On divisor labeling of co-prime order graphs of finite groups

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Abstract. The co-prime order graph of a finite group G is an undirected graph whose vertex set is G and two distinct vertices $u, v \in G$ are adjacent if $gcd(o(u), o(v)) = 1$ or a prime number. Labeling a graph is the process of assigning integers to its vertices and/or edges subject to certain conditions. In other words, vertex (edge) labeling is a function of the set of vertices (edges) to a set of labels (generally integers). A graph Γ is a divisor graph if all its vertices can be labeled with positive integers such that two distinct vertices x and y are adjacent if and only if $x|y$ or $y|x$. This paper focuses on some conditions under which the co-prime order graphs of finite groups, especially abelian groups and permutation groups, are divisor graphs.

Keywords: divisor graph, co-prime order graph, labeling.

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1. Introduction

For a graph Γ, we denote its vertex set and edge set by $V(\Gamma)$ and $E(\Gamma)$ respectively. In a directed graph, we use (u, v) for a directed edge from u to v, the in-degree of a vertex v is the number of edges coming to the vertex v and the out-degree of a vertex v is the number of edges going out from the vertex v. Further, in a digraph D, a vertex with zero in-degree (out-degree) is called a transmitter (receiver), whereas a vertex v with positive in-degree and positive out-degree is called a transitive vertex if $(u, w) \in E(D)$ whenever (u, v) and (v, w) belong to $E(D)$, see [1]. If G is a graph whose vertex set is V and S is a non-empty subset of V, then the subgraph of G having vertex set S and edge set as the set of those edges of G that have both ends in S is called the subgraph of G induced by S. For more details of graph theory, the reader may refer to Bondy and Murty [2].

Singh and Santosh [3] conceptualized divisor graphs for non-empty sets of integers. Assume that S is a finite non-empty set of integers. The divisor graph $G(S)$ of S is a graph with vertex set S such that two distinct vertices x and y are adjacent if either $x|y$ or $y|x$. Further, the divisor digraph $D(S)$ of S has vertex set S and (x, y) is an arc of $D(S)$ if $x|y$. A graph Γ is called a divisor graph if Γ is isomorphic to $G(S)$ for some finite non-empty set S of integers. Chartrand et al. [4] studied the divisor graphs in terms of non-empty sets of positive integers. The term divisor graph used in the paper is in the same sense as in [4]. Thus, if Γ is a divisor graph, then there exists a function $f : V(\Gamma) \to \mathbb{N}$ such that Γ is isomorphic to $G(f(V(\Gamma)))$. Such a function f is called a divisor labeling of the graph Γ. Divisor graphs associated with algebraic structures have also caught the attention of researchers. Osba and Alkam [5] worked on the necessary and sufficient conditions for the zero-divisor graphs of a class of rings to be divisor graph. Recently, Takshak et al. [6] showed that the power graph of a finite group is always a divisor graph but the converse is not true.

In 2021, Banerjee [7] introduced the co-prime order graph of a group G as the graph whose vertex set is G and two distinct vertices x, y are adjacent if $gcd(o(x), o(y))$ is either 1 or a prime number. Since then many researchers [8, 9, 10, 11] have studied co-prime order graphs and have shown their utility in characterizing finite groups.

In this paper, we shall find out some conditions under which the co-prime order graphs of finite groups (especially abelian groups and permutation groups) are/are not divisor graphs. All graphs considered in this paper are finite and simple.

2. Preliminaries

In this section, we state some relevant notations and basic results used in the paper. If G is a group and g is an arbitrary element of G, then their orders are denoted by $o(G)$ and $o(g)$ respectively. S_n denotes the permutation group of degree n. $\Theta(G)$ shall denote the co-prime order graph of the group G. Further, $\Gamma_1 \vee \Gamma_2$ represents the join of graphs Γ_1 and Γ_2. The complete graph on n vertices is denoted by K_n and $K_{n_1,n_2,\ldots,n_k}$ denotes the complete k-partite graph.

Now we state some well-known results on divisor graphs.

Theorem 2.1 ([4]). *Let Γ be a graph. Then Γ is a divisor graph if and only if there exists an orientation D of Γ such that every vertex of D is a transmitter, a receiver or a transitive vertex.*

Theorem 2.2 ([4]). *Every induced subgraph of a divisor graph is a divisor graph.*

Theorem 2.3 ([4]). *If Γ_1 and Γ_2 are two divisor graphs, then $\Gamma_1 \vee \Gamma_2$ is a divisor graph.*

Theorem 2.4. *Let Γ_1 and Γ_2 be two divisor graphs whose vertex sets are disjoint, then $\Gamma_1 \cup \Gamma_2$ is also a divisor graph.*

Theorem 2.5 ([4, 5]). *A graph that contains the following (Figure 1) induced subgraph is not a divisor graph.*

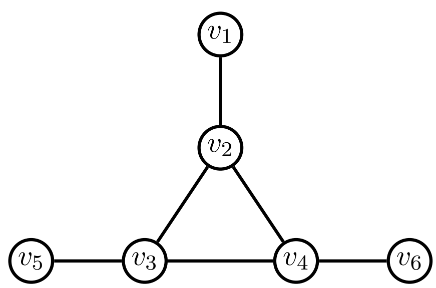

Figure 1

3. Main results

We begin this section with the following observation:

Let Γ be a graph having $\{a_1, a_2, \ldots, a_{n_1}, b_1, b_2, \ldots, b_{n_2}, c_1, c_2, \ldots, c_{n_3}, d_1, d_2, \ldots, d_{n_4}, e_1, e_2, \ldots, e_{n_5}\}$ as the vertex set s.t. its orientation is represented by Figure 2.

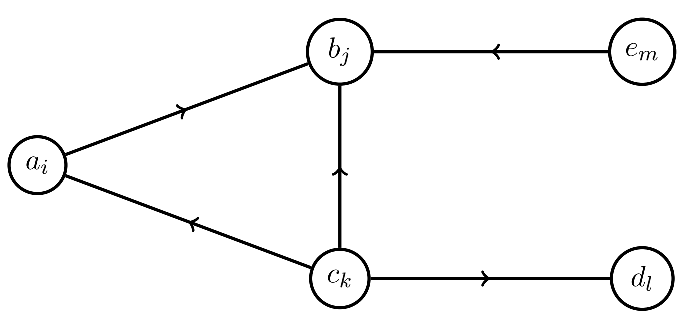

Figure 2

It is obvious that each a_i is a transitive vertex. Further, c_k's and e_m's are transmitters and b_j's and d_l's are receivers. Thus, each of the vertices of Γ is either a receiver, a transmitter or a transitive vertex. Hence, Γ is a divisor graph by Theorem 2.1.

Theorem 3.1. *Let S be a subset of a finite group such that the order of its every element divides ${p_1}^m{p_2}^n$, where p_1 and p_2 are distinct primes and $m, n \in \mathbb{N}$, then $\Theta(S)$ is a divisor graph.*

Proof. Firstly, consider the case wherein there exist $x_i, y_j, z_k, \alpha_l, \beta_r, \gamma_s, \delta_t \in S$ such that

- $o(x_i) = 1$ or p_1 or p_2, where $1 \leq i \leq n_1$;
- $o(y_j) = {p_1}^2$ or ${p_1}^3 \ldots$ or ${p_1}^m$, where $1 \leq j \leq n_2$;
- $o(z_k) = {p_2}^2$ or ${p_2}^3 \ldots$ or ${p_2}^n$, where $1 \leq k \leq n_3$;
- $o(\alpha_l) = p_1p_2$, where $1 \leq l \leq n_4$;
- $o(\beta_r) = {p_1}^2p_2$ or ${p_1}^3p_2 \ldots$ or ${p_1}^mp_2$, where $1 \leq r \leq n_5$;
- $o(\gamma_s) = p_1{p_2}^2$ or $p_1{p_2}^3 \ldots$ or $p_1{p_2}^n$, where $1 \leq s \leq n_6$;
- $o(\delta_t) = {p_1}^2{p_2}^2$ or ${p_1}^2{p_2}^3 \ldots$ or ${p_1}^m{p_2}^n$, where $1 \leq t \leq n_7$.

Now, let us partition the vertex set of graph $\Theta(S)$ into three mutually disjoint sets A, B and C, where

$A = \{x_1, x_2, \ldots, x_{n_1}\}$,
$B = \{y_1, y_2, \ldots, y_{n_2}, z_1, z_2, \ldots, z_{n_3}, \alpha_1, \alpha_2, \ldots, \alpha_{n_4}, \beta_1, \beta_2, \ldots, \beta_{n_5}, \gamma_1, \gamma_2, \ldots, \gamma_{n_6}\}$,
$C = \{\delta_1, \delta_2, \ldots, \delta_{n_7}\}$.

Let Γ_1, Γ_2 (Figure 3) and Γ_3 denote the subgraphs of $\Theta(G)$ induced by A, B and C respectively. So, we have

$$\Theta(S) = \Gamma_1 \vee (\Gamma_2 \cup \Gamma_3).$$

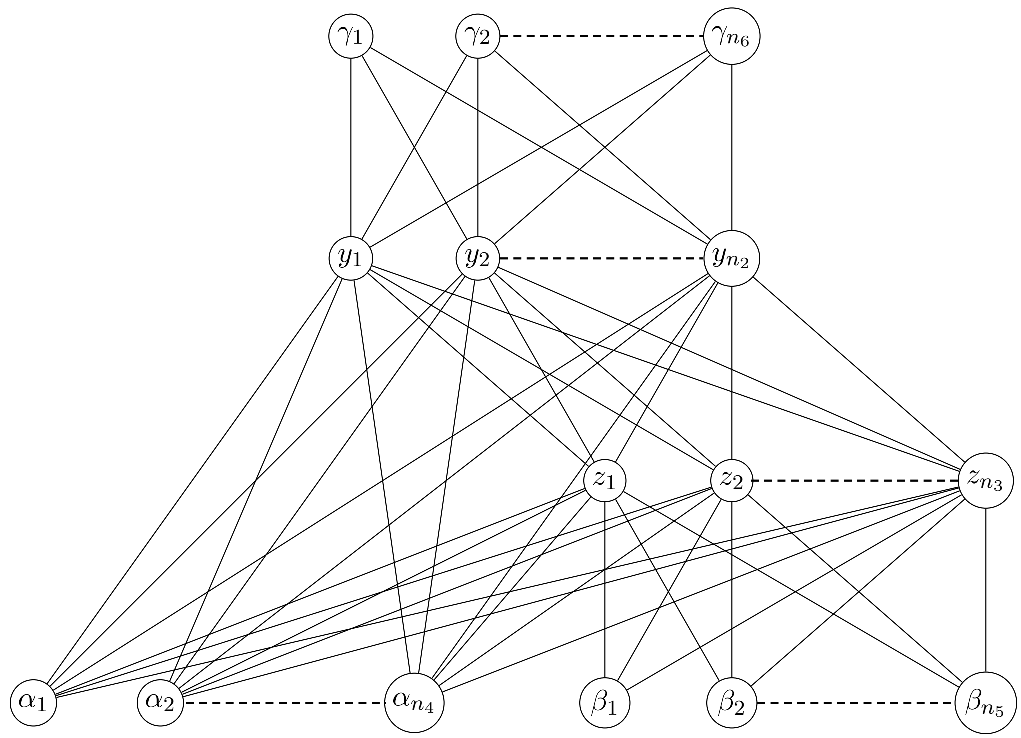

Figure 3: Subgraph Γ_2

Consider the following orientation of Γ_2 :
For $j \in \{1, 2, \ldots, n_2\}$, $k \in \{1, 2, \ldots, n_3\}$, $l \in \{1, 2, \ldots, n_4\}$, $r \in \{1, 2, \ldots, n_5\}$ and $s \in \{1, 2, \ldots, n_6\}$, we take $(y_j, z_k), (y_j, \alpha_l), (y_j, \gamma_s)$, (α_l, z_k) and (β_r, z_k) as edges of Γ_2.

As this orientation of Γ_2 is similar to that of Γ (Figure 2), it is a divisor graph. Further, as $\Gamma_1 \cong K_{n_1}$ and $\Gamma_3 \cong n_7K_1$, so Γ_1 and Γ_3 are also divisor graphs. Hence, $\Theta(S)$ is a divisor graph in this case.

In each of the remaining cases, the co-prime order graph of G is nothing but an induced subgraph of $\Theta(G)$ considered in the above case, hence a divisor graph by Theorem 2.2. □

Corollary 3.1. *If order of every element of a finite group G divides ${p_1}^m{p_2}^n$, where p_1 and p_2 are distinct prime numbers and $m, n \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

Corollary 3.2. *If G is a group of order ${p_1}^m{p_2}^n$, where p_1 and p_2 are distinct prime numbers and $m, n \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

The following result can be proved by proceeding as in Theorem 3.1:

Theorem 3.2. *Let G be a finite group s.t. $o(G) = p^m$, where p is a prime number and $m \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

Theorem 3.3. *Assume that the order of every element of a finite group G divides $p_1p_2p_3$, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is a divisor graph.*

Proof. As in Theorem 3.1, it is sufficient to prove the result in the following case:

Let there exist $x_i, y_j, z_k, \alpha_l, \beta_m \in G$ s.t.

- $o(x_i) = 1$ or p_1 or p_2 or p_3, where $1 \leq i \leq n_1$;
- $o(y_j) = p_1p_2$, where $1 \leq j \leq n_2$;
- $o(z_k) = p_1p_3$, where $1 \leq k \leq n_3$;
- $o(\alpha_l) = p_2p_3$, where $1 \leq l \leq n_4$;
- $o(\beta_m) = p_1p_2p_3$, where $1 \leq m \leq n_5$.

Now, we partition the vertex set of graph $\Theta(G)$ into three mutually disjoint subsets $\{x_1, x_2, \ldots, x_{n_1}\}$, $\{y_1, y_2, \ldots, y_{n_2}, z_1, z_2, \ldots, z_{n_3}, \alpha_1, \alpha_2, \ldots, \alpha_{n_4}\}$ and $\{\beta_1, \beta_2, \ldots, \beta_{n_5}\}$. Let Γ_4, Γ_5 and Γ_6 respectively denote the subgraphs of $\Theta(G)$ induced by these sets. It follows that $\Theta(G) = \Gamma_4 \vee (\Gamma_5 \cup \Gamma_6)$.

Further, Γ_4, Γ_5 and Γ_6 are divisor graphs as $\Gamma_4 \cong K_{n_1}, \Gamma_5 \cong K_{n_2,n_3,n_4}$ and $\Gamma_6 \cong n_5K_1$. Hence, $\Theta(G)$ is also a divisor graph. □

Corollary 3.3. *Let G be a group of order ${p_1}^{m_1}{p_2}^{m_2}{p_3}^{m_3}$ such that it has no element of order ${p_1}^2$ or ${p_2}^2$ or ${p_3}^2$, then $\Theta(G)$ is a divisor graph.*

Theorem 3.4. *Assume that a finite group G contains at least one element of order $p_1p_2, p_1p_3, p_1p_4, p_1p_2p_3, p_1p_2p_4$ and $p_1p_3p_4$ each, where p_1, p_2 and p_3 are distinct prime numbers. Then $\Theta(G)$ is not a divisor graph.*

Proof. Let x_1, x_2, x_3, x_4, x_5 and x_6 be elements of G whose orders are $p_1p_2, p_1p_3, p_1p_4, p_1p_2p_3, p_1p_2p_4$ and $p_1p_3p_4$ respectively. Then, the subgraph of $\Theta(G)$ induced by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is isomorphic to the graph represented by Figure 1. So, by Theorem 2.2, $\Theta(G)$ is not a divisor graph. □

Corollary 3.4. *Let G be an abelian group such that $o(G) = n$ and ${p_1}^{m_1}{p_2}^{m_2} \ldots {p_k}^{m_k}$ be prime power decomposition of n. Then, for $k \geq 4$, $\Theta(G)$ is not a divisor graph.*

Theorem 3.5. *If a finite group G contains at least one element of order $p_1p_2, p_1p_3, {p_1}^2, {p_1}^2p_2, {p_1}^2p_3$ and $p_1p_2p_3$ each, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is not a divisor graph.*

Proof. Let $x_1, x_2, x_3, x_4, x_5, x_6 \in G$ such that their orders are $p_1p_2, p_1p_3, {p_1}^2, {p_1}^2p_2, {p_1}^2p_3$ and $p_1p_2p_3$ respectively. Considering the subgraph of $\Theta(G)$ induced by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and proceeding as in Theorem 3.4, it follows that $\Theta(G)$ is not a divisor graph. □

Corollary 3.5. *If a group G contains at least one element of order $p_1{}^2p_2p_3$, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is not a divisor graph.*

Corollary 3.6. *Let G be an abelian group of order n and $p_1{}^{m_1}p_2{}^{m_2}p_3{}^{m_3}$ be prime power decomposition of n. If G contains at least one element of order $p_1{}^2$ or $p_2{}^2$ or $p_3{}^2$, then $\Theta(G)$ is not a divisor graph.*

Corollary 3.7. *If $n \geq 10$, then $\Theta(S_n)$ is not a divisor graph.*

Proof. Consider $x_1 = (1, 2)(3, 4, 5), x_2 = (1, 2)(3, 4, 5, 6, 7), x_3 = (1, 2, 3, 4), x_4 = (1, 2, 3, 4)(5, 6, 7), x_5 = (1, 2, 3, 4)(5, 6, 7, 8, 9)$ and $x_6 = (1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$. Then, x_1, x_2, x_3, x_4, x_5 and x_6 are elements of S_n with orders $6, 10, 4, 12, 20$ and 30 respectively and using the above theorem, it can be concluded that $\Theta(S_n)$ is not a divisor graph for $n \geq 10$. □

The following result is an implication of the results discussed above:

Theorem 3.6. *Let G be an abelian group of order n and $p_1{}^{m_1}p_2{}^{m_2}\dots p_k{}^{m_k}$ be the prime decomposition of n, then $\Theta(G)$ is divisor graph if and only if $k \leq 3$, with the condition that if $k = 3$, then G contains no element of order $p_1{}^2$ or $p_2{}^2$ or $p_3{}^3$.*

Theorem 3.7. *If a finite group G contains no element whose order is other than $1, p_1, p_2, p_3, p_4, p_1p_2, p_1p_3, p_1p_4, p_2p_3, p_1{}^2, p_2{}^2, p_1{}^3, p_1{}^2p_2, p_1{}^2p_3$, where p_1, p_2, p_3 and p_4 are distinct prime numbers, then $\Theta(G)$ is a divisor graph.*

Proof. Let there exist $x_i, y_j, z_k, w_s, \alpha_l, \beta_r, \gamma_m, \delta_n, u_q \in G$ s.t.

- $o(x_i) = 1$ or p_1 or p_2 or p_3 or p_4, where $1 \leq i \leq n_1$;
- $o(y_j) = p_1p_4$, where $1 \leq j \leq n_2$;
- $o(z_k) = p_2p_3$, where $1 \leq k \leq n_3$;
- $o(w_s) = p_2{}^2$, where $1 \leq s \leq n_4$;
- $o(\alpha_l) = p_1p_3$, where $1 \leq l \leq n_5$;
- $o(\beta_r) = p_1p_2$, where $1 \leq r \leq n_6$;
- $o(\gamma_m) = p_1{}^2p_2$, where $1 \leq m \leq n_7$;
- $o(\delta_n) = p_1{}^2p_3$, where $1 \leq n \leq n_8$;
- $o(u_q) = p_1{}^2$ or $p_1{}^3$, where $1 \leq q \leq n_9$.

We write $V(\Theta(G)) = D \cup E \cup F$, where D, E and F are three mutually disjoint sets given by:

$D = \{x_1, x_2, \dots, x_{n_1}\}$,
$E = \{y_1, y_2, \dots, y_{n_2}, z_1, z_2, \dots, z_{n_3}, w_1, w_2, \dots, w_{n_4}\}$,
$F = \{\alpha_1, \alpha_2, \dots, \alpha_{n_5}, \beta_1, \beta_2, \dots, \beta_{n_6}, \gamma_1, \gamma_2, \dots, \gamma_{n_7}, \delta_1, \delta_2, \dots, \delta_{n_8}, u_1, u_2, \dots, u_{n_9}\}$.

Consider Γ_7, Γ_8 and Γ_9 (Figure 4), the subgraphs of $\Theta(G)$ induced by D, E and F respectively. Clearly, Γ_7 and Γ_8 are divisor graphs as $\Gamma_7 \cong K_{n_1}$ and $\Gamma_8 \cong K_{n_2,n_3,n_4}$. Also, we have $\Theta(G) = (\Gamma_7 \vee \Gamma_8) \vee \Gamma_9$.

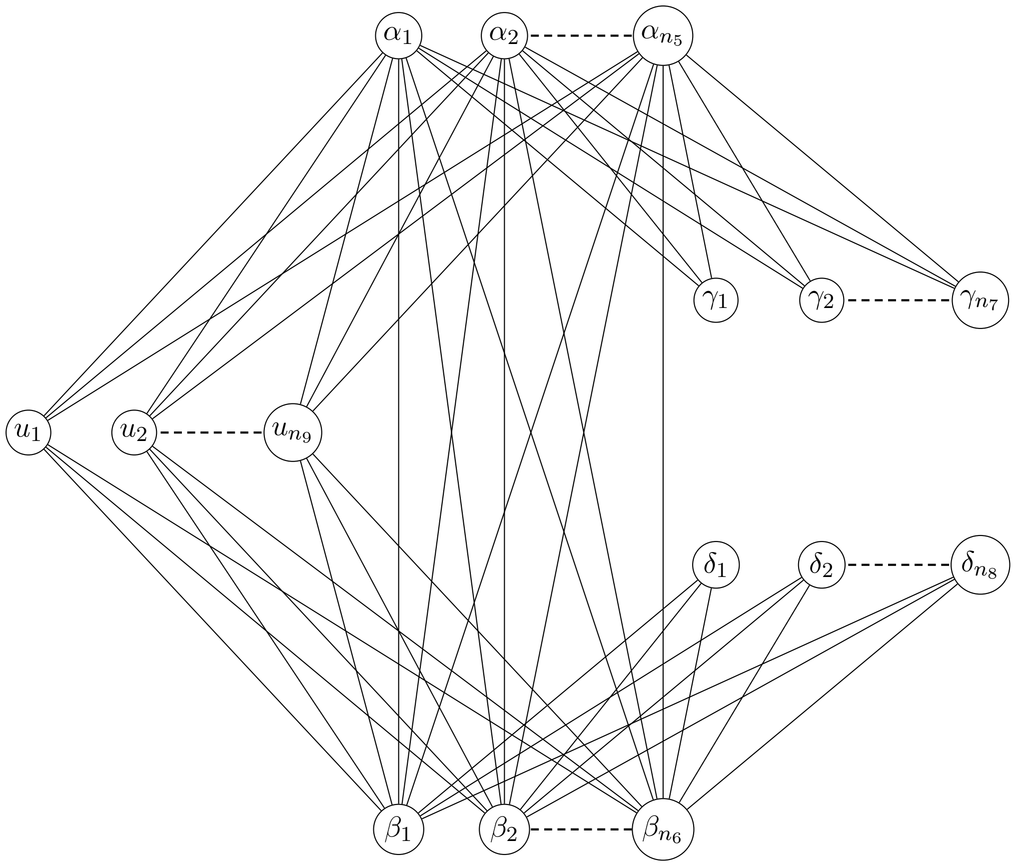

Figure 4: Subgraph Γ_9

Consider an orientation of the subgraph Γ_9 as stated below:
For every $l \in \{1, 2, \dots, n_5\}$, $r \in \{1, 2, \dots, n_6\}$, $m \in \{1, 2, \dots, n_7\}$, $n \in \{1, 2, \dots, n_8\}$ and $q \in \{1, 2, \dots, n_9\}$, we take $(\alpha_l, u_q), (\alpha_l, \beta_r), (\alpha_l, \gamma_m)$, (u_q, β_r) and (δ_n, β_r) as edges of Γ_9. Then, proceeding as in Theorem 3.1, it can be shown that the subgraph Γ_9, and hence $\Theta(G)$, is a divisor graph. □

Corollary 3.8. *For $n \leq 9$, then $\Theta(S_n)$ is a divisor graph.*

Proof. It is easy to check that for $n \leq 9$, the order of each element of S_n belongs to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 20\}$. In the above theorem, if we take $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_4 = 7$ then, $\Theta(S_n)$ becomes an induced subgraph of $\Theta(G)$. Thus, S_n is a divisor graph for $n \leq 9$. □

It follows from the Corollary 3.7 and Corollary 3.8 that:

Theorem 3.8. *$\Theta(S_n)$ is a divisor graph if and only if $n \leq 9$.*

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