Inequalities for the generalized inverse trigonometric and hyperbolic functions with one parameter

Xue-Ling Wang
College of Science
Shandong University of Aeronautics
Binzhou City, Shandong Province, 256603
China
math_2021@163.com

Li Yin*
College of Science
Shandong University of Aeronautics
Binzhou City, Shandong Province, 256603
China
yinli_79@163.com

Abstract. In this paper, we show some inequalities for the generalized inverse trigonometric and hyperbolic functions with one parameter of $(2, q)$. Especially, we also present several Shafer-Fink, Wilker and Huygens type inequalities of these functions. These results are consistent with previously known results.
Keywords: generalized inverse trigonometric function; Lerch Phi function; Shafer-Fink type inequalities; Wilker and Huygens type inequalities.

1. Introduction

For $p, q \in (1, +\infty)$ and $x \in [0, 1]$, the function $\sin_{p,q}(x)$ is defined by the inverse function of

$$\sin_{p,q}^{-1}(x) = \int_0^x (1 - t^q)^{-1/p} dt.$$

The function $\sin_{p,q}^{-1}(x)$ is increasing in $[0, 1]$ onto $[0, \pi_{p,q}/2]$ where

$$\frac{\pi_{p,q}}{2} = \sin_{p,q}^{-1}(1) = \int_0^1 (1 - t^q)^{-1/p} dt = \frac{1}{q} B(1 - \frac{1}{p}, \frac{1}{q}).$$

The function $\sin_{p,q}(x)$ is defined on $[0, \pi_{p,q}/2]$ and can be extended to $(-\infty, +\infty)$. Similarly, we can define $\cos_{p,q}(x), \tan_{p,q}(x)$ and their inverses (see [11]). In the same way, we can define the generalized hyperbolic functions as follows:

$$\sinh_{p,q}^{-1}(x) = \int_0^x (1 + t^q)^{-1/p} dt, x \in \mathbb{R}.$$

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*. Corresponding author

Recently, the arc lemniscate sine function and the hyperbolic arc lemniscate sine function defined by

$$\text{(1)} \qquad \operatorname{arcsl}(x) = \int_0^x (1-t^4)^{-1/2} dt, |x| < 1$$

and

$$\text{(2)} \qquad \operatorname{arcslh}(x) = \int_0^x (1+t^4)^{-1/2} dt, x \in \mathbb{R}$$

are deeply studied. In fact, (1) and (2) are $\sin_{2,4}^{-1}$ and $\sinh_{2,4}^{-1}$ respectively.

Neuman used the arc lemniscate sine function and the hyperbolic arc lemniscate sine function, respectively, to define the arc lemniscate tangent function and the hyperbolic arc lemniscate tangent function, as follows(see [14], [15]):

$$\text{(3)} \qquad \begin{aligned} \operatorname{arctl}(x) &= \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), x \in \mathbb{R}, \\ \operatorname{arctlh}(x) &= \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), |x| < 1. \end{aligned}$$

In [3], [4], Chen established several lemniscate function inequalities of the Wilker and Huygens type. Recently, some Shafer-Fink type inequalities for the lemniscate functions were established. In [5], inequalities of the Wilker and Huygens type involving inverse trigonometric functions were given by Chen et. al.. For more results, the reader may see references: [6], [10], [13], [16], [17]. In [18], Xu et. al. got some new bounds for the arc lemniscate functions. In particular, from the point view of bivariate means, Zhao [20, 21, 22] et. al. dealt with the arc lemniscate functions and got optimal bounds for these bivariate means.

For several functions connected to the generalized inverse lemniscate and the generalized hyperbolic inverse lemniscate functions, Yin and Lin [19] investigated monotonicity and some inequalities. By utilizing the Lerch Phi function, they provided a bound estimation of the generalized inverse lemniscate functions. Later, some inequalities of the Shafer-Fink, Wilker, and Huygens types were obtained.

The lemniscate inverse functions and the generalized inverse lemniscate functions are the generalized $(2,4)$-trigonometric and $(2,6)$-trigonometric functions respectively, thus are the special cases of the generalized $(2,q)$-trigonometric functions. Motivated by the work of references [1, 4, 17, 19], we mainly study the generalized $(2,q)$-trigonometric and hyperbolic functions:

$$\begin{aligned} \sin_{2,q}^{-1}(x) &= \int_0^x (1-t^q)^{-1/2} dt, |x| < 1, \\ \sinh_{2,q}^{-1}(x) &= \int_0^x (1+t^q)^{-1/2} dt, x \in \mathbb{R}. \end{aligned}$$

Previously, mathematicians focused on the study of generalized trigonometric and hyperbolic functions, the reader may refer to the literature [7, 8, 9, 12]. However, the generalized $(2, q)$-trigonometric and hyperbolic functions have rarely been studied. Here, we mainly showed several the Shafer-Fink, Wilker and Huygens type inequalities for the generalized $(2, q)$-trigonometric and hyperbolic functions.

2. Bounds of $\sin_{2,q}^{-1}(x)$

Lemma 2.1 ([19, Theorem 1.1]). *Let $-\infty < a < b < +\infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b) with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b), then so is f/g.*

Theorem 2.1. *For all $x \in (0, 1)$ and $q \geq 4$, we have*

$$\alpha x \Phi(x^q, 3/2, 1/q) < \sin_{2,q}^{-1}(x) < \beta x \Phi(x^q, 3/2, 1/q) \tag{4}$$

with the best possible constants $\alpha = q^{-\frac{3}{2}}$ and $\beta = \frac{B(1/2,1/q)}{q\zeta(3/2,1/q)}$ where

$$\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}, \alpha \neq 0, -1 \ldots, |z| < 1,$$
$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$
$$\zeta(s, \alpha) = \Phi(1, s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

are Lerch Phi function, classical beta function and Hurwitz zeta function respectively. If $1 < q \leq 3$, the inequalities (4) are inverse.

Proof of Theorem 2.1. Let

$$F(x) = \frac{\sin_{2,q}^{-1}(x)}{x\Phi(x^q, 3/2, 1/q)}. \tag{5}$$

Applying the Lemma 2.1 with $f(x) = \sin_{2,q}^{-1}(x)$ and $g(x) = x\Phi(x^q, 3/2, 1/q)$ and simple computation, we get

$$f(0^+) = g(0^+) = 0, f'(x) = \frac{1}{\sqrt{1-x^q}}, g'(x) = q^{\frac{3}{2}} \sum_{n=0}^{\infty} \frac{x^{qn}}{\sqrt{qn+1}}.$$

So, we obtain

$$\frac{f'(x)}{g'(x)} = \frac{1}{q^{\frac{3}{2}} h(x^q)},$$

where

$$h(t)=\sqrt{1-t}\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{qn+1}}, t\in(0,1).$$

By differentiation, we get

$$2\sqrt{1-t}h'(t)=\sum_{n=0}^{\infty}(\frac{2n+2}{\sqrt{qn+q+1}}-\frac{2n+1}{\sqrt{qn+1}})t^n.$$

Let $a_n=\frac{2n+2}{\sqrt{qn+q+1}}-\frac{2n+1}{\sqrt{qn+1}}$, then

$$a_n=\frac{(4-q)n+3-q}{(2n+2)(qn+1)\sqrt{qn+q+1}+(2n+1)(qn+q+1)\sqrt{qn+1}}.$$

If $q\geq 4$, we have $a_n<0$, thus $h'(t)<0$, it follows that $h(t)$ is strictly decreasing on $(0,1)$. This implies that $\frac{f'(x)}{g'(x)}$ is strictly increasing on$(0,1)$, by Lemma 2.1, we conclude that $F(x)$ is strictly increasing on$(0,1)$. Thus $F(0)<F(x)<F(1)$ for $x\in(0,1)$. By simple computation, we get

$$\begin{aligned}F(0^+)&=\lim_{x\to 0^+}\frac{f'(x)}{g'(x)}=q^{-\frac{3}{2}},\\ F(1^-)&=\frac{\sin_{2,q}^{-1}(1)}{\Phi(1,3/2,1/q)}=\frac{B(1/2,1/q)}{q\zeta(3/2,1/q)}.\end{aligned}\tag{6}$$

If $1<q\leq 3$, we easily complete the proof.

Remark 2.1. When $3<q<4$, the situation is more complex. Taking $q=3.1$ as an example, then by (5), we have

$$F(x)=\frac{\sin_{2,\frac{31}{10}}^{-1}(x)}{x\Phi(x^{\frac{31}{10}},\frac{3}{2},\frac{10}{31})}.$$

By (6), we get

$$F(1^-)=\frac{B(\frac{1}{2},\frac{10}{31})}{\frac{31}{10}\zeta(\frac{3}{2},\frac{10}{31})}=0.183373\ldots.$$

However, $F(0.9)=0.183419\ldots>F(1^-)$. Therefore, it is necessary to find the maximum value of $F(x)$ in $(0,1)$. This is a challenging problem and open.

3. Shafer-Fink type inequalities

Lemma 3.1. *For $q>1$, we have*

(*i*) *The function $f_1(x)=\frac{\sin_{2,q}^{-1}(x)}{x}$ is strictly increasing on $(0,1)$ with range $(1,\frac{\pi_{2,q}}{2})$, where $\frac{\pi_{2,q}}{2}=\sin_{2,q}^{-1}(1)=\frac{1}{q}B(\frac{1}{2},\frac{1}{q})$;*

(*ii*) *The function* $f_2(x)=\frac{\sinh_{2,q}^{-1}(x)}{x}$ *is strictly decreasing on* $(0,+\infty)$ *with range* $(0,1)$.

Proof of Lemma 3.1. Lemma 2.1 allows us to simply finish the proof.

Lemma 3.2. *For* $q\ge 4$, *we have*

(*i*) *The function* $g_1(x)=\frac{x-\sqrt[q]{1-x^q}\sin_{2,q}^{-1}(x)}{\sin_{2,q}^{-1}(x)-x}$ *is strictly increasing on* $(0,1)$ *with range* $(\frac{q+2}{q},\frac{2}{\pi_{2,q}-2})$;

(*ii*) *The function* $g_2(x)=\frac{\sqrt[q]{1+x^q}\sinh_{2,q}^{-1}(x)-x}{x-\sinh_{2,q}^{-1}(x)}$ *is strictly decreasing on* $(0,+\infty)$ *with range* $(\frac{\pi_{2q/(q+2),q}}{2}-1,\frac{q+2}{q})$ *where* $\frac{\pi_{2q/(q+2),q}}{2}=\frac{1}{q}B(\frac{q-2}{2q},\frac{1}{q})$.

Proof of Lemma 3.2. (i) Let $g_1(x)=\frac{g_{11}(x)}{g_{12}(x)}$ where $g_{11}(x)=x-\sqrt[q]{1-x^q}\sin_{2,q}^{-1}(x)$ and $g_{12}(x)=\sin_{2,q}^{-1}(x)-x$. Then $g_{11}(0^+)=g_{12}(0^+)=0$. By differentiation, we obtain

$$\frac{g_{11}'(x)}{g_{12}'(x)}=\frac{1+x^{q-1}(1-x^q)^{\frac{1-q}{q}}\sin_{2,q}^{-1}(x)-(1-x^q)^{\frac{2-q}{2q}}}{(1-x^q)^{-\frac{1}{2}}-1}$$

with $g_{11}'(0^+)=g_{12}(0^+)=0$. Computing once more, we obtain

$$\frac{g_{11}''(x)}{g_{12}''(x)}=\frac{2(q-1)}{q}\frac{\sin_{2,q}^{-1}(x)}{x}(1-x^q)^{\frac{2-q}{2q}}+\frac{4-q}{q}(1-x^q)^{\frac{1}{q}}.$$

As $q\ge 4$, by lemma 3.1, $\frac{g_{11}''(x)}{g_{12}''(x)}$ is strictly increasing, as a result, $g_1(x)$ strictly increases by Lemma 2.1, it follows that $g_1(0^+)<g_1(x)<g_1(1^-)$. Simple computation yields $g_1(0^+)=\frac{q+2}{q}$ and $g_1(1^-)=\frac{2}{\pi_{2,q}-2}$.

(ii) Let $g_2(x)=\frac{g_{21}(x)}{g_{22}(x)}$ where $g_{21}(x)=\sqrt[q]{1+x^q}\sinh_{2,q}^{-1}(x)-x$ and $g_{22}(x)=x-\sinh_{2,q}^{-1}(x)$. Then $g_{21}(0^+)=g_{22}(0^+)=0$. By differentiation, we obtain

$$\frac{g_{21}'(x)}{g_{22}'(x)}=\frac{x^{q-1}(1+x^q)^{\frac{1-q}{q}}\sinh_{2,q}^{-1}(x)+(1+x^q)^{\frac{2-q}{2q}}-1}{1-(1-x^q)^{-\frac{1}{2}}}$$

with $g_{11}'(0^+)=g_{12}'(0^+)=0$. Differentiating again, we get

$$\frac{g_{21}''(x)}{g_{22}''(x)}=\frac{2(q-1)}{q}\frac{\sinh_{2,q}^{-1}(x)}{x}(1+x^q)^{\frac{2-q}{2q}}+\frac{4-q}{q}(1+x^q)^{\frac{1}{q}}.$$

As $q\ge 4$, $\frac{g_{21}''(x)}{g_{22}''(x)}$ is strictly decreasing by lemma 3.1. Hence, $g_2(x)$ is strictly decreasing by Lemma 2.1, thus, it follows $g_2(0^+)>g_2(x)>g_2(+\infty)$. The limiting values read as follows

$$g_2(0^+)=\frac{q+2}{q},$$

$$g_2(+\infty) = \sinh_{2.q}^{-1}(+\infty) - 1 = \int_0^{+\infty} (1+t^q)^{-1/2} dt - 1$$
$$= \int_0^1 (1-s^q)^{\frac{-q-2}{2q}} ds - 1 = \frac{\pi_{2q/(q+2),q}}{2} - 1,$$

where we apply the substitution $1 + t^q = \frac{1}{1-s^q}$. This completes the proof.

Theorem 3.1. *For $q \geq 4$, the following inequalities exist:*

(i) $\dfrac{\pi_{2,q}}{2 + (\pi_{2,q} - 2)\sqrt[q]{1-x^q}} < \dfrac{\sin_{2,q}^{-1}(x)}{x} < \dfrac{2q+2}{q+2+q\sqrt[q]{1-x^q}}, 0 < |x| < 1;$

(ii) $\dfrac{\pi_{2q/(q+2),q}}{(\pi_{2q/(q+2),q} - 2) + 2\sqrt[q]{1+x^q}} < \dfrac{\sinh_{2,q}^{-1}(x)}{x} < \dfrac{2q+2}{q+2+q\sqrt[q]{1+x^q}}, |x| > 0.$

Proof of Theorem 3.1. *We finished the proof by utilizing Lemma 3.2.*

4. Wilker and Huygens type inequalities

The fact that the Pochhammers symbol $(a)_n$ is defined by

$$(a)_0 = 1, (a)_n = a(a+1)\dots(a+n-1), n = 1, 2, \dots,$$

and the ordinary binomial expansion can be written with the following notation,

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{7}$$

As an analogy to arc lemniscate functions which are defined in (3), $\tan_{2,q}^{-1}(x)$ and $\tanh_{2,q}^{-1}(x)$ have been defined as follows:

$$\tan_{2,q}^{-1}(x) = \sin_{2,q}^{-1}\left(\frac{x}{\sqrt[q]{1+x^q}}\right) = \int_0^{\frac{x}{\sqrt[q]{1+x^q}}} (1-t^q)^{-1/2} dt, x \in \mathbb{R},$$
$$\tanh_{2,q}^{-1}(x) = \sinh_{2,q}^{-1}\left(\frac{x}{\sqrt[q]{1-x^q}}\right) = \int_0^{\frac{x}{\sqrt[q]{1-x^q}}} (1+t^q)^{-1/2} dt, |x| < 1.$$

By using (7), we get the following power series expansions:

Lemma 4.1. *For $q > 1$, we have*

$$\sin_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(qn+1)n!} x^{qn+1}, |x| < 1, \tag{8}$$

$$\sinh_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, \tag{9}$$

$$\tan_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2} + \frac{1}{q})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, \tag{10}$$

$$\tanh_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{q})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, |x| < 1. \tag{11}$$

Proof of Lemma 4.1. We only prove (10), other proofs are completely similar. By simple computation, we get

$$\frac{d}{dx}\left(\tan_{2,q}^{-1}(x)\right) = \frac{d}{dx}\int_0^{\frac{x}{\sqrt[q]{1+x^q}}} \frac{1}{\sqrt{1-t^q}}dt = (1+x^q)^{-\frac{1}{2}-\frac{1}{q}}$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}+\frac{1}{q})_n}{n!}x^{qn}.$$

Hence,

$$\tan_{2,q}^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}+\frac{1}{q})_n}{(qn+1)n!}x^{qn+1}.$$

Lemma 4.2. *Let $q \geq 1$ be an integer. Then for all $0 < x < 1$,*

$$\text{(12)} \qquad (i) \quad \sum_{k=0}^{2p-1}(-1)^k a_k x^{qk+1} < \sinh_{2,q}^{-1}(x) < \sum_{k=0}^{2p}(-1)^k a_k x^{qk+1}$$

where

$$a_k = \frac{(\frac{1}{2})_k}{(qk+1)k!}, k = 0, 1 \dots.$$

$$\text{(13)} \qquad (ii) \quad \sum_{k=0}^{2p-1}(-1)^k b_k x^{qk+1} < \tan_{2,q}^{-1}(x) < \sum_{k=0}^{2p}(-1)^k b_k x^{qk+1}$$

where

$$b_k = \frac{(\frac{1}{2}+\frac{1}{q})_k}{(qk+1)k!}, k = 0, 1 \dots.$$

Proof of Lemma 4.2. We only prove (i). Simple computation results in

$$\frac{a_k}{a_{k+1}} = \frac{(\frac{1}{2})_k}{(qk+1)k!}\frac{(qk+q+1)(k+1)!}{(\frac{1}{2})_{k+1}}$$
$$= \frac{(qk+q+1)(2k+2)}{(qk+1)(2k+1)} > 1.$$

That is to say, $a_k > a_{k+1}$. We have

$$a_k x^{qk+1} - a_{k+1}x^{q(k+1)+1} = x^{qk+1}(a_k - a_{k+1}x^q) > 0$$

because of $a_{k+1}x^q < a_{k+1} < a_k$. According to (9), we get

$$\sinh_{2,q}^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n a_n x^{qn+1}$$
$$\text{(14)} \qquad = (a_0 x - a_1 x^{q+1}) + (a_2 x^{2q+1} - a_3 x^{3q+1}) + \dots$$
$$\text{(15)} \qquad = a_0 x - (a_1 x^{q+1} - a_2 x^{2q+1}) - (a_3 x^{3q+1} - a_4 x^{4q+1}) + \dots.$$

By using (14) and (15), we complete the proof of (i).

Theorem 4.1. *For $q \geq 2$ and $0 < x < 1$, we have*

$$\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,q}^{-1}(x)}{x} > 2. \tag{16}$$

Proof of Theorem 4.1. *For $0 < x < 1$, by using (8) and (13), we get*

$$\begin{aligned}\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 &= \left(1 + \frac{1}{2(q+1)}x^q + \frac{3}{8(2q+1)}x^{2q} + \dots\right)^2\\ &= 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} + \dots\\ &> 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} \qquad (17)\end{aligned}$$

and

$$1 - \frac{q+2}{2q(q+1)}x^q < \frac{\tan_{2,q}^{-1}(x)}{x} < 1. \tag{18}$$

So, we find

$$\begin{aligned}&\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,q}^{-1}(x)}{x} - 2\\ &> 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} + 1 - \frac{q+2}{2q(q+1)}x^q - 2\\ &> \frac{q-2}{2q(q+1)}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} > 0\end{aligned}$$

since $q \geq 2$.

Theorem 4.2. *For $q \geq 3$ and $0 < x < 1$, we have*

$$\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} < 2. \tag{19}$$

Proof of Theorem 4.2. For $0 < x < 1$, by using (17) and (18), we get

$$\begin{aligned}&\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} - 2\\ &< \frac{1}{1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q}} + \frac{1}{1 - \frac{q+2}{2q(q+1)}x^q} - 2\\ &= \frac{\begin{aligned}x^q[(q+2)(3q^2+8q+4)x^{2q} - (q+1)(3q^3-16q-8)x^q\\ - 2(q-2)(2q+1)(q+1)^2]\end{aligned}}{4q(2q+1)(q+1)^3(1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q})(1 - \frac{q+2}{2q(q+1)}x^q)}.\end{aligned}$$

Let $f(t) = at^2 + bt + c$ where

$$\begin{aligned} a &= (q+2)(3q^2+8q+4), \\ b &= -(q+1)(3q^3-16q-8), \\ c &= -2(q-2)(2q+1)(q+1)^2, \\ t &= x^q \in (0,1). \end{aligned}$$

As $q \geq 3$, so $a > 0, b < 0, c < 0$ and $f(1^-) = -7q^4 - 2q^3 + 42q^2 + 58q + 20 < 0$, Using the property of quadratic function, we get $f(t) < 0$, for all $t \in (0,1)$. Hence,

$$\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} - 2 < 0.$$

The proof is complete.

Corollary 4.1. *For $q \geq 3$ and $0 < x < 1$, we have*

$$\frac{2\sin_{2,q}^{-1}(x)}{x} + \frac{\tan_{2,q}^{-1}(x)}{x} > 3. \tag{20}$$

Proof of Corollary 4.1. Another option for inequality (19) is

$$\frac{2}{\frac{1}{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,q}^{-1}(x)}{x}}} > 1.$$

The arithmetic-geometric-harmonic mean inequality provides the following result:

$$\begin{aligned} &\frac{2\sin_{2,q}^{-1}(x)}{x} + \frac{\tan_{2,q}^{-1}(x)}{x} \geq 3\sqrt[3]{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 \frac{\tan_{2,q}^{-1}(x)}{x}} \\ &\geq 3\frac{2}{\frac{1}{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,q}^{-1}(x)}{x}}} > 3. \end{aligned}$$

In [2], Chen and Cheung proved the following inequalities:

$$\left(\frac{x}{\arcsin x}\right)^2 + \frac{x}{\arctan x} < 2, 0 < |x| < 1,$$

$$\frac{2\arcsin x}{x} + \frac{\arctan x}{x} > 3, 0 < |x| < 1.$$

So, we conject that the condition $q \geq 3$ in Theorem 4.2 and Corollary 4.1 can be changed to $q \geq 2$.

Theorem 4.3. *For $q > 1$ and $0 < x < 1$, we have*

$$\left(\frac{x}{\tanh_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\sinh_{2,q}^{-1}(x)} < 2. \tag{21}$$

Proof of Theorem 4.3. For $0 < x < 1$, by using (11) and (12), we have

$$\begin{aligned}\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2 &= \left(1 + \frac{q+2}{2q(q+1)}x^q + \dots\right)^2 \\ &= 1 + \frac{q+2}{q(q+1)}x^q + \dots \\ &> 1 + \frac{q+2}{q(q+1)}x^q\end{aligned}$$

and

$$\frac{\sinh_{2,q}^{-1}(x)}{x} > 1 - \frac{1}{2(q+1)}x^q.$$

So, we get

$$\begin{aligned}&\left(\frac{x}{\tanh_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\sinh_{2,q}^{-1}(x)} - 2 \\ &< \frac{1}{1 + \frac{q+2}{q(q+1)}x^q} + \frac{1}{1 - \frac{1}{2(q+1)}x^q} - 2 \\ &= \frac{x^q\left((2q+4)x^q - (q+1)(q+4)\right)}{(2q+2-x^q)\left(q^2+q+(q+2)x^q\right)} < 0\end{aligned}$$

since

$$(2q+4)x^q - (q+1)(q+4) < (2q+4) - (q+1)(q+4) < 0.$$

This completes the proof.

Corollary 4.2. *For $q > 1$ and $0 < x < 1$, we have*

$$\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\sinh_{2,q}^{-1}(x)}{x} > 2 \tag{22}$$

and

$$\frac{2\tanh_{2,q}^{-1}(x)}{x} + \frac{\sinh_{2,q}^{-1}(x)}{x} > 3. \tag{23}$$

Proof of Corollary 4.2. Inequality (21) can be rewritten as

$$\frac{2}{\frac{1}{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\sinh_{2,q}^{-1}(x)}{x}}} > 1.$$

The result of applying the arithmetic-geometric-harmonic mean inequality is

$$\frac{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2+\frac{\sinh_{2,q}^{-1}(x)}{x}}{2}\geq\sqrt{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2\frac{\sinh_{2,q}^{-1}(x)}{x}}$$
$$\geq\frac{2}{\frac{1}{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2}+\frac{1}{\frac{\sinh_{2,q}^{-1}(x)}{x}}}>1$$

and

$$\frac{2\tanh_{2,q}^{-1}(x)}{x}+\frac{\sinh_{2,q}^{-1}(x)}{x}\geq 3\sqrt[3]{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2\frac{\sinh_{2,q}^{-1}(x)}{x}}>3.$$

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