On nearly CAP-embedded second maximal subgroups of Sylow p-subgroups of finite groups

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Abstract. A subgroup D of a group G is called a CAP-embedded subgroup of G, if for each prime p dividing the order of D, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of D is also a Sylow p-subgroup of K. Later, we have generalized CAP-embedded subgroup to nearly CAP-embedded subgroup. A subgroup H of a group G is said to be nearly CAP-embedded in G if there is a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that G is equal to HT and the intersection of H and T is contained in H_{ce}. The main purpose of this paper is to study the p-nilpotentcy of a group which every second maximal subgroup of its Sylow p-subgroups is nearly CAP-embedded and some new results are obtained.
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1. Introduction

In this paper, all groups are finite and G stands for a finite group. Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $\mathcal{F}$ denote a formation, $\mathcal{N}_p$ the class of all p-nilpotent groups, and let us denote

$$G^{\mathcal{F}} = \cap\{N \trianglelefteq G \big| G/N \in \mathcal{F}\}$$

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as the $\mathcal{F}$-residual of G. "$H\ Char\ G$" means that H is a characteristic subgroup of G, "the group G is A_4-free" means that there are no subgroups in G for which A_4 is an isomorphic image. The other notations and terminology are standard (see, [8]).

The study of the embedding properties of subgroups of finite groups is one of the most fruitful research areas in the group theory. Many researchers use the embedding property of second maximal subgroups to describe the structure of supersolvable groups, solvable groups, p-solvable groups and other concrete groups. Given a group G, a subgroup K of G is called a second maximal subgroup if there exists a maximal subgroup M of G such that K is a maximal subgroup of M. One of the most classical results in this context is due to B. Huppert. He proved in [9] that if every second maximal subgroup of a group G is normal in G, then G is supersoluble, and if moreover the order of G is divisible by at least three distinct primes, then G is nilpotent. Agrawal in [1] generalized Huppert's result under the weaker hypothesis of permutability. A sharper insight regarding the groups such that every second maximal subgroup of G is normal in G is done by Li Shirong in [10].

Later, many authors investigated the influence of the embedding properties of second maximal subgroups of a Sylow subgroup on the structure of finite groups. For example, Adolfo et al. in [2] obtained the completely classification of finite groups in which the second maximal subgroups of the Sylow p-subgroups, p is a fixed prime, cover or avoid the chief factors of some of its chief series. Qiu et al. in [11] got the structure of finite groups in which the second maximal subgroups of the Sylow p-subgroups, p is a fixed prime, satisfy the partial Π-property (see, [3, Section 7]). Guo and Shum also in [7] proved the following result. Let G be a group and p the smallest prime number dividing the order of G. If all second maximal subgroup of every Sylow p-subgroup of G are c-normal in G and G is A_4-free, then G is p-nilpotent. In [5], Guo and Guo introduced the notion of CAP-embedded subgroup. A subgroup H of a group G is said to have the CAP-embedded property in G or is called a CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K. And they obtained the same conclusion in the case where c-normality is replaced by CAP-embedded property.

In order to generalize the c-normality and CAP-embedded property, Xu and Chen in [14] proposed the defintion of nearly CAP-embedded subgroup. A subgroup H of a group G is said to be nearly CAP-embedded in G if there is a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ce}$. Clearly a c-normal subgroup or CAP-embedded subgroup must be a nearly CAP-embedded subgroup. But the converse is not true in general.

Example 1.1. Let A_4 be the alternative group of degree 4 and $D = \langle d \rangle$ be a cyclic group of order 2. Let $G = D \times A_4$. Then, $A_4 = [K_4]C_3$ where $K_4 = \langle a, b \rangle$

is the Klein Four Group with generators a and b of order 2 and C_3 is the cyclic group of order 3. Take $H = \langle ad \rangle$ to be the cyclic subgroup of order 2 of G. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, H is nearly CAP-embedded in G. However, H is not a CAP-embedded subgroup of G as it neither covers nor avoids $(D \times K_4)/D$ and there is no the subgroup of order 6 containing H covers or avoids $(D \times K_4)/D$.

In this note, we investigate the structure of the groups in which some second maximal subgroups of a Sylow subgroup satisfy the nearly CAP-embedded property. Our results are as follows:

Theorem 1.1. *Suppose that N is a normal subgroup of a group G such that G/N is p-nilpotent and P is a Sylow p-subgroup of N, where p is the smallest prime divisor of $|G|$. If G is A_4-free and every second maximal subgroups of P is nearly CAP-embedded in G, then G is p-nilpotent.*

Theorem 1.2. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G such that $G/N \in \mathcal{F}$. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G belongs to $\mathcal{F}$.*

Remark 1.1. The hypothesis that p is the smallest prime divisor of $|G|$ in Theorem 1.1 is essential. For example, consider an elementary abelian group $U = \langle a, b | a^5 = b^5 = 1, ab = ba \rangle$ of order 25. Let α be an automorphism of U of order 3 such that $a^\alpha = b, b^\alpha = a^{-1}b^{-1}$. Let $V = \langle c, d \rangle$ be a copy of U and $G = [U \times V]\langle \alpha \rangle$. For any subgroup H of G of order 25, there exists a minimal normal subgroup K such that $H \cap K = 1$ (for details, see [11, Example 1.5]), then H satisfies the nearly CAP-embedded property in G. However, G is not p-nilpotent.

The assumption that G is A_4-free in Theorem 1.1 and Theorem 1.2 can not be removed. In fact, let $G = A_4$, then the second maximal subgroup of a Sylow 2-subgroup of G is trivial, of course, it satisfies the nearly CAP-embedded property in G, but A_4 is neither a 2-nilpotent group nor a Sylow tower group.

2. Preliminary results

For convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([5, Lemma 1]). *Suppose that U is CAP-embedded in a group G and $N \trianglelefteq G$. Then, UN/N is CAP-embedded in G/N.*

Lemma 2.2 ([14, Lemma 2.8]). *Let U be a nearly CAP-embedded subgroup and N a normal subgroup of a group G. Then*

(1) *If $N \leq U$, then U/N is nearly CAP-embedded in G/N.*

(2) *If* $(|U|, |N|) = 1$, *then* UN/N *is nearly* CAP*-embedded in* G/N.

Lemma 2.3 ([12, Lemma 1.6]). *Let* P *be a nilpotent normal subgroup of a group* G. *If* $P \cap \Phi(G) = 1$, *then* P *is the direct product of some minimal normal subgroups of* G.

Lemma 2.4 ([15, Lemma 2.6]). *Let* G *be an* A_4*-free group,* $p = min\pi(G)$, *and let* N *be a normal subgroup of* G *such that* G/N *is* p*-nilpotent. If* $p^3 \nmid |N|$, *then* G *is* p*-nilpotent.*

Lemma 2.5 ([14, Theorem 3.1]). *Let* G *be a group,* N *a normal subgroup of* G *such that* G/N *is* p*-nilpotent and* P *a Sylow* p*-subgroup of* N, *where* $p \in \pi(G)$ *with* $(|G|, p-1) = 1$. *If all maximal subgroups of* P *are nearly* CAP*-embedded in* G, *then* G *is* p*-nilpotent.*

Lemma 2.6 ([4, Lemma 2.1]). *Let* H *be a subgroups of a group* G. *Let* $1 < \cdots < N < \cdots < M < \cdots < G$ *be a normal series. If* H *covers (avoid)* M/N, *then* H *covers (avoid) any quotient factor between* M *and* N *of any refinement of the normal series.*

Lemma 2.7. *Let* N *be a normal subgroup of a group* G *and* V *a nearly* CAP*-embedded subgroup of* G. *If* $V \leq N$, *then* V *is nearly* CAP*-embedded in* N.

Proof. By the hypothesis, there is a subnormal subgroup T of G and a CAP-embedded subgroup V_{ce} of G contained in V such that $G = VT$ and $V \cap T \leq V_{ce}$, For each prime p dividing the order of V_{ce}, there exists a CAP-subgroup K of G such that a Sylow p-subgroup $(V_{ce})_p$ of V_{ce} is also a Sylow p-subgroup K_p of K. Clearly, $T \cap N$ is subnormal in N, $V(T \cap N) = N$, $V \cap T \cap N = V \cap T \leq V_{ce}$, and by the Lemma 2.6, $K \cap N$ is a CAP-subgroup of N and $(V_{ce})_p = (V_{ce})_p \cap N = K_p \cap N = (K \cap N)_p$. Hence, V is nearly CAP-embedded in N. □

3. Proofs

Now, we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Assume that the result is false. Let G be a minimal counterexample with least $|N| + |G|$.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \nleq \Phi(G)$.

Let L be a minimal normal subgroup of G contained in N. Consider the factor group $\overline{G} = G/N$. Clearly, $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P_1} = P_1L/L$ be a second maximal subgroup of $\overline{P}$. We may assume that P_1 is a second maximal subgroup of P. Then, $P_1 \cap L = P \cap L$ is a Sylow p-subgroup of L. By the hypothesis, there is a subnormal subgroup B and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1B$ and $P_1 \cap B \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. We have $P_1L \cap BL = (P_1L \cap B)L$. Let $\pi(G) = \{p_1, p_2, \cdots p_n\}$, where $p_1 = p$, and B_{p_i} be a Sylow p_i-subgroup of B $(i = 2, \cdots, n)$. Then, B_{p_i} is also a Sylow p_i-subgroup of G, hence $B_{p_i} \cap N$ is a Sylow p_i-subgroup of N $(i = 2, \cdots, n)$. Write $V = \langle L \cap B_{p_2}, \cdots, L \cap B_{p_n} \rangle$, then $V \leq B$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1L \cap BL = (P_1L \cap B)L = (P_1V \cap B)L = (P_1 \cap B)VL = (P_1 \cap B)L$. By Lemma 2.1, we get $(P_1L/L) \cap (BL/L) = (P_1 \cap B)L/L \leq (P_1)_{ce}L/L \in Syl_p(KL/L)$. Therefore, $\overline{P_1}$ is nearly CAP-embedded in $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G contained in N and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

If $E = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/E$. Clearly, $\overline{G}/\overline{N} \cong G/NE$ is p-nilpotent because G/N is, where $\overline{N} = NE/E$. Let $\overline{P_1} = P_1E/E$ be a second maximal subgroup of PE/E. We may assume that P_1 is a second maximal subgroup of P. Since P_1 is nearly CAP-embedded in G, P_1E/E is nearly CAP-embedded in G/E by Lemma 2.2 (2). The minimality of G yields that G is p-nilpotent, therefore G is p-nilpotent, a contradiction.

(3) $O_p(N) = 1$ and so L is not p-nilpotent.

If $O_p(N) \neq 1$, then by (1), $L \leq O_p(N)$ and there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. By (1) and Lemma 2.4, we get $|L| \geq p^3$. So we may choose a second maximal subgroup P_1 of P containing M_p, where $M_p \in Syl_p(M)$. Because P_1 is a nearly CAP-embedded subgroup of G, there is a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \cap (P_1)_{ce} \in Syl_p(L)$, and so $L \leq P_1$, thus $P = LM_p = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p^2$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$. Hence, G is p-nilpotent, a contradiction.

If L is p-nilpotent, then $L_{p'}$ $Char$ $L \trianglelefteq N$, so $L_{p'} \leq O_{p'}(N) \leq O_{p'}(G) = 1$ by (2). Thus L is a p-group, $L \leq O_p(N) = 1$, a contradiction. Hence, (3) holds.

(4) The final contradiction.

If $P \leq L$, then $P \in Syl_p(L)$. By (3) and Lemma 2.4, $|P| > p^2$. For every second maximal subgroup P_3 of P, P_3 is nearly CAP-embedded in G. So, there is a subnormal subgroup T_1 of G and a CAP-embedded subgroup $(P_3)_{ce}$ contained in P_3 of G such that $G = P_3T$ and $P_3 \cap T \leq (P_3)_{ce} \in Syl_p(A)$, where A is a CAP subgroup of G. Clearly $AL \neq A$ and $(P_3)_{ce} \leq L \cap A = 1$, then $p^3 \nmid |T|$, so T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then, $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$. Hence, G is p-nilpotent, a contradiction. Therefore, $P \nleq L$. If $P \cap L \leq \Phi(P)$, then L is p-nilpotent by Tate's theorem [8, IV, Th 4.7], contrary to (3). Consequently, $P \cap L \nleq \Phi(P)$.

Let P_1 be a maximal subgroup of P containing $L \cap P$. Clearly, $L \cap P \nleq \Phi(P_1)$. Hence, there exists a maximal subgroup P_2 of P_1 such that $P_1 = (L \cap P)P_2$. Let T be a subnormal supplement of P_2 in G, we have $P_2 \cap T \leq (P_2)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_2)_{ce} \in Syl_p(K)$ that $P_2 \cap K = (P_2)_{ce}$, then $P_2 \cap L \in Syl_p(L)$. Thus $L \cap P = L \cap P_2 \leq K \cap P_2 = (P_2)_{ce} \leq P_2$. We obtain $P_1 = (L \cap P)P_2 = P_2$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_2 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p^2$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.4. Hence, G is p-nilpotent by the subnormality of T, a contradiction.

This completes the proof. □

Corollary 3.1. *Let G be an A_4-free group, P a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If G is not p-nilpotent, then there is a second maximal subgroup of $P \cap G^{\mathcal{N}_p}$ which is not nearly CAP-embedded in G.*

Corollary 3.2. *Let G be an A_4-free group. If, for every prime p dividing the order of G and $P \in Syl_p(G)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G is a Sylow tower group of supersolvable type.*

Similarly, we have the following results.

Theorem 3.1. *Let N be a normal subgroup of a group G such that G/N is p-nilpotent and let P be a Sylow p-subgroup of N, where p is a prime divisor of $|G|$ with $(|G|, p^2 - 1) = 1$. If every second maximal subgroup of P is nearly CAP-embedded in G, then G is p-nilpotent.*

Proof of Theorem 1.2. By Lemma 2.7 and Corollary 3.2, we use induction on $|G|$ to see that N is a Sylow tower group of supersolvable type. Let r be the largest prime number in $\pi(N)$ and $R \in Syl_p(N)$. Then, R is normal in G and $(G/R)/(N/R) \cong G/N$ is a Sylow tower group of supersolvable type. By induction, $G/R \in \mathcal{F}$. Let q be the largest prime divisor of $|G|$ and Q a Sylow q-subgroup of G. Then, $RQ \trianglelefteq G$. If $q = r$, then G has the Sylow tower property, as desired. Hence, we may assume that $r < q$.

Case 1. $RQ < G$. In this case, we will show $G_1 = RQ$ is p-nilpotent. By the hypothesis and Lemma 2.7, G_1 is A_4-free and every second maximal subgroup R_1 of R is nearly CAP-embedded in G_1. Then, by Theorem 1.1, $Q\ Char\ RQ$ and so $Q \trianglelefteq G$. Now, consider $(G/Q, NQ/Q)$. Then, $G/Q \in \mathcal{F}$ by induction and Lemma 2.2. Thus $G \in \mathcal{F}$, as desired again.

Case 2. $G = RQ$. Let L be a minimal normal subgroup of G with $L \leq R$. Then, the quotient group G/L satisfies the hypothesis. By induction, we see that G/L is a Sylow tower group of supersolvable type. Since the class of all Sylow tower groups is a saturated formation, we have $L \nsubseteq \Phi(G)$ and L is the unique minimal normal subgroup of G which is contained in R. Therefore, $L = F(R) = R$ by Lemma 2.3. In particular, R is an abelian group. If R is a cyclic subgroup of order r, then $r < q$ implies that $G = R \times Q$. Of course, $G \in \mathcal{F}$, which completes the proof. Hence, we may assume that $|R| \geq r^2$. Let R_1 be a 2-maximal subgroup of R. By the hypothesis, R_1 is nearly CAP-embedded in G. By the proof of the step (3) in theorem 1.1, we have R is an elementary abelian group of order r^2. Now, any element g of Q induces an automorphism σ of R. When $|R| = r^2$, we know that $|Aut(R)| = (r+1)r(r-1)^2$. If $r = 2$ and some $\sigma \neq 1$, then the order of σ must be 3 as $r < q$. Thus the subgroup $R\langle g \rangle$ is not A_4-free, contrary to the hypothesis. Hence, all $\sigma = 1$, i.e., $G = R \times Q$, completing the proof. The remainder is to consider the case when $r > 2$. Noticing that $r+1$ is not a prime, so we have all $\sigma = 1$ and $G = R \times Q$, hence $G \in \mathcal{F}$. The proof is now completed. □

Corollary 3.3. *Let G be a group of odd order, N a normal subgroup of G such that G/N is a Sylow tower group of supersolvable type. If, for every prime p dividing the order of N and $P \in Sylp(N)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G is a Sylow tower group of supersolvable type.*

4. Some applications

Obviously, c-normal subgroups, CAP-subgroups and CAP-embedded subgroups are nearly CAP-embedded subgroups, a lot of results can be obtained according to our theorems.

By Theorem 1.1, Theorem 1.2 and Theorem 3.1, we have:

Corollary 4.1 ([7, Theorem 3.2])**.** *Let p is the smallest prime divisor of $|G|$, P is a Sylow p-subgroup of G. If every second maximal subgroups of P is c-normal in G and G is A_4-free, then G is p-nilpotent.*

Corollary 4.2 ([13, Theorem 4.2])**.** *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Assume that G is A_4-free and every second maximal subgroup of the Sylow p-subgroup of G is c-normal in G. Then, $G/O_p(G)$ is p-nilpotent.*

Corollary 4.3 ([6, Theorem 3.11])**.** *Let H be a normal subgroup of a group G and p the smallest prime number dividing the order of H. If all 2-maximal subgroups of every Sylow p-subgroup of H are CAP-subgroups of G and G is A_4-free, then H is p-nilpotent.*

Corollary 4.4 ([6, Corollary 3.13])**.** *Let H be a normal subgroup of a group G. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of H are CAP-subgroups of G, then H is a Sylow tower group of supersolvable type.*

Corollary 4.5 ([5, Theorem 3.3])**.** *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free, and there exists a Sylow p-subgroup P of*

H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.

Corollary 4.6 ([5, Corollary 3.4])**.** *Let p be a prime dividing the order of the group G with $(|G|, p^2 - 1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

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