Jordanian mining, construction and engineering companies.

Keywords: IFRS 15, IAS 18, revenue, stock prices, volatility, Jordan.

1. Introduction

The main objective for business entities is to generate profit, this motive, however, although this concept have been criticised widely, income is regarded as vital element to business financial performance, and also seen as the essential component for business profitability and sustainability. The treatment of revenues is crucial when recognizing income; and the primary source of revenue is the income that is generated from daily operating activities. Therefore, revenue is one of the most decisive items of financial reports as it often forms the fundamental basis for investment decision-making. The main aim of the standard IAS 18 and IFRS 15 is to give guidance on the revenue recognition and help in the application of the revenue recognition criteria. Revenue recognition has become controversial issue today, hence, the major objectives (among others) when International Accounting Standard Board (IASB) issued the IASB 18 is to overcome any problems associated with the principle of revenue recognition by providing more guidance and clarity to accountants to report reliable and relevant information concerning revenue; however, the standard worked well for more than decade (Knachel, 2016; Bloom and Kamm, 2014; Aladwan and Alsinglawi, 2016).

In order to achieve the convergence project between FASB and IASB to unify the accounting standards, on May, 2014, the FASB and IASB issued their long-awaited converged standard on revenue recognition, IFRS 15; as a replacement of previously applied IAS 18. Almost all entities will be affected to some extent by the significant increase in required disclosures. But the changes extend beyond disclosures, and the effect on entities will vary depending on industry and current accounting practices (IASB, 2014; FASB, 2015; Ernst and Young, 2014). Entities that apply this standard need to consider changes that might be necessary to information technology systems, processes, and internal controls to capture new data and address changes in financial reporting (Beaver et al., 2012; Aladwan, 2018). This new converged standard aims to eliminate the differences and inconsistencies incorporated between US GAAP and IFRS to promote a single set of high-quality globally accepted accounting standards that allows for comparability of firms within an industry and on a global financial market; furthermore, the standard targets to incorporate the changes made to the theoretical basis of IFRS that took place during recent years with the project on the Conceptual Framework (FW) and other important standards.

Most studies on IFRS have concentrated on it as a financial reporting issue. But financial reporting is one aspect of the total impact of IFRS composition. Much more significant is the impact of a set of standards on a companys organization, accountability philosophy, and business structure compliance to the standards, performance management, and internal control and so on (Hale, 2016; AICPA, 2016; Deloitte, 2014; KPMG, 2016). Therefore, the objective of the study is to examine whether the new enacted and implemented IFRS 15 has enhanced the entity financial status in comparison to IAS 18. Further, the study is promoted by the need to bring more insight to such issue by investigating the comprehensive effect of IFRS 15 on both of earnings and company value in emerged country Jordan as that early adopted this standards in some economic sectors. Another contribution to the existing literature that, this study is considered from the earliest studies in the whole region intends to examine the new effect for such standards on companies.

The remaining part of this paper is devoted for the in-depth review of the literature of IAS 18 and the IFRS 15 with proper formulation of the study hypotheses through literature review hypotheses development section followed by the methodology of the study, discussion of results and discussion and the last section for conclusion.

2. Review of literature

2.1 Financial reporting

The (IASB) Framework assured that; the main objective of financial statements is to provide a set of financial statements about entity financial position, performance and changes in financial position that is useful to a wide range of decisions makers (IASB, 2010; IASB, 2015). These financial statements are regarded traditionally as the first source of independent and true communication about the performance of managers (Sloan, 2001; Wagenhofer, 2014). And in order to be able to meet the needs of users, such financial statements must not only comply with the (IFRS), but also be beneficial for decision making (IASB, 2010). Historically, The IFRS/IAS standards consist of a set of international accounting and reporting standards established to provide a clear rules and guidance for the accounting profession. Such rules are the fundamental for accountants to draw up comparable and transparent annual reports and financial statements (Cardozza, 2008; IASB, 2010; Alsinglawi and Aladwan, 2016). Their adoption represents an essential element to the success of accounting and auditing profession (Deloitte, 2016).

Nowadays, the term (IFRS) has both a narrow and a broad meaning (Iasplus, 2010); narrowly, those IFRS refer to the any new numbered series of pronouncements that the IASB is issuing continuously, as distinct from the International Accounting Standards (IASs) series issued by its predecessor. More broadly, IFRSs are regarded as the entire body of International Accounting Standards Board (IASB) pronouncements, including standards and interpretations approved by the IASB and IASs and the Standards Interpretations Committee (SIC) (now replaced with International Financial Reporting Interpretations Committee (IFRIC) interpretations approved by the predecessor International Accounting Standards Committee (Iasplus, 2010).

2.2 IFRS 15 vs. IAS 18

The (IASB) and the US Financial Accounting Standards Board (FASB) (collectively, the Boards) have jointly issued a new revenue standard, IFRS 15 Revenue from Contracts with Customers, that will supersede virtually all revenue recognition requirements in IFRS and US GAAP. Noting several concerns with existing requirements for revenue recognition under both US GAAP and IFRS, the Boards decided to develop a joint revenue standard that would: remove inconsistencies and weaknesses in the current revenue recognition literature; provide a more robust framework for addressing revenue recognition issues; improve comparability of revenue recognition practices across industries, entities within those industries, jurisdictions and capital markets; reduce the complexity of applying revenue recognition requirements by reducing the volume of the relevant standards and interpretations; and provide more useful information to users through new disclosure requirements (Schipper et al., 2009; Prakash and

Sinha, 2012). IFRS 15 Revenue from Contracts with Customers specifies the accounting treatment for all revenue arising from contracts with customers. It applies to all entities that enter into contracts to provide goods or services to their customers (IFRS 15, 2015).

The new standard is mandatorily effective for annual periods beginning on or after January 1, 2018, with earlier application permitted starting on May 1, 2014, and to be applied retrospectively using either a full retrospective approach (subject to certain practical expedients) or a modified retrospective approach. IFRS 15 establishes a comprehensive framework for recognition of revenue from contracts with customers based on a core principle that an entity should recognize revenue representing the transfer of promised goods or services to customers in an amount that reflects the consideration to which the entity expects to be entitled in exchange for those goods or services (IFRS 15, 2015; Cohen et al., 2014). The core principle of IFRS 15 is that an entity shall recognize revenue to depict the transfer of promised goods or services to customers in an amount that reflects the consideration to which the entity expects to be entitled to in exchange for those goods or services (Deloitte, 2014; Dyson, 2015).

IFRS 15 defines revenue somewhat more simply than the existing standard IAS 18, as income arising in the course of an entitys ordinary activities. According to IFRS 15, an entity shall generally recognize revenue when (or as) the entity transfers a promised good or service (i.e. an asset) to a customer. An asset is transferred when (or as) the customer obtains control of that asset; previously, the revenue model depended on whether a contract was covered by IAS 11 or IAS 18 and was based on the type of transaction or event (i.e. whether the entity was performing under a construction contract, sold a good, rendered a service or had income from interest, royalties and dividends (Deloitte, 2014; KPMG, 2016). The amount of revenue recognized is the amount allocated to the satisfied performance obligation. A performance obligation may be satisfied at a point in time (typically for promises to transfer goods to a customer) or over time (typically for promises to transfer services to a customer (Badertscher et al., 2012). For performance obligations satisfied over time, an entity recognizes revenue over time by selecting an appropriate method for measuring its progress towards complete satisfaction of that performance obligation (IFRS 15, 2015; Dyson, 2015).

Altogether, revenue is the gross inflow of economic benefits during the period arising in the course of the ordinary activities of an entity when those inflows result in increases in equity, other than increases relating to contributions from equity participants. The objective of the revenue standard is to provide a single, comprehensive revenue recognition model for all contracts with customers to improve comparability within industries, across industries, and across capital markets. The revenue standard contains principles that an entity will apply to determine the measurement of revenue and timing of when it is recognized. The underlying principle is that an entity will recognize revenue to depict the transfer of goods or services to customers at an amount that the entity expects to be entitled to in exchange for those goods or services.

2.3 Previous literature and hypotheses development

Recent convergence in accounting standards could provide direction and guidance on how business enterprises in a globalized world could achieve the goal of proper record keeping, transparency, uniformity, comparability and enhancing public confidence in financial reporting (McCarthy and McCarthy, 2014). Moreover, the failure for such global convergence according to Altamuro et al (2005) would result in inconsistencies, lack of accountability and transparency, distortion in financial reports, which in turn results into poor financial reporting practices and dissemination of accounting information that is of less value to any particular group of users (Zang, 2012; Aladwan et al., 2018). Revenue is regarded as a type of income that arises in the course of ordinary activities of an entity and is referred to by a variety of different names including sales, fees, interest, dividends and royalties. The primary issue in accounting for revenue is determining when to recognize revenue; and according to IAS 18 and IFRS 15; revenue is recognized when it is probable that future economic benefits will flow to the entity and these benefits can be measured reliably. This definition of revenue caused debate on the exact time and conditions that allow entities to report income; this argument promoted the need for practical guidance on the application of revenue recognition (IFRS 15, 2015).

Most of theoretical literature agreed that revenue is the total of benefits that inflow during a period of time arising from the course of the ordinary activities of an business when those inflows result in increases in equity, other than increases relating to contributions from equity participants (IAS 18). According to Ernst and Young (2014) revenue does not include any gains from the sale of property plant and equipment (PPE) unless the PPE items were leased out under an operating lease - or other fixed assets and net finance income (IAS 18). As a consequence for the changes occurred on the scope of IFRS 15 the amount of revenue that recognized in financial statements is expected to be changed, therefore our first hypothesis is:

H1: There is a significant difference in the mean of revenue for companies pre and post the application of new revenue standard IFRS 15.

There is a consensus among scholars, researchers, analysts and users that financial information have to mirror that business performance reality (Atrill and Mclaney, 2015; Benedict and Elliott, 2011). Thus, the content of financial reports must give the true substance for entity economic status. One of the fundamental requirements of Generally Accepted Accounting Principles (GAAP) is that entities should prepare and disclose financial information that serve the internal and external interested parties of such information (IASB, 2010). So the standards legislators demanded all business entities to publish financial statements about their financial position (statement of financial position), performance (comprehensive income statement), monetary position (cash flow statement), and owners position (changes in owner equity). Moreover, the conceptual framework of accounting stated that the financial statements have to include useful information for the decision making process (IASB, 2010). According to Kenneth (2012) the quality of financial reporting is indispensable to the need of users who requires them for investment and other decision making purposes. Financial reports can only be regarded as useful if it represents the economic substance of an organization in terms of relevance, reliability, comparability and aids interpretation simplicity.

In Jordan, Amman stock exchange (ASE) requires from companies to fully comply with IFRSs, entities whose financial statements do comply with IFRSs are prevented from publishing their financial reports in the market (ASE, 2007). All companies that have publicly traded stocks are mandated to comply with the new standards and they should revise their periodical financial statements to comply with such standards. Duru (2012) assured that, any revisions on financial statements that are published will in turn change the opinion of internal users as well as a wide range of external parties. Likewise, Leuz and Wysocki (2016) also believed that financial statement guide the decision maker behavior.

The pioneer work of Ball and Brown (1968) was always admired as the starting point for all research concerned with the relationship between accounting information and stock price. Their study provided a strong empirical evidence for the relationship between earnings disclosures and stock prices; they pointed that, if entity reported excess earnings this could enable investors to obtain abnormal returns. Their remarkable foundings also suggests that, the financial informations usefulness relies basically on the ability of this information to guide stock price behavior. Several other studies also confirmed this assumed relationship between stock prices value and the accounting information that disclosed through regular financial reports (Safajou et al., 2005; Bohusova and Nerudova, 2015; Chandra and Ro, 2008; Pourheydari et al., 2008; Fosbre et al., 2009). Dechow et al (2010) pointed out that financial statements can take many forms; the best known is the profit or loss account and balance sheet of businesses. Decision makers attention historically approved to be mostly on income from regular operations therefore revenue gain high priority in decision maker mind relative to other accounting disclosed numbers (Chandra and Ro, 2008; Dechow et al., 2010; Srivastava, 2014).

Historically, the annual reports and accounts of companies are produced by management to serve shareholders and other people such as potential investors, tax authorities, banks, regulators, suppliers, customers and employees may also have an interest in knowing that the financial statements are presented fairly, in all material aspects (Chandra and Ro, 2008; Fosbre et al., 2009). Such produced annual reports contain or effected by the amount of revenue reported by the business; when these reports that contain information about revenue or net income are disclosed to market the stock price of entity in market is influenced and volatile depending on the amount reported (Ball and Brown, 1968). Based on previous review of literature our second hypothesis is:

H2: There is a significant difference between the relationship of stock price and revenue of companys pre and post the application of new revenue standard IFRS 15.

3. Methodology

3.1 Sample and method

The population used in this study comprises the listed mining, construction and engineering companies in Amman stock exchange. This sample sector in Jordan is very crucial part of Jordanian economic sector, the mining sector is regarded as the important sector that might be effected by the new standard IFRS 15; on the other hand construction and engineering sectors is also greatly affected by the new standard after IAS 11 on construction is eliminated and these companies are now comply with IFRS 15 concerning revenue. The study data was obtained from company quid and annual reports issued by Amman Stock Exchange (ASE) for the years from 2012 to 2017. The sample consisted of 23 company; companies that do not apply IFRS 15 were excluded from the sample. The final total sample consisted of 138 observations; these observation comprises 69 before the application of the standard and for the years from 2012-2014 and the same 69 after the standard application for the years from 2015-2017.

The methodology of the study follows two steps; firstly a comparison of means for revenue before and after the inclusion of the standard to capture if there is any difference in athematic means. Secondly, a simple regression analysis will be used Ordinary Least square (OLS) in order to examine the relationship between revenue and stock prices pre and post the IFRS 15 application. The proposed model to test this relationship is as follows:

$$P_{it} = \alpha_0 + \alpha_1 REV_{it} + E \tag{1}$$

Where Pit: Stock price Firm i at the end of year t. REVit: Revenue (sales) for firm i during period t. Eit: any other value relevant information of firm i for period t.

3.2 Results and discussion

3.2.1 Descriptive statistics

The descriptive statistics of the studys variables are over viewed in table (1), the average of market value of stocks price before the standard application was 2.78 JD, and this value fluctuated between the minimum value of 0.26 JD and the maximum value of 28.05 JD. On the other hand, the average of stocks price after the standard application was 1.85 JD, and this value fluctuated between a minimum value of 0.17 JD, and the maximum value of 21.00 JD. As appear from these results that the value of stock prices was decreased with about 51%; the possible explanation for this decrease can be attributed to the decrease of revenue for these companies after the application of the standard. As we can observe from the same table that the average of revenue before the standard application was 114,622 JD, and this value fluctuated between the minimum value of 256.77 JD and the maximum value of 938,429 JD; whilst the average of revenue after the standard application was 61,943 JD, and this value fluctuated between a minimum value of 0 JD, and the maximum value of 750,174 JD. The percentage decrease in revenue average after the application of the standard was about 85% ; and the percentage decrease in total revenue was about 25% . These results provides preliminary evidence that the new standard of revenue recognition (IFRS 15) affected negatively the reported revenues by companies, moreover, the new standard as observed from the results caused also a decrease in the stock prices of these companies after the year 2014.

Table 1: Descriptive Statistics of Model Variables

Variables	Minimum	Mean	Maximum	Std. Deviation
P_-befor	0.26	28.05	2.78	4.74
REV_-before_(000)	-256.77	938,429	114,622	164,504
P_-after	0.17	21.00	1.85	3.93
REV_-after_(000)	0	750,174	61,943	155,672
* N= 69				

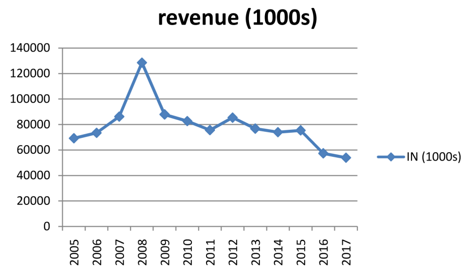

Figure 1: The average revenue for companies from 2005-2017

3.2.2 Empirical results

For further exhibition of the effect of IFRS 15 on both of revenue and stock price we demonstrated this effect using excels charts. As appear from chart 1&2 both of revenue and stock prices was continuously declined for the three years that followed the inclusion of IFRS 15 starting the year 2015.

In order to examine the effect for the inclusion of IFRS 15 on companies a compare of means test was conducted to investigate whether there is a difference

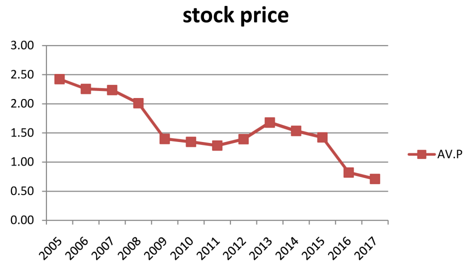

Figure 2: The average stock price of companies from 2005-2017

between the means before and after standard inclusion. But before that we tested if there is a correlation between the years before and after the inclusion of the standard to capture if there any effect for previous years before 2015 on revenue and stock prices after 2015. The results of paired samples correlation appear in table (2). The results of this test show that there is no significant correlation between the two samples before and after the reforms. Therefore we can conclude that the change in companies revenues and stock prices is devoted to the new standard application.

3.2.3 Paired samples correlation

Table 2: the correlation of results before and after IFRS 15

Variables	N	Correlation	Sig.
Pair 1 BEFOR and AFTER	69	0.009	0.916

The second step in compare of means for the new standard effect, we conducted a test of means difference between the both of revenue and stock price pre and post the inclusion of IFRS 15, a paired t-test was employed to capture this effect; and the comparison results between the two means are shown in table (3). In table (3) we can observe that the revenue mean after the standard inclusion was (55,943), while the mean before the standard was applied was (67,622), the change in mean was a decrease with about (-21)%. Similarly, the stocks price mean after the standard inclusion was (2.15) and appears less than that of mean before (2.78), the change in mean was a decrease with about (-29)%. These results prove that the new accounting standard (IFRS 15) that applied starting year 2015 negatively influenced both of revenue and stock prices for the companies under study. Moreover, the results in table (3) for parametric and non-parametric statistical tests that show the means difference for revenue and stock prices showed that, the calculated (t-stat) for both of revenues and stock price was significant; and the P-value (two-tails test) for both of variables was below 5%. Thus we can conclude that, there is a significant difference in the value of revenue and stock prices before and after the new standard inclusion. Also the two related samples test (Wilcoxon) show that, the (Z-stat) for both of revenue and stock prices was 0.000, which is significantly below 5%. Consequently, Jordanian mining, construction and engineering companies t-stat and Z-stat, parametric and non-parametric statistical tests supports the results that we obtained from our previous tests that is, there is a significant difference between the means of revenue and stock prices before and after new standard inclusion.

Table 3: compare of means for revenue & stock prices before and after inclusion of IFRS 15

Variables	**(under IAS 18)**	**(under IFRS 15)**	**change**
Revenue Mean_(000)	67,622	55,943	-21%
Revenue Variance_(000)	27,061	22,233	-22%
Stocks Mean	2.78	2.15	-29%
Observations	69	69	
Def.	68	68	
t-stat (revenue difference)	2.875		
Sig	0.000		
t-stat (stocks difference)	3.495		
Sig	0.000		
2-related-samples test (Wilcoxon)			
Z stat(revenue difference)	-2.785		
Sig	0.000		
Z stat (stock difference)	-4.557		
Sig	0.000		
* the results are sig ($p < 0.05$).			

3.2.4 Regression results

For more exploration to compare of means results, we conducted simple regression test in order to capture the change for the effect IFRS 15 before and after year 2015 on revenue. The results are shown in table (4). We recall our previously mentioned model equation as:

$$P_{it} = \alpha_0 + \alpha_1 REV_{it} + E \tag{2}$$

Our simple regression applied run twice the first run on data from 2012-2014 before the inclusion of IFRS 15, and the second run on data from years 2015-2017 after the application of IFRS15. The results of simple regression for the correlation between revenue (REV) as independent variable and the dependent variable sock price (P) are presented in table (4). As appear in table simple regression results for the effect of revenue on stock price before IFRS 15 show that; the model was fit and statically significant with an F-value of 49.788; Adj R2 was about 51 percent, Coef = 2.08, t-value = 7.056 with sig less than 5 percent. On the other hand the second run of regression show that model also was significantly fit with F-value of 30.365; the Adj R2 was about 39 percent, Coef =1.59, t-value =5.510 with sig less than 5 percent.

Table 4: regression results for revenue & stock prices before and after inclusion of IFRS 15

Ind-variable	R	R2	Adj R2	Coef	t	t- sig.	F	F-sig
Rev-before	0.721	0.520	0.509	2.08	7.056	0.000	49.788	0.000
Rev-after	0.631	0.398	0.385	1.59	5.510	0.000	30.365	0.000
change	-14%	-30%	-32%	-30%	-28%		-64%	

These results indicate that there is a statically significant effect for revenue on stock price pre and post the IFRS 15 inclusion, but this effect was decreased with about 32% percent as appear in the difference between Adj R2 before and after standard implementation. Another difference appears in the value of coefficients and t-value these two results also was decreased with about 29%. This great decrease in all results of regression indicates that the inclusion of IFRS 15 starting the year 2015 instead of AIS 18 has significant negative effect on revenue level that caused witnessed negative decrease in the stock prices for Jordanian companies. However, these results provide more indication that, in Jordan there is a gradual decrease in realization of revenue following the year 2015 and this decrease can be contributed to IFRS 15, thus, and based on previous results of compare of means test and regression test results the hypotheses of the study 1 & 2 are accepted that is, Jordanian companies revenue was gradually changed after the application of new revenue standard IFRS 15; furthermore, stock prices of Jordanian companies were also greatly negatively affected after IFRS 15 application.

4. Conclusion

This study was an attempt to contribute for accounting literature by examining the effect of the adoption of the new revenue recognition standard IFRS 15 and its economic consequences on both of revenue and stock market price. The study was conducted on a sample of Jordanian companies that highly influenced after the inclusion of the new standard. The study measured revenue and stock prices pre and post IFRS 15 implementation; results showed that there were a significant difference in the value of revenue and stock prices before and after the new standard inclusion. Moreover, both of revenue and stock prices was found negatively decreased after IFRS 15 inclusion. These findings provide supportive evidence to literature that Jordanian market have the ability to capture and reflect any changes that occurs in accounting standards. This study contributes to the knowledge through several aspects. First, it provides new evidence on the use of IFRS 15 in Jordanian context. Hence, our findings can be generalized to for developing countries that have stock exchanges with similar characteristics to more developed countries. Second, this study provides more insight for market value behavior association with revenue recognition principle. Third, higher levels of restrictions on revenue recognition could cause an increase in accounting discretion regarding the different forms of earnings. Finally, this study suggests a new avenue for future research in Jordan to investigate the influence of this standard on other economic sectors.

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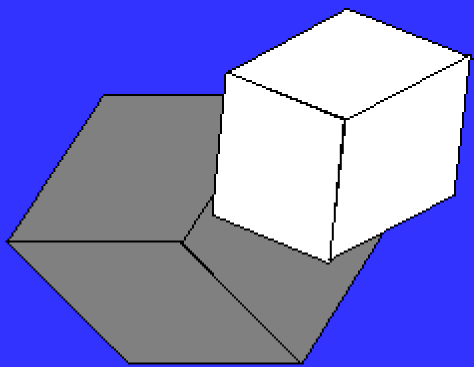

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