Structural invariants of the product maximal graph

G. Ramya*
Department of Mathematics
Bharathiar University PG Extension and Research Centre
Erode, Tamil Nadu
India
ramyamphil1989@gmail.com

D. Kalamani
Department of Mathematics
Bharathiar University PG Extension and Research Centre
Erode, Tamil Nadu
India
kalamanikec@gmail.com

Abstract. In this paper, some structural properties of the product maximal graph like matching, vertex covering, edge covering and cordial labeling are studied. Furthermore, the number of triangles of $\Gamma_{pm}(R)$ are calculated. The isomorphism between the product maximal graph of cartesian product of two commutative rings and cartesian product of two product maximal graphs of commutative rings and its relations is interpreted with an example.

Keywords: product maximal graph, covering, matching, labeling, cartesian product.
MSC 2020: 05C25, 05C78, 05C15, 05C40

1. Introduction

A graph G is an ordered pair (V, E), where $V = V(G)$ is a set of elements called vertices, $E = E(G)$ is a set of elements called edges and each edge is an unordered pair of vertices (its ends or end vertices or end-points). Graph theory has become a very popular and promptly increasing area of discrete mathematics for its numerous theoretical development and manifold applications to the practical problems. Graphs constructed from algebraic structures have been studied extensively by many authors and have become a major field of research.

Groups as graphs contain the most merging combination which is used repeatedly in the algebraic graph theory. The graphs from groups include power graph, commuting graph, non-commuting graph etc., Another important kind of graph construction is the construction of graphs from rings. The study of graphs from rings contributes to the interplay between the ring invariants and the graph structure. Graphs from rings are introduced by Beck. I [2] and it is

*. Corresponding author

named as the zero-divisor graphs of a finite commutative ring. The complete summary of graphs from rings and the results are found in [4]. Some graphs [11, 12, 13] may represent the molecular structure of certain chemical compound and it is mainly associated with the different molecular biology.

Moreover, D. Kalamani and G. Ramya [10] defined a new graph from ring called product maximal graph. It is a graph of a finite commutative ring with unity whose vertices are all the elements of ring R and two distinct vertices are adjacent if and only if the product of two vertices are in maximal ideals of R. They also extended the graph properties such as domination number [14] and some graph theoretic properties [9] of the product maximal graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labelled graphs serve useful mathematical models for a broad range of applications. Graph labeling is useful in network problems because each network node has a different transmission capacity for sending or receiving messages in wired or wireless link. Most of the details related to graph labeling and different methods of labeling like cordial labeling, graceful labeling, hormonius labeling are in [8]. G. Ramya and D. Kalamani [15] establish that the commuting graph of the subset of the dihedral group admits sum cordial, signed product cordial and divisor cordial labeling. Further notation and terminologies are followed from Frank Harary [7] and Douglas B. West [6] for graphs and from Dummit and Foote [5] for algebra concepts.

2. Preliminaries

In this section, the essential definitions of the matching, covering, labeling, cordial labeling and cartesian product are specified. Also some basic properties of the covering and product maximal graph are given.

Definition 2.1. *Let R be a finite commutative ring with unity and $M_1, M_2, \dots, M_r$ be the maximal ideals of R. The* ***product maximal graph*** *of a commutative ring R is the graph whose vertices are the elements of R and two distinct vertices u and v are adjacent if and only if the product $uv \in M_i, i = 1, 2, \dots, r$ and it is denoted by $\Gamma_{pm}(R)$.*

Definition 2.2. *A subset $\mathscr{M}$ of the edge set E is called a* ***matching or edge independent set*** *in G if no two edges of $\mathscr{M}$ are adjacent in a graph G. The two ends of an edge in $\mathscr{M}$ are said to be matched under $\mathscr{M}$.*

Definition 2.3. *A matching $\mathscr{M}$ is a* ***maximum matching*** *if a graph G has no matching $\mathscr{M}'$ with $|\mathscr{M}'| > |\mathscr{M}|$. The number of edges in a maximum matching of G is called the* ***matching number*** *(edge independent number) of a graph G. It is denoted by $\alpha'(G)$.*

Definition 2.4. *A set S of vertices which covers all the edges of a graph G is called* ***vertex cover****, in the sense that every edge of G is incident with some vertex in S. A vertex cover with minimum cardinality is the minimum vertex* *cover, the cardinality of minimum vertex cover is called the* ***vertex covering number*** *and it is denoted by* $\beta(G)$ *for the graph* G*.*

Definition 2.5. *A set* S *of edges which covers all the vertices of a graph* G *is called* ***edge cover*** *of* G*. A minimum edge cover is one with minimum cardinality. The cardinality of a minimum edge cover of a graph* G *is called the* ***edge covering number*** *and it is denoted by* $\beta'(G)$*. The graph without isolated vertices have an edge cover.*

Definition 2.6. *Let* $G = (V, E)$ *be a graph. A mapping* $f : V(G) \to \{0, 1\}$ *is called* ***binary vertex labeling*** *of* G *and* $f(v)$ *is called the label of the vertex* v *of* G *under* f*.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having label 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges having label 0 and 1 respectively under f^*.

Definition 2.7. *A vertex labeling* $f : V(G) \to \{0, 1\}$ *and the induced edge labeling* $f^* : E(G) \to \{0, 1\}$ *is given by* $f^*(uv) = |f(u) - f(v)|$*. Such labeling is called* ***cordial labeling*** *if* $|v_f(0) - v_f(1)| \leq 1$ *and* $|e_f(0) - e_f(1)| \leq 1$*. A graph* G *is cordial if it admits cordial labeling.*

Definition 2.8. *A binary vertex labeling of a graph* G *with induced edge labeling* $f^* : E(G) \to \{0, 1\}$ *is defined by* $f^*(uv) = (f(u) + f(v))(mod 2)$ *is named as* ***sum cordial labeling*** *if* $|v_f(0) - v_f(1)| \leq 1$ *and* $|e_f(0) - e_f(1)| \leq 1$*. A graph* G *is sum cordial if it admits sum cordial labeling.*

Definition 2.9. *A vertex labeling,* $f : V(G) \to \{-1, 1\}$ *of a graph* G *with induced edge labeling* $f^* : E(G) \to \{-1, 1\}$ *defined by* $f^*(uv) = f(u)f(v)$ *is called a* ***signed product cordial labeling*** *if* $|v_f(-1) - v_f(1)| \leq 1$ *and* $|e_f(-1) - e_f(1)| \leq 1$ *. A graph* G *is signed product cordial if it admits signed product cordial labeling.*

Definition 2.10. *A* ***divisor cordial labeling*** *of a graph* G *with the vertex set* V *is a bijection* $f : V \to \{1, 2, \ldots, |V|\}$ *such that if each edge* uv *is assigned the label* 1 *if* $f(u)|f(v)$ *or* $f(v)|f(u)$ *and 0 otherwise. A graph* G *is divisor cordial if it admits divisor cordial labeling.*

Definition 2.11. *An* ***isomorphism*** *from a simple graph* G *to a simple graph* H *is a bijection* $f : V(G) \to V(H)$ *such that* $uv \in E(G)$ *iff* $f(u)f(v) \in E(H)$*.*

Definition 2.12. *The* ***cartesian product*** *of* G *and* H*, written as* $G \times H$ *is the graph with vertex set* $V(G) \times V(H)$ *specified by putting* (u, v) *adjacent to* (u', v') *iff*

- $u = u'$ *and* $vv' \in E(H)$ *or*

- $v = v'$ *and* $uu' \in E(G)$.

Theorem 2.1 ([6]). *In a graph G, the subset S is an independent set iff $\overline{S}$ is a vertex cover and hence $\alpha(G) + \beta(G) = n$.*

Theorem 2.2 ([6]). *If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n$, where n is the number of vertices in G.*

Lemma 2.1 ([10]). *Let R be a finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. Let $\Gamma_{pm}(R)$ be a product maximal graph of R. Then, the degree of vertex v of the graph $\Gamma_{pm}(R)$ is given by*

$$deg(v) = \begin{cases} n-1, & v \in M_i, i = 1, 2, \ldots, r, \\ m, & otherwise, \end{cases}$$

where m and n are the cardinalities of $M = \bigcup_{i=1}^{r} M_i$ and R respectively.

Lemma 2.2 ([10]). *Let $\Gamma_{pm}(R)$ be the product maximal graph of a finite commutative ring R. Then, the independent number is $\alpha(\Gamma_{pm}(R)) = n - m$, where m and n are the cardinalities of $M = \bigcup_{i=1}^{r} M_i$ and R respectively.*

3. Main results

In this section, the number of triangles of the product maximal graph and some graph theoretic properties like matching, covering, cordial labeling are discussed. Also the product maximal graph of a finite commutative ring which is isomorphic to the product maximal graph of cartesian product of two commutative rings is found. Moreover, the relation between product maximal graph of cartesian product of two rings and cartesian product of two product maximal graphs are established.

Theorem 3.1. *Let R be a commutative ring of order n, where n is not a prime then the number of triangles for the product maximal graph $\Gamma_{pm}(R)$ is $mC_2(n-m) + mC_3$, where m is the number of elements in $\bigcup_{i=1}^{r} M_i$.*

Proof. We know that the elements which are in maximal ideals form the complete subgraph.

Note that, the number of triangles for the complete graph is mC_3,

$$\therefore \triangle(K_m) = mC_3.$$

Now, the elements which are in non-maximal ideals are adjacent to the elements in maximal ideals. Therefore, $(n-m)$ vertices of non-maximal elements form the triangles with mC_2 vertices of maximal elements. Hence, $\triangle(\Gamma_{pm}(R)) = mC_2(n-m) + mC_3$. $\square$

Corollary 3.1. *If p is a prime, then the product maximal graph of a finite commutative ring of order p has no triangles.*

3.1 Matching number of the product maximal graph

Graph matching has applications in flow networks, scheduling and planning, modeling bonds in chemistry, the stable marriage problem, neural networks in artifical intelligance. A matching is a subset of the edge set such that no two edges have a common vertex. Any matching with the largest size in G is called a maximum matching. A maximal matching in a graph is a matching that cannot be enlarged by adding an edge. Every maximum matching is a maximal matching but the converse need not hold. The matching number of $\Gamma_{pm}(R)$ is shown in Theorem 3.3.

Theorem 3.2. *Let R be the finite commuatative ring with unity and $M_1, M_2, \ldots,$ M_r be the maximal ideals of R then the matching number of the product maximal graph is*

$$\alpha'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ m, & n \text{ is odd}, \end{cases}$$

where n is the number of vertices in $\Gamma_{pm}(R)$ and m is the number of elements in $M = \bigcup_{i=1}^{r} M_i$.

Proof. Let $E(\Gamma_{pm}(R))$ be the edge set of the product maximal graph.

The matching set $\mathscr{M}$ is the subset of the edge set $E(\Gamma_{pm}(R))$ and the end points of the edges of $\mathscr{M}$ are obtained in the following ways:

(i) both ends are in maximal ideals.

(ii) one end is in maximal ideal and other end is in non-maximal ideal.

Denote the subset of $\mathscr{M}$ defined by (i) as $\mathscr{M}_1$ and the subset defined by (ii) as $\mathscr{M}_2$. Clearly the matching set $\mathscr{M}$ is the disjoint union of two subsets $\mathscr{M}_1$ and $\mathscr{M}_2$ of the edge set $E(\Gamma_{pm}(R))$. The edges in $\mathscr{M}_1$ and $\mathscr{M}_2$ are all independent.

Case 1. n is even. In this case, $|M| \geq |\overline{M}|$ and m is even.

If $|M| = |\overline{M}|$, then the number of edges in $\mathscr{M}_1$ various from 0 to $\frac{m}{2}$ and the number of edges in $\mathscr{M}_2$ various from 0 to $\frac{n}{2}$. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m}{2} \leq k \leq \frac{n}{2}$. From this, it is clear that the maximum matching set $\mathscr{M}$ has $\frac{n}{2}$ edges and the matching number is $\frac{n}{2}$

$$\alpha'(\Gamma_{pm}(R)) = \frac{n}{2}.$$

If $|M| > |\overline{M}|$, then the number of edges in $\mathscr{M}_1$ various from $m - \frac{n}{2}$ to $\frac{m}{2}$ and the number of edges in $\mathscr{M}_2$ various from 0 to $n - m$. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m}{2} \leq k \leq \frac{n}{2}$. From this, it is clear that the maximum matching set $\mathscr{M}$ has $\frac{n}{2}$ edges and the matching number is $\frac{n}{2}$

$$\alpha'(\Gamma_{pm}(R)) = \frac{n}{2}.$$

Case 1. n is odd. In this case, $|M| < |\overline{M}|$ and m is odd.

The number of edges in $\mathscr{M}_1$ various from 0 to $\frac{m-1}{2}$ and the number of edges in $\mathscr{M}_2$ various from 1 to m. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m+1}{2} \leq k \leq m$. From this, it is clear that the maximum matching set $\mathscr{M}$ has m edges and the matching number is m.

Hence, the maximum matching number of the product maximal graph is $\alpha'(\Gamma_{pm}(R)) = m$. □

Example 3.1. The maximum matching of the product maximal graph $\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z})$ are shown below:

- Let $R = \mathbb{Z}/8\mathbb{Z}$. The maximal ideal of $\mathbb{Z}/8\mathbb{Z}$ is $M_1 =< 2 >$. Here, n is even and $|M| = |\overline{M}|$ then the matching set $\mathscr{M}$ has 4 edges. $\mathscr{M} = \{(0,1),(2,7),(3,6),(4,5)\}$ is one of the maximum matching set and hence $\alpha'(\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})) = 4$ and the matching is shown in Figure 1 with colored lines.

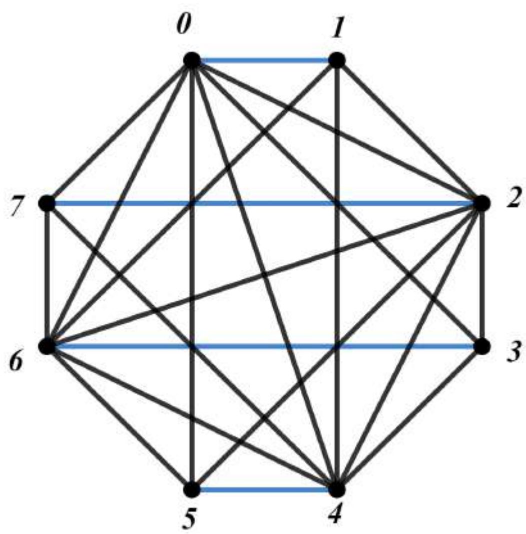

Figure 1: The Graph $\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})$ with matching.

- Let $R = \mathbb{Z}/9\mathbb{Z}$. The maximal ideal of $\mathbb{Z}/9\mathbb{Z}$ is $M_1 =< 3 >$. Here, n is odd and $|M| < |\overline{M}|$ then the matching set $\mathscr{M}$ may have either 2 or 3 edges. One of the maximum matching set of the graph is $\mathscr{M} = \{(0,1),(3,5),(6,8)\}$ and hence $\alpha'(\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z}) = 3$ and the matching is shown in Figure 2 with colored lines.

3.2 Vertex covering and edge covering of the product maximal graph

Graph covering is one of the classical topics in graph theory. The vertex covering problem, matching number problem are said to be classical optimization problem

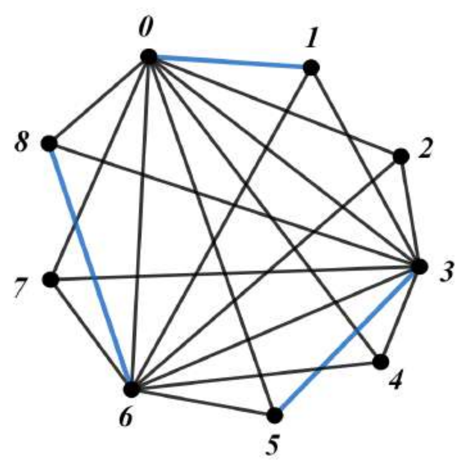

Figure 2: The Graph $\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z})$ with matching.

in computer science. A covering graph is a subgraph that has either all the vertices or all the edges belonging to another graph. Edge covering refers to a subgraph that has all of the vertices. Vertex covering number and edge covering number of the product maximal graph are shown in the following theorems.

Theorem 3.3. *Let $\Gamma_{pm}(R)$ be the product maximal graph of the finite commutative ring R with unity. The vertex covering number of the product maximal graph is $\beta(\Gamma_{pm}(R)) = m$.*

Proof. Let n be the number of vertices in the product maximal graph and m be the number of elements in $M = \bigcup_{i=1}^{r} M_i$.

The vertex cover is the subset of the vertex set of the product maximal graph which covers all the edges of the graph $\Gamma_{pm}(R)$. By Theorem 2.1, the complement of the independent set is a vertex covering set of the graph $\Gamma_{pm}(R)$. By Lemma 2.3, the independent set for the product maximal graph is $I = \{v \in \Gamma_{pm}(R) | v \notin M\}$.

The independence number of $\Gamma_{pm}(R)$ is $n - m$. i.e., $|I| = n - m$. Now, the complement of the independent set of the product maximal graph is $\overline{I} = \{v \in \Gamma_{pm}(R) | v \in M\}$ and its cardinality is m.
Hence, the vertex covering number of the product maximal graph is m.

$\beta(\Gamma_{pm}(R)) = m$. $\square$

Theorem 3.4. *Let R be the finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. The edge covering number of the product maximal graph is*

$$\beta'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ n - m, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. An edge cover is the subset of the edge set which covers all the vertices in the graph $\Gamma_{pm}(R)$. By Theorem 2.2, the edge covering number of $\Gamma_{pm}(R)$ is the complement of the matching number in the product maximal graph.

$\therefore$ The edge covering number of $\Gamma_{pm}(R)$ is

$$\beta'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ n-m, & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Example 3.2. Let $R = \mathbb{Z}/12\mathbb{Z}$ whose maximal ideals are $M_1 =< 2 >$ and $M_2 =< 3 >$.

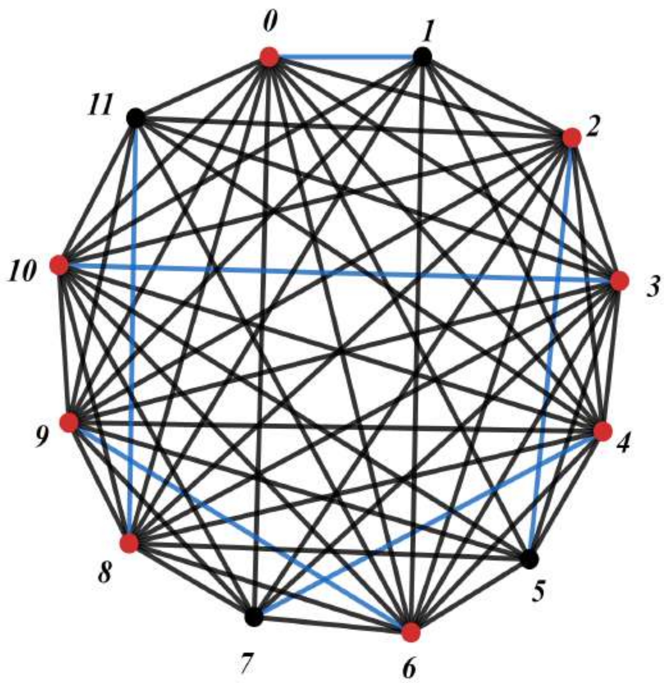

Figure 3: The Graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ with vertex covering and edge covering.

- The subset $\{0, 2, 3, 4, 6, 8, 9, 10\}$ of the vertex set $V(\Gamma_{pm}(R))$ covers all the edges of the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ and it is the minimum vertex cover. Hence, the vertex covering number $\beta(\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})) = 8$ and the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ is shown in Figure 3 with colored vertices.
- The subset $\{(0, 1), (2, 5), (4, 7), (8, 1), (10, 3), (6, 9)\}$ of the edge set $E(\Gamma_{pm}(R))$ covers all the vertices of the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ and it is the minimum edge cover. Hence, the edge covering number $\beta'(\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})) = 6$ and is shown in Figure 3 with colored lines.

3.3 Cordial labeling of the product maximal graph

Cahit [3] has introduced a weeker version of both graceful and harmonious labeling. The following theorem shows that the graph $\Gamma_{pm}(R)$ satisfies sum cordial, signed product cordial and divisor cordial labeling if $n = p^2$, where p is a prime.

Theorem 3.5. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits sum cordial labeling if $p = 2$ or $p \equiv 3(mod\, 4)$, where p is a prime number.*

Proof. Let $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ be the product maximal graph, where $\mathbb{Z}/p^2\mathbb{Z}$ is the finite commutative ring with unity and $< p >$ is the maximal ideal of $\mathbb{Z}/p^2\mathbb{Z}$. Assume that, for $i < j$ then, $v_i < v_j$.

Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ be defined as

$$f(v_i) = \begin{cases} 0, & \text{if } 1 \le i \le \lceil \frac{p^2}{2} \rceil, \\ 1, & \text{if } \lceil \frac{p^2}{2} \rceil + 1 \le i \le p^2. \end{cases}$$

Case 1. $p = 2$. Clearly $v_f(0) = v_f(1) = p$. The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_iv_j) = |f(v_i) + f(v_j)|(mod 2)$, where $1 \le i, j \le p^2$ and $i \ne j$.

We have $e_f(0) = p$ and $e_f(1) = p + 1$. Thus, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

$\therefore$ The product maximal graph satisfies the sum cordial labeling if $p = 2$.

Case 2: $p \equiv 3(mod\, 4)$. Clearly, $v_f(0) = \lceil \frac{p^2}{2} \rceil$ and $v_f(1) = \lfloor \frac{p^2}{2} \rfloor$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_iv_j) = |f(v_i) + f(v_j)|(mod 2)$, where $1 \le i, j \le p^2$ and $i \ne j$. We have $e_f(0) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$ and $e_f(1) = \lceil \frac{p(p-1)(1+2p))}{4} \rceil$. Thus, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

$\therefore$ The product maximal graph satisfies the sum cordial labeling if $p \equiv 3(mod\, 4)$. □

Theorem 3.6. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits signed product cordial labeling if $p = 2$ or $p \equiv 3(mod\, 4)$, where p is a prime number.*

Proof. Assume that $i < j$ then $v_i < v_j$.

Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ be define as

$$f(v_i) = \begin{cases} 1, & \text{if } 1 \le i \le \lceil \frac{p^2}{2} \rceil, \\ -1, & \text{if } \lceil \frac{p^2}{2} \rceil + 1 \le i \le p^2. \end{cases}$$

Case 1. $p = 2$. Clearly $v_f(1) = v_f(-1) = p$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ is given by $f^*(v_iv_j) = f(v_i)f(v_j)$, where $1 \le i, j \le p^2$ and $i \ne j$.

We have $e_f(1) = p$ and $e_f(-1) = p + 1$. Thus, $|v_f(1) - v_f(-1)| \le 1$ and $|e_f(1) - e_f(-1)| \le 1$.

$\therefore$ The product maximal graph satisfies the signed product cordial labeling if $p = 2$.

Case 2. $p \equiv 3(mod\ 4)$. Clearly, $v_f(1) = \lceil \frac{p^2}{2} \rceil$ and $v_f - (1) = \lfloor \frac{p^2}{2} \rfloor$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ is given by $f^*(v_i v_j) = f(v_i)f(v_j)$, where $1 \leq i, j \leq p^2$ and $i \neq j$.

We have $e_f(1) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$ and $e_f(-1) = \lceil \frac{p(p-1)(1+2p)}{4} \rceil$. Thus, $|v_f(1) - v_f(-1)| \leq 1$ and $|e_f(1) - e_f(-1)| \leq 1$.

$\therefore$ The product maximal graph satisfies the signed product cordial labeling if $p \equiv 3(mod\ 4)$. □

Theorem 3.7. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits divisor cordial labeling if $p = 2$ or $p \equiv 3(mod 4)$, where p is a prime number.*

Proof. Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, 2, \ldots, p^2\}$ be define as $f(v_i) = i$, where $1 \leq i \leq p^2$.

Then, the edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_i v_j) = f(v_i)|f(v_j)$ or $f(v_i)|f(v_j)$, where $1 \leq i, j \leq p^2$ and $i \neq j$.

We have $e_f(0) = \lceil \frac{p(p-1)(1+2p)}{4} \rceil$ and $e_f(1) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$.

$\therefore |e_f(1) - e_f(-1)| \leq 1$.

The product maximal graph satisfies the divisor cordial labeling if $p = 2$ and $p \equiv 3(mod\ 4)$. □

4. Cartesian product

In this section, the product maximal graph of the cartisean product $R \times S$ of two finite commutative rings R and S and the cartesian product of the product maximal graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ are studied and its relation is also discussed in the subsequent theorems.

4.1 Product maximal graph of the cartesian product of two rings

Let R and S be two finite commutative rings with unity whose orders are n_1 and n_2 respectively. Then, the cartesian product $R \times S$ of two rings is also a finite commutative ring with unity whose order is $n_1 n_2$. If I and J are the ideals of R and S respectively then every ideal of $R \times S$ is of the form $I \times J$ [1].

Let $M_i, i = 1, 2, \ldots, r$ and $N_j, j = 1, 2, \ldots, s$ be the maximal ideals of R and S respectively then the maximal ideals of $R \times S$ are of the form $R \times N_j$ and $M_i \times S$. Note that the number of maximal ideals of $R \times S$ is equal to the sum of the number of maximal ideals of R and S.

The next theorem explains the isomorphism between the product maximal graph of cartesian product of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ and the product maximal graph $\mathbb{Z}/n_1 n_2\mathbb{Z}$. It is proved that $\mathbb{Z}/n_1 n_2\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ whenever $(n_1, n_2) = 1$ in [5].

Theorem 4.1. *Let $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ be the product maximal graphs of $\mathbb{Z}/n_1n_2\mathbb{Z}$ and $\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}$ respectively then $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ whenever $(n_1, n_2) = 1$.*

Proof. Let $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ be the product maximal graph of $\mathbb{Z}/n_1n_2\mathbb{Z}$ and $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ be the product maximal graph of cartesian product of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$, where $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ are the integers modulo n_1 and n_2 respectively.

Let the vertex mapping $f : V(\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})) \longrightarrow V(\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}))$ be defined as $[x]_{n_1n_2} \longrightarrow ([x]_{n_1}, [x]_{n_2})$, where $[x]_{n_1}$ is the residue class of $x \ mod \ n_1$. Obviously the function f is bijective.

Let $M_1, M_2, \ldots, M_r$ and $N_1, N_2, \ldots, N_s$ be the maximal ideals of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ respectively and $M_1, M_2, \ldots, M_r, N_1, N_2, \ldots, N_s$ be the maximal ideals of $\mathbb{Z}/n_1n_2\mathbb{Z}$.

Let x and y be any two vertices in $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$. If x and y are adjacent in $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ then either x or y is an element of the maximal ideal of $\mathbb{Z}/n_1n_2\mathbb{Z}$. Without loss of generality, we assume that x is an element of the maximal ideal of $\mathbb{Z}/n_1n_2\mathbb{Z}$.

Since f is bijective, $f(x)$ is an element of the maximal ideal of $\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}$. This implies that $f(x)$ is adjacent to all other elements in $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z})$. It means that $f(x)$ and $f(y)$ are adjacent in $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$.

$$\therefore \Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}).$$

□

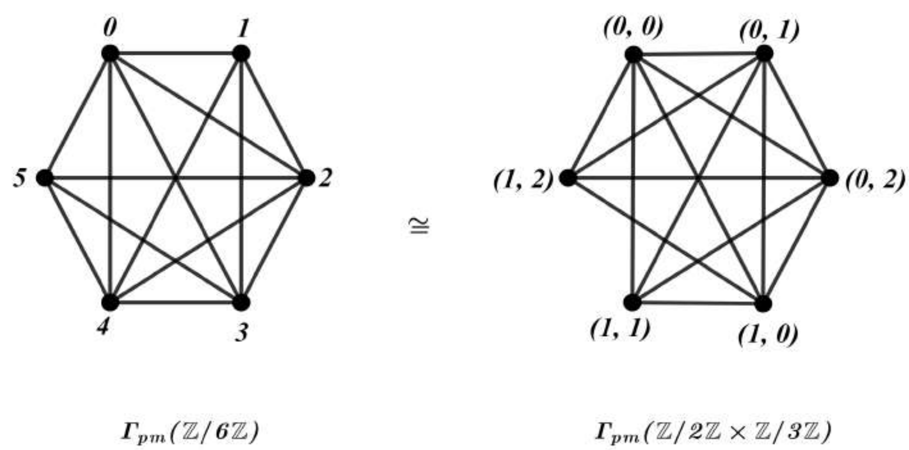

Figure 4: $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$.

Example 4.1. Consider $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$ be the product maximal graphs. The isomorphism between $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$ is shown in Figure 4.

4.2 Cartesian product of two product maximal graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$

In this section, the graph theoretic property like cartesian product of two graphs is explained and it is applied for product maximal graph of the commutative ring.

Theorem 4.2. *Let $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ be the two product maximal graphs of commutative rings R and S respectively. Then, the degree of the vertex (x, y) of the cartesian product $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is $deg(x, y) = deg(x) + deg(y)$, where $x \in \Gamma_{pm}(R)$ and $y \in \Gamma_{pm}(S)$.*

Proof. Let $M_1, M_2, \ldots, M_r$ and $N_1, N_2, \ldots, N_s$ be the maximal ideals of R and S respectively. By Lemma 2.3, the degree of the vertex x of the product maximal graph $\Gamma_{pm}(R)$ is

$$deg(x) = \begin{cases} n_1 - 1, & x \in M_i, i = 1, 2, \ldots r, \\ m, & \text{otherwise}, \end{cases} \tag{1}$$

where $m = |\bigcup_{i=1}^{r} M_i|$. Similarly, the degree of the vertex y of the product maximal graph $\Gamma_{pm}(S)$ is

$$deg(y) = \begin{cases} n_2 - 1, & y \in N_j, i = 1, 2, \ldots, s \\ n, & \text{otherwise}, \end{cases} \tag{2}$$

where $n = |\bigcup_{j=1}^{s} N_j|$.

The degree of the vertex (x, y) of the cartesian product of two graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ is

$$deg(x, y) = \begin{cases} n_1 + n_2 - 1, & x \in M_i, y \in N_j \\ n_1 - 1 + n, & x \in M_i, y \notin N_j \\ m + n_2 - 1, & x \notin M_i, y \in N_j \\ m + n, & x \notin M_i, y \notin N_j. \end{cases} \tag{3}$$

Comparing (1), (2) and (3), we conclude that

$$deg(x, y) = deg(x) + deg(y).$$

□

The following example explains the degree of the cartesian product of two graphs $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Example 4.2. Consider the cartesian product $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ of two product maximal graphs $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. The graphs of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ are shown in Figure 5. Table 1 shows the degrees of some of the vertices (x, y) of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

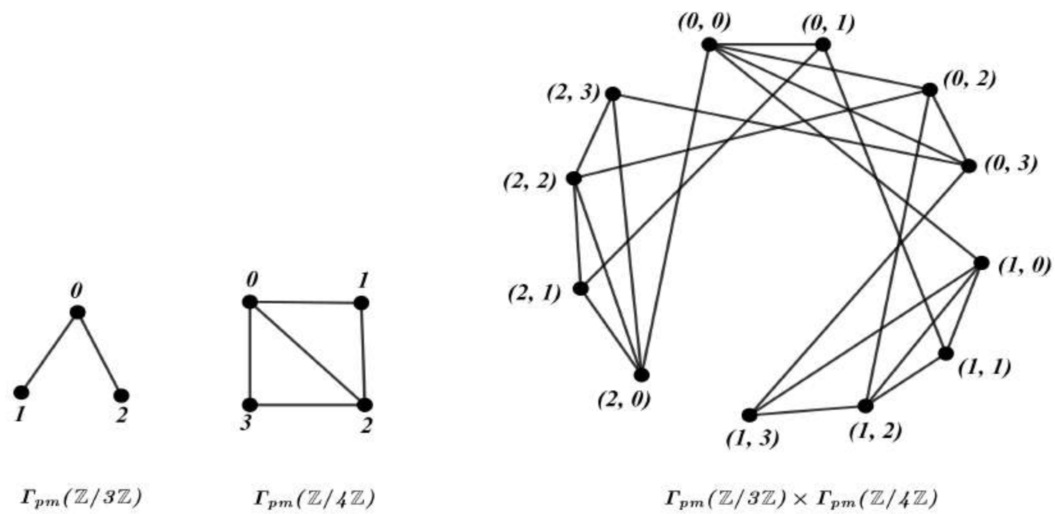

Figure 5: Cartesian product of two product maximal graph $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Table 1: Degrees of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

$\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$	$\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$	$\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$
$deg(0) = 2$	$deg(0) = 3$	$deg(0,0) = 5$
$deg(1) = 1$	$deg(1) = 2$	$deg(1,1) = 3$
$deg(2) = 1$	$deg(2) = 3$	$deg(2,2) = 4$

Theorem 4.3. *Let $\Gamma_{pm}(R \times S)$ be the product maximal graph of cartesian product R and S and $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ be the cartesian product of two product maximal graphs then $\Gamma_{pm}(R \times S)$ is not isomorphic to $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$.*

Proof. Let $M_i, i = 1, 2, \ldots, r$ and $N_j, j = 1, 2, \ldots, s$ be the maximal ideals of R and S respectively and $T_1, T_2, \ldots, T_t$ be the maximal ideals of $R \times S$, where T_k is either in $R \times N_j$ or $M_i \times S$.

Since the number of vertices of $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is $n_1 n_2$ then the degree of the vertex (x, y) in $\Gamma_{pm}(R \times S)$ is

$$deg(x, y) = \begin{cases} n_1 n_2 - 1, & (x, y) \in T_k, k = 1, 2, \ldots, t, \\ l, & \text{otherwise}, \end{cases} \tag{4}$$

where l be the number of elements in $l = |\bigcup_{k=1}^{t} T_k|$. By Theorem 4.2, the degree of the vertex (x, y) in the cartesian product of two product maximal graphs $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is

$$deg(x, y) = deg(x) + deg(y) = n_1 + n_2. \tag{5}$$

But from (4) and (5) the degrees are not equal i.e., the degree of any vertex in $\Gamma_{pm}(R \times S)$ is not the same as the degree of that vertex in $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$. Hence, the graphs $\Gamma_{pm}(R \times S)$ and $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ are not isomorphic. □

Example 4.3. Consider the product maximal graph $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. Then, Figure 6 shows that there is no isomorphism between $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. The degree of every vertex in two graphs are given in the Table 2 .

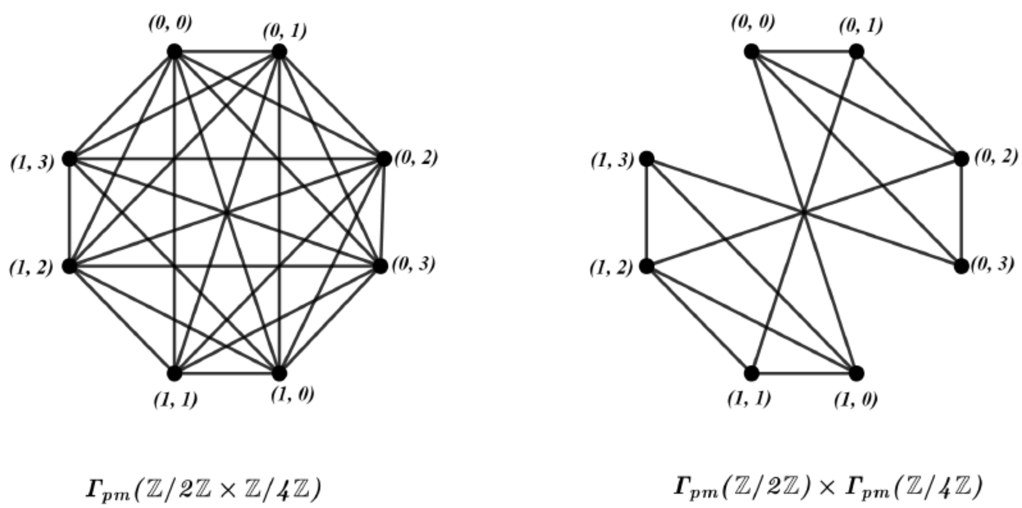

Figure 6: $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ is not isomorphic to $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Table 2: Degrees of $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$

(x, y)	**degree of** $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$	**degree of** $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$
$deg(0,0)$	7	4
$deg(0,1)$	7	3
$deg(0,2)$	7	4
$deg(0,3)$	7	3
$deg(1,0)$	7	4
$deg(1,1)$	6	3
$deg(1,2)$	7	4
$deg(1,3)$	6	3

Conclusion

In this paper, the product maximal graph and its graph theoretic properties like matching, covering and some cordial labeling are studied. Also, the relation between the product maximal graph of cartesian product of finite commutative ring and cartesian product of two product maximal graphs are discussed with theorems and suitable examples.

References

[1] D.D. Anderson, *Commutative rings*, Springer, 2006.

[2] I. Beck, *Coloring of commutative rings*, Journal of Algebra, 116 (1988), 208-226.

[3] I. Cahit, *Cordial graph: a weaker version of gracful and harmonious graph*, Ars Combinatorial, 23 (1987), 201-207.

[4] David F. Anderson, T. Asir, Ayman Badawi, T. TamizhChelvam, *Graphs from rings*, Springer, 2021.

[5] David S. Dummit, Richard M. Foote, *Abstract algebra*, John Wiley and Sons Inc, Third Edition, 2004.

[6] Douglas B. West, *Introduction to graph theoy*, Pearson, Second Edition, 2007.

[7] F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass, 1972.

[8] Joseph A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 2016.

[9] D. Kalamani, G. Ramya, *Graph theoretic properties for* $\Gamma_{pm}(R)$ *and resistance distance based indices*, Advances and Application in Mathematical Science, 21 (2022), 3213-3231.

[10] D. Kalamani, G. Ramya, *Product maximal graph of a finite commutative ring*, Bulletin of Calcutta Mathematical Society, 113 (2021), 127-134.

[11] D. Kalamani, G. Kiruthika, *Subdivision vertex corona and subdivision vertex neighbourhood corona of cyclic graphs*, Advances in Mathematics: Scientific journal, 9 (2020), 607-617.

[12] G. Kiruthika, D. Kalamani, *Computational Zagrab indices of* K*-Gamma graphs*, International Journal of Research in Advent Technology, 7 (2019), 413-417.

[13] G. Kiruthika, D. Kalamani, *Degree based partition of the power graphs of a finite abelian group*, Malaya Journal of Matematik, 1 (2020), 66-71.

[14] G. Ramya, D. Kalamani, *Domination number of the product maximal graph of the finite commutative ring*, Southeast Asian Journal Mathematics and Mathematical Sciences, Proceeding, 18 (2022), 187-194.

[15] G. Ramya, D. Kalamani, *Some cordial labelling for commuting graph of a subset of the dihedral group*, Malaya Journal of Matematik, 9 (2021), 461-464.

Accepted: March 23, 2023