A remark on relative Hilali conjectures

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Abstract. S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have recently proved, in certain cases, a relative Hilali conjecture. This is an inequality about the dimensions of the kernel of homomorphisms of rational homotopy groups and rational homology groups, hence shall be called a Kernel-relative Hilali conjecture. In this paper we add another relative Hilali conjecture with respect to the cokernel of such homomorphisms, which shall be called a Cokernel-relative Hilali conjecture. We consider some examples for these conjectures and discuss conditions under which these conjectures hold and also conditions under which they are equivalent to each other. As byproducts of these computations, we show that $\dim(\pi_*(X)\otimes\mathbb{Q})$ and the formal dimension n_X of X have the same parity and that the Hilali conjecture holds when $\dim(\pi_*(X)\otimes\mathbb{Q}) \le 4$.
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1. Introduction

The homotopy and homology ranks of a topological space X are respectively defined by $\dim(\pi_*(X)\otimes\mathbb{Q})$ and $\dim H_*(X;\mathbb{Q})$, where $\pi_*(X)\otimes\mathbb{Q} := \sum_{i\geqq 1}\pi_i(X)\otimes\mathbb{Q}$ and $H_*(X;\mathbb{Q}) := \sum_{i\geqq 0} H_i(X;\mathbb{Q})$. Since $\mathbb{Q}$ is a field, it follows from the Universal Coefficient Theorem for the homology group, involving the torsion-module $\mathrm{Tor}(A,B)$, that we have $H_i(X;\mathbb{Q}) \cong H_i(X)\otimes\mathbb{Q}$ where $H_i(X) := H_i(X;\mathbb{Z})$. So, we use $H_*(X)\otimes\mathbb{Q}$ instead of $H_*(X;\mathbb{Q})$.

A *rationally elliptic space* is a simply connected topological space X such that

$$\dim\left(\pi_*(X)\otimes\mathbb{Q}\right) < \infty \text{ and } \dim\left(H_*(X)\otimes\mathbb{Q}\right) < \infty.$$

In [9] M. R. Hilali conjectured that if X is a rationally elliptic space, then the following inequality holds:

$$\dim(\pi_*(X)\otimes\mathbb{Q}) \leqq \dim\left(H_*(X)\otimes\mathbb{Q}\right). \tag{1.1}$$

Namely, since X is simply connected, (1.1) means that

$$\dim\left(\bigoplus_{i\geqq 2}\pi_i(X)\otimes\mathbb{Q}\right) \leqq 1 + \dim\left(\bigoplus_{i\geqq 2} H_i(X)\otimes\mathbb{Q}\right).$$

Remark 1.1. Usually the Hilali conjecture is the following inequality, using the rational cohomology group:

$$\dim(\pi_*(X) \otimes \mathbb{Q}) \leqq \dim H^*(X;\mathbb{Q}). \tag{1.2}$$

However, since $\dim H_*(X;\mathbb{Q}) < \infty$ and $\dim H^*(X;\mathbb{Q}) < \infty$, namely $H_*(X;\mathbb{Q})$ and $H^*(X;\mathbb{Q})$ are finitely generated[1], we have $H^*(X;\mathbb{Q}) \cong Hom\,((H_*(X;\mathbb{Q}),\mathbb{Q})$ (more precisely, $H^i(X;\mathbb{Q}) \cong Hom\,((H_i(X;\mathbb{Q}),\mathbb{Q})$ for each i), so $H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q})$, hence $\dim H^*(X;\mathbb{Q}) = \dim H_*(X;\mathbb{Q})$. Therefore, (1.1) and (1.2) are the same. Since we use (rational) Hurewicz Theorem later, it is better to use homology groups instead of cohomology groups.

Remark 1.2. In [19] we showed the Hilali conjecture "modulo product", which is that for any rationally elliptic space X such that its fundamental group is an Abelian group, then there exists some integer n_0 such that for any $n \geq n_0$ the following strict inequality holds:

$$\dim(\pi_*(X^n) \otimes \mathbb{Q}) < \dim(H_*(X^n) \otimes \mathbb{Q}),$$

where X^n is the Cartesian product $X^n = \underbrace{X \times \cdots \times X}_{n}$. As to some work on such an integer n_0 and related topics, see [11, 12, 20].

In our previous paper [17] (also see [18]) we made the following conjecture, called *a relative Hilali conjecture*:

Conjecture 1.1. *For a continuous map $f: X \to Y$ of rationally elliptic spaces X and Y, the following inequality holds:*

$$\sum_{i\geqq 2} \dim(\mathrm{Ker}(\pi_i(f) \otimes \mathbb{Q})) \leqq 1 + \sum_{i\geqq 2} \dim(\mathrm{Ker}(H_i(f) \otimes \mathbb{Q})).$$

As remarked below (a remark right after Conjecture 2.1 below in §2), in the above conjecture it suffices to assume only that the source space X is rationally elliptic.

In [2, 21] S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have proved this relative conjecture positively in some cases. This relative conjecture is a conjecture using the kernel of the homomorphisms $\pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ and $H_*(f) \otimes \mathbb{Q} : H_*(X) \otimes \mathbb{Q} \to H_*(Y) \otimes \mathbb{Q}$. So, this shall be called *a Kernel-relative Hilali conjecture*, abusing words. In this note we add another relative conjecture, called *a Cokernel-relative Hilali conjecture*, using the cokernel of these two homomorphisms. We consider some examples for these two conjectures and we discuss conditions under which these two conjectures hold

1. If $H_*(X;\mathbb{Q})$ is not finitely generated, then we do not have $H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q})$. Indeed, if $H_*(X;\mathbb{Q}) = \oplus_{n\in\mathbb{N}}\mathbb{Q}$, which is note finitely generated, then $H^*(X;\mathbb{Q}) \cong Hom(H_*(X;\mathbb{Q}),\mathbb{Q}) = \prod_{n\in\mathbb{N}}\mathbb{Q}$. Thus, $H^*(X;\mathbb{Q}) \ncong H_*(X;\mathbb{Q})$.

and also conditions under which they are equivalent to each other. For example, if the above inequality (1.1) becomes equality for both X and Y, then for *any* continuous map $f : X \to Y$ the Kernel-relative Hilali conjecture holds if and only if the Cokernel-relative Hilali conjecture holds.

As byproducts of these computations and using the well-known stringent restrictions on homotopy groups, we show that if $\dim(\pi_*(X)\otimes\mathbb{Q})$ is odd (resp., even), the formal dimension n_X is odd (resp., even), and also we show that if $\dim(\pi_*(X)\otimes\mathbb{Q}) = 1, 2, 3, 4$, the Hilali conjecture holds.

In this paper we discuss without appealing to minimal models, although minimal models play important roles in rational homotopy theory.

2. Relative Hilali conjectures

In this section, for the sake of later presentation, we recall some basic ingredients of homotopical and homological aspects of a continuous map, for example, homotopical and homological Poincaré polynomial of a map.

Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y of finite type. For the homomorphisms $H_i(f)\otimes\mathbb{Q} : H_i(X)\otimes\mathbb{Q} \to H_i(Y)\otimes\mathbb{Q}$ and $\pi_i(f)\otimes\mathbb{Q} : \pi_i(X)\otimes\mathbb{Q} \to \pi_i(Y)\otimes\mathbb{Q}$, we have the following exact sequences of finite dimensional $\mathbb{Q}$-vector spaces[2]:

$$\begin{aligned} 0 \to \operatorname{Ker}(H_i(f)\otimes\mathbb{Q}) \to H_i(X)\otimes\mathbb{Q} \\ \to H_i(Y)\otimes\mathbb{Q} \to \operatorname{Coker}(H_i(f)\otimes\mathbb{Q}) \to 0, \ \ \forall i \geqq 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} 0 \to \operatorname{Ker}(\pi_i(f)\otimes\mathbb{Q}) \to \pi_i(X)\otimes\mathbb{Q} \\ \to \pi_i(Y)\otimes\mathbb{Q} \to \operatorname{Coker}(\pi_i(f)\otimes\mathbb{Q}) \to 0, \quad \forall i \geqq 2. \end{aligned} \tag{2.2}$$

Since X and Y are simply connected, they are path-connected as well (by the definition of simply connectedness), thus we have

$$\mathbb{Q} \cong H_0(X)\otimes\mathbb{Q} \xrightarrow[\cong]{f_*} H_0(Y)\otimes\mathbb{Q} \cong \mathbb{Q},$$

so, $\operatorname{Ker}(H_0(f)\otimes\mathbb{Q}) = \operatorname{Coker}(H_0(f)\otimes\mathbb{Q}) = 0$. It follows from (2.1) and (2.2) that we get the following equalities: for $\forall i \geqq 2$

$$\begin{aligned} \dim(\operatorname{Ker}(H_i(f)\otimes\mathbb{Q})) - \dim(H_i(X)\otimes\mathbb{Q}) + \dim(H_i(Y)\otimes\mathbb{Q}) \\ - \dim(\operatorname{Coker}(H_i(f)\otimes\mathbb{Q})) = 0, \end{aligned}$$

$$\begin{aligned} \dim(\operatorname{Ker}(\pi_i(f)\otimes\mathbb{Q})) - \dim(\pi_i(X)\otimes\mathbb{Q}) + \dim(\pi_i(Y)\otimes\mathbb{Q}) \\ - \dim(\operatorname{Coker}(\pi_i(f)\otimes\mathbb{Q})) = 0. \end{aligned}$$

2. Recall that $\operatorname{Coker}(T) := B/\operatorname{Im}(T)$ for a linear map $T : A \to B$ of vector spaces.

For later use, we use the following notation.

$$\dim\left(\operatorname{Ker}(\pi_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\operatorname{Ker}(\pi_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\operatorname{Ker}(H_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\operatorname{Ker}(H_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\operatorname{Coker}(\pi_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\operatorname{Coker}(\pi_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\operatorname{Coker}(H_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\operatorname{Coker}(H_i(f)\otimes\mathbb{Q})\right).$$

Definition 2.1. *Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y.*

1. *If $\dim\left(\operatorname{Ker}(H_*(f)\otimes\mathbb{Q})\right) < \infty$ and $\dim\left(\operatorname{Ker}(\pi_*(f)\otimes\mathbb{Q})\right) < \infty$, then f is called rationally elliptic with respect to Kernel or rationally Kernel-elliptic.*
2. *If $\dim\left(\operatorname{Coker}(H_*(f)\otimes\mathbb{Q})\right) < \infty$ and $\dim\left(\operatorname{Coker}(\pi_*(f)\otimes\mathbb{Q})\right) < \infty$, then f is called rationally elliptic with respect to Cokernel or rationally Cokernel-elliptic.*
3. *If the map f is rationally elliptic with respect to both kernel and cokernel, then f is called rationally elliptic.*

Remark 2.1. Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y.

1. If X is rationally elliptic, then f is rationally Kernel-elliptic.
2. If Y is rationally elliptic, then f is rationally Cokernel-elliptic.
3. If X and Y are both rationally elliptic, then f is rationally elliptic.

In our previous paper [17] (cf. [18]) we made the following conjecture, called *a relative Hilali conjecture*

Conjecture 2.1. *For a continuous map $f : X \to Y$ of simply connected rationally elliptic spaces X and Y, the following inequality holds:*

$$\sum_{i\geqq 2}\dim\left(\operatorname{Ker}(\pi_i(f)\otimes\mathbb{Q})\right) \leqq 1 + \sum_{i\geqq 2}\dim\left(\operatorname{Ker}(H_i(f)\otimes\mathbb{Q})\right).$$

It follows from the above Remark 2.1 that it suffices to require only the rational ellipticity of the source space X for the above Conjecture 2.1, which is a conjecture *as to the Kernel*. Due to Remark 2.1 (3), clearly Conjecture 2.1 can be modified as follows, *adding an inequality with respect to Cokernel*:

Conjecture 2.2. *For a rationally elliptic continuous map $f : X \to Y$ of simply connected spaces X and Y, the following inequalities hold:*

$$\sum_{i \geqq 2} \dim \left(\operatorname{Ker}(\pi_i(f) \otimes \mathbb{Q})\right) \leqq 1 + \sum_{i \geqq 2} \dim \left(\operatorname{Ker}(H_i(f) \otimes \mathbb{Q})\right), \tag{2.3}$$

$$\sum_{i \geqq 2} \dim \left(\operatorname{Coker}(\pi_i(f) \otimes \mathbb{Q})\right) \leqq 1 + \sum_{i \geqq 2} \dim \left(\operatorname{Coker}(H_i(f) \otimes \mathbb{Q})\right). \tag{2.4}$$

Here, we note that when the target space Y is contractible, the above conjecture (2.3) becomes the original Hilali conjecture. Similarly, when the source space X is contractible, the above conjecture (2.4) also becomes the original Hilali conjecture.

In order to make it clear, we call (2.3) and (2.4), respectively, *a Kernel-relative Hilali conjecture* and *a Cokernel-relative Hilali conjecture*, abusing words.

Remark 2.2. We note that if $f : A \to B$ is a linear map of two vector spaces, then we have

$$\begin{aligned}
\dim \operatorname{Coker} f &= \dim(B/\operatorname{im}(f)) \\
&= \dim B - \dim\left(\operatorname{im}(f)\right) \\
&= \dim B - \dim(A/\ker(f)) \\
&= \dim B - \dim A + \dim(\ker(f)).
\end{aligned}$$

Hence, the above (2.4) is also expressed as follows:

$$\begin{aligned}
&\sum_{i \geqq 2} \dim \left(\ker(\pi_i(f) \otimes \mathbb{Q})\right) + \sum_{i \geqq 2} \dim(\pi_i(Y) \otimes \mathbb{Q}) - \sum_{i \geqq 2} \dim(\pi_i(X) \otimes \mathbb{Q}) \\
&\leqq 1 + \sum_{i \geqq 2} \dim \left(\ker(H_i(f) \otimes \mathbb{Q})\right) + \sum_{i \geqq 2} \dim(H_i(Y) \otimes \mathbb{Q}) - \sum_{i \geqq 2} \dim(H_i(X) \otimes \mathbb{Q})).
\end{aligned}$$

It may be interesting to see whether these two conjectures are related to each other or not, namely whether (2.3) implies (2.4) and vice versa.

In [2, 21] S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have proved the above Kernel-relative Hilali conjecture (2.3) in some cases. Thus, it would be interesting to see whether the above Cokernel-relative Hilali conjecture also holds in these cases considered by Chouingou–Hilali–Hilali–Zaim.

3. Some examples

For discussion below, we use the following symbols for the sake of simplicity:

$$\varpi(X) := \sum_{i \geqq 2} \dim(\pi_i(X) \otimes \mathbb{Q}), \quad \eta(X) := \sum_{i \geqq 2} \dim(H_i(X) \otimes \mathbb{Q})),$$

$$\ker \varpi(f) := \sum_{i \geqq 2} \dim \left(\ker(\pi_i(f) \otimes \mathbb{Q})\right), \quad \ker \eta(f) := \sum_{i \geqq 2} \dim \left(\ker(H_i(f) \otimes \mathbb{Q})\right).$$

Thus, the Hilali conjecture is claiming

$$\varpi(X) \leq 1 + \eta(X),$$

i.e., either $\varpi(X) = 1 + \eta(X)$ or $\varpi(X) < 1 + \eta(X)$. The latter means that $\varpi(X) \leq \eta(X)$. Clearly, for any rationally elliptic space X, either $\varpi(X) \leq \eta(X)$ or $\varpi(X) > \eta(X)$. Therefore, the Hilali conjecture claims that if $\varpi(X) > \eta(X)$, then $\varpi(X)$ *exceeds* $\eta(X)$ *only by* 1; $\varpi(X) = 1 + \eta(X)$.

No counterexample to the Hilali conjecture has been found yet. If there exists a counterexample to the Hilali conjecture, then that would be a rationally elliptic space Z such that

$$\varpi(Z) = j + \eta(Z) \text{ for some integer } j \geq 2.$$

Simple typical examples for $\varpi(X) > \eta(X)$ are all the even dimensional spheres $S^{2k}(k \geq 1)$, by the following well-known results (due to Serre Finiteness Theorem [14, 15]):

$$\pi_i(S^{2k}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2k, 4k-1, \\ 0 & i \neq 2k, 4k-1, \end{cases} \qquad \pi_i(S^{2k+1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2k+1, \\ 0 & i \neq 2k+1. \end{cases}$$

$\varpi(S^{2k}) = 2$ and $\eta(S^{2k}) = 1$, thus $\varpi(S^{2k}) = 1 + \eta(S^{2k}) = 2$. $\varpi(S^{2k+1}) = \eta(S^{2k+1}) = 1$

For later computation, we recall the rational homotopy and homology groups of some familiar rationally elliptic spaces:

1. $\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$ for $k > 1$. Hence, we have

$$\pi_k(\mathbb{RP}^n) \otimes \mathbb{Q} = \pi_k(S^n) \otimes \mathbb{Q}.$$

$$H_k(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{for } k = 0, n, \\ 0, & \text{for } k \neq 0, n. \end{cases}$$

So, $\varpi(\mathbb{RP}^n) = 1$ if n is odd and $\varpi(\mathbb{RP}^n) = 2$ if n is even. $\eta(\mathbb{RP}^n) = 1$. Thus, we have

$$\varpi(\mathbb{RP}^n) = \eta(\mathbb{RP}^n) = 1 \text{ for } n \text{ odd and } \varpi(\mathbb{RP}^n) = 1 + \eta(\mathbb{RP}^n) = 2 \text{ for } n \text{ even.}$$

2.

$$\pi_k(\mathbb{CP}^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \text{for } k = 2, 2n+1, \\ 0, & \text{for } k \neq 2, 2n+1, \end{cases}$$

which follows from the long exact sequence of a fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n$:

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{CP}^n) \to \pi_{k-1}(S^1) \to \pi_{k-1}(S^{2n+1}) \to \cdots,$$

$$H_k(\mathbb{CP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{for } k = 0 \text{ and } 2 \leq k \leq 2n \text{ for even } k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\varpi(\mathbb{CP}^n) = 2$ and $\eta(\mathbb{CP}^n) = n$. Therefore, we have

$$\varpi(\mathbb{CP}^1) = 1 + \eta(\mathbb{CP}^1) = 2 \text{ and } \varpi(\mathbb{CP}^n) \leq \eta(\mathbb{CP}^n) \text{ for } n \geq 2.$$

We shall use the following terminology:

Definition 3.1. *Le X be a rationally elliptic space.*

1. *If $\varpi(X) \leq \eta(X)$, it is called a space of type $\varpi \leq \eta$, (e.g., S^{2k+1}, $\mathbb{RP}^{2k+1}$, $\mathbb{CP}^n$ for $n \geq 2$).*
2. *If $\varpi(X) = 1 + \eta(X)$, it is called a Hilali space*[3]*, (e.g., S^{2k}, $\mathbb{RP}^{2k}$. Here, $S^2 = \mathbb{CP}^1$.)*
3. *If $\varpi(X) = j + \eta(X)$ with an integer $j \geq 2$, it is called a space of type $\varpi = j + \eta (j \geq 2)$ or a non-Hilali space with $\varpi = j + \eta (j \geq 2)$.*

Or we can simplify these names as follows:

Definition 3.2. *A rationally elliptic space X such that $\varpi(X) = \gamma(X) + \eta(X)$, where $\gamma(X)$ is an integer called a homotopy–homology gap, is called a space of type $\varpi = \gamma + \eta$.*

1. *If $\gamma < 1$, it is a space of type $\varpi \leq \eta$, which shall be also called a standard space,*
2. *If $\gamma = 1$, it is a Hilali space,*
3. *If $\gamma > 1$, it is a non-Hilali space of type $\varpi = \gamma + \eta (\gamma \geq 2)$.*

Remark 3.1. The Hilali conjecture [9] claims that $\gamma \leq 1$ for any rationally elliptic space X.

Remark 3.2. Our previous result [19] about the Hilali conjecture "modulo product" (see Remark 1.2 above) means that for any rationally elliptic space X, in particular, whether it is a Hilali space or a non-Hilali space, there exists a certain integer N_0 such that for all integers $n \geq N_0$ the Cartesian product X^n of n copies of X becomes a standard space, i.e., even if $\varpi(X) > \eta(X)$, $\varpi(X^n) \leq \eta(X^n)$ for all integers $n \geq N_0$.

Example 3.1. Let Y be a rationally elliptic space. *The Kernel-relative Hilali conjecture holds for any continuous map $f : S^{2k+1} \to Y$.* Since $\pi_n(S^{2k+1}) \otimes \mathbb{Q} = 0$ for $n \neq 2k+1$,

$$\ker\Big(f_* \otimes \mathbb{Q} : \pi_n(S^{2k+1}) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}\Big) = 0,$$

3. We call it so, since Hilali made such a conjecture.

for $n \neq 2k+1$. Hence, we consider only $f_* \otimes \mathbb{Q} : \pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} \to \pi_{2k+1}(Y) \otimes \mathbb{Q}$, which is either injective or the zero homomorphism by the dimension reason since $\pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} = \mathbb{Q}$ and $f_* \otimes \mathbb{Q}$ is a linear map of vector spaces over $\mathbb{Q}$.

$\ker \varpi(f) = 0$ if $f_* \otimes \mathbb{Q}$ is injective and $\ker \varpi(f) = 1$ if $f_* \otimes \mathbb{Q}$ is the zero homomorphism. Thus, $\ker \varpi(f) = 0$ or 1. As to the rational homology, it is the same, i.e., $\ker \eta(f) = 0$ or 1, hence $1 + \ker \eta(f) = 1$ or 2. Therefor we have

$$\ker \varpi(f) \leq 1 + \ker \eta(f). \tag{3.1}$$

Thus, the Kernel-relative Hilali conjecture holds for any continuous map $f : S^{2k+1} \to Y$.

Example 3.2. For a continuous map $f : S^{2k+1} \to Y$, let us consider whether the Cokernel-relative Hilali conjecture holds or not, i.e., we consider whether the following holds or not:

$$\ker \varpi(f) + \varpi(Y) - \varpi(S^{2k+1}) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(S^{2k+1}). \tag{3.2}$$

Since $\varpi(S^{2k+1}) = \eta(S^{2k+1}) = 1$, the above (3.2) becomes

$$\ker \varpi(f) + \varpi(Y) \leq 1 + \ker \eta(f) + \eta(Y). \tag{3.3}$$

If Y is a space of type $\varpi \leq \eta$, then the inequality (3.3) holds due to the above (3.1). Therefore, if Y is a space of type $\varpi \leq \eta$, then the Cokernel-relative Hilali conjectures hold for any continuous map $f : S^{2k+1} \to Y$. This result still holds even if the source space S^{2k+1} is replaced by $S^{2k+1} \times S^{2m}$, because a key point in the above argument is the equality $\varpi(S^{2k+1}) = \eta(S^{2k+1})$, which is equal to 1 in this case, and we do have the equality $\varpi(S^{2k+1} \times S^{2m}) = \eta(S^{2k+1} \times S^{2m})$, which is equal to 3 in this case.

Remark 3.3. For a continuous map $f : S^{2k+1} \to Y$, we consider the cases when Y is not a space of type $\varpi \leq \eta$.

1. Let Y be a Hilali space, i.e., $\varpi(Y) = \eta(Y) + 1$. Then (3.3) becomes

   $$\ker \varpi(f) \leq \ker \eta(f). \tag{3.4}$$

   Since we have that $\ker \varpi(f) = 0$ or 1 and $\ker \eta(f) = 0$ or 1, we need to check only the case when $\ker \varpi(f) = 1$, namely whether $\ker \varpi(f) = 1$ automatically implies $\ker \eta(f) = 1$ or not. $\ker \varpi(f) = 1$ implies that $f_* \otimes \mathbb{Q} : \pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} \to \pi_{2k+1}(Y) \otimes \mathbb{Q}$ is the zero homomorphism. If we could claim that $f : S^{2k+1} \to Y$ is homotopic to a constant map, then $f_* : H_{2k+1}(S^{2k+1}; \mathbb{Q}) \to H_{2k+1}(Y; \mathbb{Q})$ is the zero homomorphism, thus $\ker \eta(f) = 1$, therefore we would get the above (3.4). However, $f_* \otimes \mathbb{Q} = 0$ for the homotopy groups *does not necessarily imply* that f is homotopic to a constant map[4]. So, we can say that if Y is a Hilali space and $f : S^{2k+1} \to Y$ is homotopic to a constant map, then the Cokernel-relative Hilali conjecture also holds.

2. Suppose that the Hilali conjecture does not hold, i.e., there is a non-Hilali space Y, i.e., there is a space Y such that $\varpi(Y) = \eta(Y) + j$ with $j \geq 2$. Then, (3.3) becomes

$$\ker \varpi(f) + j - 1 \leq \ker \eta(f), \tag{3.5}$$

which may not hold. If $j \geq 3$, then clearly (3.5) does not hold.

Now, by the above arguments, simply by the dimension reason, we can show the following corollary:

Corollary 3.1. *1. Let $\varpi(X) = 1$. Then, the Kernel-relative Hilali conjecture always holds for any continuous map $f : X \to Y$.*

2. Let $\varpi(X) = \eta(X) = 1$ and Y be of type $\varpi \leq \eta$. Then, the Kernel- and Cokernel-relative Hilali conjectures both hold for any continuous map $f : X \to Y$. (Note: In fact, $\varpi(X) = 1$ implies $\eta(X) = 1$ as we will see in §4 below.)

Remark 3.4. 1. A typical example for a space X such that $\varpi(X) = 1$ is the Eilenberg–Maclane space $K(\mathbb{Z}, n)$. A more general one for such a space is

$$K(\mathbb{Z} \oplus F_{01} \oplus \cdots \oplus F_{0m}, n_0) \times K(F_1, n_1) \times \cdots K(F_k, n_k), \tag{3.6}$$

where F_{0i} $(i = 1, \cdots, m)$ and F_i $(i = 1, \cdots, k)$ are finite abelian groups.

2. As to the case of $\eta(X) = 1$, as an example for such a space we can consider Moore space $M(G, n)$ (e.g., see [8, Example 2.40, p.143]), which is a homological analogue of Eilenberg–Maclane space, i.e., a CW complex X such that $H_n(X) \cong G$ and $\widetilde{H}_i(X) \cong 0$ for $i \neq n$. Here, we note that $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $\widetilde{H}_i(X) \cong H_i(X)$ for $i \geq 1$. So, by the Künneth Theorem, a more general example of X such that $\eta(X) = 1$ is a "Moore space" version of the above (3.6). i.e.,

$$M(\mathbb{Z} \oplus F_{01} \oplus \cdots \oplus F_{0m}, n_0) \times M(F_1, n_1) \times \cdots M(F_k, n_k),$$

where $n_i \geq 2$ $(i = 0, 1, \cdots, k)$.

4. According to MathOverFlow "Maps which induce the same homomorphism on homotopy and homology groups are homotopic" (answered by Allen Hatcher), the composition of a degree one map $f : T^3 \to S^3$ with the Hopf map $g : S^3 \to S^2$ is trivial on homotopy groups, but $g \circ f$ is not homotopic to a constant map.

3. In the case of $\varpi(X) = \eta(X) = 1$, which, for example, S^{2n+1} satisfies as observed above, it does not seem to be so easy to come up with a general example of such a space X. For example, although $\varpi(K(\mathbb{Z}, n)) = 1$ for any n, if $n = 2k$ is even, then $\eta(K(\mathbb{Z}, 2k)) = \infty$ since $H^*(K(\mathbb{Z}, 2k); \mathbb{Q}) = \mathbb{Q}[\alpha]$ where $\alpha \in H^{2k}(K(\mathbb{Z}, 2k); \mathbb{Q}) \cong \mathbb{Q}$ is a generator, thus $1 = \varpi(K(\mathbb{Z}, 2k)) \neq \eta(K(\mathbb{Z}, 2k)) = \infty$. Note that, the Eilenberg–Maclane space $K(\mathbb{Z}, 2n+1)$ is rationally homotopy equivalent to the sphere S^{2n+1}. Since the Eilenberg–Maclane space $K(F, n)$ for a finite abelian group is rationally homotopy equivalent to a point for any integer n, for the following space

$$X := K(\mathbb{Z}; 2n+1) \times K(F_1, n_1) \times \cdots K(F_s, n_s) \tag{3.7}$$

with finite abelian groups $F_i (i = 1, \cdots, s)$, we have $\varpi(X) = \eta(X) = 1$.

Let $X = S^{2n_1+1} \times \cdots \times S^{2n_k+1}$, where $n_i \neq n_j$ if $i \neq j$. Then, clearly we have $\varpi(X) = k$, but we have $\eta(X) = 2^k - 1$ since $\dim(H^*(S^{2n_i+1}; \mathbb{Q})) = 2$. Hence, we have that $k = \varpi(X) < \eta(X) = 2^k - 1$ for $k \geq 2$. As to the case of $\varpi(X) = \eta(X) = 2$, an example of such a space is $\mathbb{CP}^2$ as observed above. So, we pose the following problem-conjecture:

Problem 3.1. *For each $n \geq 3$, give an example of a space X satisfying the equality $\varpi(X) = \eta(X) = n$. (See also §4.2 below). Or, we conjecture that there does not exist such a space X.*

Example 3.3. Let us consider the case when $Y = S^{2m}$ in the above Remark 3.3 (1). By the above discussion, it suffices to consider the homomorphism $f_* : H_{2k+1}(S^{2k+1}; \mathbb{Q}) \to H_{2k+1}(S^{2m}; \mathbb{Q}) = 0$, which is clearly the zero homomorphism, thus $\eta(f) = 1$. Therefore, (3.4) holds, thus the Cokernel-relative Hilali conjecture holds. Namely, *for any continuous map $f : S^{2k+1} \to S^{2m}$, the Kernel- and Cokernel-relative Hilali conjectures both hold.*

Example 3.4. Let X be homotopy equivalent to (3.7) and Y be homotopy equivalent to the following space

$$\prod_{i=1}^{k} K(\mathbb{Z}, 2m_i + 1) \times K(F'_1, n'_1) \times \cdots K(F'_j, n'_j),$$

where $F'_i (i = 1, \cdots, j)$ is a finite abelian group. Note that, $\varpi(Y) \leq \eta(Y)$. Then, by Corollary 3.1, *for any continuous map $f : X \to Y$, the Kernel-and Cokernel-relative Hilali conjectures both hold.*

Example 3.5. Let us consider a continuous map $f : S^{2k} \to Y$ where $k \geq 1$ and Y is a simply connected rationally elliptic space. Since $\pi_n(S^{2k}) \otimes \mathbb{Q} = 0$ for $n \neq 2k, 4k+1$,

$$\ker\Big(f_* \otimes \mathbb{Q} : \pi_n(S^{2k+1}) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}\Big) = 0,$$

for $n \neq 2k, 4k+1$. Hence, we consider the following two cases:

$$(f_* \otimes \mathbb{Q})_{2k} : \pi_{2k}(S^{2k}) \otimes \mathbb{Q} \to \pi_{2k}(Y) \otimes \mathbb{Q},$$
$$(f_* \otimes \mathbb{Q})_{4k+1} : \pi_{4k+1}(S^{2k}) \otimes \mathbb{Q} \to \pi_{4k+1}(Y) \otimes \mathbb{Q},$$

each of which is either injective or the zero homomorphism by the dimension reason, in the same way as in Example 3.1. Hence, we have

$$\ker \pi(f) = \begin{cases} 0, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ are both injective,} \\ 1, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ is injecive and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ is the zero map,} \\ 1, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ is the zero map and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ is injective,} \\ 2, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ are both the zero map.} \end{cases}$$

Thus, $\ker \varpi(f) = 0, 1$ or 2. As to the homology, we consider

$$f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q}),$$

which is either injective or the zero map. Hence, we have $\ker \eta(f) = 0$ or 1, hence $1 + \ker \eta(f) = 1$ or 2. Therefore, unless $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map, we have

$$\ker \varpi(f) \leq 1 + \ker \eta(f). \tag{3.8}$$

Thus, the Kernel-relative Hilali conjecture holds. If $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map *and* $f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q})$ *is also the zero map*, e.g., if $f : S^{2k} \to Y$ is homotopic to a constant map, then we also have (3.8), thus the Kernel-relative Hilali conjecture holds. If $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map *and* $f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q})$ *is injective*, then $\ker \varpi(f) = 2$ and $1 + \ker \eta(f) = 1$, thus the Kernel-relative Hilali conjecture *does not* hold.

For the Cokernel-relative Hilali conjecture, we consider whether the following holds or not.

$$\ker \varpi(f) + \varpi(Y) - \varpi(S^{2k}) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(S^{2k}). \tag{3.9}$$

Since $\varpi(S^{2k}) = 2$ and $\eta(S^{2k}) = 1$, the above (3.9) becomes

$$\ker \varpi(f) + \varpi(Y) - 2 \leq \ker \eta(f) + \eta(Y), \tag{3.10}$$

in other words, we consider whether the following inequality holds or not

$$\ker \varpi(f) + \varpi(Y) \leq 2 + \ker \eta(f) + \eta(Y). \tag{3.11}$$

Here, we note that from the above discussion, for any space Y the following inequality always holds:

$$\ker \varpi(f) \leq 2 + \ker \eta(f). \tag{3.12}$$

1. If Y is a space of type $\varpi \leq \eta$, then the inequality (3.11) holds due to the above (3.12). Therefore, if Y is a space of type $\varpi \leq \eta$, then the Cokernel-relative Hilali conjecture *holds for any* continuous map $f : S^{2k} \to Y$.

2. If Y is not a space of type $\varpi \leq \eta$, say it is a Hilali space, i.e., $\varpi(Y) = \eta(Y) + 1$, then (3.11) becomes (3.8). In other words, in this case the Kernel-relative Hilali conjecture holds *if and only if* the Cokernel-relative Hilali conjecture holds.

3. Suppose that the Hilali conjecture does not hold, i.e., there is a non-Hilali space Y, i.e., there is a space Y such that $\varpi(Y) = \eta(Y) + j$ with $j \geq 2$. Then, (3.10) becomes

   $$\ker \varpi(f) + j - 2 \leq \ker \eta(f), \tag{3.13}$$

   which may not hold. If $j \geq 4$, then clearly (3.13) does not hold.

Proposition 3.1. *Let X and Y be simply connected rationally elliptic spaces of type $\varpi = \gamma_X + \eta$ and $\varpi = \gamma_Y + \eta$, respectively. Then, we have:*

1. *If $\gamma_Y \leq \gamma_X$, the Kernel-relative Hilali conjecture implies the Cokernel-relative Hilali conjecture.*

2. *If $\gamma_Y \geq \gamma_X$, the Cokernel-relative Hilali conjecture implies the Kernel-relative Hilali conjecture.*

3. *If both X and Y are Hilali spaces, the Kernel-relative Hilali conjecture holds if and only if the Cokernel-relative Hilali conjecture holds.*

Proof. The proof is simple, but we write it down.

1. Suppose that the Kernel-relative Hilali conjecture holds, i.e., $\ker \varpi(f) \leq 1 + \ker \eta(f)$. We have $\varpi(X) = \eta(X) + \gamma_X$ and $\varpi(Y) = \eta(Y) + \gamma_Y$. Since $\gamma_Y \leq \gamma_X$, we have

   $$\ker \varpi(f) + \gamma_Y \leq 1 + \ker \eta(f) + \gamma_X. \tag{3.14}$$

   Hence

   $$\ker \varpi(f) + \varpi(Y) - \eta(Y) \leq 1 + \ker \eta(f) + \varpi(X) - \eta(X),$$

   which implies

   $$\ker \varpi(f) + \varpi(Y) - \varpi(X) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(X), \tag{3.15}$$

   which is nothing but the Cokernel-relative Hilali conjecture.

2. The Cokernel-relative Hilali conjecture, i.e., (3.15) implies (3.14). Hence, we have
$$\ker \varpi(f) + \gamma_Y - \gamma_X \leq 1 + \ker \eta(f),$$
which implies $\ker \varpi(f) \leq 1 + \ker \eta(f)$ because $\gamma_Y - \gamma_X \geq 0$. Hence, the Kernel-relative Hilali conjecture holds.

3. It is due to the above results, since $\gamma_X = \gamma_Y = 1$. □

For any continuous map $f : X \to Y$, it is clear that we have $\ker \varpi(f) \leq \varpi(X)$ and $\ker \eta(f) \leq \eta(X)$, similarly we have $\operatorname{Coker} \varpi(f) \leq \varpi(Y)$ and $\operatorname{Coker} \eta(f) \leq \eta(Y)$. Let us set the gaps between these integers as follows:
$$\ker \varpi(f) + \varpi_f = \varpi(X), \quad \ker \eta(f) + \eta_f = \eta(X),$$
$$\operatorname{Coker} \varpi(f) + \operatorname{Co} \varpi_f = \varpi(Y), \quad \operatorname{Coker} \eta(f) + \operatorname{Co} \eta_f = \eta(Y).$$

Proposition 3.2. *Let $f : X \to Y$ be a continuous map of simply connected rationally elliptic spaces.*

1. *If the Hilali conjecture holds for the source space X and $\eta_f \leq \varpi_f$, then the Kernel-relative Hilali conjecture holds.*

2. *If the Hilali conjecture holds for the target space Y and $\operatorname{Co} \eta_f \leq \operatorname{Co} \varpi_f$, then the Cokernel-relative Hilali conjecture holds.*

Proof. Since the second statement is proved in the same way as in the first one, we prove the first one.
$$\begin{aligned} \ker \varpi(f) &= \varpi(X) - \varpi_f \\ &\leq 1 + \eta(X) - \varpi_f \\ &\qquad \text{(since the Hilali conjecture holds for } X\text{: } \varpi(X) \leq 1 + \eta(X)) \\ &\leq 1 + \eta(X) - \eta_f \quad (\text{since } -\varpi_f \leq -\eta_f) \\ &= 1 + \eta(f). \end{aligned}$$
□

Now, we observe that it follows from the fundamental homomorphism theorem on vector spaces that we have
$$\varpi_f = \operatorname{Co} \varpi_f, \qquad \eta_f = \operatorname{Co} \eta_f.$$

Corollary 3.2. *Suppose that the Hilali conjecture holds for any simply connected elliptic spaces. Let $f : X \to Y$ be a continuous map of simply connected elliptic spaces X and Y such that $\eta_f \leq \varpi_f$. Then, the Kernel- and Cokernel-relative Hilali conjectures both hold.*

4. Stringent restrictions on homotopy groups of rationally elliptic spaces

In this section we discuss some results which follow from some *stringent restrictions* on homotopy groups of rationally elliptic spaces (see [3], [4], [5]).

First we recall Halperin's theorems. For that, we set

$$\pi_{\mathrm{even}}(X)\otimes\mathbb{Q} := \bigoplus_{k\geq 1}\pi_{2k}(X)\otimes\mathbb{Q}, \quad \pi_{\mathrm{odd}}(X)\otimes\mathbb{Q} := \bigoplus_{k\geq 0}\pi_{2k+1}(X)\otimes\mathbb{Q},$$
$$\varpi_{even}(X) := \dim\left(\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}\right), \quad \varpi_{odd}(X) := \dim\left(\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}\right).$$

$$\chi^{\pi}(X) := \varpi_{even}(X) - \varpi_{odd}(X),$$

which is called the homotopical Euler–Poincaré characteristic of X and is a homotopical version of the (usual homological) Euler–Poincaré characteristic

$$\chi(X) = \chi_{\mathrm{even}}(X) - \chi_{\mathrm{odd}}(X),$$

where

$$\chi_{\mathrm{even}}(X) := \dim\left(H_{\mathrm{even}}(X;\mathbb{Q})\right), \quad \chi_{\mathrm{odd}}(X) := \dim\left(H_{\mathrm{odd}}(X;\mathbb{Q})\right),$$
$$H_{\mathrm{even}}(X;\mathbb{Q}) := \bigoplus_{k\geq 0} H_{2k}(X;\mathbb{Q}), \quad H_{\mathrm{odd}}(X;\mathbb{Q}) := \bigoplus_{k\geq 0} H_{2k+1}(X;\mathbb{Q}).$$

S. Halperin proved.

Theorem 4.1 ([7, Theorem 1])**.** *$\chi^{\pi}(X) \leq 0$ and $\chi(X) \geq 0$. Moreover, the following are equivalent:*

1. *$\chi^{\pi}(X) = 0$.*
2. *$\chi(X) > 0$.*
3. *$H_{odd}(X)\otimes\mathbb{Q} = 0$.*

Remark 4.1. 1. In other words, $\chi^{\pi}(X) < 0 \Longleftrightarrow \chi(X) = 0$.

2. The equivalence of the above (1), (2) and (3) was posed as a question in D. Sullivan's famous paper [16].

Let $y_1, \cdots, y_q$ be a basis of $\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}$ and $x_1, \cdots, x_r$ be a basis of $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}$. If $y_j \in \pi_{2b_j-1}(X)\otimes\mathbb{Q})$ and $x_i \in \pi_{2a_i}(X)\otimes\mathbb{Q}$, $2b_j-1$ and $2a_i$ are called the degrees of y_j and x_j. $(b_1, \cdots, b_q)$ and $(a_1, \cdots, a_r)$ are respectively called *b-exponents* and *a-exponents* of X in [6]. The largest integer n_X such that $H_{n_X}(X;\mathbb{Q}) \neq 0$ is called the formal dimension of X. Halperin showed the following:

Theorem 4.2 ([7, Theorem 3' and Corollary 2])**.**

1. *$\sum_{j=1}^{q}(2b_i - 1) \leq 2n_X - 1$ and $\sum_{i=1}^{r} 2a_j \leq n_X$.*

2. $n_X = \sum_{j=1}^q (2b_j - 1) - \sum_{i=1}^r (2a_i - 1)$.

3. *Betti numbers* $\beta_i = \dim H_i(X;\mathbb{Q})$ *satisfy Poincaré duality;* $\beta_i = \beta_{n_X - i}$.

4. *In the case when* $\chi^\pi(X) = 0$, *i.e.,* $q = r$, *Poincaré polynomial of* X *is*

$$P_X(t) = \frac{\prod_{i=1}^q (1 - t^{2b_i})}{\prod_{i=1}^q (1 - t^{2a_i})}. \tag{4.1}$$

In particular, $\chi(X) = P_X(-1) = P_X(1) = \dim(H_*(X) \otimes \mathbb{Q}) = \frac{\prod_{i=1}^q b_i}{\prod_{i=1}^q a_i}$.

Note that $\chi(X) = \frac{\prod_{i=1}^q b_i}{\prod_{i=1}^q a_i}$ follows from

$$\frac{\prod_{i=1}^q (1 - t^{2b_i})}{\prod_{i=1}^q (1 - t^{2a_i})} = \frac{\prod_{i=1}^q (1 - (t^2)^{b_i})}{\prod_{i=1}^q (1 - (t^2)^{a_i})} = \frac{\prod_{i=1}^q (1 + t^2 + \cdots + (t^2)^{b_i - 1})}{\prod_{i=1}^q (1 + t^2 + \cdots + (t^2)^{a_i - 1})}.$$

Definition 4.1 ([6, Definition, pp.117–118])**.** *Let* $B = (b_1, b_2, \cdots, b_q)$ *and* $A = (a_1, a_2, \cdots, a_r)$ *be two finite sequences of positive integers.*

1. *We say that* $(B; A)$ *satisfies strong arithmetic condition (abbr. S.A.C.) if for every subsequence* A^* *of* A *of length* s $(1 \leq s \leq r)$ *there exists at least* s *elements* b_j*'s of* B *such that*

$$b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i \tag{4.2}$$

where γ_{ij} *is a non-negative integer such that* $\sum_{a_i \in A^*} \gamma_{ij} \geq 2$.

2. *If* $\sum_{a_i \in A^*} \gamma_{ij} \geq 2$ *is not required, then we say that* (B, A) *satisfies arithmetic condition (abbr. A.C.).*

Thus, in both cases, it is necessary that $r \leq q$.

In [6, Theorem 1, p.118] J. B. Friedlander and S. Halperin show the following characterization theorem about a pair $(B; A)$ satisfying S.A.C.

Theorem 4.3 (Friedlander–Halperin Theorem)**.** *Let* $B = (b_1, b_2, \cdots, b_q)$ *and* $A = (a_1, a_2, \cdots, a_r)$ *be a pair of sequences of positive intgers. The following conditions are equivalent:*

1. (B, A) *satisfies S.A.C.*

2. *The sequences* B *and* A *are respectively the b-exponents and a-exponents of a rationally elliptic space* X.

Moreover, if $b_i \geq 2$ *for all* i *and S.A.C. holds, then* X *may be chosen to be simply connected; if in addition* $q > r$, X *may be taken to be a closed manifold.*

Remark 4.2. $(b_1, b_2, \cdots, b_q)$ and $(a_1, a_2, \cdots, a_r)$ are respectively called *"odd" exponents* and *"even" exponents* of X in Félix–Halperin–Thomas's book [4].

In fact, from S.A.C., i.e., (4.2), we get the following result:

Lemma 4.1 ([6, 2.5. Lemma])**.** *If $B = (b_1, b_2, \cdots, b_q)$; $b_1 \geq b_2 \geq \cdots \geq b_q$, and $A = (a_1, a_2, \cdots, a_r)$; $a_1 \geq a_2 \geq \cdots \geq a_r$. If $(B; A)$ satisfies S.A.C, then $b_i \geq 2a_i$ for $1 \leq i \leq r$.*

Remark 4.3. Usually we consider the following order $b_1 \leq b_2 \leq \cdots \leq b_q$ for $B = (b_1, b_2, \cdots, b_q)$ and $a_1 \leq a_2 \leq \cdots \leq a_r$ for $A = (a_1, a_2, \cdots, a_r)$, but in order to prove the above lemma and also for the description of the statement of the lemma, the above descending order in the lemma is better.

Using Lemma 4.1 we can get the following formulas:

Corollary 4.1 ([6, 1.3. Corollary, p.118])**.**

1. $n_X \geq q + r = \dim(\pi_*(X) \otimes \mathbb{Q})$.
2. $n_X \geq \sum_{j=1}^{q} b_j$.
3. $2n_X - 1 \geq \sum_{j=1}^{q} (2b_j - 1)$.
4. $n_X \geq \sum_{i=1}^{r} 2a_i$.

Corollary 4.2 ([1, Proposition 2.1])**.** *If $q = r$, then the Hilali conjecture holds.*

Proof. $\dim(\pi_*(X) \otimes \mathbb{Q}) = 2q$. It follows from Theorem 4.2 (4) and Lemma 4.1 that

$$\dim(H_*(X; \mathbb{Q})) = \frac{\prod_{i=1}^{q} b_i}{\prod_{i=1}^{q} a_i} \geq \frac{\prod_{i=1}^{q} 2a_i}{\prod_{i=1}^{q} a_i} = 2^q, \tag{4.3}$$

which is [6, 2.6. Proposition (3)]. Since $2q \leq 2^q$ $(q \geq 1)$, we have

$$\dim(\pi_*(X) \otimes \mathbb{Q}) \leq \dim(H_*(X; \mathbb{Q})).$$ □

Next, we discuss the parity of $\varpi(X)$ and $\eta(X)$. We can show the following:

Lemma 4.2. *If $\varpi(X)$ is odd, then $\eta(X)$ is also odd.*

Proof. Suppose that $\varpi(X)$ is odd. Then, the homotopical Euler characteristic $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) < 0$. Indeed, by the dichotomy, $\chi^\pi(X) \leq 0$. If $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) = 0$, then $\varpi_{even}(X) = \varpi_{odd}(X)$, thus $\varpi(X) = \varpi_{even}(X) + \varpi_{odd}(X)$ is even, which is a contradiction. Now, it follows from Theorem 4.2 that $\chi^\pi(X) < 0$ implies that the Euler–Poincaré characteristic $\chi(X) = 0$, i.e., $\chi(X) = \chi_{\text{even}}(X) - \chi_{\text{odd}}(X) = 0$, thus $\chi_{\text{even}}(X) = \chi_{\text{odd}}(X)$. Hence $1 + \eta(X) = \chi_{\text{even}}(X) + \chi_{\text{odd}}(X)$ is even, thus $\eta(X)$ is also odd. □

Remark 4.4. One might be tempted to expect that if $\varpi(X)$ is even, then $\eta(X)$ would be also even, but it is not the case. A very simple counterexample is $X = S^{2n+1} \times S^{2m+1}$. Then, $\varpi(X) = 2$, but $\eta(X) = 2^2 - 1 = 3$ is odd. In general, consider $X = S^{2n_1+1} \times \cdots \times S^{2n_k+1}$. Then, $\varpi(X) = k$ and $\eta(X) = 2^k - 1$, thus whether $\varpi(X) = k$ is even or odd, $\eta(X) = 2^k - 1$ is always odd. In fact, in the case when $\varpi(X)$ is even, $\eta(X)$ can be both even and odd. A typical example for this is the complex projective space $\mathbb{CP}^n$. $\varpi(\mathbb{CP}^n) = 2$ is even for any n, but the parity of $\eta(\mathbb{CP}^n) = n$ depends on the complex dimension n.

Corollary 4.3. *Suppose that the Hilali conjecture holds. If X is a rationally elliptic space such that $\dim(\pi_*(X) \otimes \mathbb{Q})$ is odd, then $\dim(\pi_*(X) \otimes \mathbb{Q}) < \dim H_*(X; \mathbb{Q})$.*

Proposition 4.1. *Let X be a rationally elliptic space. The parity of $\varpi(X)$ is the same as the parity of the formal dimension n_X, i.e., if $\varpi(X)$ is odd (resp., even), then its formal dimension n_X is odd (resp., even).*

Proof. Let $p := \varpi(X)$ and $e := \varpi_{even}(X)$. So, $\varpi_{odd}(X) = p - e$. Since $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) \leq 0$, thus $e \leq p - e$. It follows from Theorem 4.2 (2) that if $e = 0$, then we have

$$n_X = \sum_{i=1}^{p}(2b_i - 1) = 2\sum_{i=1}^{p} b_i - p, \tag{4.4}$$

and if $e \geq 1$, then we have

$$n_X = \sum_{i=1}^{p-e}(2b_i - 1) - \sum_{j=1}^{e}(2a_j - 1),$$

which is

$$n_X = 2\sum_{i=1}^{p-e} b_i - (p - e) - 2\sum_{j=1}^{e} a_j + e = 2\left(\sum_{i=1}^{p-e} b_i - \sum_{j=1}^{e} a_j + e\right) - p. \tag{4.5}$$

Therefore, it follows from (4.4) and (4.5) that if $p = \varpi(X)$ is odd (resp., even), then n_X is odd (resp., even). □

Remark 4.5. First we note that (4.5) can be also written as follows:

$$n_X = 2\left(\sum_{i=1}^{p-e} b_i - \sum_{j=1}^{e} a_j\right) - (p - 2e).$$

The parity of $p - 2e$, which is $-\chi^\pi(X) = \varpi_{odd}(X) - \varpi_{even}(X)$, is also the same as the parity of the formal dimension n_X. For example, in [13, §2] Nakamura and Yamaguchi call $(2a_1, 2a_2, \cdots, 2a_n : 2b_1 - 1, 2b_2 - 1, \cdots, 2b_{n+p} - 1)$ a homotopy rank type of X and all the *homotopy rank types* with the formal dimension ≤ 16 are listed in [13, §3]. In their list, if the formal dimension, denoted fd, is even (resp. odd), then p is even (resp. odd). Note that, their p is equal to our $p-2e$. Also note that clearly the parity of $p-2e$ is the same as the parity of $p=\varpi(X)$.

Finally, we discuss lower bounds of $\eta(X)$ for some cases. Before discussion, we recall Klaus–Kreck's rational Hurewicz theorem, which is a version stronger than the usual one:

Theorem 4.4 ([10, Theorem 1.1]). *Let X be a simply connected topological space with $\pi_i(X)\otimes\mathbb{Q}=0$ for $1<i<r$. Then, the Hurewicz map induces an isomorphism*

$$H:\pi_i(X)\otimes\mathbb{Q}\to H_i(X;\mathbb{Q})$$

for $1<i<2r-1$ and a surjection for $i=2r-1$.

Theorem 4.5. *For any rationally elliptic space X such that $1\leq\varpi(X)\leq 4$, the Hilali conjecture holds. To be more precise,*

1. *If $\varpi(X)=1$, then $\eta(X)=1$, thus $1=\varpi(X)<1+\eta(X)=2$.*
2. *If $\varpi(X)=2$, then $\eta(X)\geq 1$, thus $\varpi(X)\leq 1+\eta(X)$.*
3. *If $\varpi(X)=3$, then $\eta(X)\geq 3$, thus $\varpi(X)<1+\eta(X)$.*
4. *If $\varpi(X)=4$, then $\eta(X)\geq 3$, thus $\varpi(X)\leq 1+\eta(X)$.*

Proof. First we recall that

$$\varpi(X)=\varpi_{\text{even}}(X)+\varpi_{\text{odd}}(X)\quad\text{and}\quad\varpi_{\text{even}}(X)\leq\varpi_{\text{odd}}(X). \tag{4.6}$$

1. Let $\varpi(X)=1$. It follows from (4.6) that $\varpi_{\text{even}}(X)=0$ and $\varpi_{\text{odd}}(X)=1$. Hence, $\pi_{2b-1}(X)\otimes\mathbb{Q}\cong\mathbb{Q}$ for some odd integer $2b-1$ $(b\geq 2)$ (since X is simply connected) and $\pi_i(X)\otimes\mathbb{Q}\cong 0$ for $i\neq 2b-1$. It follows from Theorem 4.2 (2) that the formal dimension $n_X=2b-1$ and $H_{2b-1}(X;\mathbb{Q})\cong\mathbb{Q}$ and it also follows from the rational Hurewicz Theorem that $H_i(X;\mathbb{Q})\cong 0$ for $i\neq 0,2b-1$. Thus, $\eta(X)=1$.
2. Let $\varpi(X)=2$. It follows from (4.6) that we have two possibilities:
   (a) $\varpi_{\text{even}}(X)=0$ and $\varpi_{\text{odd}}(X)=2$. In this case we have

   $$\pi_{2b_1-1}(X)\otimes\mathbb{Q}\cong\mathbb{Q},\quad \pi_{2b_2-1}(X)\otimes\mathbb{Q}\cong\mathbb{Q},\ \text{where } 2\leq b_1\leq b_2. \tag{4.7}$$

   Here, we are a bit sloppy. If $b_1=b_2$, then the above (4.7) is really understood to mean the following:

   $$\pi_{2b_1-1}(X)\otimes\mathbb{Q}\cong\mathbb{Q}\oplus\mathbb{Q}.$$

It follows from Theorem 4.2 (2) that the formal dimension $n_X = 2b_1 - 1 + 2b_2 - 1 = 2(b_1 + b_2) - 2$, which is greater than $2b_2 - 1$ since $2b_1 - 1 \geq 0$ (in fact, $2b_1 - 1 \geq 3$.) It follows from the Hurewicz Theorem that $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ if $b_1 < b_2$ and $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}\oplus\mathbb{Q}$ if $b_1 = b_2$. In the case when $b_1 < b_2$, by the Poincaré duality (Theorem 4.2 (3)) we do have $H_{2b_2-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ since $n_X - (2b_1 - 1) = 2b_2 - 1$. Hence, in any case we can see that $\eta(X) \geq 3$.

(b) $\varpi_{\text{even}}(X) = 1$ and $\varpi_{\text{odd}}(X) = 1$. It follows from Corollary 4.2 that the Hilali conjecture holds, thus we are done. However, in this paper we take a more direct approach in order to see more information about $\eta(X)$.

$$\pi_{2b-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \text{ and } \pi_{2a}(X) \otimes \mathbb{Q} \cong \mathbb{Q}, \text{ where } b \geq 2 \text{ and } a \geq 1.$$

The formal dimension $n_X = 2b-1-(2a-1) = 2b-2a$. It follows from Theorem 4.2 that we have $2b - 1 \leq 2(2b - 2a) - 1$ and $2a \leq 2b - 2a$, both of which is the same inequality $4a \leq 2b$, i.e., $2a \leq b$. Thus, we have $2a < 2b - 1$ since $b - 1 \geq 1$. Then, we can see that we have the following orders:

$$2a \leq b \leq 2b - 2a < 2b - 1.$$

Hence, we have $H_{2b-2a}(X;\mathbb{Q}) \cong \mathbb{Q}$ and it follows from the rational Hurewicz theorem that $H_{2a}(X;\mathbb{Q}) \cong \mathbb{Q}$.

i. If $b = 2a$, then $2b - 2a = 2a$. Thus, we have $\eta(X) = 1$.

ii. If $2a < b < 3a$, then $(2b - 2a) - 2a = 2b - 4a = 2(b - 2a) > 0$ and $2b - 4a < 2a$. Thus, we have the following orders:

$$2b - 4a < 2a < 2b - 2a.$$

Then, it follows from the Poincaré duality that $H_{2b-4a}(X;\mathbb{Q}) \cong Q$. However, since $\pi_i(X) \otimes \mathbb{Q} \cong 0$ for $2 \leq i2a$, the rational Hurewicz theorem implies that $H_{2b-4a}(X;\mathbb{Q}) \cong 0$. Therefore, the case $2a < b < 3a$ is *ruled out*. This is a stringent restriction due to the Poincaré duality.

iii. If $b = 3a$, then $2b - 2a = 4a$ and $2b - 4a = 2a$, thus we have $\eta(X) = 2$.

iv. If $b > 3a$, then we have the following orders:

$$2a < 2b - 4a < 2b - 2a.$$

Then, by the Poincaré duality $H_{2b-4a}(X;\mathbb{Q}) \cong \mathbb{Q}$, hence $\eta(X) \geq 3$.

In any case we have $\eta(X) \geq 1$, thus we have $\pi(X) \leq 1 + \eta(X)$.

3. Let $\varpi(X) = 3$. In this case we have the following two possibilities:

   (a) $\varpi_{\text{even}}(X) = 0$ and $\varpi_{\text{odd}}(X) = 3$. In this case we have

   (4.8) $\pi_{2b_i-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$, where $2 \le b_1 \le b_2 \le b_3$

   As above, here we are a bit sloppy. E.g., if $b_1 = b_2$, then the above (4.8) is really understood to mean the following:

   $$\pi_{2b_1-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \text{ and } \pi_{2b_3-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$$

   The formal dimension $n_X = 2(b_1 + b_2 + b_3) - 3$. If $b_1 < b_2$, then by the Poincaré duality $H_{2(b_2+b_3)-2}(X;\mathbb{Q}) \cong \mathbb{Q}$. Hence, $\eta(X) \ge 3$. If $b_1 = b_2$, then by the Poincaré duality $H_{2(b_1+b_3)-2}(X;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$. Hence, $\eta(X) \ge 5$. In any case we have $\eta(X) \ge 3$.

   (b) $\varpi_{\text{even}}(X) = 1$ and $\varpi_{\text{odd}}(X) = 2$. In this case we have

   $$\pi_{2b_i-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \text{ and } \pi_{2a}(X) \otimes \mathbb{Q} \cong \mathbb{Q},$$

   where $2 \le b_1 \le b_2$ and $a \ge 1$. The formal dimension $n_X = 2b_1 + 2b_2 - 2a - 1$. It follows from Theorem 4.2 that $4a \le 2b_1 + 2b_2 - 1$, which is in fact $4a < 2b_1 + 2b_2 - 1$ since $4a$ is even and $2b_1 + 2b_2 - 1$ is odd. Hence, $4a \le 2b_1 + 2b_2 - 1 - 1 = (2b_1 - 1) + (2b_2 - 1) \le 2(2b_2 - 1)$ since $b_1 \le b_2$. If $b_1 < b_2$, then $2a < 2b_2 - 1$. If $b_1 = b_2$, then $2a \le 2b_1 - 1$. Thus, two possibilities: $2b_1 - 1 < 2a$ and $2a < 2b_1 - 1$. In any case by the Hurewicz theorem $H_{2a}(X;\mathbb{Q}) \cong \mathbb{Q}$ or $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}$, and by the Poincareé duality we have $H_{2b_1+2b_2-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ or $H_{2b_2-2a}(X;\mathbb{Q}) \cong \mathbb{Q}$. Hence, we can see $\eta(X) \ge 3$.

   In any case, we can see that $\eta(X) \ge 3$. The above argument is quite detailed. Here, is a very simpler argument, which is as follows. In both cases (a) and (b), we can see that $\eta(X) \ge 2$ since $\dim H_{2b_1-1}(X;\mathbb{Q}) \ge 1$ and $H_{n_X}(X;\mathbb{Q}) \cong \mathbb{Q}$ in the case (a) (note that $2b_1 - 1 < n_X$), and $\dim H_{2b_1-1}(X;\mathbb{Q}) = 1$ or $\dim H_{2a}(X;\mathbb{Q}) = 1$ and $H_{n_X}(X;\mathbb{Q}) \cong \mathbb{Q}$ in the case (b). Since $\eta(X)$ has to be odd, it follows that $\eta(X) \ge 3$.

4. $\varpi(X) = 4$. In this case we have the following cases

   (a) $\pi_{\text{even}}(X) = 0$ and $\pi_{\text{odd}}(X) = 4$: Consider the degrees:

   $$2b_1 - 1, 2b_2 - 1, 2b_3 - 1, 2b_4 - 1, \quad (2 \le b_1 \le b_2 \le b_3 \le b_4).$$

   $n_X = \sum(2b_i - 1)$. Since $\dim(\pi_{2b_1-1}(X) \otimes \mathbb{Q}) \ge 1$ (because $b_1 \le b_2 \le b_3 \le b_4$), it follows from the Hurewicz theorem and the Poincaré duality that $\eta(X) \ge 1 + 1 + 1 = 3$. If $2b_i - 1 \le 2(2b_1 - 2)$ holds, then the inequality $\eta(X) \ge 3$ can be sharpened to $\eta(X) \ge 2i + 1$, which follows from the rational Hurewicz Theorem. Here, we note that $i \ge 1$, since $2b_1 - 1 \le 2(2b_1 - 2)$, i.e., $3 \le 2b_1$, which holds since $b_1 \ge 2$. In any case, we have $\varpi(X) \le 1 + \eta(X)$.

(b) $\varpi_{\mathrm{even}}(X) = 1$ and $\varpi_{\mathrm{odd}}(X) = 3$: Consider the degrees:

$$2b_1 - 1, 2b_2 - 1, 2b_3 - 1, 2a, \quad (2 \leq b_1 \leq b_2 \leq b_3, a \geq 1).$$

$n_X = 2(b_1 + b_2 + b_3 - a) - 2$. Whether $2b_1 - 1 < 2a$ or $2a < 2b_1 - 1$, by the Hurewicz Theorem $\beta_{2b_1-1} \geq 1$ or $\beta_{2a} \geq 1$, thus by the Poincaré duality $\beta_{n_X-2b_1-1} \geq 1$ or $\beta_{n_X-2a} \geq 1$. Since $\beta_{n_X} = 1$, we have $\eta(X) \geq 3$. Therefore, $\varpi(X) \leq 1 + \eta(X)$. Here, we need to be a bit careful about β_{n_X-2a}. We need to check whether $n_X - 2a = 2a$. Namely, if $2a < 2b_1 - 1$ and $n_X - 2a = 2a$, in which case we cannot use the trick of Poincaré duality, hence $\eta(X) \geq 2$ instead of $\eta(X) \geq 3$. However, we do have $n_X - 2a > 2a$. Indeed $n_X - 2a - 2a = (2b_1 - 1) + (2b_2 - 1) + (2b_3 - 1) - (2a - 1) - 2a - 2a = \{(2b_1 - 1) - 2a\} + \{(2b_2 - 1) - 2a\} + \{(2b_3 - 1) - 2a\} + 1 \geq 4$, because $2a < 2b_1 - 1$ and $b_1 \leq b_2 \leq b_3$.

(c) $\varpi_{\mathrm{even}}(X) = 2$ and $\varpi_{\mathrm{odd}}(X) = 2$: By Corollary 4.2 we do know that the Hilali conjecture holds, thus $\eta(X) \geq 3$. However, let us see this without using this corollary. Consider the degrees:

$$2b_1 - 1, 2b_2 - 1, 2a_1, 2a_2, \quad (2 \leq b_1 \leq b_2, 1 \leq a_1 \leq a_2).$$

It follows from Lemma 4.1 that the following cases are possible:

i. $2a_1 < 2b_1 - 1 < 2a_2 < 2b_2 - 1$.
ii. $2a_1 \leq 2a_2 < 2b_1 - 1 \leq 2b_2 - 1$.

$n_X = 2(b_1 + b_2) - 2(a_1 + a_2)$. Then, in which case is it possible that $n_X - 2a_1 = 2a_1$? $n_X - 2a_1 - 2a_1 = 2(b_1 + b_2) - 2(a_1 + a_2) - 4a_1 = 2(b_1 - 3a_1) + 2(b_2 - a_2) \geq 2(2a_1 - 3a_1) + 2(2a_2 - a_2) = -2a_1 + 2a_2 = 2(a_2 - a_1)$. So, when $a_1 = a_2$, $b_1 = 2a_1$ and $b_2 = 2a_2$, we do have $n_X - 2a_1 = 2a_1$. In this case, surely we have $\eta(X) \geq 2 + 1 = 3$, since $H_{2a_1}(X; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ and $H_{n_X}(X; \mathbb{Q}) = \mathbb{Q}$. Otherwise we have $n_X - 2a_1 > 0$ and $n_X - 2a_1 \neq 2a_1$. In this case we also have $\eta(X) \geq 3$, since $H_{2a_1}(X; \mathbb{Q}) = \mathbb{Q}$, $H_{n_X-2a_1}(X; \mathbb{Q}) = \mathbb{Q}$ by the Poincaré duality and $H_{n_X}(X; \mathbb{Q}) = \mathbb{Q}$. In any case we do have $\eta(X) \geq 3$.

□

Remark 4.6. Let $\varpi(X) = 5$. Then, it follows from Lemma 4.2 that $\eta(X)$ is odd. By an analysis as above, we see that $\eta(X) \geq 2$, hence $\eta(X) \geq 3$ since $\eta(X)$ is odd. If $\eta(X) \geq 5$, then the Hilali conjecture holds. If not, there would exist a counterexample such that $\varpi(X) = 5$ and $\eta(X) = 3$, i.e., $5 = \dim(\pi_*(X) \otimes \mathbb{Q}) > \dim(H_*(X; \mathbb{Q}) = 1 + \eta(X) = 4$. It follows from [1] that the formal dimension of such a counterexample is greater than or equal to 21.

Proposition 4.2. *Let $\varpi(X) = 2m+1$ such that $\varpi_{\mathrm{even}}(X) = 0$ and $\varpi_{\mathrm{odd}}(X) = 2m+1$. Let the degrees be*

$$2b_1 - 1, \cdots, 2b_m - 1, \cdots, 2b_{2m+1} - 1,$$

where $b_1 \leq b_2 \leq \cdots \leq b_{2m+1}$*. If* $2b_m - 1 \leq 2(2b_1 - 2)$*, then we have* $\eta(X) \geq 2m+1$*. In particular the Hilali conjecture holds for such a space* X*.*

Proof. This simply follows from Klaus–Kreck's rational Hurewicz theorem, i.e., we have the Hurewicz homomorphism $\pi_i(X) \otimes \mathbb{Q} \cong H_i(X;\mathbb{Q})$ for $1 \leq i \leq 2(2b_1 - 2)$. Since the formal dimension $n_X = \sum_j^{2m+1}(2b_j - 1)$, by the Poincaré duality we see that $\eta(X) \geq 2m+1$, since $n_X - (2b_j - 1)$ is even for any b_j, thus $n_X - (2b_j - 1)$ cannot be equal to any odd integer $2b_k - 1$ $(k = 1, 2, \cdots, m)$, therefore $H_{n_X - (2b_j - 1)}(X;\mathbb{Q}) = \mathbb{Q}$ for $k = 1, 2, \cdots, m$. □

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