Simultaneous approximation of translation operators

N. Tsirivas
University of Thessaly
Department of Mathematics
3 Km P.E.O, Lamia-Athens
T.K. 35100 Lamia
Greece
ntsirivas@uth.gr

Abstract. Let $(m_n)_{n\in\mathbb{N}}$ be an unbounded sequence of complex numbers and $(a_v)_{v\in\mathbb{N}}$ be a sequence of numbers in the unit circle

$$C(0,1)=\{z\in\mathbb{C}\mid |z|=1\},$$

where $\mathbb{N}$ is the set of natural numbers.

We shall prove that there is an entire function f so that, for every entire function g there is a subsequence (λ_n), $n\in\mathbb{N}$ of $(m_n)_{n\in\mathbb{N}}$ such that, for every compact subset $L\subseteq\mathbb{C}$ and for every $v\in\mathbb{N}$,

$$\sup_{z\in L}|f(z+\lambda_n a_v)-g(z)|\to 0 \quad \text{as} \quad n\to\infty.$$

In relation with other results about hypercyclic operators, the new element in this paper is that we achieve the approximation with the same sequence (λ_n), for all numbers a_v $(v=1,2,\ldots)$.
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1. Introduction

We denote $\mathcal{H}(\mathbb{C})$ the set of entire functions endowed with the topology $\mathcal{T}_u$ of uniform convergence on compacta.

Let $a\in\mathbb{C}$. We denote $t_a:\mathbb{C}\to\mathbb{C}$ the translation function, which is given by the formula $t_a(z)=z+a$, for every $z\in\mathbb{C}$.

We consider the translation operator $T_a:\mathcal{H}(\mathbb{C})\to\mathcal{H}(\mathbb{C})$, that is, the operator defined by the formula $T_a(f)=f\circ t_a$, for every $f\in\mathcal{H}(\mathbb{C})$. The operator T_a is a linear and continuous operator.

We write $T_a^1=T_a$ and

$$T_a^{n+1}=T_a\circ T_a^n, \quad \text{for} \quad n=1,2,\ldots.$$

Birkhoff proved [4] that there is $f\in\mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_a^n(f),\ \ n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C}), \quad \text{where} \quad a\in\mathbb{C}\setminus\{0\}.$$

His proof was constructive.

Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence of complex numbers. Luh [12] proved that there is $f \in \mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_{a_n}(f), n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C}).$$

Gethner and Shapiro [8] and Grosse-Erdmann [9] have also proved the above results by using the Baire's Category Theorem. In particular, let $\mathcal{U}((T_{a_n}))$ be the set of entire functions that are universal (or hypercyclic) for the sequence (T_{a_n}), that is,

$$\mathcal{U}((T_{a_n})) = \{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{T_{a_n}(f) \mid n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\}.$$

Then, the set $\mathcal{U}((T_{a_n}))$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. Let $(b_m)_{m\in\mathbb{N}}$ be a sequence of non-zero complex numbers. Based on the previous result, the set $\bigcap_{m\in\mathbb{N}} \mathcal{U}((T_{b_m a_n}))$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$.

Costakis and Sambarino [6] established a notable strengthening of Birkhoff's result. More specifically, they proved that the set

$$\bigcap_{a\in\mathbb{C}-\{0\}} \{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{T_a^n(f), n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\}$$

contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. Note that each set in the last intersection is $\mathcal{U}(T_a) := \mathcal{U}((T_a^n)) = \mathcal{U}((T_{an}))$.

The important element here is the uncountable range of a.

Furthermore, Costakis [5] proved a more general result, that is, the set $\bigcap_{b\in C(0,1)} \mathcal{U}(T_{ba_n})$ contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$, where a_n is an unbounded and specific sequence of complex numbers.

Let us apply this result, in certain cases.

Let $(\theta_v)_{v\in\mathbb{N}}$ be a sequence of distinct numbers in $[0,1)$ and $(m_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers so that $m_n\to\infty$. We shall consider the numbers $w_n(\theta_v) = m_n e^{2\pi i\theta_v}$, $n, v \in \mathbb{N}$. That is, for every $v \in \mathbb{N}$ we shall consider the sequence $(w_n(\theta_v))_{n\in\mathbb{N}}$. Of course, we have $w_n(\theta_v)\to\infty$ as $n\to+\infty$, for every $v \in \mathbb{N}$.

We now set:

$$E_v = \{f \in \mathcal{H}(\mathbb{C}) | \overline{\{f(\cdot + w_n(\theta_v)) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\}, \quad \text{for every} \quad v \in \mathbb{N}.$$

Based on Grosse-Erdmann's result we conclude that, for every $v \in \mathbb{N}$ the set E_v is G_δ and dense in $\mathcal{H}(\mathbb{C})$. Hence, the set $E := \bigcap_{v=1}^{+\infty} E_v$ is a G_δ dense subset of $\mathcal{H}(\mathbb{C})$, so it is non-empty by Baire's Category Theorem, given that the space $\mathcal{H}(\mathbb{C})$ is a complete metric space. Let us see in more detail what this result means.

Let $f \in E$. Then, for every $v \in \mathbb{N}$ and $g \in \mathcal{H}(\mathbb{C})$ there is a subsequence $(\lambda_n^v) = (\lambda_n(v,g))$ of $(w_n(\theta_v))$, that depends on g and v so that, for every compact set $K \subseteq \mathbb{C}$ one has

$$\sup_{z\in K} |f(z+\lambda_n^v) - g(z)| \to 0 \quad \text{as} \quad n\to\infty.$$

So, this convergence depends on the specific sequence $\lambda_n^v = \lambda_n(v, g)$, $n \in \mathbb{N}$, and the sequence λ_n^v depends on the specific number $\theta_v \in [0, 1)$. In the present paper we shall examine whether we can have this convergence *without* the dependence on the specific number $\theta_v \in [0, 1)$.

With this aim, we shall introduce the set of entire functions that achieve simultaneous approximation on all numbers θ_v, $v \in \mathbb{N}$, where $\theta_v \in [0, 1)$, for every $v \in \mathbb{N}$ with the same sequence of indices. More specifically, we shall consider the set SA (standing for Simultaneous Approximation) defined as

$$SA = \{f \in \mathcal{H}(\mathbb{C}) : \text{ for every } \ g \in \mathcal{H}(\mathbb{C})$$

there is a subsequence $(\lambda_n)_{n\in\mathbb{N}}$ of $(m_n)_{n\in\mathbb{N}}$ such that

$$\sup_{z\in K} |f(z + \lambda_n e^{2\pi i\theta_v}) - g(z)| \to 0 \ \text{ as } \ n\to\infty,$$

for every compact set $K \subset \mathbb{C}$ and every $v \in \mathbb{N}\}$.

Of course $SA \subseteq E$.

We prove that the set SA is a G_δ-dense subset of $\mathcal{H}(\mathbb{C})$, so it is non-empty. In order to prove that SA is a G_δ, dense subset of $\mathcal{H}(\mathbb{C})$ we shall introduce one other set $V \subseteq \mathcal{H}(\mathbb{C})$ and we prove that V is a G_δ, dense subset of $\mathcal{H}(\mathbb{C})$ and $SA = V$. Other articles dealing with translation operators or sequences of translation operators on $\mathcal{H}(\mathbb{C})$ are [3] and [10].

Also, there are some papers concerning common hypercyclic vectors for translation operators; see the papers [1], [5], [6], [7], [14], [15], [16], as well as Chapter 11 in the book [11]. The notion of simultaneous hypercyclicity/universality was formally introduced (for finitely many operators) in [2].

Whenever we refer to a topology in the $\mathcal{H}(\mathbb{C})$ space, we always mean the topology of uniform convergence on compacta.

In the following Section 2 we prove some helpful propositions in order to prove our main result Theorem 2.6.

2. The main result

First of all, we shall prove a proposition which is the key in order to prove our main result.

We fix $g \in \mathcal{H}(\mathbb{C})$.

We also fix some natural numbers $n_0 \geq 2$, v_0, N_0, and some real numbers $\theta_1, \theta_2, \ldots, \theta_{n_0}$ where $\theta_i \in [0, 1)$ for each $i = 1, \ldots, n_0$ and $\theta_i \neq \theta_j$, for every $i, j \in A_{n_0} = \{1, \ldots, n_0\}$, $i \neq j$. For every natural number m we use the set

$$V_g(m, v_0, N_0, n_0) = \Big\{ f \in \mathcal{H}(\mathbb{C}) \Big/ \sup_{|z|\leq v_0} \Big| f(z + m e^{2\pi i\theta_j}) - g(z) \Big| < \frac{1}{N_0}, \\ \text{for every } j = 1, \ldots, n_0 \Big\}.$$

For every $m \in \mathbb{N}$, $j \in A_{n_0}$ we use the set

$$\widetilde{V}_g(m, v_0, N_0, j) = \Big\{ f \in \mathcal{H}(\mathbb{C}) \Big| \sup_{|z| \leq v_0} \Big| f(z + me^{2\pi i\theta_j}) - g(z) \Big| < \frac{1}{N_0} \Big\}.$$

Of course, we have

$$(1) \qquad V_g(m, v_0, N_0, n_0) = \bigcap_{j=1}^{n_0} \widetilde{V}_g(m, v_0, N_0, j),$$

based on the above definitions.

It is easy to see that the sets $\widetilde{V}_g(m, v_0, N_0, j)$ are open in $\mathcal{H}(\mathbb{C})$, for every $m \in \mathbb{N}$, $j = 1, \ldots, n_0$, so the set $V_g(m, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$, for every $m \in \mathbb{N}$, according to the above relation (1). Therefore, the set $\bigcup_{m=1}^{+\infty} V_g(m, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$.

For a function $h : \mathbb{C} \to \mathbb{C}$ and $A \subseteq \mathbb{C}$, we shall denote $\|h\|_A := \sup\{|h(z)| : z \in A\}$.

Proposition 2.1. *Under the above notations, we have that the set* $\bigcup_{m=1}^{+\infty} V_g(m, v_0, N_0, n_0)$ *is dense in* $\mathcal{H}(\mathbb{C})$.

Proof. We fix a function $h \in \mathcal{H}(\mathbb{C})$, a compact set $K \subseteq \mathbb{C}$ and an $\varepsilon > 0$. It suffices to show that there are $f \in \mathcal{H}(\mathbb{C})$ and $m_0 \in \mathbb{N}$, so that

$$(1) \qquad f \in V_g(m_0, v_0, N_0, n_0) \ \text{ and } \ \|f - h\|_K < \varepsilon.$$

We set $D_v = \{z \in \mathbb{C} \mid |z| \leq v\}$, for every $v \in \mathbb{N}$. We also choose $v_1 \in \mathbb{N}$ so that

$$(2) \qquad D_{v_0} \cup K \subseteq D_{v_1}.$$

Let us assume that $m \in \mathbb{N}$ satisfies

$$D_{v_1} \cap (D_{v_1} + me^{2\pi i\theta_j}) \neq \emptyset,$$

for some $j \in A_{n_0}$ (if it exists). We remind that $A_{n_0} = \{1, 2, \ldots, n_0\}$.
This means that there also exist $z_j, w_j \in D_{v_1}$, so that

$$(3) \qquad w_j = z_j + me^{2\pi i\theta_j}, \ \text{ for some } \ j \in A_{n_0}.$$

According to (3), we shall have:

$$|w_j - z_j| = m\,, \ \text{ and this gives } \ m \leq 2v_1.$$

Therefore, for every $m \in \mathbb{N}$ and $m > 2v_1$, we have

$$(4) \qquad D_{v_1} \cap (D_{v_1} + me^{2\pi i\theta_j}) = \emptyset, \ \text{ for every } \ j \in A_{n_0}.$$

Let $j_1, j_2 \in A_{n_0}$, so that $j_1 \neq j_2$.

Let $m \in \mathbb{N}$ so that

$$(D_{v_1} + me^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + me^{2\pi i\theta_{j_2}}) \neq \emptyset \quad \text{(if it exists)}.$$

This means that there are $z_1, w_1 \in D_{v_1}$ so that

$$z_1 + me^{2\pi i\theta_{j_1}} = w_1 + me^{2\pi i\theta_{j_2}}. \tag{5}$$

By (5) we have:

$$|z_1 - w_1| = m\left|e^{2\pi i(\theta_{j_2}-\theta_{j_1})} - 1\right|. \tag{6}$$

From (6) we deduce that:

$$m \leq \frac{2v_1}{\left|e^{2\pi i(\theta_{j_2}-\theta_{j_1})} - 1\right|}. \tag{7}$$

So, for every $m \in \mathbb{N}$ satisfying

$$m > \frac{2v_1}{\left|e^{2\pi i(\theta_{j_2}-\theta_{j_1})} - 1\right|}$$

we have:

$$(D_{v_1} + me^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + me^{2\pi i\theta_{j_2}}) = \emptyset. \tag{8}$$

We set

$$M_0 = \min\left\{\left|e^{2\pi i(\theta_{j_2}-\theta_{j_1})} - 1\right| : j_1, j_2 \in A_{n_0} : j_1 \neq j_2\right\}.$$

We fix now some natural number m_0 so that $m_0 > \max\{2v_1, \frac{2v_1}{M_0}\}$. Then, by (4) and (8) we derive

$$D_{v_1} \cap (D_{v_1} + m_0e^{2\pi i\theta_j}) = \emptyset, \quad \text{for every} \quad j \in A_{n_0}$$

and

$$(D_{v_1} + m_0e^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + m_0e^{2\pi i\theta_{j_2}}) = \emptyset, \quad \text{for every} \quad j_1, j_2 \in A_{n_0}, \quad j_1 \neq j_2. \tag{9}$$

Now, we set

$$L := D_{v_1} \cup \left(\bigcup_{j=1}^{n_0}(D_{v_1} + m_0e^{2\pi i\theta_j})\right).$$

Because of (9) we have that the set L is a union of $n_0 + 1$ disjoint closed discs with the same radius v_1.

This means that the set L is a compact set with connected complement. We shall consider the function $F : L \to \mathbb{C}$, defined as follows:

$$F(z) = \begin{cases} h(z), & \text{if } z \in D_{v_1} \\ g(z - m_0e^{2\pi i\theta_j}), & \text{if } z \in D_{v_1} + m_0e^{2\pi i\theta_j}, \text{ for some } j \in A_{n_0}. \end{cases}$$

Of course, F is continuous on L and holomorphic on $\overset{\circ}{L}$, the interior of L. So, according to Mergelyan's Approximation Theorem (see, e.g., [[13], Chapter 20]) there is a complex polynomial f, so that

$$\|F - f\|_L < \min\left\{\varepsilon, \frac{1}{N_0}\right\}. \tag{10}$$

Based on the definition of F and (10), we have:

$$\|f - h\|_K < \varepsilon, \tag{11}$$

because of relation (2), and the definition of L.

Let us suppose $w \in D_{v_1}$. Then, for $j \in A_{n_0}$, $w + m_0 e^{2\pi i\theta_j} \in D_{v_1} + m_0 e^{2\pi i\theta_j}$.

We also set $z = w + m_0 e^{2\pi i\theta_j}$. Then, $F(z) = g(z - m_0 e^{2\pi i\theta_j}) = g(w)$. By (10) we have, for every $w \in D_{v_1}$ and $j \in A_{n_0}$ that

$$\left|f(w + m_0 e^{2\pi i\theta_j}) - g(w)\right| < \frac{1}{N_0}.$$

This yields that $f \in V_g(m_0, v_0, N_0, n_0)$, because of relation (2) and the fact that f is a polynomial (so entire). By this fact and (11) the proof of this proposition is complete now because relation (1) is satisfied. □

Now, we shall fix an unbounded sequence $(m_s)_{s\in\mathbb{N}}$ of complex numbers. With the notation of the previous Proposition 2.1 we shall consider the set:

$$V_g(m_s, v_0, N_0, n_0) = \left\{f \in \mathcal{H}(\mathbb{C}) \,\middle|\, \sup_{|z|\le v_0} \left|f(z + m_s e^{2\pi i\theta_j}) - g(z)\right| < \frac{1}{N_0}, \right.$$
$$\left. \text{for every } \ j \in A_{n_0}\right\},$$

for every $s \in \mathbb{N}$.

The sets $V_g(m_s, v_0, N_0, n_0)$ are open for every $s \in \mathbb{N}$, so the set $\bigcup_{s=1}^{+\infty} V_g(m_s, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$.

As in Proposition 2.1, we now state the following proposition:

Proposition 2.2. *The set* $\bigcup_{s=1}^{+\infty} V_g(m_s, v_0, N_0, n_0)$ *is dense in* $\mathcal{H}(\mathbb{C})$.

Proof. The proof is similar to that of Proposition 2.1 and for this reason the proof is omitted. □

Indeed, the only property of $\{1, 2, \ldots\}$ used in the proof of the last proposition is its non-boundedness.

Recall that the space $\mathcal{H}(\mathbb{C})$ is separable, and so we can fix a dense sequence $(p_k)_{k\in\mathbb{N}}$ of $\mathcal{H}(\mathbb{C})$ (for example $(p_k)_{k\in\mathbb{N}}$ be an enumeration of all complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$). For every $v, N, k, n, s \in \mathbb{N}$, $n \ge 2$ we shall consider the set:

$$V_{p_k}(m_s, v, N, n) = \left\{ f \in \mathcal{H}(\mathbb{C}) \middle| \sup_{|z| \leq v} \left| f(z + m_s e^{2\pi i \theta_j}) - p_k(z) \right| < \frac{1}{N}, \right.$$
$$\left. \text{for every } \; j \in A_n \right\}.$$

The sets $V_{p_k}(m_s, v, N, n)$ are open in $\mathcal{H}(\mathbb{C})$, for every $v, N, k, n, s \in \mathbb{N}$, $n \geq 2$, so that the set $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ is open for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. According to Proposition 2.2, we have that the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are dense in $\mathcal{H}(\mathbb{C})$, for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$.

We shall also consider the set:

$$V = \bigcap_{v=1}^{+\infty} \bigcap_{N=1}^{+\infty} \bigcap_{k=1}^{+\infty} \bigcap_{n=2}^{+\infty} \left(\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n) \right).$$

Under the above notation, we shall establish the following assertion

Proposition 2.3. *The set V is a G_δ-dense subset of $\mathcal{H}(\mathbb{C})$, so V is non-empty.*

Proof. The set V is a G_δ subset of $\mathcal{H}(\mathbb{C})$ due to its definition, because the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are open for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. Based on Proposition 2.2, the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are dense for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. Hence, the conclusion follows from Baire's Category Theorem because the space $\mathcal{H}(\mathbb{C})$ is a complete metric space. □

We now connect the previous set V with the set of entire functions that succeed simultaneous approximation with respect to a countable set of real numbers.

We shall state here the respective data. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, so that $\theta_n \in [0, 1)$ and $\theta_{j_1} \neq \theta_{j_2}$, for every $j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$, $n \in \mathbb{N}$. Let $(m_s)_{s\in\mathbb{N}}$ be a fixed sequence of complex numbers which is unbounded.

Let $\Theta := \{\theta_n : n \in \mathbb{N}\}$. Of course, the set Θ and the set:

$$m = \{m_s : s \in \mathbb{N}\}$$

consisting of all the terms of the sequence $(m_s)_{s\in\mathbb{N}}$ are also infinite.

We shall consider the set:

$$SA = \{ f \in \mathcal{H}(\mathbb{C}) \,|, \text{ for every } g \in \mathcal{H}(\mathbb{C}),$$

there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ so that $\lambda_n \in m$, for every $n \in \mathbb{N}$, so that for every $a \in \Theta$ and for every compact set $K \subseteq \mathbb{C}$ it holds that

$$\sup_{z\in K} \left| f(z + \lambda_n e^{2\pi i a}) - g(z) \right| \to 0 \;\text{ as }\; n\to\infty \}.$$

The method to prove that $SA \neq \emptyset$ is the following:
We shall prove that $SA = V$ and given that $V \neq \emptyset$ we shall also have $SA \neq \emptyset$.

In order to prove that $SA = V$ we show that $SA \subseteq V$ and $V \subseteq SA$. This is the subject of the following two propositions.

Proposition 2.4. *It holds $SA \subseteq V$.*

Proof. If $SA = \emptyset$, then the result is obvious. We suppose that $SA \neq \emptyset$. Let $f \in SA$. We fix $v_0, N_0, n_0, k_0 \in \mathbb{N}$, $n_0 \geq 2$.

Because $f \in SA$ for $g = p_{k_0}$ there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$, so that $\lambda_n \in m$, for every $n \in \mathbb{N}$ and, for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$, we have

$$\sup_{z\in K} |f(z + \lambda_n e^{2\pi i a}) - p_{k_0}(z)| \to 0 \text{ as } n\to\infty.$$

So, for $K = D_{v_0}$ we have that

$$\sup_{|z|\leq v_0} |f(z + \lambda_n e^{2\pi i\theta_j}) - p_{k_0}(z)| \to 0 \text{ as } n\to\infty,$$

for every $j \in A_{n_0}$.

This entails that for every $j \in A_{n_0}$ there is some $n_j \in \mathbb{N}$, so that

$$\sup_{|z|\leq v_0} \left|f(z + \lambda_n e^{2\pi i\theta}) - p_{k_0}(z)\right| < \frac{1}{N_0}, \text{ for every } n \in \mathbb{N}, \ n \geq n_j.$$

Let $\widetilde{n} = \max\{n_j | j \in A_{n_0}\}$. With this selection we obtain

$$\sup_{|z|\leq v_0} \left|f(z + \lambda_n e^{2\pi i\theta_j}) - p_{n_0}(z)\right| < \frac{1}{N_0},$$

for every $j \in A_{n_0}$, for every $n \in \mathbb{N}$, $n \geq \widetilde{n}$.

This implies that $f \in V_{p_{k_0}}(\lambda_{\widetilde{n}}, v_0, N_0, n_0)$, or equivalently, $f \in \bigcup_{s=1}^{+\infty} V_{p_{k_0}}(m_s, v_0, N_0.n_0)$ because $\lambda_{\widetilde{n}} \in m$, that implies $f \in V$ and the result is proven. $\square$

Proposition 2.5. *It holds that $V \subseteq SA$.*

Proof. We know that $V \neq \emptyset$. Let $f \in V$. We shall prove that $f \in SA$.

We fix $g \in \mathcal{H}(\mathbb{C})$. We shall show that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$, so that $\lambda_n \in m$, for every $n \in \mathbb{N}$, and so that for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$

$$\sup_{z\in K} \left|f(z + \lambda_n e^{2\pi i a}) - g(z)\right| \to 0 \text{ as } n\to\infty.$$

Based on the above mentioned properties, we shall now construct the respective sequence $(\lambda_n)_{n\in\mathbb{N}}$. We shall fix some $n_0 \in \mathbb{N}$, $n_0 \geq 2$.

Given that the sequence $(p_k)_{k\in\mathbb{N}}$ of complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathcal{H}(\mathbb{C})$, there is some $k_0 \in \mathbb{N}$ so that

$$\|g - p_{k_0}\|_{D_{n_0}} < \frac{1}{2n_0}. \tag{1}$$

Since $f \in V$ we have $f \in \bigcup_{s=1}^{+\infty} V_{p_{k_0}}(m_s, n_0, 2n_0, n_0)$. This means that there is some $s_{n_0} \in \mathbb{N}$ so that $f \in V_{p_{k_0}}(m_{s_{n_0}}, n_0, 2n_0, n_0)$, or equivalently,

$$\sup_{|z|\leq n_0} \left|f(z + m_{s_{n_0}} e^{2\pi i\theta_j}) - p_{k_0}(z)\right| < \frac{1}{2n_0}, \text{ for every } j \in A_{n_0}. \tag{2}$$

By (1), (2) and the triangle inequality we have:

$$(3) \qquad \sup_{|z|\leq n_0} \left| f(z + m_{s_{n_0}} e^{2\pi i \theta_j}) - g(z) \right| < \frac{1}{n_0}, \quad \text{for every} \quad j \in A_{n_0}.$$

According to the previous procedure, for every $n \in \mathbb{N}$, $n \geq 2$, we can choose some $s_n \in \mathbb{N}$ such that

$$(4) \qquad \sup_{|z|\leq n} \left| f(z + m_{s_n} e^{2\pi i \theta_j}) - g(z) \right| < \frac{1}{n}, \quad \text{for every} \quad j \in A_n.$$

We shall now prove that for the sequence $(s_n)_{n\in\mathbb{N}}$ one has the following: $\sup_{z\in K} \left| f(z+m_{s_n} e^{2\pi i a}) - g(z) \right| \to 0$ as $n \to \infty$, for every compact set $K \subseteq \mathbb{C}$ and for every $a \in \Theta$.

With this aim, fix some $\varepsilon_0 > 0$.

There are $v_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, so that $K \subseteq D_v$, for every $v \in \mathbb{N}$, $v \geq v_0$ and $a_0 = \theta_{n_0}$. Let us choose $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \varepsilon_0$.

Let $M_0 = \max\{v_0, n_0, N_0, 2\}$. For every $n \in \mathbb{N}$, $n \geq M$, we have $n \geq v_0$, so $K \subseteq D_n$. Of course, $a_0 \in \{\theta_1, \theta_2, \ldots, \theta_n\}$, for every $n \in \mathbb{N}$, $n \geq M_0$, because $a_0 = \theta_{n_0}$ and $n_0 \leq M_0 \leq n$. With this, we also get

$$\frac{1}{n} \leq \frac{1}{M_0} \leq \frac{1}{N_0} < \varepsilon_0, \quad \text{for every} \quad n \in \mathbb{N}, \quad n \geq M_0.$$

Then, for every $n \in \mathbb{N}$, $n \geq M_0$, it follows from (4) that

$$\sup_{z\in K} \left| f(z + m_{s_n} e^{2\pi i a_0}) - g(z) \right| \leq \sup_{|z|\leq n} \left| f(z + m_{s_n} e^{2\pi i a_0}) - g(z) \right| < \frac{1}{n} < \varepsilon_0.$$

This yields that

$$\sup_{z\in K} \left| f(z + m_{s_n} e^{2\pi i a_0}) - g(z) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

So, for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$ we have:

$$\sup_{z\in K} \left| f(z + m_{s_n} e^{2\pi i a}) - g(z) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Since this is the case for arbitrary $g \in \mathcal{H}(\mathbb{C})$ we conclude that $f \in SA$ and the proof of this proposition is complete. □

Based on the above results, we are ready now to state and prove the main result of this paper, that is Theorem 2.6.

Theorem 2.6. *The set SA is a G_δ dense subset of $\mathcal{H}(\mathbb{C})$. In particular, the set SA is non-empty.*

Proof. Based on Proposition 2.4 and 2.5 we have that $SA = V$. We have also proved in Proposition 2.3 that the set V is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. So, the result follows. □

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