Extensions of singular value inequalities for sector matrices

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Abstract. In this paper, we present singular value inequalities for matrices. As a consequence, we prove singular value inequalities for sector matrices. Moreover, we give singular value inequalities involving operator concave function, which are generalizations of some existing results.

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1. Introduction

Throughout this paper, let M_n represent the set of all $n \times n$ complex matrices. I_n denotes the identity matrix. For two Hermitian matrices $A, B \in M_n$, we use $A \geq B$ to mean that $A - B$ is positive semidefinite. If the eigenvalues of matrix $A \in M_n$ are all real, the jth largest eigenvalue of A is denoted by $\lambda_j(A), j = 1, 2, \cdots, n$. The singular values $s_j(A)(j = 1, 2, \cdots, n)$ of A are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ arrange in a decreasing order. For $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$, the matrices $\mathrm{Re}(A) = \frac{A+A^*}{2}$ and $\mathrm{Im}(A) = \frac{A-A^*}{2i}$ are called the real part and imaginary part of A, respectively. A real valued continuous function f on an interval J is called matrix concave of order n if $f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha)f(B)$ for any two Hermitian matrices $A, B \in M_n$ with spectrum in J and all $\alpha \in [0, 1]$. If f is operator concave function for all n, then it is called operator concave. It is well known that a continuous non-negative function f on $[0, \infty)$ is operator monotone if and only if f is operator concave.

The numerical range of $A \in M_n$ is described by

$$W(A) = \{x^*Ax | x \in C^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, we define a sector on the complex plane

$$S_\alpha = \{z \in C : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| \leq \tan\alpha \mathrm{Re}(z)\}.$$

Clearly, for some $\alpha \in [0, \frac{\pi}{2})$, if $W(A), W(B) \subset S_\alpha$, then $W(A+B) \subset S_\alpha$. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is nonsingular. A matrix $A \in M_n$ is said to be sector matrix if its numerical range is contained in S_α, for some $\alpha \in [0, \frac{\pi}{2})$.

Garg and Aujla [1] proved that if $A, B \in M_n$ and $1 \leq r \leq 2$. Then

$$\prod_{j=1}^{k} s_j(|A+B|^r) \leq \prod_{j=1}^{k} s_j(I_n + |A|^r) \prod_{j=1}^{k} s_j(I_n + |B|^r) \tag{1.1}$$

and

$$\prod_{j=1}^{k} s_j(I_n + f(|A+B|)) \leq \prod_{j=1}^{k} s_j(I_n + f(|A|)) \prod_{j=1}^{k} s_j(I_n + f(|B|)), \tag{1.2}$$

where $f : [0, \infty) \to [0, \infty)$ is an operator concave function and $1 \leq k \leq n$.

Xue and Hu [2] showed that if $A, B \in M_n$ such that $W(A), W(B) \subset S_\alpha$. Then

$$\prod_{j=1}^{k} s_j(A+B) \leq \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)) \tag{1.3}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \leq \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)), \tag{1.4}$$

where $1 \leq k \leq n$.

Recently, Lin and Fu [3], Yang [4] and Nasiri and Furuichi [5] independently gave some singular value inequalities for sector matrices related to Garg and Aujla's results.

In this paper, we give some new singular value inequalities for sector matrices, which are generalizations of existing results.

2. Main results

We begin this section with the following lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1 ([6])**.** *Let $A, B \in M_n$. There exist unitary matrices $U, V \in M_n$ such that*

$$|A+B| \leq U^*|A|U + V^*|B|V.$$

Lemma 2.2 ([7])**.** *Let $A, B \in M_n$ be positive semidefinite matrices. Then $A \sharp B$ is the largest Hermitian matrix X such that*

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix}$$

is positive semidefinite.

Lemma 2.3 ([8])**.** *Let $A \in M_n$ be Hermitian matrix. Then*

$$\prod_{j=1}^{k} s_j(A) = \max\left|\det(U^* A U)\right|,$$

where maximum is taken over $n \times k$ matrices U for $U^*U = I_k$, $1 \leq k \leq n$.

Lemma 2.4 ([1])**.** *Let $A \in M_n$ be Hermitian matrix and B be positive definite matrix with $A < B, -A < B$. Then*

$$|\det A| < \det B.$$

Lemma 2.5 ([1])**.** *Let $A, B \in M_n$ be positive semidefinite matrices. Then*

$$\prod_{j=1}^{k} \lambda_j(A \sharp B) \leq (\prod_{j=1}^{k} \lambda_j(A)) \sharp (\prod_{j=1}^{k} \lambda_j(B)), 1 \leq k \leq n.$$

Lemma 2.6 ([6])**.** *Let $A, B \in M_n$. Then*

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A) s_j(B), 1 \leq k \leq n.$$

Lemma 2.7 ([1])**.** *The inequality*

$$(1 + x^t)^r \leq (1 + x^r)^t$$

holds, for all $x > 0$ and $0 \leq r \leq t$.

Lemma 2.8 ([9])**.** *Let $A \in M_n$ be such that $W(A) \subset S_\alpha$ and $A = U|A|$ be the polar decomposition of A. Then*

$$|A| \leq \frac{\sec(\alpha)}{2}(\mathrm{Re}(A) + U^*(\mathrm{Re}(A))U).$$

Lemma 2.9 ([10])**.** *If $f : [0, +\infty) \to [0, +\infty)$ is operator monotone. Then*

$$f(\alpha t) \leq \alpha f(t),$$

for $\alpha \geq 1$.

Theorem 2.1. *Let $A, B \in M_n$ and $\mu > 0$. Then*

$$\prod_{j=1}^{k} s_j(|A + B|^r) \leq \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}},$$

where $1 \leq r \leq 2$ and $1 \leq k \leq n$.

Proof. For $A, B \in M_n$, by Lemma 2.1, there exist unitary matrices $V_1, V_2 \in M_n$ such that

$$|A+B| \leq V_1^*|A|V_1 + V_2 * |B|V_2. \tag{2.1}$$

By $\mu > 0$, we have

$$\begin{aligned}
&\begin{bmatrix} \mu I_n & V_1^* \,|A|\, V_1 \\ V_2^* \,|B|\, V_2 & \mu I_n \end{bmatrix} \begin{bmatrix} \mu I_n & V_2^* \,|B|\, V_2 \\ V_1^* \,|A|\, V_1 & \mu I_n \end{bmatrix} \\
&= \begin{bmatrix} \mu^2 I_n + V_1^* \,|A|^2\, V_1 & \mu(V_1^* \,|A|\, V_1 + V_2^* \,|B|\, V_2) \\ \mu(V_1^* \,|A|\, V_1 \ + V_2^* \,|B|\, V_2) & \mu^2 I_n + V_2^* \,|B|^2\, V_2 \end{bmatrix} \geq 0.
\end{aligned} \tag{2.2}$$

Using (2.1), (2.2) and Lemma 2.2, we have

$$\begin{aligned}
\pm\mu \,|A+B| \ &\leq \ \mu(V_1^* \,|A|\, V_1 + V_2^* \,|B|\, V_2) \\
&\leq \ (\mu^2 I_n + V_1^* \,|A|^2\, V_1)\sharp(\mu^2 I_n + V_2^* \,|B|^2\, V_2).
\end{aligned} \tag{2.3}$$

By Lemma 2.3, there exists an $n \times k$ matrix U with $U^*U = I_k$ and

$$\prod_{j=1}^{k} s_j(|A+B|) = |\det(U^*(A+B)U)|\,, 1 \leq k \leq n. \tag{2.4}$$

By (2.3) and Lemma 2.4, we have

$$|\det(U^*\mu|A+B|U)| \leq \det[U^*((\mu^2 I_n + V_1^* \,|A|^2\, V_1)\sharp(\mu^2 I_n + V_2^* \,|B|^2\, V_2))U]. \tag{2.5}$$

Now, from (2.4), (2.5), Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
\prod_{j=1}^{k} s_j(\mu\,|A+B|) &\leq \det[U^*((\mu^2 I_n + V_1^* \,|A|^2\, V_1)\sharp(\mu^2 I_n + V_2^* \,|B|^2\, V_2))U] \\
&\leq \max|\det[V^*((\mu^2 I_n + V_1^* \,|A|^2\, V_1)\sharp(\mu^2 I_n + V_2^* \,|B|^2\, V_2))V]| \\
&= \prod_{j=1}^{k} s_j((\mu^2 I_n + V_1^* \,|A|^2\, V_1)\sharp(\mu^2 I_n + V_2^* \,|B|^2\, V_2)) \\
&\text{(by Lemma 2.3)} \\
&\leq \sqrt{\prod_{j=1}^{k} s_j(\mu^2 I_n + V_1^* \,|A|^2\, V_1) \prod_{j=1}^{k} s_j(\mu^2 I_n + V_2^* \,|B|^2\, V_2)} \\
&\leq \sqrt{\prod_{j=1}^{k} s_j(\mu^2 I_n + |A|^2) \prod_{j=1}^{k} s_j(\mu^2 I_n + |B|^2)}, 1 \leq k \leq n.
\end{aligned}$$

That is

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^2) &\le \frac{1}{\mu^{2k}} \prod_{j=1}^{k} s_j(\mu^2 I_n + |A|^2) \prod_{j=1}^{k} s_j(\mu^2 I_n + |B|^2) \\ &= \mu^{2k} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2}), 1 \le k \le n.\end{aligned}$$

Then

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}}.$$

This completes the proof. $\square$

Substituting A and B with $\frac{A}{\mu}$ and $\frac{B}{\mu}$ in the inequality (1.1), respectively, we have the following inequality

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^r}{\mu^r}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^r}{\mu^r}). \tag{2.6}$$

Applying Lemma 2.7, we can obtain the following corollary, which is sharper than inequality (2.6).

Corollary 2.1. *Let $A, B \in M_n$ and $\mu > 0$. Then*

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^r) &\le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}} \\ &\le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^r}{\mu^r}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^r}{\mu^r}),\end{aligned}$$

where $1 \le r \le 2$ and $1 \le k \le n$.

Proof. For $1 \le j \le n$ and $1 \le r \le 2$, by Lemma 2.7, we have

$$(1 + (\frac{s_j(A)}{\mu})^2)^{\frac{r}{2}} \le 1 + (\frac{s_j(A)}{\mu})^r.$$

That is

$$s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \le s_j(I_n + (\frac{|A|}{\mu})^r). \tag{2.7}$$

Now, from Theorem 2.1 and inequality (2.7), we get

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^r) &\leq \mu^{rk}\prod_{j=1}^{k} s_j(I_n+\frac{|A|^2}{\mu^2})^{\frac{r}{2}}\prod_{j=1}^{k} s_j(I_n+\frac{|B|^2}{\mu^2})^{\frac{r}{2}}\\ &\leq \mu^{rk}\prod_{j=1}^{k} s_j(I_n+\frac{|A|^r}{\mu^r})\prod_{j=1}^{k} s_j(I_n+\frac{|B|^r}{\mu^r}).\end{aligned}$$

This completes the proof. □

Remark 2.1. Let $\mu = 1$ in Corollary 2.1. Obviously, Corollary 2.1 is a generalization of the inequality (1.1).

Using Corollary 2.1, we have the following Theorem which is a generalization of the inequality (1.3).

Theorem 2.2. *Let $A, B \in M_n$ such that $W(A), W(B) \subset S_\alpha$. Then*

$$\prod_{j=1}^{k} s_j(A+B) \leq \mu^k \prod_{j=1}^{k} s_j^2(I_n+\frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))s_j^2(I_n+\frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B)),$$

where $1 \leq k \leq n$ and $\mu > 0$.

Proof. Let U, V be unitary matrices, we have the following chain of inequalities:

$$\begin{aligned}\prod_{j=1}^{k} s_j(A+B) &= \prod_{j=1}^{k} s_j(|A+B|)\\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n+\frac{|A|}{\mu})\prod_{j=1}^{k} s_j(I_n+\frac{|B|}{\mu}) \quad (\text{by Corollary 2.1})\\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n+\frac{\sec(\alpha)}{2\mu}(\mathrm{Re}(A)+U^*\mathrm{Re}(A)U))\\ &\quad s_j(I_n+\frac{\sec(\alpha)}{2\mu}(\mathrm{Re}(B)+V^*\mathrm{Re}(B)V)) \quad (\text{by Lemma 2.8})\\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n+\frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))s_j(I_n+\frac{\sec(\alpha)}{2\mu}U^*\mathrm{Re}(A)U)\\ &\quad \prod_{j=1}^{k} s_j(I_n+\frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))s_j(I_n+\frac{\sec(\alpha)}{2\mu}V^*\mathrm{Re}(B)V)\end{aligned}$$

(by (1.2))

$$= \mu^k \prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))s_j(U^*(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))U)$$

$$\prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))s_j(V^*(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))V)$$

$$\leq \mu^k \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))$$

(by Lemma 2.6).

This completes the proof. □

Theorem 2.3. *Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$ and $a, b > 0$. Then*

$$\prod_{j=1}^{k} s_j(I_n + f(|aA + bB|))$$

$$\leq \prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha) f(\frac{a\mathrm{Re}(A)}{2})) \prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha) f(\frac{b\mathrm{Re}(B)}{2})), \tag{2.8}$$

where $f : [0, +\infty) \to [0, +\infty)$ is operator concave function, $1 \leq k \leq n$.

Proof. Let U, V be unitary matrices, we have the following chain of inequalities:

$$\prod_{j=1}^{k} s_j(I_n + f(|aA + bB|))$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(a|A|)) \prod_{j=1}^{k} s_j(I_n + f(b|B|)) \quad (\text{by } (1.2))$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}(\mathrm{Re}(A) + U_1^*\mathrm{Re}(A)U_1)))$$

$$\prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}(\mathrm{Re}(B) + U_2^*\mathrm{Re}(B)U_2))) \quad (\text{by Lemma } 2.8)$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}\mathrm{Re}(A))s_j(I_n + f(\frac{a\sec(\alpha)}{2}U_1^*\mathrm{Re}(A)U_1))$$

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(I_n + f(\frac{b\sec(\alpha)}{2}\mathrm{Re}(B))s_j(I_n + f(\frac{a\sec(\alpha)}{2}U_2^*\mathrm{Re}(B)U_2)) \quad (\text{by}\,(1.2)\,)\\
&\le \prod_{j=1}^{k} s_j^2(I_n + f(\frac{a\sec(\alpha)}{2}\mathrm{Re}(A))\prod_{j=1}^{k} s_j^2(I_n + f(\frac{b\sec(\alpha)}{2}\mathrm{Re}(B))\\
&\quad (\text{by}\,\text{Lemma}\,2.6\,)\\
&\le \prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha)f(\frac{a\mathrm{Re}(A)}{2}))\prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha)f(\frac{b\mathrm{Re}(B)}{2}))\\
&\quad (\text{by}\,\text{Lemma}\,2.9\,).
\end{aligned}$$

This completes the proof. □

Remark 2.2. Let $f(t) = t$ and $a = b = 1$ in Theorem 2.3, we obtain the inequality (1.4).

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