Common fixed point for compatible self-maps in an orbitally complete b-metric space

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Abstract. A common fixed point theorem is obtained for three self-maps on a b-metric space, satisfying a rational type condition, through the notions of orbital completeness, orbital continuity and the compatibility.
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1. Introduction

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc., while the conditions on the underlying mappings are usually metrical or compact type conditions. Further, new algebraic structures were also formulated to improve the results. For instance, the following notion of b-metric space is a generalization of a metric space, due to Bakhtin [2].

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Definition 1.1. *Let $s \geq 1$, X be a nonempty set and $\rho_s : X \times X \to [0, +\infty)$ be such that*

$(b1)$ $\rho_s(x, y) = 0$ *if and only if* $x = y$

$(b2)$ $\rho_s(x, y) = \rho_s(y, x)$, *for all* $x, y \in X$

$(b3)$ $\rho_s(x, y) \leq s[\rho_s(x, z) + \rho_s(y, z)]$, *for all* $x, y, z \in X$.

Then, ρ_s is called a b-metric on X, and the pair (X, ρ_s) denotes a b-metric space.

If $s = 1$, the condition (b_3) reduces to the the triangle inequality of a metric. Thus metric space is a particular case of a b-metric space, when $s = 1$. However, a b-metric space is not necessarily a metric space. For instance, consider the pair (X, ρ_s), where $X = \mathbb{R}$ and $\rho_s(x, y) = |x - y|^2$, for all $x, y \in \mathbb{R}$. Then, the conditions (b_1) and (b_2) are obvious. Further, $\rho_s(x, y) = |x - y|^2 = |x - z + z - y|^2 \leq 2(|x - z|^2 + |z - y|^2) = 2[\rho_s(x, z) + \rho_s(y, z)]$, for all $x, y \in X$. Thus $(\mathbb{R}, \rho_s)$ is a b-metric space with $b = 2$. Since $\rho_s(1, 3) + \rho_s(1, 0) = 5$ and $\rho_s(0, 3) = 9$, the triangle inequality fails to hold good, showing that ρ_s is not a metric. Thus the class of b-metric spaces contains that of metric spaces.

Definition 1.2. *A b-ball in a b-metric space (X, ρ_s) is defined by*

$$B_{\rho_s}(x, r) = \{y \in X : \rho_s(x, y) < r\}.$$

The family of all b-balls forms a basis for topology, which is called the b-metric topology $\tau(\rho_s)$ on X.

Definition 1.3. *Let (X, ρ_s) be a b-metric space with parameter s. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X is said to be*

(a) *b-convergent, with limit p, if it converges to p in the b-metric topology $\tau(\rho_s)$*

(b) *b-Cauchy, if* $\lim_{n,m\to\infty} \rho_s(x_n, x_m) = 0$

(c) *b-complete, if every b-Cauchy sequence in X is b-convergent in it.*

Remark 1.1. A b-metric is not jointly continuous in its coordinate variables x and y, even though a metric d is known to be continuous (see, Example 2.13, [8]).

Definition 1.4. *Let (X, ρ_s) be a b-metric space with parameter s. Given $x_0 \in X$, and self-maps A, S and T on X, if there exist points $x_0, x_1, x_2, \ldots, x_n, \ldots$ such that*

$$y_{2n-1} = Sx_{2n-2} = Ax_{2n-1},\ y_{2n} = Tx_{2n-1} = Ax_{2n} \text{ for } n = 1, 2, \ldots, \tag{1.1}$$

then, the sequence $\langle Ax_n \rangle_{n=1}^{\infty}$ is called an (S, T)-orbit with respect to A at x_0 or simply an (S, T, A)-orbit at x_0, and is denoted by $\mathscr{O}_{S,T,A}(x_0)$.

The pair (S,T) is said to be asymptotically regular with respect to A at x_0 , if $\lim_{n\to\infty}\rho_s(Ax_n, Ax_{n+1}) \to 0$, and (S,T) is asymptotically regular with respect to A, if it is asymptotically regular with respect to A at each $x_0 \in X$. The b-metric space X is said to be (S,T,A)-orbitally b-complete at x_0, if every b-Cauchy sequence in $\mathscr{O}_{S,T,A}(x_0)$ converges in X. The space X is said to be (S,T,A)-orbitally b-complete, if it is (S,T,A)-orbitally b-complete at each x_0.

Definition 1.5. *Let (X,ρ_s) be a b-metric space with parameter s. A self-map $T : X \to X$ is said to be continuous at $p \in X$, if $\lim_{n\to\infty}\rho_s(Tp_n, Tp) = 0$ whenever $\langle p_n\rangle_{n=1}^{\infty} \subset X$ $\lim_{n\to\infty}\rho_s(p_n, p) = 0$. And, T is continuous on X, if it is continuous at every $x_0 \in X$.*

Definition 1.6. *The self-map A is (S,T) orbitally continuous at x_0 or simply orbitally continuous at x_0, if it is continuous on some (S,T,A)-orbit at x_0.*

Self-maps A and S on a metric space (X,d) are commuting, if $Asx = SAx$, for all $x \in X$. As a weaker form of it, Sessa [7] introduced weakly commuting maps A and S on X with the choice $d(ASx, SAx) \le d(Ax, Sx)$, for all $x \in X$. Gerald Jungck [4] introduced compatible maps as a generalization for weakly commuting maps as follows:

Definition 1.7. *Self-maps f and r on a metric space (X,d) are said to be compatible, if*

$$\lim_{n\to\infty} d(ASp_n, SAp_n) = 0, \tag{1.2}$$

whenever there exists a sequence $\langle p_n\rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = z, \text{ for some } z \in X. \tag{1.3}$$

In [1], the following notion was introduced:

Definition 1.8. *Let (X,d) be a metric space. Self-maps T and A on X are (T,A)-weak compatible, if*

$$\lim_{n\to\infty} ATp_n = Tz, \text{ and } \lim_{n\to\infty} TAp_n = \lim_{n\to\infty} T^2p_n = Tz, \tag{1.4}$$

whenever there exists a sequence $\langle p_n\rangle_{n=1}^{\infty} \subset X$ with the choice (1.3).

Note that, compatible maps T and A are (T,A)-weak compatible. However, the converse is not true. For example, let $X = (-\infty, +\infty)$ with usual metric $d(x,y) = |x-y|$, for all $x, y \in X$.

As the compatibility of a pair of self-maps on a b-metric space is just similar to that in metric space, we skip its discussion. In this paper, we establish a common fixed point theorem for three self-maps on a b-metric space, which satisfy a rational inequality, through the notions of orbital completeness, orbital continuity and the compatibility.

2. Main results

We use the following results from [5]:

Lemma 2.1. *Let (X, ρ_s) be a b-metric space with parameter s. Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is b-convergent with limit x, and $\langle y_n \rangle_{n=1}^{\infty}$ is b-convergent with limit y in X. Then*

$$\frac{1}{s^2}\rho_s(x,y) \le \liminf_{n\to\infty} \rho_s(x_n, y_n) \le \limsup_{n\to\infty} \rho_s(x_n, y_n) \le s^2 \rho_s(x,y). \tag{2.1}$$

In particular, if $x = y$, then $\lim_{n\to\infty} \rho_s(x_n, y_n) = 0$. Further, for each $z \in X$, we have

$$\frac{1}{s}\rho_s(x,z) \le \liminf_{n\to\infty} \rho_s(x_n, z) \le \limsup_{n\to\infty} \rho_s(x_n, z) \le s\rho_s(x,z). \tag{2.2}$$

The following is the main result of this paper:

Theorem 2.1. *Let A, S and T be self-maps on a b-metric space (X, ρ_s) with $s \ge 1$, satisfying the inclusions:*

$$S(X) \subset A(X) \text{ and } T(X) \subset A(X) \tag{2.3}$$

and the rational inequality

$$\begin{aligned} \rho_s(Sx, Ty) \le a\,\rho_s(Ax, Ay) &+ \beta \cdot \frac{\rho_s(Ay, Ty)[1 + \rho_s(Ax, Sx)]}{1 + \rho_s(Ax, Ay)} \\ &+ \gamma \cdot \frac{\rho_s(Ay, Ty) + \rho_s(Ay, Sx)}{1 + \rho_s(Ay, Ty)\rho_s(Ay, Sx)}, \text{ for all } x, y \in X, \end{aligned} \tag{2.4}$$

where a, β and γ are non-negative numbers, not all being zero, such that

$$s^4 a + (s^4 + 1)\beta + (s^5 + s^4 + s)\gamma < 1. \tag{2.5}$$

Then, (S, T) is asymptotically regular with respect to A at each $x_0 \in X$. Suppose that

(a) the space X is (S, T, A)-orbitally b-complete,

(b) A is orbitally continuous.

If one of the pairs (A, S) and (A, T) is compatible, then S, T and A have a unique common fixed point.

Proof. Given $x_0 \in X$, in view of (2.3), we see that $Sx_0 = Ax_1$ for some $x_1 \in X$ and $Tx_1 = Ax_2$ for some $x_2 \in X$ and so on. Thus inductively we choose points $x_1, x_2, \dots, x_n, \dots$ in X with the choice (1.1).

Writing $x = x_{2n-2}$, $y = x_{2n-1}$ in (2.4) and using (1.1),

$$\begin{aligned} \rho_s(y_{2n-1}, y_{2n}) &= \rho_s(Sx_{2n-2}, Tx_{2n-1}) \\ &\le \alpha\, \rho_s(Ax_{2n-2}, Ax_{2n-1}) \\ &\quad + \beta \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1})[1+\rho_s(Ax_{2n-2}, Sx_{2n-2})]}{1+\rho_s(Ax_{2n-2}, Ax_{2n-1})} \\ &\quad + \gamma \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1}) + \rho_s(Ax_{2n-1}, Sx_{2n-2})}{1+\rho_s(Ax_{2n-1}, Tx_{2n-1})\rho_s(Ax_{2n-1}, Sx_{2n-2})} \\ &= \alpha\, \rho_s(y_{2n-2}, y_{2n-1}) \\ &\quad + \beta \cdot \frac{\rho_s(y_{2n-1}, y_{2n})[1+\rho_s(y_{2n-2}, y_{2n-1})]}{1+\rho_s(y_{2n-2}, y_{2n-1})} \\ &\quad + \gamma \cdot \frac{\rho_s(y_{2n-1}, y_{2n}) + \rho_s(y_{2n-1}, y_{2n-1})}{1+\rho_s(y_{2n-1}, y_{2n})\rho_s(y_{2n-1}, y_{2n-1})} \\ &\le \frac{\alpha}{1-\beta-\gamma} \cdot \rho_s(y_{2n-2}, y_{2n-1}) < q.\rho_s(y_{2n-2}, y_{2n-1}), \end{aligned} \tag{2.6}$$

where

$$q = \frac{\alpha+\beta+(s+1)\gamma}{1-\beta-s\gamma}. \tag{2.7}$$

Similarly, in view of (1.1), the inequality (2.4) with $x = x_{2n-2}$ and $y = x_{2n-3}$, gives

$$\begin{aligned} \rho_s(y_{2n-2}, y_{2n-1}) &= \rho_s(y_{2n-1}, y_{2n-2}) \\ &= \rho_s(Sx_{2n-2}, Tx_{2n-3}) \\ &\le \alpha\, \rho_s(Ax_{2n-2}, Ax_{2n-3}) \\ &\quad + \beta \cdot \frac{\rho_s(Ax_{2n-3}, Tx_{2n-3})[1+\rho_s(Ax_{2n-2}, Sx_{2n-2})]}{1+\rho_s(Ax_{2n-2}, Ax_{2n-3})} \\ &\quad + \gamma \cdot \frac{\rho_s(Ax_{2n-3}, Tx_{2n-3}) + \rho_s(Ax_{2n-3}, Sx_{2n-2})}{1+\rho_s(Ax_{2n-3}, Tx_{2n-3})\rho_s(Ax_{2n-3}, Sx_{2n-2})} \\ &= \alpha\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta \cdot \frac{\rho_s(y_{2n-3}, y_{2n-2})[1+\rho_s(y_{2n-2}, y_{2n-1})]}{1+\rho_s(y_{2n-2}, y_{2n-3})} \\ &\quad + \gamma \cdot \frac{\rho_s(y_{2n-3}, y_{2n-2}) + \rho_s(y_{2n-3}, y_{2n-1})}{1+\rho_s(y_{2n-3}, y_{2n-2})\rho_s(y_{2n-3}, y_{2n-1})} \\ &\le \alpha\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta\, \rho_s(y_{2n-2}, y_{2n-1}) \\ &\quad + \gamma\rho_s(y_{2n-3}, y_{2n-2}) + s\gamma[\rho_s(y_{2n-3}, y_{2n-2}) + \rho_s(y_{2n-2}, y_{2n-1})] \end{aligned}$$

so, that

$$\rho_s(y_{2n-2}, y_{2n-1}) \le q.\rho_s(y_{2n-3}, y_{2n-2}). \tag{2.8}$$

Thus from (2.6) and (2.8), it follows that

$$\rho_s(y_{n-1}, y_n) \le q\rho_s(y_{n-2}, y_{n-1}), \text{ for all } n.$$

By induction,

(2.9)
$$\rho_s(y_n, y_{n+1}) \le q\rho_s(y_{n-1}, y_n) \le q^2\rho_s(y_{n-2}, y_{n-1}) \le \cdots \le q^{n-1}\rho_s(y_1, y_2),\ n \ge 1.$$

Since $q < 1/s^4 < 1$, (2.9) implies that $\rho_s(y_n, y_{n+1}) \to 0$ as $n \to \infty$. Thus, (S, T) is asymptotically regular with respect to A at x_0.

Now, for all $m > n$, employing the condition (b_3) repeatedly and using (2.9),

$$\begin{aligned}
\rho_s(y_n, y_m) &\le s[\rho_s(y_n, y_{n+1}) + \rho_s(y_{n+1}, y_m)] \\
&\le s\rho_s(y_n, y_{n+1}) + s^2[\rho_s(y_{n+1}, y_{n+2}) + \rho_s(y_{n+2}, y_m)] \\
&\le s\rho_s(y_n, y_{n+1}) + s^2\rho_s(y_{n+1}, y_{n+2}) + s^3[\rho_s(y_{n+2}, y_{n+3}) + \rho_s(y_{n+3}, y_m)] \\
&\dots \\
&\le s\rho_s(y_n, y_{n+1}) + s^2\rho_s(y_{n+1}, y_{n+2}) + \cdots + s^{m-n}\rho_s(y_{m-1}, y_m) \\
&\le \left[sq^{n-1} + s^2q^n + \cdots + s^{m-n}q^{m-2}\right]\rho_s(y_1, y_2) \\
&= sq^{n-1}\left[1 + sq + \cdots + (sq)^{m-n-1}\right]\rho_s(y_1, y_2) \\
&\le \frac{sq^{n-1}}{1 - sq} \cdot \rho_s(y_1, y_2).
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, we see that $\rho_s(y_n, y_m) \to 0$. Thus $\langle y_n \rangle_{n=1}^{\infty}$ is a b-Cauchy sequence.

Since X is (S, T, A)-orbitally b-complete at x_0, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. That is,

$$\lim_{n\to\infty} Ax_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{2.10}$$

In view of the condition (b) of the theorem, from (2.10) we get that

$$\lim_{n\to\infty} A^2x_{2n+1} = \lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n+2} = \lim_{n\to\infty} ATx_{2n+1} = Az. \tag{2.11}$$

First, we suppose that (A, S) is compatible. Then, from (2.11), it follows that

$$\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n} = Az. \tag{2.12}$$

Now, from (2.4) with $x = Ax_{2n}$ and $y = x_{2n-1}$,

$$\begin{aligned}
\rho_s(SAx_{2n}, Tx_{2n-1}) \le{}& \alpha\,\rho_s(A^2x_{2n}, Ax_{2n-1}) \\
&+ \beta \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1})[1 + \rho_s(A^2x_{2n}, SAx_{2n})]}{1 + \rho_s(A^2x_{2n}, Ax_{2n-1})} \\
&+ \gamma \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1}) + \rho_s(Ax_{2n-1}, SAx_{2n})}{1 + \rho_s(Ax_{2n-1}, Tx_{2n-1})\rho_s(Ax_{2n-1}, SAx_{2n})},
\end{aligned}$$

which, in view of (2.1), (2.10), (2.11) and (2.12), gives

$$\begin{aligned}
\frac{1}{s^2}\rho_s(Az,z) &\leq \liminf_{n\to\infty}\rho_s(SAx_{2n},Tx_{2n-1}) \leq \limsup_{n\to\infty}\rho_s(SAx_{2n},Tx_{2n-1}) \\
&\leq \limsup_{n\to\infty}\Bigg[\alpha\,\rho_s(A^2x_{2n},Ax_{2n-1}) \\
&\qquad + \beta\cdot\frac{\rho_s(Ax_{2n-1},Tx_{2n-1})[1+\rho_s(A^2x_{2n},SAx_{2n})]}{1+\rho_s(A^2x_{2n},Ax_{2n-1})} \\
&\qquad\quad + \gamma\cdot\frac{\rho_s(Ax_{2n-1},Tx_{2n-1})+\rho_s(Ax_{2n-1},SAx_{2n})}{1+\rho_s(Ax_{2n-1},Tx_{2n-1})\rho_s(Ax_{2n-1},SAx_{2n})}\Bigg] \\
&\leq s^2\Bigg[\alpha\,\rho_s(Az,z)+\beta\cdot\frac{\rho_s(z,z)[1+\rho_s(Az,Az)]}{1+\rho_s(Az,z)} \\
&\qquad + \gamma\cdot\frac{\rho_s(z,z)+\rho_s(z,Az)}{1+\rho_s(z,z)\rho_s(z,Az)}\Bigg] \\
&= s^2(\alpha+\gamma)\rho_s(z,Az)
\end{aligned}$$

so that $\rho_s(Az,z) \leq s^4(\alpha+\gamma)\rho_s(z,Az)$ and hence $Az = z$.

On one hand, writing $x = Ax_{2n}$ and $y = z$ in (2.4),

$$\begin{aligned}
\rho_s(SAx_{2n},Tz) \leq \alpha\,\rho_s(A^2x_{2n},Az) &+ \beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(A^2x_{2n},SAx_{2n})]}{1+\rho_s(A^2x_{2n},Az)} \\
&+ \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,SAx_{2n})}{1+\rho_s(Az,Tz)\rho_s(Az,SAx_{2n})}
\end{aligned}$$

Using (2.2), (2.10), (2.11) and (2.12), this gives

$$\begin{aligned}
\frac{1}{s}\rho_s(Az,Tz) &\leq \liminf_{n\to\infty}\rho_s(SAx_{2n},Tz) \\
&\leq \limsup_{n\to\infty}\rho_s(SAx_{2n},Tz) \\
&\leq s\Bigg[\alpha\rho_s(Az,Tz)+\beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(Az,Az)]}{1+\rho_s(Az,Az)} \\
&\qquad + \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,Az)}{1+\rho_s(Az,Tz)\rho_s(Az,Az)}\Bigg]
\end{aligned}$$

so that $\rho_s(Az,Tz) \leq s^2(\alpha+\beta+\gamma)\rho_s(Az,Tz)$ or $\rho_s(Az,Tz) = 0$ and hence $Az = Tz$. Thus

$$Az = Tz = z. \tag{2.13}$$

On the other hand, writing $x = z$ and $y = z$ in (2.4), and using (2.13),

$$\begin{aligned}
\rho_s(Sz,z) = \rho_s(Sz,Tz) &\leq \alpha\,\rho_s(Az,Az)+\beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(Az,Sz)]}{1+\rho_s(Az,Az)} \\
&\qquad + \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,Sz)}{1+\rho_s(Az,Tz)\rho_s(Az,Sz)} \\
&= \rho_s(Az,Sz)
\end{aligned}$$

so that $\rho_s(Sz, z) = 0$ or $Sz = z$. In other words, z is a common fixed point of A, S and T. Similarly, a common fixed point of A, S and T is obtained, if (A, T) is compatible.

It is not hard to establish the uniqueness of the common fixed point. □

Corollary 2.1. *Let T be a self-map on a b-metric space (X, ρ_s) with $s \geq 1$, satisfying the inequality*

$$\rho_s(Tx, Ty) \leq \alpha\,\rho_s(x, y) + \beta \cdot \frac{\rho_s(y, Ty)[1 + \rho_s(x, Tx)]}{1 + \rho_s(x, y)} + \gamma \cdot \frac{\rho_s(y, Ty) + \rho_s(y, Tx)}{1 + \rho_s(y, Ty)\rho_s(y, Tx)}, \text{ for all } x, y \in X, \tag{2.14}$$

where α, β and γ are non-negative numbers, not all being zero, such that

$$s^4\alpha + (s^4 + 1)\beta + (s^5 + s^4 + s)\gamma < 1. \tag{2.15}$$

If the space X is T-orbitally b-complete, then T has a unique fixed point.

Proof. We write $S = T$ and $A = I_X$ in Theorem 2.1, where I_X is the identity self-map on X. Note that I_X commutes with every map and hence (I, T) is compatible. Since every continuous function is T-orbitally continuous, by Theorem 2.1, T has a unique fixed point. □

The following result was proved in [6]:

Theorem 2.2. *Let T be a self-map on a complete b-metric space (X, ρ_s) with $s \geq 1$, satisfying the inequality* (2.14)*, where α, β and γ are non-negative numbers, not all being zero, such that*

$$s\alpha + \beta + \gamma < 1. \tag{2.16}$$

Then, T has a unique fixed point.

Remark 2.1. It may be noted that a complete b-metric space is T-orbitally b-complete at each of its points, and $s\alpha+\beta+\gamma < s^4\alpha+(s^4+1)\beta+(s^5+s^4+s)\gamma < 1$, a unique fixed point of T follows from Corollary 2.1. Therefore, Corollary 2.1 is a generalization of Theorem 2.2.

Since every complete metric space is orbitally complete, the following result of Dass and Gupta [3] follows from Corollary 2.1 with $s = 1$ and $\gamma = 0$:

Corollary 2.2. *Let T be a self-map on a complete metric space (X, d) satisfying the inequality*

$$d(Tx, Ty) \leq \alpha\,d(x, y) + \beta \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \text{ for all } x, y \in X, \tag{2.17}$$

where α and β are non-negative numbers, not both being zero, such that

$$\alpha + 2\beta < 1. \tag{2.18}$$

Then, T has a unique fixed point.

3. Conclusions

In the introductory section of this paper, a brief account of b-metric space and its relation with metric space is presented along with its topological properties. The highlights of Theorem 2.1 for three compatible self-maps on a b-metric space satisfying a rational type condition are the notions of asymptotic regularity, orbital completeness and orbital continuity. Also, the main result of this paper is an elegant extension of theorems of Sarwar and Rahman [6], and Dass and Gupta [3].

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References

[1] Antony Jessey, *Studies in fixed points and quasi-gauges,* thesis, Indian Institute of Tecnology, Madras, 1991.

[2] I.A. Bakhtin, *The contraction mapping principle in quasi-metric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., 30 (1989), 26-37.

[3] B.K. Dass, S. Gupta, *An extension of Banach contraction principle through rational expressions*, Indian J. Pure Appl. Math., 6 (1975), 1455–1458.

[4] Gerald Jungck, *Compatible self-maps and common fixed point*, Int. J. Math. & Math. Sci., 9 (1986), 771-779.

[5] J.R. Roshan, N. Shobkolaei, S. Sedghi, M. Abbas, *Common fixed point of four maps in b-metric spaces*, Hacettepe Journal of Mathematics and Statistics, 43 (2014), 613-624.

[6] Muhammad Sarwar, Mujeeb Ur Rahman, *Fixed point theorems for Ciric's and generalized contractions in b-metric spaces*, Int. Jour. Anal. & Appl., 7 (2015), 70-78.

[7] S. Sessa, *On weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. Debre., 32 (1982), 149-153.

[8] B. Wu, F. He, T. Xu, *Common fixed point theorems for Ciric type mappings in b-metric spaces without any completeness assumption*, J. Nonlinear Sci. Appl., 10 (2017), 3180-3190.

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